

# Generalized augmented alternating links and hyperbolic volumes

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Augmented alternating links are links obtained by adding trivial components that bound twice-punctured disks to nonsplit reduced non-2-braid prime alternating projections. These links are known to be hyperbolic. Here, we extend to show that generalized augmented alternating links, which allow for new trivial components that bound  $n$ -punctured disks, are also hyperbolic. As an application we consider generalized belted sums of links and compute their volumes.

57M50; 57M25

## 1 Introduction

By the work of W Thurston [15], a nonsplit link in  $S^3$  is known to be either hyperbolic or to contain an essential torus or annulus in its complement. When the link is hyperbolic, its complement admits a hyperbolic metric that is uniquely determined and, hence, the hyperbolic volume of its complement becomes an invariant that can be used to distinguish it from other links.

Menasco [9] proved that prime alternating non-2-braid links are hyperbolic. In Adams [2], it was further proven that augmented alternating links are hyperbolic. These links are obtained from a prime non-2-braid alternating link projection by adding trivial “vertical” components perpendicular to the projection plane that bound a disk punctured twice by the alternating link. These augmented alternating links have proved useful in a variety of contexts. In particular, they appear as the geometric limits of alternating links that correspond to twisting the two strands around which the augmenting components wrap. As such, together with the alternating links, they form the closure of the collection of alternating links in the geometric topology (see Corollaries 2 and 3 of Lackenby [8]). In particular, the volumes of the links in such a sequence must approach the volume of the augmented link from below.

If a link is not alternating, one can augment it at a subset of the crossings that would need to be changed to make it alternating, and then the result has the same complement as an augmented alternating link, since a full twist along one of the twice-punctured disks can reverse that crossing.

Augmented alternating links are useful in a variety of settings. See for instance Blair, Futer and Tomova [4], Dasbach and Tsietkova [5], Futer, Kalfagianni and Purcell [6], Lackenby [8] and Purcell [10; 13]. In some papers, the links considered are fully augmented. That is to say, every crossing in the original knot is in a twist sequence around which a vertical component has been added. In that case, one can use Andreev's theorem to prove hyperbolicity (see [13]).

In this paper, we extend the results of [2] to allow the vertical components to bound disks that are punctured more than twice by the alternating link in the projection plane. These new links are called generalized augmented alternating links. A precise definition appears in Section 2. Our main theorem is to prove that, indeed, their complements are always hyperbolic.

In particular, since  $(1, q)$ -Dehn filling of a vertical component corresponds to adding  $q$  full twists to the strands of the original link, Thurston's hyperbolic Dehn surgery theorem and the fact the augmented link is hyperbolic imply that the resulting links are always hyperbolic for high enough values of  $q$ .

Note that in several papers, authors have considered generalized augmented links that were also obtained by adding vertical components to a projection, but in this case, a not necessarily alternating projection such that the projection breaks up into generalized twist regions as in Figure 1. But here again, each twist region has to receive a crossing circle. These links have a variety of interesting properties, as discussed in Futer, Kalfagianni and Purcell [6] and Purcell [11; 12; 13].

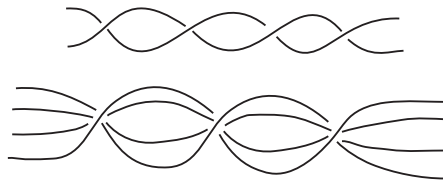


Figure 1: A traditional twist region and a generalized twist region

The presence of twice-punctured disks in link complements and the fact that twice-punctured disks are totally geodesic with a unique hyperbolic structure (see Adams [1]) implies that one can take belted sums of the links, which is the operation depicted in Figure 2, top row. The resulting link  $L_1 \#_b L_2$  has volume equal to the sum of the volumes of the two links  $L_1$  and  $L_2$  that are summed. Note that a belt is any component bounding a twice-punctured disk in a hyperbolic link complement.

In Section 3, we generalize the notion of belted sum, and find an explicit formula between the volumes of the two links and their summand. Specifically, we show

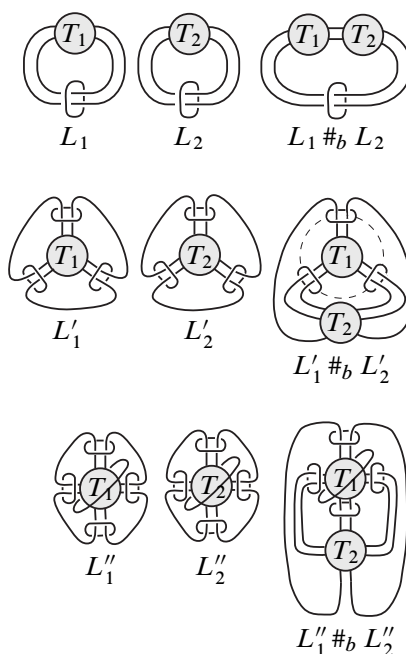


Figure 2: Generalized belted sums of links

that if a hyperbolic link denoted by  $L'_1 \#_b L'_2$  is constructed from two links  $L'_1$  and  $L'_2$  as in Figure 2, center, then  $\text{vol}(L'_1 \#_b L'_2) = \text{vol}(L'_1) + \text{vol}(L'_2) - 4(3.6638 \dots)$ . Similarly, if  $L''_1 \#_b L''_2$  is a link constructed as in Figure 2, bottom, then  $\text{vol}(L''_1 \#_b L''_2) = \text{vol}(L''_1) + \text{vol}(L''_2) - 8(3.6638 \dots)$ . In this case, there are two distinct options for the central belt, wrapping either lower left to upper right or lower right to upper left.

More generally, let  $L_1$  and  $L_2$  be two links, each with a  $2n$ -string tangle at center, with  $n$  belts around adjacent pairs of the exiting strings and  $n - 3$  belts around the central tangle in the same pattern, as for instance appears in Figure 3 in the case  $n = 5$ . Then  $\text{vol}(L_1 \#_b L_2) = \text{vol}(L_1) + \text{vol}(L_2) - 4(n - 2)3.6638 \dots$ . This construction

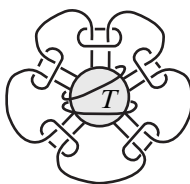


Figure 3: A potential generalized belted sum factor link

answers a question asked by Oliver Dasbach about the behavior of the volumes in Figure 2, center row, and was motivated by that question.

Of course, given a specific link, we would like to add vertical components to obtain either a link that can be decomposed via belted sum into simpler links or composed via belted sum into more complicated links. But we need to know that the resulting link is hyperbolic. This is what the main theorem provides when the initial link is alternating.

In the case of augmented alternating links, twisting a half-twist on the twice-punctured disk bounded by a vertical component, which adds or subtracts a crossing, will yield a new link complement that is hyperbolic with the same volume as the original link. This follows because the links are hyperbolic and the twice-punctured disks are totally geodesic with a unique hyperbolic structure (see Adams [1]). However, in the case of generalized augmented alternating links, if we twist a half-twist on an  $n$ -punctured disk bounded by a vertical component for  $n \geq 3$ , the result need not be hyperbolic, and even if it is, the volume is generally not preserved. As an example, adding one vertical component bounding a thrice-punctured disk in the figure-eight knot complement and then twisting a half-twist yields a Seifert fibered space. Further applications of generalized augmented alternating links to volume bounds for links appear in Adams [3].

