Equivariant iterated loop space theory
and permutative G–categories

BERTRAND J GUILLOU

J PETER MAY

We set up operadic foundations for equivariant iterated loop space theory. We start by building up from a discussion of the approximation theorem and recognition principle for V–fold loop G–spaces to several avatars of a recognition principle for infinite loop G–spaces. We then explain what genuine permutative G–categories are and, more generally, what E_∞–G–categories are, giving examples showing how they arise. As an application, we prove the equivariant Barratt–Priddy–Quillen theorem as a statement about genuine G–spectra and use it to give a new, categorical proof of the tom Dieck splitting theorem for suspension G–spectra. Other examples are geared towards equivariant algebraic K–theory.

Introduction

Let G be a finite group. We will develop equivariant infinite loop space theory in a series of papers. In this introductory one, we focus on the operadic equivariant infinite loop space machine. This is the most topologically grounded machine, as we illustrate by first focusing on its relationship to V–fold deloopings for G–representations V. Genuine permutative G–categories and, more generally, E_∞–G–categories are also defined operadically. They provide the simplest categorical input needed to construct genuine G–spectra from categorical input.

For background, naive G–spectra are just spectra with actions by G. They have their uses, but they are not adequate for serious work in equivariant stable homotopy theory. The naive suspension G–spectra of spheres S^n with trivial G–action are invertible in the naive equivariant stable homotopy category. In contrast, for all real orthogonal G–representations V, the genuine suspension G–spectra of G–spheres S^V are invertible in the genuine equivariant stable homotopy category, where S^V is the one-point compactification of V. Naive G–spectra represent $\mathbb{Z}$–graded cohomology theories, whereas genuine G–spectra represent cohomology theories graded on the real

representation ring $RO(G)$. The $RO(G)$–grading is essential for Poincaré duality and, surprisingly, for many nonequivariant applications.

The zeroth space $E_0 = \Omega^\infty E$ of a naive Ω–G–spectrum is an infinite loop G–space in the sense that it is equivalent to an n–fold loop G–space $\Omega^n E_n$ for each $n \geq 0$. The zeroth space E_0 of a genuine Ω–G–spectrum E is an infinite loop G–space in the sense that it is equivalent to a V–fold loop G–space $\Omega^V E(V)$ for all real representations V. The essential point of equivariant infinite loop space theory is to construct G–spectra from space or category level data. Such a result is called a recognition principle since it allows us to recognize infinite loop G–spaces when we see them. A functor that constructs G–spectra from G–space or G–category level input is called an equivariant infinite loop space machine.

As we shall see, a recognition principle for naive G–spectra is obtained simply by letting G act in the obvious way on the input data familiar from the nonequivariant theory. One of our main interests is to construct and apply an equivariant infinite loop space machine that constructs genuine G–spectra from categorical input.

A permutative category is a symmetric strictly associative and unital monoidal category, and any symmetric monoidal category is equivalent to a permutative category. The classifying space of a permutative category $\mathscr{A}$ is rarely an infinite loop space, but infinite loop space theory constructs an Ω–spectrum $\mathbb{K}\mathscr{A}$ whose zeroth space is a group completion of the classifying space $B\mathscr{A}$. A naive permutative G–category is a permutative category that is a G–category with equivariant structure data. It is a straightforward adaptation of the nonequivariant theory to construct naive G–spectra $\mathbb{K}\mathscr{A}$ from naive permutative G–categories $\mathscr{A}$ in such a way that $\mathbb{K}_0\mathscr{A}$ is a group completion of $B\mathscr{A}$, meaning that $(\mathbb{K}_0\mathscr{A})^H$ is a nonequivariant group completion of $B(\mathscr{A}^H)$ for all subgroups H of G.

In this paper, we explain what genuine permutative G–categories are and what E_∞–G–categories are, and we explain how to construct a genuine G–spectrum $\mathbb{K}_G\mathscr{A}$ from a genuine permutative G–category $\mathscr{A}$ or, more generally, from an E_∞–G–category $\mathscr{A}$. A genuine G–spectrum has an underlying naive G–spectrum, and the underlying naive G–spectrum of $\mathbb{K}_G\mathscr{A}$ will be $\mathbb{K}\mathscr{A}$. Therefore, we still have the crucial group completion property relating $B\mathscr{A}$ to the zeroth G–space of $\mathbb{K}_G\mathscr{A}$.

We use this theory to show how to construct suspension G–spectra from categorical data, giving a new equivariant version of the classical Barratt–Priddy–Quillen (BPQ) theorem for the construction of the sphere spectrum from symmetric groups. In Guillou, May, Merling and Osorno [13], we shall use this version of the BPQ theorem as input to a proof of the results from equivariant infinite loop space theory that were promised in Guillou and May [10], where we described the category of G–spectra as an easily

understood category of spectral presheaves. Here we use this version of the BPQ theorem to give a new categorical proof of the tom Dieck splitting theorem for the fixed-point spectra of suspension G–spectra. The new proof is simpler and gives more precise information than the classical proof by induction up orbit types.

A complementary interest is to understand the geometry of V–fold loop G–spaces. As we shall explain in this paper, these interests lead to quite different perspectives. They are manifested in point-set level distinctions that would be invisible to a more abstract approach. One way of pinpointing these differences is to emphasize the distinction between the role played by E_V–operads for representations V, which are the equivariant generalizations of E_n–operads, and the role played by (genuine) E_∞–operads of G–spaces.

An E_V–space is a G–space with an action by an E_V–operad. We here develop a machine that constructs V–fold loop G–spaces from E_V–spaces. For future perspective, we envision the possibility of an equivariant version of factorization homology in which E_V–operads will govern local structure of G–manifolds in analogy with the role played by E_n–operads in the existing nonequivariant theory. For such a theory, E_∞–operads would be essentially irrelevant.

In contrast, for infinite loop space theory, E_V–operads serve merely as scaffolding used to build a machine that constructs genuine G–spectra from E_∞–G–spaces, which are spaces with an action by some E_∞–operad. The classifying G–spaces of genuine permutative G–categories are examples of E_∞–G–spaces with actions by a particular E_∞–operad $\mathscr{P}_G$, but E_∞–G–spaces with actions by quite different E_∞–operads abound. We concentrate on such an operadic machine in this paper. The machine we concentrate on in the sequels (with Merling and Osorno [31; 12; 13]) makes no use of E_V–operads and does not recognize V–fold loop G–spaces, but it allows a level of categorical power and multiplicative control that is unobtainable with the machine built here.

This paper offers a number of variant perspectives on the topics it studies. We give recognition principles for V–fold loop spaces (Theorem 1.14), for orthogonal G–spectra (Theorem 1.25 and Definition 2.7) and, preserving space level structure invisible in orthogonal G–spectra, for Lewis–May G–spectra (Definition 2.11 and Theorem 2.13). The geometric input data for Theorem 1.14 consists of algebras over the little disks or Steiner operad, $\mathscr{D}_V$ or $\mathscr{K}_V$. For Theorem 1.25, it consists of compatible algebras over the $\mathscr{K}_V$ for all finite-dimensional V.

In both Definitions 2.7 and 2.11, the input data consists of algebras over an E_∞–operad of G–spaces. These algebras may come by applying the classifying-space functor B to algebras over an E_∞–operad of G–categories. The orthogonal spectrum machine

and the Lewis–May spectrum machine are shown to be equivalent by comparing them both to a machine landing in the S_G–modules of Elmendorf, Kriz, Mandell and May (EKMM) [7] and Mandell and May [19]. In effect, the machines landing in Lewis–May G–spectra and in S_G–modules provide highly structured fibrant approximations of the machine landing in orthogonal G–spectra. In retrospect, such fibrant approximation is central to nonequivariant calculational understanding, and one can hope that the same will eventually prove true equivariantly.

The variants have alternative and contradictory good features, which become particularly apparent and relevant when specialized to free E_∞–algebras, where they are all viewed as giving variants of the equivariant BPQ theorem. Thinking unstably and geometrically, Theorem 1.21 shows how the machine recognizes V–fold suspensions $\Sigma^V X$ and shows that the recognition is precisely compatible with the evident G–homeomorphisms $\Sigma^V X \wedge \Sigma^W Y \cong \Sigma^{V \oplus W} Y$. Thinking stably and geometrically, Theorems 1.31 and 2.18 show how the machine recognizes orthogonal or Lewis–May suspension G–spectra $\Sigma_G^\infty X$. In both cases, the recognition is precisely compatible with the standard G–isomorphisms $\Sigma_G^\infty X \wedge \Sigma_G^\infty Y \cong \Sigma_G^\infty (X \wedge Y)$. However, the meaning of Σ_G^∞ is quite different in the two cases. For orthogonal G–spectra, $\Sigma_G^\infty X$ is cofibrant if G is cofibrant as a G–space, but it is never fibrant. For Lewis–May or EKMM G–spectra, $\Sigma_G^\infty X$ is always fibrant and often bifibrant.

Theorems 6.1 and 9.9 show how the machine recognizes suspension G–spectra from two variant categorical inputs. Here we do *not* have precise compatibility with smash products, a failure that will be rectified with a hefty dose of 2–category theory in the sequel [13], but instead we have structure that allows our new proof of the tom Dieck splitting theorem.

As already mentioned, there are three sequels to this paper. The first [31] develops a new version of the Segal–Shimakawa infinite loop space machine and proves among other things that it is equivalent both to the original Segal–Shimakawa machine and to the machine landing in orthogonal G–spectra that we develop here. That requires a generalization of the present machine from operads to categories of operators, about which we say nothing here. The second [12] gives a multiplicative elaboration of the Segal–Shimakawa machine, starting from space level input. The third [13] gives a more categorically sophisticated machine. It starts with more general categorical input than we deal with here, and it gives new information even nonequivariantly.

Outline We begin with a machine for recognizing iterated equivariant loop spaces in Section 1. All versions of our iterated loop space machine are based on use of the Steiner operads, whose equivariant versions have not previously appeared. We define them and compare them to the little disks operads in Section 1.1. All versions

are also based on an approximation theorem, which is explained in Section 1.2. We use a strengthened version due to Rourke and Sanderson [37], and that allows us to obtain slightly stronger versions of the recognition principle than might be expected. The compatibility with smash products of the geometric versions of the recognition principle is based on pairings between Steiner operads that are defined in Section 1.4; the relevant definition of a pairing is recalled in Appendix A. The promised variants of the recognition principle starting from space level input data are given in Sections 1.3, 1.5, 2.2, and 2.3.

Section 2 gives our machines for recognizing infinite loop G–spaces. After recalling the notion of E_∞–G–operad and giving some examples in Section 2.1, the orthogonal and Lewis–May machines are defined and compared in Sections 2.2–2.4. Examples of E_∞–G–spaces are given in Section 2.5. General properties that must hold for any equivariant infinite loop space machine are described in Section 2.6. A recognition principle for naive G–spectra, with G not necessarily finite, is given in Section 2.7. An interesting detail there shows how to use the recognition principle to construct change of universe functors on the space level. The proof uses a double bar construction described in Appendix B.

The recognition principle starting from categorical input is given in Section 4.5. It is preceded by preliminaries about equivariant universal bundles and equivariant E_∞– operads in Section 3 and by a discussion of operadic definitions of naive and genuine permutative G–categories in Section 4. In the brief and parenthetical Section 4.4, we point out how these ideas and our prequel [11] with Merling specialize to give a starting point for equivariant algebraic K–theory; see also Dress and Kuku [6], Fiedorowicz, Hauschild and May [9], Kuku [17] and Merling [33]. We give an alternative and equivalent starting point in the case of G–rings R in Section 8.2.

We give a precise description of the G–fixed E_∞–categories of free $\mathscr{P}_G$–categories in Section 5. This is a precursor of our first categorical version of the BPQ theorem, which we prove in Section 6.1, and of the tom Dieck splitting theorem for suspension G–spectra, which we reprove in Section 6.2.

Changing focus, in Sections 7 and 8 we give three interrelated examples of E_∞–G– operads, denoted by $\mathscr{V}_G$, $\mathscr{V}_G^\times$, and $\mathscr{W}_G$, and give examples of their algebras. This approach to examples is more intuitive than the approach based on genuine permutative G–categories, and it has some technical advantages. It is new and illuminating even nonequivariantly. It gives a more intuitive categorical hold on the BPQ theorem than does the treatment starting from genuine permutative G–categories, as we explain in Section 9.3. It also gives a new starting point for multiplicative infinite loop space theory, both equivariantly and nonequivariantly, but that is work in progress.

Notational preliminaries A dichotomy between Hom objects with G–actions and Hom objects of equivariant morphisms, often denoted using a G in front, is omnipresent. We start with an underlying category $\mathscr{V}$. A G–object X in $\mathscr{V}$ can be defined to be a group homomorphism $G \to \operatorname{Aut} X$. We have the category $\mathscr{V}_G$ of G–objects in $\mathscr{V}$ and all morphisms in $\mathscr{V}$ between them, with G acting by conjugation. We denote the morphism objects of $\mathscr{V}_G$ simply by $\mathscr{V}(X, Y)$.[1] We also have the category $G\mathscr{V}$ of G–objects in $\mathscr{V}$ and G–maps in $\mathscr{V}$. Since objects are fixed by G, we see that $G\mathscr{V}$ is in fact the G–fixed category $(\mathscr{V}_G)^G$, although we shall not use that notation. Thus the Hom object $G\mathscr{V}(X, Y)$ in $\mathscr{V}$ of G–morphisms between G–objects X and Y is the fixed-point object $\mathscr{V}(X, Y)^G$.

One frequently used choice of $\mathscr{V}$ is $\mathscr{U}$, the category of unbased (compactly generated) spaces. We let $\mathscr{T}$ denote the category of based spaces. We assume once and for all that the basepoints $*$ of all given based G–spaces X (or spaces X when $G = e$) are nondegenerate. This means that $* \to X$ is a G–cofibration (satisfies the G–HEP). It follows that $* \to X^H$ is a cofibration for all $H \subset G$.

By an equivalence $f \colon X \to Y$ of G–spaces, we understand a G–map whose fixed-point maps $f^H \colon X^H \to Y^H$ are weak homotopy equivalences for all subgroups H of G. When X and Y have the homotopy types of G–CW complexes, such an f is a G–homotopy equivalence.

By a topological category $\mathscr{C}$, we understand a category internal to $\mathscr{U}$; thus it has an object space and a morphism space such that the structural maps I, S, T, and C are continuous. This is more structure than a topologically enriched category, which would have a discrete space of objects. We also have the based variant of categories internal to $\mathscr{T}$, but $\mathscr{U}$ will be the default.

We let $\mathscr{C}at$ denote the category of (small) topological categories. As above, starting from $\mathscr{C}at$, we obtain the concomitant categories $G\mathscr{C}at$ and $\mathscr{C}at_G$ of G–categories. A G–category is a topological category equipped with an action of G through natural isomorphisms. This is the same structure as a category internal to $G\mathscr{U}$. Similarly, a based G–category is a category internal to $G\mathscr{T}$. That is, an action of G on a topological category $\mathscr{C}$ is given by actions of G on both the object space and the morphism space such that I, S, T, and C are G–maps. In particular, G can and often will act nontrivially on the space of objects. That may be unfamiliar (as the referee noted), but in many of our examples it is essential for proper behavior on passage to H–fixed subcategories for $H \subset G$.

---

[1] In [19] and elsewhere, we used the notation $\mathscr{V}_G(X, Y)$ instead of $\mathscr{V}(X, Y)$, but some readers found that misleadingly analogous to $\operatorname{Hom}_G(X, Y)$.

For brevity of notation, we shall often but not always write $|-|$ for the composite classifying-space functor $B = |N-|$ from topological categories through simplicial spaces to spaces. It works equally well to construct G–spaces from topological G–categories. We assume that the reader is familiar with operads (as originally defined in May [21]) and especially with the fact that operads can be defined in any symmetric monoidal category $\mathcal{V}$. Brief modernized expositions are given in May [27; 28]. Since it is product preserving, the functor $|-|$ takes operads in $\mathcal{C}at$ or in $G\mathcal{C}at$ to operads in $\mathcal{U}$ or in $G\mathcal{U}$, and it takes algebras over an operad $\mathcal{C}$ in $\mathcal{C}at$ or in $G\mathcal{C}at$ to algebras over the operad $|\mathcal{C}|$ in $\mathcal{U}$ or in $G\mathcal{U}$.

To avoid proliferation of letters, we shall write $\mathbb{O}_G$ for the monad on based G–categories constructed from an operad $\mathcal{O}_G$ of G–categories. We shall write $\boldsymbol{O}_G$ for the monad on based G–spaces constructed from the operad $|\mathcal{O}_G|$ of G–spaces. More generally, for an operad $\mathcal{C}_G$ of unbased G–spaces, we write $\boldsymbol{C}_G$ for the associated monad on based G–spaces.

1 E_V–operads and V–fold loop G–spaces

In this geometrically focused chapter, we first define E_V–operads and give two examples. We then relate E_V–spaces to V–fold loop G–spaces via the equivariant approximation theorem and recognition principle. The approximation theorem shows how to approximate "free" V–fold loop G–spaces $\Omega^V \Sigma^V X$ in terms of free algebras $\mathbb{D}_V X$ or $\mathbb{K}_V X$ over the E_V–operad $\mathcal{D}_V$ or $\mathcal{K}_V$. The recognition principle shows how to construct V–fold loop spaces from E_V–algebras. We elaborate multiplicatively by showing how machine-built pairings relate to evident pairings between iterated loop G–spaces. We then give a geometric version of a concrete spacewise infinite loop G–space machine that does not use E_∞–operads and is new even nonequivariantly. This gives a geometric precursor of the BPQ theorem that relates well to smash products. As already noted, we envision that the theory here can provide the local data for an as yet undeveloped equivariant factorization homology theory.

1.1 The little disks and Steiner operads

Definition 1.1 Let $D(V)$ be the open unit disk in V. A little V–disk is a map $d\colon D(V) \to D(V)$ of the form $d(v) = rv + v_0$ for some $r \in [0, 1)$ and some $v_0 \in V$; $c(d) = v_0$ is the center point of d and r is the radius. For $g \in G$, we have $(gd)(v) = rv + gv_0$. Define $\mathscr{D}_V(j)$ to be the G–space of (ordered) j–tuples of little V–disks whose images have empty pairwise intersections. With the evident structure maps determined by disjoint union and composites of little disks, the $\mathscr{D}_V(j)$ form an operad $\mathscr{D}_V$, called the little disks operad.

For a G–space V, let $F(V, j) \subset V^j$ be the configuration space of (ordered) j–tuples of distinct points of V, with G acting by restriction of the diagonal action on V^j. By convention, $F(V, 0)$ is a point, the empty 0–tuple of points in V. We are interested in the special case when V is a real representation of G, by which we understand an orthogonal action of G on a real inner product space. In contrast to the nonequivariant case, very little is known about the (Bredon) homology and cohomology of the G–spaces $F(V, j)$, but we have the following result.

Lemma 1.2 *There is a $(G \times \Sigma_j)$–homotopy equivalence $\mathscr{D}_V(j) \to F(V, j)$ for each $j \geq 0$.*

Proof Choose a decreasing rescaling homeomorphism $\zeta\colon [0, \infty) \to [0, 1)$ and also denote by ζ the rescaling homeomorphism $V \to D(V)$ that sends v to $\zeta(|v|/|v|)v$, where $D(V)$ is the open unit disc in V. Then ζ induces a rescaling homeomorphism $\zeta\colon F(V, j) \to F(D(V), j)$. Define a map $c\colon \mathscr{D}_V(j) \to F(D(V), j)$ by sending little disks to their center points. For a point $\underline{v} = (v_1, \ldots, v_j)$ in $F(D(V), j)$, define

$$\delta(\underline{v}) = \tfrac{1}{2} \min\{|v_i - v_j| \,\big|\, i \neq j\}.$$

Define $s\colon F(D(V), j) \to \mathscr{D}_V(j)$ by $s(\underline{v}) = (d_1, \ldots, d_j)$, where $d_j(v) = \delta(\underline{v})v + v_i$. Then s and c are $(G \times \Sigma_j)$–maps, $c \circ s = \mathrm{id}$, and there is a $(G \times \Sigma_j)$–homotopy $h\colon s \circ c \simeq \mathrm{id}$. If $\underline{d} = (d_1, \ldots, d_j) \in \mathscr{D}(j)$, where $d_i(v) = r_i v + v_i$, then $c(\underline{d}) = \underline{v}$ and $h(\underline{d}, t)$ has i^{th} little V–disk $d_i(t)$ given by $d_i(t)(v) = ((1-t)\delta(\underline{v}) + tr_i)v + v_i$. $\square$

The following definition is the equivariant generalization of the usual definition of an E_n–operad. We say that a map of operads of G–spaces is a weak equivalence if its jth map is a weak $(G \times \Sigma_j)$–equivalence.

Definition 1.3 An operad $\mathscr{C}_G$ of G–spaces is an E_V–operad if there is a chain of weak equivalences of operads connecting $\mathscr{C}_G$ to $\mathscr{D}_V$.

Of course, we could use any operad weakly equivalent to $\mathscr{D}_V$ as a reference operad in the definition. As explained in [30, Section 3], for inclusions $V \subset W$ of inner product spaces, there is no map of operads $\mathscr{D}_V \to \mathscr{D}_W$ that is compatible with suspension, so that use of the little disks operads is inappropriate for iterated loop space theory. The Steiner operads remedy the defect and will be used in [31] to compare the operadic and Segalic equivariant infinite loop space machines. Their equivariant definition is little different from their nonequivariant definition given in [30], following Steiner [46].

Definition 1.4 Let E_V be the space of embeddings $V \to V$, with G acting by conjugation, and let $\mathrm{Emb}_V(j) \subset E_V^j$ be the G–subspace of (ordered) j–tuples of embeddings with pairwise disjoint images. Regard such a j–tuple as an embedding $^jV \to V$, where ^jV denotes the disjoint union of j copies of V (where ^0V is empty). The element id in $\mathrm{Emb}_V(1)$ is the identity embedding, the group Σ_j acts on $\mathrm{Emb}_V(j)$ by permuting embeddings, and the structure maps

$$\gamma\colon \mathrm{Emb}_V(k) \times \mathrm{Emb}_V(j_1) \times \cdots \times \mathrm{Emb}_V(j_k) \to \mathrm{Emb}_V(j_1 + \cdots + j_k)$$

are defined by composition and disjoint union in the evident way [30, Section 3]. This gives an operad Emb_V of G–spaces.

Define $R_V \subset E_V = \mathrm{Emb}_V(1)$ to be the G–subspace of distance-reducing embeddings $f\colon V \to V$. This means that $|f(v) - f(w)| \leq |v - w|$ for all $v, w \in V$. Define a Steiner path to be a map $h\colon I \to R_V$ such that $h(1) = \mathrm{id}$ and let P_V be the G–space of Steiner paths, with action of G induced by the action on R_V. Define $\pi\colon P_V \to R_V$ by evaluation at 0; that is, $\pi(h) = h(0)$.

Define $\mathscr{K}_V(j)$ to be the G–space of (ordered) j–tuples $(h_1, \ldots, h_j)$ of Steiner paths such that the $\pi(h_i)$ have disjoint images. The element id in $\mathscr{K}_V(1)$ is the constant path at the identity embedding, the group Σ_j acts on $\mathscr{K}_V(j)$ by permutations, and the structure maps γ are defined pointwise in the same way as those of Emb_V. This gives an operad of G–spaces, and application of π to Steiner paths gives a map of operads $\pi\colon \mathscr{K}_V \to \mathrm{Emb}_V$. Evaluation of embeddings at $0 \in V$ gives center point $(G \times \Sigma_j)$–maps $c\colon \mathrm{Emb}_V(j) \to F(V, j)$.

The Steiner operads $\mathscr{K}_V$ are reduced, meaning that $\mathscr{K}_V(0)$ is a point, and $\mathscr{K}_0$ is the trivial operad with $\mathscr{K}_0(1) = \mathrm{id}$ and $\mathscr{K}_0(j) = \varnothing$ for $j > 1$. By pullback along π, any space with an action by Emb_V inherits an action by $\mathscr{K}_V$. As in [21, Section 5], [24, Section VII.2], or [30, Section 3], Emb_V acts naturally on $\Omega^V X$ for based G–spaces X.

Proposition 1.5 [46] *There is a weak equivalence of operads $\iota\colon \mathscr{D}_V \to \mathscr{K}_V$.*

Proof For each j, we have a composite $(G \times \Sigma_j)$–map

$$c \circ \pi \colon \mathcal{K}_V(j) \to \mathrm{Emb}_V(j) \to F(V, j).$$

Steiner's nonequivariant proof that $c \circ \pi$ is a Σ_j–homotopy equivalence applies to prove that it is a $(G \times \Sigma_j)$–homotopy equivalence. The argument is a clever and nontrivial variant on the proof above for $\mathscr{D}_V$, but for us the essential point is that it uses the metric on V and the contractibility of I and V in such a way that the construction is clearly G–equivariant.

For a little disk $d(v) = rv + v_0$, define a path of little disks from d to the identity map of $D(V)$ by sending $s \in I$ to the little disk

$$d(s)(v) = (s - rs + r)v + (1 - s)v_0.$$

Conjugating d by the rescaling ζ of Lemma 1.2 gives a distance-reducing embedding $\zeta^{-1} d \zeta \colon V \to V$, and conjugating paths pointwise gives an embedding ι of $\mathscr{D}_V$ as a suboperad of $\mathscr{K}_V$. Composing the inverse $(G \times \Sigma_j)$–homotopy equivalence $F(V, j) \to \mathscr{D}_V(j)$ with $\iota \colon \mathscr{D}_V(j) \to \mathscr{K}_V(j)$ gives an inverse $(G \times \Sigma_j)$–homotopy equivalence to $c \circ \pi$, by Steiner's proof, and it follows that ι is a $(G \times \Sigma_j)$–homotopy equivalence. $\square$

Again, one key advantage of the Steiner operads over the little disks operads is that, for an inclusion $V \subset W$ of G–inner product spaces, there is an induced inclusion $\mathscr{K}_V \to \mathscr{K}_W$ of G–operads such that the map

$$\Omega^V \eta \colon \Omega^V X \to \Omega^V \Omega^{W-V} \Sigma^{W-V} X \cong \Omega^W \Sigma^{W-V} X$$

is a map of $\mathscr{K}_V$–spaces for any G–space X. Here $W - V$ is the orthogonal complement of V in W. If $f \colon V \to V$ is a distance-reducing embedding, then $f \oplus \mathrm{id}_{W-V} \colon W \to W$ is also distance reducing, and this construction induces the inclusion.

1.2 The approximation theorem

Write $\boldsymbol{K}_V$ for the monad on based G–spaces associated to the operad $\mathscr{K}_V$. For a G–space X, we have $\boldsymbol{K}_V X = \bigsqcup \mathscr{K}_V(j) \times_{\Sigma_j} X^j / (\sim)$. If $\sigma_i \colon \mathscr{K}_V(j) \to \mathscr{K}_V(j-1)$ deletes the ith Steiner path and $s_i \colon X^{j-1} \to X^j$ inserts the basepoint in the ith position, then $(\sigma^i k, y) \sim (k, s_i y)$ for $k \in \mathscr{K}_V(j)$ and $y \in X^{j-1}$. The monad $\boldsymbol{D}_V$ arising from the operad $\mathscr{D}_V$ is defined the same way.

The unit $\eta \colon \mathrm{Id} \to \Omega^V \Sigma^V$ of the monad $\Omega^V \Sigma^V$ and the action θ of $\boldsymbol{K}_V$ on the G–spaces $\Omega^V \Sigma^V X$ induce a composite natural map

$$\alpha_V \colon \boldsymbol{K}_V X \xrightarrow{\boldsymbol{K}_V \eta} \boldsymbol{K}_V \Omega^V \Sigma^V X \xrightarrow{\theta} \Omega^V \Sigma^V X,$$

and $\alpha_V\colon \boldsymbol{K}_V \to \Omega^V \Sigma^V$ is a map of monads whose adjoint defines a right action of $\boldsymbol{K}_V$ on the functor Σ^V, just as in [21]. The restriction to $\boldsymbol{D}_V$ gives the corresponding map $\alpha_V\colon \boldsymbol{D}_V X \to \Omega^V \Sigma^V X$.

The heart of the operadic recognition principle is the approximation theorem that says that α_V is a group completion. However, already nonequivariantly, we have two variants of what it means for a map $X \to Y$ to be a group completion. Recall that Hopf spaces are spaces with a product with a two-sided unit element up to homotopy.

Definition 1.6　A Hopf space Y is grouplike if $\pi_0(Y)$ is a group. Let X and Y be homotopy associative and commutative Hopf spaces, where Y is grouplike, and let $f\colon X \to Y$ be a Hopf map. Then f is a group completion if $f_*\colon \pi_0(X) \to \pi_0(Y)$ is the Grothendieck construction converting a commutative monoid to an abelian group and if, for any field of coefficients k, the map of commutative k–algebras $H_*(X)[\pi_0(X)^{-1}] \to H_*(Y)$ induced by f_* is an isomorphism.

The second version of group completion drops the commutativity assumption and lives in the setting of A_∞–spaces. For us, an A_∞–space will mean a space with an action of the Steiner operad $\mathscr{K}_\mathbb{R}$. An A_∞–map will mean either a map homotopic to a map of $\mathscr{K}_\mathbb{R}$–spaces or the homotopy inverse of a map of $\mathscr{K}_\mathbb{R}$–spaces that is an underlying homotopy equivalence.

Definition 1.7　An A_∞–map $f\colon X \to Y$ of $\boldsymbol{K}_\mathbb{R}$–spaces is a weak group completion if it is equivalent under a chain of A_∞–maps to the natural map $\eta\colon M \to \Omega BM$ for some topological monoid M.

The following classical result has several proofs; see [23, Section 15] for discussion in slightly greater generality.

Theorem 1.8　*If a topological monoid M is homotopy commutative, then the natural map $\eta\colon M \to \Omega BM$ is a group completion.*

Returning to our equivariant context, we have the following definition.

Definition 1.9　A Hopf G–space Y is grouplike if each $\pi_0(Y^H)$ is a group. Let X and Y be homotopy associative and commutative Hopf G–spaces, where Y is grouplike, and let $f\colon X \to Y$ be a Hopf G–map. Then f is a group completion if $f^H\colon X^H \to Y^H$ is a group completion for all subgroups H of G.

For the equivariant notion of weak group completion, note that if X is a $\boldsymbol{K}_\mathbb{R}$–$G$–space and $H \subset G$ is a subgroup, then X^H inherits an action of $\boldsymbol{K}_\mathbb{R}$.

Definition 1.10 A map $f\colon X \to Y$ of $\boldsymbol{K}_{\mathbb{R}}$–$G$–spaces is a weak group completion if f^H is a weak group completion for all H. By Theorem 1.8, f is then a group completion if X and Y are homotopy commutative.

In the weak case we require no compatibility between the monoids $M(H) \simeq X^H$ as H varies. Recall that we understand equivalences of G–spaces to mean maps that induce (weak) equivalences on passage to fixed points and observe that a group completion is an equivalence if X is grouplike, for example if X is G–connected in the sense that each X^H is (path) connected.

Theorem 1.11 (the approximation theorem) *Let V be a representation of G. If X is G–connected, then $\alpha_V\colon \boldsymbol{K}_V X \to \Omega^V \Sigma^V X$ is an equivalence. If V contains a copy of the trivial representation $\mathbb{R}$, then α_V is a weak group completion. Therefore, if V contains a copy of $\mathbb{R}^2$, then α_V is a group completion.*

We shall not give a proof, only a commentary on the existing proofs. The group completion version was first proven by Hauschild in his unpublished Habilitationschrift [14], but the shorter published version [15] restricts to the case $X = S^0$, remarking that the proof in the general case is essentially the same. Assuming that V contains $\mathbb{R}^\infty$ and not just $\mathbb{R}^2$, Caruso and Waner [3, Theorem 1.18] gave a shorter proof in a paper that concentrated on compact Lie groups G, rather than just finite groups.

