Algebraic & Geometric Topology 17 (2017) 3213–3257

3–manifolds built from injective handlebodies

JAMES COFFEY
HYAM RUBINSTEIN

This paper studies a class of closed orientable 3–manifolds constructed from a gluing of three handlebodies, such that the inclusion of each handlebody is π_1–injective. This construction is the generalisation to handlebodies of the construction for gluing three solid tori to produce non-Haken Seifert fibred 3–manifolds with infinite fundamental group. It is shown that there is an efficient algorithm to decide if a gluing of handlebodies satisfies the disk-condition. Also, an outline for the construction of the characteristic variety (JSJ decomposition) in such manifolds is given. Some non-Haken and atoroidal examples are given.

57N10, 57M10, 57M50

1 Introduction

This paper is concerned with the class of 3–manifolds that meet the disk-condition. These are closed orientable 3–manifolds constructed from the gluing of three handlebodies, such that the induced map on the fundamental group of each of the handlebodies is injective. Thus all manifolds that meet the disk-condition have infinite fundamental group. The disk-condition is an extension to handlebodies of conditions for the gluing of three solid tori to produce non-Haken Seifert fibred manifolds with infinite fundamental group. These manifolds appear to have many nice properties. In this paper, some tools for understanding manifolds that meet the disk-condition are investigated. A number of constructions are given for this class, including some manifolds that are non-Haken and some that are atoroidal. The characteristic variety of manifolds that meet the disk-condition is also investigated. It is shown that the handlebody structure carries all the information for building the characteristic variety.

In Section 2, standard definitions that are used throughout this paper are given. Also, the "disk-condition" is defined and discussed. In particular, it is shown how this condition is a generalisation of the construction of non-Haken Seifert fibred manifolds with infinite fundamental group. We also discuss how, on an intuitive level, the class of manifolds that meet the disk-condition contains many other non-Haken examples.

Published: 4 October 2017 DOI: 10.2140/agt.2017.17.3213

Section 3 is divided into three subsections. The first develops some basic tools and also shows that all 3–manifolds that meet the disk-condition have infinite fundamental group and are irreducible. In the second subsection, a sufficient condition is given for gluings of handlebodies to meet the disk-condition. This condition is easily checked and useful for constructing examples. We then give a necessary and sufficient condition and an algorithm that can be checked in bounded time. The final part gives some constructions of manifolds that meet the disk-condition, using Dehn fillings along knots in $\mathbb{S}^3$ and n–fold cyclic branched covers of knots in $\mathbb{S}^3$. Some non-Haken examples are produced.

Section 4 is concerned with the construction of the characteristic variety Σ in a manifold M that satisfies the disk-condition. The main theorem proved in Section 4 is:

Theorem 1.1 *Let M be a closed orientable 3–manifold that satisfies the disk-condition, and let T be a torus. If $f\colon T \to M$ is a π_1–injective map, then there is $\Sigma \subseteq M$ a Seifert fibred submanifold with essential boundary and a map $g\colon T \to M$ homotopic to f such that $g(T) \subset \Sigma$.*

If the characteristic variety Σ has nonempty boundary, then the boundary components are essential embedded tori. Therefore, a direct corollary of the above theorem is:

Corollary 1.2 *If M is a closed orientable 3–manifold that satisfies the disk-condition and there is a π_1–injective map of the torus into M, then either there is a π_1–injective embedding of a torus in M, or M is a non-Haken Seifert fibred manifold.*

These are not new results. However, the aim is to examine how the characteristic variety behaves in manifolds that meet the disk-condition. The proof of the torus theorem (Theorem 1.1) is constructive and gives an algorithm for finding the characteristic variety of manifolds that meet the disk-condition. In the construction of the characteristic variety, the components come in two "flavours". The intersection of all three handlebodies in the manifold is a set of injective simple closed curves, called the triple curves. The first flavour is a component which is disjoint from the triple curves. These components are similar to the constructions used by W Jaco and P Shalen to prove the torus theorem for Haken manifolds; see Jaco [6]. The intersections of the components of the characteristic variety with each handlebody are either essential Seifert fibred submanifolds or I–bundles. If we remove an open neighbourhood of the triple curves, we get a manifold with incompressible boundary, which is therefore Haken. What remains of the boundaries of the handlebodies after the triple curves are removed is a set of disjoint spanning surfaces. Therefore, the fact that these carry all the information for the characteristic variety components disjoint from the triple curves is not surprising.

We will refer to the second flavour of characteristic variety as the disk components. The intersections of these disk components with the handlebodies are regular neighbourhoods of intersecting meridian disks. For this flavour of characteristic variety components to occur, the manifold must meet a minimal disk-condition, as described in Section 2. The two flavours of characteristic variety components are not necessarily disjoint. If two such components intersect, their fibrings can always be made to agree. In fact, when they intersect, the disk components are thickened compressing annuli of the characteristic variety components disjoint from the triple curves.

Acknowledgements The authors would like to thank Ian Agol for a very helpful comment on this project. We would also like to thank the referee for extremely diligent work, which has greatly improved this paper. This research was partially supported by the Australian Research Council.

2 Definitions and preliminaries

Throughout this paper, we will assume that, unless stated otherwise, we are working in the PL category of manifolds and maps. We will use standard PL constructions, such as regular neighbourhoods and transversality, defined by C Rourke and B Sanderson in [12]. Other definitions relating to 3–manifolds are given by J Hempel in [5] or Jaco in [6].

A manifold M is *closed* if it is compact and $\partial M = \varnothing$. Also, M is *irreducible* if every embedded S^2 bounds a ball. We will assume, unless otherwise stated, that all 3–manifolds are orientable. The reason for this is that all closed nonorientable $\mathbb{P}^2$–irreducible 3–manifolds are Haken. (A manifold is $\mathbb{P}^2$–irreducible if it is irreducible and does not contain any embedded 2–sided projective planes.) A main motivation for our approach is to find constructions of non-Haken 3–manifolds.

A map $f\colon S \to M$ is *proper* if $f^{-1}(\partial M) = \partial S$. If $F\colon S \times I \to M$ is a homotopy/isotopy such that $F|_{S\times 0}$ is a proper map, then it is assumed, unless otherwise stated, that $F|_{S\times t}$ is a proper map for all $t \in I$. To simplify notation, an isotopy/homotopy of a surface $S \subset M$ is used without defining the map. Here we are assuming that there is a map $f\colon S \to M$, and we are referring to an isotopy/homotopy of f. If M is a 3–manifold and S is a compact surface which is not a sphere, disk or projective plane, the proper map $f\colon S \to M$ is called π_1*–injective* if the induced map $f_*\colon \pi_1(S) \to \pi_1(M)$ is injective. If a π_1–injective map f is not homotopic as a map of pairs $(S, \partial S) \to (M, \partial M)$ into ∂M, then the map is called *essential*.

If H is a handlebody and D is a properly embedded disk in H such that ∂D is essential in ∂H, then D is a *meridian disk* of H. If D is a proper singular disk in H such that ∂D is essential in ∂H, then it is called a *singular meridian disk.*

In this paper, normal curve theory, as defined by S Matveev in [9], is used to list finite classes of curves in surfaces. A triangulation of the surface is required to define normal curves. The surfaces may have polygonal faces. However, a barycentric subdivision will produce the required triangulation.

2.1 The disk-condition

Before we discuss the disk-condition in closed 3–manifolds, we define some useful objects and the disk-condition in handlebodies.

Definition 2.1 Let H be a handlebody, $\mathcal{T}$ a set of curves in ∂H and D a meridian disk. Assuming that ∂D and $\mathcal{T}$ are transverse, $|D|$ will denote the number of intersection points of ∂D and $\mathcal{T}$.

Definition 2.2 If H is a handlebody and $\mathcal{T}$ is a set of essential disjoint simple closed curves in ∂H, then $\mathcal{T}$ satisfies the n *disk-condition* in H if $|D| \geq n$ for every meridian disk D.

This seems a difficult condition to verify, for if H has genus two or higher, there are an infinite number of meridian disks to check. However, later we give some sufficient conditions that are easily checked and an algorithm that determines if the disk-condition is satisfied.

Next we give a construction of 3–manifolds that meet the disk-condition. Please note that even though this description is technically correct, it is not enlightening, so later we discuss different ways of describing these manifolds that are much more useful.

Let H_1, H_2 and H_3 be three handlebodies. Let $S_{i,j}$, for $i \neq j$, be a subsurface of ∂H_i such that:

(1) $\partial S_{i,j} \neq \varnothing$.

(2) The induced map of $\pi_1(S_{i,j})$ into $\pi_1(H_i)$ is injective.

(3) $S_{i,j} \cup S_{i,k} = \partial H_i$ for $j \neq k$.

(4) $\mathcal{T}_i = S_{i,j} \cap S_{i,k} = \partial S_{i,j} = \partial S_{i,k}$ is a set of disjoint essential simple closed curves that meet the n_i disk-condition in H_i.

(5) $S_{i,j} \subset \partial H_i$ is homeomorphic to $S_{j,i} \subset \partial H_j$.

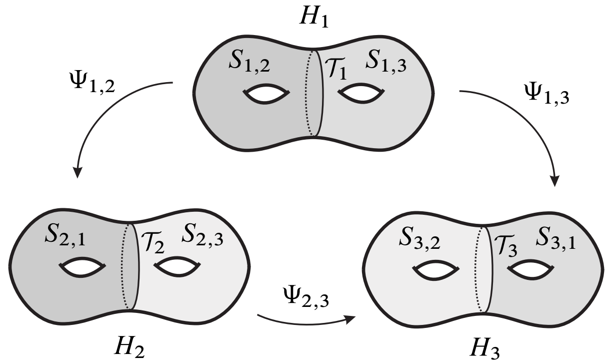

Figure 1: Homeomorphisms between boundaries of handlebodies

Note that $S_{i,j}$ need not be connected. Given that the boundary of each handlebody is cut up into π_1–injective regions, we glue the handlebodies together by homeomorphisms $\Psi_{i,j}\colon S_{i,j} \to S_{j,i}$ that agree along the $\mathcal{T}_i$; see Figure 1. The result is a closed 3–manifold M for which the image of each handlebody is embedded.

Definition 2.3 If M is a manifold constructed from three handlebodies as above such that $\mathcal{T}_i$ satisfies the n_i disk-condition in H_i and

$$\sum_{i=1,2,3} \frac{1}{n_i} \leq \frac{1}{2}, \tag{1}$$

then M satisfies the (n_1, n_2, n_3) *disk-condition*. M is simply said to meet the *disk-condition* if the specific (n_1, n_2, n_3) is understood from the context.

As said previously, the above definition is not very enlightening. Thus, from now on, we view 3–manifolds that meet the disk-condition in the following way. Assume that M is a manifold that satisfies the disk-condition and H_1, H_2 and H_3 are the images of the handlebodies in M. Then $M = \bigcup_{i=1,2,3} H_i$, and each H_i is embedded in M. Then $X = \bigcup_{i=1,2,3} \partial H_i$ cuts M up into handlebodies. X can be viewed as a 2–complex by splitting up each of the surfaces forming X into cells. Also, $\mathcal{T} = \bigcap_{i=1,2,3} H_i$ is a set of essential disjoint simple closed curves in M that satisfies the n_i disk-condition in H_i where $\sum_{i=1,2,3} 1/n_i \leq \frac{1}{2}$.

It may seem confusing that we are using the same name for the conditions for the construction of 3–manifolds and the curves in the boundary of handlebodies. However, the curve condition is the restriction of the condition on closed 3–manifolds to each of its component handlebodies. When we have an equality in (1), the result is the three "minimal" cases for the disk-condition. These are: $(6,6,6)$, $(4,8,8)$ or $(4,6,12)$. These three cases are of special interest since if a manifold satisfies the disk-condition,

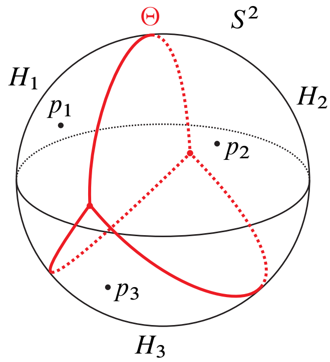

Figure 2: Base space of non-Haken Seifert fibred space with infinite π_1

then it meets at least one of these three conditions. Therefore, these are the key cases to consider. It is also worth noting that unlike Heegaard splittings, we don't require the three handlebodies to have the same genera.

Another way of viewing a 3–manifold M that satisfies the disk-condition is that $X = \bigcup \partial H_i$ is a 2–complex such that the triple curves $\mathcal{T}$ consist of essential curves in X. Therefore, we obtain a manifold M that satisfies the disk condition by gluing handlebodies to X such that each meridian disk of the handlebodies intersects $\mathcal{T}$ enough times. In fact, the disk-condition is an extension of the construction of non-Haken Seifert fibred 3–manifolds with infinite fundamental group. In the latter case, if a Seifert fibred space is non-Haken with infinite fundamental group, then it has a fibring with base space a 2–sphere, and it has three exceptional fibres of multiplicity p_i, where $\sum 1/p_i \leq 1$ (*), as in Figure 2. For more details, see P Scott in [13]. If the inequality (*) is made an equality, the exceptional fibres have indices $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$. Another way of viewing this construction is if Θ is the graph in S^2 shown in Figure 2, then $\Theta \times \mathbb{S}^1$ is a 2–complex X consisting of three annuli glued together along two triple curves $\mathcal{T}$. Then glue three solid tori H_i to X so that the boundaries of the meridian disks meet each triple curve p_i times. As there are two triple curves in $\mathcal{T}$, each meridian disk has $2p_i$ intersections with $\mathcal{T}$. Thus, as $\sum 1/(2p_i) \leq \frac{1}{2}$, all non-Haken Seifert fibred manifolds with infinite π_1 are in the class of manifolds that meet the disk-condition.

Yet another way of viewing 3–manifolds that meet the disk-condition is if we glue two handlebodies together to form a 3–manifold with a single incompressible boundary component. Then glue a handlebody to this boundary component. A very short hierarchy in a closed Haken manifold, as defined by I Aitchison and H Rubinstein in [1], can be built from a set of handlebodies, gluing each handlebody to itself so that each of the resulting manifolds has incompressible boundary. Then glue these incompressible boundaries together to produce the closed manifold. So the incompressible boundaries become incompressible surfaces in the Haken manifold. This suggests that the disk-condition is a weaker condition than the manifold being Haken. In fact, we already know that the class of manifolds satisfying the disk-condition contains all non-Haken Seifert fibred manifolds with infinite π_1, but it also contains examples of other non-Haken manifolds.

The disk-condition can be easily extended to gluings of four or more handlebodies such that all the statements in this paper follow. Construct a closed manifold M by gluing together $r \geq 3$ handlebodies $H_1, \dots, H_r$ such that, for i, j, k and l different,

- H_i is embedded,
- $H_i \cap H_j \subset \partial H_i \cap \partial H_j$ is a subsurface,
- $H_i \cap H_j \cap H_k$ is a possibly empty set of pairwise disjoint curves, and
- $H_i \cap H_j \cap H_k \cap H_l = \varnothing$.

Then $X = \bigcup_{1 \leq i < j \leq r} H_i \cap H_j$ is a 2–complex which cuts M up into the H_i, and $\mathcal{T} = \bigcup_{1 \leq i < j < k \leq r} H_i \cap H_j \cap H_k$ is a union of pairwise disjoint simple closed curves. Suppose α is a component of $\mathcal{T}$. Let H_{α_1}, H_{α_2} and H_{α_3} be the three handlebodies around α and suppose that $\mathcal{T}$ satisfies the n_{α_i} disk-condition in H_{α_i}. Then M satisfies the *generalised disk-condition* if $\sum_{i=1,2,3} 1/n_{\alpha_i} \leq \frac{1}{2}$ for each $\alpha \in \mathcal{T}$. For the purposes of this paper, we will not consider such manifolds for $r \geq 4$ as they are all Haken. To see this, if $r \geq 4$, then we can choose H_i and H_j such that $H_i \cap H_j \neq \varnothing$ and there is a component M' of $\overline{M - (H_i \cup H_j)}$ that contains at least two of the handlebodies. Let S be the boundary surface between $H_i \cup H_j$ and M'. Then the proof of Lemma 3.2 can be modified to show that no essential simple closed curve in S bounds a disk, and thus S is an embedded incompressible surface. Therefore, the manifold is Haken as claimed.

