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The unstabilized canonical Heegaard splitting of a mapping torus

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Let S be a closed orientable surface of genus at least 2. The action of an automorphism f on the curve complex of S is an isometry. Via this isometric action on the curve complex, a translation length is defined on f. The geometry of the mapping torus M_f depends on f. As it turns out, the structure of the minimal-genus Heegaard splitting also depends on f: the canonical Heegaard splitting of M_f, constructed from two parallel copies of S, is sometimes stabilized and sometimes unstabilized. We give an example of an infinite family of automorphisms for which the canonical Heegaard splitting of the mapping torus is stabilized. Interestingly, complexity bounds on f provide insight into the stability of the canonical Heegaard splitting of M_f. Using combinatorial techniques developed on 3–manifolds, we prove that if the translation length of f is at least 8, then the canonical Heegaard splitting of M_f is unstabilized.

57M27; 57M50

1 Introduction

Let S be a closed orientable surface of genus at least 2. Then there is a curve complex $\mathcal{C}(S)$ defined by Harvey [5]. Later, Masur and Minsky [6; 7] assigned a metric d on it and then proved that under this metric, the curve complex is δ–hyperbolic. Assume that f is an automorphism of S. Then f is extended to be an isomorphism of $\mathcal{C}(S)$ and hence an isometry on $(\mathcal{C}(S), d_{\mathcal{C}(S)})$. For simplicity, this isometry is still denoted by f. Then there is a translation length $d(f) = \min\{d_{\mathcal{C}(S)}(C, f(C)) \mid C \in \mathcal{C}^0(S)\}$ defined on f. If f is either reducible or periodic, there is an universal upper bound on the translation length of f^n for any $n \in N$. But if f is a pseudo-Anosov map, $d(f^n)$ goes to infinity as n goes to infinity; see [7, Proposition 7.6]. Conversely, if there is an universal upper bound on the translation length of f^n for any n, then by Thurston's result (see Casson and Bleiler [3]), f is either reducible or periodic. Otherwise, f is a pseudo-Anosov map.

Let $M = S \times I$ be an I–bundle of S. It is known that there are two standard Heegaard splittings for M; see Scharlemann and Thompson [8]. One, called the trivial Heegaard

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splitting, is $S\times[0,0.5]\cup_{S\times\{0.5\}} S\times[0.5,1]$. The other one is as follows. Assume that there are a point $p\in S$ and an arc $a=p\times I$ in $S\times I$. Let $N(a)$ be the closed regular neighborhood of a in $S\times I$, $V_1=\overline{S\times[0.3,0.6]-N(a)}$ and $V_2=\overline{S\times I-V_1}$. Then both V_1 and V_2 are compression bodies. Hence $V_1\cup_{\partial_+V_1} V_2$ is a Heegaard splitting of M.

For the 3–manifold $M=S\times I$, its boundary components consist of two homeomorphic surfaces, $S\times\{0\}$ and $S\times\{1\}$. Thus, gluing these two components by a homeomorphism $f\colon S\times\{1\}\to S\times\{0\}$ produces a closed 3–manifold M_f, called a mapping torus. Here there is a small change in the definition of the translation length of f in M_f, which is $d(f)=\min\{d_{\mathcal{C}(S\times\{0\})}(C\times\{0\}, f(C\times\{1\}))\}$, where $C\times\{0\}$ is an essential simple closed curve in $S\times\{0\}$.

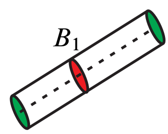

Figure 1: A core disk

It is not hard to see that there is a canonical Heegaard splitting for M_f, as follows. Let $V_2^f=V_2/f$ and let B_1 be the core disk of $N(a)$, as shown in Figure 1. Then V_2^f is homeomorphic to $S\times[0.5,1]\cup_f B_1\times[0,0.5]$, where f maps a disk in $S\times\{1\}$ to $B_1\times\{0\}$. Let $b\subset V_2^f$ be a properly embedded and unknotted arc connecting $S\times\{0.5\}$ and $S\times\{1\}$ and B_2 be the core disk of $N(b)$. Then

$$H_2=\overline{V_2^f-N(b)}$$

is a handlebody. Equivalently,

$$H_2=\overline{S-B_2}\times[0.5,1]\cup_f B_1\times[0,0.5].$$

Moreover, H_1, the complement of H_2 in M_f, is given by

$$H_1=\overline{S-B_1}\times[0,0.5]\cup_f B_2\times[0.5,1].$$

So it is also a handlebody. Since $\partial H_1=\Sigma=\partial H_2$, $H_1\cup_\Sigma H_2$ is a Heegaard splitting of M_f, called the canonical Heegaard splitting.