Nonequivariantly, there is a proof by direct calculation due to Fred Cohen [18] and a geometric proof due to Segal [42]. Starting from Segal's proof, Rourke and Sanderson [38; 39; 40] gave an elegant proof using their "compression theorem". Following up a suggestion of May, they generalized that proof to give the stated version of the theorem in [37]. However, their notation is quite different from ours. They never work equivariantly and focus instead on G–fixed-point spaces. They use the notation $\Omega^V \Sigma^V X$ for the G–fixed-point space $(\Omega^V \Sigma^V X)^G$. One can replace G by a subgroup H in their proof, and it works just as well.

All known proofs are manifold-theoretic in nature and start with the G–space $\boldsymbol{F}_V X$ of (unordered) configurations of points in V with labels in X. More precisely, $\boldsymbol{F}_V X = \bigsqcup F(V, j) \times_{\Sigma_j} X^j/(\sim)$ is defined in the same way as $\boldsymbol{K}_V X$. In the notation of [37], their $C_V X$ is our $(\boldsymbol{F}_V X)^G$. They work with little disks, and their $C_V^o X$ is our $(\boldsymbol{D}_V X)^G$. Their map j_V is the restriction to $\boldsymbol{F}_V(X)^G$ of our map α_V^G.

Translated to our notation, [37, Theorem 1] proves the first statement of Theorem 1.11, taking X to be G–connected; here there are no Hopf G–space structures in sight. When $W = V \oplus \mathbb{R}$, Rourke and Sanderson observe that $(\boldsymbol{D}_W X)^G$ is equivalent to a monoid,

and their [37, Theorem 2] proves that its classifying space is weak homotopy equivalent to $(\Omega^V \Sigma^W X)^G$. The approximation theorem as stated follows by applying Ω as in [37, Corollary 1].

1.3 The recognition principle for V–fold loop spaces

We explain how K_V–spaces, which are based spaces with an action of $\mathscr{K}_V$, give rise to V–fold loop spaces. For fixed V, we can work equally well with $\mathscr{D}_V$. For compatibility as V varies, $\mathscr{K}_V$ is required. The two-sided monadic bar construction is described in [21; 30] and works exactly the same way equivariantly as nonequivariantly.[2] The adjoint of α_V gives a right action $\tilde{\alpha}_V \colon \Sigma^V K_V \to \Sigma^V$ of the monad K_V on the functor Σ^V.

Definition 1.12 Let Y be a $\mathscr{K}_V$–space. We define

$$\mathbb{E}_V Y = B(\Sigma^V, K_V, Y).$$

We have the diagram of $\mathscr{K}_V$–spaces and $\mathscr{K}_V$–maps

$$(1.13) \qquad Y \xleftarrow{\varepsilon} B(K_V, K_V, Y) \xrightarrow{\bar{\alpha}_V} B(\Omega^V \Sigma^V, K_V, Y) \xrightarrow{\zeta} \Omega^V B(\Sigma^V, K_V, Y),$$

where $\bar{\alpha}_V = B(\alpha_V, \mathrm{id}, \mathrm{id})$ and ζ will be defined in the following sketch proof, which is based on arguments in [4; 21; 22].

Theorem 1.14 (from $\mathscr{K}_V$–spaces to V–fold loop spaces) *The following statements hold relating a $\mathscr{K}_V$–space Y to its **V–fold delooping** $\mathbb{E}_V Y$:*

(i) *The map ε is a G–homotopy equivalence with a natural homotopy inverse ν.*

(ii) *The map $\bar{\alpha}_V$ is an equivalence when Y is G–connected and is a weak group completion when $V \supset \mathbb{R}$.*

(iii) *The map ζ is an equivalence.*

Therefore, the composite

$$(1.15) \qquad\qquad \xi = \zeta \circ \bar{\alpha}_V \circ \nu \colon Y \to \Omega^V \mathbb{E}_V Y$$

is an equivalence if Y is G–connected, a weak group completion if $V \supset \mathbb{R}$, and a group completion if $V \supset \mathbb{R}^2$.

Proof The proof of (i) uses an "extra-degeneracy argument" explained in [21, Proposition 9.8]; note that the homotopy equivalence ν is not a $\mathscr{K}_V$–map. For (ii), it is shown nonequivariantly in [22, Theorem 2.3], that $\bar{\alpha}_V$ is an equivalence when Y is connected

---

[2] In particular, Reedy cofibrancy (or properness) works the same way; see [31].

and is a group completion when $V = \mathbb{R}^n$ with $n \geq 2$. We use Theorem 1.11 to improve on that equivariantly. Geometric realization of simplicial G–spaces commutes with passage to H–fixed points, so we can work nonequivariantly, one fixed-point space at a time. If Y is G–connected, each $(\mathbb{K}_V^q Y)^H$ is connected, hence $\bar{\alpha}^H$ is the realization of a levelwise equivalence of simplicial spaces and hence an equivalence.

Now assume $V \supset \mathbb{R}$ and let $\mathscr{K} = \mathscr{K}_\mathbb{R}$, with associated monad $\boldsymbol{K}$. We then have an inclusion of the nonequivariant A_∞–operad $\mathscr{K}$ in $\mathscr{K}_V$ and can regard Y and each $(\mathbb{K}_V^q Y)^H$ as a $\mathscr{K}$–space. From here we combine arguments from [21, Section 13] and the proof of [22, Theorem 2.3] with the Rourke–Sanderson proof of the approximation theorem. Let $\mathscr{M}$ be the associativity operad that defines monoids; we have a weak equivalence of (G–fixed) operads $\delta\colon \mathscr{K} \to \mathscr{M}$. For a $\mathscr{K}$–space X, we define a topological monoid $\Lambda(X) = B(\boldsymbol{M}, \boldsymbol{K}, X)$, where the monad $\boldsymbol{M}$ is a $\boldsymbol{K}$–functor via δ. We have a zigzag

$$X \xleftarrow{\varepsilon} B(\boldsymbol{K}, \boldsymbol{K}, X) \xrightarrow{\bar{\delta}} B(\boldsymbol{M}, \boldsymbol{K}, X) = \Lambda X$$

in which ε is a $\mathscr{K}$–map and a G–homotopy equivalence and $\bar{\delta} = B(\delta, \mathrm{id}, \mathrm{id})$ is an equivalence. Define $\Gamma(X) = \Omega B \Lambda(X)$ and $\gamma = \eta \circ \bar{\delta}\colon B(\boldsymbol{K}, \boldsymbol{K}, X) \to \Gamma X$. We view γ as a natural choice of a weak group completion. Moreover, γ is an equivalence if X is grouplike. If $f\colon X \to Y$ is a weak group completion between $\mathscr{K}$–spaces, then Γf is an equivalence. To see this, note that by the definition of weak group completion, we may assume without loss of generality that f is the map $\eta\colon M \to \Omega BM$ for some topological monoid M. It suffices to show that $B\Lambda(\eta)\colon B\Lambda M \to B\Lambda(\Omega BM)$ is an equivalence. This follows from [47, Proposition 3.9 and Theorem 3.11].

Now consider the following commutative diagram:

$$
\begin{array}{ccccc}
Y & \xleftarrow{\varepsilon} & B(\boldsymbol{K}_V, \boldsymbol{K}_V, Y) & \xrightarrow{\bar{\alpha}_V} & B(\Omega^V \Sigma^V, \boldsymbol{K}_V, Y) \\
\varepsilon \uparrow & & \varepsilon \uparrow & & \varepsilon \uparrow \\
B(\boldsymbol{K}, \boldsymbol{K}, Y) & \xleftarrow{\varepsilon} & B(B(\boldsymbol{K}, \boldsymbol{K}, \boldsymbol{K}_V), \boldsymbol{K}_V, Y) & \xrightarrow{\bar{\alpha}_V} & B(B(\boldsymbol{K}, \boldsymbol{K}, \Omega^V \Sigma^V), \boldsymbol{K}_V, Y) \\
\gamma \downarrow & & \bar{\gamma} \downarrow & & \bar{\gamma} \downarrow \\
\Gamma Y & \xleftarrow{\varepsilon} & B(\Gamma \boldsymbol{K}_V, \boldsymbol{K}_V, Y) & \xrightarrow{\overline{\Gamma \alpha}_V} & B(\Gamma \Omega^V \Sigma^V, \boldsymbol{K}_V, Y)
\end{array}
$$

The maps ε are G–homotopy equivalences, hence the middle map $\bar{\gamma} = B(\gamma, \mathrm{id}, \mathrm{id})$ is a weak group completion since γ is so. The right map $\bar{\gamma}$ and the bottom map $\overline{\Gamma \alpha}_V$ are equivalences since realization preserves levelwise equivalences. Therefore, $\bar{\alpha}_V$ is a weak group completion.

In (iii), ζ is an instance of the natural G–map $\zeta\colon |\Omega^V K| \to \Omega^V |K|$ for simplicial based G–spaces K; suspensions commute with realization, and the adjoint of ζ is the evident evaluation G–map $\Sigma^V |\Omega^V K| \cong |\Sigma^V \Omega^V K| \to |K|$. The proof of (iii) is due to Hauschild [14] and appears in [4, pages 495–496]. We will not repeat the argument, which reduces the proof to the nonequivariant case treated in [21, Section 12]. The main equivariant input that allows the reduction is the fact if $S(V)$ is the unit sphere in V, then the space $\mathrm{Map}_H(S(V), K_n)$ of H–maps is connected, where $K_n = \Sigma^V K_V^n Y$ is the G–space of n–simplices of the simplicial G–space $B_*(\Sigma^V, \boldsymbol{K}_V, Y)$. This holds since K_n^J is $(\dim(V^J)-1)$–connected for each subgroup $J \subset G$, while $S(V)$ regarded as an H–CW complex only has cells of type $H/J \times e^n$ where $n < \dim(V^J)$. $\qquad \square$

Remark 1.16 Equivariant homotopy theory often admits varying generalizations of nonequivariant theorems. A very different and very interesting equivariant recognition principle was proven by Salvatore and Wahl [41].

1.4 The pairing $(\mathscr{K}_V, \mathscr{K}_W) \to \mathscr{K}_{V \oplus W}$ and the recognition principle

The general notion of a pairing of operads is recalled in Appendix A. In [21, Proposition 8.3], a pairing
$$\boxtimes\colon \boldsymbol{C}_m X \wedge \boldsymbol{C}_n Y \to \boldsymbol{C}_{m+n}(X \wedge Y)$$

is defined for based spaces X and Y, where $\boldsymbol{C}_n$ denotes the monad on based spaces induced from the little n–cubes operad $\mathscr{C}_n$. Implicitly, it comes from a pairing of operads $\boxtimes\colon (\mathscr{C}_m, \mathscr{C}_n) \to \mathscr{C}_{m+n}$. The Steiner operad analogue appears in [25, page 337], and we recall it here.

Proposition 1.17 *For finite-dimensional real inner product G–spaces V and W, there is a unital, associative, and commutative system of pairings*
$$\boxtimes\colon (\mathscr{K}_V, \mathscr{K}_W) \to \mathscr{K}_{V \oplus W}$$

of Steiner operads of G–spaces.

Proof The required maps
$$\boxtimes\colon \mathscr{K}_V(j) \times \mathscr{K}_W(k) \to \mathscr{K}_{V \oplus W}(jk)$$

are given by $(c \otimes d) = e$, where, writing $c = (f_1, \ldots, f_j)$ and $d = (g_1, \ldots, g_k)$, e is the jk–tuple of Steiner paths
$$(f_q, g_r)\colon I \to R_V \times R_W \subset R_{V \oplus W}$$

for $1 \le q \le j$ and $1 \le r \le k$, ordered lexicographically. The formulas required in Definition A.1 are easily verified, as we illustrate in Example A.4.

The pairing is unital in the sense that $\boxtimes\colon \mathcal{K}_V(j) \cong \mathcal{K}_0(1) \times \mathcal{K}_V(j) \to \mathcal{K}_V(j)$ is the identity map. It is associative in the sense that the following diagram commutes for a triple (V, W, Z) of inner product G–spaces and a triple (i, j, k):

$$
\begin{array}{ccc}
\mathcal{K}_V(i) \times \mathcal{K}_W(j) \times \mathcal{K}_Z(k) & \xrightarrow{\ \boxtimes \times \mathrm{id}\ } & \mathcal{K}_{V \oplus W}(ij) \times \mathcal{K}_Z(k) \\
{\scriptstyle \mathrm{id} \times \boxtimes} \downarrow & & \downarrow {\scriptstyle \boxtimes} \\
\mathcal{K}_V(i) \times \mathcal{K}_{W \oplus Z}(jk) & \xrightarrow{\ \boxtimes\ } & \mathcal{K}_{V \oplus W \oplus Z}(ijk)
\end{array}
$$

It is commutative in the sense that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{K}_V(j) \times \mathcal{K}_W(k) & \xrightarrow{\ \boxtimes\ } & \mathcal{K}_{V \oplus W}(jk) \\
{\scriptstyle t} \downarrow & & \downarrow {\scriptstyle \tau(j,k)} \\
\mathcal{K}_W(k) \times \mathcal{K}_V(j) & \xrightarrow{\ \boxtimes\ } & \mathcal{K}_{W \oplus V}(kj)
\end{array}
$$

Here t is the interchange map and $\tau(j, k)$ is determined in an evident way by the interchange map for V and W and the permutation $\tau(j, k)$ of jk–letters. $\qquad\square$

Passing to monads as in Proposition A.3 below, we obtain a unital, associative, and commutative system of pairings

$$(1.18) \qquad\qquad \boxtimes\colon \boldsymbol{K}_V X \wedge \boldsymbol{K}_W Y \to \boldsymbol{K}_{V \oplus W}(X \wedge Y).$$

For the unit property, when $V = 0$ the map $\boxtimes\colon X \wedge \boldsymbol{K}_W Y \to \boldsymbol{K}(X \wedge Y)$ is induced by the maps $X \times Y^j \to (X \times Y)^j$ obtained from the diagonal map on X and shuffling. We have the following key observation. Its analogue for the little cubes operads is [21, Proposition 8.3].

Lemma 1.19 *The following diagram commutes:*

$$
\begin{array}{ccc}
\boldsymbol{K}_V X \wedge \boldsymbol{K}_W Y & \xrightarrow{\ \boxtimes\ } & \boldsymbol{K}_{V \oplus W}(X \wedge Y) \\
{\scriptstyle \alpha_V \wedge \alpha_W} \downarrow & & \downarrow {\scriptstyle \alpha_{V \oplus W}} \\
\Omega^V \Sigma^V X \wedge \Omega^W \Sigma^W Y & \xrightarrow[\wedge]{} & \Omega^{V \oplus W} \Sigma^{V \oplus W}(X \wedge Y)
\end{array}
$$

The notion of a pairing of a $\mathcal{K}_V$–space X and a $\mathcal{K}_W$–space Y to a $\mathcal{K}_{V \oplus W}$–space Z is defined in Definition A.2, and we have the following recognition principle for pairings.

Note that smashing maps out of spheres gives a natural map

$$\Omega^V X \wedge \Omega^W Y \to \Omega^{V \oplus W}(X \wedge Y).$$

Proposition 1.20 *A pairing* $f \colon X \wedge Y \to Z$ *of a* $\mathscr{K}_V$–space X *and a* $\mathscr{K}_W$–space Y *to a* $\mathscr{K}_{V \oplus W}$–space Z *induces a* G–map

$$\mathbb{E} f \colon \mathbb{E}_V X \wedge \mathbb{E}_W Y \to \mathbb{E}_{V \oplus W} Z$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
X \wedge Y & \xrightarrow{\ \xi \wedge \xi\ } \Omega^V \mathbb{E}_V X \wedge \Omega^W \mathbb{E}_W Y \longrightarrow & \Omega^{V \oplus W}(\mathbb{E}_V X \wedge \mathbb{E}_W Y) \\
{\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \mathbb{E} f} \\
Z & \xrightarrow[\ \xi\ ]{} & \Omega^{V \oplus W} \mathbb{E}_{V \oplus W} Z
\end{array}
$$

Proof By convention, $\boldsymbol{K}_V^0 = \mathrm{Id}$ for any V. Starting at $q = 0$ with the identity map on $X \wedge Y$, the map $\boxtimes$ inductively determines a pairing $\boxtimes^q$ for all q, namely the composite

$$\boldsymbol{K}_V^q X \wedge \boldsymbol{K}_W^q Y \xrightarrow{\ \boxtimes\ } \boldsymbol{K}_{V \oplus W}(\boldsymbol{K}_V^{q-1} X \wedge \boldsymbol{K}_W^{q-1} Y) \xrightarrow{\ \boldsymbol{K}_{V \oplus W} \boxtimes^{q-1}\ } \boldsymbol{K}_{V \oplus W}^q(X \wedge Y).$$

The map $\mathbb{E} f$ is the geometric realization of a map of simplicial topological spaces that is given on q–simplices by

$$\Sigma^V \boldsymbol{K}_V^q X \wedge \Sigma^W \boldsymbol{K}_W^q Y \cong \Sigma^{V \oplus W}(\boldsymbol{K}_V^q X \wedge \boldsymbol{K}_W^q Y) \xrightarrow{\ \Sigma^{V \oplus W} \boxtimes^q\ } \Sigma^{V \oplus W} \boldsymbol{K}_{V \oplus W}^q(X \wedge Y).$$

Commutation with face and degeneracy operators follows from Proposition A.3. The diagram in the statement commutes by a diagram chase from Lemma 1.19, Definition A.2, and the description of ξ given in (1.15).   □

We have an unstable precursor of the BPQ theorem.

Theorem 1.21 (the BPQ theorem for V–fold suspensions) *For based* G–spaces X, *there is a natural* G–homotopy equivalence

$$\omega \colon \Sigma^V X \to \mathbb{E}_V \boldsymbol{K}_V X$$

such that the following diagram commutes for based G–spaces X *and* Y :

$$
\begin{array}{ccc}
\Sigma^V X \wedge \Sigma^W Y & \xrightarrow{\ \omega \wedge \omega\ } & \mathbb{E}_V \boldsymbol{K}_V X \wedge \mathbb{E}_W \boldsymbol{K}_W Y \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \mathbb{E}(\boxtimes)} \\
\Sigma^{V \oplus W}(X \wedge Y) & \xrightarrow[\ \omega\ ]{} & \mathbb{E}_{V \oplus W} \boldsymbol{K}_{V \oplus W}(X \wedge Y)
\end{array}
$$

Therefore, $\mathbb{E}(\boxtimes)$ *is an equivalence.*

Proof Since $\mathbb{E}_V K_V X = B(\Sigma^V, K_V, K_V X)$, another extra-degeneracy argument explained in [21, Proposition 9.8] gives the natural homotopy equivalence ω. For the diagram, it suffices to prove commutativity of the adjoint diagram, which features two adjoint maps $X \wedge Y \to \Omega^{V \oplus W}(-)$. These maps are equal by inspection of definitions. $\square$

1.5 The geometric recognition principle for orthogonal G–spectra

As in [10], we let $G\mathcal{S}$ denote the category of orthogonal G–spectra. Briefly, these start with $\mathscr{I}_G$–spaces E, which are continuous functors $E: \mathscr{I}_G \to \mathscr{T}_G$, where $\mathscr{I}_G$ is the category of finite-dimensional G–inner product spaces and linear isometric isomorphisms, with G acting by conjugation on morphism spaces $\mathscr{I}_G(V, V')$. The continuous G–maps $E: \mathscr{I}_G(V, V') \to \mathscr{T}_G(E(V), E(V'))$ can be specified via adjoint evaluation G–maps $\mathscr{I}_G(V, V')_+ \wedge E(V) \to E(V')$.

An $\mathscr{I}_G$–space E is an orthogonal G–spectrum if there exist structure G–maps $\Sigma^W E(V) \to E(V \oplus W)$ that give a natural transformation $E \overline{\wedge} S_G \to E \circ \oplus$ of functors $\mathscr{I}_G \times \mathscr{I}_G \to \mathscr{T}_G$, where $S_G = \{S^V\}$ is the sphere G–spectrum, $\overline{\wedge}$ is the external smash product specified by $(D \overline{\wedge} E)(V, W) = D(V) \wedge E(W)$ for $\mathscr{I}_G$–spaces D and E, and $\oplus: \mathscr{I}_G \times \mathscr{I}_G \to \mathscr{I}_G$ is the direct sum of G–inner product spaces functor. See [19, Section II.2] for details.

Definition 1.22 We define a continuous G–functor $\mathscr{K}_*$ from $\mathscr{I}_G$ to G–operads. It takes a G–inner product space V to the Steiner operad $\mathscr{K}_V$. Linear isometric isomorphisms $i: V \to V'$ act by conjugation of embeddings to send R_V to $R_{V'}$. The action extends pointwise to Steiner paths and then applies one at a time to j–tuples of Steiner paths to give G–maps $\mathscr{K}_V(j)$ to $\mathscr{K}_{V'}(j)$. Compatibility with the operad structure is immediate. Composing with the functor that sends the operad $\mathscr{K}_V$ to the associated monad K_V on based G–spaces gives a functor K from $\mathscr{I}_G$ to the category of monads in the category of $\mathscr{I}_G$–spaces. In more detail, for an $\mathscr{I}_G$–space $\mathscr{X}$ with V^{th} space $\mathscr{X}(V)$, we have based evaluation G–maps

$$\mathscr{I}(V, V')_+ \wedge K_V \mathscr{X}(V) \to K_{V'} \mathscr{X}(V').$$

Using the diagonal action of $\mathscr{I}_G(V, V')$, we obtain G–maps

$$\mathscr{I}_G(V, V') \times \mathscr{K}_V(k) \times \mathscr{K}_V(j_1) \times \cdots \times \mathscr{K}_V(j_k) \times \mathscr{X}(V)$$
$$\downarrow$$
$$\mathscr{K}_{V'}(k) \times \mathscr{K}_{V'}(j_1) \times \cdots \times \mathscr{K}_{V'}(j_k) \times \mathscr{X}(V'),$$

and these give evaluation G–maps

$$\mathscr{I}_G(V, V')_+ \wedge K_V K_V \mathscr{X}(V) \to K_{V'} K_{V'} \mathscr{X}(V').$$

The product and unit maps are compatible with these maps in the sense that the following diagrams commute, where the unlabeled arrows are evaluation G–maps:

$$
\begin{array}{ccc}
\mathscr{I}_G(V, V')_+ \wedge \mathscr{X}(V) & \xrightarrow{\ \pi\ } & \mathscr{X}(V') \\
{\scriptstyle \mathrm{id} \wedge \eta} \downarrow & & \downarrow {\scriptstyle \eta} \\
\mathscr{I}_G(V, V')_+ \wedge \boldsymbol{K}_V \mathscr{X}(V) & \longrightarrow & \boldsymbol{K}_{V'} \mathscr{X}(V')
\end{array}
$$

(1.23)

$$
\begin{array}{ccc}
\mathscr{I}_G(V, V')_+ \wedge \boldsymbol{K}_V \boldsymbol{K}_V \mathscr{X}(V) & \longrightarrow & \boldsymbol{K}_{V'} \boldsymbol{K}_{V'} \mathscr{X}(V) \\
{\scriptstyle \mathrm{id} \wedge \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\
\mathscr{I}_G(V, V')_+ \wedge \boldsymbol{K}_V \mathscr{X} & \longrightarrow & \boldsymbol{K}_{V'} \mathscr{X}(V')
\end{array}
$$

Note that we can regard based G–spaces X as constant $\mathscr{I}_G$–spaces, $X(V) = X$; the evaluation G–maps $\mathscr{I}_G(V, V')_+ \wedge X \to X$ are then the projections.

Definition 1.24 Define a $\mathscr{K}_*$–G–space $\mathscr{Y}$ to be an $\mathscr{I}_G$–space $\mathscr{Y}$ with a structure of $\mathscr{K}_V$–algebra on $\mathscr{Y}(V)$ for each V together with G–maps $i: \mathscr{Y}(V) \to \mathscr{Y}(V \oplus W)$ such that the following diagrams commute, where the θ are monad action maps:

$$
\begin{array}{ccc}
\mathscr{I}_G(V, V')_+ \wedge \boldsymbol{K}_V \mathscr{Y}(V) & \longrightarrow & \boldsymbol{K}_{V'} \mathscr{Y}(V') \\
{\scriptstyle \mathrm{id} \wedge \theta} \downarrow & & \downarrow {\scriptstyle \theta} \\
\mathscr{I}_G(V, V')_+ \wedge \mathscr{Y}(V) & \longrightarrow & \mathscr{Y}(V')
\end{array}
$$

In the second diagram, we identify $S^V \wedge S^W$ with $S^{V \oplus W}$:

$$
\begin{array}{ccc}
(\mathscr{I}_G(V, V') \times \mathscr{I}_G(W, W'))_+ \wedge \mathscr{Y}(V) \wedge S^V \wedge S^W & \longrightarrow & \mathscr{Y}(V') \wedge S^{V'} \wedge S^{W'} \\
{\scriptstyle \oplus \wedge i \wedge \mathrm{id}} \downarrow & & \downarrow {\scriptstyle i \wedge \mathrm{id}} \\
(\mathscr{I}_G(V \oplus W, V' \oplus W'))_+ \wedge \mathscr{Y}(V \oplus W) \wedge S^{V \oplus W} & \longrightarrow & \mathscr{Y}(V' \oplus W') \wedge S^{V' \oplus W'}
\end{array}
$$

The first diagram says that θ is a map of $\mathscr{I}_G$–spaces and, ignoring the sphere coordinates, the second diagram says that $i: \mathscr{Y} \circ \pi_1 \Rightarrow \mathscr{Y} \circ \oplus$ is a natural transformation of functors $\mathscr{I}_G \times \mathscr{I}_G \to \mathscr{T}_G$.

Theorem 1.25 (from $\mathscr{K}_*$–G–spaces to orthogonal G–spectra) For a $\mathscr{K}_*$–G–space $\mathscr{Y}$, the based G–spaces $\mathbb{E}_V \mathscr{Y}(V)$ and the based G–maps

$$
\Sigma^W \mathbb{E}_V \mathscr{Y}(V) \to \mathbb{E}_{V \oplus W} \mathscr{Y}(V \oplus W)
$$

determined by $i: \mathscr{Y} \circ \pi_1 \Rightarrow \mathscr{Y} \circ \oplus$ specify an orthogonal G–spectrum $\mathbb{E}_G^{\mathrm{geo}} \mathscr{Y}$.

Proof Regarding the $\mathscr{I}_G(V, V')$ as constant simplicial G–spaces, we see by diagram chases from the definitions that the data of the previous definitions determine G–maps

$$\mathscr{I}_G(V, V')_+ \wedge \Sigma^V K_V^q \mathscr{Y}(V) \to \Sigma^{V'} K_{V'}^q \mathscr{Y}(V')$$

and

$$\Sigma^W \Sigma^V K_V^q \mathscr{Y}(V) \to \Sigma^{V \oplus W} K_{V \oplus W}^q \mathscr{Y}(V \oplus W).$$

On passage to geometric realization, these give the required $\mathscr{I}_G$–space $\mathbb{E}_G^{\mathrm{geo}} \mathscr{Y}$ and the required natural transformation $\mathbb{E}_G^{\mathrm{geo}} \mathscr{Y} \,\overline{\wedge}\, S_G \to \mathbb{E}_G^{\mathrm{geo}} \mathscr{Y} \circ \oplus$. ☐

Of course, the recognition principle of (1.13) and Theorem 1.14 applies to describe the relationship between the G–spaces $\mathscr{Y}(V)$ and $\Omega^V (\mathbb{E}_G^{\mathrm{geo}} \mathscr{Y})(V)$. The recognition principle for pairings also adapts directly.

Definition 1.26 Let $\mathscr{X}$, $\mathscr{Y}$, and $\mathscr{Z}$ be $\mathscr{K}_*$–G–spaces. A pairing

$$f \colon \mathscr{X} \,\overline{\wedge}\, \mathscr{Y} \to \mathscr{Z} \circ \oplus$$

is a natural transformation of continuous functors $\mathscr{I}_G \times \mathscr{I}_G \to \mathscr{T}_G$ such that each $f \colon \mathscr{X}(V) \wedge \mathscr{Y}(W) \to \mathscr{Z}(V \oplus W)$ is a pairing as in Definition A.2 and the following diagram commutes for all U, V, W:

$$(1.27)$$

This diagram expresses that the three composite natural transformations of functors $\mathscr{I}_G^3 \to \mathscr{T}_G$ in sight agree.

The smash product of orthogonal G–spectra is obtained by first applying Day convolution to the external smash product $\overline{\wedge}$ and then coequalizing the action of the sphere G–spectrum on the two variables. See [19, Section II.3] for details.

Proposition 1.28 *A pairing $f \colon \mathscr{X} \,\overline{\wedge}\, \mathscr{Y} \to \mathscr{Z} \circ \oplus$ of $\mathscr{K}_*$–G–spaces induces a map*

$$\mathbb{E}_G^{\mathrm{geo}} f \colon \mathbb{E}_G^{\mathrm{geo}} \mathscr{X} \wedge \mathbb{E}_G^{\mathrm{geo}} \mathscr{Y} \to \mathbb{E}_G^{\mathrm{geo}} \mathscr{Z}$$

of orthogonal G–spectra that is given levelwise by specialization of Proposition 1.20.

Proof The definition of a pairing immediately implies that f induces an external pairing

$$\mathbb{E}_G^{\text{geo}}\,\mathscr{X}\,\overline{\wedge}\,\mathbb{E}_G^{\text{geo}}\,\mathscr{Y}\to\mathbb{E}_G^{\text{geo}}\,\mathscr{Z}\circ\oplus,$$

and the diagram (1.27) ensures that the resulting map from the Day convolution to $\mathbb{E}_G^{\text{geo}}\,\mathscr{Z}$ factors through the coequalizer defining $\mathbb{E}_G^{\text{geo}}\,\mathscr{X}\wedge\mathbb{E}_G^{\text{geo}}\,\mathscr{Y}$. $\qquad\square$

The suspension G–spectrum $\Sigma_G^\infty X$ of a based G–space X is given by the G–spaces $\Sigma^V X$; its structure maps are the evident identifications $\Sigma^W\Sigma^V X\cong\Sigma^{V\oplus W}X$. The unstable BPQ theorem of Theorem 1.21 leads to the following "geometric" version of the BPQ theorem.

Definition 1.29 For a based G–space X, define $\boldsymbol{K}_*X$ to be the $\mathscr{K}_*$–G–space given by the $\mathscr{K}_V$–spaces $\boldsymbol{K}_V X$ and the maps $i\colon\boldsymbol{K}_V X\to\boldsymbol{K}_{V\oplus W}X$ induced by the map of operads $\mathscr{K}_V\to\mathscr{K}_{V\oplus W}$ obtained by sending embeddings $e\colon V\to V$ to $e\times\mathrm{id}\colon V\times W\to V\times W$.