## 2 Hyperbolicity

Given an alternating link  $J$  in a reduced alternating projection  $P$ , we will often consider it as a 4-regular graph on the projection sphere. That graph cuts the sphere up into complementary regions.

Let  $J$  be a prime nonsplit non-2-braid alternating link. Let  $P$  be a reduced alternating projection of  $J$ . Note that by results of Menasco [9], the projection is connected and there are no simple closed curves in the plane that intersect the projection transversely twice such that there are crossings to either side of the curve. Choose two complementary regions in the projection plane that do not share an edge. Take a trivial component  $C$  that intersects the projection sphere in precisely one point in each of the complementary regions. Then we say that  $J' = J \cup C$  is a *generalized singly augmented alternating link*. We call the additional component a *vertical component*.

In the projection plane, we keep track of the vertical component as a gray arc  $\gamma$ . While fixing endpoints, we can isotope  $\gamma$  to minimize the number of intersections with the link  $J$ . This corresponds to an isotopy of the vertical component in the complement of  $J$ . We will assume that such an isotopy has already taken place, and call the corresponding vertical component a *minimal representative* of the isotopy class.

For any other pair of nonadjacent complementary regions, we allow the introduction of additional vertical components, as long as there are minimal representations of all the individual vertical components that are disjoint as arcs in the plane. We call the resulting link a *generalized augmented alternating link*. Note that for any pair of nonadjacent regions there can be at most one corresponding vertical component. There can be quite a few vertical components, as for instance occurs in the generalized augmented figure-eight knot in Figure 4. This link is maximally augmented in the sense that every pair of possible nonadjacent regions corresponds to a vertical component. Note that for the figure-eight knot, there is more than one option for the maximally augmented link that results.

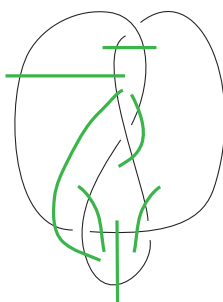


Figure 4: A generalized augmented projection of the figure-eight knot with the maximum possible number of vertical components

It should be noted that there are reduced alternating projections such that not all of the possible vertical components can be added since minimal representations overlap, as in Figure 5.

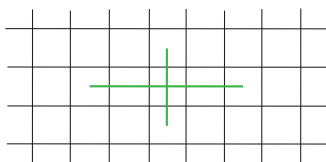


Figure 5: These two vertical components on this alternating grid cannot be made to avoid intersecting while in minimal representations.

**Theorem 2.1** *Let  $J$  be a prime non-2-braid nonsplit alternating link. Then any generalized augmented alternating link  $J'$  constructed from a reduced alternating projection of  $J$  is hyperbolic.*

In fact, even in the case of a 2-braid link in a reduced alternating projection, if we add a vertical component that does not correspond to the axis around which the knot is braided, the result is hyperbolic. However, we will not include that case here.

It was proved in [9] that  $J$  is hyperbolic. Here, we are proving that the addition of these new vertical component preserves hyperbolicity. We use the machinery developed in [9], as described below.

We consider the projection plane  $P$  as a subspace of the projection sphere obtained by the 1-point compactification. We will move from the plane to the sphere without comment.

Let  $L$  be a link in a projection. At each crossing of the projection, a bubble is inserted, with the overstrand going over the top of the bubble and the understrand going under the bubble. We let  $S_+$  be the sphere obtained by replacing each equatorial disk of a bubble in the projection sphere with the top hemisphere of the corresponding bubble. We define  $B_+$  to be the ball bounded by  $S_+$ . Similarly, we define  $S_-$  and  $B_-$ , using the bottom hemispheres. In Lemma 1 of [9], Menasco showed that for any projection, and any closed surface that is incompressible and not boundary parallel, the surface can be isotoped so that it intersects bubbles in saddles and each intersection curve with  $S_{\pm}$  intersects each bubble at most once and intersects at least one bubble. We will use a version of this that applies to surfaces with boundary, as appears below. We begin with the following lemma.

**Lemma 2.2** *If  $C$  is a vertical component in a generalized augmented alternating link  $J'$  such that  $C$  is a minimal representative of its isotopy class, then the vertical punctured disk  $D$  that is bounded by  $C$  is incompressible.*

**Proof** We can treat  $D$  as a surface in either  $S^3 - (J \cup C)$  or  $S^3 - J'$ . If there are no compression disks for  $D$  in  $S^3 - (J \cup C)$ , then there are certainly no compression disks in  $S^3 - J'$ . Hence we can drop all the vertical components except for  $C$ , since incompressibility of  $D$  in  $S^3 - (J \cup C)$  implies incompressibility in  $S^3 - J'$ . Let  $\gamma$  denote the arc in the projection plane that is the projection of  $C$ .

Suppose that  $D$  compresses. Let  $E$  be a compressing disk, in general position with respect to the projection of  $J$ , chosen to have the minimum possible number of saddles for such a compressing disk. We will show, as Menasco did with closed and punctured surfaces in alternating knot and link complements, that the disk  $E$  behaves appropriately with respect to the bubbles inserted at each crossing.

As in [9], we isotope  $E$  to be transverse to a vertical edge in each bubble, and then push  $E$  out from that edge so that  $E$  only intersects each bubble in a possibly empty collection of saddle disks. We assume that  $E$  has the minimum number of saddles for a compressing disk of  $D$ . If the intersection of  $E$  with either  $B_+$  or  $B_-$ , say  $B_+$  for the argument, contains a subsurface  $W$  of  $E'$  other than a disk, then there is a

nontrivial curve in  $E'$  (take an innermost boundary component of  $W$ ) that bounds a disk in  $B_+$ . We can perform surgery on  $E$  using this disk to lower the number of saddles in  $E$ , contradicting our assumption that  $E$  has a minimal number of saddles. Thus, we can assume that the intersections of  $E$  with the bubbles and spheres  $S_+$  and  $S_-$  decomposes  $E$  into saddle disks corresponding to where it intersects bubbles, and over-disks which have interior above  $S_+$  and boundary on  $S_+$  and under-disks, which have interior below  $S_-$  and boundary on  $S_-$ . We call the resultant graph on  $E$ , where saddles are treated as vertices, the *intersection graph*.

We can immediately eliminate any simple closed curves that do not intersect a bubble. Such a curve must bound a disk in either  $B_+$  or  $B_-$  and again we can perform surgery on  $E$  to eliminate all such curves.

As in Lemma 1 of [9], we show that no intersection arc intersects a bubble more than once. For, if an intersection curve hits both sides of a bubble, we can choose an innermost such intersection curve  $\alpha$  on that bubble, which must hit both sides of a single saddle. By taking an arc crossing the saddle and an arc on the disk bounded by the intersection curve, we obtain a closed curve  $\delta$  on  $E$  bounding a disk  $E'$  on  $E$ , whose boundary can be isotoped to a meridian of  $J$  as in Figure 6, left. But this creates a once-punctured sphere in the complement of  $J$ , a contradiction.