A Heegaard splitting is *stabilized* if there is a pair of essential disks in two compression bodies such that their boundaries intersect in one point. If a Heegaard splitting is stabilized, then there is a move called a *destabilization* on it, which produces a smaller-genus Heegaard splitting. Thus, to study a Heegaard splitting of a 3–manifold, it is sufficient to study the destabilized one. Furthermore, there are some problems related to a Heegaard splitting, which all require that the Heegaard splitting is unstabilized. For example, the rank-versus-genus problem of a 3–manifold, ie when is $r(M) = g(M)$? Hence, for a given Heegaard splitting, it is a priority to determine its stability.

If f is periodic, then $S \times S^1$ is a finite covering of M_f, so M_f has the geometry of $H^2 \times R$; if f is reducible, then M_f contains at least one essential torus; if f is a pseudo-Anosov map, then Thurston [11, Theorem 0.1] proved that M_f is a hyperbolic 3–manifold. From this point of view, the geometry of M_f is determined by f. Moreover, the stability of its canonical Heegaard splitting is also influenced by f. For example, Schultens [9, Theorem 5.7] proved that if f is isotopic to an identity map, then the canonical Heegaard splitting of M_f is unstabilized; Souto and Biringer [10, Theorem 1.1; 2, Theorem 1.1] proved that if the pseudo-Anosov map f is complicated enough, the canonical Heegaard splitting is unstabilized; Bachmann and Schleimer [1, Corollary 3.2] proved that if the $d(f) \geq 2g(S)$, then the canonical Heegaard splitting is unstabilized and minimal.

With all these supporting results, it seems that the canonical Heegaard splitting of every mapping torus is unstabilized. However, this is not true in general; see Example 1.1.

Example 1.1 Let α and β be two essential simple closed curves in S, where $\alpha \cap \beta$ is a point p. It is known that $\tau_\alpha \circ \tau_\beta$, the concatenation of the two Dehn twists τ_α and τ_β, maps α to β. Let $S_\beta = \overline{S - \beta}$. By Thurston's classification [3] of automorphisms of a surface, there is a pseudo-Anosov map g on S_β fixing its boundary pointwise such that the translation length satisfies $d(g)|_{S_\beta} \geq 6$. Naturally g induces an automorphism on S, still denoted by g. Then $f = g \circ (\tau_\alpha \circ \tau_\beta)$.

Since $\alpha \times [0, 0.5]$ intersects $\beta \times [0.5, 1]$ in one point p on $S \times \{0.5\}$, there are two points $p_1, p_2 \in \alpha \times \{0.5\}$ disjoint from p such that $f(p_2 \times \{1\}) \neq p_1 \times \{0\}$. Let $a = p_1 \times [0, 0.5]$ and $b = p_2 \times [0.5, 1]$. Then both

$$H_1 = \overline{S \times \{0, 0.5\} - N(a)} \cup_f N(b) \quad \text{and} \quad H_2 = \overline{S \times \{0.5, 1\} - N(b)} \cup_f N(a)$$

are handlebodies. Moreover,

$$\overline{\alpha \times [0, 0.5] - N(a)} \quad \text{and} \quad \overline{\beta \times [0.5, 1] - N(b)}$$

are essential disks in H_1 and H_2, respectively, where they intersect in one point p. This means that the Heegaard splitting $H_1 \cup_\Sigma H_2$ is stabilized.

Remark 1.2 In Example 1.1, the translation length of g in S_β is at least 6. It is known that for any $n \in N$, there is an automorphism g of S_β whose translation length restricted to S_β is larger than n. So there are infinitely many choices of g in Example 1.1. Hence there are infinitely many choices of f on S.

So there is a question:

Question 1.3 What is the least value of $d(f)$ such that the canonical Heegaard splitting of M_f is unstabilized?

With tools developed in the curve complex, we give a partial answer to this question.

Theorem 1.4 *If the translation length satisfies $d(f) \geq 8$, then the canonical Heegaard splitting of M_f is unstabilized.*

This paper is organized as follows. We introduce some lemmas in Section 2, and prove the main theorem in Section 3.