It is easily verified that $\boldsymbol{K}_*X$ is a $\mathscr{K}_*$–G–space and the pairings $\boxtimes$ of (1.18) prescribe pairings

(1.30) $$\boxtimes\colon\mathbb{K}_*X\,\overline{\wedge}\,\mathbb{K}_*Y\to\mathbb{K}_*(X\wedge Y)\circ\oplus.$$

Theorem 1.31 (the geometric BPQ theorem for orthogonal suspension G–spectra)
For based G–spaces X, there is a natural equivalence

$$\omega\colon\Sigma_G^\infty X\to\mathbb{E}_G^{\text{geo}}\,\boldsymbol{K}_*X$$

such that the following diagram commutes for based G–spaces X and Y:

$$
\begin{array}{ccc}
\Sigma_G^\infty X\wedge\Sigma_G^\infty Y & \xrightarrow{\ \omega\wedge\omega\ } & \mathbb{E}_G^{\text{geo}}\,\boldsymbol{K}_*X\wedge\mathbb{E}_G^{\text{geo}}\,\boldsymbol{K}_*Y \\
{\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\mathbb{E}_G^{\text{geo}}(\boxtimes)} \\
\Sigma_G^\infty(X\wedge Y) & \xrightarrow[\ \ \omega\ \ ]{} & \mathbb{E}_G^{\text{geo}}\,\boldsymbol{K}_*(X\wedge Y)
\end{array}
$$

Proof The levelwise equivalence follows from Theorem 1.21. For the diagram, the functor Σ_G^∞ is left adjoint to the 0th G–space functor, and inspection of definitions shows that the adjoint diagram starting with $X\wedge Y$ commutes. $\qquad\square$

1.6   A configuration space model for free $\mathscr{K}_V$–spaces

The free $\mathscr{K}_V$–spaces $\mathbb{K}_V X$ can be modeled more geometrically by configuration spaces. To explain this, we first record the nonequivariant analogue in terms of the little cubes operads, since that is relevant folklore which is not in the literature.

Consider the little n–cubes operads $\mathscr{C}_n$ and their associated monads C_n. Let $J = (0, 1)$ be the interior of I. We have the configuration spaces $F(J^n, j)$ of j–tuples of distinct points in J^n. Sending little n–cubes $c\colon J^n \to J^n$ to their center points $c\big(\frac{1}{2}, \dots, \frac{1}{2}\big)$ gives a Σ_n–homotopy equivalence $f\colon \mathscr{C}_n(j) \to F(J^n, j)$.

For based spaces X, we construct spaces $F_n X$ by replacing $\mathscr{C}_n(j)$ by $F(J^n, j)$ in the construction of $C_n X$ as the quotient of $\bigsqcup \mathscr{C}_n(j) \times_{\Sigma_j} X^j$ by basepoint identifications; we now use the evident omit a point projections $F(j, n) \to F(j, n-1)$ rather than the analogous maps $\mathscr{C}_n(j) \to \mathscr{C}_n(j-1)$. The maps f induce a homotopy equivalence

$$f\colon C_n X \to F_n X.$$

That much has been known since [21].

The folklore observation is that although the $F(J^n, j)$ do not form an operad, $\mathscr{C}_n$ acts on $F_n X$ in such a way that f is a map of $\mathscr{C}_n$–spaces. Indeed, we can evaluate little n–cubes $J^n \to J^n$ on points of J^n to obtain maps

$$\mathscr{C}_n(j) \times F(J^n, j) \to F(J^n, j),$$

and any reader of [21] will see how to proceed from there. Moreover, we have pairings

$$\boxtimes\colon F_m X \wedge F_n Y \to F_{m+n}(X \wedge Y)$$

defined as in Definition A.2 and Proposition A.3, starting from the maps

$$F(J^n, j) \times F(J^n, k) \to F(J^n, jk)$$

that send (x, y), where $x = (x_1, \dots, x_j)$ and $y = (y_1, \dots, y_k)$, to the set of pairs (x_q, y_r), for $1 \le q \le j$ and $1 \le r \le k$, ordered lexicographically.

Nonequivariantly, we put this together to obtain an analogue of Theorem 1.31, using the evident variant of the geometric recognition principle that is obtained from the operads $\mathscr{C}_n$ as n varies. Here it is more natural to use symmetric spectra rather than orthogonal spectra, since it is natural to deal with sequences rather than inner product spaces. The relationship between the little cubes operads and symmetric spectra is explained in [19, Section I.8], and we leave details of the relevant retooling of the previous subsections to the interested reader.

Theorem 1.32 (the configuration space BPQ theorem for symmetric spectra)   *For based spaces X, there is a natural equivalence*

$$\omega \colon \Sigma^\infty X \to \mathbb{E}^{\mathrm{geo}} \boldsymbol{F}_* X$$

such that the following diagram commutes for based spaces X and Y:

$$
\begin{array}{ccc}
\Sigma^\infty X \wedge \Sigma^\infty Y & \xrightarrow{\ \omega \wedge \omega\ } & \mathbb{E}^{\mathrm{geo}} \boldsymbol{F}_* X \wedge \mathbb{E}^{\mathrm{geo}} \boldsymbol{F}_* Y \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \mathbb{E}^{\mathrm{geo}}_G(\boxtimes)} \\
\Sigma^\infty (X \wedge Y) & \xrightarrow[\ \ \omega\ \ ]{} & \mathbb{E}^{\mathrm{geo}} \boldsymbol{F}_* (X \wedge Y)
\end{array}
$$

For fixed V, the discussion generalizes equivariantly to relate $\boldsymbol{D}_V X$ or $\boldsymbol{K}_V X$ to $\boldsymbol{F}_V X$ for based G–spaces X. In the case of $\boldsymbol{K}_V X$, we use the time-0 projections from Steiner paths to embeddings $V \to V$ and the centerpoint map from $\mathrm{Emb}_V(j)$ to $F(V, j)$. Letting V vary, we obtain the following equivariant version of Theorem 1.32.

Theorem 1.33 (the configuration space BPQ theorem for orthogonal G–spectra)   *For based G–spaces X, there is a natural equivalence*

$$\omega \colon \Sigma^\infty_G X \to \mathbb{E}^{\mathrm{geo}}_G \boldsymbol{F}_* X$$

such that the following diagram commutes for based G–spaces X and Y:

$$
\begin{array}{ccc}
\Sigma^\infty_G X \wedge \Sigma^\infty_G Y & \xrightarrow{\ \omega \wedge \omega\ } & \mathbb{E}^{\mathrm{geo}}_G \boldsymbol{F}_* X \wedge \mathbb{E}^{\mathrm{geo}}_G \boldsymbol{F}_* Y \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \mathbb{E}^{\mathrm{geo}}_G(\boxtimes)} \\
\Sigma^\infty_G (X \wedge Y) & \xrightarrow[\ \ \omega\ \ ]{} & \mathbb{E}^{\mathrm{geo}}_G \boldsymbol{F}_* (X \wedge Y)
\end{array}
$$

2   The recognition principle for infinite loop G–spaces

The equivariant recognition principle shows how to recognize (genuine) G–spectra in terms of category or space level information. It comes in various versions. We shall give two modernized variants of the machine from [21], differing in their choice of the output category of G–spectra. In contrast with the previous section, we are now concerned with infinite loop space machines with input given by E_∞–G–spaces (or G–categories) defined over any (genuine) E_∞–operad. A G–spectrum E is connective if the negative homotopy groups of each of its fixed point spectra E^H are zero, and all infinite loop space machines take values in connective G–spectra.

As in [10], we let $\mathscr{S}$, $\mathscr{S}p$, and $\mathscr{Z}$ denote the categories of orthogonal spectra [20], Lewis–May spectra [18], and EKMM S–modules [7]. Similarly, we let $G\mathscr{S}$, $G\mathscr{S}p$

and $G\mathscr{Z}$ denote the corresponding categories of genuine G–spectra from [19], [18], and again [19]. We start with a machine that lands in $G\mathscr{S}$. It is related to but different from the geometric machine of the previous section, and it is the choice preferred in [10] and in the sequels [31; 12; 13]. The sphere G–spectrum S_G in $G\mathscr{S}$ is cofibrant, and so are the suspension G–spectra $\Sigma_G X$ of cofibrant based G–spaces X. We then give the variant machine that lands in $G\mathscr{S}p$ or $G\mathscr{Z}$, where every object is fibrant, and give a comparison that illuminates homotopical properties of the first machine via its comparison with the second.

2.1 Equivariant E_∞–operads

Since operads make sense in any symmetric monoidal category, we have operads of categories, spaces, G–categories, and G–spaces. Operads in $G\mathscr{U}$ were first used in [18, Chapter VII]. Although we are only interested in finite groups G in this paper, the following definition makes sense for any topological group G and is of interest in at least the generality of compact Lie groups.

Definition 2.1 An E_∞–operad $\mathscr{C}_G$ of G–spaces is an operad in the cartesian monoidal category $G\mathscr{U}$ such that $\mathscr{C}_G(0)$ is a contractible G–space and the $(G\times\Sigma_j)$–space $\mathscr{C}_G(j)$ is a universal principal (G,Σ_j)–bundle for each $j\geq 1$. Equivalently, for a subgroup Λ of $G\times\Sigma_j$, the Λ–fixed-point space $\mathscr{C}_G(j)^\Lambda$ is contractible if $\Lambda\cap\Sigma_j=\{e\}$ and is empty otherwise. We say that $\mathscr{C}_G$ is reduced if $\mathscr{C}_G(0)$ is a point.

As is usual in equivariant bundle theory, we think of G as acting from the left and Σ_j as acting from the right on the spaces $\mathscr{C}_G(j)$. These actions must commute and so define an action of $G\times\Sigma_j$. We shall say nothing more about equivariant bundle theory except to note the following parallel. In [21], an operad $\mathscr{C}$ of spaces was defined to be an E_∞–operad if $\mathscr{C}(j)$ is a free contractible Σ_j–space. Effectively, $\mathscr{C}(j)$ is then a universal principal Σ_j–bundle. If we regard each $\mathscr{C}(j)$ as a G–trivial G–space, such an operad is called a naive E_∞–operad of G–spaces. Analogously, we have defined genuine E_∞–operads by requiring the $\mathscr{C}_G(j)$ to be universal principal (G,Σ_j)–bundles. That dictates the appropriate homotopical properties of the $\mathscr{C}_G(j)$, and it is only those homotopical properties and not their bundle-theoretic consequences that concern us in the theory of operads. The bundle theory implicitly tells us which homotopical properties are relevant to equivariant infinite loop space theory. Our default is that E_∞–operads are understood to be genuine unless otherwise specified.

We give two well-known examples. Recall that a complete G–universe U is a G–inner product space that contains countably many copies of each irreducible representation of G; a canonical choice is the sum of countably many copies of the regular representation ρ_G.

Example 2.2 (the Steiner operad $\mathscr{K}_U$) Inclusions $V \subset W$ induce inclusions of operads $\mathscr{K}_V \to \mathscr{K}_W$. Let $\mathscr{K}_U$ be the union over $V \subset U$ of the operads $\mathscr{K}_V$, where U is a complete G–universe.[3] This is the infinite Steiner operad of G–spaces. It is an E_∞–operad since Σ_j–acts freely on $\mathscr{K}_U(j)$ and $\mathscr{K}_U(j)^\Lambda$ is contractible if $\Lambda \subset G \times \Sigma_j$ and $\Lambda \cap \Sigma_j = e$. Indeed, such a Λ is isomorphic to a subgroup H of G via the projection $G \times \Sigma_j \to G$, and if we let H act on U through the isomorphism, then U is a complete H–universe and U^H is isomorphic to $\mathbb{R}^\infty$. Therefore, by the proof of Proposition 1.5, $\mathscr{K}_U(j)^\Lambda$ is equivalent to the configuration space $F(\mathbb{R}^\infty, j)$, which is contractible.

Example 2.3 (linear isometries operad) The equivariant linear isometries operad $\mathscr{L}_U$ was first used in [18, Section VII.1] and is defined just as nonequivariantly (eg [30, Section 2]). The $(G \times \Sigma_j)$–space $\mathscr{L}_U(j)$ is the space of linear isometries $U^j \to U$, with G acting by conjugation, and $\mathscr{L}_U$ is an E_∞–operad of G–spaces if U is a complete G–universe. Indeed, Σ_j acts freely on $\mathscr{L}_U(j)$ and $\mathscr{L}_U(j)^\Lambda$ is contractible if $\Lambda \subset G \times \Sigma_j$ and $\Lambda \cap \Sigma_j = e$. If $\Lambda \cong H$ and H acts on U through the isomorphism, then U is a complete H–universe and $\mathscr{L}_U(j)^H$ is isomorphic to the space of H–linear isometries $U^j \to U$. The usual argument that $\mathscr{L}(j)$ is contractible (eg [24, Lemma I.1.3]) adapts to prove that this space is contractible.

We define E_∞–operads in G–categories in Section 3.3 and give examples in Section 4.2 and Section 7.

Remark 2.4 We will encounter one naturally occurring operad that is not reduced. When an operad $\mathscr{C}$ acts on a space X via maps θ_i and we choose points $c_i \in \mathscr{C}(i)$, we have a map $\theta_0 \colon \mathscr{C}(0) \to X$ and the relation

$$\theta_2(c_2; \theta_0(c_0), \theta_1(c_1, x)) = \theta_1(\gamma(c_2; c_0, c_1), x)$$

for $x \in X$. When the $\mathscr{C}(i)$ are connected, this says that $\theta_0(c_0)$ is a unit element for the product determined by c_2. Reduced operads give a single unit element. The original definition [21, 1.1] required operads to be reduced.

Lemma 2.5 Let $\mathscr{C}_G$ be an E_∞–operad of G–spaces and define $\mathscr{C} = (\mathscr{C}_G)^G$. Then $\mathscr{C}$ is an E_∞–operad of spaces. If Y is a $\mathscr{C}_G$–space, then Y^G is a $\mathscr{C}$–space.

Proof $(\mathscr{C}_G)^G$ is an operad since the fixed-point functor commutes with products, and it is an E_∞–operad since the space $\mathscr{C}_G(j)^G$ is contractible and Σ_j–free.   □

---

[3] We denoted the nonequivariant version as $\mathscr{C}$ in [30], but we prefer the notation $\mathscr{K}_U$ here.

2.2   The infinite loop space machine: orthogonal G–spectrum version

In brief, we have a functor $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{S}}$ that assigns an orthogonal G–spectrum $\mathbb{E}_G Y$ to a G–space Y with an action by some chosen E_∞–operad $\mathscr{C}_G$ of G–spaces. We want to start with $\mathscr{C}_G$–algebras and still exploit the Steiner operads, and we use the product of operads trick recalled in Section 2.3 to allow this; compare [30, Section 9]. For simplicity of notation, define $\mathscr{C}_V = \mathscr{C}_G \times \mathscr{K}_V$. We use the following observation.

Lemma 2.6   *If $\mathscr{C}_G$ is an E_∞–operad of G–spaces, then the projection*

$$\mathscr{C}_V(j) = \mathscr{C}_G(j) \times \mathscr{K}_V(j) \to \mathscr{K}_V(j)$$

is a $(G \times \Sigma_j)$–equivalence for each j.

Proof   We must show that for each subgroup $\Lambda \subseteq G \times \Sigma_j$, the induced map on fixed points

$$\mathscr{C}_G(j)^\Lambda \times \mathscr{K}_V(j)^\Lambda \to \mathscr{K}_V(j)^\Lambda$$

is an equivalence. If $\Lambda \cap \{e\} \times \Sigma_j = \{e\}$, then $\mathscr{C}(j)^\Lambda \simeq *$, so the projection is an equivalence. If Λ contains a nonidentity permutation, then the fixed points on both sides are empty. Both sides are trivial if $j = 0$. $\qquad\square$

We view $\mathscr{C}_G$–spaces as $\mathscr{C}_V$–spaces for all V via the projections $\mathscr{C}_V \to \mathscr{C}_G$, and $\mathscr{C}_V$ acts on V–fold loop spaces via its projection to $\mathscr{K}_V$. Write C_V for the monad on based G–spaces associated to the operad $\mathscr{C}_V$. The categories of $\mathscr{C}_V$–spaces and C_V–algebras are isomorphic. As in the V–fold delooping argument, the unit $\eta\colon \mathrm{Id} \to \Omega^V \Sigma^V$ of the monad $\Omega^V \Sigma^V$ and the action θ of C_V on the G–spaces $\Omega^V \Sigma^V X$ induce a composite natural map

$$\alpha_V \colon C_V X \xrightarrow{\;C_V \eta\;} C_V \Omega^V \Sigma^V X \xrightarrow{\;\theta\;} \Omega^V \Sigma^V X,$$

and $\alpha_V \colon C_V \to \Omega^V \Sigma^V$ is a map of monads whose adjoint defines a right action of C_V on the functor Σ^V.

Definition 2.7   (from $\mathscr{C}_G$–spaces to orthogonal G–spectra)   Let Y be a $\mathscr{C}_G$–space. We define an orthogonal G–spectrum $\mathbb{E}_G Y$, which we denote by $\mathbb{E}_G^{\mathscr{S}} Y$ when necessary for clarity. Let

$$\mathbb{E}_G Y(V) = B(\Sigma^V, C_V, Y).$$

Using the action of isometric isomorphisms on the $\mathscr{K}_V$ and Σ^V, as in the previous section but starting with Y regarded as a constant $\mathscr{I}_G$–functor, as we can since its action by $\mathscr{C}_G$ is independent of V, this defines an $\mathscr{I}_G$–space. The structure G–map

$$\sigma \colon \Sigma^W \mathbb{E}_G Y(V) \to \mathbb{E}_G Y(V \oplus W)$$

is the composite

$$\Sigma^W B(\Sigma^V, C_V, Y) \cong B(\Sigma^{V \oplus W}, C_V, Y) \to B(\Sigma^{V \oplus W}, C_{V \oplus W}, Y).$$

obtained by commuting Σ^W with geometric realization and using the map of monads $C_V \to C_{V \oplus W}$ induced by the inclusion $i \colon \mathcal{K}_V \to \mathcal{K}_{V \oplus W}$.

Just as in (1.13), we have the diagram of $\mathscr{C}_V$–spaces and $\mathscr{C}_V$–maps

$$(2.8) \qquad Y \xleftarrow{\varepsilon} B(C_V, C_V, Y) \xrightarrow{\bar{\alpha}} B(\Omega^V \Sigma^V, C_V, Y) \xrightarrow{\zeta} \Omega^V B(\Sigma^V, C_V, Y),$$

where $\bar{\alpha} = B(\alpha, \mathrm{id}, \mathrm{id})$. Theorem 1.14 applies verbatim, with the same proof. We let $\xi_V = \zeta \circ \bar{\alpha} \circ \nu$, where ν is the canonical homotopy inverse to ε. Then the following diagram commutes, where $\tilde{\sigma}$ is adjoint to σ:

$$
\begin{array}{ccc}
& Y & \\
{\scriptstyle \xi_V} \swarrow & & \searrow {\scriptstyle \xi_{V \oplus W}} \\
\Omega^V \mathbb{E}_G Y(V) & \xrightarrow{\quad \Omega^V \tilde{\sigma} \quad} & \Omega^{V \oplus W} \mathbb{E}_G Y(V \oplus W)
\end{array}
$$

Therefore, $\Omega^V \tilde{\sigma}$ is a weak equivalence if $V \supset \mathbb{R}$. If we replace $\mathbb{E}_G Y$ by a fibrant approximation $\mathbf{R} \mathbb{E}_G Y$, there results a group completion $\xi \colon Y \to (\mathbf{R} \mathbb{E}_G Y)_0$. We shall shortly use the category $\mathscr{S}p$ to give an explicit way to think about this.

Remark 2.9 Since $\mathcal{K}_0(0) = \{*\}$, $\mathcal{K}_0(1) = \{\mathrm{id}\}$, and $\mathcal{K}_0(j) = \varnothing$ for $j > 1$, we have that C_0 is the identity functor if $\mathscr{C}_G(0) = \{*\}$ and $\mathscr{C}_G(1) = \{\mathrm{id}\}$. In that case,

$$\mathbb{E}_G Y(0) = B(\Sigma^0, C_0, Y) = B(\mathrm{Id}, \mathrm{Id}, Y) \cong Y.$$

We comment on an alternative point of view not taken above but relevant below. We can use the product of operads trick from [21] to replace a $\mathscr{C}_G$–space Y by the equivalent $\mathcal{K}_U$–space $B(K_U, C_U, Y)$, where C_U is the monad associated to the E_∞–operad $\mathscr{C}_U = \mathscr{C}_G \times \mathcal{K}_U$ and from there only use Steiner operads. However, there is a catch. A $\mathcal{K}_U$–algebra Y is a $\mathcal{K}_V$–algebra by restriction, but the constant $\mathscr{I}_G$–space Y is *not* a $\mathcal{K}_*$–G–space in the sense of Definition 1.24 since conjugation by isometries is not compatible with the inclusions used to define $\mathcal{K}_U$. Therefore, the $B(\Sigma^V, \mathcal{K}_V, Y)$ do not define an $\mathscr{I}_G$–space. However, ignoring isometries, they do define a coordinate free G–prespectrum, as defined in [19, II.1.2]. That can be viewed as the starting point for the alternative machine we construct next.

2.3 The infinite loop space machine: Lewis–May G–spectrum version

A Lewis–May (henceforward LM) G–spectrum E consists of G–spaces EV for each finite-dimensional G–inner product subspace V in a complete G–universe U together with G–homeomorphisms $EV \to \Omega^{W-V}EW$ whenever $V \subset W$. For a based G–space X we define $Q_G X = \operatorname{colim} \Omega^V \Sigma^V X$. The suspension LM G–spectrum $\Sigma_G^\infty X$ has V^{th} G–space $Q_G \Sigma^V X$, and the functor Σ_G^∞ is left adjoint to the zeroth space functor Ω_G^∞. We sometimes change notation to Σ_U^∞ and Ω_U^∞, allowing change of universe. While $G\mathscr{S}p$ is not symmetric monoidal, that is rectified by passage to the S_G–modules of [7], at the inevitable price of losing the adjunction; see [30, Section 11].

The operad $\mathscr{K}_U$ acts on $\Omega_U^\infty E$ for any LM G–spectrum E. One could not expect such precise structure when working with orthogonal G–spectra. Nonequivariantly, such highly structured infinite loop spaces are central to calculations, and it is to be hoped that the equivariant theory will eventually reach a comparable state. Therefore, it is natural to want an infinite loop space machine that lands in the category $G\mathscr{S}p$ of LM G–spectra.

The operad $\mathscr{K}_U$ plays a privileged role. As noted above, if $\mathscr{C}_G$ is an E_∞–G–operad, we can convert $\mathscr{C}_G$–spaces to equivalent $\mathscr{K}_U$–spaces, so that it suffices to build a machine for $\mathscr{K}_U$–spaces. On the other hand, $\mathscr{C}_G$ spaces inherit actions of $\mathscr{C}_U = \mathscr{C}_G \times \mathscr{K}_U$, so that it suffices to build a machine for $\mathscr{C}_U$–spaces. To encompass both of these approaches in a single machine, we suppose given a map (necessarily an equivalence) of E_∞–G–operads $\mathscr{O}_G \to \mathscr{K}_U$. We can take $\mathscr{O}_G = \mathscr{C}_U$ or $\mathscr{O}_G = \mathscr{K}_U$, but both here and in [12; 13; 31], our primary interest is in $\mathscr{C}_U$. Formally, the equivariant theory now works in the same way as the nonequivariant theory, and we follow the summary in [30, Section 9]. An early version of this machine is in the paper [4] of Costenoble and Waner.

Scholium 2.10 We must use the Steiner operads $\mathscr{K}_V$ and $\mathscr{K}_U$ rather than the little disks operads $\mathscr{D}_V$ and $\mathscr{D}_U$, which was the choice in [4], and our notion of an E_∞–operad of G–spaces should replace the notion of a complete operad used there.

Definition 2.11 (from $\mathscr{O}_G$–spaces to Lewis–May G–spectra) Let Y be an $\mathscr{O}_G$–space. We define a LM G–spectrum $\mathbb{E}_G Y$, which we denote by $\mathbb{E}_G^{\text{Sp}}$ when necessary for clarity, by

$$\mathbb{E}_G Y = B(\Sigma_G^\infty, \boldsymbol{O}_G, Y).$$

Here $\boldsymbol{O}_G$ acts on Σ_G^∞ through its projection to $\mathbb{K}_U$.

We have the diagram of $\mathscr{O}_G$–spaces and $\mathscr{O}_G$–maps

$$(2.12) \quad Y \xleftarrow{\varepsilon} B(\boldsymbol{O}_G, \boldsymbol{O}_G, Y) \xrightarrow{\bar{\alpha}_U} B(Q_G, \boldsymbol{O}_G, Y) \xrightarrow{\xi} \Omega_G^\infty B(\Sigma_G^\infty, \boldsymbol{O}_G, Y) = \Omega_G^\infty \mathbb{E}_G Y,$$

where $\bar{\alpha}_U = B(\alpha_U, \operatorname{id}, \operatorname{id})$. As explained nonequivariantly in [30, Section 9], the following analogue of Theorem 1.14 holds.

Theorem 2.13 *Let $\mathscr{O}_G$ be an E_∞–G–operad with a map of operads $\mathscr{O}_G \to \mathscr{K}_U$. The following statements hold for an $\mathscr{O}_G$–space Y:*

   (i)   *The map ε is a G–homotopy equivalence with a natural homotopy inverse ν.*

   (ii) *The map $\bar{\alpha}_U$ is an equivalence when Y is connected and is a group completion otherwise.*

   (iii) *The map ζ is an equivalence.*

Therefore, the composite

$$\xi = \zeta \circ \bar{\alpha}_U \circ \nu \colon Y \to \Omega_G^\infty \mathbb{E}_G Y$$

is an equivalence if Y is grouplike and is a group completion otherwise.

We shall not pursue this variant of the recognition principle in further detail, but we reemphasize that its much tighter relationship with space level data may eventually aid equivariant calculation. However, it is worth stating the alternative geometric version of the stable BPQ theorem to which it leads. Here we specialize to the case $\mathscr{O}_G = \mathscr{K}_U$. This allows us to use the pairings of Steiner operads described in Section 1.4, which are not available for other E_∞–operads. By passage to colimits, we obtain the following analogue of Proposition 1.17.

Proposition 2.14 *For G–universes U and U', there is a unital, associative, and commutative pairing*

$$\boxtimes \colon (\mathscr{K}_U, \mathscr{K}_{U'}) \to \mathscr{K}_{U \oplus U'}$$

of Steiner operads of G–spaces.

Passing to monads, we obtain a unital, associative, and commutative system of pairings

(2.15) $$\boxtimes \colon \mathbb{K}_U X \wedge \mathbb{K}_{U'} Y \to \mathbb{K}_{U \oplus U'}(X \wedge Y).$$

Passage to colimits from Lemma 1.19 gives the following analogue of that result.

Lemma 2.16 *The following diagram commutes:*

$$
\begin{array}{ccc}
\boldsymbol{K}_U X \wedge \boldsymbol{K}_U Y & \xrightarrow{\ \boxtimes\ } & \boldsymbol{K}_{U \oplus U}(X \wedge Y) \\
{\scriptstyle \alpha_U \wedge \alpha_U} \downarrow & & \downarrow {\scriptstyle \alpha_{U \oplus U}} \\
\Omega_U^\infty \Sigma_U^\infty X \wedge \Omega_U^\infty \Sigma_U^\infty Y & \xrightarrow[\wedge]{} & \Omega_{U \oplus U}^\infty \Sigma_{U \oplus U}^\infty (X \wedge Y)
\end{array}
$$

The following recognition principle for pairings can by derived from Proposition 1.20 by passage to colimits or can be proven by the same argument as there. We note that

our definition of the machine $\mathbb{E}_G$ depends on a choice of complete G–universe U, and we sometimes write $\mathbb{E}_U$ to indicate that choice.

Proposition 2.17 *A pairing* $f\colon X \wedge Y \to Z$ *of a* $\mathscr{K}_U$*–space* X *and a* $\mathscr{K}_U$*–space* Y *to a* $\mathscr{K}_{U \oplus U}$*–space* Z *induces a map*

$$\mathbb{E}f\colon \mathbb{E}_U X \,\overline{\wedge}\, \mathbb{E}_U Y \to \mathbb{E}_{U \oplus U} Z$$

of LM G–spectra indexed on $U \oplus U$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X \wedge Y & \xrightarrow{\xi \wedge \xi} \Omega^\infty_U \mathbb{E}_U X \wedge \Omega^\infty_U \mathbb{E}_U Y \longrightarrow & \Omega^\infty_{U \oplus U}(\mathbb{E}_U X \,\overline{\wedge}\, \mathbb{E}_U Y) \\
f \downarrow & & \downarrow \Omega^\infty_{U \oplus U} \mathbb{E}f \\
Z & \xrightarrow{\qquad\qquad\qquad \xi \qquad\qquad\qquad} & \Omega^\infty_{U \oplus U} \mathbb{E}_{U \oplus U} Z
\end{array}
$$

We can internalize the external smash product, as in [18], by choosing a linear isometry $\phi\colon U \oplus U \to U$. Then ϕ induces a change of universe functor ϕ_* which allows us to replace the right arrow by $\Omega^\infty_U \phi_* \mathbb{E}f$. In the following result we can either stick with Lewis–May G–spectra or pass to the S_G–modules of [7; 19]. We interpret the smash product according to choice.

Theorem 2.18 (the $\mathscr{K}_U$–space BPQ theorem for Lewis–May G–spectra) *For based G–spaces X, there is a natural equivalence*

$$\omega\colon \Sigma^\infty_U X \to \mathbb{E}_U \boldsymbol{K}_U X$$

such that the following diagram commutes for based G–spaces X and Y:

$$
\begin{array}{ccc}
\Sigma^\infty_U X \wedge \Sigma^\infty_U Y & \xrightarrow{\omega \wedge \omega} & \mathbb{E}_U \boldsymbol{K}_U X \wedge \mathbb{E}_U \boldsymbol{K}_U Y \\
\cong \downarrow & & \downarrow \mathbb{E}(\boxtimes) \\
\Sigma^\infty_U (X \wedge Y) & \xrightarrow{\quad \omega \quad} & \mathbb{E}_U \boldsymbol{K}_U (X \wedge Y)
\end{array}
$$

Sketch proof The first statement is the usual extra-degeneracy argument [21, Proposition 9.8]. We comment on the diagram. In either $G\mathscr{S}p$ or $G\mathscr{Z}$, it is an internalization of a diagram of G–spectra indexed on $U \oplus U$:

$$
\begin{array}{ccc}
\Sigma^\infty_U X \,\overline{\wedge}\, \Sigma^\infty_U Y & \xrightarrow{\omega \overline{\wedge} \omega} & \mathbb{E}_U \boldsymbol{K}_U X \,\overline{\wedge}\, \mathbb{E}_U \boldsymbol{K}_U Y \\
\cong \downarrow & & \downarrow \mathbb{E}(\boxtimes) \\
\Sigma^\infty_{U \oplus U}(X \wedge Y) & \xrightarrow{\quad \omega \quad} & \mathbb{E}_{U \oplus U} \boldsymbol{K}_{U \oplus U}(X \wedge Y)
\end{array}
$$

The isomorphism on the left is trivial on the prespectrum level (indexing on inner product G–spaces of the form $V \oplus W$) and follows on the spectrum level. After passage to adjoints, to check commutativity it suffices to check starting from $X \wedge Y$ on the bottom left, where an inspection of definitions gives the conclusion. If in $G\mathscr{S}p$, this is internalized by use of a linear isometry $\phi \colon U \oplus U \to U$. If in $G\mathscr{Z}$, this is internalized by use of the definition of the smash product in terms of the linear isometries operad $\mathscr{L}_U$, as in [7; 19]. $\qquad\square$

In fact, with the model-theoretic modernization of the original version of the theory that is given nonequivariantly in [1], one can redefine the restriction of $\mathbb{E}_U$ to cofibrant $\mathscr{K}_U$–spaces Y to be

$$\mathbb{E}_U Y = \Sigma_G^\infty \otimes_{\boldsymbol{K}_U} Y,$$

where $\otimes_{\boldsymbol{K}_U}$ is the evident coequalizer. With that reinterpretation and taking X to be a G–CW complex, $\mathbb{E}_U \boldsymbol{K}_U X$ is actually isomorphic to $\Sigma_G^\infty X$.