If an intersection curve hits the same side of a bubble twice, we can again take an innermost such curve  $\alpha$  and then perform an isotopy to eliminate the two saddles that touch the curve in this bubble as in Figure 6, right, pulling the blue band through the bubble, again contradicting the minimality of the number of saddles in  $E$ . Thus, no intersection curve hits a bubble more than once.

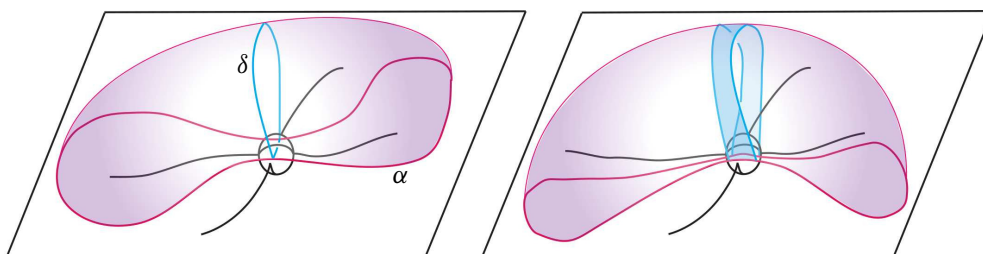


Figure 6: Intersection curves intersecting a bubble more than once

Suppose now that there is a simple closed intersection curve  $\alpha$  on  $E$ . By this we mean a simple closed curve in the intersection graph that avoids  $\partial E$  and that forms a component of the boundary of a region containing no other intersection curves on  $E$ . Then it is also a simple closed curve on either  $S_+$  or  $S_-$ ; for convenience, assume  $S_+$ .

Note that all intersection arcs that begin and end on  $\partial E$  must begin and end on the same side of  $\gamma$  in the projection plane since the boundary of  $E$  must occur on only one of the two sides of  $D$ . Since a simple closed intersection curve does not intersect  $\gamma$ , in  $S_+$  all the intersection arcs lie to the side of  $\alpha$  containing  $\gamma$ . Take an innermost intersection curve  $\alpha'$  to the other side of  $\alpha$ . If there are none, take  $\alpha$  itself considered innermost to the outside. Since the projection is alternating, each intersection curve must intersect bubbles such that the overstrand of the bubble is alternately on the left and right of the curve. As in [9], this forces  $\alpha'$  to hit a bubble twice, a contradiction. Thus, any such simple closed curve must avoid all bubbles. But then we could replace the disk it bounds on  $E$  with the disk it bounds on  $S_+$ , and push off to lower the number of intersection curves.

Hence, there are no simple closed intersection curves on  $E$ . All intersection curves are arcs that begin and end on  $\partial E$ . Note that when viewed in the plane, no such intersection arc crosses  $\gamma$  but all intersection arcs start and end on  $\gamma$ , all coming out one side of it. See Figure 7 for the picture.

A *fork* of the intersection graph is a vertex with at least three edges ending on  $\partial E$ , keeping in mind that all interior vertices are 4-valent. We show that every intersection graph has at least one fork. Since every complementary region must intersect  $\partial E$  in its boundary, the graph obtained by throwing away all edges with an endpoint on the boundary of  $E$  is a collection of trees. Every tree of two or more vertices always has at least two leaves, and those leaves will have three edges that must all end on the boundary. So in this case, there are at least two forks. The one exception is if there is only one vertex to one tree, which coincides with the case of there being only one saddle in  $E$ . However, then we still have a fork.

We consider what a fork tells us about the projection  $P$ . See Figure 7. The two disks bounded by the one saddle and the three curves ending on the boundary of  $E$  cause there to be exactly two arcs of the knot that come out of the crossing in question and then pass through  $E$  as punctures, without crossing any other arcs of the knot in between. By Theorem 1(b) of [9], which shows that an alternating knot is prime if and only if it is obviously so in any alternating projection, these arcs cannot contain any crossings between when they puncture  $E$  and when they pass through the crossing in question.

Fixing its endpoints, we can isotope  $\gamma$  past the resultant crossing. There is a corresponding isotopy of  $C$ ,  $D$  and  $E$ , changing the pattern of intersections on  $E$  and eliminating at least one saddle on  $E$ . We repeat this process until either all saddles are eliminated from  $E$ , leaving only simple arcs of intersection, or there is only one saddle remaining. In the first case, taking an outermost arc  $\alpha''$  on  $E$ , we cut off a disk  $E''$



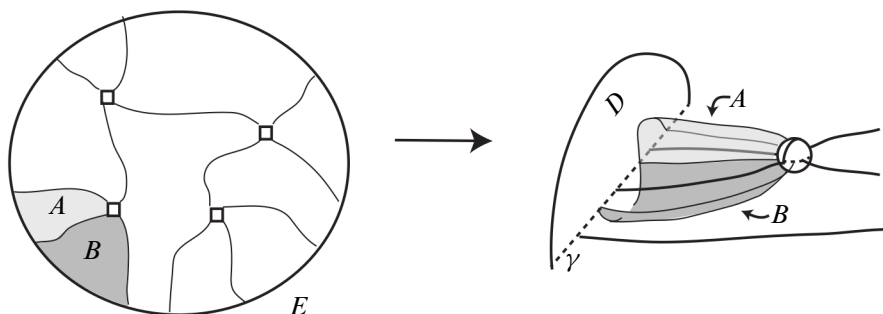


Figure 7: A fork in the compression disk allows us to isotope  $\gamma$  past a crossing.

which intersects the projection plane in an arc that does not intersect a bubble. Hence we can either isotope an arc on  $\gamma$  to this arc, and lower the number of intersections in  $\gamma \cap J$ , a contradiction to  $\gamma$  corresponding to a minimal representative of  $C$ , or if no part of  $J$  lies in the region in the projection plane cut off by  $\gamma \cup \alpha''$ , we can isotope  $E$  to lower the number of intersection arcs. In either case, repeating the process if necessary, we obtain a contradiction.

In the second case, if only one saddle remains, then, as in Figure 8, the arc  $\gamma$  can be isotoped with endpoints fixed to lower its number of punctures, and it is therefore not a minimal representative.  $\square$

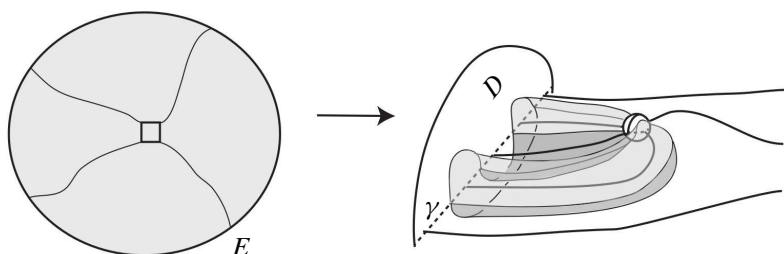


Figure 8: A single saddle in the compression disk implies  $\gamma$  is not a minimal representative.