3 Conditions and examples

For later use, we state a special case of Dehn's lemma and the loop theorem:

Lemma 3.1 *Let H be a handlebody and $\mathcal{T}$ a collection of essential curves in ∂H. If there is a singular meridian disk D of H such that D has n intersections with $\mathcal{T}$, then there exists an embedded meridian disk of H that intersects $\mathcal{T}$ at most n times.*

Let H be a handlebody and $\mathcal{T}$ be a set of disjoint essential simple closed curves in ∂H that satisfies the n disk-condition. A direct result of this lemma is that if α is a possibly singular loop in ∂H that intersects $\mathcal{T}$ less than n times and α contracts in H, then by Lemma 3.1 it follows that α is inessential in ∂H.

Lemma 3.2 *Let M be a manifold that satisfies the disk-condition. If $f\colon D \to M$ is a map of a disk D such that $f(\partial D) \subset \mathrm{int}(H_i)$ for some i, then f can be homotoped to g, keeping the boundary fixed, so that $g(D) \subset \mathrm{int}(H_i)$.*

Proof We can assume that $f(D)$ is transverse to X, where X is the union of the boundaries of the three handlebodies making up M and f is the disk map as in the lemma. Thus $\Gamma = f^{-1}(X)$ is a set of trivalent graphs and simple closed curves Γ_j, $1 \le j \le m$, in D. Note that $\partial D \cap \Gamma = \varnothing$. An *innermost* component of Γ is a component Γ_j such that there is a subdisk $D^* \subset D$ where $\partial D^* \subset \Gamma_j$ and $D^* \cap \Gamma = \Gamma_j$. An easy argument shows that if Γ is nonempty, then it must have at least one innermost component. The reason is that the closure of a component of the complement of Γ_j which does not contain ∂D is a subdisk D'. Clearly we can define a partial order on the components of Γ by $\Gamma_r < \Gamma_j$ if Γ_j has a complementary component which does not meet ∂D and contains Γ_r. A smallest component is then innermost.

If Γ_j is a simple loop, then $\Gamma_j = \partial D'$ and $f(D') \subset H_k$ for $k = 1, 2$ or 3. By the disk-condition, we know that $f(\partial D')$ must be nonessential in ∂H_k as $f(\partial D')$ doesn't intersect $\mathcal{T}$ and thus $f(D')$ is homotopic into ∂H_k. We can thus homotope f so that $f(D') \subset \partial H_k$ and then push $f(D')$ through to remove the component Γ_j altogether.

If Γ_j is a graph, then as it is innermost, there is a disk D^* with $\partial D^* \subset \Gamma_j$ and $\Gamma_j = \Gamma \cap D^*$. Thus any face F bounded by a subset of Γ_j in D^* is an (m,n)–gon, where F has m vertices in its boundary and is mapped by f to a handlebody H_k such that $\mathcal{T}$ satisfies the n disk-condition in H_k. We can put a PL metric on D^* by assuming that all the edges are geodesic arcs of unit length, that the internal angle at each vertex of an (m,n)–gon F is $\pi(1-2/n)$ and all the curvature of F is at a cone point in $\mathrm{int}(F)$. For example, if H_k satisfies the 6 disk-condition, the angle at each corner of an $(m,6)$–gon will be $\frac{2\pi}{3}$. Note that as each vertex of Γ_j in the interior of D^* is adjacent to three faces, each of these faces is mapped to a different handlebody. Assuming that M satisfies the $(6,6,6)$, $(4,6,12)$ or $(4,8,8)$ disk-conditions, then the total angle around each such interior vertex is 2π. If F is an (m,n)–gon, then $\chi(F) = 1$ and the exterior angle sum is $m(2\pi/n)$. If $\boldsymbol{K}(F)$ is the curvature of the cone point in $\mathrm{int}(F)$, then by the Gauss–Bonnet theorem,

$$\boldsymbol{K}(F) = 2\pi - m(2\pi/n) = 2\pi(1-m/n).$$

Thus if F is an (m,n)–gon and $m<n$, then $\boldsymbol{K}(F) > 0$, and if $m \ge n$, then $\boldsymbol{K}(F) \le 0$. Let $\boldsymbol{F}$ be the set of faces of D^* and $\boldsymbol{v}$ be the vertices in ∂D^*. For $v \in \boldsymbol{v}$, there are two faces $F_1, F_2 \in \boldsymbol{F}$ adjacent to v. Let F_i be an (m_i, n_i)–gon. Let the *jump* angle at v be $\theta_v = \pi - \sum_{i=1,2} \pi(1-2/n_i)$. By the disk-condition, $n_i = 4, 6, 8$ or 12, and it is

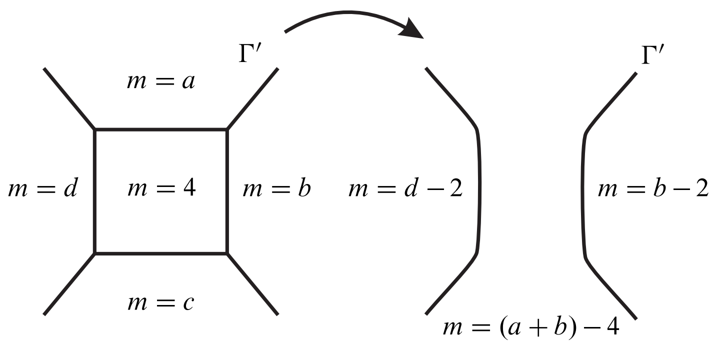

Figure 3: Removing a $(4, n)$–gon from Γ' by homotopy

not possible to have $n_1 = n_2 = 4$. Thus $\theta_v \leq -\frac{\pi}{6}$. Then once again by Gauss–Bonnet we know that

$$\sum_{F \in \boldsymbol{F}} \boldsymbol{K}(F) = 2\pi - \sum_{v \in \boldsymbol{v}} \theta_v > 2\pi.$$

This implies that D^* must always have some (m, n)–gon faces such that $m < n$. For example, if the manifold satisfies the $(6, 6, 6)$ disk-condition, then D^* would have some $(2, 6)$–gons and/or some $(4, 6)$–gons, since m is even. If F is an (m, n)–gon of D^* such that $m < n$ and $f(F) \subset H_k$, then by the disk-condition and Lemma 3.1, we know that $f(\partial F)$ is not essential in ∂H_k. Thus we can homotope f so that $f(F)$ lies in ∂H_k. We can then homotope f so $f(F)$ is pushed off ∂H_k. This decreases the total number of faces of D^*, as shown in Figure 3. Thus in a finite number of steps, Γ_j will become a simple closed curve, and we can then homotope f to remove the component Γ_j entirely.

As Γ always contains an innermost component, we can continue this process until all of Γ has been removed, and thus $f(D) \subset \operatorname{int}(H_i)$. □

This lemma yields important corollaries about 3–manifolds that meet the disk-condition.

Corollary 3.3 *Let M be a 3–manifold that satisfies the disk-condition. Then, for any $1 \leq i \leq 3$, the induced map of $\pi_1(H_i)$ into $\pi_1(M)$ is injective.*

Remark 3.4 Note that $\pi_1(H_i)$ is the free group on g generators, where $g > 0$ is the genus of H_i. This corollary implies that if a 3–manifold satisfies the disk-condition, then its fundamental group is infinite.

Proof Let D be a disk and γ be a simple closed curve in H_i that represents a nontrivial element of $\pi_1(H_i)$. If the element is trivial in $\pi_1(M)$, then there is a map $f \colon D \to M$ such that $f(\partial D) = \gamma$. By Lemma 3.2, we can homotope f so that $f(D) \subset \operatorname{int}(H_i)$, giving us a contradiction. □

Corollary 3.5 *If M is a 3–manifold that satisfies the disk-condition, it is irreducible.*

Proof Let S be a 2–sphere and $f\colon S \to M$ be an embedding. Note that f is an embedding and all the moves in the proof of Lemma 3.2 can be performed as isotopies. Thus we can isotope f so that $f(S) \cap X = \varnothing$; that is, for some i, $f(S) \subset H_i$. Then, as handlebodies are irreducible, $f(S)$ must bound a 3–ball. □

3.1 Test for the n disk-condition in handlebodies

It is not necessary to check every meridian disk of a handlebody H to find out if a set of curves $\mathcal{T}$ in ∂H satisfies the n disk-condition. Let $\mathcal{D}$ be a set made up of a single representative from each isotopy class of meridian disk of H.

The first test is that $\mathcal{T}$ must separate ∂H into subsurfaces that can be 2–coloured. Therefore, all meridian disks must intersect $\mathcal{T}$ an even number of times. From this point on we will assume that $\mathcal{T}$ is separating in ∂H.

Put a Riemannian metric on ∂H. We will assume that the loops in $\mathcal{T}$ are length minimizing geodesics. Note that if $\mathcal{T}$ contains parallel curves, the neighbourhood of the corresponding length minimizing geodesic can be "flattened", so we can have parallel length minimizing geodesics. We will also assume the boundaries of the disks in $\mathcal{D}$ are length minimizing geodesics. Both of these can be done simultaneously. From M Freedman, J Hass and Scott [2], we know that this implies that the number of intersections between the boundary of a disk in $\mathcal{D}$ and $\mathcal{T}$ is minimal, as is the intersection between the boundaries of any two disks in $\mathcal{D}$, after possibly a small perturbation to make these intersections transverse. For any disk $D \in \mathcal{D}$, let $|D|$ be the number of intersections of ∂D with $\mathcal{T}$ and for any set of meridian disks $\boldsymbol{D} = \{D_i\} \subset \mathcal{D}$, let $|\boldsymbol{D}| = \sum_i |D_i|$. From this point on, unless otherwise stated, when discussing meridian disks, we will assume that the number of intersections between their boundaries is minimal.

Lemma 3.6 *Any two disks of $\mathcal{D}$ can be isotoped, leaving their boundaries fixed, so that any curves of intersection are properly embedded arcs.*

Proof This proof uses the standard innermost argument and the fact that handlebodies are irreducible to remove all the components of intersection between two disks that are simple closed curves. □

Definition 3.7 Let H be a genus-g handlebody. We shall call $\boldsymbol{D} \subset \mathcal{D}$ a *system of meridian disks* if all the disks are disjoint, nonparallel and cut H up into a set of 3–balls. If $\partial \boldsymbol{D}$ cuts ∂H up into $2g-2$ pairs of pants (thrice punctured 2–spheres), then it is a *basis* for H.

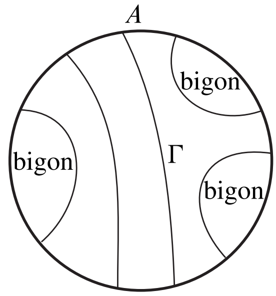

Figure 4: Meridian disk cut up by arcs of intersection

If H has genus g, then a minimal system of meridian disks for H consists of g disjoint meridian disks which cut H up into a single ball.

Definition 3.8 Let P be a punctured sphere and γ be a properly embedded arc in P. If both ends of γ are in one component of ∂P and the arc is not isotopic into ∂P, then it is called a *wave*.

Let H be a handlebody, $\mathcal{T}$ a set of essential disjoint simple closed curves in ∂H, $\boldsymbol{D}$ a system of meridian disks for H and $\{P_1, \dots, P_l\}$ the resulting set of punctured spheres produced when we cut ∂H along $\partial \boldsymbol{D}$. Also, let $\mathcal{T}_i = P_i \cap \mathcal{T}$. Thus $\mathcal{T}_i$ is a set of properly embedded disjoint arcs in P_i.

Definition 3.9 If each $\mathcal{T}_i$ contains no waves, then $\boldsymbol{D}$ is said to be a *waveless* system of meridian disks for H.

Definition 3.10 Let $\boldsymbol{D}$ be a waveless system of disks. If every wave in each P_i intersects $\mathcal{T}_i$ at least $\frac{1}{2}n$ times, then $\boldsymbol{D}$ is called an *n–waveless* system of meridian disks.

If $\boldsymbol{D}$ is an n–waveless basis, then each $\mathcal{T}_i$ has at least $\frac{1}{2}n$ parallel arcs running between each pair of boundaries in P_i.

Lemma 3.11 *Let H be a handlebody, $\mathcal{T} \subset \partial H$ a separating set of essential simple closed curves and $\boldsymbol{D}$ a basis for H. If $\boldsymbol{D}$ is an n–waveless basis, then $\mathcal{T}$ satisfies the n disk-condition in H.*

Proof From the definition of the n–waveless condition we know that $\mathcal{T}$ intersects each disk in $\boldsymbol{D}$ at least $\frac{3}{2}n$ times. If $C \in \mathcal{D}$ is a meridian disk not in $\boldsymbol{D}$, then $C \cap \boldsymbol{D} \neq \varnothing$. By Lemma 3.6, we can isotope C so that $C \cap \boldsymbol{D}$ is a set of disjoint properly embedded arcs. Therefore, if we cut C along $C \cap \boldsymbol{D}$ the faces produced must all be disks and contain at least two bigons, as shown in Figure 4. Therefore, the set $\{P_i \cap \partial C\}$ must contain

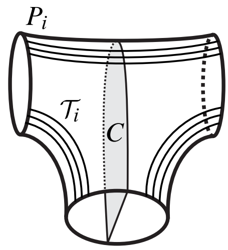

Figure 5: Bigon in a pair of pants

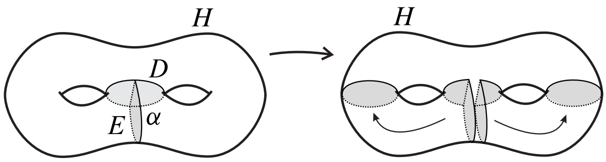

Figure 6: Boundary compressing a meridian disk

at least two waves, coming from bigons. As $\boldsymbol{D}$ satisfies the n–waveless condition, any wave must intersect $\mathcal{T}$ at least $\frac{1}{2}n$ times; see Figure 5. Therefore, ∂C must intersect $\mathcal{T}$ at least n times. □

If $\mathcal{T}$ intersects each disk in $\boldsymbol{D}$ exactly n times, then it must be an n–waveless basis. The reason is that the only pattern of arcs in a pair of pants, where there are the same number n of endpoints on each boundary curve, consists of $\frac{1}{2}n$ arcs joining each pair of boundary loops. This gives us the following corollary.

Corollary 3.12 *Let H be a handlebody, $\mathcal{T} \subset \partial H$ a separating set of simple closed curves and $\boldsymbol{D}$ a basis for H. If $\mathcal{T}$ intersects each disk in $\boldsymbol{D}$ exactly n times, then $\mathcal{T}$ satisfies the n disk-condition in H.*

This test for the n disk-condition is a significant restriction. However, it is an easy enough condition to verify when constructing examples.

Next we describe a specific type of surgery of meridian disks. Let D be a meridian disk of H and let E be an embedded disk in H such that $\partial E \subset D \cup \partial H$, $\partial E \cap \partial D$ is two points, a_1 and a_2 in ∂H, $\alpha = E \cap \partial H$ is an arc in ∂H which is not homotopic through ∂H into ∂D and $D \cap E$ is an arc properly embedded in D, as shown in Figure 6. If we then surger D along E, we produce two disks. As α is an arc which is not homotopic through ∂H into ∂D, both resulting disks are meridian disks isotopic to disks in $\mathcal{D}$. We shall call this surgery a *boundary compression* of a meridian disk.

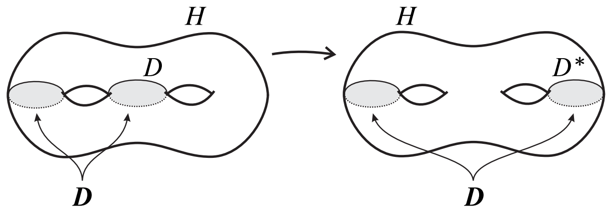

Figure 7: Disk-swap move

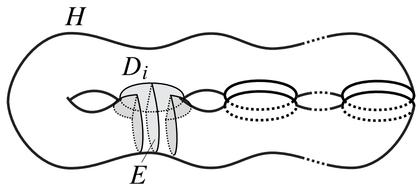

Figure 8: Boundary compressing a disk from a system of meridian disks

Let $\boldsymbol{D}$ be a system of disks for the handlebody H. Let $D^* \in \mathcal{D}$ be a meridian disk disjoint from $\boldsymbol{D}$ such that $(\boldsymbol{D} \setminus D) \cup D^*$ is a system of meridian disks for some $D \in \boldsymbol{D}$. Then if we remove D from $\boldsymbol{D}$ and replace it with D^*, this is called a *disk-swap move* on $\boldsymbol{D}$ as shown in Figure 7.