The nonequivariant statement is often restricted to the case $Y = S^0$. Then $\boldsymbol{K}_U S^0$ is the disjoint union of operadic models for the classifying spaces $B\Sigma_j$. Similarly, $\boldsymbol{K}_U S^0$ is the disjoint union of operadic models for the classifying G–spaces $B(G, \Sigma_j)$.

2.4 A comparison of infinite loop space machines

We compare the $\mathscr{S}$ and $\mathscr{S}p$ machines $\mathbb{E}_G^{\mathscr{S}}$ and $\mathbb{E}_G^{\mathscr{S}p}$ by transporting both of them to the category $G\mathscr{Z}$ of S_G–modules, following [19]. As discussed in [19, Section IV.4] with slightly different notation, there is a diagram of Quillen equivalences:

$$
\begin{array}{ccc}
G\mathscr{P} & \underset{\ell}{\overset{L}{\rightleftarrows}} & G\mathscr{S}p \\[4pt]
\mathbb{U} \Big\Updownarrow \mathbb{P} & & \mathbb{V} \Big\Updownarrow \mathbb{F} \\[4pt]
G\mathscr{S} & \underset{\mathbb{N}^{\#}}{\overset{\mathbb{N}}{\rightleftarrows}} & G\mathscr{Z}
\end{array}
$$

Here $G\mathscr{P}$ is the category of coordinate-free G–prespectra. The left adjoint $\mathbb{N}$ is strong symmetric monoidal, and the unit map $\eta \colon X \to \mathbb{N}^{\#}\mathbb{N}X$ is a weak equivalence for all cofibrant orthogonal G–spectra X. It can be viewed as a fibrant approximation in the stable model structure on $G\mathscr{S}$. The pair $(\mathbb{N}, \mathbb{N}^{\#})$ is a Quillen equivalence with the positive stable model structure on $G\mathscr{S}$; see [19, Sections III.4–5].

We can compare machines using the diagram. In fact, by a direct inspection of definitions, we see the following result, which is essentially a reinterpretation of the original construction of [21] that becomes visible as soon as one introduces orthogonal spectra.

Lemma 2.19 *The functor* $\mathbb{E}_G^{\mathscr{S}p}$ *from* $\mathscr{C}_G$*–spaces to the category* $G\mathscr{S}p$ *of Lewis–May* G*–spectra is naturally isomorphic to the composite functor* $L \circ \mathbb{U} \circ \mathbb{E}_G^{\mathscr{S}}$.

As explained in [19, Section IV.5], there is a monad $\mathbb{L}$ on $G\mathscr{S}p$ and a category $G\mathscr{S}p[\mathbb{L}]$ of $\mathbb{L}$–algebras. The left adjoint $\mathbb{F}$ in the diagram is the composite of left adjoints

$$\mathbb{L}\colon G\mathscr{S}p \to G\mathscr{S}p[\mathbb{L}] \quad \text{and} \quad \mathbb{J}\colon G\mathscr{S}p[\mathbb{L}] \to G\mathscr{Z}.$$

The functor $L \circ \mathbb{U}\colon G\mathscr{S} \to G\mathscr{S}p$ lands naturally in $G\mathscr{S}p[\mathbb{L}]$, so that we can define

$$\mathbb{M} = \mathbb{J} \circ L \circ \mathbb{U}\colon G\mathscr{S} \to G\mathscr{Z}.$$

By [19, Lemma IV.5.2 and Theorem IV.5.4], $\mathbb{M}$ is lax symmetric monoidal and there is a natural lax symmetric monoidal map $\alpha\colon \mathbb{N}X \to \mathbb{M}X$ that is a weak equivalence when X is cofibrant. Effectively, we have two infinite loop space machines landing in $G\mathscr{Z}$, namely $\mathbb{N} \circ \mathbb{E}_G^{\mathscr{S}}$ and $\mathbb{J} \circ \mathbb{E}_G^{\mathscr{S}p}$. In view of the lemma, the latter is isomorphic to $\mathbb{M} \circ \mathbb{E}_G^{\mathscr{S}}$; hence

$$\alpha\colon \mathbb{N} \circ \mathbb{E}_G^{\mathscr{S}} \to \mathbb{M} \circ \mathbb{E}_G^{\mathscr{S}} \cong \mathbb{J} \circ \mathbb{E}_G^{\mathscr{S}p}$$

compares the two machines, showing that they are equivalent for all practical purposes. Homotopically, these categorical distinctions are irrelevant, and we can use whichever machine we prefer, deducing properties of one from the other.

2.5 Examples of E_∞–spaces and E_∞–ring spaces

Many of the examples from the nonequivariant theory generalize directly to the equivariant setting. To illustrate the point of using varying E_∞–operads and their natural actions on spaces of interest, rather than just using $\mathscr{K}_U$, we focus on actions of the linear isometries operad $\mathscr{L}_U$.

Nonequivariantly, taking $U \cong \mathbb{R}^\infty$, a systematic account of naturally occurring examples of $\mathscr{L}_U$–spaces was already given in [24, Section I.1]. It was revisited briefly in more modern language [30, Section 2]. It includes the infinite classical groups O, SO, Spin, U, SU, Sp, their classifying spaces, constructed either using Grassmannian manifolds or the standard classifying-space functor B, and all of their associated infinite homogeneous spaces. All of these examples are grouplike, and all of them are given infinite loop spaces by application of the nonequivariant infinite loop space machine. The discussion in [24; 30] was in terms of inner-product subspaces V of a universe U. The point to make here is that the entire exposition works verbatim equivariantly, with the V being G–inner-product subspaces of our complete G–universe U. We give a brief account to show the idea.

As explained in [30, Section 2], an $\mathscr{I}_G$–FCP (functor with cartesian product) is a lax symmetric monoidal functor $\mathscr{I}_G \to \mathscr{T}_G$. We say that an $\mathscr{I}_G$–FCP is monoid valued if it factors through the category of equivariant topological monoids and monoid homomorphisms. The classical groups all give group-valued $\mathscr{I}_G$–FCPs:

$$V \mapsto O(V), \quad V \mapsto \mathrm{SO}(V), \quad V \mapsto U(\mathbb{C} \otimes_{\mathbb{R}} V), \quad V \mapsto \mathrm{SU}(\mathbb{C} \otimes_{\mathbb{R}} V), \quad \text{etc.}$$

Any $\mathscr{I}_G$–FCP X extends to a functor on all isometries (not just isometric isomorphisms) as follows: an isometry $\alpha \colon V \to W$ yields an identification $W \cong \alpha(V) \oplus \alpha(V)^{\perp}$. Then $X(\alpha)$ is the composite

$$X(V) \xrightarrow{\;X(\alpha) \times 0\;} X(\alpha(V)) \times X(\alpha(V)^{\perp}) \to X(\alpha(V) \oplus \alpha(V)^{\perp}).$$

Then the colimit $X(U) = \mathrm{colim}_V X(V)$ inherits an action of $\mathscr{L}_U$. The classifying space BF of a monoid-valued $\mathscr{I}_G$–FCP F is an $\mathscr{I}_G$–space, and the cited sources show that F is equivalent to ΩBF as an $\mathscr{L}_U$–space when F is group valued.

The formal structure of the operad pair $(\mathscr{K}_U, \mathscr{L}_U)$ works the same way equivariantly as nonequivariantly. It is an E_∞–operad pair in the sense originally defined in [24, VI.1.2] and reviewed in [30, Section 1] and, in more detail, [29, 4.2]. See Section 7.2 below for an example of an operad pair in G–categories. The action of $\mathscr{L}_U$ on $\mathscr{K}_U$ is defined nonequivariantly in [30, Section 3], and it works the same way equivariantly.

From here, multiplicative infinite loop space theory works equivariantly to construct E_∞–ring G–spectra from $(\mathscr{K}_U, \mathscr{L}_U)$–spaces, alias E_∞–ring G–spaces, in exactly the same way as nonequivariantly [24; 30; 29]. In particular, for any $\mathscr{L}_U$–algebra X, the free $\mathscr{K}_U$–algebra $\boldsymbol{K}_U X_+$ is an E_∞–ring G–space, where X_+ is obtained from X by adjoining an additive G–fixed basepoint 0. The group completion $\alpha_U \colon \boldsymbol{K}_U X_+ \to Q_G X_+$ is a map of E_∞–ring G–spaces, and $\mathbb{E}_G \boldsymbol{K}_U X_+$ is equivalent to $\Sigma_G^\infty X_+$ as E_∞–ring G–spectra.

As we intend to show elsewhere, the passage from category level data to E_∞–ring G–spaces, in analogy with [26; 29], generalizes to equivariant multicategories.

We remark that the usual construction of Thom G–spectra, such as MO_G and MU_G, already presents them as E_∞–ring G–spectra, without use of infinite loop space theory, as was explained and generalized in [18, Chapter X].

2.6 Some properties of equivariant infinite loop space machines

Many properties of the infinite loop space machine $\mathbb{E}_G$ follow directly from the group completion property, independent of how the machine is constructed, but it is notationally convenient to work with the machine $\mathbb{E}_G^{\mathscr{S}^p}$, for which ξ is a natural

group completion without any bother with fibrant approximation. The results apply equally well to $\mathbb{E}_G^{\mathscr{S}}$. It is plausible to hope that the group completion property actually characterizes the machine up to homotopy, as in [32], but the proof there fails equivariantly. A direct point-set level comparison of our machine with a new version of the Segal–Shimakawa machine will be given in [31]. We illustrate with the following two results, some version of which must hold for any equivariant infinite loop space machine $\mathbb{E}_G$. The first says that it commutes with passage to fixed points and the second says that it commutes with products, both up to weak equivalence.

Theorem 2.20 *For $\mathscr{C}_G$–spaces Y, there is a natural map of spectra*

$$\phi \colon \mathbb{E}(Y^G) \to (\mathbb{E}_G Y)^G$$

that induces a natural map of spaces under Y^G

in which the diagonal arrows are both group completions. Therefore, the horizontal arrow is a weak equivalence of spaces, and ϕ is a weak equivalence of spectra.

Proof For based G–spaces X, we have natural inclusions $\boldsymbol{C}_{U^G}(X^G) \to (\boldsymbol{C}_U X)^G$ and $\Sigma^\infty(X^G) \to (\Sigma_G^\infty X)^G$. For G–spectra E, we have a natural isomorphism $\Omega^\infty(E^G) \cong (\Omega_G^\infty E)^G$. This gives the required natural map of spectra

$$\mathbb{E}(Y^G) = B(\Sigma^\infty, \boldsymbol{C}_{U^G}, Y^G) \xrightarrow{\phi} (B(\Sigma_G^\infty, \boldsymbol{C}_U, Y))^G = (\mathbb{E}_G Y)^G$$

and the induced natural map of spaces under Y^G. Since the diagonal arrows in the diagram are group completions, the horizontal arrow must be a homology isomorphism and hence a weak equivalence. Since our spectra are connective, ϕ must also be a weak equivalence. □

Theorem 2.21 *Let X and Y be $\mathscr{C}_G$–spaces. Then the map*

$$\mathbb{E}_G(X \times Y) \to \mathbb{E}_G X \times \mathbb{E}_G Y$$

induced by the projections is a weak equivalence of G–spectra.

Proof We are using that the product of $\mathscr{C}_G$–spaces is a $\mathscr{C}_G$–space, the proof of which uses that the category of operads is cartesian monoidal. Working in $G\mathscr{S}p$, the

functor Ω_G^∞ commutes with products and passage to fixed points, and we have the commutative diagram:

$$(X \times Y)^H \cong X^H \times Y^H$$

$$\xi^H \qquad\qquad \xi^H \times \xi^H$$

$$(\Omega_G^\infty \mathbb{E}_G(X \times Y))^H \longrightarrow (\Omega_G^\infty \mathbb{E}_G X)^H \times (\Omega_G^\infty \mathbb{E}_G Y)^H$$

Since the product of group completions is a group completion, the diagonal arrows are both group completions. Therefore, the horizontal arrow is a weak equivalence. Since our spectra are connective, the conclusion follows. $\qquad\square$

2.7 The recognition principle for naive G–spectra

We elaborate on Theorem 2.20. The functor $\mathbb{E} = \mathbb{E}_e$ in that result is the nonequivariant infinite loop space machine, which is defined using the product of the nonequivariant Steiner operad $\mathscr{K} = \mathscr{K}_{U^G}$ and the fixed-point operad $\mathscr{C} = (\mathscr{C}_G)^G$. We may think of U^G as $\mathbb{R}^\infty$, without reference to U, and start with any (naive) E_∞–operad $\mathscr{C}$ to obtain a recognition principle for naive G–spectra, which are just spectra with G–actions. Again we can use either the category $\mathscr{S}$ of orthogonal spectra or the category $\mathscr{S}p$ of Lewis–May spectra, comparing the two by mapping to the category $\mathscr{Z}$ of EKMM S–modules, but letting G act on objects in all three. We continue to write $\mathbb{E}$ for this construction since it is exactly the same construction as the nonequivariant one, but applied to G–spaces with an action by the G–trivial E_∞–operad $\mathscr{C}$.

It is worth emphasizing that when working with naive G–spectra, there is no need to restrict to finite groups. We can just as well work with general topological groups G. The machine $\mathbb{E}$ still enjoys the same properties, including the group completion property. Working with Lewis–May spectra, the adjunction $(\Sigma^\infty, \Omega^\infty)$ relating spaces and spectra applies just as well to give an adjunction relating based G–spaces and naive G–spectra. For based G–spaces X, the map $\alpha\colon \boldsymbol{C}X \to \Omega^\infty \Sigma^\infty X$ is a group completion of Hopf G–spaces by the nonequivariant special case since $(\boldsymbol{C}X)^H = \boldsymbol{C}(X^H)$ and $(\Omega^\infty \Sigma^\infty X)^H = \Omega^\infty \Sigma^\infty (X^H)$.

Returning to finite groups, we work with Lewis–May spectra and G–spectra in the rest of this section in order to exploit the more precise relationship between spaces and spectra that holds in that context. However, the conclusions can easily be transported to orthogonal spectra. We index genuine G–spectra on a complete G–universe U and we index naive G–spectra on the trivial G–universe $U^G \cong \mathbb{R}^\infty$. The inclusion of universes $i\colon U^G \to U$ induces a forgetful functor $i^*\colon G\mathscr{S}p^U \to G\mathscr{S}p^{U^G}$ from genuine G–spectra to naive G–spectra. It represents the forgetful functor from $RO(G)$–graded

cohomology theories to $\mathbb{Z}$–graded cohomology theories. The functor i^* has a left adjoint i_*. The following observations are trivial but important.

Lemma 2.22 *The functors $i_*\Sigma^\infty$ and Σ_G^∞ from based G–spaces to genuine G–spectra are isomorphic.*

Proof Clearly $\Omega^\infty \iota^* = \Omega_G^\infty$, since both are evaluation at $V = 0$, hence their left adjoints are isomorphic. $\square$

Remark 2.23 For G–spaces X, the unit of the (i_*, i^*) adjunction gives a natural map $\Sigma^\infty X \to i^*i_*\Sigma^\infty X \cong \iota^*\Sigma_G^\infty$ of naive G–spectra. It is very far from being an equivalence, as the tom Dieck splitting theorem shows; see Theorem 6.5.

The inclusion of universes $i\colon U^G \to U$ induces an inclusion of operads of G–spaces $\iota\colon \mathscr{K}_{U^G} \to \mathscr{K}_U$, where G acts trivially on $\mathscr{K}_{U^G}$. The product of this inclusion and the inclusion $\iota\colon \mathscr{C} = (\mathscr{C}_G)^G \to \mathscr{C}_G$ is an inclusion

$$\iota\colon \mathscr{C}_{U^G} \equiv \mathscr{C} \times \mathscr{K}_{U^G} \to \mathscr{C}_G \times \mathscr{K}_U \equiv \mathscr{C}_U.$$

Pulling actions back along ι gives a functor ι^* from $\mathscr{C}_U$–spaces to $\mathscr{C}_{U^G}$–spaces. The following consistency statement is important since, by definition, the H–fixed-point spectrum E^H of a genuine G–spectrum E is $(i^*E)^H$ and the homotopy groups of E are $\pi_*^H(E) \equiv \pi_*(E^H)$.

Theorem 2.24 *Let Y be a $\mathscr{C}_G$–space. Then there is a natural weak equivalence of naive G–spectra $\mathbb{E}\iota^*Y \to i^*\mathbb{E}_G Y$.*

Proof Again, although we work with $\mathbb{E}_G^{\mathscr{S}p}$, the conclusion carries over to $\mathbb{E}_G^{\mathscr{S}}$. It is easy to check from the definitions that, for G–spaces X, we have a natural commutative diagram of G–spaces:

$$
\begin{array}{ccc}
C_{U^G} X & \xrightarrow{\;\alpha\;} & \Omega^\infty \Sigma^\infty X \\
\downarrow & & \downarrow \\
C_U X & \xrightarrow{\;\alpha\;} & \Omega_G^\infty \Sigma_G^\infty X
\end{array}
$$

The vertical arrows both restrict colimits over representations to colimits over trivial representations. Passing to adjoints, we obtain a natural commutative diagram:

$$
\begin{array}{ccc}
\Sigma_G^\infty C_{U^G} X & \longrightarrow & \Sigma^\infty X \\
\downarrow & & \downarrow \\
\Sigma_G^\infty C_U X & \longrightarrow & \Sigma_G^\infty X
\end{array}
$$

The composite gives a right action of C_{UG} on Σ_G^∞ that is compatible with the right action of C_U. Using the natural map $\Sigma^\infty \to i^* \Sigma_G^\infty$ of Remark 2.23, there results a natural map

$$\mu: \mathbb{E}\iota^* Y = B(\Sigma^\infty, C_{UG}, \iota^* Y) \to B(i^* \Sigma_G^\infty, C_U, Y) \cong i^* \mathbb{E}_G Y$$

of naive G–spectra. The following diagram commutes by a check of definitions:

$$
\begin{array}{ccccccc}
Y & \xleftarrow{\varepsilon} & B(C_{UG}, C_{UG}, \iota^* Y) & \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} & B(Q, C_{UG}, Y) & \xrightarrow{\zeta} & \Omega^\infty B(\Sigma^\infty, C_{UG}, Y) \\
{\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega^\infty \mu} \\
Y & \xleftarrow{\varepsilon} & B(C_G, C_G, Y) & \xrightarrow{B(\alpha,\mathrm{id},\mathrm{id})} & B(Q_G, C_U, Y) & \xrightarrow{\zeta} & \Omega_G^\infty B(\Sigma_G^\infty, C_U, Y)
\end{array}
$$

Here the right vertical map is the map of zeroth spaces given by μ. Replacing the maps ε with their homotopy inverses, the horizontal composites become group completions. Therefore, $\Omega^\infty \mu$ is a weak equivalence, hence so is μ. $\qquad\square$

We also have the corresponding statement for the left adjoint i_* of i^*. In effect, it gives a space level construction of the change of universe functor i_* on connective G–spectra. We need a homotopically well-behaved version of the left adjoint of the functor ι^* from $\mathscr{C}$–spaces to $\mathscr{C}_G$–spaces, and we define it by $\iota_! X = B(C_G, C, X)$.

Theorem 2.25 *Let X be a $\mathscr{C}$–space. Then there is a natural weak equivalence of genuine G–spectra $\mathbb{E}_G(i_! X) \simeq i_* \mathbb{E}(X)$.*

We give the proof in Appendix B, using a construction that is of independent interest.

3 Categorical preliminaries on classifying G–spaces and G–operads

We recall an elementary functor $\mathscr{C}at(\mathcal{E}G, -)$ from G–categories to G–categories from our paper [11] with Mona Merling. We explored this functor in detail in the context of equivariant bundle theory in [11], and we refer the reader there for proofs. In Section 4, we shall use it to define a certain operad $\mathscr{P}_G$ of G–categories. The $\mathscr{P}_G$–algebras will be the genuine permutative G–categories.

3.1 Chaotic topological categories and equivariant classifying spaces

For (small) categories $\mathscr{A}$ and $\mathscr{B}$, we let $\mathscr{C}at(\mathscr{A}, \mathscr{B})$ denote the category whose objects are the functors $\mathscr{A} \to \mathscr{B}$ and whose morphisms are the natural transformations between

them. When $\mathscr{B}$ has a right action by some group Π, then $\mathscr{C}\!at(\mathscr{A},\mathscr{B})$ inherits a right Π–action. When a group G acts from the left on $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{C}\!at(\mathscr{A},\mathscr{B})$ inherits a left G–action by conjugation on objects and morphisms. Then $G\mathscr{C}\!at(\mathscr{A},\mathscr{B})$ is alternative notation for the G–fixed category $\mathscr{C}\!at(\mathscr{A},\mathscr{B})^G$ of G–functors and G–natural transformations. We have the G–equivariant version of the standard adjunction

$$(3.1) \qquad \mathscr{C}\!at(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathscr{C}\!at(\mathscr{A}, \mathscr{C}\!at(\mathscr{B}, \mathscr{C})).$$

Definition 3.2 For a space X, the chaotic (topological) category $\mathcal{E}X$ has object space X, morphism space $X \times X$, and structure maps I, S, T, and C given by $I(x) = (x,x)$, $S(y,x) = x$, $T(y,x) = y$, and $C((z,y),(y,x)) = (z,x)$. For any point $* \in X$, the map $\eta \colon X \to X \times X$ specified by $\eta(x) = (*,x)$ is a continuous natural isomorphism from the identity functor to the trivial functor $\mathcal{E}X \to * \to \mathcal{E}X$, hence $\mathcal{E}X$ is equivalent to $*$. When $X = G$ is a topological group, $\mathcal{E}G$ is isomorphic to the translation category of G, but the isomorphism encodes information about the group action and should not be viewed as an identification; see [11, Remark 1.7]. We say that a topological category with object space X is chaotic if it is isomorphic to $\mathcal{E}X$.

Definition 3.3 Without changing notation, we regard a topological group Π as a topological category with a single object $*$ and morphism space Π, with composition given by multiplication. Then Π is isomorphic to the orbit category $\mathcal{E}\Pi/\Pi$, where Π acts from the right on $\mathcal{E}\Pi$ via right multiplication on objects and diagonal right multiplication on morphisms. The resulting functor $p \colon \mathcal{E}\Pi \to \Pi$ is given by the trivial map $\Pi \to *$ of object spaces and the map $p \colon \Pi \times \Pi \to \Pi \times \Pi/\Pi \cong \Pi$ on morphism spaces specified by $p(\tau,\sigma) = \tau\sigma^{-1}$.

Theorem 3.4 [11, Theorem 2.7] *For a G–space X and a topological group Π, regarded as a G–trivial G–space, the functor $p \colon \mathcal{E}\Pi \to \Pi$ induces an isomorphism of topological G–categories*

$$\xi \colon \mathscr{C}\!at(\mathcal{E}X, \mathcal{E}\Pi)/\Pi \to \mathscr{C}\!at_G(\mathcal{E}X, \Pi).$$

Therefore, passing to G–fixed-point categories,

$$(\mathscr{C}\!at(\mathcal{E}X, \mathcal{E}\Pi)/\Pi)^G \cong \mathscr{C}\!at(\mathcal{E}X, \Pi)^G \cong \mathscr{C}\!at(\mathcal{E}X/G, \Pi).$$

The last isomorphism is clear since G acts trivially on Π. Situations where G is allowed to act nontrivially on Π are of considerable interest, as we shall see in Section 4.4, but otherwise they will only appear peripherally in this paper. The paper [11] works throughout in that more general context. The previous result will not be used directly, but it is the key underpinning for the results of the next section.

3.2 The functor $\mathscr{C}\!at(\mathcal{E}G, -)$

The functor $\mathscr{C}\!at(\mathcal{E}G, -)$ from G–categories to G–categories is a right adjoint (3.1), hence it preserves limits and in particular products. The projection $\mathcal{E}G \to *$ to the trivial G–category induces a natural map

$$(3.5) \qquad \iota\colon \mathscr{A} = \mathscr{C}\!at(*, \mathscr{A}) \to \mathscr{C}\!at(\mathcal{E}G, \mathscr{A}).$$

The map ι is not an equivalence of G–categories in general [11, Proposition 4.19], but the functor $\mathscr{C}\!at(\mathcal{E}G, -)$ is idempotent in the sense that the following result holds.

Lemma 3.6 *For any G–category $\mathscr{A}$,*

$$\iota\colon \mathscr{C}\!at(\mathcal{E}G, \mathscr{A}) \to \mathscr{C}\!at(\mathcal{E}G, \mathscr{C}\!at(\mathcal{E}G, \mathscr{A}))$$

is an equivalence of G–categories.

Proof This follows from the adjunction (3.1) using that the diagonal $\mathcal{E}G \to \mathcal{E}G \times \mathcal{E}G$ is an equivalence with inverse given by either projection and that the specialization of ι here is induced by the first projection. $\qquad\square$

Lemma 3.7 [11, Lemma 3.7] *Let Λ be a subgroup of $G \times \Pi$. The Λ–fixed category $\mathscr{C}\!at(\mathcal{E}G, \mathcal{E}\Pi)^\Lambda$ is empty if $\Lambda \cap \Pi \neq e$ and is nonempty and chaotic if $\Lambda \cap \Pi = e$.*

With G acting trivially on Π, let $H^1(G; \Pi)$ denote the set of isomorphism classes of homomorphisms $\alpha\colon G \to \Pi$. Equivalently, it is the set of Π–conjugacy classes of subgroups $\Lambda = \{(g, \alpha(g)) \mid g \in G\}$ of $G \times \Pi$. Define $\Pi^\alpha \subset \Pi$ to be the subgroup of elements σ that commute with $\alpha(g)$ for all $g \in G$.

Theorem 3.8 [11, Theorems 4.14 and 4.18] *For $H \subset G$, the H–fixed category $\mathscr{C}\!at(\mathcal{E}G, \Pi)^H$ is equivalent to the coproduct of the groups Π^α (regarded as categories), where the coproduct runs over $[\alpha] \in H^1(H; \Pi)$.*

Definition 3.9 Define $E(G, \Pi) = |\mathscr{C}\!at(\mathcal{E}G, \mathcal{E}\Pi)|$ and $B(G, \Pi) = |\mathscr{C}\!at(\mathcal{E}G, \Pi)|$. Let

$$p\colon E(G, \Pi) \to B(G, \Pi)$$

be induced by the passage to orbits functor $\mathcal{E}\Pi \to \Pi$.

Theorem 3.10 [11, Theorems 3.11, 4.23, 4.24] *Let Π be a discrete or compact Lie group and let G be a discrete group. Then $p\colon E(G, \Pi) \to B(G, \Pi)$ is a universal principal (G, Π)–bundle. For a subgroup H of G,*

$$B(G, \Pi)^H \simeq \bigsqcup B(\Pi^\alpha),$$

where the union runs over $[\alpha] \in H^1(H; \Pi)$.

3.3 E_∞–operads of G–categories

The definition of an E_∞–operad of G–spaces given in Section 2.1 has the following categorical analogue.

Definition 3.11 An E_∞–operad $\mathscr{O}_G$ of (topological) G–categories is an operad in the cartesian monoidal category $G\mathscr{C}at$ such that $|\mathscr{O}_G|$ is an E_∞–operad of G–spaces. We say that $\mathscr{O}_G$ is reduced if $\mathscr{O}_G(0)$ is the trivial category. In practice, the $\mathscr{O}_G(j)$ are groupoids.

The proof of Lemma 2.5 works just as well to give the following analogue.

Lemma 3.12 Let $\mathscr{O}_G$ be an E_∞–operad of G–categories. Then $\mathscr{O} = (\mathscr{O}_G)^G$ is an E_∞–operad of categories. If $\mathscr{A}$ is an $\mathscr{O}_G$–category, then $\mathscr{A}^G$ is an $\mathscr{O}$–category.

4 Categorical philosophy: what is a permutative G–category?

4.1 Naive permutative G–categories

We have a notion of a monoidal category $\mathscr{A}$ internal to a cartesian monoidal category $\mathscr{V}$. It is a category internal to $\mathscr{V}$ together with a coherently associative and unital product $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$. It is strict monoidal if the product is strictly associative and unital. It is symmetric monoidal if it has an equivariant symmetry isomorphism satisfying the usual coherence properties. A functor $F\colon \mathscr{A} \to \mathscr{B}$ between symmetric monoidal categories is strict monoidal if $F(A \otimes A') = FA \otimes FA'$ for $A, A' \in \mathscr{A}$ and $FI = J$, where I and J are the unit objects of $\mathscr{A}$ and $\mathscr{B}$.

A permutative category is a symmetric strict monoidal category.[4] Taking $\mathscr{V}$ to be $\mathscr{U}$, these are the topological permutative categories. Taking $\mathscr{V}$ to be $G\mathscr{U}$, these are the *naive* topological permutative G–categories.

Nonequivariantly, there is a standard E_∞–operad of spaces that is obtained by applying the classifying-space functor to an E_∞–operad $\mathscr{P}$ of categories. The following definition goes back to Barratt and Eccles, thought of simplicially [2], and to [22], thought of categorically.

---

[4]In interesting examples, the product cannot be strictly commutative.

Definition 4.1 We define an E_∞–operad $\mathscr{P}$ of categories. Let $\mathscr{P}(j) = \mathcal{E}\Sigma_j$. Since Σ_j acts freely and $\mathcal{E}\Sigma_j$ is chaotic, the classifying space $|\mathscr{P}(j)|$ is Σ_j–free and contractible, as required of an E_∞–operad. The structure maps

$$\gamma\colon \mathcal{E}\Sigma_k \times \mathcal{E}\Sigma_{j_1} \times \cdots \times \mathcal{E}\Sigma_{j_k} \to \mathcal{E}\Sigma_j,$$

where $j = j_1 + \cdots + j_k$, are dictated on objects by the definition of an operad. If we view the object sets of the $\mathscr{P}(j)$ as discrete categories (identity morphisms only), then they form the associativity operad $\mathscr{M}$.

We can define $\mathscr{M}$–algebras and $\mathscr{P}$–algebras in $\mathscr{C}\!at$ or in $G\mathscr{C}\!at$. In the latter case, we regard $\mathscr{M}$ and $\mathscr{P}$ as operads with trivial G–action. The following result characterizes naive permutative G–categories operadically. The proof is easy [22].

Proposition 4.2 *The category of strict monoidal G–categories and strict monoidal G–functors is isomorphic to the category of $\mathscr{M}$–algebras in G$\mathscr{C}\!at$. The category of naive permutative G–categories and strict symmetric monoidal G–functors is isomorphic to the category of $\mathscr{P}$–algebras in G$\mathscr{C}\!at$.*

The term "naive" is appropriate since naive permutative G–categories give rise to naive G–spectra on application of an infinite loop space machine. Genuine permutative G–categories need more structure, especially precursors of transfer maps, to give rise to genuine G–spectra. Nonequivariantly, there is no distinction.

4.2 Genuine permutative G–categories

The following observation will play a helpful role in our work. Recall the natural map $\iota\colon \mathscr{A} \to \mathscr{C}\!at(\mathcal{E}G, \mathscr{A})$ of (3.5).

