Algebraic & Geometric Topology 17 (2017) 3341–3373

The localized skein algebra is Frobenius

NEL ABDIEL
CHARLES FROHMAN

When A in the Kauffman bracket skein relation is set equal to a primitive n^{th} root of unity ζ with n not divisible by 4, the Kauffman bracket skein algebra $K_\zeta(F)$ of a finite-type surface F is a ring extension of the $\mathrm{SL}_2\mathbb{C}$–character ring of the fundamental group of F. We localize by inverting the nonzero characters to get an algebra $S^{-1}K_\zeta(F)$ over the function field of the corresponding character variety. We prove that if F is noncompact, the algebra $S^{-1}K_\zeta(F)$ is a symmetric Frobenius algebra. Along the way we prove $K(F)$ is finitely generated, $K_\zeta(F)$ is a finite-rank module over the coordinate ring of the corresponding character variety, and learn to compute the trace that makes the algebra Frobenius.

57M27

1 Introduction

This paper is a step in a program to build a 4–dimensional extended field theory that assigns invariants to manifolds equipped with a homomorphism of their fundamental group into $\mathrm{SL}_2\mathbb{C}$. A *symmetric Frobenius algebra* A over a field k is a k–algebra equipped with a k–linear map Tr: $A \to k$ that is cyclic in the sense that for all $\alpha, \beta \in A$, $\mathrm{Tr}(\alpha\beta) = \mathrm{Tr}(\beta\alpha)$, and for all nonzero $\alpha \in A$, there exists $\beta \in A$ with $\mathrm{Tr}(\alpha\beta) \neq 0$. Frobenius algebras are central to the construction of field theories.

We show that the Kauffman bracket skein algebra of a compact surface with nonempty boundary can be localized to give a symmetric Frobenius algebra over the function field a character variety of the fundamental group of the surface. The trace that makes the localized skein algebra Frobenius is a potent tool for explicating the algebraic structure of $K_\zeta(F)$, as seen in Frohman and Kania-Bartoszynska [10].

A surface F is of finite type if there is a closed oriented surface $\widehat{F}$ and a finite set of points $\{p_i\} \in \widehat{F}$ such that $F = \widehat{F} - \{p_i\}$. In this paper all surfaces are either compact oriented (possibly with boundary) or of finite type. If F is a compact, connected, oriented surface, a punctured disk can be glued into each boundary component to obtain a finite-type surface. There is a one-to-one correspondence between disjoint families of simple closed curves in the two surfaces, so the theorems we prove working with finite-type surfaces apply to surfaces having finitely many boundary components.

Published: 4 October 2017 DOI: 10.2140/agt.2017.17.3341

A central result of this paper is:

Theorem 3.7 *The Kauffman bracket skein algebra $K(F)$ of a finite-type surface with coefficients in $\mathbb{Z}[A, A^{-1}]$ is finitely generated as an algebra by a finite family of simple closed curves S_i. In fact,*

$$\{S_{\sigma(1)}^{k_1} * S_{\sigma(2)}^{k_2} * \cdots * S_{\sigma(n)}^{k_n}\}, \tag{1-1}$$

where $k_i \in \mathbb{Z}_{\geq 0}$, spans $K(F)$ for any permutation $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$.

This is an extension of a theorem of Bullock [3]. Key to the proof is a well ordering of the simple diagrams on the surface. Given an ideal triangulation with edges E of the surface F, there is an embedding of the isotopy classes of simple diagrams on F into $\mathbb{Z}_{\geq 0}^E$. Letting $\mathcal{S}$ denote the simple diagrams on F, there is an injective map,

$$\iota\colon \mathcal{S} \to \mathbb{Z}_{\geq 0}^E, \tag{1-2}$$

which assigns the tuple $\prod_c i(S, c)$ to the simple diagram S, where $i(S, c)$ is the geometric intersection number of S with the edge c. Choosing an order on E gives rise to the lexicographic ordering of $\mathbb{Z}_{\geq 0}^E$ which in turn induces a well ordering of $\mathcal{S}$. The *geometric sum* of two simple diagrams S and S' is the simple diagram $S + S'$ such that $\iota(S + S') = \iota(S) + \iota(S')$.

Any skein has a unique expression as a linear combination of simple diagrams with nonzero coefficients. The *lead term* of a skein is the term in that expression involving the largest simple diagram. Suppose that we have defined the Kauffman bracket skein algebra $K_{\mathfrak{D}}(F)$ over an integral domain $\mathfrak{D}$, so that the variable ζ in the Kauffman bracket skein relation is a unit in $\mathfrak{D}$. The central tool for proving the theorems in this paper is:

Theorem 3.4 *Let F be a finite-type surface with at least one puncture and negative Euler characteristic. Choose an ideal triangulation with edges E, and order E in order to define the lead term of a skein. The lead term of the product of two simple diagrams S and S' in $K_{\mathfrak{D}}(F)$ is ζ raised to a power times the geometric sum of S and S'.*

If A is set equal to -1 in $K(F)$, the corresponding skein algebra is canonically isomorphic to the coordinate ring of the $\mathrm{SL}_2\mathbb{C}$–character variety of the fundamental group of F; see Bullock [4] and Przytycki and Sikora [15]. If A is set equal to 1, then the corresponding skein algebra is still isomorphic to the $\mathrm{SL}_2\mathbb{C}$–character variety of $\pi_1(F)$, just not canonically; see Barrett [1].

Let ζ be an n^{th} root of unity, and let $m = n/\gcd(n, 4)$. For reasons that will become clear, we call m the *index of threading*. Let $\epsilon = \zeta^{m^2}$. Throughout this paper we assume

$n \neq 0 \bmod 4$. In the case that n is odd, $\epsilon = 1$. If $n = 2 \bmod 4$ then $\epsilon = -1$. There is a theorem of Bonahon and Wong that, when $n \neq 0 \bmod 4$, there is a natural embedding

$$\text{Ch}\colon K_\epsilon(F) \to K_\zeta(F). \tag{1-3}$$

We denote the image of Ch with its coefficients extended to $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$ by $\chi(F)$. This is to remind us that it is canonically isomorphic to the coordinate ring of a character variety. We use the finite generation of $K(F)$ to prove that $K_\zeta(F)$ is a finitely generated module over $K_\epsilon(F)$. Localizing at $S = \chi(F) - \{0\}$, the algebra $S^{-1}K_\zeta(F)$ is finite-dimensional over $S^{-1}\chi(F)$.

Theorem 3.9 *Suppose that ζ is an n^{th} root of unity with $n \neq 0 \bmod 4$, and let m be the index of threading. Let F be a finite-type surface. If S_i is any system of simple diagrams corresponding to an integral basis of the cone of admissible colorings, then the skeins $\prod_i T_{k_i}(S_i)$, where the $k_i \in \{0, 1, \dots, m-1\}$ span $K_\zeta(F)$ over $\chi(F)$. In particular, $K_\zeta(F)$ is a finite ring extension of $\chi(F)$.*

Frohman and Kania-Bartoszynska [10] prove that $S^{-1}K_\zeta(F)$ is a vector space of dimension $m^{-3e(F)}$ over $S^{-1}\chi(F)$, where $e(F)$ is the Euler characteristic of F and m is the index of threading. Next we prove:

Theorem 3.10 *If F is closed and ζ is a primitive n^{th} root of unity with $n \neq 0 \bmod 4$, then $K_\zeta(F)$ is a finite-rank module over $\chi(F)$.*

This means that all irreducible representations of $K_\zeta(F)$ over the complex numbers are of bounded dimension.

Each $\alpha \in S^{-1}K_\zeta(F)$ induces an $S^{-1}\chi(F)$–linear endomorphism

$$L_\alpha\colon S^{-1}K_\zeta(F) \to S^{-1}K_\zeta(F) \tag{1-4}$$

by left multiplication. The *normalized trace* $\text{Tr}(\alpha)$ of α is the trace of L_α as a linear endomorphism divided by the dimension of the vector space $S^{-1}K_\zeta(F)$ over the field $S^{-1}\chi(F)$. The normalized trace has the properties:

- $\text{Tr}(1) = 1$.
- $\text{Tr}(\alpha * \beta) = \text{Tr}(\beta * \alpha)$.
- Tr is $S^{-1}\chi(F)$–linear.

Hence, if Tr is nondegenerate then $S^{-1}K_\zeta(F)$ equipped with the normalized trace is a symmetric Frobenius algebra over the function field of the character variety of the fundamental group of the surface F.

Along the way we learn to compute Tr: $S^{-1}K_\zeta(F) \to S^{-1}\chi(F)$ with respect to a special basis. A primitive diagram on F is a system of disjoint simple closed curves S_i such that no S_i bounds a disk and no two curves in the system cobound an annulus. The skein $\prod_i T_{k_i}(S_i)$ is the product over all i of the result of threading S_i with the k_i^{th} Chebyshev polynomial of the first kind. These span $S^{-1}K_\zeta(F)$ over $S^{-1}\chi(F)$.

Theorem 4.13 *Suppose that $s = \sum_i \beta_i P_i$, where the β_i are in $S^{-1}\chi(F)$ and the P_i are primitive diagrams whose components have been threaded with Chebyshev polynomials of the first kind. Let J be those indices i such that the components of P_i have only been threaded with Chebyshev polynomials whose index is divisible by the index of threading. Then*

$$\mathrm{Tr}(s) = \sum_{i \in J} \beta_i P_i. \tag{1-5}$$

The derivation of the formula for the trace depends on the following surprising fact. Let $\bigcup_i S_i$ be a simple diagram, made up of the simple closed curves S_i. The extension $S^{-1}\chi(F)[S_1, \dots, S_n]$ of $S^{-1}\chi(F)$ obtained by adjoining the S_i is a field. This extends a result of Muller [14], which says that simple closed curves are not zero divisors.

Since the value of the formula for the trace doesn't have any denominators that didn't appear in the input, the trace is actually defined as a $\chi(F)$–linear map

$$\mathrm{Tr}\colon K_\zeta(F) \to \chi(F). \tag{1-6}$$

Next, the formula for the trace is used to prove that there are no nontrivial principal ideals in the kernel of Tr: $K_\zeta(F) \to \chi(F)$, completing the proof that $S^{-1}K_\zeta(F)$ is a symmetric Frobenius algebra. Essential to the proof is the fact that, given a primitive diagram $\bigcup S_i$, the skeins $\prod_i T_{k_i}(S_i)$ with $0 \le k_i \le m-1$ generate a field extension of $S^{-1}\chi(F)$ in $S^{-1}K_\zeta(F)$.

Acknowledgements The authors would like to thank Pat Gilmer, Thang Lê and Joanna Kania-Bartoszynska for helpful input.