Lemma 3.13 *For a minimal system of meridian disks $\boldsymbol{D} = \{D_1, \dots, D_n\}$, if we perform a boundary compression on any D_i along a disk disjoint from $\boldsymbol{D} \setminus \{D_i\}$, then one of the resulting disks can be used for a disk-swap move on $\boldsymbol{D}$ removing D_i.*

Remark 3.14 Note that an essential wave in $\overline{\partial H - \boldsymbol{D}}$ defines a disk-swap move on $\boldsymbol{D}$.

Proof Let $\boldsymbol{D}^*$ be the set of all meridian disks disjoint from $\boldsymbol{D}$. Then if a disk $D_i \in \boldsymbol{D}$ is boundary compressed along a disk E disjoint from $\boldsymbol{D} - D_i$, one of the resulting disks will be isotopic to a disk in $\boldsymbol{D} \cup \boldsymbol{D}^*$. If we cut H along $\{D_1, \dots, D_{i-1}, D_{i+1}, \dots, D_n\}$ the result is a solid torus T. Then D_i is a meridian disk of T. Thus a boundary compression on D_i along E will produce two disks, one of which is a meridian disk of T and the other is boundary parallel, as shown in Figure 8. □

Let $\boldsymbol{D} \subset \mathcal{D}$ be a minimal system of meridian disks for the handlebody H. That is, $\boldsymbol{D}$ cuts H up into a single ball. Let $\boldsymbol{D}^* \subset \mathcal{D}$ be the set of disks disjoint from $\boldsymbol{D}$.

Lemma 3.15 *$\mathcal{T}$ satisfies the n disk-condition if and only if there is a minimal system of meridian disks $\boldsymbol{D}$ such that $|D| \geq n$ for all disks $D \in \boldsymbol{D} \cup \boldsymbol{D}^*$ and there are no disk-swap moves between $\boldsymbol{D}$ and $\boldsymbol{D}^*$ that reduce $|\boldsymbol{D}|$.*

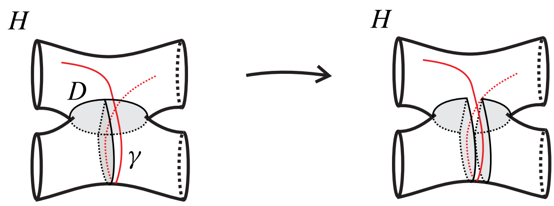

Figure 9: Boundary compression to remove a wave

Proof In the "only if" direction, $\mathcal{T}$ satisfying the n disk-condition in H implies that $|D| \geq n$ for any meridian disk. Given any initial $\boldsymbol{D} \cup \boldsymbol{D}^*$ such that there are disk-swap moves to reduce $|\boldsymbol{D}|$, we can construct a sequence of disk-swaps that reduce $|\boldsymbol{D}|$ with each move. If $\mathcal{T}$ satisfies the n disk-condition, then such a sequence must terminate, thus giving the required basis.

For the proof in the "if" direction, the first thing to note is that if there are no disk-swap moves to reduce $|\boldsymbol{D}|$, then every essential wave in $\overline{\partial H - \boldsymbol{D}}$ must intersect $\mathcal{T}$ at least $\frac{1}{2}n$ times. Let $D \in \mathcal{D}$ be a meridian disk such that $D \notin \boldsymbol{D} \cup \boldsymbol{D}^*$. Then $\Gamma = D \cap \boldsymbol{D} \neq \varnothing$. We are assuming that the intersection between the boundaries of disks is minimal. Thus by Lemma 3.6 we can assume that Γ is a set of pairwise disjoint properly embedded arcs in D, as shown in Figure 4. Thus all the faces of the meridian D, when D is cut along Γ, are disks. Also, there must be at least two bigons, D_1 and D_2 in this meridian disk. $D_i \cap \overline{\partial H - \boldsymbol{D}}$ are essential waves in $\overline{\partial H - \boldsymbol{D}}$ and thus intersect $\mathcal{T}$ at least $\frac{1}{2}n$ times. □

Next we want to use Lemma 3.15 to produce an algorithm to determine whether a boundary pattern satisfies the n disk-condition. Here by a boundary pattern, we mean a family of disjoint essential closed curves in the boundary of a handlebody.

Lemma 3.16 *Assume we are given a handlebody H and a set $\mathcal{T}$ of essential curves in ∂H. There is an algorithm to find, in finite time, a waveless minimal system of meridian disks.*

Proof Suppose we start with an arbitrary minimal system of meridian disks $\boldsymbol{D}$ for H. If $\mathcal{T}$ has a wave when H is cut along $\boldsymbol{D}$, then there is a subarc $\gamma \subset \mathcal{T}$ with both ends in some disk $D \in \boldsymbol{D}$ and $\mathrm{int}(\gamma) \cap \boldsymbol{D} = \varnothing$. Then D has a boundary compression disk E such that the arc $E \cap \partial H = \gamma$. Let D_1 and D_2 be the disks produced by compressing D along E. Then $\Sigma_i |D_i| \leq |D| - 2$, as shown in Figure 9. Thus when a disk-swap move is done swapping D for one of the D_i, we see that $|\boldsymbol{D}|$ will decrease by at least two. Note also that the number of waves does not go up. If there is another wave we can always do another boundary disk compression and a disk-swap move to reduce $|\boldsymbol{D}|$, thus this process must terminate in a finite number of moves. □

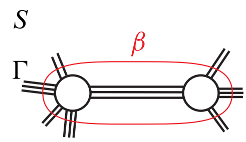

Figure 10: Boundary of meridian disk to add to $\boldsymbol{D}$

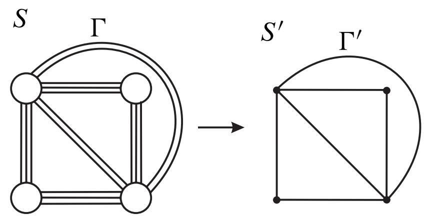

Figure 11: Γ and Γ'

Given that it is possible to find a waveless minimal system of meridian disks $\boldsymbol{D}$, to show that we can find a waveless basis, we proceed as follows. Suppose we have already found a waveless system of disks and want to add new waveless disks, until we get a basis. We can use our initial set of boundary curves of disks to cut ∂H to obtain a punctured sphere $S = \overline{\partial H - \boldsymbol{D}}$. Suppose that there is at least one pair of boundary curves of S such that all the arcs of $\Gamma = \mathcal{T} \cap S$ running between them are parallel. Then there is a simple closed curve β which is essential in S, is not boundary parallel and each curve in Γ intersects β at most once, as shown in Figure 10. Then we can add a disk with boundary β to enlarge our system of waveless disks.

To simplify this problem, collapse each boundary component of S to a vertex and identify parallel copies of edges of Γ. This produces a graph Γ' embedded in a 2–sphere S' such that Γ' is connected, no two edges are parallel and no edge has both ends at one vertex. This means that if we cut S' along Γ' all the resulting faces will be disks and will have degree at least 3.

Definition 3.17 A 2–*cycle* in a graph is a simple closed curve that is the union of two edges.

The problem of finding a waveless basis is now to show that we can always find two vertices of Γ' that are joined by exactly one edge. This means finding a vertex not contained in a 2–cycle. Let c be a 2–cycle in Γ', thus c cuts S' into two disks and as Γ' does not contain any parallel edges, the interior of both disks must contain at least one vertex of Γ'. We now want to show that there is a vertex of Γ' that is not part of a 2–cycle. Let c and c' be two 2–cycles in Γ'. If $c \cap c'$ is empty, a single vertex or edge, then the interior of one of the disks produced when we cut S' along c must be disjoint from c'. If $c \cap c'$ is two vertices, then we can construct a third 2–cycle c'' such that when we cut S' along c'', the interior of one of the disks produced is disjoint from both c and c'. (We obtain c'' by taking one edge from each of c, c'.) By induction on the number of 2–cycles in C, the set of all 2–cycles in Γ', it follows that there must be a 2–cycle $c \in C$ such that when S' is cut along c we get a disk D for which there are no 2–cycles intersecting $\mathrm{int}(D)$. As there are no parallel edges in Γ', we have $\Gamma' \cap \mathrm{int}(D) \neq \varnothing$. Therefore, Γ' has to have a vertex in $\mathrm{int}(D)$ that is not in a 2–cycle. This gives us the following lemma.

Lemma 3.18 *Assume we are given a handlebody H and a set $\mathcal{T}$ of essential curves in ∂H. There is an algorithm to find, in finite time, a waveless basis.*

Note that this means that once a minimal waveless system of meridian disks has been found, most of the work has been done and that to produce a waveless basis, suitable meridian disks are added to the system. This lemma is not expressly used in the rest of this paper, but waveless bases are used in Section 4 in a condition for manifolds to be atoroidal. Thus it is nice to know that given a 3–manifold that satisfies the disk-condition, we can always find a waveless basis for each of its handlebodies.

Lemma 3.19 *Let H be a handlebody and $\mathcal{T}$ a set of essential curves in ∂H. Then there is an algorithm to determine, in finite time, if $\mathcal{T}$ satisfies the n disk-condition.*

Proof Once again let $\boldsymbol{D}$ be a minimal system of disks and $N(\boldsymbol{D})$ be a regular neighbourhood of $\boldsymbol{D}$. Let $S = \overline{\partial H - N(\boldsymbol{D})}$ and $\Gamma = \mathcal{T} \cap S$. Then S is a $2g$–punctured sphere, where g is the genus of H. Also, Γ is a set of arcs properly embedded in S. By Lemma 3.16, we can assume that Γ does not contain any waves. Therefore, Γ cuts S up into polygonal disks of degree at least four. As above let $\boldsymbol{D}^* \subset \mathcal{D}$ be the set of meridian disks disjoint from $\boldsymbol{D}$. For any $D^* \in \boldsymbol{D}^*$, we have that $D^* \cap S = \alpha$ is a simple closed curve in $\mathrm{int}(S)$. Let $|\alpha|$ be the number of times that α intersects Γ. Note that $|\alpha| = |D^*|$. We have therefore reduced the question of looking for meridian disks disjoint from $\boldsymbol{D}$ to studying essential simple closed curves in S. For $D \in \boldsymbol{D}$, we have that $N(D) \cap S$ is two boundary curves, ∂D_1 and ∂D_2, of S. Then if γ is an essential simple closed curve in S that separates ∂D_1 from ∂D_2, the disk bounded by γ can be used for a disk-swap move on D. Let $N = \max\{|D| : D \in \boldsymbol{D}\}$ and L be the set of essential simple closed curves in S of length at most N. Thus as L is a finite set of curves and as each face of S is a polygon, we can list all the elements of L using normal curve theory, using the polygonal disk structure or a triangular subdivision. Therefore, to test whether $\boldsymbol{D}$ satisfies Lemma 3.15 we need to check that; all disks in $\boldsymbol{D}$ intersect $\mathcal{T}$ at least n times, all the curves in L have length at least n, and $|\gamma| \geq |D|$ for $\gamma \in L$ and $D \in \boldsymbol{D}$ such that γ separates the two curves $D \cap S$ in S. If a disk-swap move is found, then we perform the move and then test the new system. As $|D|$ decreases by at least two with each move, the algorithm will terminate in finite time, either when a suitable system is found, meaning $\mathcal{T}$ satisfies the n disk-condition or when a meridian disk is found that intersects $\mathcal{T}$ less than n times. □

Note that this algorithm can be continued until a system is found which has a "locally minimal" intersection. If $n = \min\{|D| : D \in \boldsymbol{D}\}$, then n is the supremum disk-condition satisfied by $\mathcal{T}$. For if there is a meridian disk that intersects $\mathcal{T}$ less than n times that is not in $\boldsymbol{D}$, then the algorithm would not have terminated. An equivalent statement is that $\boldsymbol{D}$ is an n–waveless system of disks. Clearly if there is an essential wave in $\overline{\partial H - \boldsymbol{D}}$ that intersects $\mathcal{T}$ less than $\frac{1}{2}n$ times, then there is a disk-swap move to reduce $|\boldsymbol{D}|$. In the other direction, if $\boldsymbol{D}$ is an n–waveless system and there is a meridian disk $D \in \mathcal{D}$ such that $|D| < n$, then clearly $D \cap \boldsymbol{D} \neq \varnothing$. Thus D gives a boundary compressing disk for some disk in $\boldsymbol{D}$ and thus a wave in $\overline{\partial H - \boldsymbol{D}}$, that intersects $\mathcal{T}$ at less than $\frac{1}{2}n$ points. Therefore, there is an alternative algorithm to test the disk-condition, giving the corollary:

Corollary 3.20 *If H is a handlebody and $\mathcal{T} \subset \partial H$ is a set of essential curves that meet the n disk-condition, then there is an algorithm to find an n–waveless minimal system of meridian disks.*

3.2 Examples

To construct manifolds that meet the disk-condition, we use Dehn surgery or branched covers to build a manifold M which contains a 2–complex that cuts M up into three injective handlebodies.

3.2.1 Dehn filling examples The first class of examples of manifolds that meet the disk-condition are constructed by performing Dehn surgery along suitable knots in $\mathbb{S}^3$. Let $K \subset \mathbb{S}^3$ be the $(3,3,3)$–pretzel knot and F the free spanning surface shown in Figure 12. For $A \subset \mathbb{S}^3$, let $N(A)$ be a regular neighbourhood of A. Let $H_3 = N(K)$ and $H_1 = \overline{N(F) - H_3}$, as shown in Figure 13. Then H_1 is a genus-2 handlebody, and $\mathcal{T} = \partial(H_1 \cap H_3)$ is two copies of K. Furthermore, H_1 is homeomorphic to an I–bundle over F and $\mathcal{T}$ to the boundary curves of the vertical boundary of the I–bundle structure. Given the arcs $\beta_1, \beta_2, \beta_3$ in Figure 12, $\bigcup_i (\beta_i \times I)$ is a basis for H_1. Each wave in the pairs of pants produced when ∂H_1 is cut along the basis intersects $\mathcal{T}$ at least twice. Therefore, the basis is 4–waveless, and by Lemma 3.11, $\mathcal{T}$ satisfies the 4 disk-condition in H_1. Also, $H_2 = \overline{\mathbb{S}^3 - (H_1 \cup H_3)}$ is a genus-2

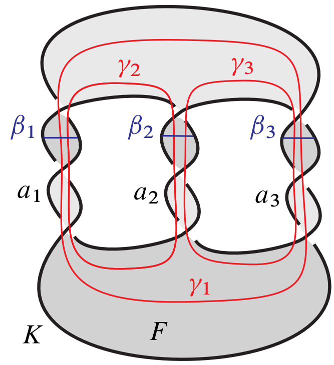

Figure 12: $(3,3,3)$–pretzel knot

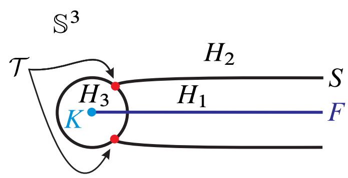

Figure 13: Handlebodies in Dehn filling construction

handlebody, and the curves $\gamma_1, \gamma_2, \gamma_3$ in Figure 12 bound meridian disks of a basis $\boldsymbol{D}$ for H_2. As $\mathcal{T}$ is two copies of K each wave in the two pairs of pants, produced by cutting ∂H_2 along the γ_i, intersects $\mathcal{T}$ six times. Thus $\boldsymbol{D}$ is a 12–waveless basis for H_2, and by Lemma 3.11, $\mathcal{T}$ satisfies the 12 disk-condition in H_2. Therefore, if a Dehn surgery along K is performed such that the meridian disk of the solid torus glued back in intersects $\mathcal{T}$ at least six times, a manifold that satisfies the $(4,6,12)$ disk-condition is produced. U Oertel showed in [10] that all but finitely many Dehn surgeries on such pretzel knots produce non-Haken 3–manifolds.

This construction can be generalised to any knot $K \subset \mathbb{S}^3$, that has a free spanning surface F, such that K satisfies the 6 disk-condition in $\overline{\mathbb{S}^3 - F}$. Then any Dehn surgery of type (p,q) with $|p| \geq 6$ will produce a manifold meeting the disk-condition.