Lemma 4.3 *For any space X regarded as a G–trivial G–space, $\iota\colon \mathcal{E}X \to \mathscr{C}\!at(\mathcal{E}G, \mathcal{E}X)$ is the inclusion of the G–fixed category $G\mathscr{C}\!at(\mathcal{E}G, \mathcal{E}X)$.*

Proof Since $\mathcal{E}X$ is chaotic, functors $\mathcal{E}G \to \mathcal{E}X$ are determined by their object map $G \to X$ and are G–fixed if and only if the object map factors through $G/G = *$. $\square$

Definition 4.4 Let $\mathscr{P}_G$ be the (reduced) operad of G–categories whose j^{th} G–category is $\mathscr{P}_G(j) = \mathscr{C}\!at(\mathcal{E}G, \mathscr{P}(j))$, where $\mathscr{P}(j) = \mathcal{E}\Sigma_j$ is viewed as a G–category with trivial G–action and is given its usual right Σ_j–action. The unit in $\mathscr{P}_G(1)$ is the unique functor from $\mathcal{E}G$ to the trivial category $\mathscr{P}(1) = \mathscr{P}_G(1)$. The structure maps γ of $\mathscr{P}_G$ are induced from those of $\mathscr{P}$, using that the functor $\mathscr{C}\!at(\mathcal{E}G, -)$ preserves products. By Theorem 3.10, $\mathscr{P}_G$ is an E_∞–operad of G–categories. The natural map ι of (3.5) induces an inclusion $\iota\colon \mathscr{P} = (\mathscr{P}_G)^G \to \mathscr{P}_G$ of operads of G–categories.

Definition 4.5 A *genuine* permutative G–category is a $\mathscr{P}_G$–algebra in $G\mathscr{C}at$. A map of genuine permutative G–categories is a map of $\mathscr{P}_G$–algebras.

We usually call these $\mathscr{P}_G$–categories. We have an immediate source of examples. Let ι^* be the functor from genuine permutative G–categories to naive permutative G–categories that is obtained by restricting actions by $\mathscr{P}_G$ to its suboperad $\mathscr{P}$.

Proposition 4.6 *The action of $\mathscr{P}$ on a naive permutative G–category $\mathscr{A}$ induces an action of $\mathscr{P}_G$ on $\mathscr{C}at(\mathcal{E}G, \mathscr{A})$. Therefore, $\mathscr{C}at(\mathcal{E}G, -)$ restricts to a functor from naive permutative G–categories to genuine permutative G–categories.*

Proof This holds since the functor $\mathscr{C}at(\mathcal{E}G, -)$ preserves products. □

Proposition 4.7 *The map ι of (3.5) restricts to a natural map $\mathscr{A} \to \iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{A})$ of naive permutative G–categories, and ι is an equivalence when $\mathscr{A} = \iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{B})$ for a naive permutative G–category $\mathscr{B}$.*

Proof Since ι is induced by the projection $\mathcal{E}G \to \mathcal{E}\{e\} = *$, the first claim is clear, and the second holds by Lemma 3.6. □

As noted before, the map $\iota\colon \mathscr{A} \to \iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{A})$ is not an equivalence in general [11, Proposition 4.19]. The $\mathscr{P}_G$–categories of interest in this paper are of the form $\mathscr{C}at(\mathcal{E}G, \mathscr{A})$ for a naive permutative G–category $\mathscr{A}$. In fact, we do not yet know how to construct other examples, although we believe that they exist.

Remark 4.8 Shimakawa [43, page 256] introduced the E_∞–operad $\mathscr{P}_G$ under the name $\mathscr{D}$ and demonstrated the first part of Proposition 4.6.

Remark 4.9 One might hope that $(\mathscr{C}at(\mathcal{E}G, -), \iota^*)$ is an adjoint pair. However, regarding ι^* monadically as the forgetful functor from $\mathbb{P}_G$–algebras to $\mathbb{P}$–algebras, its left adjoint is the coend that sends a naive permutative G–category $\mathscr{A}$ to the genuine permutative G–category $\mathbb{P}_G \otimes_{\mathbb{P}} \mathscr{A}$, which is the coequalizer in $G\mathscr{C}at$ of the maps $\mathbb{P}_G\mathbb{P}\mathscr{A} \rightrightarrows \mathbb{P}_G\mathscr{A}$ induced by the action map $\mathbb{P}\mathscr{A} \to \mathscr{A}$ and by the map $\mathbb{P}_G\mathbb{P} \to \mathbb{P}_G\mathbb{P}_G \to \mathbb{P}_G$ induced by the inclusion $\mathbb{P} \to \mathbb{P}_G$ and the product on $\mathbb{P}_G$. The universal property of the coequalizer gives a natural map

$$\widetilde{\iota}\colon \mathbb{P}_G \otimes_{\mathbb{P}} \mathscr{A} \to \mathscr{C}at(\mathcal{E}G, \mathscr{A})$$

of genuine permutative G–categories that restricts to ι on $\mathscr{A}$, but $\widetilde{\iota}$ is not an isomorphism. We shall say a bit more about this in Remark 4.20.

4.3 E_∞–G–categories

We can generalize the notion of a genuine permutative G–category by allowing the use of E_∞–operads other than $\mathscr{P}_G$. In fact, thinking as algebraic topologists rather than category theorists, there is no need to give the particular E_∞–operad $\mathscr{P}_G$ a privileged role.

Definition 4.10 An E_∞–G–category $\mathscr{A}$ is a G–category together with an action of some E_∞–operad $\mathscr{O}_G$ of G–categories. The classifying space $B\mathscr{A} = |\mathscr{A}|$ is then an $|\mathscr{O}_G|$–space and thus an E_∞–G–space.

We may think of E_∞–G–categories as generalized kinds of genuine permutative G–categories. The point of the generalization is that we have interesting examples of E_∞–operads of G–categories with easily recognizable algebras. We shall later define E_∞–operads $\mathscr{V}_G$, $\mathscr{V}_G^\times$, and $\mathscr{W}_G$ that are interrelated in a way that illuminates the study of multiplicative structures.

Observe that $\mathscr{P}_G$–algebras, like nonequivariant permutative categories, have a canonical product, whereas E_∞–G–categories over other operads do not. The general philosophy of operad theory is that algebras over an operad $\mathscr{C}$ in any suitable category $\mathscr{V}$ have j–fold operations parametrized by the objects $\mathscr{C}(j)$. Homotopical properties of $\mathscr{C}$ relate these operations. In general, in an E_∞ space, there is no preferred choice of a product on its underlying H–space, and none relevant to the applications; E_∞–G–categories work similarly.

Remark 4.11 Symmetric monoidal categories occur more often "in nature" than permutative categories. We have not specified a notion of a genuine symmetric monoidal G–category in this paper. One approach is to apply the construction $\mathscr{C}at(\mathscr{E}G, -)$ to the tree operad that defines symmetric monoidal categories. Another approach, which we find more useful, is to define a genuine symmetric monoidal G–category to be a pseudoalgebra over $\mathscr{P}_G$. That approach is developed and applied in the categorical sequels [12; 13]. We shall not pursue the topic further here. A first comparison between symmetric monoidal G–categories and (genuine) G–symmetric monoidal categories, whose definition is a priori quite different, is given in Hill and Hopkins [16, Section 3.2], but work in progress shows that there is a good deal more to be said about that comparison and about the comparison between these notions and Tambara functors that is given in [16, Section 5.1].

Up to homotopy, any two choices of E_∞–operads give rise to equivalent categories of E_∞–G–spaces. To see that, we apply the trick from [21] of using products of operads to transport operadic algebras from one E_∞–operad to another. The product of operads $\mathscr{C}$ and $\mathscr{D}$ in any cartesian monoidal category $\mathscr{V}$ is given by

$$(\mathscr{C} \times \mathscr{D})(j) = \mathscr{C}(j) \times \mathscr{D}(j),$$

with the evident permutations and structure maps. With the choices of $\mathscr{V}$ of interest to us, the product of E_∞–operads is an E_∞–operad. The projections

$$\mathscr{C} \leftarrow \mathscr{C} \times \mathscr{D} \rightarrow \mathscr{D}$$

allow us to construct $(\mathscr{C} \times \mathscr{D})$–algebras in $\mathscr{V}$ from either $\mathscr{C}$–algebras or $\mathscr{D}$–algebras in $\mathscr{V}$, by pullback of action maps along the projections.

More generally, for any map $\mu \colon \mathscr{C} \to \mathscr{D}$ of operads in $\mathscr{V}$, the pullback functor μ^* from $\mathscr{D}$–algebras to $\mathscr{C}$–algebras has a left adjoint pushforward functor $\mu_!$ from $\mathscr{C}$–algebras to $\mathscr{D}$–algebras. One can work out a homotopical comparison model categorically. Pragmatically, use of the two-sided bar construction as in [21; 30] gives all that is needed. One redefines $\mu_! X = B(\mathbb{D}, \mathbb{C}, X)$, where $\mathbb{C}$ and $\mathbb{D}$ are the monads whose algebras are the $\mathscr{C}$–algebras and $\mathscr{D}$–algebras.[5] In spaces, or equally well G–spaces, μ^* and $\mu_!$ give inverse equivalences of homotopy categories between $\mathscr{C}$–algebras and $\mathscr{D}$–algebras when $\mathscr{C}$ and $\mathscr{D}$ are E_∞–operads.

Starting with operads in $\mathscr{C}at$ or in $G\mathscr{C}at$ we can first apply the classifying-space functor and then apply this trick. The conclusion is that all E_∞–categories and E_∞–G–categories give equivalent inputs for infinite loop space machines. In particular, for example, letting $\boldsymbol{O}_G$, $\boldsymbol{P}_G$, and $\boldsymbol{O}_G \times \boldsymbol{P}_G$ denote the monads in the category of G–spaces whose algebras are $|\mathscr{O}_G|$–algebras, $|\mathscr{P}_G|$–algebras, and $|\mathscr{O}_G \times \mathscr{P}_G|$–algebras, we see that after passage to classifying spaces, every $\boldsymbol{P}_G$–algebra Y determines an $\boldsymbol{O}_G$–algebra $X = B(\boldsymbol{O}_G, \boldsymbol{O}_G \times \boldsymbol{P}_G, Y)$ such that X and Y are weakly equivalent as $(\boldsymbol{O}_G \times \boldsymbol{P}_G)$–algebras (and conversely). This says that for purposes of equivariant infinite loop space theory, $\mathscr{P}_G$ and any other E_∞–operad $\mathscr{O}_G$ can be used interchangeably, regardless of how their algebras compare categorically.

4.4 Equivariant algebraic K–theory

The most interesting nonequivariant permutative categories are given by categories $\mathscr{A} = \bigsqcup \Pi_n$, where $\{\Pi_n \mid n \geq 0\}$ is a sequence of groups (regarded as categories with a single object) and where the permutative structure is given by an associative and unital system of pairings $\Pi_m \times \Pi_n \to \Pi_{m+n}$. Then the pairings give the classifying space $B\mathscr{A} = \bigsqcup B\Pi_n$ a structure of topological monoid, and one definition of the algebraic K–groups of $\mathscr{A}$ is the homotopy groups of the space $\Omega B(B\mathscr{A})$.

Equivariantly, it is sensible to replace the spaces $B\Pi_n$ by the classifying G–spaces $B(G, \Pi_n)$ and proceed by analogy. This definition of equivariant algebraic K–groups was introduced and studied calculationally in [9]. It is the equivariant analogue of

---

[5]Of course, this is an abuse of notation, since $\mu_!$ here is really a derived functor.

Quillen's original definition in terms of the plus construction. With essentially the same level of generality, the analogue of Quillen's definition in terms of the Q–construction has been studied by Dress and Kuku [6; 17]. Shimada [44] has given an equivariant version of Quillen's "plus $=Q$" theorem in this context.

Regarding $\mathscr{A}$ as a G–trivial naive permutative G–category, we see that the classifying G–space of the genuine permutative G–category $\mathscr{C}at(\mathcal{E}G, \mathscr{A})$ is the disjoint union of classifying spaces $B(G, \Pi_n)$. Just as nonequivariantly, the functor ΩB can be replaced by the zeroth space functor $\Omega_G^\infty \mathbb{E}_G$ of an infinite loop G–space machine $\mathbb{E}_G$. The underlying equivariant homotopy type is unchanged. Therefore, we may redefine the algebraic K–groups to be the homotopy groups of the genuine G–spectrum $\mathbb{K}_G\mathscr{A} \equiv \mathbb{E}_G B\mathscr{C}at(\mathcal{E}G, \mathscr{A})$. Essentially the same definition is implicit in Shimakawa [43], who focused on an equivariant version of Segal's infinite loop space machine. A different equivariant version of Segal's machine is developed and compared to Shimakawa's in [31]. It is generalized categorically in [12; 13].

Applying the functor $\mathscr{C}at(\mathcal{E}G, -)$ to naive permutative G–categories $\mathscr{A}$ with nontrivial G–actions gives more general input for equivariant algebraic K–theory than has been studied in the literature. This allows for G–actions on the groups Π_n, and we then replace $B(G, \Pi_n)$ by classifying G–spaces $B(G, (\Pi_n)_G)$ for the $(G, (\Pi_n)_G)$–bundles associated to the split extensions $\Pi_n \rtimes G$. Such classifying spaces are studied in [11]. Alternative but equivalent constructions of the associated G–spectra $\mathbb{K}_G(\mathscr{A})$ are given in Section 4.5 and Section 8.2 below. The resulting generalization of equivariant algebraic K–theory is studied in [33].

4.5 The recognition principle for permutative G–categories

We may start with any E_∞–operad $\mathcal{O}_G$ of G–categories and apply the classifying-space functor to obtain an E_∞–operad $|\mathcal{O}_G|$ of G–spaces. If $\mathcal{O}_G$ acts on a category $\mathscr{A}$, then $|\mathcal{O}_G|$ acts on $|\mathscr{A}| = B\mathscr{A}$. We can replace $|\mathcal{O}_G|$ by its product with the Steiner operads $\mathscr{K}_V$ or with the Steiner operad $\mathscr{K}_U$ and apply the functor $\mathbb{E}_G^{\mathscr{S}}$ or $\mathbb{E}_G^{\mathscr{S}p}$ to obtain a (genuine) associated G–spectrum, which we denote ambiguously by $\mathbb{E}_G(B\mathscr{A})$.

Definition 4.12 Define the (genuine) algebraic K–theory G–spectrum of an $\mathcal{O}_G$–category $\mathscr{A}$ by $\mathbb{K}_G(\mathscr{A}) = \mathbb{E}_G(B\mathscr{A})$.

We might also start with an operad $\mathcal{O}$ of categories such that $|\mathcal{O}|$ is an E_∞–operad of spaces and regard these as G–objects with trivial action. Following up the previous section, we then have the following related but less interesting notion.

Definition 4.13 The (naive) algebraic K–theory G–spectrum of an $\mathcal{O}$–category $\mathscr{A}$ is defined by $\mathbb{K}(\mathscr{A}) = \mathbb{E}(B\mathscr{A})$.

Until Section 7, we restrict attention to the cases $\mathcal{O}_G = \mathscr{P}_G$ and $\mathcal{O} = \mathscr{P}$, recalling that the $\mathscr{P}_G$–categories are the genuine permutative G–categories, the $\mathscr{P}$–categories are the naive permutative G–categories, and the inclusion $\iota\colon \mathscr{P} \to \mathscr{P}_G$ induces a forgetful functor ι^* from genuine to naive permutative G–categories. Since the classifying-space functor commutes with products, passage to fixed points, and the functors ι^*, Theorems 2.20, 2.21, and 2.24 have the following immediate corollaries. The first was promised in [10, Theorem 2.2].

Theorem 4.14 *For $\mathscr{P}_G$–categories $\mathscr{A}$, there is a natural weak equivalence of spectra*

$$\mathbb{K}(\mathscr{A}^G) \to (\mathbb{K}_G \mathscr{A})^G.$$

Theorem 4.15 *Let $\mathscr{A}$ and $\mathscr{B}$ be $\mathscr{P}_G$–categories. Then the map*

$$\mathbb{K}_G(\mathscr{A} \times \mathscr{B}) \to \mathbb{K}_G \mathscr{A} \times \mathbb{K}_G \mathscr{B}$$

induced by the projections is a weak equivalence of G–spectra.

Theorem 4.16 *For $\mathscr{P}_G$–categories $\mathscr{A}$, there is a natural weak equivalence of naive G–spectra $\mathbb{K}\iota^*\mathscr{A} \to i^*\mathbb{K}_G\mathscr{A}$.*

The algebraic K–groups of $\mathscr{A}$ are defined to be the groups

$$(4.17) \qquad K_*^H \mathscr{A} = \pi_*^H(\mathbb{K}\iota^*\mathscr{A}) \cong \pi_*^H(\mathbb{K}_G\mathscr{A}).$$

We are particularly interested in examples of the form $\mathscr{C}at(\mathcal{E}G, \mathscr{A})$, where $\mathscr{A}$ is a naive permutative G–category. As noted in Proposition 4.6, we then have a natural map $\iota\colon \mathscr{A} \to \iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{A})$ of naive permutative G–categories. We can pass to classifying spaces and apply the functor $\mathbb{E}$ to obtain a natural map

$$(4.18) \qquad \mathbb{K}\mathscr{A} \xrightarrow{\mathbb{K}\iota} \mathbb{K}\iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{A}) \xrightarrow[\simeq]{\mu} i^*\mathbb{K}_G\mathscr{C}at(\mathcal{E}G, \mathscr{A}).$$

This map is a weak equivalence when $\iota^H\colon \mathscr{A}^H \to (\iota^*\mathscr{C}at(\mathcal{E}G, \mathscr{A}))^H$ is an equivalence of categories for all $H \subset G$. The following example where this holds is important in equivariant algebraic K–theory.

Example 4.19 Let E be a Galois extension of F with Galois group G and let G act entrywise on $\mathrm{GL}(n, E)$ for $n \geq 0$. The disjoint union of the $\mathrm{GL}(n, E)$ is a naive permutative G–category that we denote by $\mathrm{GL}(E_G)$. Its product is given by the block sum of matrices. Write $\mathrm{GL}(R)$ for the nonequivariant permutative general linear category of a ring R. As we proved in [11, Example 4.20], Serre's version of Hilbert's Theorem 90 implies that

$$\iota^H\colon \mathrm{GL}(E^H) \cong \mathrm{GL}(E_G)^H \to (\iota^*\mathscr{C}at(\mathcal{E}G, \mathrm{GL}(E_G))^H$$

is an equivalence of categories for $H \subset G$. This identifies the equivariant algebraic K–groups of E with the nonequivariant algebraic K–groups of its fixed fields E^H.

Remark 4.20 In the list above of theorems about permutative categories, a consequence of Theorem 2.25 is conspicuous by its absence. Letting $\iota_! \mathscr{A} \equiv \mathbb{P}_G \otimes_{\mathbb{P}} \mathscr{A}$ denote the left adjoint of ι^*, as defined in Remark 4.9, one might hope that $B\iota_! \mathscr{A}$ is equivalent as an $|\mathscr{P}_G|$–space to $\iota_! B\mathscr{A}$ for a naive permutative G–category $\mathscr{A}$. We do not know whether or not that is true.

5 The free $|\mathscr{P}_G|$–space generated by a G–space X

The goal of this section is to obtain a decomposition of the fixed point categories of free permutative G–categories. This decomposition will be the crux of the proof of the tom Dieck splitting theorem given in Section 5.2.

5.1 The monads $\mathbb{P}_G$ and P_G associated to $\mathscr{P}_G$

Recall that $\mathscr{P}_G$ is reduced. In fact, both $\mathscr{P}_G(0)$ and $\mathscr{P}_G(1)$ are trivial categories. As discussed for spaces in [30, Section 4], there are two monads on G–categories whose algebras are the genuine permutative G–categories. The unit object of an $\mathscr{P}_G$–category can be preassigned, resulting in a monad $\mathbb{P}_G$ on based G–categories, or it can be viewed as part of the $\mathscr{P}_G$–algebra structure, resulting in a monad $\mathbb{P}_{G+}$ on unbased G–categories. Just as in [30], these monads are related by

$$\mathbb{P}_G(\mathscr{A}_+) \cong \mathbb{P}_{G+}\mathscr{A},$$

where $\mathscr{A}_+ = \mathscr{A} \sqcup *$ is obtained from an unbased G–category $\mathscr{A}$ by adjoining a disjoint copy of the trivial G–category $*$. Explicitly,

$$(5.1) \qquad \mathbb{P}_G(\mathscr{A}_+) = \bigsqcup_{j \geq 0} \mathscr{P}_G(j) \times_{\Sigma_j} \mathscr{A}^j.$$

The term with $j = 0$ is $*$ and accounts for the copy of $*$ on the left. The unit $\eta \colon \mathscr{A} \to \mathbb{P}_G(\mathscr{A}_+)$ identifies $\mathscr{A}$ with the term with $j = 1$. The product $\mu \colon \mathbb{P}_G \mathbb{P}_G \mathscr{A}_+ \to \mathbb{P}_G \mathscr{A}_+$ is induced by the operad structure maps γ. We are only concerned with based G–categories that can be written in the form $\mathscr{A}_+$.

Since we are concerned with the precise point-set relationship between an infinite loop space machine defined on G–categories and suspension G–spectra, it is useful to think of (unbased) G–spaces X as categories. Thus we also let X denote the topological G–category with object and morphism G–space X and with I, S, T, and C all given by the identity map $X \to X$; this makes sense for C since we can identify $X \times_X X$ with X. We can also identify the classifying G–space $|X|$ with X.

By specialization of (5.1), we have an identification of (topological) G–categories

$$(5.2) \qquad \mathbb{P}_G(X_+) = \bigsqcup_{j \geq 0} \mathscr{P}_G(j) \times_{\Sigma_j} X^j.$$

The following illuminating result gives another description of $\mathbb{P}_G(X_+)$.

Proposition 5.3 *For G–spaces X, there is a natural isomorphism of genuine permutative G–categories*

$$\mathbb{P}_G(X_+) = \bigsqcup_j \mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j) \times_{\Sigma_j} X^j \to \bigsqcup_j \mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j \times_{\Sigma_j} X^j) = \mathscr{C}at(\mathcal{E}G, \mathbb{P}(X_+)).$$

Proof For each j and for $(G \times \Sigma_j)$–spaces Y, such as $Y = X^j$, we construct a natural isomorphism of $(G \times \Sigma_j)$–categories

$$\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j) \times Y \to \mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j \times Y).$$

Here Y is viewed as the constant $(G \times \Sigma_j)$–category at Y. The target is

$$\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j) \times \mathscr{C}at(\mathcal{E}G, Y).$$

Since there is a map between any two objects of $\mathcal{E}G$ but the only maps in Y are identity maps $i_y \colon y \to y$ for $y \in Y$, the only functors $\mathcal{E}G \to Y$ are the constant functors c_y at $y \in Y$ and the only natural transformations between them are the identity transformations $\mathrm{id}_y \colon c_y \to c_y$. Sending y to c_y on objects and i_y to id_y on morphisms specifies an identification of $(G \times \Sigma_j)$–categories $Y \to \mathscr{C}at(\mathcal{E}G, Y)$. The product of the identity functor on $\mathscr{C}at(\mathcal{E}G, \mathcal{E}\Sigma_j)$ and this identification gives the desired natural equivalence. With $Y = X^j$, passage to orbits over Σ_j gives the jth component of the claimed isomorphism of G–categories. It is an isomorphism of $\mathscr{P}_G$–categories since on both sides the action maps are induced by the structure maps of the operad $\mathscr{P}$. $\quad\square$

Recall that we write $\boldsymbol{P}_G$ for the monad on based G–spaces associated to the operad $|\mathscr{P}_G|$. Thus $\boldsymbol{P}_G(X_+)$ is the free $|\mathscr{P}_G|$–space generated by the G–space X.

Proposition 5.4 *For G–spaces X, there is a natural isomorphism*

$$\boldsymbol{P}_G(X_+) = \bigsqcup_{j \geq 0} |\mathscr{P}_G(j)| \times_{\Sigma_j} X^j \cong |\mathbb{P}_G X_+|.$$

Proof For a $(G \times \Sigma_j)$–space Y viewed as a G–category, the nerve NY can be identified with the constant simplicial space Y_* with $Y_q = Y$. The nerve functor N

does not commute with passage to orbits in general, but arguing as in [11, Section 2.3] we see that

$$N(\mathscr{P}_G(j) \times_{\Sigma_j} Y) \cong (N\mathscr{P}_G(j)) \times_{\Sigma_j} Y_* = N(\mathscr{P}_G(j) \times_{\Sigma_j} NY).$$

Therefore, the classifying-space functor commutes with coproducts, products, and the passage to orbits that we see here. □

5.2 The identification of $(\mathbb{P}_G X_+)^G$

The functor $|-|$ commutes with passage to G–fixed points, and we shall prove the following identification. Let $\mathbb{P}$ denote the monad on nonequivariant based categories associated to the operad $\mathscr{P}$ that defines permutative categories.

Theorem 5.5 *For G–spaces X, there is a natural equivalence of $\mathscr{P}$–categories*

$$\mathbb{P}_G(X_+)^G \simeq \prod_{(H)} \mathbb{P}(\mathcal{E}WH \times_{WH} X^H)_+,$$

where (H) runs over the conjugacy classes of subgroups of G and $WH = NH/H$.

We are regarding $\mathscr{P}$ as the suboperad $(\mathscr{P}_G)^G$ of $\mathscr{P}_G$, and the identification of categories will make clear that the identification preserves the action by $\mathscr{P}$. Of course,

$$(5.6) \qquad \mathbb{P}_G(X_+)^G = \bigsqcup_{j \geq 0} (\mathscr{P}_G(j) \times_{\Sigma_j} X^j)^G$$

and

$$(5.7) \qquad \mathbb{P}(\mathcal{E}WH \times_{WH} X^H)_+ = \bigsqcup_{k \geq 0} \mathcal{E}\Sigma_k \times_{\Sigma_k} (\mathcal{E}WH \times_{WH} X^H)^k.$$

We shall prove Theorem 5.5 by identifying both (5.6) and (5.7) with a small (but not skeletal) model $\mathscr{F}_G(X)^G$ for the category of finite G–sets over X and their isomorphisms over X. We give the relevant definitions and describe these identifications here, and we fill in the easy proofs in Sections 5.3 and 5.4.

A homomorphism $\alpha\colon G \to \Sigma_j$ is equivalent to the left action of G on the set $\boldsymbol{j} = \{1, \ldots, j\}$ specified by $g \cdot i = \alpha(g)(i)$ for $i \in \boldsymbol{j}$. Similarly, an antihomomorphism $\alpha\colon G \to \Sigma_j$ is equivalent to the right action of G on $\boldsymbol{j}$ specified by $i \cdot g = \alpha(g)(i)$ or, equivalently, the left action specified by $g \cdot i = \alpha(g^{-1})(i)$; of course, if we set $\alpha^{-1}(g) = \alpha(g)^{-1}$, then α^{-1} is a homomorphism. We focus on homomorphisms and left actions, and we denote such G–spaces by $(\boldsymbol{j}, \alpha)$. When we say that A is a finite G–set, we agree to mean that $A = (\boldsymbol{j}, \alpha)$ for a given homomorphism $\alpha\colon G \to \Sigma_j$. That convention has the effect of fixing a small groupoid $G\mathscr{F}$ equivalent to the groupoid of all finite G–sets and isomorphisms of finite G–sets. By a j–pointed G–set, we mean a G–set with j elements.

Definition 5.8 Let X be a G–space and $j \geq 0$.

(i) Let $\mathscr{F}_G(j)$ be the G–groupoid whose objects are the j–pointed G–sets A and whose morphisms $\sigma\colon A \to B$ are the bijections, with G acting by conjugation. Then $\mathscr{F}_G(j)^G$ is the category with the same objects and with morphisms the isomorphisms of G–sets $\sigma\colon A \to B$.

(ii) Let $\mathscr{F}_G(j, X)$ be the G–groupoid whose objects are the maps (not G–maps) $p\colon A \to X$ and whose morphisms $f\colon p \to q$, for $q\colon B \to X$, are the bijections $f\colon A \to B$ such that $q \circ f = p$; the action of G is by conjugation on all maps p, q, and f. We view $\mathscr{F}_G(j, X)^G$ as the category of j–pointed G–sets over X and isomorphisms of j–pointed G–sets over X.

(iii) Let $\mathscr{F}_G = \bigsqcup_{j \geq 0} \mathscr{F}_G(j)$ and $\mathscr{F}_G(X) = \bigsqcup_{j \geq 0} \mathscr{F}_G(j, X)$.

(iv) Let $\mathscr{F}_G^{\mathscr{P}}(j)$ be the full G–subcategory of G–fixed objects of $\mathscr{P}_G(j)/\Sigma_j$ and let $\mathscr{F}_G^{\mathscr{P}}(j, X)$ be the full G–subcategory of G–fixed objects of $\mathscr{P}_G(j) \times_{\Sigma_j} X^j$. Then

$$\mathscr{F}_G^{\mathscr{P}}(j)^G = (\mathscr{P}_G(j)/\Sigma_j)^G \quad \text{and} \quad \mathscr{F}_G^{\mathscr{P}}(j, X)^G = (\mathscr{P}_G(j) \times_{\Sigma_j} X^j)^G.$$

In Section 5.3, we prove that the right side of (5.6) can be identified with $\mathscr{F}_G(X)^G$.

Theorem 5.9 *There is a natural isomorphism of permutative categories*

$$(\mathbb{P}_G(X_+))^G = \bigsqcup_{j \geq 0} \mathscr{F}_G^{\mathscr{P}}(j, X)^G \cong \bigsqcup_{j \geq 0} \mathscr{F}_G(j, X)^G = \mathscr{F}_G(X)^G.$$

We will prove an equivariant variant of this result, before passage to fixed points, in Theorem 9.6. In Section 5.4, we prove that the right side of (5.7) can also be identified with $\mathscr{F}_G(X)^G$. At least implicitly, this identification of fixed-point categories has been known since the 1970s; see for example Nishida [36, Appendix A].

Theorem 5.10 *There is a natural equivalence of categories*

$$\prod_{(H)} \bigsqcup_{k \geq 0} \mathcal{E}\Sigma_k \times_{\Sigma_k} (\mathcal{E}WH \times_{WH} X^H)^k \to \bigsqcup_{j \geq 0} \mathscr{F}_G(j, X)^G = \mathscr{F}_G(X)^G.$$

These two results prove Theorem 5.5.

Remark 5.11 With our specification of finite G–sets as $A = (\boldsymbol{j}, \alpha)$, the disjoint union of A and $B = (\boldsymbol{k}, \beta)$ is obtained via the obvious identification of $\boldsymbol{j} \sqcup \boldsymbol{k}$ with $\boldsymbol{j} + \boldsymbol{k}$. The disjoint union of finite G–sets over a G–space X gives $\mathscr{F}_G(X)$ a structure of naive permutative G–category. By Theorem 5.9, its fixed-point category $\mathscr{F}_G(X)^G$ is a $\mathscr{P}$–category equivalent to $(\mathbb{P}_G(X_+))^G$. One might think that $\mathscr{F}_G(X)$ is a genuine

permutative G–category equivalent to the free $\mathscr{P}_G$–category $\mathscr{P}_G(X_+)$. However, its H–fixed subcategory for $H \neq G$ is not equivalent to $\mathscr{F}_H(X)^H$, and one cannot expect an action of $\mathscr{P}_G$ (or any other E_∞–G–operad) on $\mathscr{F}_G(X)$. To see the point, let G be the quaternion group of order 8: $G = \{\pm 1, \pm i, \pm j, \pm k\}$, and let $X = *$. Every nontrivial subgroup of G contains the center $H = Z = \pm 1$. Therefore, the H–set H cannot be obtained by starting with a G–set (a disjoint union of orbits G/K) and restricting along the inclusion $H \to G$.

To compare with our paper [10], we offer some alternative notation.

Definition 5.12 For an unbased G–space X, let $\mathscr{E}_G(X) = \mathscr{E}_G^{\mathscr{P}}(X) = \mathbb{P}_G(X_+)$. It is a genuine permutative G–category, and its H–fixed subcategory $\mathscr{E}_G(X)^H$ is equivalent to $\mathscr{E}_H(X)^H$ and therefore to $\mathscr{F}_H(X)^H$.