The following results will put us in position to prove Theorem 2.1. Throughout the rest of this section,  $J$  is a prime non-2-braid nonsplit alternating link, and  $J'$  is a generalized augmented alternating link complement obtained from  $J$ .

**Lemma 2.3** *The link complement  $S^3 - J'$  is irreducible.*

**Proof** Proving irreducibility is equivalent to showing that  $S^3 - J'$  is not splittable. Since the alternating projection of  $J$  is connected, Theorem 1(a) from [9] shows that

$J$  is nonsplittable. Hence, if  $J'$  is splittable, there must be a sphere with  $J$  to one side and at least one vertical component  $C$  to the other side. Let  $D$  be the vertical punctured disk bounded by  $C$ . Discarding the other vertical components, we work with just this one vertical component. But if  $C$  is contained in a sphere, it bounds a disk in the sphere. We can use this disk to obtain a compression disk for  $D$ , contradicting Lemma 2.2. Hence,  $J'$  is nonsplittable.  $\square$

**Lemma 2.4** *Any essential torus  $T$  in  $S^3 - J'$  is meridionally compressible.*

**Proof** By definition,  $T$  is incompressible and not boundary-parallel. To prove meridional compressibility, we must show there is a nontrivial simple closed curve on  $T$  that bounds a disk punctured once by the link  $J'$ . So suppose  $T$  is meridionally incompressible.

We again apply the techniques of [9], which Menasco utilized to prove a similar result for alternating links. First, we flatten each vertical component into the projection plane as in Figure 9.

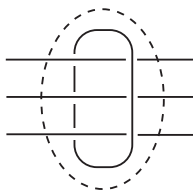


Figure 9: Projecting the vertical components

Since  $T$  is incompressible and meridionally incompressible, Lemma 1 of [9] tells us there exists a realization of  $T$  such that the intersection curves with  $S_+$  and  $S_-$  do not intersect the same crossing bubble more than once. Moreover, every curve must intersect at least one bubble.

We now eliminate the vertical components, without isotoping the surface  $T$ . Saddles that appeared in crossing bubbles involving the vertical components disappear and the intersection curves that entered a dotted region as in Figure 9 now connect to one another.

There are two fundamental changes in the system of intersection curves. First of all, each intersection curve need no longer bound a disk above  $S_+$  in the case of  $B_+$  and below  $S_-$  in the case of  $B_-$ . Instead, a collection of intersection curves can bound a subsurface of  $T$  above  $S_+$  or below  $S_-$ . Second, for each of the resulting intersection curves on  $S_+$  and  $S_-$ , it can either be the case that the curve does or does not intersect a bubble more than once.

We add the vertical components back in, but now they are once again vertical, perpendicular to the projection plane, each puncturing the projection plane in two points. For convenience, we consider intersection curves on  $S_+$ , but everything works just as well for intersection curves on  $S_-$ . We assume that  $T$  is chosen to minimize the resultant number of saddles.

The collection of intersection curves and saddles decompose  $T$  into squares on its surface corresponding to the saddles, and the components of intersections with  $B_+$  and  $B_-$ . We first show that, with the possible exception of a single  $n$ -punctured torus, all of these components are either disks or annuli.

Let  $R$  be a planar component of  $T \cap B_+$ . Suppose an intersection curve  $\alpha$  that forms one of the boundaries of  $R$  on  $S_+$  bounds a disk  $F$  on  $T$ . We show that then  $R$  is a disk. Let  $D'$  be the disk bounded by  $\alpha$  on  $S_+$ . It may or may not contain additional intersection curves. If there are no vertical components with endpoints in  $D'$ , then we can isotope the disk  $F$  bounded by  $\alpha$  on  $T$  to  $D'$ , pushing any other parts of  $T$  out of the way in the process. After this isotopy, we have either eliminated  $\alpha$  as an intersection curve, simplifying the intersection graph, or  $R$  was a disk in  $T \cap S_+$ . If  $D'$  is punctured by some vertical component once, then that vertical component will be nontrivially linked with  $\alpha$ , contradicting the fact that  $\alpha$  bounds  $F$ . Hence, any vertical component  $C$  that intersects  $D'$  must do so with both of its endpoints. No other vertical components can be linked with this one above the projection plane, as in Figure 10, since if they were, they would also have to be similarly linked below the projection plane, which the existence of  $F$  prevents.

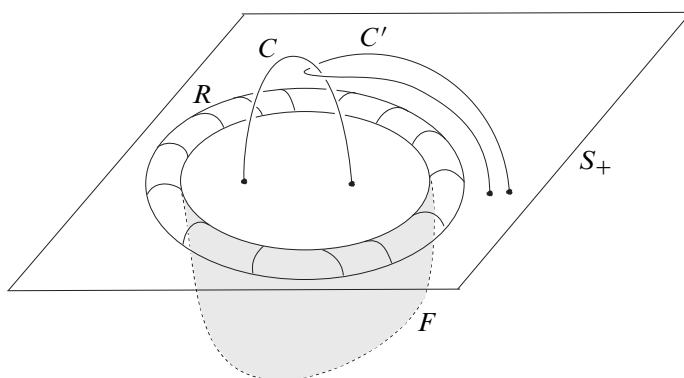


Figure 10: Vertical components cannot be linked above the projection plane when an intersection curve bounds a disk on  $T$ .

We can then take a disk  $D''$  that is horizontal and has boundary in  $R$  that is a curve parallel to  $\alpha$  and slightly above it. Then we can isotope  $F$  to  $D'$ , eliminating the

intersection curve  $\alpha$  and any saddles that it touches, a contradiction to minimality. Note we are using the fact  $J'$  is nonsplittable here. So, the only planar components of  $T \cap S_+$  are disks and annuli, with all boundary components of the annuli appearing as parallel nontrivial curves on the torus, as in Figure 11, right.

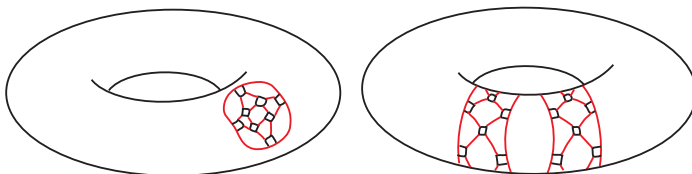


Figure 11: Islands as on the left can only occur on  $T$  if there is a component in  $T \cap B_{\pm}$  that is an  $n$ -punctured torus. Annular regions appear as on the right.

We first show that there are no innermost curves on  $S_+$  that bound disks in  $B_+$ . An innermost curve is one that bounds a disk  $R$  in  $S_+$  that contains no other intersection curves. A bubble that intersects an innermost curve is called an inner (outer) bubble if its overstrand lies inside (outside) of  $R$ .

Such an innermost curve  $\alpha$  bounding a disk  $G$  in  $T \cap B_+$  must intersect bubbles more than once, as  $J$  is alternating, so the bubbles must alternate between bubbles with their overstrand to the right and bubbles with their overstrand to the left as we travel around the curve. Since there are no other curves inside  $\alpha$ , the other side of each inner bubble must be hit by  $\alpha$ , as in Figure 12.