2 Some lemmas

Let $\mathcal{C}(S)$ be the curve complex of S. Masur and Minsky proved:

Lemma 2.1 [6, Proposition 4.6] *$(\mathcal{C}(S), d)$ is connected and the diameter is infinite.*

Let $F \subset S$ be a subsurface. Then F is *essential* if there is no incompressible simple closed curve in F bounding a disk in S. If the subsurface F is essential, then Masur and Minsky [7, Section 2.2] introduced the subsurface projection on F for all of those vertices in the curve complex, as follows. For any vertex $\alpha \subset \mathcal{C}^0(S)$, by the bigon criterion [4, Proposition 1.7], there is a representative curve in its isotopy class that intersects ∂F essentially, ie there is no bigon capped by them in S. So the subsurface projection $\pi_F(\alpha)$ is defined to be one essential component of $\partial N(\alpha \cup \partial F)$ in F depending on choice.

An essential simple closed curve α cuts F if $\pi_F(\alpha) \neq \varnothing$. For any two given disjoint essential simple closed curves α and β, if they both cut F, then

$$d_{\mathcal{C}(F)}(\pi_F(\alpha), \pi_F(\beta)) \leq 2.$$

In general, Masur and Minsky proved:

Lemma 2.2 [7, Lemma 2.2] *Let F and S be as above, and let $\mathcal{G} = \{\alpha_0, \dots, \alpha_k\}$ be a geodesic in $\mathcal{C}(S)$ such that α_i cuts F for each $0 \leq i \leq k$. Then $d_{\mathcal{C}(F)}(\alpha_0, \alpha_k) \leq 2k$.*

It is known that when ∂F is connected, no component of $\pi_F(\alpha)$ cuts out a planar surface in F. But if ∂F is not connected, it is possible that some element of $\pi_F(\alpha)$ does cut out a planar subsurface of F. In this case, we introduce the definition of a strongly essential curve in F, which is defined in [12].

Definition 2.3 An essential simple arc or simple closed curve $c \subset F$ is *strongly essential* if no component of $\pi_F(c)$ cuts out a planar subsurface in F.

Let F be a compact orientable surface of genus at least 1 with connected boundary. For the handlebody $F \times [0,1]$, each essential disk intersects ∂F nontrivially. Moreover:

Lemma 2.4 *For any essential disk $D \subset F \times [0,1]$, there is an essential disk D_1 such that:*

(1) $\partial D_1 \cap F \times \{1\}$ *is connected and isotopic to a component of* $\partial D \cap F \times \{1\}$*;*

(2) $D_1 = (\partial D_1 \cap F \times \{1\}) \times [0,1]$*;*

(3) $\partial D_1 \cap F \times \{0\}$ *is disjoint from some component of* $\partial D \cap F \times \{0\}$*.*

Proof Without loss of generality, for any two essential disks in H, it is assumed that their intersection consists of arcs. Since ∂D intersects $F \times \{1\}$ nontrivially, there is an arc $a \subset \partial D \cap F \times \{1\}$ such that the number of components of $(a \times I) \cap D$ is minimal among all arcs in $\partial D \cap F \times \{1\}$.

Let $D_a = a \times I$. An essential arc $\alpha \subset F \times \{0\}$ is called a 0–arc. Similarly, an essential arc $\beta \subset (\partial H - F \times \{0\})$ is called an 1–arc. It is not hard to see that the boundary curve of D consists of alternating 1–arcs and 0–arcs while the boundary curve of D_a consists of one 1–arc and one 0–arc.

If $D_a \cap D = \varnothing$, then the proof is finished. So suppose that $D_a \cap D \neq \varnothing$. Then there is an outermost disk B in D where $B \cap D_a$ is an arc. Since $a \subset D \cap F \times \{1\}$, all of those intersecting arcs between D_a and D have ends in $\partial D_a \cap F \times \{0\}$. Therefore there is a 0–arc of $\partial B \cap F \times \{0\}$ in $\partial D \cap F \times \{0\}$ disjoint from $\partial D_a \cap F \times \{0\}$, for if not, then ∂B contains only one 1–arc and no 0–arc. Doing a boundary compression on D_a along B, D_a is changed into two disks $D_{a,1}$ and $D_{a,2}$. Since D intersects D_a essentially, these two disks are both essential. As one of $D_{a,1}$ and $D_{a,2}$ intersects $F \times \{0\}$ in one arc, one of these two disks is an I–bundle of the 1–arc of $\partial B \cap F \times \{1\}$. Without loss of generality, let $D_{a,1}$ be this disk. By the boundary compression surgery, the 1–arc of $\partial B \cap F \times \{1\}$ lies in ∂B and therefore in $\partial D \cap F \times \{1\}$. So $D_{a,1}$ is an I–bundle of some component of $\partial D \cap F \times \{1\}$. Moreover,

$$|\partial D_{a,1} \cap \partial D| \leq |\partial D_a \cap \partial D| - 2.$$

But this contradicts the choice of D_a. Then $\alpha \times \{0\}$ is disjoint from some 0–arc of ∂B and hence some 0–arc of ∂D. □

Similarly, there is also an essential disk $D_2 \subset F \times [0,1]$ such that:

(1) $\partial D_2 \cap F \times \{0\}$ is connected and isotopic to a component of $\partial D \cap F \times \{0\}$;
(2) $D_2 = (\partial D_2 \cap F \times \{0\}) \times [0, 1]$;
(3) $\partial D_2 \cap F \times \{1\}$ is disjoint from some component of $\partial D \cap F \times \{1\}$.