Remark 5.13 In [10], we gave a more intuitive definition of a G–category $\mathscr{E}_G(X)$. It will reappear in Section 9, where it will be given the alternative notation $\mathscr{E}_G^U(X)$. It is acted on by an E_∞–operad $\mathscr{V}_G$ of G–categories, and, again, its fixed-point category $\mathscr{E}_G^U(X)^H$ is equivalent to $\mathscr{E}_H^U(X)^H$ and therefore to $\mathscr{F}_H(X)^H$.

5.3 The proof of Theorem 5.9

We first use Theorem 3.8 to identify (5.6) when X is a point. The proof of Theorem 3.8 compares several equivalent categories, and antihomomorphisms appear naturally. To control details of equivariance, it is helpful to describe the relevant categories implicit in our operad $\mathscr{P}_G$ in their simplest forms up to isomorphism. Details are in [11, Sections 2.1, 2.2, 4.1, 4.2].

Lemma 5.14 *The objects of the chaotic $(G \times \Sigma_j)$–category $\mathscr{P}_G(j)$ are the functions $\phi \colon G \to \Sigma_j$. The (left) action of G on $\mathscr{P}_G(j)$ is given by $(g\phi)(h) = \phi(g^{-1}h)$ on objects and the diagonal action on morphisms. The (right) action of Σ_j is given by $(\phi\sigma)(h) = \phi(h)\sigma$ on objects and the diagonal action on morphisms.*

Lemma 5.15 *The objects of the G–category $\mathscr{P}_G(j)/\Sigma_j$ are the functions $\alpha \colon G \to \Sigma_j$ such that $\alpha(e) = e$. The morphisms $\sigma \colon \alpha \to \beta$ are the elements $\sigma \in \Sigma_j$, thought of as the functions $G \to \Sigma_j$ specified by $\sigma(h) = \beta(h)\sigma\alpha(h)^{-1}$. The composite of σ with $\tau \colon \beta \to \gamma$ is $\tau\sigma \colon \alpha \to \gamma$. The action of G is given on objects by*

$$(g\alpha)(h) = \alpha(g^{-1}h)\alpha(g^{-1})^{-1}.$$

In particular, $(g\alpha)(e) = e$. The action on morphisms is given by

$$g(\sigma \colon \alpha \to \beta) = \sigma \colon g\alpha \to g\beta.$$

Lemma 5.16 *For $\Lambda \subset G \times \Sigma_j$, the subcategory $\mathscr{P}_G(j)^\Lambda$ is empty if $\Lambda \cap \Sigma_j \neq e$. It is a nonempty and hence chaotic subcategory of $\mathscr{P}_G(j)$ if $\Lambda \cap \Sigma_j = e$.*

Lemma 5.17 *The objects of $(\mathscr{P}_G(j)/\Sigma_j)^G$ are the antihomomorphisms $\alpha\colon G \to \Sigma_j$. Its morphisms $\sigma\colon \alpha \to \beta$ are the conjugacy relations $\beta = \sigma\alpha\sigma^{-1}$, where $\sigma \in \Sigma_j$. For $H \subset G$, restriction of functions gives an equivalence of categories*

$$(\mathscr{P}_G(j)/\Sigma_j)^H \to (\mathscr{P}_H(j)/\Sigma_j)^H.$$

Now return to a general G–space X. To prove Theorem 5.9, it suffices to prove that $(\mathscr{P}_G(j) \times_{\Sigma_j} X^j)^G$ is isomorphic to $\mathscr{F}_G(j, X)^G$ for all j. Passage to orbits here means that for $\phi \in \mathscr{P}_G(j)$, $y \in X^j$, and $\sigma \in \Sigma_j$ (thought of as acting on the left on $\mathbf{j}$ and therefore on j–tuples of elements of X), $(\phi\sigma, y) = (\phi, \sigma y)$ in $\mathscr{P}_G(j) \times_{\Sigma_j} X^j$. Observe that an object $(\phi, z_1, \dots, z_j) \in \mathscr{P}_G(j) \times_{\Sigma_j} X^j$ has a unique representative in the same orbit under Σ_j of the form $(\alpha, x_1, \dots, x_j)$ where $\alpha(e) = e$. It is obtained by replacing ϕ by $\phi\tau$, where $\tau = \phi(e)^{-1}$, and replacing z_i by $x_i = z_{\tau(i)}$.

Lemma 5.18 *An object $(\alpha, y) \in \mathscr{P}_G(j) \times_{\Sigma_j} X^j$, where $\alpha(e) = e$ and $y \in X^j$, is G–fixed if and only if $\alpha\colon G \to \Sigma_j$ is an antihomomorphism and $\alpha(g^{-1})y = gy$ for all $g \in G$.*

Proof Assume that $(\alpha, y) = (g\alpha, gy)$ for all $g \in G$. Then each $g\alpha$ must be in the same Σ_j–orbit as α, where α is regarded as an object of $\mathscr{P}_G(j)$ and not $\mathscr{P}_G(j)/\Sigma_j$, so that $(g\alpha)(h) = \alpha(g^{-1}h)$. Then $(g\alpha)(h) = \alpha(h)\sigma$ for all $h \in G$ and some $\sigma \in \Pi$. Taking $h = e$ shows that $\sigma = \alpha(g^{-1})$. The resulting formula $\alpha(g^{-1}h) = \alpha(h)\alpha(g^{-1})$ implies that α is an antihomomorphism. Now

$$(\alpha, y) = (g\alpha, gy) = (\alpha\alpha(g^{-1}), gy) = (\alpha, \alpha(g)gy),$$

which means that $\alpha(g)gy = y$ and thus $gy = \alpha(g^{-1})y$. $\qquad\square$

Use α^{-1} to define a left action of G on $\mathbf{j}$ and define $p\colon \mathbf{j} \to X$ by $p(i) = x_i$. Then the lemma shows that the G–fixed elements (α, y) are in bijective correspondence with the maps of G–sets $p\colon A \to X$, where A is a j–pointed G–set. Using Lemma 5.17, we see similarly that maps $f\colon A \to B$ of j–pointed G–sets over X correspond bijectively to morphisms in $(\mathscr{P}_G(j) \times_{\Sigma_j} X^j)^G$. These bijections specify the required isomorphism between $\mathscr{F}_G(j, X)^G$ and $(\mathscr{P}_G(j) \times_{\Sigma_j} X^j)^G$.

5.4 The proof of Theorem 5.10

This decomposition is best proven by a simple thought exercise. Every finite G–set A decomposes nonuniquely as a disjoint union of orbits G/H, and orbits G/H and G/J

are isomorphic if and only if H and J are conjugate. Choose one H in each conjugacy class. Then A decomposes uniquely as the disjoint union of the G–sets A_H, where A_H is the set of elements of A with isotropy group conjugate to H. This decomposes the category $G\mathscr{F} \equiv (\mathscr{F}_G)^G$ as the product over H of the categories $G\mathscr{F}(H)$ of finite G–sets all of whose isotropy groups are conjugate to H.

In turn, $G\mathscr{F}(H)$ decomposes uniquely as the coproduct over $k \geq 0$ of the categories $G\mathscr{F}(H,k)$ whose objects are isomorphic to the disjoint union, denoted by kG/H, of k copies of G/H. Up to isomorphism, kG/H is the only object of $G\mathscr{F}(H,k)$. The automorphism group of the G–set G/H is WH, hence the automorphism group of kG/H is the wreath product $\Sigma_k \int WH$. Viewed as a category with a single object, we may identify this group with the category $\mathcal{E}\Sigma_k \times_{\Sigma_k} (WH)^k$. This proves the following result.

Proposition 5.19 *The category $G\mathscr{F}$ is equivalent to the category*

$$\prod_{(H)} \bigsqcup_{k \geq 0} \mathcal{E}\Sigma_k \times_{\Sigma_k} (WH)^k.$$

The displayed category is a skeleton of $G\mathscr{F}$. As written, its objects are sets of numbers $\{k_H\}$, one for each (H), but they are thought of as the finite G–sets $\bigsqcup_H k_H G/H$. Its morphism groups specify the automorphisms of these objects. On objects, the equivalence sends a finite G–set A to the unique finite G–set of the form $\bigsqcup_{(H)} kG/H$ in the same isomorphism class as A. Via chosen isomorphisms, this specifies the inverse equivalence to the inclusion of the chosen skeleton in $G\mathscr{F}$.

We parametrize this equivalence to obtain a description of the category $G\mathscr{F}(X)$ of finite G–sets over X. Given any H and k, a k–tuple of elements $\{x_1, \ldots, x_k\}$ of X^H determines the G–map $p\colon kG/H \to X$ that sends eH in the ith copy of G/H to x_i, and it is clear that every finite G–set A over X is isomorphic to one of this form. Similarly, for a finite G–set $q\colon B \to X$ over X and an isomorphism $f\colon A \to B$, we have that f is an isomorphism over X from q to $p = q \circ f$, and every isomorphism over X can be constructed in this fashion. Since we may as well choose A and B to be in our chosen skeleton of $G\mathscr{F}$, this argument proves Theorem 5.10.

6 The Barratt–Priddy–Quillen and tom Dieck splitting theorems

6.1 The Barratt–Priddy–Quillen theorem revisited

The BPQ theorem shows how to model suspension G–spectra in terms of free E_∞–G–categories and G–spaces. It is built tautologically into the equivariant infinite loop

space machine in the same way as it is nonequivariantly [22, Theorem 2.3(vii)] or [30, Section 10]. The following result works for either $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{S}p}$ or $\mathbb{E}_G = \mathbb{E}_G^{\mathscr{S}}$, but note that the interpretation of both the source and target are different in the two cases. The proof shows consistency with the versions of the BPQ theorem in Theorems 1.31 and 2.18.

Theorem 6.1 (the E_∞–operad BPQ theorem) *For an E_∞–operad $\mathscr{C}_G$ of G–spaces and based G–spaces X, there is a natural weak equivalence of G–spectra*

$$\Sigma_G^\infty X \to \mathbb{E}_G \mathbf{C}_G X.$$

Proof For $\mathbb{E}_G^{\mathscr{S}p}$, recall that $\mathscr{C}_U = \mathscr{K}_U \times \mathscr{C}_G$. The same formal argument as for Theorem 2.18 and use of the projections to $\mathscr{C}_G$ and to $\mathscr{K}_U$ give equivalences of LM G–spectra:

$$\begin{array}{ccc}
\Sigma_G^\infty X & \longrightarrow & B(\Sigma_G^\infty, \mathbf{C}_U, \mathbf{C}_U X) \longrightarrow B(\Sigma_G^\infty, \mathbf{C}_U, \mathbf{C}_G X) \\
& \searrow & \downarrow \\
& & B(\Sigma_G^\infty, \mathbf{K}_U, \mathbf{K}_U X)
\end{array}$$

For $\mathbb{E}_G^{\mathscr{S}}$, recall that $\mathscr{C}_V = \mathscr{K}_V \times \mathscr{C}_G$. Analogously to Theorem 1.31, there is an orthogonal G–spectrum with V^{th} space $B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_V X)$. The usual formal argument and the projections to $\mathscr{C}_G$ and $\mathscr{K}_V$ give diagrams

$$\begin{array}{ccc}
\Sigma^V X & \longrightarrow & B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_V X) \longrightarrow B(\Sigma^V, \mathbf{C}_V, \mathbf{C}_G X) \\
& \searrow & \downarrow \\
& & B(\Sigma^V, \mathbf{K}_V, \mathbf{K}_V X)
\end{array}$$

for all V in which the left horizontal arrow and the vertical arrow are level equivalences of orthogonal G–spectra, and the right horizontal arrow is a weak equivalence (π_*–isomorphism) of orthogonal G–spectra, as we see by forgetting to G–prespectra and passing to colimits over $V \subset U$, where U is a complete G–universe. $\qquad\square$

Taking $Y = X_+$ for an unbased G–space X and using (5.2), we can rewrite this version of the BPQ theorem using the infinite loop space machine defined on permutative G–categories.

Theorem 6.2 (the categorical BPQ theorem: first version) *For unbased G–spaces X, there is a natural weak equivalence of G–spectra*

$$\Sigma_G^\infty X_+ \to \mathbb{K}_G \mathbb{P}_G(X_+).$$

Remark 6.3 Diagrams showing compatibility with smash products, like those in Theorems 1.31 and 2.18 are conspicuous by their absence from Theorems 6.1 and 6.2. A previous version of this article erroneously claimed that the operad $\mathscr{P}$ has a self pairing $(\mathscr{P}, \mathscr{P}) \to \mathscr{P}$ induced by the homomorphisms

$$(6.4) \qquad\qquad \otimes \colon \Sigma_j \times \Sigma_k \to \Sigma_{jk},$$

which are made precise in Appendix A by use of lexicographic ordering. However, these do not satisfy the condition in Definition A.1(iii); see Counterexample A.5. For a conceptual understanding of why $\mathscr{P}$ cannot have a self-pairing, consider the free $\mathscr{P}$–algebra $\mathbb{P}(S^0)$. This is a model for the groupoid of finite sets. As explained in [26, Appendix A], a self-pairing on $\mathscr{P}$ would give strict distributivity on both sides in $\mathbb{P}(S^0)$. But the lexicographic ordering on $j \times (k \sqcup m)$ does not agree with the lexicographic ordering on $(j \times k) \sqcup (j \times m)$.

As we explain in [13], the homomorphisms $\otimes$ exhibit a product that exists in any operad. The categorical operads $\mathscr{P}$ and $\mathscr{P}_G$ are *pseudocommutative*, meaning that certain diagrams of functors defined using these products commute up to natural isomorphism. Putting together Theorem 6.2, the comparison of operadic and Segalic machines in [31], and 2–category machinery developed in [12], we will obtain multicategorical generalizations of the missing diagrams in [13], where we complete the proofs from equivariant infinite loop space theory promised in [10].

6.2 The tom Dieck splitting theorem

The G–fixed-point spectra of suspension G–spectra have a well-known splitting. It is due to tom Dieck [5] on the level of homotopy groups and was lifted to the spectrum level in [18, Section V.11]. The tom Dieck splitting actually works for all compact Lie groups G, but we have nothing helpful to add in that generality. Our group G is always finite. In that case, we have already given the ingredients for a new categorical proof, as we now explain.

Theorem 6.5 *For a based G–space Y,*

$$(\Sigma_G^\infty Y)^G \simeq \bigvee_{(H)} \Sigma^\infty(EWH_+ \wedge_{WH} Y^H).$$

The wedge runs over the conjugacy classes of subgroups H of G, and $WH = NH/H$.

Theorem 6.5 and the evident natural identifications

$$(6.6) \qquad\qquad EWH_+ \wedge_{WH} X_+^H \cong (EWH \times_{WH} X^H)_+$$

imply the following version for unbased G–spaces X.

Theorem 6.7 *For an unbased G–space X,*

$$(\Sigma_G^\infty X_+)^G \simeq \bigvee_{(H)} \Sigma^\infty (EWH \times_{WH} X^H)_+.$$

Conversely, we can easily deduce Theorem 6.5 from Theorem 6.7. Viewing S^0 as $\{1\}_+$ with trivial G action, our standing assumption that basepoints are nondegenerate gives a based G–cofibration $S^0 \to Y_+$ that sends 1 to the basepoint of Y, and $Y = Y_+/S^0$. The functors appearing in Theorem 6.7 preserve cofiber sequences, and the identifications (6.6) imply identifications

(6.8) $(EWH \times_{WH} Y^H)_+/(EWH \times_{WH} \{1\})_+ \cong EWH_+ \wedge_{WH} Y^H.$

Therefore, Theorem 6.7 implies Theorem 6.5.

We explain these splittings in terms of the categorical BPQ theorem. We begin in the based setting. The nonequivariant case $G = e$ of the BPQ theorem relates to the equivariant case through Theorem 2.20. Explicitly, Theorems 2.20 and 6.1 give a pair of weak equivalences

(6.9) $(\Sigma_G^\infty Y)^G \to (\mathbb{E}_G C_G Y)^G \leftarrow \mathbb{E}((C_G Y)^G).$

Since the functor Σ^∞ commutes with wedges, the nonequivariant BPQ theorem gives a weak equivalence

(6.10) $\bigvee_{(H)} \Sigma^\infty (EWH_+ \wedge_{WH} Y^H) \to \mathbb{E}C\big(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H)\big).$

If we could prove that there is a natural weak equivalence of $\mathscr{C}$–spaces

$$(C_G Y)^G \simeq C\big(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H)\big),$$

that would imply a natural weak equivalence

(6.11) $\mathbb{E}((C_G Y)^G) \simeq \mathbb{E}C\big(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H)\big)$

and complete the proof of Theorem 6.5. However, the combinatorial study of the behavior of C on wedges is complicated by the obvious fact that wedges of based spaces do not commute with products.

We use the following consequence of Theorem 5.5 and the relationship between wedges and products of spectra to get around this. Recall that P_G is the monad on based G–spaces obtained from the operad $|\mathscr{P}_G|$ of G–spaces.

Theorem 6.12 *For unbased G–spaces X, there is a natural equivalence of $|\mathscr{P}|$–spaces*

$$(P_G X_+)^G \simeq \prod_{(H)} P(EWH \times_{WH} X^H)_+,$$

where (H) runs over the conjugacy classes of subgroups of G and $WH = NH/H$.

Proof Remembering that $|\mathcal{E}G| = EG$, we see that the classifying space of the category $\mathcal{E}WH \times_{WH} X^H$ can be identified with $EWH \times_{WH} X^H$. The commutation relations between $|-|$ and the constituent functors used to construct the monads $\boldsymbol{P}_G$ on G–spaces and $\mathbb{P}_G$ on G–categories make the identification clear. $\square$

Remark 6.13 Of course, we can and must replace $\mathscr{P}_G$ and $\mathscr{P}$ by their products with the equivariant and nonequivariant Steiner operad to fit into the infinite loop space machine. There is no harm in doing so since if we denote the product operads by $\mathscr{O}_G$ and $\mathscr{O}$, as before, the projections $\mathscr{O}_G \to \mathscr{P}_G$ and $\mathscr{O} \to \mathscr{P}$ induce weak equivalences of monads that fit into the following commutative diagram:

$$
\begin{array}{ccc}
(\boldsymbol{O}_G X_+)^G & \xrightarrow{\simeq} & \prod_{(H)} \boldsymbol{O}(EWH \times_{WH} X^H)_+ \\
{\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} \\
(\boldsymbol{P}_G X_+)^G & \xrightarrow{\simeq} & \prod_{(H)} \boldsymbol{P}(EWH \times_{WH} X^H)_+
\end{array}
$$

The functor Σ_G^∞ commutes with wedges, and the natural map of G–spectra

$$E \vee F \to E \times F$$

is a weak equivalence. Theorems 2.21 and 6.1 have the following implication. We state it equivariantly, but we shall apply its nonequivariant special case.

Proposition 6.14 *For based G–spaces X and Y, the natural map*

$$\mathbb{E}_G \boldsymbol{O}_G(X \vee Y) \to \mathbb{E}_G(\boldsymbol{O}_G X \times \boldsymbol{O}_G Y)$$

is a weak equivalence of G–spectra.

Proof The following diagram commutes by the universal property of products:

$$
\begin{array}{ccc}
\Sigma_G^\infty(X \vee Y) & \longrightarrow & \mathbb{E}_G \boldsymbol{O}_G(X \vee Y) \\
{\scriptstyle\cong}\downarrow & & \downarrow \\
\Sigma_G^\infty X \vee \Sigma_G^\infty Y & & \mathbb{E}_G(\boldsymbol{O}_G X \times \boldsymbol{O}_G Y) \\
\downarrow & & \downarrow \\
\Sigma_G^\infty X \times \Sigma_G^\infty Y & \longrightarrow & \mathbb{E}_G \boldsymbol{O}_G X \times \mathbb{E}_G \boldsymbol{O}_G Y
\end{array}
$$

All arrows except the upper right vertical one are weak equivalences, hence that arrow is also a weak equivalence. $\square$

For any nonequivariant E_∞–operad $\mathscr{C}$, we therefore have a weak equivalence

$$(6.15) \qquad \mathbb{E} C\big(\bigvee_{(H)}(EWH_+ \wedge_{WH} Y^H)\big) \to \mathbb{E} \prod_{(H)} C(EWH_+ \wedge_{WH} Y^H).$$

Together with (6.15), Theorem 6.12 and Remark 6.13 give a weak equivalence (6.11) in the case $Y = X_+$. Together with (6.9) and (6.10), this completes the proof of Theorem 6.7, and Theorem 6.5 follows.

7   The E_∞–operads $\mathscr{V}_G$, $\mathscr{V}_G^\times$, and $\mathscr{W}_G$

The operad $\mathscr{P}_G$ has a privileged conceptual role, but there are other categorical E_∞–G–operads with different good properties. We define three interrelated examples. The objects of the chaotic category $\mathscr{P}_G(j)$ are functions $G \to \Sigma_j$. We give analogous chaotic G–categories in which the objects are suitable functions between well chosen infinite G–sets, with G again acting by conjugation. Their main advantage over $\mathscr{P}_G$ is that it is easier to recognize G–categories on which they act.

7.1   The definitions of $\mathscr{V}_G$ and $\mathscr{V}_G^\times$

We start with what we would like to take as a particularly natural choice for the jth category of an E_∞–G–operad. It is described in more detail in [11, Section 6.1].

Definition 7.1   Let U be a countable ambient G–set that contains countably many copies of each orbit G/H. Let U^j be the product of j copies of U with diagonal action by G, and let ^jU be the disjoint union of j copies of the G–set U. Here U^0 is a one-point set, sometimes denoted by 1, and ^0U is the empty set, sometimes denoted by $\varnothing$ and sometimes denoted by 0.

Let $\mathbf{j} = \{1, \ldots, j\}$ with its natural left action by Σ_j, written $\sigma\colon \mathbf{j} \to \mathbf{j}$.

Definition 7.2   For $j \geq 0$, let $\widetilde{\mathscr{E}}_G^U(j)$ be the chaotic $(G \times \Sigma_j)$–category whose objects are the pairs (A, ι), where A is a j–element subset of U and $\iota\colon \mathbf{j} \to A$ is a bijection. The group G acts on objects by $g(A, \iota) = (gA, g\iota)$, where $(g\iota)(i) = g \cdot \iota(i)$. The group Σ_j acts on objects by $(A, \iota)\sigma = (A, \iota \circ \sigma)$ for $\sigma \in \Sigma_j$. Since $\widetilde{\mathscr{E}}_G^U(j)$ is chaotic, this determines the actions on morphisms.

Proposition 7.3   [11, Proposition 6.3]   *For each j, the classifying space $|\widetilde{\mathscr{E}}_G^U(j)|$ is a universal principal (G, Σ_j)–bundle.*

Therefore, $\widetilde{\mathcal{E}}_G^U(j)$ satisfies the properties required of the jth category of an E_∞–G–operad. However, these categories as j varies do not form an operad. The problem is a familiar one. These categories can be thought of as analogous to configuration spaces. Just as we fattened up the configuration space models of Section 1.6 to the little discs operads of Section 1.1, we must fatten up these categories to provide enough room for an operad structure.

Definition 7.4　We define a reduced operad $\mathcal{V}_G$ of G–categories. Let $\mathcal{V}_G(j)$ be the chaotic G–category whose set of objects is the set of injective functions $^jU \to U$. Let G act by conjugation and let Σ_j have the right action induced by its left action on ^jU. Let $\mathrm{id} \in \mathcal{V}_G(1)$ be the identity function $U \to U$. Define

$$\gamma \colon \mathcal{V}_G(k) \times \mathcal{V}_G(j_1) \times \cdots \times \mathcal{V}_G(j_k) \to \mathcal{V}_G(j),$$

where $j = j_1 + \cdots + j_k$, to be the composite

$$\mathcal{V}_G(k) \times \mathcal{V}_G(j_1) \times \cdots \times \mathcal{V}_G(j_k) \to \mathcal{V}_G(k) \times \mathcal{V}_G(^jU, {}^kU) \to \mathcal{V}_G(j)$$

obtained by first taking coproducts of maps and then composing. Here $\mathcal{V}(^jU, {}^kU)$ is the set of injections $^jU \to {}^kU$. The operad axioms [21, Definition 1.1] are easily verified.

Remembering that taking sets to the free $\mathbb{R}$–modules they generate gives a coproduct-preserving functor from sets to $\mathbb{R}$–modules, we see that $\mathcal{V}_G$ is a categorical analogue of the linear isometries operad $\mathcal{L}_U$.

There is a parallel definition that uses products instead of coproducts.

Definition 7.5　We define an unreduced operad $\overline{\mathcal{V}}_G^\times$ of G–categories. Let $\overline{\mathcal{V}}_G^\times(j)$ be the chaotic G–category whose set of objects is the set of injective functions $U^j \to U$. Let G act by conjugation and let Σ_j have the right action induced by its left action on U^j. Let $\mathrm{id} \in \overline{\mathcal{V}}_G^\times(1)$ be the identity function. Define

$$\gamma \colon \overline{\mathcal{V}}_G^\times(k) \times \overline{\mathcal{V}}_G^\times(j_1) \times \cdots \times \overline{\mathcal{V}}_G^\times(j_k) \to \overline{\mathcal{V}}_G^\times(j),$$

where $j = j_1 + \cdots + j_k$, to be the composite

$$\overline{\mathcal{V}}_G^\times(k) \times \overline{\mathcal{V}}_G^\times(j_1) \times \cdots \times \overline{\mathcal{V}}_G^\times(j_k) \to \overline{\mathcal{V}}_G^\times(k) \times \overline{\mathcal{V}}_G^\times(U^j, U^k) \to \overline{\mathcal{V}}_G^\times(j)$$

obtained by first taking products of maps and then composing. Here $\overline{\mathcal{V}}_G^\times(U^j, U^k)$ is the set of injections $U^j \to U^k$. Again, the operad axioms are easily verified.

Observe that the objects of $\overline{\mathscr{V}}_G^\times(0)$ are the injections from the point U^0 into U and can be identified with the set U, whereas $\mathscr{V}_G(0)$ is the trivial category given by the injection of the empty set ^0U into U. As in Remark 2.4, the objects of the zeroth category give unit objects for operad actions, and it is convenient to restrict attention to a reduced variant of $\overline{\mathscr{V}}_G^\times$.

Definition 7.6 Choose a G–fixed point $1 \in U$ (or, equivalently, adjoin a G–fixed basepoint 1 to U) and also write 1 for the single point in U^0. Give U^j, $j \geq 0$, the basepoint whose coordinates are all 1. The reduced variant of $\overline{\mathscr{V}}_G^\times$ is the operad $\mathscr{V}_G^\times$ of G–categories that is obtained by restricting the objects of the $\overline{\mathscr{V}}_G^\times(j)$ to consist only of the basepoint-preserving injections $U^j \to U$ for all $j \geq 0$.

Remark 7.7 If $\overline{\mathscr{V}}_G^\times$ acts on a category $\mathscr{A}$, then $\mathscr{V}_G^\times$ acts on $\mathscr{A}$ by restriction of the action. However, $\mathscr{V}_G^\times$ can act even though $\overline{\mathscr{V}}_G^\times$ does not. This happens when the structure of $\mathscr{A}$ encodes a particular unit object and the operad action conditions fail for other choices of objects in $\mathscr{A}$.

Proposition 7.8 *The classifying spaces* $|\mathscr{V}_G(j)|$, $|\overline{\mathscr{V}}_G^\times(j)|$, *and* $|\mathscr{V}_G^\times(j)|$ *are universal principal* (G, Σ_j)*–bundles; hence* $\mathscr{V}_G$, $\overline{\mathscr{V}}_G^\times$, *and* $\mathscr{V}_G^\times$ *are* E_∞*–operads.*

Proof Since the objects of our categories are given by injective functions, Σ_j acts freely on the objects of $\mathscr{V}_G(j)$ and $\mathscr{V}_G^\times(j)$. Since our categories are chaotic, it suffices to show that if $\Lambda \cap \Sigma_j = \{e\}$, where $\Lambda \subset G \times \Sigma_j$, then the object sets $\mathscr{V}_G(j)^\Lambda$ and $\mathscr{V}_G^\times(j)^\Lambda$ are nonempty. This means that there are Λ–equivariant injections $^jU \to U$ and $U^j \to U$, and in fact, there are Λ–equivariant bijections. We have $\Lambda = \{(h, \alpha(h)) \mid h \in H\}$ for a subgroup H of G and a homomorphism $\alpha \colon H \to \Sigma_j$, and we may regard U as an H–set via the canonical isomorphism $H \cong \Lambda$. Since countably many copies of every orbit of H embed in U, ^jU, and U^j for $j \geq 1$, these sets are all isomorphic as H–sets and therefore as Λ–sets. $\square$

7.2 The definition of $\mathscr{W}_G$ and its action on $\mathscr{V}_G$

This section is parenthetical, aimed towards work in progress on a new version of multiplicative infinite loop space theory. The notion of an action of a *multiplicative* operad $\mathscr{G}$ on an *additive* operad $\mathscr{C}$ was defined in [24, VI.1.6], and $(\mathscr{C}, \mathscr{G})$ was then said to be an *operad pair*. This notion was redefined and discussed in [30; 29]. Expressed in terms of diagrams rather than elements, it makes sense for operads in any cartesian monoidal category, such as the categories of G–categories and of G–spaces. As is emphasized in the cited papers, although this notion is the essential starting point for

the theory of E_∞–ring spaces, the only interesting nonequivariant example we know is $(\mathcal{K}, \mathcal{L})$, where $\mathcal{K}$ is the Steiner operad. As pointed out in Section 2.5, this example works equally well equivariantly.

The pair of operads $(\mathcal{V}_G, \mathcal{V}_G^\times)$ very nearly gives another example, but we must shrink $\mathcal{V}_G^\times$ and drop its unit object to obtain this.

Definition 7.9 Define $\mathscr{W}_G \subset \mathcal{V}_G^\times$ to be the suboperad such that $\mathscr{W}_G(j)$ is the full subcategory of $\mathcal{V}_G^\times(j)$ whose objects are the based bijections $U^j \to U$. In particular, $\mathscr{W}_G(0)$ is the empty category, so that the operad $\mathscr{W}_G$ does not encode unit object information. By the proof of Proposition 7.8, $\mathscr{W}_G(j)$ for $j \geq 1$ is again a universal principal (G, Σ_j)–bundle. We view $\mathscr{W}_G$ as a restricted E_∞–operad, namely one without unit objects.

Proposition 7.10 *The restricted operad $\mathscr{W}_G$ acts on the operad $\mathcal{V}_G$.*

Proof We must specify action maps

$$\lambda\colon \mathscr{W}_G(k) \times \mathcal{V}_G(j_1) \times \cdots \times \mathcal{V}_G(j_k) \to \mathcal{V}_G(j),$$

where $j = j_1 \cdots j_k$ and $k \geq 1$. To define them, consider the set of sequences $I = \{i_1, \ldots, i_k\}$, ordered lexicographically, where $1 \leq i_r \leq j_r$ and $1 \leq r \leq k$. For an injection $\phi_r\colon {}^{j_r}U \to U$, let $\phi_{i_r}\colon U \to U$ denote the restriction of ϕ_r to the i_r^{th} copy of U in ${}^{j_r}U$. Then let

$$\phi_I = \phi_{i_1} \times \cdots \times \phi_{i_k}\colon U^k \to U^k.$$

For a bijection $\psi\colon U^k \to U$, define

$$\lambda(\psi; \phi_1, \ldots, \phi_k)\colon {}^j U \to U$$

to be the injection which restricts on the Ith copy of U to the composite

$$U \xrightarrow{\psi^{-1}} U^k \xrightarrow{\phi_I} U^k \xrightarrow{\psi} U.$$

It is tedious but straightforward to verify that all conditions specified in [24, Definition VI.1.6], [29, Definition 4.2] that make sense are satisfied.[6] $\qquad\square$

Remark 7.11 When all $j_i = 1$, so that there is only one sequence I, we can define λ more generally, with $\mathcal{V}_G^\times(k)$ replacing $\mathscr{W}_G(k)$, by letting

$$\lambda(\psi; \phi_1, \ldots, \phi_k)\colon U \to U$$

---

[6]In fact, with the details of [29, Definition 4.2], the only condition that does not make sense would require $\lambda(1) = \mathrm{id} \in \mathcal{V}_G(1)$, where $\{1\} = \mathscr{W}(0)$, and that condition lacks force since it does not interact with the remaining conditions.

be the identity on the complement of the image of the injection $\psi\colon U^k \to U$ and

$$\psi(U) \xrightarrow{\psi^{-1}} U^k \xrightarrow{\phi_I} U^k \xrightarrow{\psi} \psi(U)$$

on the image of ψ. Clearly we can replace $\mathscr{V}_G(1)$ by $\mathscr{V}_G^\times(1)$ here.