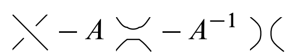
2 Preliminaries

2.1 Kauffman bracket skein module

Let M be an orientable 3–manifold. A *framed link* in M is an embedding of a disjoint union of annuli into M. Throughout this paper $M = F \times [0,1]$ for an orientable surface F. Diagrammatically we depict framed links by showing the core of the annuli lying parallel to F. Two framed links in M are equivalent if they are isotopic. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. By $\mathbb{Z}[A, A^{-1}]$ we mean Laurent polynomials with integral coefficients in the formal variable A. Consider the free module over $\mathbb{Z}[A, A^{-1}]$,

(2-1) $$\mathbb{Z}[A, A^{-1}]\mathcal{L},$$

with basis $\mathcal{L}$. Let S be the submodule spanned by the Kauffman bracket skein relations,

(2-2) 

and

$$\bigcirc \cup L + (A^2 + A^{-2})L.$$

The framed links in each expression are identical outside the balls pictured in the diagrams, and when both arcs in a diagram lie in the same component of the link, the same side of the annulus is up. The Kauffman bracket skein module $K(M)$ is the quotient

(2-3) $$\mathbb{Z}[A, A^{-1}]\mathcal{L}/S(M).$$

A *skein* is an element of $K(M)$. Let F be a compact orientable surface and let $I = [0, 1]$. There is an algebra structure on $K(F \times I)$ that comes from laying one framed link over the other. The resulting algebra is denoted by $K(F)$ to emphasize that it comes from the particular structure as a cylinder over F. Denote the stacking product with a $*$, so $\alpha * \beta$ means α stacked over β. If it is known the two skeins commute, the $*$ will be omitted.

A *simple diagram* D on the surface F is a system of disjoint simple closed curves such that none of the curves bounds a disk. A simple diagram D is *primitive* if no two curves in the diagram cobound an annulus. A simple diagram can be made into a framed link by choosing a system of disjoint annuli in F so that each annulus has a single curve in the diagram as its core. This is sometimes called the *blackboard framing*. The set of isotopy classes of blackboard framed simple diagrams form a basis for $K(F)$ [5; 12; 16].

2.2 Specializing A

If R is a commutative ring and $\zeta \in R$ is a unit, then R is a $\mathbb{Z}[A, A^{-1}]$–module, where the action

(2-4) $$\mathbb{Z}[A, A^{-1}] \otimes R \to R$$

is given by letting $p \in \mathbb{Z}[A, A^{-1}]$ act by multiplication by the result of evaluating p at ζ. The skein module specialized at $\zeta \in R$ is

$$K_R(M) = K(M) \otimes_{\mathbb{Z}[A,A^{-1}]} R. \tag{2-5}$$

You can think of the specialization as setting A equal to ζ in the Kauffman bracket skein relations.

This is much too general a setting to get nice structure theorems for $K_R(M)$, so we restrict our attention to when the ring R is an integral domain. To emphasize that we are working with an integral domain we denote the ring by $\mathfrak{D}$. Since $\mathbb{Z}[A, A^{-1}]$ is an integral domain and A is a unit, our results hold for $K(A)$ as a special case. For that reason the theorems in this paper are all stated in terms of $K_{\mathfrak{D}}(M)$, the skein module specialized at a unit ζ in an integral domain $\mathfrak{D}$.

We are most interested in the case when ζ is a primitive n^{th} root of unity, where $n \neq 0 \bmod 4$. The integral domain is $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$. We need 2 to be a unit so that a collection of skeins that are adapted to the computation of the trace will be a basis.

2.3 Threading

The Chebyshev polynomials of the first type T_k are defined recursively by

- $T_0(x) = 2$,
- $T_1(x) = x$, and
- $T_{n+1}(x) = T_1(x) \cdot T_n(x) - T_{n-1}(x)$.

They satisfy some nice properties.

Proposition 2.1 *For $m, n > 0$, $T_m(T_n(x)) = T_{mn}(x)$. Furthermore, for all $m, n \geq 0$, $T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$.*

For a proof see [8].

We denote the oriented surface of genus g with p punctures by $\Sigma_{g,p}$. It is easy to see that $K_{\mathfrak{D}}(\Sigma_{0,2})$ is isomorphic to $\mathfrak{D}[x]$, where x is the framed link coming from the blackboard framing of the core of the annulus. Hence $1, x, x^2, \dots, x^n, \dots$ is a basis for $K_{\mathfrak{D}}(\Sigma_{0,2})$. Since $T_0(x) = 2$, in order to use the $T_k(x)$ as a basis for $\mathfrak{D}$, 2 must be a unit in $\mathfrak{D}$. If $2 \in \mathfrak{D}$ is a unit then $\{T_k(x) \mid k \in \mathbb{Z}_{\geq 0}\}$ is a basis for $K_{\mathfrak{D}}(\Sigma_{0,2})$.

If the components of the primitive diagram on a finite-type surface F are the simple closed curves S_i and $k_i \in \mathbb{Z}_{\geq 0}$ has been chosen for each component, the result of threading each of the curves S_i with the T_{k_i} is $\prod_i T_{k_i}(S_i)$. Since the S_i are disjoint from one another, they commute, so order doesn't matter in the product. For any compact or finite-type surface F, the primitive diagrams on F up to isotopy, with their components threaded with all possible choices of Chebyshev polynomials, form a basis for $K_{\mathfrak{D}}(F)$ so long as $2 \in \mathfrak{D}$ is a unit. This basis is becoming more commonly used in the study of skein algebras [9; 17; 13].

The following theorem of Bonahon and Wong is the starting point for this investigation. The convention for defining $K_\zeta(M)$ means that it is a module over $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$. This means that $K_\epsilon(M)$ is a module over $\mathbb{Z}\left[\frac{1}{2}\right]$. For the sake of the following theorem, after choosing an m^{th} root of unity ζ we interpret $K_\epsilon(M)$ to have its coefficients extended to include $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$; that way we don't have to mess around with extending coefficients in the range. More formally, let

(2-6) $$\overline{K}_\epsilon(M) = K_\epsilon(M) \otimes_{\mathbb{Z}\left[\frac{1}{2}\right]} \mathbb{Z}\left[\tfrac{1}{2}, \zeta\right]$$

Theorem 2.2 (Bonahon and Wong [2; 13]) *If M is a compact oriented three-manifold and we specialize at an n^{th} root of unity ζ such that $n \neq 0 \bmod 4$, there is a $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$–linear map*

$$\mathrm{Ch}\colon \overline{K}_\epsilon(M) \to K_\zeta(M)$$

given by threading framed links with T_m, where $m = n/\gcd(n, 4)$ is the index of threading. Any framed link in the image of Ch *is central in the sense that if $L' \cup K$ differs from $L \cup K$ by a crossing change of L and L' with K, then $T_m(L) \cup K = T_m(L') \cup K$. In the case that $M = F \times [0, 1]$, the map*

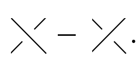
$$\mathrm{Ch}\colon \overline{K}_\epsilon(F) \to K_\zeta(F)$$

is an injective homomorphism of algebras such that the image of Ch *lies in the center of $K_\zeta(F)$.*

The skein module $\overline{K}_\epsilon(M)$ is a ring under disjoint union. At $A = \pm 1$, the Kauffman bracket skein relation

(2-7) 

can be rotated 90 degrees and then subtracted from itself to yield

(2-8) 

This means that, in $K_{\pm 1}(M)$, changing crossings does not change the skein. To take the product of two equivalence classes of framed links, choose representatives that are disjoint from one another and take their union. The product is independent of the representatives chosen, since the results differ by isotopy and changing crossings. The product can be extended distributively to give a product on $K_{\pm 1}(M)$. Let $\sqrt{0}$ denote the nilradical of $K_{-1}(M)$. It is a theorem of Bullock [3], proved independently in [15], that, for any oriented compact 3–manifold, $K_{-1}(M)/\sqrt{0}$ is canonically isomorphic to the coordinate ring of the $\mathrm{SL}_2\mathbb{C}$–character variety of the fundamental group of M. In the case that $M = F \times [0,1]$, the disjoint union product coincides with the stacking product, as stacking is one way to perturb the components of the two links so that they are disjoint. For any oriented finite-type surface F, the ring $\chi(F)$ has basis the isotopy classes of primitive diagrams threaded with $T_{k_i m}$ for all choices $k_i \in \mathbb{Z}_{\geq 0}$.

In Sections 2.4–2.6, some algebraic background is presented that will be applied to $K_\zeta(F)$ in Section 4.

2.4 Specializing at a place

A place of $\chi(F)$ is a homomorphism $\phi\colon \chi(F) \to \mathbb{C}$. The places correspond to evaluation at a point on the character variety. A place defines a $\chi(F)$–module structure on $\mathbb{C}$, where the action

(2-9) $$\chi(F) \otimes \mathbb{C} \to \mathbb{C}$$

is defined by letting $s \in \chi(F)$ act as multiplication by $\phi(s)$ on $\mathbb{C}$. We define the *specialization* of $K_\zeta(F)$ at ϕ to be

(2-10) $$K_\zeta(F)_\phi = K_\zeta(F) \otimes_{\chi(F)} \mathbb{C}.$$

The specialization at a place is an algebra over the complex numbers.

2.5 Localization

Let $R \subset J$ be a ring extension, where R is an integral domain, J is an associative ring with unit and R is a subring of the center of J. Since R has no zero divisors, $S = R - \{0\}$ is multiplicatively closed. Start with the set of ordered pairs $J \times S$, and place an equivalence relation on $J \times S$ by saying (a,s) is equivalent to (b,t) if $at = bs$. Denote the equivalence class of (a,s) under this relation by $[a,s]$. The set of equivalence classes is denoted by $S^{-1}J$, and called the localization of J with respect to S. Denote the set of equivalence classes $[a,s]$ for $a \in R$ by $S^{-1}R$. Define multiplication of equivalence classes by $[a,s][b,t] = [ab,st]$ and addition by $[a,s]+[b,t] = [at+bs,st]$. Under these operations, $S^{-1}R$ is a field, and $S^{-1}J$ is an algebra over that field.

In this paper, J is a subalgebra of $K_\zeta(F)$ and R is $\chi(F)$. This means that $S^{-1}R$ is the function field of the character variety associated to $K_\epsilon(F)$.