3 Proof of Theorem 1.4

Let f, d, S, M_f, a, b, Σ, H_1, H_2, $H_1 \cup_\Sigma H_2$, B_1 and B_2 be as in Section 1. Then the main theorem is written as follows:

Proposition 3.1 *If the translation length satisfies $d(f) \geq 8$, then $H_1 \cup_\Sigma H_2$ is unstabilized.*

Before proving Proposition 3.1, we need the following lemma:

Lemma 3.2 *For any essential simple closed curve C bounding two essential disks in H_1 and H_2 simultaneously, both $C \cap \partial B_1 \neq \varnothing$ and $C \cap \partial B_2 \neq \varnothing$.*

Proof Since $S \times I$ is irreducible and its boundary components are incompressible, M_f is irreducible and not homeomorphic to S^3.

The construction of $H_1 \cup_\Sigma H_2$ in Section 1 says that

$$H_1 = \overline{S - B_1} \times [0, 0.5] \cup B_2 \times [0.5, 1]$$

and

$$H_2 = \overline{S - B_2} \times [0.5, 1] \cup B_1 \times [0, 0.5].$$

Assume that C bounds an essential disk D (resp. E) in H_1 (resp. H_2). If we consider the intersection between E and B_1 in H_2, then:

Fact 3.3 $$C \cap \partial B_1 \neq \varnothing.$$

Proof Suppose the conclusion is false. Then C is either isotopic to ∂B_1 or disjoint from ∂B_1. Since ∂B_1 bounds no disk in H_1, C is not isotopic to ∂B_1. Thus C is disjoint from ∂B_1. Moreover, C is strongly essential in $\Sigma_{B_1} = \overline{\Sigma - \partial B_1}$, for if not, then C cuts out a pair of pants P in Σ_{B_1} such that ∂P consists of two copies of ∂B_1 and C. Since C bounds an essential disk E in H_2, E cuts out a solid torus $ST \subset H_2$ containing B_1. Similarly, the essential disk D also cuts out a solid torus in H_1. Then the Heegaard splitting $H_1 \cup_\Sigma H_2$ is a connected sum of a genus-1 Heegaard splitting and a smaller-genus Heegaard splitting. Because M_f is irreducible, one of these two Heegaard splittings is of S^3, which implies that the genus-1 Heegaard splitting is not of S^3. The reason is that since the longitude l of the solid torus ST intersects ∂B_1 in one point, l intersects $S\times\{t\}$ in one point for some $t\in(0,0.5)$. So the representative of l in $\pi_1 M_f$ is nontrivial. Then the Heegaard splittings of genus $(g(\Sigma)-1)$ belongs to S^3. Hence, under this circumstance, M_f is a lens space. Moreover, it contains a closed embedded genus at least 1 incompressible surface. But it contradicts the fact that there is no positive genus closed incompressible surface in a lens space.

After removing $N(a)$ from H_2, H_2 is changed into

$$H_2^{B_1}=\overline{S\times[0.5,1]-N(b)}.$$

Let

$$H_1^*=\overline{M_f-H_2^{B_1}}.$$

Equivalently, $H_1^*=S\times[0,0.5]\cup N(b)$. Since C is strongly essential in Σ_{B_1} and $C\cap\partial B_1=\varnothing$, C is essential in $\partial H_2^{B_1}$. So E is also an essential disk in $H_2^{B_1}$. The I–bundle structure of $H_2^{B_1}$ implies that $C=\partial E$ intersects ∂B_2 nontrivially. Since C (resp. ∂B_2) bounds an essential disk D (resp. B_2) in H_1^*, by the standard outermost disk argument, there is an outermost disk of D in $S\times[0,0.5]=\overline{H_1^*-B_2}$. By the proof of Lemma 2.6 in [12], this outermost disk is a properly embedded essential disk of $S\times[0,0.5]$. But this contradicts the fact that $\partial(S\times[0,0.5])$ is incompressible in $S\times[0,0.5]$. □

Similarly, $C\cap\partial B_2\neq\varnothing$. This completes the proof of Lemma 3.2. □

Then the proof of the Proposition 3.1 is written as follows:

Proof of Proposition 3.1 Since $S\times I$ is irreducible and its boundary components are incompressible, M_f is irreducible and not homeomorphic to S^3.