This allows us to give the following speculative analogue of Definition 4.10. An E_∞–ring space is defined to be a $(\mathscr{C}, \mathscr{G})$–space, where $(\mathscr{C}, \mathscr{G})$ is an operad pair such that $\mathscr{C}$ and $\mathscr{G}$ are E_∞–operads of spaces. Briefly, a $(\mathscr{C}, \mathscr{G})$–space X is a $\mathscr{C}$–space and a $\mathscr{G}$–space with respective basepoints 0 and 1 such that 0 is a zero element for the $\mathscr{G}$–action and the action $\boldsymbol{C}X \to X$ is a map of $\mathscr{G}$–spaces with zero, where $\boldsymbol{C}$ denotes the monad associated to the operad $\mathscr{C}$. Here the action of $\mathscr{G}$ on $\mathscr{C}$ induces an action of $\mathscr{G}$ on the free $\mathscr{C}$–spaces $\boldsymbol{C}X$, so that $\boldsymbol{C}$ restricts to a monad in the category of $\mathscr{G}$–spaces. These notions are redefined in the more recent papers [30; 29]. The definitions are formal and apply equally well to spaces, G–spaces, categories, and G–categories.

Definition 7.12 An E_∞–ring G–category $\mathscr{A}$ is a G–category together with an action by the E_∞–operad pair $(\mathscr{V}_G, \mathscr{W}_G)$ such that the multiplicative action extends from the restricted E_∞–operad $\mathscr{W}_G$ to an action of the E_∞–operad $\mathscr{V}_G^\times$.

The notion of a bipermutative category, or symmetric strict bimonoidal category, was specified in [24, Definition VI.3.3]. With the standard skeletal model, the direct sum and tensor product on the category of finite-dimensional free modules over a commutative ring R gives a typical example. Without any categorical justification, we allow ourselves to think of E_∞–ring G–categories as an E_∞ version of genuine operadic bipermutative G–categories. A less concrete but more general version of this notion is defined and developed in [13].

Our notion of an E_∞–G–category $\mathscr{A}$ implies that $B\mathscr{A}$ is an E_∞–G–space. We would like to say that our notion of an E_∞–ring G–category $\mathscr{A}$ implies that $B\mathscr{A}$ is an E_∞–ring G–space, but that is not quite true. However, we believe there is a way to prove the following conjecture that avoids the categorical work of [8; 12; 13; 26; 29]. However, that proof is work in progress.

Conjecture 7.13 *There is an infinite loop space machine that carries E_∞–ring G–categories to E_∞–ring G–spectra.*

8   Examples of E_∞ and E_∞–ring G–categories

We have several interesting examples. We emphasize that these particular constructions are new even when $G = e$. In that case, we may take U to be the set of positive integers, with 1 as basepoint.

We have the notion of a genuine permutative G–category, which comes with a preferred product, and the notion of a $\mathscr{V}_G$–category, which does not. It seems plausible that the latter notion is more general, but to verify that we would have to show how to regard a permutative category as a $\mathscr{V}_G$–algebra. One natural way to do so would be to construct a map of operads $\mathscr{V}_G \to \mathscr{P}_G$, but we do not know how to do that. Of course, the equivalence of $\mathscr{V}_G$–categories and $\mathscr{P}_G$–categories shows that genuine permutative categories give a plethora of examples of $\mathscr{V}_G$–algebras up to homotopy. However, the most important examples can easily be displayed directly, without recourse to the theory of permutative categories.

8.1 The G–category $\mathscr{E}_G^U = \mathscr{E}_G^{\mathscr{V}}$ of finite sets

Recall Remark 5.13. Intuitively, we would like to have a genuine permutative G–category whose product is given by disjoint unions of finite sets, with G relating finite sets (not G–sets) by translations. Even nonequivariantly, this is imprecise due to both size issues and the fact that categorical coproducts are not strictly associative. We make it precise by taking coproducts of finite subsets of our ambient G–set U, but we must do so without assuming that our given finite subsets are disjoint. We achieve this by using injections ${}^jU \to U$ to separate them. We do not have canonical choices for the injections, hence we have assembled them into our categorical E_∞–operad $\mathscr{V}_G$. Recall Definition 7.2 and Proposition 7.3.

Definition 8.1 The G–category $\widetilde{\mathscr{E}}_G^U$ of finite ordered sets is the coproduct over $n \geq 0$ of the G–categories $\widetilde{\mathscr{E}}_G^U(n)$. The G–category $\mathscr{E}_G^U = \mathscr{E}_G^{\mathscr{V}}$ of finite sets is the coproduct over $n \geq 0$ of the orbit categories $\widetilde{\mathscr{E}}_G^U(n)/\Sigma_n$. By Proposition 7.3, $B\mathscr{E}_G^U$ is the coproduct over $n \geq 0$ of classifying spaces $B(G, \Sigma_n)$. Explicitly, by [11, Lemma 6.5], the objects of $\mathscr{E}_G^U$ are the finite subsets (not G–subsets) A of U. Its morphisms are the bijections $v \colon A \to B$; if A has n points, the morphisms $A \to A$ give a copy of the set Σ_n. The group G acts by translation on objects, so that $gA = \{ga \mid a \in A\}$, and by conjugation on morphisms, so that $gv \colon gA \to gB$ is given by $(gv)(g \cdot a) = g \cdot v(a)$.

Proposition 8.2 The G–categories $\widetilde{\mathscr{E}}_G^U$ and $\mathscr{E}_G^U$ are $\mathscr{V}_G$–categories, and passage to orbits over symmetric groups defines a map $\widetilde{\mathscr{E}}_G^U \to \mathscr{E}_G^U$ of $\mathscr{V}_G$–categories.

Proof Define a G–functor
$$\theta_j \colon \mathscr{V}_G(j) \times (\mathscr{E}_G^U)^j \to \mathscr{E}_G^U$$
as follows. On objects, for $\phi \in \mathscr{V}_G(j)$ and $A_i \in \mathrm{Ob}\,\mathscr{E}_G^U$, $1 \leq i \leq j$, define
$$\theta_j(\phi; A_1, \ldots, A_j) = \phi(A_1 \sqcup \cdots \sqcup A_j),$$

where A_i is viewed as a subset of the ith copy of U in ^jU. For a morphism

$$(\zeta; \nu_1, \ldots, \nu_j)\colon (\phi; A_1, \ldots, A_j) \to (\psi; B_1, \ldots, B_j),$$

where $\zeta\colon \phi \to \psi$ is the unique morphism, define $\theta_j(\zeta; \nu_1, \ldots, \nu_j)$ to be the unique bijection that makes the following diagram commute:

$$
\begin{array}{ccc}
A_1 \sqcup \cdots \sqcup A_j & \xrightarrow{\phi} & \phi(A_1 \sqcup \cdots \sqcup A_j) \\
{\scriptstyle \nu_1 \sqcup \cdots \sqcup \nu_j} \downarrow & & \downarrow {\scriptstyle \theta_j(\zeta; \nu_1, \ldots, \nu_j)} \\
B_1 \sqcup \cdots \sqcup B_j & \xrightarrow{\psi} & \psi(B_1 \sqcup \cdots \sqcup B_j)
\end{array}
$$

Then the θ_j specify an action of $\mathcal{V}_G$ on $\mathcal{E}_G^U$.

Since the $\widetilde{\mathcal{E}}_G^U(n)$ are chaotic, to define an action of $\mathcal{V}_G$ on $\widetilde{\mathcal{E}}_G^U$ we need only specify the required G–functors

$$\widetilde{\theta}_j\colon \mathcal{V}_G(j) \times (\widetilde{\mathcal{E}}_G^U)^j \to \widetilde{\mathcal{E}}_G^U$$

on objects. A typical object has the form $(\phi; (A_1, \iota_1), \ldots, (A_j, \iota_j))$ for $\iota_i\colon \boldsymbol{n}_i \to A_i$. We have the canonical isomorphism $\boldsymbol{n}_1 \sqcup \cdots \sqcup \boldsymbol{n}_j \cong \boldsymbol{n}$, where $n = n_1 + \cdots + n_j$, and $\widetilde{\theta}_j$ sends our typical object to

$$(\phi(A_1 \sqcup \cdots \sqcup A_j), \phi \circ (\iota_1 \sqcup \cdots \sqcup \iota_j)).$$

Again, the $\widetilde{\theta}_j$ specify an action. The compatibility with passage to orbits is verified by use of canonical orbit representatives for objects A that are obtained by choosing fixed reference maps $\eta_A\colon \boldsymbol{n} \to A$ for each n–point set $A \subset U$; compare [11, Proposition 6.3 and Lemma 6.5]. $\qquad\square$

Remark 8.3 If we restrict to the full G–subcategory of $\mathcal{E}_G^U$ of G–fixed sets A of cardinality n, we obtain an equivalent analogue of the category $\mathcal{F}_G(n)$ of Definition 5.8: these are two small models of the G–category of all G–sets with n elements and the bijections between them, and they have isomorphic skeleta. Thus the restriction of $\mathcal{E}_G^U$ to its full G–subcategory of G–fixed sets A is an equivalent analogue of $\mathcal{F}_G$. Remember from Remark 5.11 that no E_∞–operad can be expected to act on $\mathcal{F}_G$. The $\mathcal{V}_G$–category $\mathcal{E}_G^U$ gives a convenient substitute.

8.2 The G–category $\mathcal{GL}_G(R)$ for a G–ring R

Let R be a G–ring, that is a ring with an action of G through automorphisms of R. We have analogues of Definitions 7.2 and 8.1 that can be used in equivariant algebraic K–theory. For a set A, let $R[A]$ denote the free R–module on the basis A. Let G act entrywise on the matrix group $\mathrm{GL}(n, R)$ and diagonally on R^n. Our conventions on

semidirect products and their universal principal $(G, \mathrm{GL}(n, R)_G)$–bundles are in [11], and [11, Section 6.3] gives more details on the following definitions.

Definition 8.4   We define the chaotic general linear category $\widetilde{\mathscr{GL}}_G(n, R)$. Its objects are the monomorphisms of (left) R–modules $\iota\colon R^n \to R[U]$. The group G acts on objects by $g\iota = g \circ \iota \circ g^{-1}$. The group $\mathrm{GL}(n, R)$ acts on objects by $\iota\tau = \iota \circ \tau\colon R^n \to R[U]$. Since $\widetilde{\mathscr{GL}}_G(n, R)$ is chaotic, this determines the actions on morphisms.

Proposition 8.5   [11, Proposition 6.18] *The actions of G and $\mathrm{GL}(n, R)$ on $\widetilde{\mathscr{GL}}_G(n, R)$ determine an action of $\mathrm{GL}(n, R) \rtimes G$, and the classifying space $|\widetilde{\mathscr{GL}}_G(n, R)|$ is a universal principal $(G, \mathrm{GL}(n, R)_G)$–bundle.*

Definition 8.6   The general linear G–category $\mathscr{GL}_G(R)$ of finite-dimensional free R–modules is the coproduct over $n \geq 0$ of the orbit categories $\widetilde{\mathscr{GL}}_G(n, R)/\mathrm{GL}(n, R)$. By Proposition 8.5, $B\mathscr{GL}_G(R)$ is the coproduct over $n \geq 0$ of classifying spaces $B(G, \mathrm{GL}(n, R)_G)$. Explicitly, by [11, Lemma 6.20], the objects of $\mathscr{GL}_G(R)$ are the finite-dimensional free R–submodules M of $R[U]$. The morphisms $\nu\colon M \to N$ are the isomorphisms of R–modules. The group G acts by translation on objects, so that $gM = \{gm \mid m \in M\}$, and by conjugation on morphisms, so that $(g\nu)(gm) = \nu(m)$ for $m \in M$ and $g \in G$.

Proposition 8.7   *The G–categories $\widetilde{\mathscr{GL}}_G(R)$ and $\mathscr{GL}_G(R)$ are $\mathscr{V}_G$–categories and passage to orbits over general linear groups defines a map $\widetilde{\mathscr{GL}}_G(R) \to \mathscr{GL}_G(R)$ of $\mathscr{V}_G$–categories.*

Proof   Define a functor

$$\theta_j\colon \mathscr{V}_G(j) \times \mathscr{GL}_G(R)^j \to \mathscr{GL}_G(R)$$

as follows. On objects, for $\phi \in \mathscr{V}_G(j)$ and $M_i \in \mathscr{O}b\,\mathscr{GL}_G(R)$, $1 \leq i \leq j$, define

$$\theta_j(\phi; M_1, \ldots, M_j) = R[\phi](M_1 \oplus \cdots \oplus M_j),$$

where $R[\phi]\colon R[^jU] \to R[U]$ is induced by $\phi\colon {}^jU \to U$ and M_i is viewed as a submodule of the ith copy of $R[U]$ in $R[^jU] = \bigoplus_j R[U]$. For a morphism

$$(\zeta; \nu_1, \ldots, \nu_j)\colon (\phi; M_1, \ldots, M_j) \to (\psi; N_1, \ldots, N_j),$$

define $\theta_j(\iota; \nu_1, \ldots, \nu_j)$ to be the unique isomorphism of R–modules that makes the following diagram commute:

$$
\begin{array}{ccc}
M_1 \oplus \cdots \oplus M_j & \xrightarrow{\ R[\phi]\ } & R[\phi](M_1 \oplus \cdots \oplus M_j) \\
{\scriptstyle \nu_1 \oplus \cdots \oplus \nu_j}\Big\downarrow & & \Big\downarrow{\scriptstyle \theta_j(\zeta;\nu_1,\ldots,\nu_j)} \\
N_1 \oplus \cdots \oplus N_j & \xrightarrow[\ R[\psi]\ ]{} & R[\psi](N_1 \oplus \cdots \oplus N_j)
\end{array}
$$

Then the θ_j specify an action of $\mathscr{V}_G$ on $\mathscr{GL}_G(R)$. Since the $\widetilde{\mathscr{GL}}_G(R,n)$ are chaotic, to define an action of $\mathscr{V}_G$ on $\widetilde{\mathscr{GL}}_G(R)$, we need only specify the required G–functors

$$\widetilde{\theta}_j \colon \mathscr{V}_G(j) \times \widetilde{\mathscr{GL}}_G(R)^j \to \widetilde{\mathscr{GL}}_G(R)$$

on objects. A typical object has the form $(\phi; \iota_1, \ldots, \iota_j)$ for $\iota_i \colon R^{n_i} \to R[U]$, and with $n = n_1 + \cdots + n_j$, we have that $\widetilde{\theta}_j$ sends it to

$$R[\phi] \circ (\iota_1 \oplus \cdots \oplus \iota_j) \colon R^n \to R[U].$$

Again, the $\widetilde{\theta}_j$ specify an action. The compatibility with passage to orbits is verified by use of canonical orbit representatives for objects that are obtained by choosing reference maps $\eta_M \colon R^n \to M$ for each M–dimensional free R–module $M \subset R[U]$; compare [11, Proposition 6.18, Lemma 6.20]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

On passage to classifying spaces and then to G–spectra via our infinite loop space machine $\mathbb{E}_G$, we obtain a model $\mathbb{E}_G B\mathscr{GL}_G(R)$ for the K–theory spectrum $\mathbb{K}_G(R)$ of R. The following result compares the two evident models in sight.

Definition 8.8 Define the naive permutative G–category $\mathrm{GL}_G(R)$ to be the G–groupoid whose objects are the $n \geq 0$ and whose set of morphisms $m \to n$ is empty if $m \neq n$ and is the G–group $\mathrm{GL}(n, R)$ if $m = n$, where G acts entrywise. The product is given by block sum of matrices. Applying the chaotic groupoid functor to the groups $\mathrm{GL}(n, R)$ we obtain another naive permutative G–category $\mathcal{E}\mathrm{GL}_G(R)$ and a map $\mathcal{E}\mathrm{GL}_G(R) \to \mathrm{GL}_G(R)$ of naive permutative G–categories. Applying the functor $\mathscr{C}at(\mathcal{E}G, -)$ from Proposition 4.6, we obtain a map of genuine permutative G–categories $\mathscr{C}at(\mathcal{E}G, (\mathcal{E}\mathrm{GL}_G(R))) \to \mathscr{C}at(\mathcal{E}G, (\mathrm{GL}_G(R)))$.

It is convenient to write $\mathscr{GL}_G^{\mathscr{P}}(R)$ for the $\mathscr{P}_G$–category $\mathscr{C}at(\mathcal{E}G, (\mathrm{GL}_G(R)))$ and $\mathscr{GL}_G^{\mathscr{V}}(R)$ for the $\mathscr{V}_G$–category $\mathscr{GL}_G(R)$, and similarly for their total space variants $\mathscr{C}at(\mathcal{E}G, (\mathcal{E}\mathrm{GL}_G(R)))$ and $\widetilde{\mathscr{GL}}_G(R)$. We have the following comparison theorem.

Theorem 8.9 *The G–spectra $\mathbb{K}_G \mathscr{GL}_G^{\mathscr{P}}(R)$ and $\mathbb{K}_G \mathscr{GL}_G^{\mathscr{V}}(R)$ are weakly equivalent, functorially in G–rings R.*

Proof We again use the product of operads trick from [21]. Projections and quotient maps give the following commutative diagram of $(\mathscr{P}_G \times \mathscr{V}_G)$–categories:

$$
\begin{array}{ccccc}
\widetilde{\mathscr{GL}}_G^{\mathscr{P}}(R) & \longleftarrow & \widetilde{\mathscr{GL}}_G^{\mathscr{P}}(R) \times \widetilde{\mathscr{GL}}_G^{\mathscr{V}}(R) & \longrightarrow & \widetilde{\mathscr{GL}}_G^{\mathscr{V}}(R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{GL}_G^{\mathscr{P}}(R) & \longleftarrow & \mathscr{GL}_G^{\mathscr{P}\times\mathscr{V}}(R) & \longrightarrow & \mathscr{GL}_G^{\mathscr{V}}(R)
\end{array}
$$

The middle term at the top denotes the diagonal product, namely

$$\bigsqcup_n \widetilde{\mathscr{GL}}_G^{\mathscr{P}}(n, R) \times \widetilde{\mathscr{GL}}_G^{\mathscr{V}}(n, R).$$

The middle term on the bottom is the coproduct over n of the orbits of these products under the diagonal action of $\mathrm{GL}(n, R)$. The product of total spaces of universal principal $(G, \mathrm{GL}(R, n)_G)$–bundles is the total space of another universal principal $(G, \mathrm{GL}(R, n)_G)$–bundle. Therefore, after application of the classifying-space functor, the horizontal projections display two equivalences between universal principal $(G, \mathrm{GL}(R, n)_G)$–bundles. The conclusion follows by hitting the resulting diagram with the functor $\mathbb{K}_G$ defined with respect to $(\mathscr{P}_G \times \mathscr{V}_G)$–categories and using evident equivalences to the functors $\mathbb{K}_G$ defined with respect to $\mathscr{P}_G$–categories and $\mathscr{V}_G$–categories when the input is given by $\mathscr{P}_G$ or $\mathscr{V}_G$–categories.   □

8.3 Multiplicative actions on $\mathscr{E}_G^U$ and $\mathscr{GL}_G(R)$

We agree to think of $\mathscr{V}_G^\times$–categories as multiplicative, whereas we think of $\mathscr{V}_G$–categories as additive.

Proposition 8.10 *The G–category $\mathscr{E}_G^U$ is a $\mathscr{V}_G^\times$–category.*

Proof Define a G–functor

$$\xi_j \colon \mathscr{V}_G^\times(j) \times (\mathscr{E}_G^U)^j \to \mathscr{E}_G^U$$

as follows. On objects, for $\phi \in \mathscr{V}_G^\times(j)$ and $A_i \in \mathscr{E}_G$, $1 \leq i \leq j$, define

$$\xi_j(\phi; A_1, \dots, A_j) = \phi(A_1 \times \cdots \times A_j).$$

For a morphism

$$(\zeta; \nu_1, \dots, \nu_j) \colon (\phi; A_1, \dots, A_j) \to (\psi; B_1, \dots, B_j),$$

define $\xi_j(\zeta; \nu_1, \dots, \nu_j)$ to be the unique bijection that makes the following diagram commute:

$$
\begin{array}{ccc}
A_1 \times \cdots \times A_j & \xrightarrow{\ \phi\ } & \phi(A_1 \times \cdots \times A_j) \\[4pt]
{\scriptstyle \nu_1 \times \cdots \times \nu_j} \downarrow & & \downarrow {\scriptstyle \xi_j(\zeta; \nu_1, \dots, \nu_j)} \\[4pt]
B_1 \times \cdots \times B_j & \xrightarrow[\ \psi\ ]{} & \psi(B_1 \times \cdots \times B_j)
\end{array}
$$

Then the ξ_j specify an action of $\mathscr{V}_G^\times$ on $\mathscr{E}_G^U$.   □

Proposition 8.11 *If R is a commutative G–ring, then $\mathscr{GL}_G(R)$ is a $\mathscr{V}_G^\times$–category.*

Proof Define a functor

$$\xi_j \colon \mathscr{V}_G^{\times}(j) \times \mathscr{GL}(R)_G^j \to \mathscr{GL}_G(R)$$

as follows. Identify $R[U^j]$ with $\bigotimes_j R[U]$, where $\otimes = \otimes_R$. On objects, for $\phi \in \mathscr{V}_G(j)$ and R–modules $M_i \subset R[U]$, $1 \leq i \leq j$, define

$$\xi_j(\phi; M_1, \ldots, M_j) = R[\phi](M_1 \times \cdots \times M_j).$$

For a morphism

$$(\zeta; \nu_1, \ldots, \nu_j) \colon (\phi; M_1, \ldots, M_j) \to (\psi; N_1, \ldots, N_j),$$

define $\xi_j(\zeta; \nu_1, \ldots, \nu_j)$ to be the unique isomorphism of R–modules that makes the following diagram commute:

$$
\begin{CD}
M_1 \otimes \cdots \otimes M_j @>{R[\phi]}>> \phi(M_1 \otimes \cdots \otimes M_j)] \\
@V{\nu_1 \otimes \cdots \otimes \nu_j}VV @VV{\xi_j(\zeta; \nu_1, \ldots, \nu_j)}V \\
N_1 \otimes \cdots \otimes N_j @>{R[\psi]}>> \psi(N_1 \otimes \cdots \otimes N_j)
\end{CD}
$$

Then the ξ_j specify an action of $\mathscr{V}_G^{\times}$ on $\mathscr{GL}_G(R)$. $\qquad\square$

Restricting the action from $\mathscr{V}_G^{\times}$ to $\mathscr{W}_G$, the examples above and easy diagram chases prove that the operad pair $(\mathscr{V}_G, \mathscr{W}_G)$ acts on the categories $\mathscr{E}_G$ and $\mathscr{GL}_G(R)$. This proves the following result.

Theorem 8.12 *The categories $\mathscr{E}_G^U$ and $\mathscr{GL}_G(R)$ for a commutative G–ring R are E_∞–ring G–categories in the sense of Definition 7.12.*

Although we have a definition of a genuine permutative G–category, we do not have a comparably simple definition of a genuine bipermutative G–category. The previous examples show that we do have examples of E_∞–ring G–categories. In [13], we will show how to construct E_∞–ring G–categories from general naive bipermutative G–categories, in particular nonequivariant bipermutative categories, and we will show how to construct genuine commutative ring G–spectra from them.

9 The $\mathscr{V}_G$–category $\mathscr{E}_G^U(X)$ and the BPQ theorem

We now return to the categorical BPQ theorem, but thinking in terms of $\mathscr{V}_G$–categories rather than $\mathscr{P}_G$–categories. This gives a more intuitive approach to the G–category of finite sets over a G–space X.

9.1 The G–category $\mathscr{E}_G^U(X)$ of finite sets over X

Definition 9.1 Let X be a G–space. We define the G–groupoid $\mathscr{E}_G^U(X) = \mathscr{E}_G^{\mathcal{V}}(X)$ of finite sets over X. Its objects are the functions $p\colon A \to X$, where A is a finite subset of our ambient G–set U. For a second function $q\colon B \to X$, a map $\nu\colon p \to q$ is a bijection $\nu\colon A \to B$ such that $q \circ \nu = p$. Composition is given by composition of functions over X. The group G acts by translation of G–sets and conjugation on all maps in sight. Thus, for an object $p\colon A \to X$, we have $gp\colon gA \to X$ given by $(gp)(ga) = g(p(a))$. For a map $\nu\colon p \to q$, we have $g\nu\colon gA \to gB$ given by $(g\nu)(ga) = g(\nu(a))$.

To topologize $\mathscr{E}_G^U(X)$, give U and X disjoint basepoints $*$.[7] View the set $\mathscr{O}b$ of objects of $\mathscr{E}_G^U(X)$ as the set of based functions $p\colon U_+ \to X_+$ such that $p^{-1}(*)$ is the complement of a finite set $A \subset U$. Topologize $\mathscr{O}b$ as a subspace of $X_+^{U_+}$. View the set $\mathscr{M}or$ of morphisms of $\mathscr{E}_G^U(X)$ as a subset of the set of functions $\mu\colon U_+ \to U_+$ that send the complement of some finite set $A \subset U$ to $*$ and map A bijectively to some finite set $B \subset U$. Topologize $\mathscr{M}or$ as the subspace of points (p, μ, q) in $\mathscr{O}b \times U_+^{U_+} \times \mathscr{O}b$, where $U_+^{U_+}$ is discrete. When X is a finite set and thus a discrete space (since points are closed in spaces in the category $\mathscr{U}$), $\mathscr{E}_G^U(X)$ is discrete.

Let $\mathscr{E}_G^U(n, X)$ denote the full subcategory of $\mathscr{E}_G^U(X)$ of maps $p\colon A \to X$ such that A has n elements. Then $\mathscr{E}_G^U(X)$ is the coproduct of the groupoids $\mathscr{E}_G^U(n, X)$.

Proposition 9.2 *The operad $\mathscr{V}_G$ acts naturally on the categories $\mathscr{E}_G^U(X)$.*

Proof For $j \geq 0$, we must define functors

$$\theta_j\colon \mathscr{V}_G(j) \times \mathscr{E}_G^U(X)^j \to \mathscr{E}_G^U(X).$$

To define θ_j on objects, let $\phi\colon {}^jU \to U$ be an injective function and $p_i\colon A_i \to X$ be a function, $1 \leq i \leq j$, where A_i is a finite subset of U. We define $\theta_j(\phi; p_1, \ldots, p_j)$ to be the composite

$$\phi(A_1 \sqcup \cdots \sqcup A_j) \xrightarrow{\phi^{-1}} A_1 \sqcup \cdots \sqcup A_j \xrightarrow{\sqcup p_i} {}^jX \xrightarrow{\nabla} X,$$

where ∇ is the fold map, the identity on each of the j copies of X. To define θ on morphisms, let $\psi\colon {}^jU \to U$ be another injective function, and let $\zeta\colon \phi \to \psi$ be the unique map in $\mathscr{V}_G(j)$. For functions $q_i\colon B_i \to X$ and bijections $\nu_i\colon A_i \to B_i$ such that $q_i\nu_i = p_i$, define $\theta_j(\zeta; \nu_1, \ldots, \nu_j)$ to be the unique dotted arrow bijection that

---

[7]These basepoints are just a convenience for specifying the topology; they play no other role.

makes the following diagram commute:

$$
\begin{array}{ccc}
\phi(A_1 \sqcup \cdots \sqcup A_j) & \xrightarrow{\phi^{-1}} & A_1 \sqcup \cdots \sqcup A_j \\
\theta(\zeta;v_1,\ldots,v_j) \Big\downarrow & & \Big\downarrow \sqcup v_i \\
\psi(B_1 \sqcup \cdots \sqcup B_j) & \xrightarrow[\psi^{-1}]{} & B_1 \sqcup \cdots \sqcup B_j
\end{array}
$$

with the maps $\sqcup p_i$, $\sqcup q_i$ to $jX \xrightarrow{\nabla} X$.

Then the maps θ_j specify an action of $\mathscr{V}_G$ on the category $\mathscr{E}_G^U(X)$. $\qquad\square$

We have a multiplicative elaboration, which is similar to [24, Proposition VI.1.9] but curiously restricted. Regarding a G–space X as a constant G–category with object and morphism space both X, it makes sense to speak of an action of the operad $\mathscr{V}_G^\times$ on the G–category X. For example, $\mathscr{V}_G^\times$ acts on X if X is a commutative topological G–monoid. The following result is closely related to Proposition 7.10. It has the minor advantage that restriction from $\mathscr{V}_G^\times$ to $\mathscr{W}_G$ is unnecessary but the major limitation that it only applies to commutative G–monoids, not to general $\mathscr{V}_G^\times$–algebras.

Proposition 9.3 *If X is a commutative topological G–monoid, then $\mathscr{E}_G^U(X)$ is an E_∞–ring G–category.*

Proof By analogy with the previous proof, for $k \geq 0$, we have functors

$$
\xi \colon \mathscr{V}_G^\times(k) \times \mathscr{E}_G^U(X)^k \to \mathscr{E}_G^U(X^k).
$$

With notation as in the previous proof, on objects $(\phi; p_1, \ldots, p_k)$ for $p_r \colon A_r \to X$, we define $\xi(\phi; p_1, \ldots, p_k)$ to be the composite

$$
\phi(A_1 \times \cdots \times A_k) \xrightarrow{\phi^{-1}} A_1 \times \cdots \times A_k \xrightarrow{\times p_k} X^k \xrightarrow{\pi} X,
$$

where π is the k–fold product on X. On morphisms $(\zeta; v_1, \ldots, v_k)$, where the $v_r \colon p_r \to q_r$ are understood to be order preserving, $\xi(\zeta; v_1, \ldots, v_k)$ is defined to be the unique dotted arrow that makes the following diagram commute:

$$
\begin{array}{ccc}
\phi(A_1 \times \cdots \times A_k) & \xrightarrow{\phi^{-1}} & A_1 \times \cdots \times A_k \\
\xi(\zeta;v_1,\ldots,v_k) \Big\downarrow & & \Big\downarrow \times v_i \\
\psi(B_1 \times \cdots \times B_k) & \xrightarrow[\psi^{-1}]{} & B_1 \times \cdots \times B_k
\end{array}
$$

with the maps $\times p_i$, $\times q_i$ to $X^k \xrightarrow{\pi} X$.

Further details are similar to those in the proof of [24, Proposition VI.1.9] or [30, Proposition 4.9]. $\qquad\square$

9.2 Free $\mathscr{V}_G$–categories and the $\mathscr{V}_G$–categories $\mathscr{E}_G^U(X)$

The categories $\mathscr{E}_G^U(X)$ are conceptually simple, and they allow us to give the promised genuinely equivariant variant of Theorem 5.9. To see that, we give a reinterpretation of $\mathscr{E}_G^U(X)$. Regarding X as a topological G–category as before, we have the topological G–category $\widetilde{\mathscr{E}}_G^U(j) \times_{\Sigma_j} X^j$.