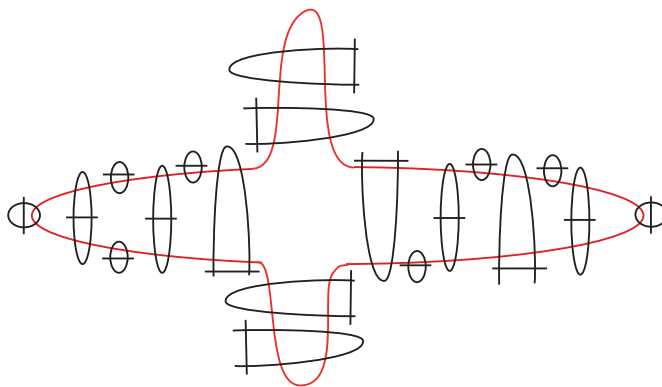


Figure 12: Inner bubbles must be intersected on both sides by  $\alpha$ .

There are three types of bubble intersections with  $\alpha$ . A bubble of type I intersects  $\alpha$  on both sides of the bubble and the overstrand lies to the inside of  $\alpha$ . A type II bubble

is one that is intersected by  $\alpha$  at least twice on one side and the overstrand lies to the outside of  $\alpha$ . A bubble of type III intersects  $\alpha$  once and has its overstrand to the outside of  $\alpha$ . If a bubble is of type II or III, we consider only multiple intersections of that bubble occurring to the inside of the curve. We do not care if distinct bubbles appearing inside  $\alpha$  are in fact the same bubble when also considered outside  $\alpha$ .

Choosing an innermost pair of intersections of  $\alpha$  with a bubble of type I, one on each side of the bubble, we can form a loop  $\delta$  out of an arc on the corresponding saddle and an arc on  $G$  that forms the boundary for a meridional compression to the overstrand of the bubble. Since there are no meridional compressions, it must be the case that one or more vertical components  $C$  block this meridional compression, as in Figure 13, left.

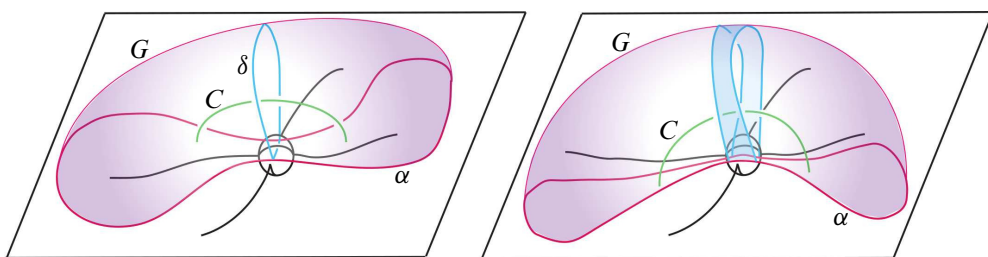


Figure 13: A vertical component blocking meridional compressions and saddle-reducing isotopies

Similarly, for a bubble of type II, we can isotope a band on the disk bounded by  $\alpha$  to eliminate two saddles unless the isotopy is blocked by one or more vertical components  $C$ , as in Figure 13, right.

Relative to the vertical components, the intersection curve  $\alpha$  can wind around the curves, as in Figure 14, so when  $\alpha$  is drawn uncomplicated, it could be the case that the vertical components are tangled with one another, as if in a plat.

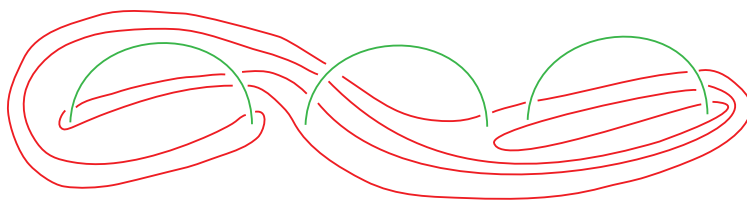


Figure 14: The intersection curve  $\alpha$  can wind around the vertical components.

We note the following *pairing property*: Consider all intersection curves on  $S_+$  that intersect a vertical disk  $D$  bounded by vertical component  $C$ , which is not necessarily

in a minimal representation. In  $B_+$ , they must form the boundaries of surfaces in  $T \cap B_+$ . Then  $T \cap B_+ \cap D$  is a collection of arcs that pair the points in  $\alpha \cap D$ , some potentially nested. However, there must be an innermost arc such that it cuts a disk from  $D \cap B_+$  that contains no other such arcs. Hence two adjacent intersection curves are connected by an innermost arc on  $D$ . We consider those two curves an *innermost pair*.

We now utilize the pairing property. Suppose  $C$  is a vertical component bounding a vertical punctured disk  $D$  that blocks one or more meridional compressions or saddle-reducing isotopies caused by bubbles to the inside of the innermost curve  $\alpha$  bounding the disk  $G$  in  $T \cap S_+$ .

Choose any vertical component  $C$  with endpoints in  $\alpha$  that blocks crossings from generating either a meridional compression or an isotopy lowering the number of saddles. Regions between bubbles inside  $\alpha$  fall into six types, denoted by  $H$ ,  $K$ ,  $L$ ,  $M$ ,  $N$  and  $P$  in Figure 15. A region of type  $H$  has two bubbles on its boundary, one of type I and one of type III. A region of type  $K$  has two bubbles of type I and two bubbles of type III, appearing alternately around its boundary. A region of type  $L$  has a bubble of type I, a bubble of type II and a bubble of type III around its boundary. A region of type  $M$  has only one bubble of type II on its boundary. A region of type  $N$  has two bubbles of type II on its boundary. A region of type  $P$ , which we call a *junction*, has three or more type II bubbles on its boundary (four in the case of Figure 15).

Notice that regions  $H$  and  $M$  must each have at least one vertical component that has exactly one endpoint in the region, since only two adjacent complementary regions of the projection intersect a region  $H$  and only one complementary region of the projection intersects a region of type  $M$ . Isotope  $C$  so that the arc that represents its projection follows  $\alpha$  as closely as possible, as in Figure 15, and let  $D$  be the punctured disk that it bounds. Note that we may have to carry along other vertical components with which it is entangled.

Shifting from the intersection curves on  $S_+$  to the corresponding intersection curves on  $S_-$ , we find that each pair of adjacent intersection curves passing through  $D$  share a bubble. See Figure 16. In every such diagram, each of the six types of regions will have exactly the same pattern of intersection curves shown here, with only a junction region that could look different, depending on the number of type II crossing around its boundary. If we had left our arc representing the vertical component going straight across a junction, there would have been a pair of adjacent curves passing under the vertical component that would not have shared a bubble. But, instead, we have  $C$  follow  $\alpha$  around the outside. This ensures that all pairs of adjacent curves passing through  $D$  share a bubble.