Suppose that the conclusion is false. Then $H_1\cup_\Sigma H_2$ is stabilized. It is known that each stabilized Heegaard splitting is either reducible or a genus-1 Heegaard splitting of S^3. Since M_f is not homeomorphic to the S^3, the canonical Heegaard splitting $H_1\cup_\Sigma H_2$ is reducible. Therefore, there is an essential simple closed curve $C\subset\Sigma$ such that C bounds an essential disk D (resp. E) in H_1 (resp. H_2).

It is not hard to see that there is an isotopy on D such that $\partial D\cap\partial E=\varnothing$ (just pushing ∂D away from ∂E). Without loss of generality, it is assumed that ∂D intersects $\partial B_1\sqcup\partial B_2$ essentially, ie there is no bigon capped by any two of them in Σ. By Lemma 3.2, neither ∂B_1 nor ∂B_2 is disjoint from C. Then $D\cap B_2\neq\varnothing$. Furthermore, we assume that D intersects B_2 minimally. So $D\cap B_2$ consists of arcs and no closed circle. By the standard outermost disk argument, there is an outermost disk

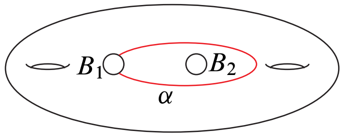

Figure 2: A one-hole bigon

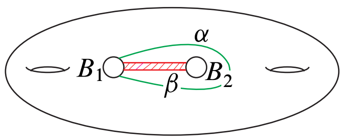

Figure 3: The case where ∂B_1, ∂B_2 and β bound a rectangle

in D bounded by a component $\alpha \subset \partial D$ and an arc of $D \cap B_2$. Similarly, there is an outermost disk in E bounded by a component $\beta \subset \partial E$ and an arc of $E \cap B_1$.

Let $\Sigma_{B_1} = \overline{\Sigma - \partial B_1}$ and $\Sigma_{B_2} = \overline{\Sigma - \partial B_2}$. Then:

Claim 3.4 *The arc α (resp. β) is strongly essential in Σ_{B_2} (resp. Σ_{B_1}).*

Proof We prove this claim for α only; the other case is similar.

Since ∂B_2 is nonseparating in Σ, Σ_{B_2} has two boundary curves C_1 and C_2. Suppose α is not strongly essential in Σ_{B_2}. Then α cuts out an annulus in Σ_{B_2} which contains one boundary component of Σ_{B_2}, for example, C_2. So

$$|C \cap C_2| \leq |C \cap C_1| - 2.$$

But it contradicts the fact that C_1 and C_2 are isotopic in Σ. □

Let $H_1^{B_2} = \overline{H_1 - B_2}$ and $H_2^{B_1} = \overline{H_2 - B_1}$. Since C intersects both ∂B_1 and ∂B_2 essentially, there is no bigon capped by α and ∂B_1 (resp. β and ∂B_2) in Σ_{B_2} (resp. Σ_{B_1}). Furthermore:

Claim 3.5 *There is no one-hole bigon capped by α and ∂B_1 in Σ_{B_2}.*

Note 3.6 A one-hole bigon is shown in Figure 2.

Proof of Claim 3.5 Suppose that the conclusion is false. Then there is a one-hole bigon capped by α and ∂B_1 in Σ_{B_2}. Since $\beta \cap \alpha = \varnothing$ and $\partial \beta \subset \partial B_1$, either $\beta \cap \partial B_2 = \varnothing$ or β intersects ∂B_2 in at most two points. In the latter case, there is a rectangle bounded by ∂B_1, ∂B_2 and β; see Figure 3. For both of these two cases, it is not hard to see that

$\pi_{\Sigma_{B_1}}(\beta)$ is disjoint from $\partial B_1 \cup \partial B_2$ up to isotopy. But since β is in the boundary of the outermost disk in E and strongly essential in Σ_{B_1}, $\pi_{\Sigma_{B_1}}(\beta)$ bounds an essential disk in $H_2^{B_1}$. So $\pi_{\Sigma_{B_1}}(\beta) \cap \partial B_1 \neq \varnothing$ up to isotopy. This is a contradiction. □

Similarly, there is no one-hole bigon capped by β and ∂B_2 in Σ_{B_1}.

Although ∂D intersects ∂B_1 and ∂B_2 minimally, it is possible there is a rectangle bounded by ∂B_1, ∂B_2 and α in Σ_{B_2}; see Figure 4.