3.2.2 Branched cover examples The next method for constructing manifolds which meet the disk-condition is taking cyclic branched covers over knots in $\mathbb{S}^3$. We look at two conditions on knots that are sufficient for the resulting manifolds to meet the disk-condition.

Let B_i, for $i = 1, 2$ or 3, be 3–balls and $\gamma_i = \{\gamma_i^1, \dots, \gamma_i^k\}$, for $k \geq 2$, be a set of properly unknotted pairwise disjoint embedded arcs in B_i. Unknotted means that there

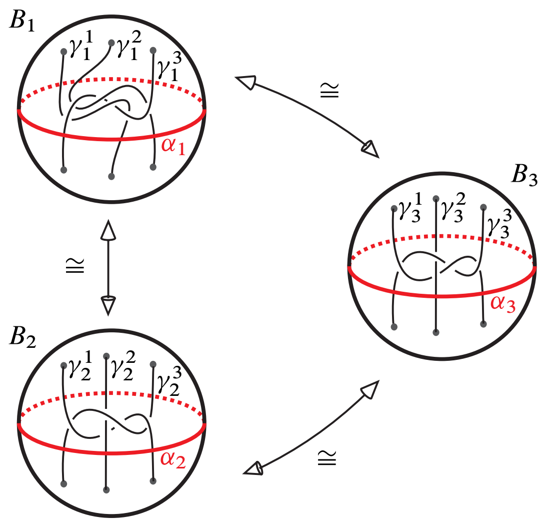

Figure 14: Bubble construction

is a set of pairwise disjoint embedded disks, $D_i = \{D_i^1, \dots, D_i^k\}$, such that

$$\gamma_i^j \subset \partial D_i^j \quad \text{and} \quad \overline{\partial D_i^j - \gamma_i^j} = D_i^j \cap \partial B.$$

Therefore, if we take the p–fold cyclic branched cover of B_i, with γ_i as the branch set, then the result will be a genus-$(p-1)(k-1)$ handlebody H_i. Let $r_i\colon H_i \to B_i$ be the branched covering map and $\alpha_i \subset \partial B_i$ be a simple closed curve disjoint from γ_i such that $\mathcal{T}_i = r^{-1}(\alpha_i)$ satisfies the n_i disk-condition in H_i. Note that α_i can be thought of as cutting ∂D_i up into two hemispheres.

Now glue the three balls by homeomorphisms between their hemispheres, as shown in Figure 14, so that the resulting manifold is $\mathbb{S}^3$ and the endpoints of the γ_i match up. Thus $K = \bigcup \gamma_i$ is a link and $C = \bigcup \partial B_i$ is a 2–complex of three disks glued along a triple curve α, which is the image of the α_i. Let M be the p–fold cyclic branched cover of $\mathbb{S}^3$ with K as the branch set. Let $r\colon M \to \mathbb{S}^3$ be the branched covering map. Then $X = r^{-1}(C)$ is a 2–complex that cuts M up into handlebodies and $\mathcal{T} = r^{-1}(\alpha)$ is a set of triple curves that satisfies the n_i disk-condition in H_i. Thus if $\sum 1/n_i \le \frac{1}{2}$, then M satisfies the disk-condition.

If $k = 2$ or 3 and the intersection of α_i with D_i is minimal under isotopy in $\partial B_i - \gamma_i$, then a sufficient condition for the lift of γ_i to the p–fold cyclic branched cover of B_i to meet the n disk-condition is that any essential wave in $\overline{\partial B_i - D_i}$ intersects $\gamma \cap \overline{\partial B_i - D_i}$ at least $\frac{1}{2}n$ times. Note that this is a slight variation of Lemma 3.11 and the proof is essentially the same. Given the 2–complex shown in Figure 15, it can be seen that any p–fold cyclic branched cover over an (a_1, a_2, a_3)–pretzel knot in $\mathbb{S}^3$ such that $|a_i| \ge 2$ will produce a manifold that satisfies the disk-condition.

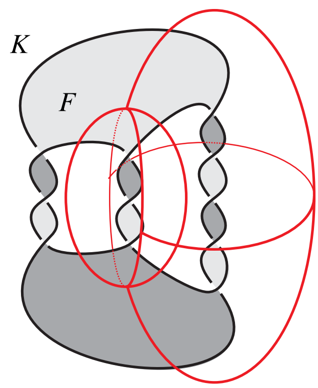

Figure 15: $(3,3,5)$–pretzel knot

Let M be a manifold that satisfies the disk-condition and can be constructed from the gluing of three genus-2 injective handlebodies. Then a simple Euler characteristic argument shows that all the faces of the 2–complex X must either be once punctured tori or twice punctured disks. If all the faces are once punctured tori, then the set of triple curves, $\mathcal{T}$, is a single curve. Thus a free involution of $\mathcal{T}$ can be canonically extended, up to isotopy, to an involution on each of the faces of X with three fixed points. Using a waveless basis for each handlebody, the involution on X can be extended to the whole of M. This means that any such manifold has a $\mathbb{Z}_2$ symmetry and is the 2–fold cyclic branched cover of $\mathbb{S}^3$ over some knot or link. In fact, the quotient of M by the involution is three balls glued together along hemispheres as in Figure 14. If all the faces of X are pairs of pants, then there is no corresponding involution of M.

The second construction involves the 3–fold cyclic branched cover of a knot that meets essentially the same condition as in the Dehn filling construction, so that the lift of the Seifert surface gives the 2–complex X. Let K be a knot in $\mathbb{S}^3$ and F be a free Seifert surface for K. This means that $\overline{\mathbb{S}^3 - F}$ is a handlebody. We construct the 3–fold cyclic branched cover over the knot K in $\mathbb{S}^3$ given by D Rolfsen in [11]. Let $N(K)$ be a regular neighbourhood of K, $\alpha \subset \partial N(K)$ the meridian curve of $N(K)$ and $N = \overline{\mathbb{S}^3 - N(K)}$. Let $\widetilde{N}$ be the 3–fold cyclic cover of N and $p\colon \widetilde{N} \to N$ the covering projection. That is, let $G \subset \pi_1(N)$ be the kernel of the homomorphism mapping $\pi_1(N)$ onto $\mathbb{Z}_3$, where the meridian of $N(K)$ is sent to a generator of $\mathbb{Z}_3$. Then $\widetilde{N}$ is the cover corresponding to G. So $\widetilde{N}$ has a single torus boundary and $\widetilde{\alpha} = p^{-1}(\alpha)$ is a single curve that covers α three times. Therefore, $\widetilde{F} = p^{-1}(F)$ is a set of three properly embedded spanning surfaces in $\widetilde{N}$. As F is free, $\overline{\widetilde{N} - \widetilde{F}}$ is three handlebodies. Let M be the 3–fold cyclic branched cover of $\mathbb{S}^3$ with K as the branch set. Then M can be constructed by gluing a solid torus T to $\partial\widetilde{N}$ so that its meridian matches $\widetilde{\alpha}$. Next extend each surface in $\widetilde{F}$ along an annulus to the spine $\mathcal{T}$ of T to produce a 2–complex X. Thus X is a 2–complex that cuts M into three handlebodies. Thus for M to meet the disk-condition it is sufficient for K to meet the 6 disk-condition in $\overline{\mathbb{S}^3 - F}$. An obvious example of such a knot is the $(3,3,3)$ pretzel knot in Figure 12.

The 3–fold cyclic branched cover of the $(3,3,5)$ pretzel knot K gives an example of a manifold with two distinct splitting 2–complexes that meet the disk-condition. Let M be the 3–fold cyclic branched cover of $\mathbb{S}^3$ with K as the branch set. Let X be the 2–complex produced by lifting the Seifert surface F to M and let X' be the 2–complex produced by lifting the "bubble" 2–complex shown in Figure 15. X and X' are distinct 2–complexes meeting the disk-condition. That is there is no homeomorphism of M that sends X to X', for if there was, M would have a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry and thus K would have a $\mathbb{Z}_3$ symmetry, which is clearly not the case. Note that if each twisted band in K has the same number of crossings, for example the $(3,3,3)$ pretzel knot, then the 3–fold cyclic branched cover does have a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry.

4 Characteristic variety

In this section we prove the torus theorem and construct the characteristic variety in 3–manifolds that meet the disk-condition. The first step is to look at how, in the component handlebodies, properly embedded essential annuli disjoint from the triple curves intersect and how meridian disks that intersect $\mathcal{T}$ exactly n_i times intersect. This allows us to build a picture of the characteristic variety in each of the handlebodies, which we then use to construct the characteristic variety of the manifold.

4.1 Handlebodies, embedded annuli and meridian disks

Throughout this section, let H be a handlebody and $\mathcal{T}$ be a set of disjoint essential simple closed curves in ∂H that meet the n disk-condition in H. We will assume that all intersections between surfaces are transverse. Before we look at the components of the characteristic variety in each handlebody, we need to look at some properties of embedded essential annuli that are disjoint from $\mathcal{T}$.

4.2 Essential annuli

In this section we investigate intersections between embedded essential proper annuli.

Definition 4.1 An intersection curve between two annuli is said to be *vertical* if it is a properly embedded arc which is not boundary parallel in either annulus. The intersection curve is *horizontal* if it is an essential simple closed curve in both annuli.

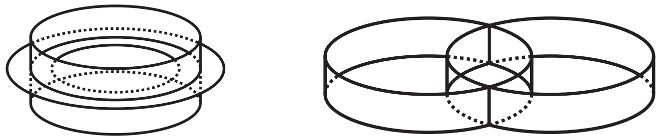

Figure 16: Intersecting embedded annuli: horizontal (left) and vertical (right)

If there is a proper isotopy in $H - \mathcal{T}$ of two annuli which removes their intersections, then the annuli will be said to have *trivial intersection* and if the intersection cannot be removed, the annuli have *nontrivial intersection*. This means that if two embedded annuli have nontrivial intersection they cannot be isotopically parallel. The disk-condition restricts how properly embedded annuli can intersect.

Lemma 4.2 *Let A_1 and A_2 be two essential properly embedded annuli in $H - \mathcal{T}$. Then there is a proper isotopy of them in $H - \mathcal{T}$ such that all their intersections are either vertical or horizontal.*

Remark 4.3 This means that nontrivial intersections between embedded annuli must either be all horizontal or all vertical.

Proof This uses standard innermost curve arguments and the following observations. Let A_1 and A_2 be essential properly embedded annuli in $H - \mathcal{T}$ and let $\Gamma = A_1 \cap A_2$. First note that as the A_i are embedded they cannot have both horizontal and vertical intersections. As H is irreducible there is an isotopy of A_1 to remove components of Γ that are simple closed curves and inessential in both A_i. Also, by irreducibility of H and the disk-condition, there is an isotopy of A_1 to remove components of Γ which are properly embedded arcs and boundary parallel in both A_i. Let γ be a component of Γ which is a simple closed curve and is essential in A_1 and not essential in A_2. Then the disk in A_2 bounded by γ implies that A_1 is not π_1–injective, which is a contradiction. Now let γ be a component of Γ which is a properly embedded arc which has both ends in the same boundary curve of A_1 and runs between the boundary curves of A_2. Then the disk bounded by γ in A_1 is a boundary compression disk for A_2 and the disk produced by compressing A_2 is disjoint from $\mathcal{T}$, thus implying that A_2 is boundary parallel in $H - \mathcal{T}$. □

Lemma 4.4 *Let H be a handlebody and $\mathcal{T}$ a set of curves in ∂H that meet the n disk-condition. Assume a properly embedded essential annulus in $H - \mathcal{T}$ intersects two other properly embedded essential annuli in $H - \mathcal{T}$, one vertically and the other horizontally. Then if there is a nontrivial horizontal intersection, the vertical intersections can be removed by an isotopy.*

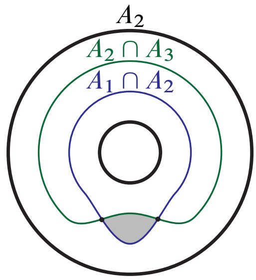

Figure 17: Curves of intersection in A_2

Remark 4.5 This indicates there are three types of essential embedded annuli in $H - \mathcal{T}$: those that have nontrivial horizontal intersections with other annuli, those that have nontrivial vertical intersections with other annuli and those that have no nontrivial intersections with other annuli. Later in this section, we will see that these types of annuli correspond to the flavours of characteristic variety in $H - \mathcal{T}$.

We could follow a least-area argument using a suitable Riemannian metric on the handlebody but use instead a more elementary direct cut-and-paste approach.

Proof Let A_1, A_2 be two properly embedded essential annuli in $H - \mathcal{T}$ that have nontrivial horizontal intersection. Let A_3 be a third embedded essential annulus in $H - \mathcal{T}$ that intersects A_1 vertically. If the vertical intersection between A_1 and A_3 is nonempty, then $(A_1 \cap A_2) \cap A_3 \neq \varnothing$ and thus the intersection between A_2 and A_3 is nonempty. By Lemma 4.2, we can isotope A_3 so that its intersection with A_2 is either vertical or horizontal and its intersection with A_1 is vertical. We will assume that the vertical intersection between A_1 and A_3 is still nonempty. If the intersection between A_2 and A_3 is horizontal, then ∂A_3 is disjoint from ∂A_2, as both $A_2 \cap A_1$ and $A_2 \cap A_3$ are essential simple closed curves in A_2. There is an innermost bigon on A_2 bounded by one arc from each of $A_2 \cap A_1$ and $A_2 \cap A_3$ with common endpoints; see Figure 17. This is clear because each arc of $A_1 \cap A_3$ has to have at least one corresponding vertex of $(A_2 \cap A_1) \cap (A_2 \cap A_3)$. If we assume there is a single vertical arc of $A_1 \cap A_3$ which contains both vertices of the bigon, then by the irreducibility of H there is an isotopy of A_2 over a ball in H bounded by the bigon and disks in A_1 and A_3 to remove the bigon. It is then straightforward to see that A_2 can be isotoped so for any bigon bounded by an arc of $A_2 \cap A_1$ and $A_2 \cap A_3$ there are two vertical arcs of intersection of $A_1 \cap A_3$ which contain the two vertices of this bigon; see Figure 18. We can then isotope A_3 across this bigon to convert these two vertical arcs into two boundary parallel arcs of $A_1 \cap A_3$ which can be removed by a further isotopy. In this way, eventually all the vertical arcs of $A_1 \cap A_3$ can be removed. Thus we can assume that A_3 intersects both A_1 and A_2 vertically.

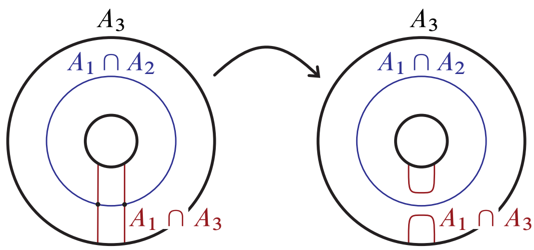

Figure 18: Curves of intersection in A_3

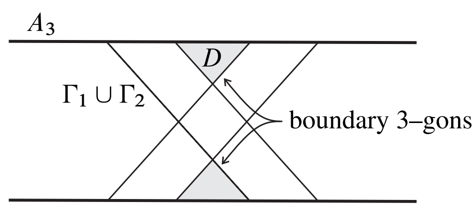

Figure 19: Component of the pullback graph $\Gamma_1 \cup \Gamma_2$

Let $\Gamma_i = A_3 \cap A_i$ for $i \neq 3$. Then Γ_i is a set of properly embedded pairwise disjoint spanning arcs in A_3, where each arc from Γ_1 intersects at least one arc from Γ_2. The faces produced when A_3 is cut up along $\Gamma_1 \cup \Gamma_2$ are all disks. As each connected component of $\Gamma_1 \cup \Gamma_2$ contains at least two arcs, each component will have a boundary 3–gon, D, as shown in Figure 19, such that subarcs of ∂A_3, Γ_1 and Γ_2 make up its three edges. Then the disk D gives an isotopy of A_1 that converts the corresponding essential closed curve of $A_1 \cap A_2$ into a boundary parallel arc. Thus there is a further isotopy to remove the intersection altogether. This process can be repeated to remove all the intersections of $A_1 \cap A_2$, giving a contradiction. □

Therefore, if a proper essential annulus in $H - \mathcal{T}$ has a nontrivial horizontal/vertical intersection with one annulus, then we can arrange that all its nontrivial intersections with all other essential annuli must be horizontal/vertical.