2.6 Trace and extension of scalars

Let f be a field and suppose that V is a finite-dimensional vector space over f. If $L \in \mathrm{End}_f(V)$, we use $\mathrm{tr}(L)$ to denote the unnormalized trace of L. The linear map L can be represented with respect to a basis $\{v_j\}$ by a matrix (l_i^j). The trace of L is given by

$$\mathrm{tr}(L) = \sum_i l_i^i. \tag{2-11}$$

If W is also a finite-dimensional vector space over f and $M\colon W \to W$ is an f–linear map, then

$$\mathrm{tr}(L \otimes_f M) = \mathrm{tr}(L)\,\mathrm{tr}(M) \quad \text{and} \quad \mathrm{tr}(L \oplus M) = \mathrm{tr}(L) + \mathrm{tr}(M). \tag{2-12}$$

Suppose that $f \leq a$ is a field extension and V is a vector space of dimension n over f; then

$$V \otimes_f a \tag{2-13}$$

is a vector space of dimension n over a. In fact, if $\{v_j\}$ is a basis for V then $\{v_j \otimes 1\}$ is a basis for $V \otimes_f a$ over a.

Under extension of scalars, $L\colon V \to V$ gets sent to $L \otimes_f 1_a$. The matrix of $L \otimes_f 1_a$ with respect to the basis $\{v_j \otimes 1\}$ is the same as the matrix of L with respect to $\{v_j\}$, so

$$\mathrm{tr}(L \otimes_f 1_a) = \mathrm{tr}(L), \tag{2-14}$$

where the trace on the left is taken as an a–linear map, and the trace on the right is taken as an f–linear map, and we are using $f \leq a$ to make the identification.

The next proposition gives the method by which we will be computing the trace.

Proposition 2.3 *Suppose that $K \leq P \leq J$, where K and P are fields, J is an algebra over K and J is a finite-dimensional vector space over k. Thus J is a finite-dimensional vector space over P, and P is a finite field extension of K. If $s \in P$, then it defines a K–linear maps $l_s\colon P \to P$, and $L_s\colon J \to J$ by left multiplication. If d is the dimension of J over P, then*

$$\mathrm{tr}(L_s) = d\,\mathrm{tr}(l_s), \tag{2-15}$$

where the traces are both taken as linear maps over K.

Proof Since $K \leq P$ is finite-dimensional it has basis $p_1, \dots p_n$ over K. Since J is a finite-dimensional vector space of dimension d over P it has basis $j_1, \dots j_d$ over P. This implies that $p_a j_c$ is a basis of J over K. Expressing l_s with respect to the basis p_a, we get

$$l_s(p_a) = \sum_b l_b^a p_b. \tag{2-16}$$

Since s acts as scalar multiplication on J,

$$L_s(p_a j_c) = \sum_b l_b^a p_b j_c. \tag{2-17}$$

Hence the matrix for L_s decomposes into d blocks that are all copies of the matrix for l_s. Therefore the trace of L_s is equal to d times the trace of l_s. □

2.7 Geometric intersection numbers

Suppose that X and Z are properly embedded 1–manifolds in the finite-type surface F, where X is compact. We say that X' is a transverse representative of X if X' is ambiently isotopic to X via a compactly supported isotopy and $X' \pitchfork Z$. Define the *geometric intersection number* of X and Z, denoted by $i(X,Z)$, to be the minimum cardinality of $X' \cap Z$ over all transverse representatives of X. We could have instead worked with Z up to compactly supported ambient isotopy and taken the minimum over all Z' isotopic to Z and transverse to X and gotten the same number, so $i(X,Z) = i(Z,X)$.

It is a theorem that a transverse representative of X realizes the geometric intersection number $i(X,Z)$ if and only if there are no *bigons*. A bigon is a disk D embedded in F so that the boundary of D consists of the union of two arcs $a \subset X$ and $b \subset Z$ [7]. If there is a bigon, there is always an innermost bigon, whose interior is disjoint from $X \cup Z$.

3 $K_{\mathfrak{D}}(F)$ is finitely generated

3.1 Parametrizing the simple diagrams

An ideal triangle is a triangle with its vertices removed. An ideal triangulation of a finite-type surface F consists of finitely many ideal triangles Δ_i with their edges identified pairwise, along with a homeomorphism from the resulting quotient space to F. Alternatively an ideal triangulation is defined by a family E of properly embedded lines that cuts F into finitely many ideal triangles. The surface F needs to have at least

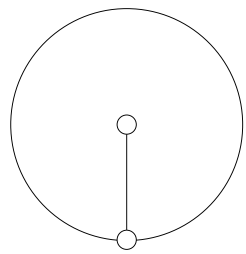


Figure 1: A folded triangle

one puncture, and negative Euler characteristic or it doesn't admit an ideal triangulation. If the Euler characteristic of the surface F is $-e(F)$ then any ideal triangulation of F consists of $2e(F)$ ideal triangles. The cardinality of a set of lines E defining an ideal triangulation is $3e(F)$.

If Δ is an ideal triangle in an ideal triangulation then $\partial\Delta = \{a, b, c\}$, where a, b and c are homeomorphic to $\mathbb{R}$. The lines a, b, and c are the *sides* of Δ. There is a map of Δ to the closure of a component D of the complement of E into F. If this map is an embedding, then Δ is an *embedded ideal triangle*. It could be that two sides c_1 and c_2 of the ideal triangle Δ get mapped to the same line c; in this case Δ is a *folded ideal triangle*. Figure 1 is a picture of a folded ideal triangle. There are two punctures in the picture, and the mapping is $2-1$ along the vertical line joining them. The edge that is covered twice by the mapping has *multiplicity* 2.

Let E denote a disjoint family of properly embedded lines that defines an ideal triangulation of F, and suppose the triangles are the set $\{\Delta_j\}$. An *admissible coloring* $f\colon E \to \mathbb{Z}_{\geq 0}$ is an assignment of a nonnegative integer $f(c)$ to each $c \in E$ such that the following conditions hold:

- If $\{a, b, c\}$ form the boundary of an embedded ideal triangle Δ_j then the sum $f(a) + f(b) + f(c)$ is even and the triple $\{f(a), f(b), f(c)\}$ satisfies the triangle inequality

$$f(a) \leq f(b) + f(c), \quad f(b) \leq f(a) + f(c) \quad \text{and} \quad f(c) \leq f(a) + f(b). \tag{3-1}$$

- If $\{a, b\}$ are the image of the boundary of a folded ideal triangle Δ_j, where b has multiplicity 2, we require that $f(a) + 2f(b)$ be even and $f(a) \leq 2f(b)$.

If $S \subset F$ is a simple diagram then $f_S\colon E \to \mathbb{Z}_{\geq 0}$ given by $f_S(c) = i(S, c)$ is an admissible coloring. Conversely, for each admissible coloring $f\colon E \to \mathbb{Z}_{\geq 0}$ there is an isotopy class of simple diagrams having geometric intersection numbers with the edges given by f. We denote a representative of this isotopy class by $[f]$. In particular, $[f_s] = S$. We use $\mathcal{A}$ to denote the set of all admissible colorings $f\colon E \to \mathbb{Z}_{\geq 0}$.

Proposition 3.1 *The admissible colorings of the edges of an ideal triangulation of F are in one-to-one correspondence with isotopy classes of simple diagrams on F.* □

A *pointed integral polyhedral cone* is a subset $\mathcal{A}$ of some $\mathbb{Z}^k$ that is defined by finitely many equations and inequalities with $\vec{0} \in \mathcal{A}$.

Proposition 3.2 *The admissible colorings of an ideal triangulation of F form a pointed integral polyhedral cone.*

Proof If E is the set of edges of the ideal triangulation then there is a map

$$\mathcal{A} = \{f\colon E \to \mathbb{Z}_{\geq 0} \mid f \text{ is admissible}\} \to \mathbb{Z}^E \tag{3-2}$$

that sends each f to its tuple of values. We still denote the image of this map by $\mathcal{A}$.

The only part of recognizing $\mathcal{A}$ as an integral cone that is tricky is the condition that the sum of colors over the sides of a triangle needs to be even. This can be avoided by using a linearly equivalent description of the admissible colorings via *corner numbers*. An ideal triangle has three corners, determined by a choice of two of the three sides. For instance, if a triangle has three sides a, b and c, then the corners correspond to $\{a,b\}$, $\{b,c\}$ and $\{a,c\}$. If $f\colon E \to \mathbb{Z}_{\geq 0}$ is an admissible coloring, the three corner numbers of this triangle are

$$\tfrac{1}{2}(f(a)+f(b)-f(c)), \quad \tfrac{1}{2}(f(b)+f(c)-f(a)), \quad \tfrac{1}{2}(f(a)+f(c)-f(b)). \tag{3-3}$$

It is easy to see that the corner numbers determine the admissible coloring and vice versa. An assignment of corner numbers corresponds to an admissible coloring of the edges if and only if all the corner numbers are nonnegative and if, for each edge, the sum of the two corner numbers on one side of the edge is equal to the sum of the corner numbers on the other side of that edge. The description in terms of corner numbers allows us to conclude that the admissible colorings are a pointed integral cone. □

An *integral basis* of a pointed integral polyhedral cone is a subset of the cone that has minimal cardinality among all subsets that span the cone additively. It is a classical result [11] that any pointed integral polyhedral cone admits a finite integral basis. The integral basis is unique. If P is a pointed integral polyhedral cone, $p \in P$ is *indivisible* if $s = 0$ or $p = 0$ whenever $s, t \in P$ and $s + t = p$. The set of indivisible elements of P is the integral basis [18]. In the case of the cone of admissible colorings, the diagrams corresponding to indivisible colorings are simple closed curves.

Forgetting positivity, and the triangle inequality, the admissible colorings generate a free module over $\mathbb{Z}$. It makes sense to ask whether a collection $f_{S_i}\colon E \to \mathbb{Z}_{\geq 0}$ are linearly independent. Oddly, the integral basis need not be linearly independent.

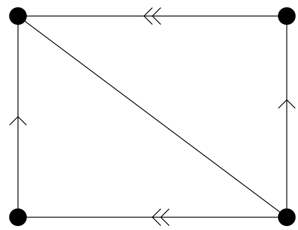


Figure 2: An ideal triangulation of $\Sigma_{1,1}$

Remark 3.3 Decompose the punctured torus $\Sigma_{1,1}$ into two ideal triangles. This requires three edges, which form the boundary of both triangles. In the diagram below we identify the left- and right-hand sides of the rectangle, and the top and bottom of the rectangle with the vertices deleted to obtain a once-punctured torus. The lines defining the triangulation come from the sides of the rectangle and the diagonal, as shown in Figure 2.

The admissible colorings can be seen as triples of counting numbers (m, n, p) whose sum is even that satisfy the triangle inequality. The nonzero indecomposable admissible colorings are $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$. This set is an integral basis. Notice that if (a,b,c) is an admissible coloring and one of the triangle inequalities is strict, say $a < b+c$, we can subtract the corresponding indecomposable $(0,1,1)$ to get a triple $(a, b-1, c-1)$ that still satisfies the triangle inequality and the sum of the colors $a+b+c-2 < a+b+c$. If all three triangle inequalities are equalities $a = b+c$, $b = a+c$ and $c = a+b$, then $(a,b,c) = (0,0,0)$. The three curves corresponding to $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are the generators that Bullock and Przytycki [6] obtained for $K(\Sigma_{1,1})$. There are infinitely many ideal triangulations of $\Sigma_{1,1}$ but Euler characteristic forces them all to be two triangles that share all their edges. The argument above goes through, even though the curves on the torus will be different. Since the integral basis is unique, any set of skeins that generates $K_{\mathfrak{D}}(\Sigma_{1,1})$ must have at least three elements.