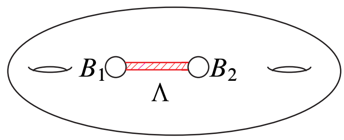

Figure 4: A rectangle

Let

$$S_1 = S_1 \times \{0.5\} = \overline{S \times \{0.5\} - B_1},$$
$$S_3 = S_3 \times \{0.5\} = \overline{S \times \{0.5\} - B_2},$$
$$S_2 = S_1 \cap S_3.$$

Then $H_1^{B_2} = S_1 \times [0, 0.5]$ and $H_2^{B_1} = S_3 \times [0.5, 1]$.

Claim 3.7 *There is no rectangle bounded by ∂B_1, ∂B_2 and α in Σ_{B_2}.*

Proof Without loss of generality, we assume that both $\partial\alpha$ and $\partial\beta$ are in S_2. The other cases are similar, so we omit them here.

Suppose the conclusion is false. Then there is a rectangle Λ bounded by ∂B_1, ∂B_2 and α in Σ. Although the proof is similar to the proof of Lemma 3.9 in [13], for integrity, it is written here. If $\beta \cap \Lambda \neq \varnothing$, then $\Lambda \cap \beta$ is one or two arcs connecting ∂B_1 and ∂B_2. Otherwise there is at least one point in $\alpha \cap \beta$. Since $\beta \cap \partial B_1 = \partial\beta$ and $\alpha \cap \partial B_2 = \partial\alpha$, there is an isotopy on β such that β is pushed away from Λ. Moreover, $\alpha \cap \beta = \varnothing$. Therefore we may assume that β is disjoint from Λ.

For simplicity, $\pi_{\Sigma_{B_2}}(\alpha)$, disjoint from β, is abbreviated by α. It is not hard to see that there is a bigon capped by α and ∂B_1. Then there is an isotopy on α such that there is no bigon capped by α and ∂B_1 anymore. As a result of this process, by the proof of Claim 3.5, there is no one-hole bigon bounded by α and ∂B_1 in Σ_{B_2}. At the end, there is no bigon or one-hole bigon capped by α and ∂B_1 in Σ_{B_2}. So α intersects ∂B_1 in $\partial H_1^{B_2}$ essentially (for if not, then there is a bigon capped by them, which corresponds to a one-hole bigon or a bigon in Σ_{B_2}). On one hand, since $H_1^{B_2} = S_1 \times [0, 0.5]$, by Lemma 2.4, there is one essential arc $a \subset \alpha \cap S_1 \times \{0.5\}$ such that $a \times \{0\}$ is disjoint from some component $c \subset \alpha \cap S_1 \times \{0\}$. On the other hand, for the subsurface $S_2 \subset \Sigma_{B_2}$, since $S_1 = S_2 \cup B_2$, we have $\alpha \cap S_1 = \alpha \cap S_2$. Then $a \subset S_2$. Since β intersects no bigon bounded by α and ∂B_1 in this isotopy, $\alpha \cap \beta = \varnothing$. Hence $a \cap \beta = \varnothing$.

If the union of β, ∂B_1 and ∂B_2 bound a rectangle in Σ_{B_1}, then $\pi_{\Sigma_{B_1}}(\beta)$, still denoted by β, misses α. Otherwise $\alpha \cap \beta \neq \varnothing$. By the same argument as above, there is also an isotopy on β such that there is no bigon bounded by β and ∂B_2 anymore. As a result of this process, by the proof of Claim 3.5, there is no one-hole bigon bounded by β and ∂B_2 in Σ_{B_1}. Therefore there is no bigon or one-hole bigon capped by β and ∂B_2 in $\partial H_2^{B_1}$. So β intersects ∂B_2 in $\partial H_2^{B_1}$ essentially. On one hand, since $H_2^{B_1} = S_3 \times [0.5, 1]$, by Lemma 2.4, there is one essential arc $b \subset \beta \cap S_1 \times \{0.5\}$ such that $b \times \{1\}$ is disjoint from some component $d \subset \beta \cap S_3 \times \{1\}$. On the other hand, for the subsurface $S_2 \subset \Sigma_{B_1}$, since $S_3 = S_2 \cup B_1$, we have $\beta \cap S_3 = \beta \cap S_2$. Then $b \subset S_2$. Since α intersects no bigon bounded by β and ∂B_2 in the isotopy, $\alpha \cap \beta = \varnothing$. Hence $a \cap b = \varnothing$.