4.3 Meridian disks

Next we want to examine intersecting meridian disks. In particular, if $\mathcal{T}$ satisfies the n disk-condition in H, then there may be meridian disks that intersect $\mathcal{T}$ exactly n times. These disks are important when we are considering the disk flavour of characteristic variety.

Definition 4.6 If F is an n–gon and γ is a properly embedded arc in F such that if F is cut along γ, the result is two disks that have $\frac{1}{2}n$ intersections with $\mathcal{T}$, then γ is said to be a *bisecting* arc of F.

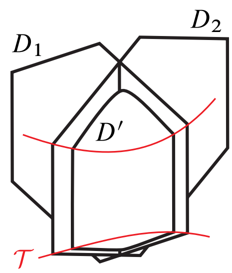

Figure 20: Two trivially intersecting 6–gons

Lemma 4.7 *Let H be a handlebody and $\mathcal{T}$ a set of curves in ∂H that satisfies the n disk-condition. If D_1 and D_2 are meridian disks that have n intersections with $\mathcal{T}$, then there is an isotopy of the disks such that $\Gamma = D_1 \cap D_2$ is a set of properly embedded disjoint bisecting arcs in both D_i or the intersection Γ can be removed.*

Proof This proof uses the usual innermost curve arguments and the following observations, to construct an isotopy to remove arcs of Γ that are not bisecting in both disks. By Lemma 3.6, we can assume that all components of Γ are properly embedded arcs. If such an arc is not bisecting in D_1, it is easy to see there is an arc γ of Γ which bounds an innermost subdisk D in D_1 which intersects $\mathcal{T}$ less than $\frac{1}{2}n$ times. Then one of the disks D' produced by surgering D_2 along D must intersect $\mathcal{T}$ in less than n points, as shown in Figure 20, and thus is boundary parallel in H. So there is an isotopy of D_1 to remove γ. □

Lemma 4.8 *Let H be a handlebody, $\mathcal{T}$ a set of curves in ∂H that meet the n disk-condition and D_1, D_2 and D_3 a set of meridian disks that all have n intersections with $\mathcal{T}$. Then there is an isotopy of the D_i such that $\bigcap D_i = \varnothing$.*

Proof By the previous lemma, we can isotope D_1 and D_2 so that their intersection is a set of parallel arcs in both disks. Assume that D_1 and D_2 have been isotoped so that their intersection has the least possible number of components and that $D_1 \cap D_2 \neq \varnothing$. Let A be a regular neighbourhood of $D_1 \cup D_2$ and B be the frontier of A in H. As no annulus component of B intersects $\mathcal{T}$, B consists of meridian disks that intersect $\mathcal{T}$ exactly n times and essential annuli whose boundary compressing disks intersect $\mathcal{T}$ at least $\frac{1}{2}n$ times.

Let D be a disk and $f\colon D \to H$ be an embedding such that $f(D) = D_3$. Then f can be isotoped so that $\Gamma = f^{-1}(B)$ is a set of properly embedded pairwise disjoint curves. As usual there is an isotopy of f to remove components of Γ that are simple closed curves. If D_3 intersects an annulus of B, then from above, either the intersections are parallel arcs or there is an isotopy of f to remove them. Similarly from Lemma 4.7 if D_3 intersects a disk of B, then either the intersections are bisecting parallel arcs or there is an isotopy of f to remove them. Therefore, there is an isotopy of f such that Γ is a set of parallel bisecting arcs. Thus $f^{-1}(A)$ is a set of 4–gons. Let D' be a 4–gon in $f^{-1}(A)$. Then using the same arguments as in the final step of the proof of Lemma 4.4, there is an isotopy of f such that $D' \cap f^{-1}(D_2 \cup D_1)$ is a set of parallel bisecting arcs. Moreover $f(D') \cap D_1 \cap D_2 = \varnothing$. This process can be repeated for each component of $f^{-1}(A)$ and thus $D_1 \cap D_2 \cap D_3 = \varnothing$. □

4.4 Flavours of characteristic variety in the handlebodies

4.4.1 I–bundle regions Let H be a handlebody and $\mathcal{T}$ a set of essential simple closed curves in ∂H, that meet the n disk-condition in H. Let N be a maximal, up to isotopy, I–bundle in H disjoint from $\mathcal{T}$, with its horizontal boundaries embedded in $\partial H - \mathcal{T}$, each component of N has nontrivial fundamental group and the induced map on the fundamental group is injective. Thus N is an I–bundle with a base space which is an embedded surface in H. Let S be a component of this embedded surface. If S is orientable, then the corresponding component of N has a product structure and its horizontal surface consists of two copies of S embedded in $\partial H - \mathcal{T}$. Alternatively, if S is nonorientable, then the corresponding component of N has a horizontal boundary which is a double cover of S embedded in $\partial H - \mathcal{T}$. In both cases the vertical boundary is a set of essential properly embedded annuli. From this point on these surfaces will be called frontier annuli. Also note that none of the base surfaces can be disks. This means that N is a set of embedded handlebodies in H with genus ≥ 1. N is not unique, for if H contains two embedded annuli that intersect horizontally, in a nontrivial way, then N can contain the regular neighbourhood of one or the other annulus but not both.

Definition 4.9 Let the *I–bundle region*, N_I, be the set of all components N_i from N which have base spaces that are not annuli or Möbius bands.

Later the I–bundle region is shown to be unique up to isotopy.

Lemma 4.10 *If A is a properly embedded essential annulus in $H - \mathcal{T}$ that has a nontrivial vertical intersection with another properly embedded essential annulus, then it is isotopic into N_I.*

Proof Let the map $f_i \colon A \to H - \mathcal{T}$, for $i = 1$ or 2, be an essential proper embedding of an annulus A such that $f_1(A) = A_1$ and $f_2(A) = A_2$ have nontrivial vertical intersections. Let B be the set of frontier annuli of N_I. If $A_1 \cap N_I \neq \varnothing$, then by Lemmas 4.2 and 4.4 we know that there is an isotopy of f_1 such that the intersection between A_1 and the annuli in B is vertical. Thus the pullback $\Gamma_1 = f_1^{-1}(B)$ is a set of properly embedded nonboundary parallel arcs in A and, as B is separating in H, there must be an even number of them. Thus Γ_1 cuts A up into quadrilaterals and every alternate one is mapped by f_1 into $\overline{(H-N_I)}$. Let $A' \subset A$ be a quadrilateral such that $f(A') \subset \overline{(H-N_I)}$. Also, let $N(f_1(A'))$ be the regular neighbourhood of $f_1(A')$ in $\overline{(H-N_I)}$ disjoint from $\mathcal{T}$. Note that $N(f_1(A'))$ can be fibred as an I–bundle over a quadrilateral. Then there must be an isotopy of f_1 to remove the curves $\Gamma_1 \cap A'$ otherwise $N(f_1(A')) \cup N_I$ would be larger than N_I, contradicting maximality. We can repeat this process until $\Gamma_1 = \varnothing$, thus $A_1 \cap B = \varnothing$. This process can be repeated for A_2 so that it is disjoint from B. If $A_1 \cap A_2$ is disjoint from N_I and the annuli have been isotoped so that their intersection is a minimal set of essential arcs, then $N(A_1 \cup A_2)$ can be fibred as an I–bundle and added to N_I, contradicting maximality. Thus $A_1 \cup A_2 \subset N_I$. □

Note that in distinction to the above lemma, if an annulus A meets another annulus horizontally, it may not be possible to isotope A into N_I.

Now let $\breve{H}$ be a regular finite-sheeted cover of H and $\breve{\mathcal{T}}$ be the lift of $\mathcal{T}$. Thus $\breve{H}$ also is a handlebody with $\breve{\mathcal{T}}$ satisfying the n disk-condition. Now let $N_I \subset \breve{H}$ be the I–bundle region, as described above. Also, let G be the group of covering translations of $\breve{H}$, so $\breve{H}/G = H$. Let N_i, for $1 \le i \le n$, be the connected subhandlebodies of N_I and S_i be the base-surface corresponding to N_i.

Lemma 4.11 *If N_i is a component of N_I, then $g(N_i)$ is isotopic to a component of N_I for any $g \in G$.*

Proof Let $\boldsymbol{A}$ be the set of frontier annuli of $g(N_i)$ and $\boldsymbol{B}$ the set of frontier annuli of N_I. If $g(N_i)$ and N_I have a nontrivial intersection, then by Lemma 4.2 there is an isotopy of g such that if any annuli in $\boldsymbol{A}$ and any annuli in $\boldsymbol{B}$ intersect, then the intersection curves are all either vertical or horizontal. Now isotope g to remove all trivial intersections between annuli in $\boldsymbol{A}$ and $\boldsymbol{B}$.

Let $B \in \boldsymbol{B}$ be an annulus such that it intersects at least one annulus in $\boldsymbol{A}$ horizontally. By Lemma 4.4, it can only intersect the other annuli in $\boldsymbol{A}$ horizontally. Thus $B \cap g(N_i)$ is a set of annuli properly embedded in $g(N_i)$. Let $B' \subset B$ be one such annulus.

Isotope B' so that it is transverse to the I–bundle structure. As intersections of B with annuli in $\boldsymbol{A}$ are minimal, B' either projects one-to-one onto the base space or double covers it. This depends on whether the two boundary curves of B' are in different annuli in A or in the same annulus, respectively. Therefore, the base space of $g(N_i)$ and thus N_i is either an annulus or a Möbius band, giving us a contradiction. This means that all horizontal intersections between annuli in $\boldsymbol{A}$ and $\boldsymbol{B}$ can be removed.

Therefore, all intersections between annuli in $\boldsymbol{A}$ and $\boldsymbol{B}$ that are nontrivial are vertical. But by Lemma 4.10 we can isotope all such annuli in $\boldsymbol{A}$ into N_I. Therefore, there is an isotopy of g such that $g(N_i) \cap N_I \neq \varnothing$ and $\boldsymbol{A} \cap \boldsymbol{B} = \varnothing$. Thus we know that we can isotope g so that $g(N_i)$ lies inside N_I, otherwise $g(N_i) \cup N_I$ would be a larger I–bundle than N_I, contradicting maximality.

As $g(N_i)$ is connected we know that it lies in a single component, N_k, of N_I. If $g(N_i)$ is not isotopic to N_k, then $g^{-1}(N_k - g(N_i)) \cup N_I$ is a larger I–bundle region, contradicting maximality. □

From the previous lemma we get the following corollary.

Corollary 4.12 *The regions N_I and $g(N_I)$ are isotopic for any $g \in G$.*

This corollary can be used to show that N_I can be isotoped so that it is preserved by G. Put a Riemannian metric on H, lift it to $\breve{H}$ and then isotope N_I so that the frontier annuli of the N_I are least area. Let $g \in G$ and A be a frontier annulus of N_I. By the arguments used by Freedman, Hass and Scott in [3], $g(A)$ is either a frontier annulus of N_I or disjoint from all frontier annuli of N_I. Let N_I' and N_I'' be components of N_I such that $g(N_I')$ is isotopic to N_I''. If $N_I' \neq N_I''$, then replace N_I'' by $g(N_I')$. Now assume that $N_I' = N_I''$. We need to look at what happens to the frontier annuli under g. Let A and A' be frontier annuli of N_I' such that $g(A)$ is isotopic to A'. If $A \neq A'$, then replace A' by $g(A)$. Now assume that $A = A'$ and $g(A) \neq A$. As each element of G is a periodic homeomorphism, $g(N_I') \not\subset \operatorname{int}(N_I')$. Then by this observation and maximality of N_I, <u>either $g(N_I') \cap N_I'$ is empty</u> or it is isotopic to N_I'. Another way of saying this is that $g(N_I') - N_I'$ and $N_I' - g(N_I')$ are sets of thickened annuli. We can then assume that <u>$g(A)$ is disjoint from</u> N_I'. Let U_i, for $i \in \mathbb{N}$, be the thickened annulus component of $g^i(N_I') - g^{i-1}(N_I')$, where g^0 is the identity. As $\breve{H}$ is a finite-sheeted normal cover, there is some $m \in \mathbb{N}$ such that g^m is the identity. Therefore, $U_1 \cup \cdots \cup U_m$ is an annulus bundle over $\mathbb{S}^1$ properly embedded in $\breve{H}$, which cannot happen, thus $g(A) = A$. This gives us the following corollary.

Corollary 4.13 *There is an isotopy of $N_I \subset \breve{H}$ such that it is preserved by all the covering transformations.*

Lemma 4.10 implies that if H contains two embedded annuli that have nontrivial vertical intersection, then N_I is not empty. Note this is a sufficient condition not a necessary one. For example, if N_I is an I–bundle over a twice punctured disk, then any two embedded annuli contained in N_I are parallel to frontier annuli and thus their intersections can be removed isotopically.

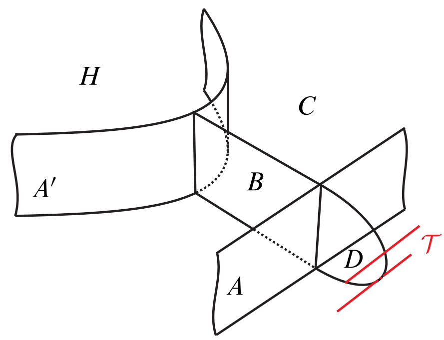

Figure 21: Extending boundary compression disk through an I–bundle component

Lemma 4.14 *N_I is unique up to ambient isotopy of H.*

We will not give the proof for this lemma as the method is the same as Lemma 4.11, the idea being that if we assume that we have two I–bundle regions N_I and N_I' that are not isotopic, then we get a contradiction to their maximality. Another property of N_I we need later is this lemma:

Lemma 4.15 *Let H be a handlebody, $\mathcal{T}$ a set of pairwise disjoint essential simple closed curves in ∂H that meet the n disk-condition and N_I the I–bundle region in H. Then if A is a frontier annulus of N_I and D is a boundary compression disk for A, then $|D| \geq \frac{1}{2}n$.*

Proof Assume that N_I has a frontier annulus A with a boundary compressing disk D such that $|D| < \frac{1}{2}n$. Also, let N_i be the component of N_I that has A as a frontier annulus. If we compress A along D to get a disk E, then $|E| < n$. Therefore, A must be boundary parallel, meaning there is a proper isotopy of A into ∂H. Note that this does not mean there is a proper isotopy of A into $\partial H - \mathcal{T}$. First assume that N_i has more than one frontier annulus. Let A' be another frontier annulus of N_i. As N_i is an I–bundle there is a 4–gon B, properly embedded in N_i, such that $B \cap A = D \cap A$ and $A' \cap B$ is a properly embedded arc in A' that is not boundary parallel, as shown in Figure 21, for suitable choice of D. Let $D' = D \cup B$. Then $|D'| < \frac{1}{2}n$, and if we compress A' along D', we get a disk E' with $|E'| < n$. Therefore, A' is boundary parallel through a region containing A. So A and A' must be parallel and N_i is the regular neighbourhood of a properly embedded annulus and thus can not be contained in N_I. If N_i has a single frontier annulus A, then similarly by the I–bundle structure, there is a properly embedded 4–gon $B \subset N_I$ such that it is not boundary parallel and $A \cap B$ is two arcs that are not parallel into ∂A. Then there are two boundary compression disks for A that can be glued to B along $A \cap B$. This produces a meridian disk that intersects $\mathcal{T}$ less than n times, contradicting the disk-condition. □

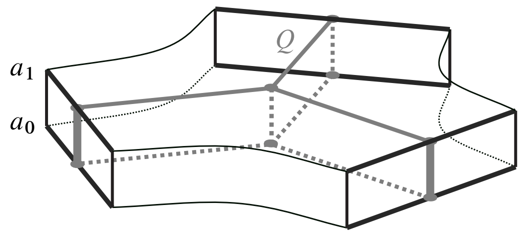

Figure 22: An example of an A_q

4.4.2 Tree regions Now let $N = \{N_i\}$ be a maximal set, up to isotopy, of fibred solid tori embedded in $H - \mathcal{T}$ such that $N_i \cap N_j = \varnothing$ for $i \neq j$ and $\partial H \cap N_i$ is a nonempty set of annuli that are π_1–injective in both ∂N_i and $\partial H - \mathcal{T}$, and the frontier of N_i in H is a nonempty set of annuli essential in $H - \mathcal{T}$ for each i. Then N is a maximal tree region of $H - \mathcal{T}$. The reason for this name will become clearer when we describe it further. Note that by Haken–Kneser finiteness arguments, we can see that N has a finite number of components.