Choose an ordering of E. Use this to order $\mathbb{Z}_{\geq 0}^E$ lexicographically. Notice that $\mathbb{Z}_{\geq 0}^E$ in the lexicographic ordering is a well-ordered monoid. By that we mean $\mathbb{Z}_{\geq 0}^E$ is well-ordered and, if $a, b \in \mathbb{Z}_{\geq 0}$ have $a < b$, then $a+c < b+c$ for any $c \in \mathbb{Z}_{\geq 0}$. Since $\mathcal{A}$ is a submonoid of $\mathbb{Z}_{\geq 0}^E$, we have that $\mathcal{A}$ is a well-ordered monoid.

If $\alpha \in K_\zeta(F)$ then we can write α as a finite linear combination of simple diagrams with complex coefficients,

$$\alpha = \sum_S z_S S, \tag{3-4}$$

where the S are simple diagrams and the z_S are nonzero elements of $\mathfrak{D}$. The *lead term* of α is $z_S S$, where S is the largest diagram appearing in the sum. We denote the lead term of the skein α as $\mathrm{ld}(\alpha)$.

3.2 The algebra $K_{\mathfrak{D}}(F)$ is finitely generated over $\mathfrak{D}$

If f_S and $f_{S'}$ are admissible colorings, choose simple diagrams S and S' that realize the colorings as the cardinality of their intersections with the $c_i \in E$ and such that S and S' realize their geometric intersection number and $S \cap S'$ is disjoint from all c_i. Up to isotopy there is a unique simple diagram whose associated coloring is $f_S + f_{S'}$, called the *geometric sum* of S and S'. Since addition of admissible colorings is associative, so is the geometric sum. It is worth noting, the geometric sum of two diagrams depends on the choice of ideal triangulation.

Suppose that S and S' transversely represent $i(S, S')$. Furthermore assume that $S \cap S' \cap E = \varnothing$. If there are n points of intersection in $S \cap S'$, there are 2^n ways of smoothing all the crossings of S and S' to get a system of simple closed curves. We call a system of simple closed curves obtained by smoothing all crossings s a *state*. A state might not be a simple diagram as it may contain some trivial simple closed curves. There is a process for writing the product $S * S'$ as a linear combination of simple diagrams. First expand the product as a sum of states using the Kauffman bracket skein relation for crossings, then delete the trivial components of each state, and for each trivial component deleted from a state multiply the coefficient of the state by $-\zeta^2 - \zeta^{-2}$. Order the crossings of $S * S'$. Based on the ordering there is a rooted tree, where the root is the diagram $S * S'$, the vertices are partial smoothings (resolvents) of the diagram, and the directed edges correspond to smoothing the crossings in order. The states are the leaves of this tree. If the shortest path from the root to a state s passes through a resolvent r, we say that s is a *descendent* of r.

Theorem 3.4 *Let S and S' be simple diagrams associated to admissible colorings $f_S, f_{S'}\colon E \to \mathbb{Z}_{\geq 0}$. Assume the product $S * S' \in K_{\mathfrak{D}}(F)$ has been written as $\sum_D z_D D$, where the D are simple diagrams that are distinct up to isotopy and the $z_D \in \mathfrak{D}$ are nonzero. There exists a unique simple diagram E in this sum, so that $f_E = f_S + f'_S$, and all the other simple diagrams appearing with nonzero coefficient in the sum are strictly smaller in the well ordering of diagrams. Furthermore, the coefficient z_E is a power of ζ.*

Lemma 3.5 *Let G be a four-valent graph with at least one vertex, embedded in a disk D^2. Assume that G is the union of two families of properly embedded arcs $A_1 \cup A_2$ and that there are three special points p, q and r in ∂D^2 such that*

- *the endpoints of the A_1 and A_2 are disjoint from one another and $\{p,q,r\}$ in ∂D^2,*
- *if $a_1 \in A_1$ and $a_2 \in A_2$, then a_1 and a_2 intersect transversely, and realize their geometric intersection number relative to their boundaries,*
- *if $a, b \in A_i$ then $a \cap b = \varnothing$, and*
- *for any arc $a \in A_1 \cup A_2$, the endpoints of a are separated by $\{p,q,r\}$,*

If $A_1 \cap A_2$ is nonempty, then there is an embedded triangle Δ whose sides consist of an arc of ∂D^2 that is disjoint from $\{p,q,r\}$, an arc contained in some $a \in A_1$ that only intersects A_2 in a single point which is one of its endpoints, and an arc in some $b \in A_2$ that only intersects A_1 in a single point which is one of its endpoints.

(We call this an *outermost triangle*.)

Proof The graph dissects the disk into vertices, edges and faces. The alternating sum of the numbers of vertices, edges and faces is 1, as that is the Euler characteristic of the disk. A face f has two kinds of sides, sides in ∂D^2 and sides in the interior of D^2. Let $e_\partial(f)$ denote the number of sides of f lying in ∂D^2 and $e_i(f)$ the number of sides of f in the interior. Similarly, let $v_\partial(f)$ be the number of vertices of the face that lie in ∂D^2, and $v_i(f)$ be the number of vertices of f that lie in the interior of D^2. The contribution of the face f to the Euler characteristic of the disk is

$$c(f) = 1 - \tfrac{1}{2}e_i(f) - e_\partial(f) + \tfrac{1}{4}v_i(f) + \tfrac{1}{2}v_{\partial(f)}. \tag{3-5}$$

We have that $\sum_f c(f) = 1$. The faces that are contained in the interior of the disk have an even number of sides, as their edges are partitioned into arcs of A_1 and arcs of A_2. Since the arcs of A_1 and A_2 realize their geometric intersection number, the interior faces have at least four sides. Hence the largest contribution of an interior face is 0. A face touching the boundary can have two sides, but these faces are cut off by a single component of A_1 or A_2, and contain a point of $\{p,q,r\}$ in their boundary face by the last condition. They contribute $\frac{1}{2}$ to the Euler characteristic of the disk. However, the edges involved in these pieces can be removed from the families A_1 and A_2 and the remaining curves still satisfy the hypotheses, so do this. Now, the only faces that contribute positively to the Euler characteristic of the disk are triangles with one edge on the boundary. These contribute $\frac{1}{4}$ to the Euler characteristic. There must be at least 4 such triangles. That means if the intersection of A_1 and A_2 is nonempty, then one of those triangles does not contain a point from $\{p,q,r\}$, so it is an outermost triangle. □

Proof of Theorem 3.4 Let S and S' be two simple diagrams, with associated colorings $f_S, f_{S'}\colon E \to \mathbb{Z}_{\geq 0}$, where E is the system of proper lines defining an ideal

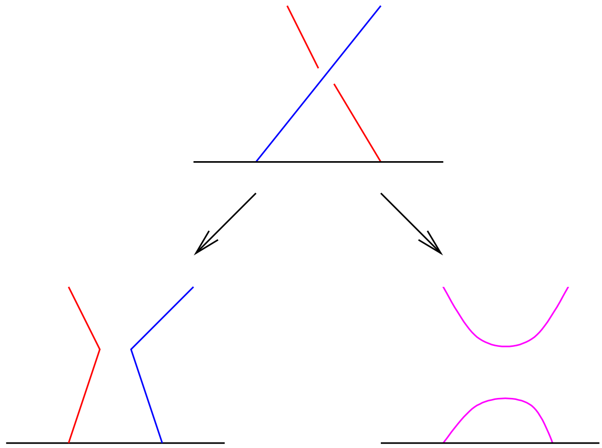


Figure 3: Resolving at an outermost triangle

triangulation with ideal triangles Δ_j. We do not need to distinguish between embedded and folded triangles for this proof, because the combinatorial lemma above is applied in the completed components of the complement of E. Isotope S and S' so that they are transverse to one another, and the lines in E, and realize all geometric intersection numbers $i(S,S')$, $i(S,c)$ and $i(S',c)$ for $c \in E$. Also make sure that $S \cap S' \cap E = \varnothing$. We resolve $S * S'$ one ideal triangle at a time. The four-valent graph $(S \cup S') \cap \Delta_j$ for each Δ_j satisfies the hypotheses of the lemma. To start with, A_1 is made up of the components of $S \cap \Delta_j$ and A_2 is made up of the components of $S' \cap \Delta_j$. Therefore we can find an outermost triangle $\Delta \subset \Delta_j$. If we resolve the crossing of $S * S'$ at the apex of the triangle there are two resolvents. One resolvent forms a bigon with the edge of the triangle, and hence any simple diagram descendent from this resolvent is strictly smaller in the ordering of diagrams than $[f_S + f_{S'}]$. This is shown in Figure 3

The other resolvent doesn't have a bigon. Any state resulting in a simple diagram whose coloring is $f_S + f_{S'}$ is a descendent of this resolvent. The triangle Δ has a face $p \subset S$ and a face $q \subset S'$. Assume that p lies in the component a of the family A_1 and q lies in the component b of A_2. We smooth by forming arcs $a - p \cup q$ and $b - q \cup p$ and then perturb them slightly so that they are disjoint. To continue on inductively, we declare that the perturbed version of $a - p \cup q$ is in A_1, whilst removing a, and the perturbation of $b - q \cup p$ is in A_2 and discard b. This operation does not produce any components of A_1 or A_2 that are simple closed curves inside the triangle Δ, because every component of the new families A_1 and A_2 still have two endpoints. Notice that the assignment of A_1 and A_2 is now just local to the ideal triangle instead of corresponding to the diagrams S and S'. However, we work ideal triangle by ideal triangle, so this isn't a problem. If the new graph has a crossing, it still satisfies the hypotheses of the lemma, so we can continue resolving crossings at the apex of an outermost triangle. There is a unique resolvent that can have a descendent with coloring $f_S + f_{S'}$. Continue until there are no crossings in Δ_j. There is a single resolvent with no bigons in Δ_j, so all the crossings in Δ_j have been resolved. All the other resolvents with no crossings in Δ_j have bigons in Δ_j and will lead to simple diagrams that are strictly smaller in the ordering of diagrams. Do this for each triangle. In the end, there is a single state with no bigons. The state must be a simple diagram. The construction did not produce any simple closed curves contained in a triangle. A simple closed curve that bounds a disk and has nonempty intersection with the edges of the triangulation must have a bigon, since a proper arc in a disk always separates the disk into two subdisks. Since there are no bigons between E and the edges of the triangulation, the admissible coloring associated to E is $f_S + f_{S'}$, so E is the geometric sum of S and S'. The coefficient of E is $\zeta^{p(E)-n(E)}$, where $p(E)$ is the number of positive smoothings and $n(E)$ is the number of negative smoothings that gave rise to the state E. The rest of the expansion is a linear combination of simple diagrams that are strictly smaller. □

Remark 3.6 A collection of skeins $\beta \in B$ spans $K_{\mathfrak{D}}(F)$ over $\mathfrak{D}$ if and only if the lead terms of the elements in β consist of units in $\mathfrak{D}$ times simple diagrams, and each isotopy class of simple diagrams appears at least once in the lead term of some $\beta \in B$.