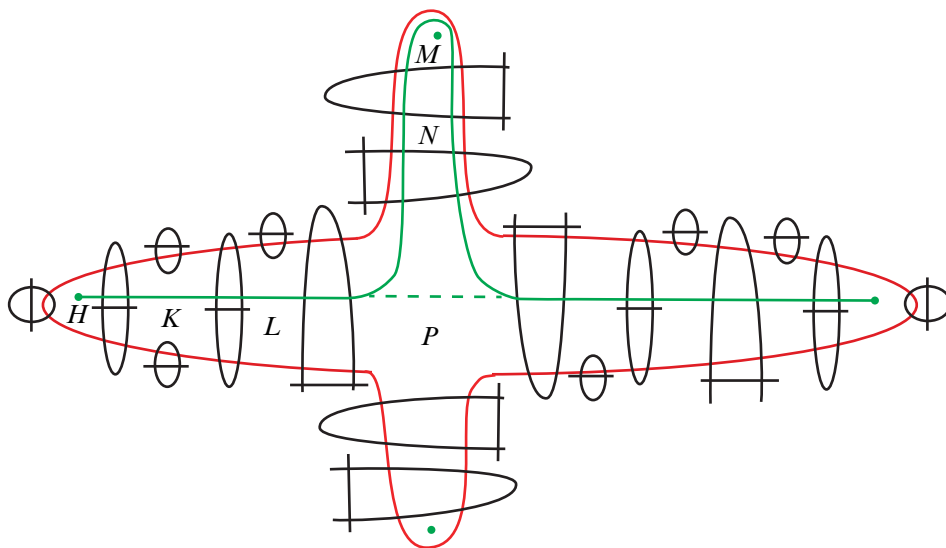


Figure 15: A vertical component blocking a meridional compression

So there exists an innermost pair of adjacent curves passing through  $D$ . They must share a bubble. That bubble cannot be blocked by a vertical component because the disk  $G$  prevents any such vertical component. Thus, we obtain either a meridional compression if

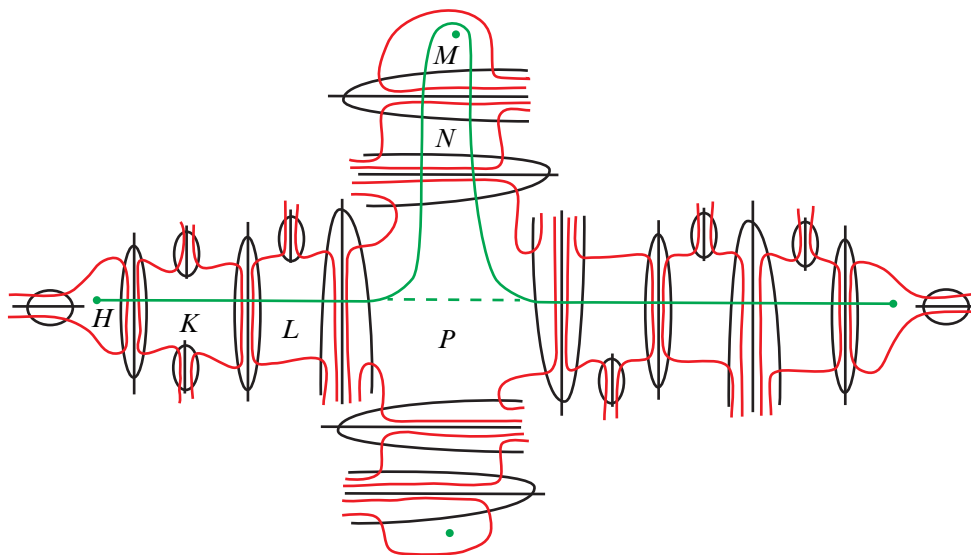


Figure 16: The view from  $S_-$ , where all adjacent curves passing under the vertical component share a bubble.

the bubble is a type I or type III bubble, or a saddle reducing isotopy if the bubble is a type II bubble. Hence, there can be no innermost intersection curve bounding a disk in  $B_+$ .

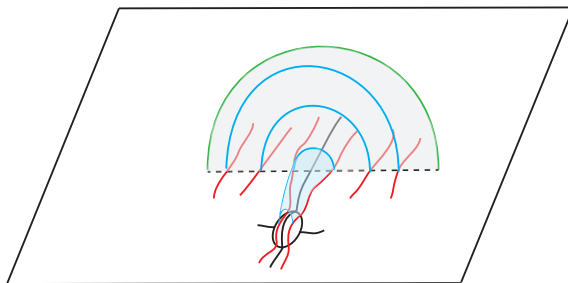


Figure 17: Two intersection curves share both a bubble and a surface, yielding a meridional compression.

We now prove there are no annular components in  $T \cap B_+$ . Suppose there were such an annulus  $A$ . Note that the existence of  $A$  precludes the possibility of a punctured torus component in either  $T \cap B_+$  or  $T \cap B_-$ . Each of its boundary components bounds a disjoint disk in  $S_+$  and those disks, together with  $A$  bound a ball in  $B_+$ . Choose an annulus that is innermost in the sense that its ball contains no other ball corresponding to such an annulus. Note that  $T$  then bounds a solid torus  $V$  to the ball side of  $A$ . Because  $T$  is incompressible and meridionally incompressible, there must be a set  $\mathcal{C}$  of at least two vertical components that together prevent  $A$  from being compressible or meridionally compressible in  $V$ . One possibility is that one or more of these components intersect the projection plane in the distinct disks bounded by  $\partial A$ . But it could also be the case that two or more vertical components are linked together above the projection plane inside the ball bounded by  $A$ . In this case, the same vertical components must be similarly linked beneath the projection plane, since ultimately these vertical components form an unlink when considered as a whole. In either case, the solid torus  $V$  must be unknotted, as  $\mathcal{C}$  intersects every meridional disk in  $V$  and, if  $V$  were knotted, this would make  $\mathcal{C}$  a nontrivial link by itself, when all other components are dropped, which is a contradiction to how we constructed  $J'$ .

Again, in either case, there is a compression disk for  $A$  that intersects only vertical components. Dropping all vertical components momentarily, that compression yields a sphere and, since  $J$  is nonsplittable, it must lie to one or the other side of that sphere. Hence,  $J$  lies to one or the other side of  $T$ . If  $J$  is in  $V$ , then to avoid compressions and meridional compressions, there must be vertical components to the other side. However, then again, we find that by dropping  $J$  temporarily, the collection of vertical components forms a nontrivial link, a contradiction.



Hence, it must be the case that  $J$  lies to the outside of  $V$ . Let  $A'$  be any other annulus in  $T \cap B_+$ . Then its ball must also intersect  $\mathcal{C}$ , since  $\mathcal{C}$  is the only collection of vertical components that together wrap all the way around  $V$ . Hence the curves  $\partial A'$  must be parallel to the curves in  $\partial A$ . This implies that there are no disk components in  $T \cap B_+$ , as, if there were such, there would need to be an annular component to each side of its boundary on  $S_+$  to not be innermost, which the existence of  $\mathcal{C}$  prevents.