Definition 4.16 Let a *simple q–tree* be a tree that is the cone on $q \geq 2$ points. A vertex of valency one is called an *end vertex*.

Let Q be a simple q–tree. Embed Q in $\mathbb{R}^2 \subset \mathbb{R}^3$. Let P^Q be a $2q$ polygon embedded in $\mathbb{R}^2$ such that every alternate edge intersects Q at an end vertex. Colour the edges of P^Q containing an end vertex of Q thick and all the others thin. Then let $A_q = P^Q \times [0,1]$ and $a_t = P^Q \times \{t\}$, for $t = 0$ or 1. Let Φ_p be a homeomorphism between a_0 and a_1 that twists by $2\pi/p$, such that it maps thick edges to thick edges and thin to thin. This means that $p = q/n$ for $n \in \mathbb{Z}$. Let $A_{(p,q)}$ be A_q with the faces a_0 and a_1 glued according to Φ_p. Therefore, $A_{(p,q)}$ is a solid torus fibred by $\mathbb{S}^1$ with an exceptional fibre of order (p,q). For each $N_i \in N$, there is a unique (p_i, q_i) such that there is a fibre-preserving homeomorphism from $A_{(p_i,q_i)}$ to N_i where the fibring agrees with the boundary curves of the frontier annuli.

Let A_1 and A_2 be two properly embedded essential annuli in $H - \mathcal{T}$ that intersect horizontally and $N(A_1 \cup A_2)$ be a regular neighbourhood disjoint from $\mathcal{T}$. Then the frontier of $N(A_1 \cup A_2)$ in H is a set of properly embedded annuli and tori. Let T be such a torus. The induced map on $\pi_1(T)$ has nontrivial image and $\pi_1(H)$ does not contain any free abelian subgroups of rank 2. Therefore, T bounds a solid torus whose intersection with $N(A_1 \cup A_2)$ is T. Glue solid tori to each torus in the frontier of $N(A_1 \cup A_2)$ in H to produce a submanifold P. Now the frontier of P in H is a set of properly embedded essential annuli and P is a solid torus. Note there is a homeomorphism from P to some $A_{(p,q)}$ that sends the boundary curves of $P \cap \partial H$ to fibres of $A_{(p,q)}$.

Definition 4.17 Let the *tree region* N_T be the union of all components $N_i \in N$ such that $p_i > 2$.

As with the I–bundle region, we are removing the components of N that are homeomorphic to $A_{(1,2)}$ or $A_{(2,2)}$, that is, regular neighbourhoods of properly embedded annuli or Möbius bands, to get N_T. This is because if there are two annuli in $H - \mathcal{T}$ that have a nontrivial vertical intersection, then a maximal tree region can contain the regular neighbourhood of only one of the annuli. Therefore, $H - \mathcal{T}$ may have a number of maximal tree regions. Later it is shown that the tree region is unique up to isotopy.

Lemma 4.18 *If A is a properly embedded annulus in $H - \mathcal{T}$ that has at least one nontrivial horizontal intersection with another properly embedded annulus in $H - \mathcal{T}$, then there is an isotopy of A into N_T.*

This proof is similar to Lemma 4.10.

Proof Let the map $f_i\colon A \to H$, for $i = 1$ or 2, be an essential proper embedding of an annulus A such that $f_i(A) = A_i$ is disjoint from $\mathcal{T}$ for each i and A_1 and A_2 have nontrivial horizontal intersections. Let B be the set of frontier annuli of N_T. If $A_1 \cap N_T \neq \varnothing$, then by Lemmas 4.2 and 4.4, we know that there is an isotopy of f_1 such that the intersection curves between A_1 and the annuli in B are horizontal. Thus the pullback $\Gamma_1 = f_1^{-1}(B)$ is a set of essential simple closed curves in A. Therefore, Γ_1 cuts A up into essential annuli. Let $A' \subset A$ be one of these annuli such that $f_1(A') \subset \overline{H - N_T}$ and let $N(f_1(A'))$ be a regular neighbourhood of $f(A')$ disjoint from $\mathcal{T}$. Then $N(f_1(A'))$ can be fibred as an $A_{(1,2)}$ fibred torus. Thus there must be an isotopy of f_1 to remove the curves $A' \cap \Gamma_1$ (there may be just one if $\partial A \cap \partial A' \neq \varnothing$) otherwise $N_T \cup N(f_1(A'))$ would be larger than N_T, contradicting maximality. So by repeating this process, there is an isotopy of f_1 such that $A_1 \cap B = \varnothing$. This same process produces an isotopy of f_2 so that $A_2 \cap B = \varnothing$. If $A_1 \cup A_2$ is disjoint from N_T, then as above, the torus boundaries of $N(A_1 \cup A_2)$ can be filled in with solid tori so the resulting manifold P is a solid torus. Then $N_T \cup P$ will be a larger tree region contradicting maximality, thus $A_1 \cup A_2 \subset N_T$. □

Once again let $\breve{H}$ be a finite-sheeted normal cover of H, $\breve{\mathcal{T}}$ the lift of $\mathcal{T}$ and G the group of covering translations of $\breve{H}$ such that $\breve{H}/G = H$. Also, let N_T be the tree region in $\breve{H}$. We then get the following lemma.

Lemma 4.19 *Let N_i be a component of N_T. For any $g \in G$, we have that $g(N_i)$ is isotopic to an element of N_T.*

Proof Assume that N_i is a component of N_T and, for some $g \in G$, that $g(N_i)$ is not isotopic to an element of N_T. Let $\boldsymbol{A}$ be the set of frontier annuli of $g(N_i)$ and $\boldsymbol{B}$

be the set of frontier annuli of N_T. By Lemma 4.2, we know that there is an isotopy of g such that any annuli from $\boldsymbol{A}$ and $\boldsymbol{B}$ intersect vertically or horizontally. Also, all trivial intersections are then removed.

Let B be an annulus in $\boldsymbol{B}$ that intersects some annuli from $\boldsymbol{A}$ vertically. Then $B \cap g(N_i)$ is a set of properly embedded squares in $g(N_i)$. Let B' be one such square. As the number of intersections between B and $\boldsymbol{A}$ has been minimized $\partial B'$ is essential in $\partial g(N_i)$. Therefore, $g(N_i)$, and thus N_i, is the regular neighbourhood of an annulus or Möbius band. This implies that $p_i = 2$, contradicting that N_i is a component of N_T. Then any intersections between annuli from $\boldsymbol{A}$ and $\boldsymbol{B}$ must be nontrivial and horizontal. By Lemma 4.18, we can isotope all such annuli from $\boldsymbol{A}$ into N_T.

We have now isotoped g so that $\boldsymbol{A} \cap \boldsymbol{B} = \varnothing$. We can thus isotope g so that $g(N_i)$ lies inside a single component of N_T, otherwise $g(N_i) \cup N_T$ would be a larger tree region, contradicting maximality of N_T. Let $g(N_i)$ lie in $N_k \in N_T$. If $g(N_i)$ is not isotopic to N_k, then $g^{-1}(N_k - g(N_i)) \cup N_T$ is a larger tree region. □

From the previous lemma we get the following corollary.

Corollary 4.20 *For any $g \in G$, we have that $g(N_T)$ is isotopic to N_T.*

From the above corollary and using the same least area arguments as we did with I–bundle regions we get the following corollary.

Corollary 4.21 *There is an isotopy of N_T in $\breve{H}$ that is preserved by the covering transformations.*

This means that N_T will project down to a nontrivial tree region in H. If H contains two embedded annuli that have a nontrivial horizontal intersection, then H has a nonempty tree region. Note this is a sufficient condition but not a necessary one.

Lemma 4.22 *N_T is unique up to ambient isotopy of H.*

We will not give the proof for this lemma as the argument is the same as Lemma 4.11. The idea is that if we assume that there are two tree regions N_T and N'_T that are not isotopic, then we get a contradiction to their maximality.

4.4.3 Annulus regions It is clear from the definitions of N_I and N_T that:

Lemma 4.23 *If H is a handlebody and $\mathcal{T}$ is a set of curves in ∂H that meet the n disk-condition, then there is an isotopy of N_I and N_T such that $N_I \cap N_T = \varnothing$.*

Let A_I be the set of I–bundles in a maximal I–bundle region but not in N_I. That is, they have base spaces that are either annuli or Möbius bands. Let A_T be the set of fibred solid tori that are in a maximal tree region but not in N_T. That is, they are all the components of the maximal tree region whose associated trees have two end vertices. Let N_A be those components of A_T which are ambient isotopic to components of A_I. Components of N_A are regular neighbourhoods of properly embedded annuli or Möbius bands and they can be fibred by intervals or circles. The components of $A_I - N_A$ ($A_T - N_A$) are the components of the maximal I–bundle (maximal tree region) that cause the maximal I–bundle (maximal tree region) to be not unique and, in fact, the components of $A_I - N_A$ ($A_T - N_A$) can be isotoped into N_T (N_I).

Clearly by the definition, N_A can be isotoped to be disjoint from N_I and N_T. Therefore, it is contained in the set of handlebodies $H' = \overline{H - (N_I \cup N_T)}$. Any annulus that can be made to intersect another nonparallel annulus either vertically or horizontally is isotopic into $N_I \cup N_T$. Thus any nonparallel annuli in H' cannot be isotoped to intersect either vertically or horizontally. Therefore, by the maximality of the maximal I–bundle region and the maximal tree region we know that N_A is isotopic to the regular neighbourhood of the maximal set of disjoint and nonparallel properly embedded annuli in H'. Thus we get the following lemma.

Lemma 4.24 *N_A is unique up to ambient isotopy of H and can be isotoped to be disjoint from $N_I \cup N_T$.*

Definition 4.25 If H is a handlebody and $\mathcal{T}$ is a set of essential disjoint simple curves in its boundary that satisfies the n disk-condition, then for the pair $\{H, \mathcal{T}\}$, let the *maximal annulus region* be $N = N_I \cup N_T \cup N_A$, where N_I, N_T and N_A are as defined above.

4.4.4 Disk regions In this section, we want to define the building blocks for the flavour of characteristic variety that intersects the triple curves. In each handlebody H_i, these blocks look like the regular neighbourhood of meridian disks that intersect the triple curves exactly n_i times, where $\sum 1/n_i = \frac{1}{2}$. Hence we will refer to them as *disk regions*. Let H be a handlebody and $\mathcal{T}$ a set of essential curves in its boundary that meet the n disk-condition in H. Let $\boldsymbol{D}$ be a set made up of a single representative from each isotopy class of meridian disks that intersect $\mathcal{T}$ exactly n times. Let S be the resulting punctured sphere when ∂H is cut along a waveless basis for $\mathcal{T}$. Then $\Gamma = \mathcal{T} \cap S$ is a set of pairwise disjoint properly embedded arcs that cut S into n–gons. Therefore, by normal curve theory up to isotopy there is a finite number of simple closed curves in the interior of S that have n intersections with Γ and waves that have $\frac{1}{2}n$ intersections with Γ. Thus $\boldsymbol{D}$ contains a finite number of disks.

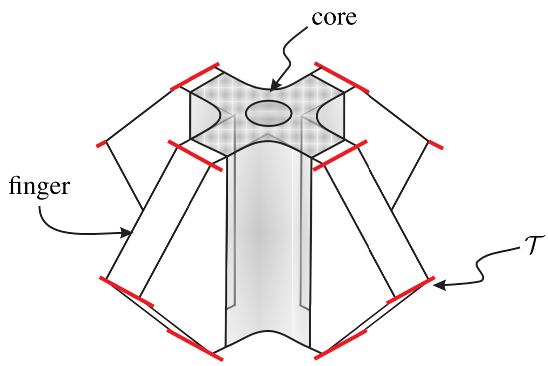

Figure 23: A component of the disk region

Assume that the disks in $\boldsymbol{D}$ have been isotoped so that the intersection between any pair of disks is a set of bisecting arcs and the intersection between any three disks is empty. Let $N(\boldsymbol{D})$ be the regular neighbourhood of $\boldsymbol{D}$. Then the frontier of $N(\boldsymbol{D})$ in H is a set of properly embedded disks that have n intersections with $\mathcal{T}$ and annuli that are disjoint from $\mathcal{T}$. For any of the boundary components that are either nonmeridian disks or nonessential annuli, add the appropriate 3–cell to $N(\boldsymbol{D})$. The resulting submanifold P is the *disk region*.

By Lemma 4.8, we can isotope the disks in $\boldsymbol{D}$ so that the intersection between any pair of disks is a set of parallel bisecting arcs and the intersection between any three is empty. Therefore, for any disk $D_i \in \boldsymbol{D}$, the intersection $\Gamma_i = D_i \cap (\boldsymbol{D} \backslash D_i)$ is a set of parallel bisecting arcs.

Then there are two innermost bisecting arcs in D_i. Therefore, when D_i is cut along the innermost bisecting arcs the result is three disks: two bigons and a third quadrilateral. Let D_i' be the third disk. Let $\boldsymbol{D}'$ be the set of disks produced when this is done to all disks in $\boldsymbol{D}$. Then $\bigcup D_i'$ is an I–bundle over a graph. This fibring can then be extended to the "core" of each component of P. The unfibred parts of each component are the regular neighbourhoods of disks that have $\frac{1}{2}n$ intersections with $\mathcal{T}$ and which boundary compress the frontier annuli of the core. We will call these *fingers*; see Figure 23. Note that each component has at least one finger. Unlike the I–bundle regions defined earlier, the core may have a disk as its base space. The fibring of each component is unique, up to isotopy, except if the component is the regular neighbourhood of a single meridian disk. In the latter case we do not fibre the core until later.

Lemma 4.26 *All possibly singular meridian disks which have n intersections with $\mathcal{T}$ can be homotoped into P.*

Proof Let D be a disk and $f\colon D \to H$ be a possibly singular map such that $A = f(D)$ is a meridian disk, ie has essential boundary. Let P be the maximal disk region, as defined above and $f^{-1}(\mathcal{T})$ be n vertices in ∂D. Then B, the frontier of P in H, is a set of meridian disks and annuli essential in $H - \mathcal{T}$. Then $\Gamma = f^{-1}(B)$ is a set of properly embedded arcs and simple closed curves in D. As H is irreducible there is a homotopy of f to remove all simple closed curves from Γ. Thus Γ is a set of properly embedded disjoint simple arcs in D.

By maximality of P, any boundary compressing disks of a component of B, as described in Section 3.1, must intersect $\mathcal{T}$ more than $\frac{1}{2}n$ times. There must be an innermost disk $D_1 \subset D$ such that $f(D_1)$ intersects $\mathcal{T}$ at most $\frac{1}{2}n$ times. Thus by Dehn's lemma and the loop theorem — see Lemma 3.1 — we can remove any arc from Γ which is in the image of ∂D_1. We can repeat this process until A is disjoint from B. Thus either A is contained in P or disjoint from P. If it is disjoint, then there must be a homotopy of f such that $A \subset P$. Otherwise, using Dehn's lemma and the loop theorem, we get a contradiction to the maximality of P. □

4.5 Handlebodies and singular annuli

In Jaco and Shalen's [7] and K Johannson's [8] proofs of the torus theorem, an essential step is the annulus theorem. In fact, the torus theorem is a consequence of the annulus theorem. Similarly, a lemma that is a slight variation of the annulus theorem is required here. Our annulus theorem is simpler as it is restricted to handlebodies. Namely, suppose a handlebody H has a set of curves in its boundary, $\mathcal{T}$, that satisfies the n disk-condition. Assume also there is a proper essential (possibly singular) map f of an annulus into $H - \mathcal{T}$. Then f is properly homotopic to an essential (possibly singular) map of an annulus into the maximal annulus region. There are two main steps to prove this lemma. The first is to show that if there is a proper singular essential map of an annulus into $H - \mathcal{T}$, then there is a similar embedded one. Next we show any proper essential embedding of an annulus in $H - \mathcal{T}$ is properly isotopic into one of its maximal annulus regions.

Lemma 4.27 *Let H be a handlebody and $\mathcal{T}$ a set of simple closed curves in ∂H that meet the n disk-condition. Let A be an annulus and $f\colon A \to H - \mathcal{T}$ a proper immersion. If f is not properly homotopic into $\partial H - \mathcal{T}$ and the curves $f(\partial A)$ are essential in ∂H, then there is a properly embedded essential annulus in $\partial H - \mathcal{T}$.*

Remark 4.28 The proof for this lemma uses a simplified version of the covering space argument used by Freedman, Hass and Scott [3]. The argument is easier, since we are operating in a handlebody.