Theorem 3.7 *Suppose that $\mathfrak{D}$ is an integral domain and $\zeta \in \mathfrak{D}$ is a unit and $2 \in \mathfrak{D}$ is a unit. Let S_i be a family of simple diagrams corresponding to the integral basis of the admissible colorings of an ideal triangulation. The skeins $\left\{\prod_i T_{k_i}(S_i)\right\}$, where the k_i range over all nonnegative integers, spans $K_{\mathfrak{D}}(F)$ over $\mathfrak{D}$.*

Proof The lead term of $T_{k_1}(S_1) * T_{k_2}(S_2) * \cdots * T_{k_n}(S_n)$ is a power of ζ times a simple diagram corresponding to the admissible coloring $\sum_i k_i f_{S_i}$, where f_{S_i} is the admissible coloring corresponding to S_i. Since the lead terms of these skeins correspond to all simple diagrams, we can inductively rewrite any skein as a linear combination of these by starting at the terms of highest weight. □

This extends a theorem of Bullock [4]. In that paper it is proved that the arbitrary products of a finite collection of curves S_i spans. Our theorem is stronger because we can specify the order of the product of the S_i, as no matter what order we work in, the leading terms are the same, though maybe with different powers of ζ as the lead coefficient. It could be that the integral basis of the space of admissible colorings is not linearly independent over $\mathbb{Z}$, so we don't have that the products form a basis.

3.3 The case when ζ is a primitive n^{th} root of unity

Now we go on to study $K_\zeta(F)$, meaning the coefficients are $\mathbb{Z}\left[\frac{1}{2},\zeta\right]$, where ζ is a primitive n^{th} root of unity, $n \neq 0 \bmod 4$, and A is set equal to ζ. Recall $\chi(F)$ is the image of the threading map

$$\mathrm{Ch}\colon \overline{K}_\epsilon(F) \to K_\zeta(F). \tag{3-6}$$

Recall the overline is to indicate that we have extended the scalars of $K_\epsilon(F)$ to include $\frac{1}{2}$ and ζ. The map Ch threads every component of a framed link corresponding to a simple diagram with $T_m(x)$, where $m = n/\gcd(n,4)$, the index of threading. Since

$$T_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{k}{k-i} \binom{k-i}{i} x^{k-2i}, \tag{3-7}$$

the lead term of $\mathrm{Ch}(S)$ where the simple diagram has admissible coloring $f_S\colon E \to \mathbb{Z}_{\geq 0}$ of weight i is $[mf_S]$.

Let S be a simple diagram with associated coloring $f_S\colon E \to \mathbb{Z}_{\geq 0}$. Assume that f_S is not identically zero. The integers $\{f_S(c)\}_{c \in C}$ generate a subgroup of $\mathbb{Z}$, which, being cyclic, has a smallest positive generator, denoted by $\gcd(f_S)$.

Proposition 3.8 *If $n > 0$ is odd and $n \mid \gcd(f_S)$ then $\frac{1}{n} f_S\colon E \to \mathbb{Z}_{\geq 0}$ is an admissible coloring with associated simple diagram S' and $(S')^n = S \in K_\zeta(F)$.*

Proof Since F is orientable, the diagram S' is two-sided, meaning that we can push it completely off of itself to take the product. This means that the admissible coloring of $(S')^n$ is $nf_{S'}\colon E \to \mathbb{Z}_{\geq 0}$. If $f_S\colon E \to \mathbb{Z}_{\geq 0}$ is an admissible coloring, and for all $c \in C$, the odd integer $n \mid f(c)$, then, for any $\{a,b,c\} = \partial\Delta$ of an embedded ideal triangle in the triangulation, $\left\{\frac{1}{n} f_S(a), \frac{1}{n} f_S(b), \frac{1}{n} f_S(c)\right\}$ satisfy all three triangle inequalities as the triangle inequality is linear. The sum $\frac{1}{n}(f_S(a) + f_S(b) + f_S(c))$ is even, as an even number divided by an odd number is even. Similarly, $\frac{1}{n} f_S\colon E \to \mathbb{Z}_{\geq 0}$ satisfies the conditions to be admissible for folded triangles. □

The restriction to $n \neq 0 \bmod 4$ means that m is always odd, so Proposition 3.8 applies. If S is a simple diagram associated to the admissible coloring

$$f_S\colon E \to \mathbb{Z}_{\geq 0} \tag{3-8}$$

then, as noted above, the lead term of $\mathrm{Ch}(S)$ is $[mf_S]$, and m divides $\gcd(mf_S)$.

Theorem 3.9 *Let F be a finite-type surface with at least one puncture, that has been ideally triangulated. If S_i is any system of simple diagrams corresponding to an integral basis of the cone of admissible colorings of the triangulation, then the skeins $\prod_i T_{k_i}(S_i)$, where the $k_i \in \{0, 1, \dots, m-1\}$, span $K_\zeta(F)$ over $\chi(F)$. In particular, $K_\zeta(F)$ is a finite ring extension of $\chi(F)$.*

Proof The proof is by induction on largest diagram appearing with nonzero coefficient in a skein. Start with a skein written in terms of the basis over $\mathbb{Z}\left[\frac{1}{2}, \zeta\right]$ of simple diagrams,

$$\sum_j \alpha_j \prod_i T_{k_{i,j}}(S_{i,j}), \tag{3-9}$$

with $\alpha_j \in \mathbb{Z}\left[\frac{1}{2}, \zeta\right]$. Suppose the lead term of the skein is indexed by j. Since

$$T_{m+k}(x) = T_m(x) * T_k(x) - T_{|m-k|}(x) \tag{3-10}$$

if some $k_{i,j} \geq m$, then, as $\chi(F)$ is central, we can factor out an element of $\chi(F)$ from the term to get a simple diagram of lower weight. Continue on inductively till the skein is written as

$$\sum_j \beta_j \prod_i T_{k_{i,j}}(S_{i,j}), \tag{3-11}$$

where all $k_{i,j} \in \{0, 1, \dots, m-1\}$ and $\beta_j \in \chi(F)$. □

Theorem 3.10 *If F is closed, and ζ is a primitive n^{th} root of unity, where $n \neq 0 \bmod 4$, then $K_\zeta(F)$ is a finite-rank module over $\chi(F)$.*

Proof If F is closed and $p \in F$, then the inclusions $K_\zeta(F - \{p\}) \to K_\zeta(F)$ and $\chi(F - \{p\}) \to \chi(F)$ are surjective homomorphisms that fit into a commutative square

$$\begin{array}{ccc} K_\zeta(F-\{p\}) & \longrightarrow & K_\zeta(F) \\ \uparrow & & \uparrow \\ \chi(F-\{p\}) & \longrightarrow & \chi(F) \end{array} \tag{3-12}$$

After choosing an ideal triangulation for $F - \{p\}$, if the admissible colorings associated with S_i form an integral basis then $T_{k_1}(S_1) * \cdots * T_{k_n}(S_n)$, where the $k_i \in \{0, \dots, m-1\}$, span $K_\zeta(F)$ over $\chi(F)$. □

In [8] we prove that $K_\zeta(\Sigma_{1,0})$ is not free over $\chi(\Sigma_{1,0})$, so there are definitely linear dependencies between the elements of the spanning set produced this way.

Theorem 3.11 *For every $\phi\colon \chi(F)\to\mathbb{C}$, $K_\zeta(F)_\phi$ is a finite-dimensional algebra over the complex numbers.*

Proof This follows from the definition of specialization. □

Let F be a finite-type surface of negative Euler characteristic and at least one puncture. Let E be the edges of an ideal triangulation of F. Recall that $\mathcal{A}$ denotes the admissible colorings of E. After ordering the set E, you can view $\mathcal{A}\subset\mathbb{Z}^E_{\geq 0}$. This allows us to define a map

$$\text{res}\colon \mathcal{A}\to\mathbb{Z}^E_m, \tag{3-13}$$

by sending each admissible coloring to the tuple of residues of its values modulo m. This is used to define

$$\text{res}\colon K_\zeta(F)-\{0\}\to\mathbb{Z}^E_m. \tag{3-14}$$

Every nonzero skein α can be written $\sum_S z_S S$ where the z_S are nonzero complex numbers and the S are simple diagrams, and the sum is nonempty. Let $f_S\colon E\to\mathbb{Z}$ be the admissible coloring of the diagram appearing in the lead term of α. Define $\text{res}(\alpha)$ to be $\text{res}(f_S)$.

Lemma 3.12 *If $\alpha,\beta\in K_\zeta(F)-\{0\}$ then $\text{res}(\alpha*\beta)=\text{res}(\alpha)+\text{res}(\beta)$.*

Proof Suppose that the lead term of α is $z_S S$ and the lead term of β is $w_T T$. If the admissible colorings corresponding to S and T are f_S and f_T, then, by Theorem 3.4, the diagram underlying the lead term of $\alpha*\beta$ has coloring f_S+f_T. Hence $\text{res}(\alpha*\beta)=\text{res}(f_S+f_T)=\text{res}(f_S)+\text{res}(f_T)$. □

Theorem 3.13 *Suppose that $\{\alpha_i\}$ is a collection of nonzero skeins and the restriction of $\text{res}\colon K_\zeta(F)-\{0\}\to\mathbb{Z}^E_m$ to $\{\alpha_i\}$ is one-to-one. The collection of skeins $\{\alpha_i\}$ is linearly independent over $\chi(F)$.*

Proof Suppose that $\sum_i \beta_i\alpha_i=0$ with the $\beta_i\in\chi(F)$. This means that the lead term of $\sum_i\beta_i\alpha_i$ is equal to zero. However if $\beta_i\in\chi(F)-\{0\}$ the coloring corresponding to its lead term is divisible by m by Proposition 3.8. Hence $\text{res}(\beta_i*\alpha_i)=\text{res}(\beta_i)+\text{res}(\alpha_i)=\text{res}(\alpha_i)$. Since the $\text{res}(\alpha_i)$ are all distinct, there can be no cancellation among the leading terms of the sum, and in fact all the $\beta_i=0$. □

4 Computing the trace

Recall if $\alpha\in K_\zeta(F)$, there is a $S^{-1}\chi(F)$–linear map,

$$L_\alpha\colon S^{-1}K_\zeta(F)\to S^{-1}K_\zeta(F), \tag{4-1}$$

given by left multiplication. If d is the dimension of $S^{-1}K_\zeta(F)$ as a vector space over $S^{-1}\overline{K}_\epsilon(F)$ then, by definition, the normalized trace of L_α is $\frac{1}{d}\,\mathrm{tr}(L_\alpha)$, where tr denotes the standard trace. The goal of this section is to compute $\mathrm{Tr}(L_\alpha)$. The correct basis for $K_\zeta(F)$ over $\mathcal{D}$ is the primitive diagrams P whose components S_i have been threaded by $T_{k_i}(x)$. If α has been written as a linear combination of such P, then the normalized trace of L_α is the result of crossing out all terms where some component of the diagram has been threaded with a k_i that is not divisible by m.