Since $\alpha \cap \beta = \varnothing$, $c \cap f(d) = \varnothing$. Hence

$$\begin{aligned}
\pi_{S_2}(a) \cap \pi_{S_2}(b) &= \varnothing;\\
\pi_{S_1 \times \{0\}}(c) \cap f(\pi_{S_3 \times \{1\}}(d)) &= \varnothing;\\
d_{\mathcal{C}(S_3 \times \{1\})}(\pi_{S_3 \times \{1\}}(b \times \{1\}), \pi_{S_3 \times \{1\}}(d)) &\leq 2;\\
d_{\mathcal{C}(S_1 \times \{0\})}(\pi_{S_1 \times \{0\}}(a \times \{0\}), \pi_{S_1 \times \{0\}}(c)) &\leq 2.
\end{aligned}$$

For simplicity, $\pi_{S_2}(a)$ (resp. $\pi_{S_2}(b)$) is abbreviated by a (resp. b). Since $a \times \{1\} \subset S_3 \times \{1\}$ intersects $b \times \{1\}$ trivially, the above equations and inequalities are changed as follows:

$$\begin{aligned}
d_{\mathcal{C}(S_1 \times \{0\})}(a \times \{0\}, \pi_{S_1 \times \{0\}}(c)) &\leq 2;\\
d_{\mathcal{C}(S_1 \times \{0\})}(\pi_{S_1 \times \{0\}}(c), f(\pi_{S_3 \times \{1\}}(d))) &\leq 1;\\
d_{\mathcal{C}(S_3 \times \{1\})}(\pi_{S_3 \times \{1\}}(d), b \times \{1\})) &\leq 2;\\
d_{\mathcal{C}(S_3 \times \{1\})}(b \times \{1\}, a \times \{1\}) &\leq 1.
\end{aligned}$$

It is known that every essential simple closed curve of $S_1 = \overline{S - B_1}$ is essential in S, and similarly for $S_3 = \overline{S - B_2}$. Then by the triangle inequality,

$$\begin{aligned}
d_{\mathcal{C}(S\times\{0\})}(a\times\{0\}, f(a\times\{1\})) &\le d_{\mathcal{C}(S\times\{0\})}(a\times\{0\}, \pi_{S_1\times\{0\}}(c)) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(\pi_{S_1\times\{0\}}(c), f(\pi_{S_3\times\{1\}}(d))) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(b\times\{1\}), f(a\times\{1\})) \\
&\le d_{\mathcal{C}(S_1\times\{0\})}(a\times\{0\}, \pi_{S_1\times\{0\}}(c)) \\
&\quad + d_{\mathcal{C}(S_1\times\{0\})}(\pi_{S_1\times\{0\}}(c), f(\pi_{S_3\times\{1\}}(d))) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(b\times\{1\}), f(a\times\{1\})) \\
&\le 2+1+d_{\mathcal{C}(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(b\times\{1\}), f(a\times\{1\})) \\
&\le 3+d_{\mathcal{C}(S\times\{1\})}(\pi_{S_3\times\{1\}}(d), b\times\{1\}) \\
&\quad + d_{\mathcal{C}(S\times\{1\})}(b\times\{1\}, a\times\{1\}) \\
&\le 3+d_{\mathcal{C}(S_3\times\{1\})}(\pi_{S_3\times\{1\}}(d), b\times\{1\}) \\
&\quad + d_{\mathcal{C}(S_3\times\{1\})}(b\times\{1\}, a\times\{1\}) \\
&\le 6.
\end{aligned}$$

But this contradicts the choice of f.

So there is no rectangle bounded by β, ∂B_1 and ∂B_2 in Σ_{B_1}. Moreover, there is no one-hole bigon or bigon capped by $\pi_{\Sigma_{B_1}}(\beta)$ and ∂B_2 in Σ_{B_1}. Otherwise, there is either a rectangle bounded by β, ∂B_1 and ∂B_2 in Σ_{B_1} or a one-hole bigon bounded by β and ∂B_1, which is prohibited by Claim 3.5. Then each component of $\pi_{\Sigma_{B_1}}(\beta)$ intersects ∂B_2 essentially in $\partial H_2^{B_1}$. On one hand, since $H_2^{B_1} = S_3\times[0.5, 1]$, by Lemma 2.4, there is one component $b \subset \pi_{\Sigma_{B_1}}(\beta)\cap S_3$ such that $b\times\{1\}$ is disjoint from one component d of $\pi_{\Sigma_{B_1}}(\beta)\cap S_3\times\{1\}$. On the other hand, since $\pi_{\Sigma_{B_1}}(\beta)\cap S_3 = \pi_{\Sigma_{B_1}}(\beta)\cap S_2$, we have $b\subset \pi_{\Sigma_{B_1}}(\beta)\cap S_2$.