Proof The first step is to find another f such that all the lifts of $f(A)$ in the universal cover are embedded. We then use subgroup separability to produce a finite-sheeted cover of H which contains a lift of $f(A)$ that is embedded and does not intersect any of its translates. From this cover we find a regular cover, in which all the lifts of $f(A)$ are embedded. This then implies that the finite regular cover has a nontrivial annulus region and thus so does the original handlebody.

We will assume that the map f is transverse at all times. Let $G = \pi_1(H)$, f_* be the induced map on $\pi_1(A)$ and $f_*(\pi_1(A)) = B \subseteq G$. Therefore, B is a cyclic subgroup generated by some $z \in G$.

Let $\overline{H}$ be the cover of H with the projection $\overline{p}\colon \overline{H} \to H$ such that $\overline{p}_*(\pi_1(\overline{H})) = B$. This means there is a lift, $\overline{f}$ of f, which is an immersed annulus such that $\pi_1(\overline{H}) \cong \overline{f}_*\pi_1(A)$. Let $\overline{\mathcal{T}} = \overline{p}^{-1}(\mathcal{T})$. As f is not properly homotopic into $\partial H - \mathcal{T}$, we have that $\overline{f}$ is not properly homotopic into $\partial \overline{H} - \overline{\mathcal{T}}$.

We now want to find an embedded annulus in $\overline{H}$ which is π_1–injective and not properly homotopic into $\partial \overline{H} - \overline{\mathcal{T}}$. Let $N(\overline{f}(A))$ be a regular neighbourhood of $\overline{f}(A)$ such that $N(\overline{f}(A)) \cap \overline{\mathcal{T}} = \varnothing$. Then the frontier of $N(\overline{f}(A))$ in $\overline{H}$ is a set of embedded surfaces. As $\pi_1(\overline{A}) \cong \pi_1(\overline{H})$, we can find two of these embedded surfaces in $\overline{H}$ both of which have (at least) two essential boundary curves in $\overline{H} - \overline{\mathcal{T}}$. (Note that $\overline{H}$ is a missing boundary solid torus, ie has interior which is an open solid torus and compactifies to a solid torus.) Let one of these surfaces be $\overline{A}'$. The boundary curves of $\overline{f}(A)$ are not homotopic in $\partial \overline{H} - \overline{\mathcal{T}}$, that is, $\overline{f}$ is not homotopic into $\partial \overline{H} - \overline{\mathcal{T}}$; thus the two essential boundary curves of $\overline{A}'$ can be chosen to be not homotopic in $\partial \overline{H} - \overline{\mathcal{T}}$.

By Dehn's lemma and the loop theorem, since $\pi_1(\overline{H})$ is infinite cyclic, we know that any handles in $\overline{A}'$ can be compressed until $\overline{A}'$ is an essential embedded annulus in $\partial \overline{H} - \overline{\mathcal{T}}$. Now let $A' = \overline{p}(\overline{A}')$. We can assume that $\overline{p}$ restricted to $\overline{A}'$ is transverse. Let $\overline{A}_i$, for $1 \leq i \leq n$, be the lifts of A' in $\overline{H}$ that intersect $\overline{A}'$ and $\overline{\alpha}_i = \overline{A}' \cap \overline{A}_i$. Thus each $\overline{\alpha}_i$ is a set of singular curves in $\overline{A}'$.

Let $\widetilde{H}$ be the universal cover of H and therefore also the universal cover of $\overline{H}$ with the projections $p\colon \widetilde{H} \to H$ and $\widetilde{p}\colon \widetilde{H} \to \overline{H}$, such that $p = \overline{p}\widetilde{p}$. As H is a handlebody, $\widetilde{H}$ is a missing boundary ball, that is, a ball with a compact set removed from its boundary. As A' is π_1–injective in H, each pullback to $\widetilde{H}$ is a universal cover of A', an infinite strip. As $\overline{A}'$ is embedded in $\overline{H}$, each pullback to $\widetilde{H}$ is embedded. Then by applying the covering transformation group to $\widetilde{H}$ we know that all the lifts of A' in $\widetilde{H}$ are infinite strips.

Let $\widetilde{A}$ be a lift of $\overline{A}'$ in $\widetilde{H}$. Then any lift of A' in $\widetilde{H}$, that intersects $\widetilde{A}$ must be a lift of one of the $\overline{A}_i$ in $\overline{H}$. Let $\widetilde{A}_i$ be some lift of $\overline{A}_i$ that intersects $\widetilde{A}$ and $\widetilde{\alpha}_i = \widetilde{A} \cap \widetilde{A}_i$.

Note this means that $\widetilde{p}(\widetilde{\alpha}_i) = \overline{\alpha}_i$. Also, let $\widetilde{G}$ be the group of deck transformations on $\widetilde{H}$ and $\widetilde{B} \subset \widetilde{G}$ the stabilizer of $\widetilde{A}$. Therefore, $\widetilde{G} \cong G$ and $\widetilde{B}$ is the cyclic subgroup of translations along $\widetilde{A}$. Also, let $g_i \in \widetilde{G}$ where $g_i(\widetilde{A}) = \widetilde{A}_i$. This means that $g_i \notin \widetilde{B}$ and that $\widetilde{B}_i = g_i\widetilde{B}$ is the set of transformations taking $\widetilde{A}$ to $\widetilde{A}_i$. So for all $b \in \widetilde{B}$, $\widetilde{p}(b(\widetilde{\alpha}_i)) = \overline{\alpha}_i$.

By Hall [4], we know there is a finite index subgroup $\widetilde{L}_i \subseteq \widetilde{G}$ such that $\widetilde{B} \subseteq \widetilde{L}_i$ but $g_i \notin \widetilde{L}_i$. This property is called subgroup separability. For all $b \in \widetilde{B}$, we have that $bg_i\widetilde{A}$ is a translate that intersects $\widetilde{A}$ and $bg_i \notin \widetilde{L}_i$. This means for any $l \in \widetilde{L}_i$ that $l(\widetilde{A}) \neq b(\widetilde{A}_i) = bg_i(\widetilde{A})$ for all $b \in \widetilde{B}$. In other words none of the deck transformations in $\widetilde{L}_i$ map $\widetilde{A}$ to the lift of $\overline{A}_i$ that intersects $\widetilde{A}$. Let $\widehat{H}_i = \widetilde{H}/\widetilde{L}_i$ be the cover of H with the fundamental group corresponding to $\widetilde{L}_i$ such that $\widehat{p}_i\colon \widetilde{H} \to \widehat{H}_i$. Therefore, $\widehat{p}_i(\widetilde{A})$ is an embedded annulus in $\widehat{H}_i$. Also, $\widehat{p}_i(b\widetilde{A}_i) \cap \widehat{p}_i(\widetilde{A}) = \varnothing$ for any $b \in B$, and as $\widetilde{L}_i$ has finite index in G, we have that $\widetilde{H}_i$ is a finite-sheeted cover of H.

Therefore, $L = \widetilde{L}_1 \cap \cdots \cap \widetilde{L}_n$ is a finite index subgroup of $\widetilde{G}$ such that for $l \in L$, either $\widetilde{A} = l(\widetilde{A})$ or $\widetilde{A} \cap l(\widetilde{A}) = \varnothing$. Let $\widetilde{H}/L = \widehat{H}$ be the finite-sheeted cover of H with the projection $\widehat{p}\colon \widetilde{H} \to \widehat{H}$. Then $\widehat{p}(\widetilde{A}) = \widehat{A}$ is an embedded annulus in $\widehat{H}$ that does not intersect any other lifts of A'.

As L has finite index, it must have a finite number of right cosets, $\{Lx_1, \ldots, Lx_n\}$, for $x_1, \ldots, x_n \in G$. Assume that $Lx_1 = L$. Thus if S_n is the group of permutations of n elements, there is a map $\phi\colon G \to S_n$, where $\phi(g)$, for $g \in G$, is the element of S_n that sends $\{Lx_i\}$ to $\{Lx_i g\}$. Both $\phi(g_1)\phi(g_2)$ and $\phi(g_1g_2)$ send $\{Lx_i\}$ to $\{Lx_i g_1 g_2\}$, so ϕ is a homomorphism. Let $K \subseteq G$ be the kernel of ϕ. If $g \in K$, then $Lx_i = Lx_i g = Lgx_i$, thus $K \subseteq L$. As S_n has a finite number of elements, the kernel K is a finite index normal subgroup. Therefore, $\breve{H} = \widetilde{H}/K$ is a finite-sheeted normal cover of H. Let $\breve{p}\colon \breve{H} \to H$ be the covering projection. Then $\breve{H}$ is a handlebody and $\breve{\mathcal{T}} = \breve{p}^{-1}(\mathcal{T})$ is a set of curves in $\partial\breve{H}$ that meet the n disk-condition in $\breve{H}$. Also, $\breve{H}$ is a cover of $\widehat{H}$; thus all the lifts of A' are properly embedded essential annuli in $\partial\breve{H} - \breve{\mathcal{T}}$.

Then by Freedman, Hass and Scott [3], if we put a Riemannian metric on H and properly homotope A' to be of least area, then all trivial self intersections between lifts of A' will be removed, and thus by Lemmas 4.2 and 4.4 all the lifts of A' in $\breve{H}$ are either pairwise disjoint or intersect each other vertically or horizontally. If the lifts of A' are pairwise disjoint, A' must be a properly embedded essential annulus in $\partial H - \mathcal{T}$. Otherwise, by Lemmas 4.10 and 4.18, we know that $\breve{H}$ must have a nontrivial region $N_I \cup N_T$. By Lemmas 4.11 and 4.19, we know that $N_I \cup N_T$ can be isotoped so that its frontier annuli are preserved under K and thus project to properly embedded essential annuli in $\partial H - \mathcal{T}$. □

Lemma 4.29 *If H is a handlebody, $\mathcal{T}$ is a set of triple curves in its boundary that satisfies the n disk-condition and $f\colon A \to H$ is a properly embedded annulus, then f is properly isotopic into the maximal annulus region N.*

Proof Let $f\colon A \to H$ be a properly embedded annulus that cannot be properly isotoped into N. By Lemmas 4.18 and 4.10, we know that if $f(A)$ has a nontrivial intersection with another embedded annulus, then f can be isotoped into N_I or N_T. Therefore, we can isotope f so that its image is disjoint from all the frontier annuli of N. This contradicts maximality of N, thus we must be able to properly isotope $f(A)$ into N. □

Lemma 4.30 *Let H be a handlebody, $\mathcal{T}$ a set of curves in its boundary that satisfies the n disk-condition and N the annulus region in H. If A is an annulus and $f\colon A \to H - \mathcal{T}$ is a proper singular essential map, then there is a proper homotopy of f such that $f(A)$ is in N.*

Proof To save on notation, we will refer to $f(A)$ by A as well. Let B be the set of frontier annuli of N and $\mathcal{T}' = \mathcal{T} \cup \partial B$. Then $H' = \overline{H - N}$ is a set of handlebodies such that for any component H'_j, the set of essential simple closed curves $\mathcal{T}' \cap H'_j$ satisfies the 4 disk-condition in H'_j. Also, there is a proper homotopy of f such that $f^{-1}(N)$ is either a set of 4–gons (case 1) or essential embedded annuli (case 2).

Case 1 All the components of N that A intersects are either in N_I or N_A. Assume the singular 4–gons $H' \cap A$ are essential in H'. Then by Dehn's lemma and the loop theorem, we know that there is an embedded essential 4–gon with two boundary arcs in the frontier annuli of N. This contradicts maximality of N.

Case 2 Here, all the components of N that A intersects are either in N_T or N_A. Then by Lemma 4.27 we know that H' must contain an essential properly embedded annulus, contradicting maximality of N.

Thus there must be a proper homotopy of f such that A is disjoint from B. If A is not contained in N, then once again by Lemma 4.27, H' contains essential embedded annuli, contradicting maximality of N. □

4.6 Torus theorem

Let M be a 3–manifold that satisfies the (n_1, n_2, n_3) disk-condition. That is, $H_i \subset M$ is an embedded handlebody for $1 \le i \le 3$ such that $\bigcup H_i = M$, $\bigcup \partial H_i = X$ is a 2–complex that cuts M up into the H_i, and $\bigcap H_i = \mathcal{T}$ is a set of essential simple closed curves that meet the n_i disk-condition in H_i. We will assume that (n_1, n_2, n_3) is either $(6, 6, 6)$, $(4, 6, 12)$ or $(4, 8, 8)$, for if the gluing of the three handlebodies meets some disk-condition, it meets one of these three.

Lemma 4.31 *Let M be a closed 3–manifold that satisfies the disk-condition as described above. Suppose T is a torus and $f\colon T \to M$ is an essential possibly singular map. Then there is a homotopy of f such that either $f(T)$ is disjoint from $\mathcal{T}$ and $H_i \cap f(T)$ is a set of essential annuli for each i, or $H_i \cap f(T)$ is a set of singular disks for each i with essential boundaries that each intersect $\mathcal{T}$ exactly n_i times.*

Proof Assume that f is transverse to X. Thus $\Gamma = f^{-1}(X)$ is a set of simple closed curves and trivalent embedded graphs which separates T. Define an (m,n)–gon to be a face of T that is a disk, has m vertices in its boundary and is mapped by f into the handlebody in which $\mathcal{T}$ satisfies the n disk-condition. Let the Γ_j be the components of Γ. Then Γ_i is a nonessential component if there is a disk $D \subset T$ such that $\Gamma_i \subset D$. So by Lemma 3.2, we know that there is a homotopy of f to remove Γ_i and hence all nonessential components of Γ.

Consequently, there are two cases. Either all faces of Γ are disks or Γ has faces which are essential annuli. Note that $f(T) \cap X \neq \varnothing$ as f is π_1–injective and $\pi_1(H_i)$ doesn't have a free abelian subgroup of rank 2.

If Γ is connected, then all the faces must be (m,n)–gons and all the vertices have order three. Let $\boldsymbol{F}$ be the set of faces of T. We can then put a metric on T, as we did in the proof of Lemma 3.2. So all the edges are geodesics of unit length, and if $F \in \boldsymbol{F}$ is an (m,n)–gon, then the angle at each vertex is $\pi(1-2/n)$ and there is a cone point in $\operatorname{int}(F)$. Once again this means that the curvature around each vertex is 2π. Let $\boldsymbol{K}(F)$ be the curvature at the cone point in F. By the Gauss–Bonnet theorem, we know that

$$\boldsymbol{K}(F) = 2\pi(1-m/n).$$

Therefore, if $m > n$ then $\boldsymbol{K}(F) < 0$, if $m = n$ then $\boldsymbol{K}(F) = 0$ and if $m < n$ then $\boldsymbol{K}(F) > 0$. Also, by the Gauss–Bonnet theorem, we know that

$$\sum_{F \in \boldsymbol{F}} \boldsymbol{K}(F) = 0.$$

Therefore, if $\boldsymbol{F}$ contains an (m,n)–gon such that $m > n$, then it must also contain a face F such that $m < n$. Thus by the disk-condition we know that $f(\partial F)$ is not essential in ∂H_k. So there is a homotopy of f such that $f(F) \subset \partial H_k$. We can then push F off ∂H_k removing the face F from $\boldsymbol{F}$. Note that when we do this, the order of the faces adjacent to F either decreases by two or an (m,n)–gon and an (m',n)–gon merge to become an $(m+m'-4,n)$–gon, as shown in Figure 3. We can repeat this process as long as $\boldsymbol{F}$ contains faces with positive curvature. Each time we do this move, we reduce the number of faces in $\boldsymbol{F}$ by at least one. Therefore, this process must terminate after a finite number of moves, when all the faces are (m,n)–gons such that $m = n$.

Now let's look at the case where Γ contains more than one component. Let Γ_i be a component of Γ. Then Γ_i cuts T up into faces that are a single annulus and a number of disks. Let A be the union of Γ_i and the faces which are disks. Now we know that the Euler characteristic of A is 0. Put a metric on A as we did above. Γ_i must have boundary vertices, that is vertices adjacent to less than three faces of A. Thus using the same arguments using the Gauss–Bonnet theorem we know that A must have some face with positive curvature. This means that such faces are boundary parallel in the handlebody and there is a homotopy of f to remove them. As before this process can be repeated until all the components are simple closed essential loops. □

We are now ready to prove the torus theorem.