Hence, we have only annular components remaining in either  $T \cap B_+$  or  $T \cap B_-$ . However, as in Figure 11, right, if any intersection curve bounding an annulus intersects a bubble, there must be disk components, since each vertex is 4-valent, so an annular boundary hitting a saddle cannot also be an annular boundary of a second annulus. Hence, all intersection curves avoid bubbles. But then the boundaries of the annuli separate the projection of  $J$ . We could isotope any annuli away that are not parallel to  $A$ , so it must be the case that all annuli are parallel to  $A$ . Similarly in  $B_-$ , all annuli must be parallel. Hence, there can be only one annulus to either side for  $T$  to be connected. Since  $J$  is to the outside of  $V$ , each disjoint disk on  $S_+$  bounded by  $\partial A$  lies in a single complementary region of  $J$ . But, since the endpoints of the vertical components in  $\mathcal{C}$  lie in these two disks, each such component either has both of its endpoints in the same complementary region or its two endpoints share the same complementary regions as another vertical component. In either case, this contradicts our construction of  $J'$ .

The last case to consider is when there is a component of  $T \cap B_+$  that is an  $n$ -punctured torus. In this case, all other components of  $T \cap B_+$  and  $T \cap B_-$  must be disks. However, then all components to the  $B_-$  side are disks, a possibility we have eliminated.

Thus, we have shown that an essential torus  $T$  is meridionally compressible.  $\square$

**Lemma 2.5** *The link  $J'$  is prime.*

**Proof** That is to say, we show that there are no essential annuli with both boundaries appearing as meridional curves for link components.

We first suppose that the boundary components of  $A$  are meridians on  $\partial N(J)$ . So we think of  $A$  as a twice-punctured sphere. Then  $A$  demonstrates that  $J'$  is a composite link. Since  $J'$  is not composite, it must be that the addition of the vertical components prevents  $A$  from being boundary-parallel. We assume that  $A$  is not meridionally compressible by doing any compressions first and taking only one of the resultant annuli to consider. As we did with  $T$ , we use the results of [9] to put the punctured sphere in standard position relative to  $S_+$  and  $S_-$  so that no intersection curve intersects a bubble more than once and every curve either has a puncture or a bubble on it. We assume that  $A$  has been chosen to minimize the number of saddles in crossing bubbles

only involving  $J$ . We again throw away the vertical components and obtain intersection curves that no longer need to bound disks in  $S_+$  and  $S_-$ , and that can intersect a given bubble more than once. As in (\*) preceding the proof of Theorem 2 in [9], since  $J$  is alternating, a curve that crosses a bubble with its overstrand to one side of the curve must pass through an odd number of punctures (there are only two total) if it subsequently passes through another bubble with its overstrand to the same side.

The same proof we used in the case of an essential meridionally incompressible torus shows that no components of  $A \cap B_{\pm}$  can be other than disks and also annuli with nontrivial boundaries on  $A$ . In fact, no component of  $A \cap B_{\pm}$  can be such an annulus, as any compression disk for such an annulus would intersect only vertical components, but every compression disk for an essential annulus in  $A$  must intersect  $J$ , since  $J$  punctures the sphere corresponding to  $A$  twice.

We could have up to four disks in  $A \cap B_{\pm}$  with punctures on their boundaries, which, in addition to the types of regions depicted in Figure 15, also allow for regions  $Q$ ,  $R$  and  $S$ , depicted in Figure 18, where the intersection curve (in red) crosses the link at a puncture, and a shaded disk represents a tangle. But again, the argument given previously applies to show that there are no disks innermost on  $S_{\pm}$  in  $A \cap B_{\pm}$  that intersect bubbles, either with or without punctures on their boundary.

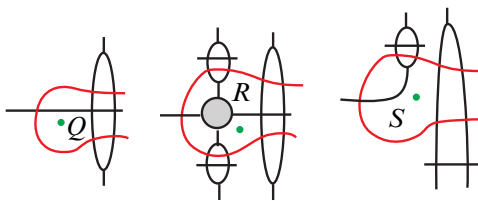


Figure 18: Additional possible regions for an innermost disk with a puncture on its boundary

But then  $A$  intersects  $S_+$  and  $S_-$  in the same pair of arcs from the first puncture to the second puncture, each of which does not intersect the projection of  $J$ . So, to one side, the projection of  $J$  is a trivial arc. There can be no vertical components to this side of  $A$  since there are only two adjacent regions of the projection plane to this side. This shows that  $A$  cannot be an essential annulus with both boundaries meridional on  $N(J)$ .

We now consider an essential annulus  $A$  with both boundary components meridians on boundaries of neighborhoods of the vertical components. It is appropriate to consider  $A$  as a twice-punctured sphere. But then both punctures must be from one vertical component  $C$ , and  $J$  must lie to one side. To the other side is a trivial arc of  $C$

and additional vertical components that prevent  $A$  from being boundary-parallel. But individual vertical components inside this twice-punctured sphere bound disks there, contradicting the fact that the  $n$ -punctured disks they bound are incompressible.

Note that there can be no annulus with one boundary a meridian on  $J$  and another boundary a meridian on a vertical component, as a sphere in  $S^3$  cannot be punctured once by a simple closed curve. Thus, there can be no essential annuli with meridional boundary components in  $S^3 - J'$  and  $J'$  is prime.  $\square$

**Corollary 2.6** *There are no essential tori in  $S^3 - J'$ .*

**Proof** If an essential torus  $T$  exists, Lemma 2.4 implies we can meridionally compress it to obtain a twice-punctured sphere. Since  $T$  is not boundary-parallel, this twice-punctured sphere shows that the link  $J'$  is composite, contradicting Lemma 2.5.  $\square$

**Lemma 2.7** *There are no essential annuli in  $S^3 - J'$ .*

**Proof** Although we have eliminated essential annuli with both boundaries meridians, we now consider the possibility of other essential annuli. Lemma 1.16 of [7] implies that if there is an essential annulus at all, then  $S^3 - J'$  is Seifert fibered, with the boundaries of the annulus as fibers. Moreover, there are either a total of one, two or three torus boundary components in  $S^3 - N(J')$ . If one boundary of  $A$  is a meridian on a vertical component  $C$ , and the other is not, then if we fill  $C$  in, the annulus becomes a compressing disk on the boundary of the resulting link. If the other boundary component lies on the boundary of a neighborhood of  $J$ , this contradicts the hyperbolicity of  $J$ . If the other boundary is on a boundary of a neighborhood of a different vertical component, we have that a nontrivial curve on its boundary is trivial in the complement of  $J$ . However, this can never occur for a trivial link component unless the curve is a longitude, but then we are contradicting the incompressibility of the vertical disk bounded by the component.

If a boundary component of  $A$  is a nonmeridian on a vertical component  $C$ , then we can fill in  $C$  and extend the Seifert fibration to the solid torus that we filled in, making the result Seifert fibered, which is a contradiction to the hyperbolicity of  $S^3 - J$  unless there is a second vertical component. If there is a second vertical component and the other boundary of  $A$  is a nonmeridian upon it (as it must be by our previous considerations), then we can fill it in also, and we obtain a Seifert fibration for  $S^3 - J$ , a contradiction to its hyperbolicity.

The only possibility left is that the boundaries of  $A$  is a nonmeridian upon both  $C_1$  and a component  $K$  of  $J$ , which is all of  $J$ , and there is a second vertical component  $C_2$ .