Since the trace is linear, we only need to compute the trace of skeins of the form $\prod_i T_{k_i}(S_i)$, where the $\{S_i\}$ for $i = 1, \dots, n$ are the components of a primitive diagram. The strategy is to prove that the subalgebra of $K_\zeta(F)$ obtained by adjoining the curves S_i to $\chi(F)$, denoted by $\chi(F)[S_1, \dots, S_n]$, is isomorphic to the tensor product of n copies of the skein algebra of the annulus $\Sigma_{0,2}$ with its coefficients extended to $\chi(F)$. The trace of the tensor product of linear endomorphisms is the product of the traces of the endomorphisms. Therefore the trace of $\prod_i T_{k_i}(S_i)$ is the product of the traces of its individual factors. Once we localize at $S = \chi(F) - \{0\}$, the subalgebra $S^{-1}\chi(F)[S_1, \dots, S_n]$ is a commutative ring that is a finite extension of the field $S^{-1}\chi(F)$, having no zero divisors. This means that $S^{-1}\chi(F)[S_1, \dots, S_n]$ is a field. Hence $S^{-1}K_\zeta(F)$ is a finite-dimensional vector space over $S^{-1}\chi(F)[S_1, \dots, S_n]$, which in turn is a finite-dimensional vector space over $S^{-1}\chi(F)$. This is exactly the computational setting for Proposition 2.3.

Since m is not necessarily prime, $\mathbb{Z}_m$ might have zero divisors. Hence, linear independence in a module over $\mathbb{Z}_m$ is subtle. Since $\mathbb{Z}_m^E$ is a free module over $\mathbb{Z}_m$, there are linearly independent subsets. Let $\vec{e}_c \in \mathbb{Z}_m^E$ be the vector whose entries are all 0 except for a 1 in the c^{th} entry. The vectors $\vec{e}_c$ form a basis for $\mathbb{Z}_m$.

Lemma 4.1 *Choose an ordering of E. Let $V = \{\vec{v}_i\} \in \mathbb{Z}_m^E$ be a collection of vectors indexed by an initial segment of the counting numbers. There is a map $I\colon V \to E$ the sends each $\vec{v}_i$ to the index of its first nonzero entry. If I is increasing and the first nonzero entry of each $\vec{v}_i$ is a unit in $\mathbb{Z}_m$, then V is linearly independent over $\mathbb{Z}_m$.*

Proof Adjoin those $\vec{e}_c$ to V that don't appear as a first nonzero entry. The determinant of the $n \times n$ matrix you get this way is a unit. Therefore it is a basis of $\mathbb{Z}_m^E$. The original $\{\vec{v}_i\}$ is independent as any subset of a basis is independent. □

Proposition 4.2 *Suppose that S_i is an ordered collection of disjoint simple closed curves on the finite-type surface F. Suppose further that there is an ideal triangulation of F with ordered set of edges E such that the map $I\colon \{S_i\} \to E$ that sends each curve S_i to the smallest edge in E that it has nonzero geometric intersection number*

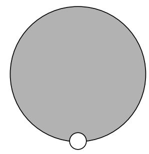


Figure 4: A monogon

with is increasing, and the geometric intersection number of S_i *with* $I(S_i)$ *is always* 1 *or* 2*. The set of skeins* $\left\{\prod_i T_{k_i}(S_i)\right\}$*, where the* k_i *range from* 0 *to* $m-1$*, are linearly independent in* $K_\zeta(F)$*.*

Proof By Lemma 4.1, the vectors $\{\mathrm{res}(S_i)\}$ are linearly independent in $\mathbb{Z}_m^E$. This implies that the vectors $\left\{\sum_i k_i \,\mathrm{res}(S_i)\right\}$, where the k_i range over 0 to $m-1$, are all distinct. However, the residue of $\prod_i T_{k_i}(S_i)$ is equal to $\sum_i k_i \,\mathrm{res}(S_i)$, hence the residues of the $\left\{\prod_i T_{k_i}(S_i)\right\}$ are distinct. By Theorem 3.13, the set $\left\{\prod_i T_{k_i}(S_i)\right\}$ is independent in $K_\zeta(F)$. □

The next several paragraphs are to prove that if the $\{S_i\}$ are the components of simple diagram, then we can find a triangulation E such that the hypotheses of the last proposition hold true for a choice of orderings for $\{S_i\}$ and E.

Suppose that E is a properly embedded system of disjoint lines in the finite-type surface F. A *monogon* is a component of the complement of E that completes to a closed disk with a single point removed from its boundary. We show a monogon in Figure 4.

A *bigon* is a component of the complement of E that completes to a closed disk with two points removed from its boundary. We show a bigon in Figure 5.

Proposition 4.3 *Suppose that* E *is a properly embedded system of disjoint lines in the finite-type surface* F *whose complement has no monogons or bigons. There exists a collection* D *of properly embedded lines such that* $C \cup D$ *defines an ideal triangulation of* F*.* □

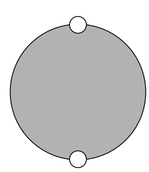


Figure 5: A bigon

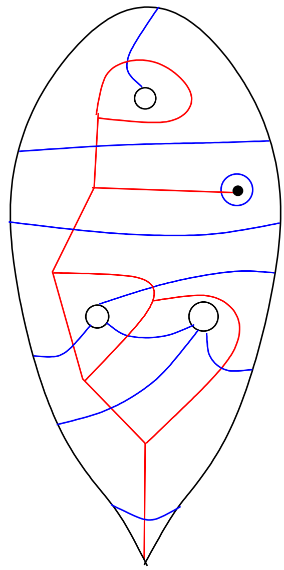


Figure 6: A filling diagram and its dual graph

Suppose that $P \subset F$ is a primitive diagram. We say that P *fills* F if the components of $F - P$ consist of once-punctured disks about the punctures of F and planar surfaces of Euler characteristic -1.

Theorem 4.4 *Suppose that F is a finite-type surface of negative Euler characteristic with at least one puncture and P fills F. There is an ordering of the disjoint curves S_i that make up P and a collection of disjoint embedded lines c_i such that if $i < j$ then $i(S_j, c_i) = 0$, and $i(S_i, c_i)$ is 1 or 2. Since no two of the c_i are parallel and all of the c_i are essential, the collection c_i can be built up to be an ideal triangulation of F.*

If P fills F, there is a dual 1–dimensional CW–complex, with a 0–cell for every component of the complement of P and a 1–cell for every component of P. The trivalent 0–cells of the CW–complex correspond to components of the complement that complete to pants. The monovalent 0–cells correspond to components of the complement that complete to a punctured disk. If a 1–cell has both its endpoints at the same 0–cell, the corresponding simple closed curve is a nonseparating curve lying in the closure of a component of the complement of P that is homeomorphic to $\Sigma_{1,1}$. The CW–complex minus its valence-one vertices can be properly embedded in the surface F, where each edge intersects the corresponding simple closed curve once in a transverse point of intersection and the trivalent vertices embedded in the

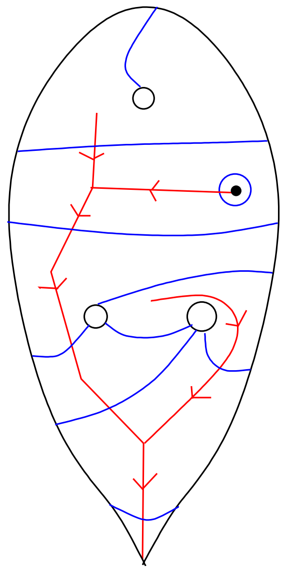


Figure 7: A maximal rooted tree

corresponding components of the complement of P, and the ends of the deleted CW–complex mapped to the ends of the corresponding disk with a point deleted. The edges of the CW–complex are in one-to-one correspondence with the components of P. If the edge e and the component S intersect one another, we say that they are *dual*. The intersection is necessarily a single point of transverse intersection.

In Figure 6 we show a twice-punctured surface of genus three. The filling diagram is in blue and the embedded dual graph is red.

Choose a maximal tree of the CW–complex and a valence-one 0–cell. Orient the tree so that it is rooted at the chosen 0–cell. That is, every edge is oriented so that it points towards the root. The monovalent 0–cells of the tree that are sources are the *leaves* of the tree. The rooted tree is in red.

We will build a train track from this tree as shown in Figure 7.

Figure 8 is color-coded so that each of the following steps is visible. First smooth the vertices of the tree so that the two edges pointing into each interior 0–cell have the same outward-pointing tangent vector. Next, for each component of the diagram that doesn't bound a punctured disk and is dual to an edge of the tree, push it off itself towards the root, and then put a kink in it where it intersects the edge dual to it and smooth the kink to get a switch where both outward normals of the curve at the kink point towards the root. These are in magenta. Next, add the remaining edges of the

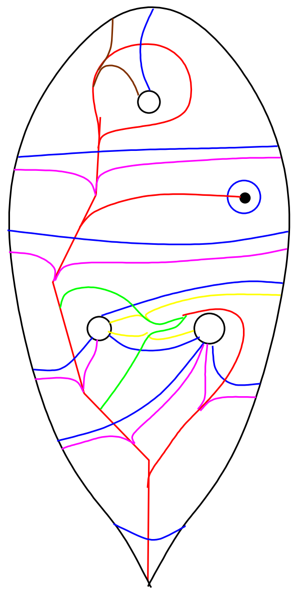


Figure 8: The train track

CW–complex, so that their outward normals, at the switches created, point towards the root. These are in green. If both endpoints of the edge are attached at the same 0–cell, that edge e lies in the closure of a component of the complement of P that is a torus. If S is the dual edge, push it off of itself and add a kink where it intersects e so that the outward tangent vectors point towards the vertex in the torus component. This is in brown. Suppose now that the 0–cells of the tree that e is attached at are distinct. For each one of those 0–cells that is a leaf, add a branch to the track, which is a pushoff of the dual component of P, with a kink in it that makes a switch in the train track pointing at that 0–cell. These are in yellow.