Proof of Theorem 1.1 Let N_i be the maximal annulus region for H_i and P_i be the maximal disk region for H_i. The idea of this proof is to find submanifolds of either the N_i or the P_i such that when glued together, the resulting embedded submanifold can be fibred by $\mathbb{S}^1$ and either has essential tori boundary or the fibring can be extended to the whole of M. In the interest of reducing notation, the image of $f(T)$ in M will be denoted as T. Thus when we talk about a homotopy of T, we are implying a homotopy of f.

By Lemma 4.31, there is a homotopy such that either T is disjoint from $\mathcal{T}$ and for each i, $H_i \cap T$ is a set of essential singular annuli not properly homotopic into $\partial H - \mathcal{T}$ or, for each i, $H_i \cap T$ is a set of singular meridian disks that intersect $\mathcal{T}$ exactly n_i times.

The first case is therefore that T is disjoint from the triple curves and $H_i \cap T$ is a set of singular essential annuli. We can also assume that no components of $H_i \cap T$ are properly homotopic into $\partial H_i - \mathcal{T}$. By Lemma 4.30, we can isotope each N_i so that $H_i \cap T \subset N_i$.

Let $A_i = X \cap N_i$, where $X = \bigcup \partial H_j$. Then A_i is a set of essential surfaces in ∂H_i and the boundary of the maximal annulus region N_i. Note that $T \cap \partial H_i \subset A_i$ and thus $T \cap X \subset \bigcup_{i \neq j} (A_i \cap A_j)$. We will first shrink N_1. Let $S_i = A_i \cap (A_j \cup A_k)$, where i, j and k are different. Let N_1' be the maximal subset of N_1 such that $N_1' \cap X \subseteq S_1$ and each component of the frontier of N_1' in H_1 is an essential annulus parallel to the fibring of N_1. There are three cases to discuss corresponding to components of N_I, N_T and N_A.

Let B be a component of N_1 such that B is an I–bundle region and F is its base space. Then let $F' \subseteq F$ be the maximal subsurface such that $B' \cap \partial H_1 \subseteq S_1$, where B' is the I–bundle over F'. Then B' is a component of N_1'. Note that components that do not intersect S_1 are removed.

If B is a tree region, then it is a fibred solid torus and $B \cap \partial H_1$ is a set of essential annuli. Then there is an isotopy of B such that each annulus in $B \cap \partial H_1$ is either contained in S_1 or in $\mathrm{int}(H_1)$. Note that some annuli in ∂H_1 may get pushed into $\mathrm{int}(H_1)$. Let B' be the resulting fibred torus. Note that when the number of annuli in $B \cap \partial H$ is reduced to produce B', the fibring of the torus is still parallel to the boundary curves of the frontier annuli. Then B' is a component of N_1'. If $B' \cap H_1 = \varnothing$ we remove it from N_1'.

If B is a component of N_A, as defined in Section 4.4.3, then either it can be isotoped so that $B \cap H_1 \subseteq S_1$ or it is removed. As $T \cap X \subset \bigcup_{i \neq j}(A_i \cap A_j)$ we know that $N_1' \neq \varnothing$. We now let $N_1 = N_1'$.

We now repeat this process for each N_i in turn until the process stabilises. That is, for $i \neq j$, $i \neq k$ and $k \neq j$, we have $A_i = \partial H_i \cap (A_j \cup A_k)$. We know that it stabilises before $\bigcup N_i = \varnothing$ because $T \subset \bigcup N_i$.

Next we want to change the fibrings of the N_i so that all components that are regular neighbourhoods of embedded annuli or Möbius bands are fibred by $\mathbb{S}^1$. This means that for any component B of N_i such that $B \cap \partial H_i$ is a set of annuli, then B is a fibred solid torus, or an I–bundle. Now when we let $\boldsymbol{N} = \bigcup N_i$ and all the fibrings of components match, then N is a Seifert fibred submanifold of M and $\partial \boldsymbol{N}$ is a set of embedded tori.

By Lemma 4.15, if $\boldsymbol{N}_j$ is a component of $\boldsymbol{N}$ such that $H_i \cap \boldsymbol{N}_j$ is an I–bundle with a base space that is not an annulus or a Möbius band, then the boundary tori of $\boldsymbol{N}_j$ are essential in $\boldsymbol{M}$. The final step in this case is to either make all the boundary tori of $\boldsymbol{N}$ essential or expand $\boldsymbol{N}$ so that $\boldsymbol{N} = \boldsymbol{M}$. If $\boldsymbol{N}_j$ is a component of $\boldsymbol{N}$ and $F \subset M$ is an embedded solid torus such that $\partial F \subseteq \boldsymbol{N}_j$, then either $F \cap \boldsymbol{N}_j = \partial F$ or $F \cap \boldsymbol{N}_j = \boldsymbol{N}_j$. If $F \cap \boldsymbol{N}_j = \partial F$, we then add F to $\boldsymbol{N}$ and extend the fibring to it. This can always be done as the fibres of the component are essential in $\boldsymbol{M}$. Therefore, the meridian disk of the solid torus being added cannot be parallel to the fibring of $\boldsymbol{N}_j$. If $\boldsymbol{N}_j$ is contained in F we remove $\boldsymbol{N}_j$ from $\boldsymbol{N}$. This process is repeated until either all boundary tori are essential or $\boldsymbol{N} = \boldsymbol{M}$. We know the process will terminate before all of $\boldsymbol{N}$ has been removed because $T \subset \boldsymbol{N}$ and T is essential. Thus the component containing T cannot be contained in a solid torus.

The next case is when $H_i \cap T$ is a set of singular n_i–gons. Let P_i be the disk region in the handlebody H_i. Next we want to define a process for shrinking components of P_i until all their boundaries coincide in X and then show that we can expand the "core" fibring to the whole submanifold. Let $A_i = X \cap P_i$. By Lemma 4.26, we know that we can isotope each P_i so that $H_i \cap g(T) \subset P_i$. Thus $T \cap \partial H_i \subset P_i \cap (P_j \cup P_k)$, for $i \neq j$, $j \neq k$ and $k \neq i$.

Reduce P_1 so that $P_1 \subseteq P_2 \cup P_3$. By reducing, we mean chop off fingers that don't match up, reduce base spaces of the cores and possible remove entire components of P_1. This process finishes before P_1 is entirely removed as $T \cap \partial H_i \subset P_i \cap (P_j \cup P_k)$.

Note that if a component of P_1 is reduced to the regular neighbourhood of a single meridian disk we forget the fibring of its core. As we reduce P_1, the frontier of P_1 in H remains a set of essential annuli and meridian disks.

This process is repeated in turn for each P_i. Once again we know that the process stabilises before all the P_i are removed as $T \cap \partial H_i \subset P_i \cap (P_j \cup P_k)$. All the components with fibred cores obviously match up to be fibred tori in $\boldsymbol{P} = \bigcup P_i$. Clearly these do not intersect so the fibring can be extended across $\boldsymbol{P}$. Also, $\boldsymbol{P}$ is a Seifert fibred submanifold of M and each of the boundary tori of $\boldsymbol{P}$ is tiled by either meridian disks or essential annuli that are essential in T. As before if any of the torus boundaries of $\boldsymbol{P}$ are not essential, they are either filled in with a solid torus or removed. □

4.7 Characteristic variety

Finally we show that both flavours of characteristic variety fit together nicely. That is, if the flavours intersect, their $\mathbb{S}^1$ fibrings can always be made to agree. If either component is a $T^2 \times I$, this is easy. Thus we want to study the case where each component has a unique fibring.

Let $\boldsymbol{N}$ be the maximal annulus region in M and $\boldsymbol{P}$ be the maximal disk region. By the usual arguments, we can see that both are unique up to isotopy. We can also assume that $\boldsymbol{N}$ is disjoint from $\mathcal{T}$ and that both flavours have nonempty boundary. Thus $\partial \boldsymbol{N} \cup \partial \boldsymbol{P}$ is a set of essential embedded tori. If $\boldsymbol{N} \cap \boldsymbol{P} = \varnothing$, then there is no problem. Therefore, we can assume that $\boldsymbol{N} \cap \boldsymbol{P} \neq \varnothing$. Let N' be a component of $\boldsymbol{N}$ and P' be a component of $\boldsymbol{P}$ such that $N' \cap P' \neq \varnothing$. It is not possible for $P' \subset N'$ and if $N' \subset P'$ there is no problem. Therefore, we can assume that there is a boundary torus $B \subset \partial P'$ such that $B \cap N' \neq \varnothing$. As $\partial N'$ is a set of essential tori, $B \cap N'$ is a set of essential annuli in N'. Thus $H_i \cap (B \cap N')$, for any i, is a set of quadrilaterals. Therefore, if the components of $H_i \cap N'$ are fibred by $\mathbb{S}^1$, then $N' \cong T^2 \times I$. Thus we can assume that N' is fibred such that $N' \cap H_i$ is a set of I–bundles. Therefore, it just remains to show that $H_i \cap (N' \cap P')$ is an I–bundle.

Let F and F' be two meridian disks in H_i that have n_i intersections with $\mathcal{T}$ and have a nontrivial intersection and A be an essential properly embedded annulus in $H_i - \mathcal{T}$. We can assume that A has been isotoped so that $F \cap A$ is a set of disjoint properly embedded arcs in F. If any of the arcs in $F \cap A$ are not bisecting, then A is boundary parallel. In this case $F' \cap A$ cannot contain any properly embedded arcs, for if it did, this would provide an isotopy of F to remove that intersection between F and F'. Thus $F \cap A$ must be a set of bisecting arcs in F, similarly $F' \cap A$ is a set of properly embedded bisecting arcs in F' and A is not boundary parallel. If we then let Q be the regular neighbourhood of $F \cup F'$, then B, the frontier of Q in H, is a set of properly embedded annuli and meridian disks that intersect $\mathcal{T}$ exactly n_i times. As in the proof of Lemma 4.8, there is an isotopy of A such that $A \cap B$ is a set of properly embedded parallel arcs that are not boundary parallel in A. Thus there is an isotopy to remove any triple points.

The components of $P' \cap H_i$ can be thought of as regular neighbourhoods of a set of meridian disks that intersect $\mathcal{T}$ exactly n_i times. From above, if there are two meridian disks in H_i that have a nontrivial intersection and that have n_i intersections with $\mathcal{T}$, then any essential annulus can be isotoped so that it is disjoint from their intersection. Lemma 4.15 says any boundary compressing disk of the annuli $N' \cap H_i$ has order at least $\frac{1}{2}n_i$. Therefore, the intersection between frontier annuli of $N' \cap H_i$ and a meridian disk of order n_i must be bisecting in the meridian disk. By these two observations, we can see that $H_i \cap (N' \cap P')$ is an I–bundle.

4.8 Atoroidal manifolds

An interesting question asked us by Cameron Gordon, is to find an additional condition that would result in manifolds satisfying the n disk-condition being atoroidal. By Lemma 4.31, a sufficient condition for a manifold M that satisfies the disk-condition to not contain any essential tori that intersect the triple curves, is the manifold meets a stronger disk-condition with $\sum 1/n_i < \frac{1}{2}$. A sufficient condition that M does not contain any essential tori disjoint from the triple curves is that in at least two of the handlebodies, any essential annuli disjoint from $\mathcal{T}$ are boundary parallel.

Let H be a handlebody and $\mathcal{T}$ an essential set of disjoint simple closed curves in ∂H that meet the n disk-condition. Let A be a properly embedded essential annulus in H disjoint from $\mathcal{T}$. Then by Lemma 3.16, H has a waveless minimal system of disks, $\mathbb{D}$; see Definition 3.9. Let B be the 3–ball produced when H is cut along $\mathbb{D}$, let $S \subset \partial B$ be the punctured sphere produced when ∂H is cut along $\mathbb{D}$ and let $\Gamma = \mathcal{T} \cap S$. As in the proof for Lemma 3.18, let $\Gamma' \subset \mathbb{S}^2$ be the graph produced by letting components of ∂S correspond to vertices and parallel components of Γ correspond to single edges; see Figure 11.

As A is a properly embedded essential annulus, $B \cap A = \{A_1, \dots, A_k\}$ is a set of properly embedded quadrilaterals in B such that $A_i \cap S$ is two properly embedded arcs in S for any i. An equivalent statement to A being boundary parallel is that the curves ∂A are parallel in ∂H or that for each i, the arcs $A_i \cap S$ are parallel in S.

Lemma 4.32 *If Γ' is maximal and contains no 2–cycles* (Definition 3.17), *then all properly embedded annuli in H disjoint from $\mathcal{T}$ are boundary parallel.*

Proof By maximality of Γ', the arcs of $A_i \cap S$, for all i, must be parallel to some arc of Γ and as Γ' contains no 2–cycles, both arcs of $A_i \cap S$ must be parallel to the same arc of Γ and thus parallel. Therefore, from above, any properly embedded essential annulus in $H - \mathcal{T}$ must be boundary parallel. □

Let $K \subset \mathbb{S}^3$ be an (a_1, a_2, a_3) pretzel link such that, for each i, $a_i \geq 4$ and the spanning surface F shown in Figure 12 is orientable. As in Section 3.2.1, let M be the manifold produced by taking the 3–fold branched cover of $\mathbb{S}^3$ with K as the branch set and X be the 2–complex produced by gluing the lifts F in M. Then M satisfies the disk-condition and X is a 2–complex that cuts it up into injective handlebodies. As $a_i \geq 4$, the basis bounded by the curves shown in Figure 12 is an 8–waveless basis (Definition 3.10) for K in the handlebody $\overline{\mathbb{S}^3 - S}$. Therefore, all meridian disks in the handlebody $\overline{\mathbb{S}^3 - S}$ intersect K at least eight times. We can produce a waveless minimal system of meridian disks for the handlebody $\overline{\mathbb{S}^3 - f}$ by removing any one of the disks from the basis. The associated graph Γ', as constructed above satisfies the conditions of Lemma 4.32. Thus the 3–fold branched cover of such a pretzel link satisfies the disk-condition and is atoroidal.

References

[1] **I R Aitchison**, **J H Rubinstein**, *Localising Dehn's lemma and the loop theorem in 3–manifolds*, Math. Proc. Cambridge Philos. Soc. 137 (2004) 281–292 MR

[2] **M Freedman**, **J Hass**, **P Scott**, *Closed geodesics on surfaces*, Bull. London Math. Soc. 14 (1982) 385–391 MR

[3] **M Freedman**, **J Hass**, **P Scott**, *Least area incompressible surfaces in 3–manifolds*, Invent. Math. 71 (1983) 609–642 MR

[4] **M Hall, Jr**, *Coset representations in free groups*, Trans. Amer. Math. Soc. 67 (1949) 421–432 MR

[5] **J Hempel**, *3–Manifolds*, Ann. of Math. Studies 86, Princeton Univ. Press (1976) MR

[6] **W Jaco**, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Mathematics 43, Amer. Math. Soc., Providence, RI (1980) MR

[7] **W Jaco**, **P B Shalen**, *A new decomposition theorem for irreducible sufficiently-large 3–manifolds*, from "Algebraic and geometric topology, part 2" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 71–84 MR

[8] **K Johannson**, *Homotopy equivalences of 3–manifolds with boundaries*, Lecture Notes in Mathematics 761, Springer (1979) MR

[9] **S Matveev**, *Algorithmic topology and classification of 3–manifolds*, Algorithms and Computation in Mathematics 9, Springer (2003) MR

[10] **U Oertel**, *Closed incompressible surfaces in complements of star links*, Pacific J. Math. 111 (1984) 209–230 MR

[11] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR

[12] **C P Rourke**, **B J Sanderson**, *Introduction to piecewise-linear topology*, Ergeb. Math. Grenzgeb. 69, Springer (1972) MR

[13] **P Scott**, *There are no fake Seifert fibre spaces with infinite* π_1, Ann. of Math. 117 (1983) 35–70 MR

Northcote, Victoria, Australia

Department of Mathematics and Statistics, The University of Melbourne
Parkville, Victoria, Australia

jadcoffey@gmail.com, joachim@unimelb.edu.au

http://www.ms.unimelb.edu.au/~rubin/

Received: 21 February 2006 Revised: 19 January 2017