Then  $S^3 - J'$  is a twice-punctured disk crossed with a circle obtained by taking a regular neighborhood of  $A \cup \partial N(K)$ . This is embedded in  $S^3$  so that each boundary torus bounds a solid torus to the exterior, which is a neighborhood of the corresponding link component. But then the component corresponding to the outer boundary of the disk, which is one of the vertical components, links both of the other components, one of which is also vertical. However, then two vertical components are linked, a contradiction to the construction of  $J'$ .

We now consider an essential annulus  $A$  with both boundary components on  $N(J)$  but not meridians. Then they must both be on the same component or else  $A$  would be essential in the complement of  $J$ , a contradiction. In this case, each boundary of  $A$  is a  $(p, q)$ -curve on the boundary of a neighborhood of the link component  $K$ , with  $|q| \geq 1$ . Hence,  $C$  is a  $(p, q)$ -cable of  $K$ . But then there is an essential annulus with one boundary on  $\partial N(C)$  and a second boundary on  $N(K)$ , a possibility we have already eliminated.  $\square$

**Proof of Theorem 2.1** By work of Thurston [15], in order to prove that  $S^3 - J'$  is hyperbolic, it is enough to show that  $S^3 - J'$  is irreducible, and to show there are no essential tori or annuli in  $S^3 - J'$ . This is the content of Lemma 2.3, Corollary 2.6 and Lemma 2.7.  $\square$

### 3 Generalized belted sums

In this section, we consider generalized belted sums as in Figure 2. We show that if  $L_1 \#_b L_2$  is constructed from two links with a  $2n$ -string tangle at center, with  $n$  belts around adjacent pairs of the exiting strings and  $n - 3$  belts around the central tangle, no two of which are parallel, then  $\text{vol}(L_1 \#_b L_2) = \text{vol}(L_1) + \text{vol}(L_2) - 4(n - 2)3.6638 \dots$ . To see this, we utilize the thrice-punctured spheres that appear in the link complement. Thrice-punctured spheres are known to be totally geodesic with a rigid structure in a hyperbolic 3-manifold (see for instance [1]). In particular, any two are isometric. In the case of a  $2n$ -string tangle  $T_1$  as in Figure 3, there is a collection of thrice-punctured spheres that shield the part of the manifold corresponding to the  $2n$ -string tangle from the rest of the manifold. Cutting the manifold open along this collection of thrice-punctured spheres and then for each resulting piece, doubling across the thrice-punctured spheres yields two link complements, one with the  $2n$ -string tangle appearing twice, once reflected, and the other an untwisted daisy chain with additional components, as appear in Figure 19. Further examples of the untwisted daisy chain with additional components appear in Figure 20. The two halves of the original manifold must have volume exactly half of these, since reflecting across totally geodesic surfaces

(the thrice-punctured spheres; see [1]) doubles the volume. So the original manifold has volume exactly half the sum of these two volumes. The same is true for the link with  $2n$ -string tangle  $T_2$ . Now, when we take the two link complements, cut them both open along the collection of thrice-punctured spheres, and throw away the two halves of the untwisted daisy chain, we obtain the volume of the first link plus the volume of the second link minus the volume of the untwisted daisy chain with additional components.

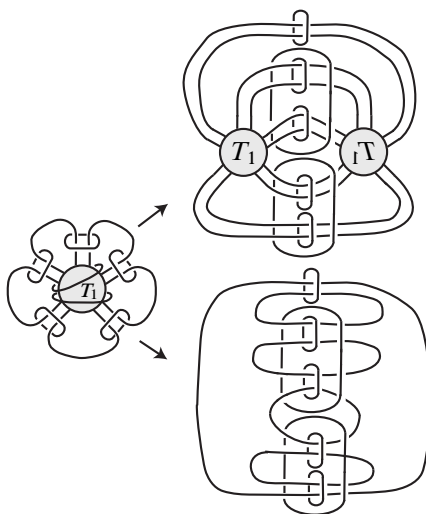
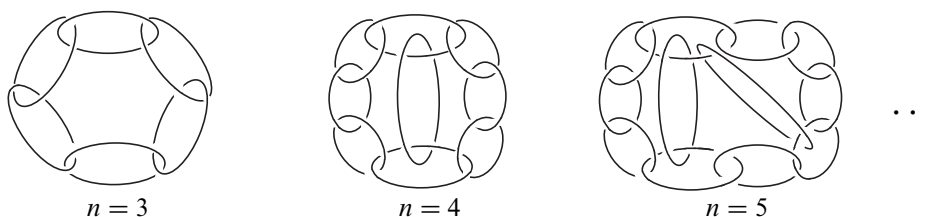


Figure 19: Cutting and doubling

In the case  $n = 3$ , the volume of the untwisted daisy chain is  $4(3.6638\dots)$ , where  $3.6638\dots$  is the volume of an ideal regular octahedron. The manifold is commensurable with the Whitehead link. (See Example 6.8.7 of [14].) For  $n > 3$ , we can cut the link complement open along the twice-punctured disks bounded by components that are not in the untwisted daisy chain to obtain  $n - 2$  pieces, each of volume  $\frac{1}{2}(4(3.6638\dots))$ , as in Figure 20. When we take the belted sum of the two links, we discard all of these pieces from both link complements, meaning we lose a volume of  $4(n - 2)3.6638\dots$ .

Figure 20: Links of volume  $4(n - 2)3.6638\dots$

We can further start with any link and add components to decompose it into pieces, each of which has a volume we can determine as in the case of a generalized belted sum, to obtain the volume of the augmented link, which will bound the volume of the original link since hyperbolic Dehn filling always decreases volume. In the case of an alternating link, Theorem 2.1 tells us that the generalized augmented link that we produce will be hyperbolic, which we need to know for the procedure to apply. As an example, consider the link appearing in Figure 21. We denote the link obtained from a  $2n$ -tangle  $T_i$  by completing it as in Figure 3 by  $L_i$ .

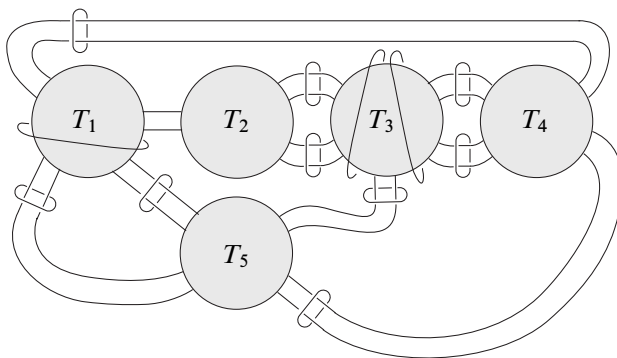


Figure 21: Finding the volume of this link

We can cut along the various thrice-punctured spheres, and realize each of the resulting pieces as a link of the appropriate type, where we have thrown away a volume  $(n-2)3.6638\dots$ . In this case, we decompose the link complement into three 8-tangles, one 6-tangle and one 10-tangle. There is also another piece remaining which is a copy of the Borromean rings. Hence the volume is  $\text{vol}(L_1) + \text{vol}(L_2) + \text{vol}(L_3) + \text{vol}(L_4) + \text{vol}(L_5) - 20(3.6638\dots) + 7.32772\dots$ .

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Received: 20 October 2015      Revised: 2 March 2017