We produce a family of disjoint properly embedded lines by splitting the tree at the switches and cutting open all the way to the root. The switches in the tree point towards the roots, and the switches in the additional edges point towards the tree, so the process of cutting open along switches terminates at the root, and we have produced a family of disjoint properly embedded lines. The train track does not carry any simple closed curves.

Order the components of P so that S and T are dual to edges in the tree then their relative order is consistent with their distance from the root of the tree, and if they aren't dual to edges of the tree then they come after all the components that are dual to edges of the tree. Working in order we prove that, given a component S of P, there is

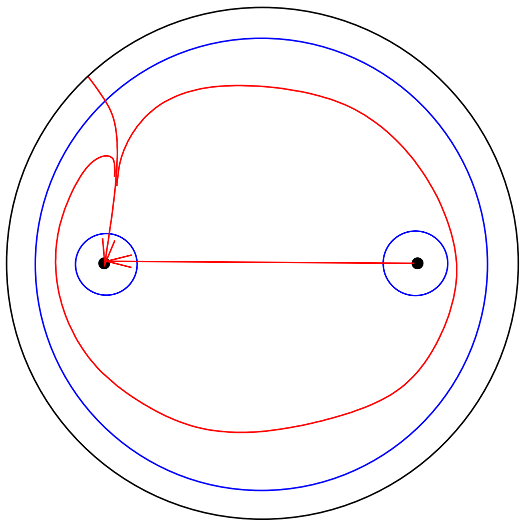


Figure 9: An edge that is simultaneously initial and a leaf

a line c_S in our family such that $i(X, c_S)$ is 1 or 2, and if $T > S$ then $c_S \cap T = \varnothing$, or we exchange order so that we can do so.

Since the lines c_S are indexed by S, the condition on intersections implies that no line is homotopically trivial (bounds a monogon) and no two lines are parallel (cobound a bigon), so the family c_S can be built up to a triangulation. The complication of the construction is that to construct the line for a given edge in the tree we need to understand what immediately follows the edge in the ordering.

We start at the root. If an edge leaving the root is a leaf in the tree, there are three possible cases. The surface could be a once-punctured torus, or a thrice-punctured sphere, or the terminal points of the edge are at punctures, and the punctured disks containing those punctures abut the same pair of pants. The construction for the punctured torus, and thrice-punctured pair of pants can be done by inspection. We focus on the last case, shown in Figure 9.

According to our rules either the edge of the tree dual to the blue curve parallel to the outer boundary or the line joining to the two punctures could come first. You really want the edge dual to the curve parallel to the outer boundary component to be first. If S_1 is the component of P that bounds the punctured disk at the root, let c_{S_1} be the line built from the branch of the track that follows the outer boundary component before heading to the puncture. Notice $i(c_{S_1}, S_1) = 2$. The circle S_2 surrounding the other puncture has geometric intersection number 1 with the line c_{S_2} having one end

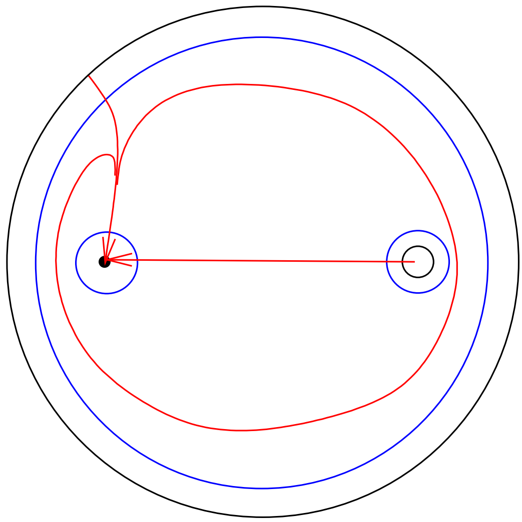


Figure 10: The case when the edge leaving the root is not a leaf

at each puncture. Since the line c_{S_1} is completely inside the diagram we have that it has geometric intersection number 0 with all later curves.

Now suppose that the edge leaving the root is not a leaf. In Figure 10 we show the situation. The line c_{S_1} coming from the branch of the track that runs around the outer boundary component has geometric intersection number 2 with S_1 and misses all the other components of the filling diagram, and for all later components it has geometric intersection number 0.

An edge dual to S_i of the tree is intermediate if there is an edge dual to S_{i-1} before it and an edge dual to S_{i+1} after it in the tree from the ordering. Let c_{S_i} be the line coming from the branch of the track that was built by perturbing S_{i+1}. Notice $i(c_{S_i}, S_i) = 2$. Do all intermediate edges before doing the leaves.

If an edge is a leaf, then it could end at a puncture, it could have both ends of an edge not part of the tree attached at its terminal 0–cell, or it could have two different edges not in the tree attached at its terminal end. These both occur in Figure 7. The highest leaf in the diagram is of the first type, and the lower leaf is of the second kind.

In the first case, the component S of P bounds a punctured disk. The line c_S of the train track that emanates from that puncture and ends at the root has geometric intersection number 1 with S.

In the second case, the vertex of the edge lies in a torus that is the closure of a component of $F-P$. Call the curve dual to the edge with both its ends attached at that 0–cell S'.

The pushoff of S' gives rise to the line c_S that has geometric intersection 2 number with S that is dual to the edge.

In the third case, let S' be the component of P that is dual to one of the edges attached at the leaf. The pushoff of S' towards the leaf gives rise to an embedded line that has geometric intersection number 2 with the curve S dual to the edge.

Throw out any curves that weren't used. Augment to form an ideal triangulation. □

Given a primitive diagram $P = \{S_1, \dots, S_n\}$, we can form the subalgebra of $K_\zeta(F)$,

$$\mathcal{P} = \chi(F)[S_1, \dots, S_n]. \tag{4-2}$$

This means we are taking the smallest subalgebra of $K_\zeta(F)$ that contains all $\chi(F)$–linear combinations of the S_i. Notice that $\mathcal{P}$ is a commutative ring, since the S_i are disjoint from one another. Also,

Form the ring

$$\bigotimes_{i=1}^{n} \chi(F)[S_i], \tag{4-3}$$

where the tensor product is as $\chi(F)$–modules. There is a ring homomorphism

$$\Psi\colon \bigotimes_{i=1}^{n} \chi(F)[S_i] \to \chi(F)[S_1, \dots, S_n] \tag{4-4}$$

given by

$$\Psi(\alpha_1 \otimes \cdots \otimes \alpha_n) = \alpha_1 * \alpha_2 * \cdots * \alpha_n. \tag{4-5}$$

Proof This is an immediate consequence of Proposition 4.2. □

Theorem 4.5 *If S_i for $i \in \{1, \dots, n\}$ is a system of simple closed curves on F that forms a primitive diagram then $\chi(F)[S_1, \dots, S_n]$ is a field extension of $\chi(F)$ of dimension m^n, then*

$$\Psi\colon \bigotimes_{\chi(F)} \chi(F)[S_i] \to \chi(F)[S_1, \dots, S_n] \tag{4-6}$$

is an isomorphism

Proof We can complete $\{S_i\}$ to a filling diagram. We apply Theorem 4.4 to get a system of curves of $\{S_i\}$ that satisfies the hypotheses of Proposition 4.2. □

Corollary 4.6 *The ring $S^{-1}\chi(F)[S_1, \dots, S_n]$, is a field extension of degree m^n over $S^{-1}\chi(F)$.*

Proof By Theorem 4.5, the ring $S^{-1}\chi(F)[S_1,\dots,S_n]$ is a commutative algebra over $S^{-1}\chi(F)$ that has dimension m^n as a vector space. Since $K_\zeta(F)$ has no zero divisors, neither does $S^{-1}\chi(F)[S_1,\dots,S_n]$. A finite commutative extension of a field is a field. □

If $S\subset F$ is a nontrivial simple closed curve, let $\Sigma_{0,2}(S)$ be an annular neighborhood of S in F. There is a left action of $K_\zeta(\Sigma_{0,2}(S))\otimes K_\zeta(F)\to K_\zeta(F)$ by gluing a copy of $\Sigma_{0,2}(S)\times[0,1]$ onto the top of $F\times[0,1]$. Notice that it restricts to give an action $\chi(\Sigma_{0,2}(S))$ on $\chi(F)$ making $S^{-1}\chi(\Sigma_{0,2}(S))\le S^{-1}\chi(F)$ a field extension.

Remark 4.7 It is worth mentioning that

$$S^{-1}\chi(\Sigma_{0,2}(S))[S]=S^{-1}K_\zeta(\Sigma_{0,2}(S)). \tag{4-7}$$

Theorem 4.8 *$S^{-1}\chi(F)[S]$ is the result of extending the coefficients of the ring $S^{-1}\chi(\Sigma_{0,2}(S))[S]$ as a vector space over $S^{-1}\chi(\Sigma_{0,2}(S))$ to a vector space over $S^{-1}\chi(F)$.*

Proof The dimension of $S^{-1}\chi(\Sigma_{0,2}(S))[S]$ over $S^{-1}\chi(\Sigma_{0,2}(S))$ is equal to the dimension of $S^{-1}\chi(F)[S]$ over $S^{-1}\chi(F)$, so the map

$$S^{-1}\chi(\Sigma_{0,2}(S))[S]\otimes_{S^{-1}\chi(\Sigma_{0,2})}S^{-1}\chi(F)\to\chi(F)[S] \tag{4-8}$$

that sends $S\otimes 1$ to S is a linear isomorphism. □

From our last paper:

Proposition 4.9 [8] *If $\Sigma_{0,2}$ is an annulus and x is the skein at its core and*

$$\mathrm{tr}\colon K_\zeta(\Sigma_{0,2})\to\chi(\Sigma_{0,2}) \tag{4-9}$$

is the unnormalized trace, $\mathrm{tr}(L_{T_k(x)})=0$ unless $m\,|\,k$, at which point $\mathrm{tr}(L_{T_k(x)})=mT_k(x)$. □

This implies the same result for $T_k(S)\colon S^{-1}K_\zeta(\Sigma_{0,2}(S))\to S^{-1}K_\zeta(\Sigma_{0,2}(S))$.