Lemma 9.4 *The topological G–categories $\mathscr{E}_G^U(j, X)$ and $\widetilde{\mathscr{E}}_G^U(j) \times_{\Sigma_j} X^j$ are naturally isomorphic.*

Proof For an ordered set $A = (a_1, \ldots, a_j)$ of points of U, let a point $(A; x_1, \ldots, x_j)$ of $\mathscr{O}b(\widetilde{\mathscr{E}}_G(j) \times_{\Sigma_j} X^j)$ correspond to the function $p: A \to X$ given by $p(a_i) = x_i$. Similarly, let a point $(v: A \to B; x_1, \ldots, x_j)$ of $\mathscr{M}or(\widetilde{\mathscr{E}}_G(j) \times_{\Sigma_j} X^j)$ correspond to the bijection $v: p \to q$ over X, where $qv(a_i) = p(a_i) = x_i$. Since we have passed to orbits over Σ_j, our specifications are independent of the ordering of A. These correspondences identify the two categories. $\qquad\square$

Recall that we write $\mathbb{V}_G$ for the monad on based G–categories associated to the operad $\mathscr{V}_G$, we write $|\mathscr{V}_G|$ for the operad of G–spaces obtained by applying the classifying-space functor B to $\mathscr{V}_G$, and we write V_G for the monad on based G–spaces associated to $|\mathscr{V}_G|$. Recall too that X_+ denotes the union of the G–category X with a disjoint trivial basepoint category $*$ and that

$$(9.5) \qquad \mathbb{V}_G(X_+) = \bigsqcup_{j \geq 0} \mathscr{V}_G(j) \times_{\Sigma_j} X^j.$$

Theorem 9.6 *There is a natural map*

$$\omega: \mathbb{V}_G(X_+) \to \mathscr{E}_G^U(X)$$

of $\mathscr{V}_G$–categories, and it induces a weak equivalence

$$B\omega: V_G(X_+) \to B\mathscr{E}_G^U(X)$$

of $|\mathscr{V}_G|$–spaces on passage to classifying spaces.

Proof Pick any G–fixed point $1 \in U$.[8] Define an inclusion $i: X_+ \to \mathscr{E}_G^U(X)$ of based G–categories by identifying $*$ with $\mathscr{E}_G^U(0, X)$ and mapping X to $\mathscr{E}_G^U(1, X)$ by sending x to the map $1 \to x$ from the 1–point subset 1 of U to X. Since $\mathbb{V}_G(X_+)$ is

---

[8]This must not be confused with the convenience basepoint $*$ used to define the topology.

the free (based) $\mathscr{V}_G$–category generated by X_+, the inclusion i induces the required natural map ω. Explicitly, it is the composite

$$\mathbb{V}_G(X_+) \xrightarrow{\mathbb{V}_G i} \mathbb{V}_G \mathscr{E}_G^U(X)) \xrightarrow{\theta} \mathscr{E}_G^U(X).$$

More explicitly still, let $\underline{1} \subset {}^j U$ be the j–point subset consisting of the elements 1 in the j summands. Then ω is the coproduct of the maps

$$\omega_j = i_j \times_{\Sigma_j} \mathrm{id} \colon \mathscr{V}_G(j) \times_{\Sigma_j} X^j \to \widetilde{\mathscr{E}}_G^U(j) \times_{\Sigma_j} X^j,$$

where $i_j \colon \mathscr{V}_G(j) \to \widetilde{\mathscr{E}}_G^U(j)$ is the $(G \times \Sigma_j)$–functor that sends an object $\phi \colon {}^j U \to U$ to the set $\phi(\underline{1}) \subset U$ and sends the morphism $\nu \colon \phi \to \psi$ to the bijection

$$\phi(\underline{1}) \xrightarrow{\phi^{-1}} \underline{1} \xrightarrow{\psi} \psi(\underline{1}).$$

Passing to classifying spaces, $|i_j|$ is a map between universal principal (G, Σ_j)–bundles, both of which are $(G \times \Sigma_j)$–CW complexes. Therefore, $|i_j|$ is a $(G \times \Sigma_j)$–equivariant homotopy equivalence. The conclusion follows. $\square$

9.3 The categorical BPQ theorem: second version

We begin by comparing Theorem 9.6, which is about G–categories, with Theorems 5.5, 5.9 and 5.10, which are about G–fixed categories. Clearly $\mathscr{E}_G(X)^G$ is a $\mathscr{V}$–category, where $\mathscr{V} = (\mathscr{V}_G)^G$. By Theorem 9.6, it is weakly equivalent (in the homotopical sense) to the $\mathscr{V}$–category $(\mathbb{V}_G X_+)^G$. We also have the $\mathscr{P}$–category $\mathscr{F}_G(X)^G$, which by Theorem 5.9 and Remark 5.11 is equivalent (in the categorical sense) to the $\mathscr{P}$–category $(\mathbb{P}_G X_+)^G$. Elaborating Remark 8.3, $\mathscr{E}_G^U(X)^G$ and $\mathscr{F}_G(X)^G$ are two small models for the category of all finite G–sets and G–isomorphisms over X and are therefore equivalent. To take the operad actions into account, recall the discussion in Section 4.3. We say that a map of topological G–categories is a weak equivalence if its induced map of classifying G–spaces is a weak equivalence.

Lemma 9.7 *The $\mathscr{P}_G$–category $\mathbb{P}_G X_+$ and the $\mathscr{V}_G$–category $\mathbb{V}_G X_+$ are weakly equivalent as $(\mathscr{P}_G \times \mathscr{V}_G)$–categories. Therefore, the $\mathscr{P}$–category $(\mathbb{P}_G X_+)^G$ and the $\mathscr{V}$–category $(\mathbb{V}_G X_+)^G$–categories are weakly equivalent.*

Proof The projections

$$\mathbb{P}_G X_+ \leftarrow (\mathbb{P}_G \times \mathbb{V}_G)(X_+) \to \mathbb{V}_G X_+$$

are maps of $(\mathscr{P}_G \times \mathscr{V}_G)$–categories that induce weak equivalences of $|\mathscr{P}_G \times \mathscr{V}_G|$–spaces on passage to classifying spaces. $\square$

Theorem 9.8 *The classifying spaces of the $\mathscr{P}$–category $\mathscr{F}_G(X)^G$ and the $\mathscr{V}$–category $\mathscr{E}_G^U(X)^G$ are weakly equivalent as $|\mathscr{P} \times \mathscr{V}|$–spaces.*

The conclusion is that, on the G–fixed level, the categories $\mathscr{E}_G^U(X)^G$ and $\mathscr{F}_G(X)^G$ can be used interchangeably as operadically structured versions of the category of finite G–sets over X. On the equivariant level, $\mathscr{E}_G^U(X)$ but not $\mathscr{F}_G(X)$ is operadically structured. It is considerably more convenient than the categories $\mathbb{P}_G(X_+)$ or $\mathbb{V}_G(X_+)$. With the notation $\mathbb{K}_G\mathbb{V}_G(X_+) = \mathbb{E}_G B\mathbb{V}_G(X_+) = \mathbb{E}_G V_G(X_+)$ and $\mathbb{K}_G\mathscr{E}_G^U(X) = \mathbb{E}_G B\mathscr{E}_G^U(X)$, we have the following immediate consequence of Theorems 6.2 and 9.6. It is our preferred version of the categorical BPQ theorem, since it uses the most intuitive categorical input.

Theorem 9.9 (categorical Barratt–Priddy–Quillen theorem) *For G–spaces X, there is a composite natural weak equivalence*

$$\alpha\colon \Sigma_G^\infty X_+ \to \mathbb{K}_G\mathbb{V}_G X_+ \to \mathbb{K}_G\mathscr{E}_G^U(X).$$

Remark 9.10 It is not known how the tom Dieck splitting theorem behaves with respect to the Mackey functor structure on homotopy groups. It seems likely to us that this could be analyzed using this version of the BPQ theorem and our categorical proof of the splitting.

Appendix A: Pairings of operads

We recall the following definition from [25, 1.4]. It applies equally well equivariantly. We write it elementwise, but written diagrammatically it applies to operads in any symmetric monoidal category $\mathscr{V}$. Write $\boldsymbol{j} = \{1, \dots, j\}$ and let

$$\otimes\colon \Sigma_j \times \Sigma_k \to \Sigma_{jk}$$

be the homomorphism obtained by identifying $\boldsymbol{j} \times \boldsymbol{k}$ with $\boldsymbol{jk}$ by ordering the set of jk elements (q, r), $1 \le q \le j$ and $1 \le r \le k$, lexicographically. More precisely, let $\lambda_{j,k}\colon \boldsymbol{jk} \to \boldsymbol{j} \times \boldsymbol{k}$ be the lexicographic ordering. Then, given $\rho \in \Sigma_j$ and $\sigma \in \Sigma_k$, $\rho \otimes \sigma$ is defined by

$$\boldsymbol{jk} \xrightarrow{\lambda_{j,k}} \boldsymbol{j} \times \boldsymbol{k} \xrightarrow{\rho \times \sigma} \boldsymbol{j} \times \boldsymbol{k} \xrightarrow{\lambda_{j,k}^{-1}} \boldsymbol{jk}.$$

For nonnegative integers h_q and i_r, let

$$\delta\colon \bigsqcup_{(q,r)} (\boldsymbol{h_q} \times \boldsymbol{i_r}) \to \left(\bigsqcup_q \boldsymbol{h_q}\right) \times \left(\bigsqcup_r \boldsymbol{i_r}\right)$$

be the distributivity isomorphism viewed as a permutation via block and lexicographic identifications of the source and target sets with the appropriate set $\boldsymbol{n}$. A little more

precisely, we define the permutation δ to be the composite

$$\sum_{q,r} h_q i_r \xrightarrow{\cong} h_1 i_1 \sqcup h_1 i_2 \sqcup \cdots \sqcup h_j i_k \xrightarrow{\lambda \sqcup \cdots \sqcup \lambda} h_1 \times i_1 \sqcup \cdots \sqcup h_j \times i_k$$

$$\xrightarrow{\text{dist}} (h_1 \sqcup \cdots \sqcup h_j) \times (i_1 \sqcup \cdots \sqcup i_k) \xrightarrow{\cong} h \times i \xrightarrow{\lambda_{h,i}^{-1}} h i .$$

Definition A.1 Let $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{E}$ be operads in a symmetric monoidal category $\mathscr{V}$ (with product denoted by $\otimes$). A pairing of operads

$$\boxtimes \colon (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$$

consists of maps

$$\boxtimes \colon \mathscr{C}(j) \otimes \mathscr{D}(k) \to \mathscr{E}(jk)$$

in $\mathscr{V}$ for $j \geq 0$ and $k \geq 0$ such that the diagrammatic versions of the following properties hold, where $c \in \mathscr{C}(j)$ and $d \in \mathscr{D}(k)$:

(i) If $\mu \in \Sigma_j$ and $\nu \in \Sigma_k$, then

$$c\mu \boxtimes d\nu = (c \boxtimes d)(\mu \otimes \nu).$$

(ii) With $j = k = 1$, we have $\text{id} \boxtimes \text{id} = \text{id}$.

(iii) If $c_q \in \mathscr{C}(h_q)$ for $1 \leq q \leq j$ and $d_r \in \mathscr{D}(i_r)$ for $1 \leq r \leq k$, then[9]

$$\gamma(c \boxtimes d; \times_{(q,r)} c_q \boxtimes d_r) = [\gamma(c; \times_q c_q) \boxtimes \gamma(d; \times_r d_r)]\delta.$$

When specialized to spaces, the following definition (which is a variant of [25, 1.2]) gives one possible starting point for multiplicative infinite loop space theory.

Definition A.2 Let $\boxtimes \colon (\mathscr{C}, \mathscr{D}) \to \mathscr{E}$ be a pairing of operads in $\mathscr{V}$. A pairing of a $\mathscr{C}$–algebra X and a $\mathscr{D}$–algebra Y to an $\mathscr{E}$–algebra Z is a map $f \colon X \otimes Y \to Z$ such that the following diagram commutes for all j and k, where X^j denotes the jth tensor power in $\mathscr{V}$ and we write θ generically for action maps:

$$
\begin{array}{ccc}
\mathscr{C}(j) \otimes X^j \otimes \mathscr{D}(k) \otimes Y^k & \xrightarrow{\quad\theta \otimes \theta\quad} & X \otimes Y \\
{\scriptstyle \boxtimes}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\
\mathscr{E}(jk) \otimes (X \otimes Y)^{jk} \xrightarrow{\text{id} \otimes f^{jk}} \mathscr{E}(jk) \otimes Z^{jk} & \xrightarrow{\quad\theta\quad} & Z
\end{array}
$$

On the left, $\boxtimes$ denotes the composite

$$\mathscr{C}(j) \otimes X^j \otimes \mathscr{D}(k) \otimes Y^k \xrightarrow{\text{id} \otimes t \otimes \text{id}} \mathscr{C}(j) \otimes \mathscr{D}(k) \otimes X^j \otimes Y^k \xrightarrow{\boxtimes \otimes \lambda} \mathscr{E}(jk) \otimes Z^{jk}.$$

---

[9]The original definition in [25] had δ on the other side in this condition.

Here, in elementwise notation,

$$\lambda((x_1 \otimes \cdots \otimes x_j) \otimes (y_1 \otimes \cdots \otimes y_k)) = ((x_1 \otimes y_1) \otimes \cdots \otimes (x_j \otimes y_k)),$$

where we order the pairs $(x_q \otimes y_r)$, $1 \le q \le j$ and $1 \le r \le k$, lexicographically.

Letting $\mathcal{V}$ be the category of unbased G–spaces, with $\otimes = \times$, but then passing to monads on based G–spaces, we obtain the following observations.

Proposition A.3 *For based G–spaces X and Y, a pairing $\boxtimes \colon (\mathscr{C}_G, \mathscr{D}_G) \to \mathscr{E}_G$ of operads of G–spaces induces a natural pairing*

$$\boxtimes \colon C_G X \wedge D_G Y \to E_G(X \wedge Y)$$

such that the following diagrams commute:

$$
\begin{array}{ccc}
X \wedge Y & \xrightarrow{\eta \wedge \eta} & C_G X \wedge D_G Y \\
 & \searrow{\scriptstyle \eta} & \downarrow{\scriptstyle \boxtimes} \\
 & & E_G(X \wedge Y)
\end{array}
$$

$$
\begin{array}{ccccc}
C_G C_G X \wedge D_G D_G Y & \xrightarrow{\mu \wedge \mu} & & & C_G X \wedge D_G Y \\
\downarrow{\scriptstyle \boxtimes} & & & & \downarrow{\scriptstyle \boxtimes} \\
E_G(C_G X \wedge D_G Y) & \xrightarrow[E_G \boxtimes]{} & E_G E_G(X \wedge Y) & \xrightarrow[\mu]{} & E_G(X \wedge Y)
\end{array}
$$

The following diagram commutes for any pairing $f \colon X \otimes Y \to Z$ of a $\mathscr{C}_G$–algebra X and a $\mathscr{D}_G$–algebra Y to an $\mathscr{E}_G$–algebra Z:

$$
\begin{array}{ccc}
C_G X \wedge D_G Y & \xrightarrow{\theta \wedge \theta} & X \wedge Y \\
\downarrow{\scriptstyle \boxtimes} & & \downarrow{\scriptstyle f} \\
E_G(X \wedge Y) \xrightarrow[E_G f]{} E_G Z & \xrightarrow[\theta]{} & Z
\end{array}
$$

Proof The map $\boxtimes$ is induced from the map $\boxtimes$ of the previous definition and the commutativity of the first two diagrams is checked by chases from Definition A.1. The commutativity of the second implies that $\boxtimes$ is a pairing in the sense of Definition A.2. The commutativity of the third follows from Definition A.2. □

Example A.4 The following commutative diagram, in which we ignore the path space variable for simplicity, shows that condition (iii) is satisfied by the pairing $(\mathscr{K}_V, \mathscr{K}_W) \to \mathscr{K}_{V \oplus W}$ defined in Proposition 1.17; this completes the proof of that result:
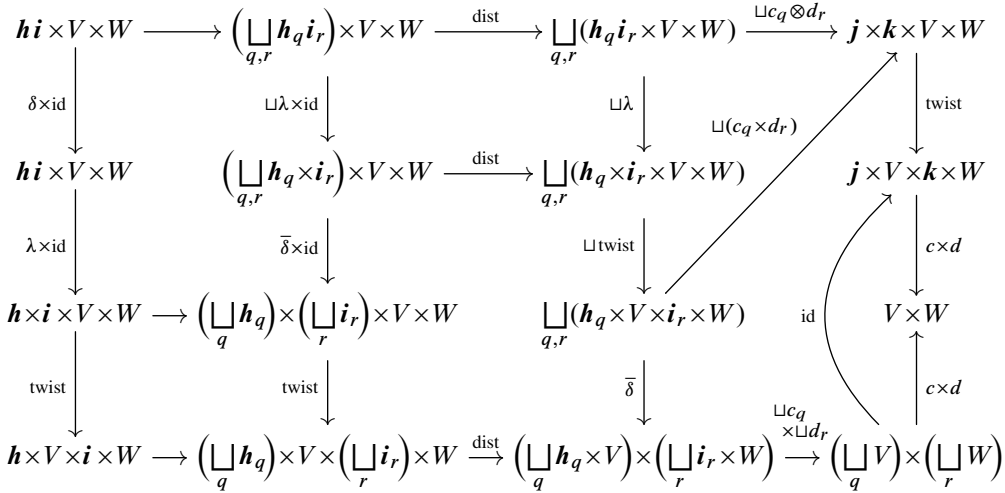
The following counterexample was pointed out to us by Anna Marie Bohmann and Angelica Osorno. Using a more sophisticated categorical framework, we shall explain how to get around the difficulty in [12; 13].

Counterexample A.5 We show that the pairing (6.4) is not a self-pairing of $\mathscr{P}$. Letting $\tau \in \mathscr{P}(2)$ be the transposition $\tau = (12)$, we calculate

$$\gamma(\tau \otimes \tau; \text{id}_2 \otimes \text{id}_1, \text{id}_2 \otimes \text{id}_1, \text{id}_1 \otimes \text{id}_1, \text{id}_1 \otimes \text{id}_1) = (1526)(3)(4)$$

whereas

$$[\gamma(\tau; \text{id}_2, \text{id}_1) \otimes \gamma(\tau; \text{id}_1, \text{id}_1)]\delta = (14526)(3).$$

In this case δ is the transposition (23). Thus condition (iii) fails.

Appendix B: The double bar construction and the proof of Theorem 2.25

The proof of Theorem 2.25 is based on a construction that the senior author has used for decades in unpublished work and whose algebraic analogue has also long been used. Heretofore he has always found alternative arguments that avoid its use in published work, and the topological version seems not to have appeared in print. The construction works in great generality with different kinds of bar constructions,

as described in [34; 35; 45], for example. We restrict attention to the monadic bar construction used in this paper. We shall be informal, since it is routine to fill in the missing details.

We assume given two monads C and D in some reasonable category $\mathscr{U}$, and we assume given a morphism of monads $\iota\colon C \to D$. We also assume given a right D–functor $\Sigma\colon \mathscr{U} \to \mathscr{V}$ for some other reasonable category $\mathscr{V}$. Then Σ is a right D–functor with the pullback action

$$\Sigma C \to \Sigma D \to \Sigma.$$

Let X be a C–algebra in $\mathscr{U}$. Reasonable means in particular that we can form "geometric realizations" of simplicial objects X as usual, tensoring X over the category Δ with a canonical (covariant) simplex functor from Δ to $\mathscr{U}$ or $\mathscr{V}$.

We assume that the functor D commutes with geometric realization, so that the realization of a simplicial D–algebra is a D–algebra. Then the bar construction

$$\iota_! X = B(D, C, X)$$

in $\mathscr{U}$ specifies an "extension of scalars" functor that converts C–algebras X to D–algebras in a homotopically well-behaved fashion. Since D acts on Σ, we have the bar construction $B(\Sigma, D, \iota_! X)$, and we also have the bar construction $B(\Sigma, C, X)$, both with values in $\mathscr{V}$. Under these assumptions, we have the following result.

Theorem B.1   *There is a natural equivalence* $B(\Sigma, D, \iota_! X) \simeq B(\Sigma, C, X)$.

Proof of Theorem 2.25   We replace $\mathscr{U}$ by $G\mathscr{U}$ and $\mathscr{V}$ by $G\mathscr{Sp}$. We take C to be the monad associated to the operad $\mathscr{C}_{UG} = (\mathscr{C}_G)^G \times \mathscr{K}_{UG}$ and D to be the monad associated to $\mathscr{C}_U = \mathscr{C}_G \times \mathscr{K}_U$. We take Σ to be Σ_G^∞, and we recall that $\Sigma_G^\infty = i_* \Sigma^\infty$ by Lemma 2.22. By inspection or a commutation of left adjoints argument, the functor i_* commutes with geometric realization. Therefore,

$$E_G(\iota_! X) \equiv B(\Sigma_G^\infty, C_U, \iota_! X) \simeq B(\Sigma_G^\infty, C_{UG}, X) \cong i_* B(\Sigma^\infty, C_{UG}, X) \equiv i_* E X,$$

where Theorem B.1 gives the equivalence.   □

Proof of Theorem B.1   We construct the double bar construction

$$B(\Sigma, D, D, C, X)$$

as the geometric realization of the bisimplicial object $B_{\bullet,\bullet}(\Sigma, D, D, C, X)$ in $\mathscr{V}$ whose (p, q)–simplex object is $\Sigma D^p D C^q X$. The horizontal face and degeneracy operations are those obtained by applying the simplicial bar construction $B_\bullet(\Sigma, D, Y)$ to the D–algebras $Y = D C^q X$. The vertical face and degeneracy operations are those

obtained by applying the simplicial bar construction $B_\bullet(\Upsilon, \boldsymbol{C}, X)$ to the $\boldsymbol{C}$–functors $\Upsilon = \Sigma \boldsymbol{D}^p \boldsymbol{D}$. The geometric realization of a bisimplicial object is obtained equivalently as the realization of the diagonal simplicial object, the horizontal realization of its vertical realization, and the vertical realization of its horizontal realization. Realizing first vertically and then horizontally, we obtain

$$B(\Sigma, \boldsymbol{D}, B(\boldsymbol{D}, \boldsymbol{C}, X)) = B(\Sigma, \boldsymbol{D}, i_! X).$$

Realizing first horizontally and then vertically, we obtain the bar construction

$$B(B(\Sigma, \boldsymbol{D}, \boldsymbol{D}), \boldsymbol{C}, X) \simeq B(\Sigma, \boldsymbol{C}, X).$$

Here $B(\Sigma, \boldsymbol{D}, \boldsymbol{D})$ is the right $\boldsymbol{C}$–functor whose value on a $\boldsymbol{C}$–algebra Y is $B(\Sigma, \boldsymbol{D}, \boldsymbol{D}Y)$ with right $\boldsymbol{C}$–action induced by the $\boldsymbol{C}$–action $\boldsymbol{C}Y \to Y$. The equivalence is induced by the standard natural equivalence $B(\Sigma, \boldsymbol{D}, \boldsymbol{D}Y) \to \Sigma Y$. □

Remark B.2 The double bar construction can be defined more generally and symmetrically. Dropping the assumption that there is a map of monads $\boldsymbol{C} \to \boldsymbol{D}$, we have that $B(\Sigma, \boldsymbol{D}, \boldsymbol{F}, \boldsymbol{C}, X)$ is defined if F is a left $\boldsymbol{D}$–functor and a right $\boldsymbol{C}$–functor $\mathbb{U} \to \mathbb{U}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\boldsymbol{DFC} & \longrightarrow & \boldsymbol{DF} \\
\downarrow & & \downarrow \\
\boldsymbol{FC} & \longrightarrow & \boldsymbol{F}
\end{array}
$$

This can even work when the domain and target categories of $\boldsymbol{F}$ differ but agree with the categories on which $\boldsymbol{C}$ and $\boldsymbol{D}$ are defined.

References

[1]   **M Ando**, **A J Blumberg**, **D Gepner**, **M J Hopkins**, **C Rezk**, *Units of ring spectra, orientations and Thom spectra via rigid infinite loop space theory*, J. Topol. 7 (2014) 1077–1117 MR

[2]   **M G Barratt**, **P J Eccles**, Γ^+*–structures, I: A free group functor for stable homotopy theory*, Topology 13 (1974) 25–45 MR

[3]   **J Caruso**, **S Waner**, *An approximation theorem for equivariant loop spaces in the compact Lie case*, Pacific J. Math. 117 (1985) 27–49 MR

[4]   **S R Costenoble**, **S Waner**, *Fixed set systems of equivariant infinite loop spaces*, Trans. Amer. Math. Soc. 326 (1991) 485–505 MR

[5]   **T tom Dieck**, *Orbittypen und äquivariante Homologie, II*, Arch. Math. (Basel) 26 (1975) 650–662 MR

[6] **A W M Dress**, **A O Kuku**, *A convenient setting for equivariant higher algebraic K–theory*, from "Algebraic K–theory, Part I" (R K Dennis, editor), Lecture Notes in Math. 966, Springer (1982) 59–68 MR

[7] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs 47, Amer. Math. Soc., Providence, RI (1997) MR

[8] **A D Elmendorf**, **M A Mandell**, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. 205 (2006) 163–228 MR

[9] **Z Fiedorowicz**, **H Hauschild**, **J P May**, *Equivariant algebraic K–theory*, from "Algebraic K–theory, Part II" (R K Dennis, editor), Lecture Notes in Math. 967, Springer (1982) 23–80 MR

[10] **B Guillou**, **J P May**, *Models of G–spectra as presheaves of spectra*, preprint (2011) arXiv

[11] **B Guillou**, **J P May**, **M Merling**, *Categorical models for equivariant classifying spaces*, Algebr. Geom. Topol. 17 (2017) 2565–2602 MR

[12] **B Guillou**, **J P May**, **M Merling**, **A Osorno**, *Equivariant infinite loop space theory, II: The multiplicative Segal machine* In preparation

[13] **B Guillou**, **J P May**, **M Merling**, **A Osorno**, *Equivariant infinite loop space theory, III: The categorical story* In preparation

[14] **H Hauschild**, *Klassifizierende Räume, G–Konfigurations-räume und äquivariante Schleifenräume*, Habilitationsschrift, Göttingen (1978)

[15] **H Hauschild**, *Äquivariante Konfigurationsräume und Abbildungsräume*, from "Topology Symposium" (U Koschorke, W D Neumann, editors), Lecture Notes in Math. 788, Springer (1980) 281–315 MR

[16] **M A Hill**, **M J Hopkins**, *Equivariant symmetric monoidal structures*, preprint (2016) arXiv

[17] **A Kuku**, *Representation theory and higher algebraic K–theory*, Pure and Applied Mathematics (Boca Raton) 287, Chapman & Hall, Boca Raton, FL (2007) MR

[18] **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer (1986) MR

[19] **M A Mandell**, **J P May**, *Equivariant orthogonal spectra and S–modules*, Mem. Amer. Math. Soc. 755, Amer. Math. Soc., Providence, RI (2002) MR

[20] **M A Mandell**, **J P May**, **S Schwede**, **B Shipley**, *Model categories of diagram spectra*, Proc. London Math. Soc. 82 (2001) 441–512 MR

[21] **J P May**, *The geometry of iterated loop spaces*, Lectures Notes in Math. 271, Springer (1972) MR

[22] **J P May**, *E_∞ spaces, group completions, and permutative categories*, from "New developments in topology" (G Segal, editor), London Math. Soc. Lecture Note Ser. 11, Cambridge Univ. Press (1974) 61–93 MR

[23] **J P May**, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. 155, Amer. Math. Soc., Providence, RI (1975) MR

[24] **J P May**, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Math. 577, Springer (1977) MR

[25] **J P May**, *Pairings of categories and spectra*, J. Pure Appl. Algebra 19 (1980) 299–346 MR

[26] **J P May**, *Multiplicative infinite loop space theory*, J. Pure Appl. Algebra 26 (1982) 1–69 MR

[27] **J P May**, *Definitions: operads, algebras and modules*, from "Operads: Proceedings of Renaissance Conferences" (J-L Loday, J D Stasheff, A A Voronov, editors), Contemp. Math. 202, Amer. Math. Soc., Providence, RI (1997) 1–7 MR

[28] **J P May**, *Operads, algebras and modules*, from "Operads: Proceedings of Renaissance Conferences" (J-L Loday, J D Stasheff, A A Voronov, editors), Contemp. Math. 202, Amer. Math. Soc., Providence, RI (1997) 15–31 MR

[29] **J P May**, *The construction of E_∞ ring spaces from bipermutative categories*, from "New topological contexts for Galois theory and algebraic geometry" (A Baker, B Richter, editors), Geom. Topol. Monogr. 16, Geom. Topol. Publ., Coventry (2009) 283–330 MR

[30] **J P May**, *What precisely are E_∞ ring spaces and E_∞ ring spectra?*, from "New topological contexts for Galois theory and algebraic geometry" (A Baker, B Richter, editors), Geom. Topol. Monogr. 16, Geom. Topol. Publ., Coventry (2009) 215–282 MR

[31] **J P May**, **M Merling**, **A Osorno**, *Equivariant infinite loop space theory, I: The space level story*, preprint (2017) arXiv

[32] **J P May**, **R Thomason**, *The uniqueness of infinite loop space machines*, Topology 17 (1978) 205–224 MR

[33] **M Merling**, *Equivariant algebraic K-theory of G–rings*, Math. Z. 285 (2017) 1205–1248 MR

[34] **J-P Meyer**, *Bar and cobar constructions, I*, J. Pure Appl. Algebra 33 (1984) 163–207 MR

[35] **J-P Meyer**, *Bar and cobar constructions, II*, J. Pure Appl. Algebra 43 (1986) 179–210 MR

[36] **G Nishida**, *The transfer homomorphism in equivariant generalized cohomology theories*, J. Math. Kyoto Univ. 18 (1978) 435–451 MR

[37] **C Rourke**, **B Sanderson**, *Equivariant configuration spaces*, J. London Math. Soc. 62 (2000) 544–552 MR

[38]   **C Rourke**, **B Sanderson**, *The compression theorem, I*, Geom. Topol. 5 (2001) 399–429 MR

[39]   **C Rourke**, **B Sanderson**, *The compression theorem, II: Directed embeddings*, Geom. Topol. 5 (2001) 431–440 MR

[40]   **C Rourke**, **B Sanderson**, *The compression theorem, III: Applications*, Algebr. Geom. Topol. 3 (2003) 857–872 MR

[41]   **P Salvatore**, **N Wahl**, *Framed discs operads and Batalin–Vilkovisky algebras*, Q. J. Math. 54 (2003) 213–231 MR

[42]   **G Segal**, *Configuration-spaces and iterated loop-spaces*, Invent. Math. 21 (1973) 213–221 MR

[43]   **K Shimakawa**, *Infinite loop G–spaces associated to monoidal G–graded categories*, Publ. Res. Inst. Math. Sci. 25 (1989) 239–262 MR

[44]   **K Shimakawa**, *Mackey structures on equivariant algebraic K–theory*, K–Theory 5 (1991/92) 355–371 MR

[45]   **M Shulman**, *Homotopy limits and colimits and enriched homotopy theory*, preprint (2006) arXiv

[46]   **R Steiner**, *A canonical operad pair*, Math. Proc. Cambridge Philos. Soc. 86 (1979) 443–449 MR

[47]   **R W Thomason**, *Uniqueness of delooping machines*, Duke Math. J. 46 (1979) 217–252 MR

Department of Mathematics, The University of Kentucky
Lexington, KY, United States

Department of Mathematics, The University of Chicago
Chicago, IL, United States

bertguillou@uky.edu, may@math.uchicago.edu