Proposition 4.10 *Let $S\subset F$ be a nontrivial simple closed curve. Define the map $L_{T_k(S)}\colon S^{-1}\chi(F)[S]\to S^{-1}\chi(F)[S]$ by left multiplication; then $\mathrm{tr}(L_{T_k(S)})=0$ unless $m\,|\,k$, at which point $\mathrm{tr}(L_{T_k(S)})=mT_k(S)$.*

Proof The map $L_{T_k(S)}\colon S^{-1}\chi(F)[S]\to S^{-1}\chi(F)[S]$ comes from

$$L_{T_k(S)}\colon S^{-1}\chi(\Sigma_{0,2}(S))[S]\to S^{-1}\chi(\Sigma_{0,2}(S))[S] \tag{4-10}$$

by extension of scalars and the fact that $\chi(\Sigma_{0,2}(S))[S]=K_\zeta(\Sigma_{0,2}(S))$. □

Proposition 4.11 *Let $\prod_i T_{k_i}(S_i)$ act on $S^{-1}\chi(F)[S_1,\dots,S_n]$ by multiplication,*

$$L_{\prod_i T_{k_i}(S_i)}\colon S^{-1}\chi(F)[S_1,\dots,S_n]\to S^{-1}\chi(F)[S_1,\dots,S_n]. \tag{4-11}$$

Then the unnormalized trace of $L_{k_1,\dots,k_n}$ is zero unless $m\,|\,k_i$ for all i, in which case it is $m^n\prod_i T_{k_i}(S_i)$.

Proof The diagram

$$\begin{array}{ccc} \bigotimes_{\chi(F)}\chi(\Sigma_{0,2}(S_i))[S_i]\otimes_{\chi(\Sigma_{0,2}(S_i))}\chi(F) & \xrightarrow{\psi} & \chi(F)[S_1,\dots,S_n] \\ \downarrow{\scriptstyle \otimes L_{T_{k_i}(S_i)}} & & \downarrow{\scriptstyle L_{\prod_i T_{k_i}(S_i)}} \\ \bigotimes_{\chi(F)}\chi(\Sigma_{0,2}(S_i))[S_i]\otimes_{\chi(\Sigma_{0,2}(S_i))}\chi(F) & \xrightarrow{\psi} & \chi(F)[S_1,\dots,S_n] \end{array} \tag{4-12}$$

where ψ is the natural isomorphism, commutes. This means that the trace of $L_{\prod_i T_{k_i}(S_i)}$ is the product of the traces of the

$$L_{T_{k_i}(S_i)}\colon \chi(\Sigma_{0,2}(S_i))[S_i]\otimes_{\chi(\Sigma_{0,2}(S_i))}\chi(F) \to \chi(\Sigma_{0,2}(S_i))[S_i]\otimes_{\chi(\Sigma_{0,2}(S_i))}\chi(F) \tag{4-13}$$

which are obtained by extension of scalars from

$$L_{T_{k_i}(S_i)}\colon K_\zeta(\Sigma_{0,2}(S_i))\to K_\zeta(\Sigma_{0,2}(S_i)).$$

□

Theorem 4.12 *Suppose that $d=[S^{-1}\chi(F)[S_1,\dots,S_n]:S^{-1}K_\zeta(F)]$. The unnormalized trace of*

$$L_{k_1,\dots,k_n}\colon S^{-1}K_\zeta(F)\to S^{-1}K_\zeta(F) \tag{4-14}$$

is zero unless $m\,|\,k_i$ for all i, in which case it is $dm^n\prod_i T_{k_i}(S_i)$.

Proof By Theorem 3.9, $S^{-1}K_\zeta(F)$ is a finite-dimensional vector space over

$$S^{-1}\chi(F)\le S^{-1}\chi(F)[S_1,\dots,S_n], \tag{4-15}$$

so Proposition 2.3 applies. □

We define the normalized trace

$$\mathrm{Tr}\colon S^{-1}K_\zeta(F)\to S^{-1}\chi(F) \tag{4-16}$$

to be the trace divided by dm^n. The map Tr is $S^{-1}\chi(F)$–linear, cyclic, and $\mathrm{Tr}(1)=1$.

Theorem 4.13 *Suppose that $s=\sum_i \beta_i P_i$ where the $\beta \in S^{-1}\chi(F)$ and the P_i are primitive diagrams whose components have been threaded with T_k. Let J be those indices i such that the components of P_i have only been threaded with T_k for $m|k$; then*

$$\mathrm{Tr}(s)=\sum_{i\in J}\beta_i P_i. \tag{4-17}$$

Theorem 4.14 *The restriction of* Tr: $S^{-1}K_\zeta(F)\to S^{-1}\chi(F)$ *to* $K_\zeta(F)$*, embedded in* $S^{-1}K_\zeta(F)$ *as fractions having denominator* 1*, yields*

$$\mathrm{Tr}\colon K_\zeta(F)\to\chi(F), \tag{4-18}$$

which is a $\chi(F)$*–linear map, so that* $\mathrm{Tr}(1)=1$ *and, for every* $\alpha,\beta\in K_\zeta(F)$,

$$\mathrm{Tr}(\alpha*\beta)=\mathrm{Tr}(\beta*\alpha). \tag{4-19}$$

Proof From the formula for Tr, the only fractions that appear in the coefficients in the trace come from fractions that are in the coefficients of the skein. □

5 The trace is nondegenerate

Lemma 5.1 *Let F be a finite-type surface with an ideal triangulation cut out by E. Suppose that $\sum_i z_i S_i\in K_\zeta(F)$, where the $z_i\in\mathbb{Z}\left[\frac{1}{2},\zeta\right]$ and the S_i are distinct simple diagrams. If some $[f_S]$ appearing in the symbol of $\sum_i z_i S_i$ with nonzero coefficient z has $m\,|\,\gcd(f_S)$, then*

$$\mathrm{Tr}\left(\sum_i z_i S_i\right)\neq 0. \tag{5-1}$$

Proof Suppose that the primitive diagram P underlying S is made up of simple closed curves S'_j. The threaded diagram having lead coefficient S is $\prod_j T_{mk_j}(S'_j)$ for some $k_j\in\mathbb{Z}_{\geq 0}$. Rewriting $\sum_i z_i S_i$ in terms of threaded diagrams, the threaded diagrams appearing in the symbol appear with the same coefficients and are distinct from one another in the sum. Hence $\prod_j T_{mk_j}(S'_j)$ appears in the trace with coefficient $z\neq 0$. This term can't cancel with other highest-weight terms in the trace, as the S_i were distinct, nor can it cancel with lower-weight terms, as that would violate the filtration of $\chi(F)$, so $\mathrm{Tr}\left(\sum_i z_i S_i\right)\neq 0$. □

Theorem 5.2 *Let F be a noncompact, finite-type surface. There are no nontrivial principal ideals in the kernel of*

$$\mathrm{Tr}\colon K_\zeta(F)\to\chi(F). \tag{5-2}$$

Proof Let $\alpha \in K_\zeta(F)$ be nonzero. Choosing an ideal triangulation and an ordering of edges, we can write $\alpha = \sum_i z_i P_i$ where the z_i are nonzero complex numbers and the P_i are threaded primitive diagrams. Suppose that the lead term of α is $z_* P_*$. Let P' be a threaded primitive diagram such that the residue of P' in $\mathbb{Z}_m^E$ is the additive inverse of the residue of P_*. If I is the principal ideal generated by α, then $P' * \alpha$ is in the principal ideal generated by α, and the residue of its lead term is zero. Since α was an arbitrary nonzero skein, there does not exist a nontrivial principal ideal of $K_\zeta(F)$ contained in the kernel of Tr. □

Corollary 5.3 *$S^{-1}K_\zeta(F)$ is a symmetric Frobenius algebra over $S^{-1}\chi(F)$.* □

Corollary 5.4 *There is a proper subvariety of the character variety of $\pi_1(F)$ away from which $K_\zeta(F)_\phi$ is a symmetric Frobenius algebra over $\mathbb{C}$.* □

References

[1] **J W Barrett**, *Skein spaces and spin structures*, Math. Proc. Cambridge Philos. Soc. 126 (1999) 267–275 MR

[2] **F Bonahon**, **H Wong**, *Representations of the Kauffman bracket skein algebra, I: Invariants and miraculous cancellations*, Invent. Math. 204 (2016) 195–243 MR

[3] **D Bullock**, *Rings of* $\mathrm{SL}_2(\mathbf{C})$*–characters and the Kauffman bracket skein module*, Comment. Math. Helv. 72 (1997) 521–542 MR

[4] **D Bullock**, *A finite set of generators for the Kauffman bracket skein algebra*, Math. Z. 231 (1999) 91–101 MR

[5] **D Bullock**, **C Frohman**, **J Kania-Bartoszyńska**, *Understanding the Kauffman bracket skein module*, J. Knot Theory Ramifications 8 (1999) 265–277 MR

[6] **D Bullock**, **J H Przytycki**, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. 128 (2000) 923–931 MR

[7] **A Fathi**, **F Laudenbach**, **V Poénaru**, *Travaux de Thurston sur les surfaces*, Astérisque 66, Soc. Math. France, Paris (1979) MR

[8] **C Frohman**, **N Abdiel**, *Frobenius algebras derived from the Kauffman bracket skein algebra*, J. Knot Theory Ramifications 25 (2016) art. id. 1650016 MR

[9] **C Frohman**, **R Gelca**, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. 352 (2000) 4877–4888 MR

[10] **C Frohman**, **J Kania-Bartoszynska**, *The structure of the Kauffman bracket Skein algebra at roots of unity*, preprint (2016) arXiv

[11] **P Gordan**, *Ueber die Auflösung linearer Gleichungen mit reellen Coefficienten*, Math. Ann. 6 (1873) 23–28 MR

[12] **J Hoste**, **J H Przytycki**, *The* $(2, \infty)$*–skein module of lens spaces; a generalization of the Jones polynomial*, J. Knot Theory Ramifications 2 (1993) 321–333 MR

[13] **T T Q Lê**, *On Kauffman bracket skein modules at roots of unity*, Algebr. Geom. Topol. 15 (2015) 1093–1117 MR

[14] **G Muller**, *Skein and cluster algebras of marked surfaces*, Quantum Topol. 7 (2016) 435–503 MR

[15] **J H Przytycki**, **A S Sikora**, *On skein algebras and* $\mathrm{Sl}_2(\mathbf{C})$*–character varieties*, Topology 39 (2000) 115–148 MR

[16] **A S Sikora**, **B W Westbury**, *Confluence theory for graphs*, Algebr. Geom. Topol. 7 (2007) 439–478 MR

[17] **D P Thurston**, *Positive basis for surface skein algebras*, Proc. Natl. Acad. Sci. USA 111 (2014) 9725–9732 MR

[18] **J G van der Corput**, *Über Systeme von linear-homogenen Gleichungen und Ungleichungen*, Proc. Akad. Wet. Amsterdam 34 (1931) 368–371

Department of Mathematics, University of Iowa
Iowa City, IA, United States

nel.abdiel@gmail.com, charles-frohman@uiowa.edu

Received: 11 January 2015 Revised: 11 May 2017