Since $\alpha\cap\beta=\varnothing$, $a\cap b$ consists of at most two points, where the worst scenario is that ∂a is not separated by β in ∂B_1. Since $\partial a\subset\partial B_1$ and $\partial b\subset \partial B_2$, $\pi_{S_2}(a)\cap\pi_{S_2}(b)$ consists of at most two points. For simplicity, $\pi_{S_2}(a)$ (resp. $\pi_{S_2}(b)$) is abbreviated by a (resp. b). Then

$$d_{\mathcal{C}(S_3\times\{1\})}(b\times\{1\}, a\times\{1\}) \le 2.$$

By the same argument as above, $d(f)\le 7$. □

Similarly, there is no rectangle bounded by ∂B_1, ∂B_2 and β in Σ.

By Claims 3.5 and 3.7, there is neither a one-hole bigon nor a bigon capped by $\pi_{\Sigma_{B_2}}(\alpha)$ and ∂B_1 in Σ_{B_2}. Otherwise there is a rectangle bounded by the union of α, ∂B_1

and ∂B_2. This means that $\pi_{\Sigma_{B_2}}(\alpha)$ intersects ∂B_1 in $\partial H_1^{B_2}$ essentially without doing any further isotopy. Similarly, $\pi_{\Sigma_{B_1}}(\beta)$ intersects ∂B_2 in $\partial H_2^{B_1}$ essentially without doing any further isotopy too.

Then it is not hard to see that:

Fact 3.8 *Each component of $\pi_{\Sigma_{B_2}}\alpha \cap S_2$ intersects every component of $\pi_{\Sigma_{B_1}}\beta \cap S_2$ in at most two points.*

Proof It is sufficient to prove that there are at most two points in $\pi_{\Sigma_{B_2}}\alpha \cap \pi_{\Sigma_{B_1}}\beta$. Since α is disjoint from β, the worst scenario is that $\alpha \cap \partial B_1$ is separated by $\partial\beta$ while $\beta \cap \partial B_2$ is separated by $\partial\alpha$. Then there are two points in $\pi_{\Sigma_{B_2}}\alpha \cap \pi_{\Sigma_{B_1}}\beta$. So the conclusion holds. □

For simplicity, $\pi_{\Sigma_{B_2}}(\alpha)$ (resp. $\pi_{\Sigma_{B_1}}(\beta)$) is abbreviated by α (resp. β). Then:

Claim 3.9 *There is an essential simple closed curve γ in S such that*

$$d_{C(S\times\{0\})}(f(\gamma\times\{1\}), \gamma\times\{0\}) \leq 7.$$

Proof Since α bounds an essential disk in $S_1 \times [0, 0.5]$, by Lemma 2.4, there is a component a of $\alpha \cap S_1 \times \{0.5\}$ such that $a \times \{0\} \subset S_1 \times \{0\}$ is disjoint from some component $c \subset \alpha \cap S_1 \times \{0\}$. Similarly, there are two such components b and d for β.

By Fact 3.8, a intersects b in at most two points. Since $\partial a \subset \partial B_1$ and $\partial b \subset \partial B_2$, $\pi_{S_2}(a)$ intersects $\pi_{S_2}(b)$ in at most two points. Then since $g(S) \geq 2$, there is a strongly essential simple closed curve γ in S_2 disjoint from both α and β and hence from both $\pi_{S_2}(a)$ and $\pi_{S_2}(b)$. Let $\gamma \times [0, 5, 1]$ and $\gamma \times [0, 0.5]$ be the product I–bundles in $S \times [0.5, 1]$ and $S \times [0, 0.5]$, respectively. Then

$$\gamma \times \{1\} \cap \pi_{S_3}(b \times \{1\}) = \varnothing \quad \text{and} \quad \gamma \times \{0\} \cap \pi_{S_1}(a \times \{0\}) = \varnothing.$$

For simplicity, $\pi_{S_2}(a)$ (resp. $\pi_{S_2}(b)$) is abbreviated by a (resp. b). Therefore $\pi_{S_3} b \times \{1\}$ (resp. $\pi_{S_1} a \times \{0\}$) is isotopic to $b \times \{1\}$ (resp. $a \times \{0\}$). Then by the proof of Claim 3.7,

$$\begin{aligned} d_{C(S\times\{0\})}(\gamma\times\{0\}, f(\gamma\times\{1\})) \leq\ & d_{C(S\times\{0\})}(\gamma\times\{0\}, a\times\{0\}) \\ &+ d_{C(S\times\{0\})}(a\times\{0\}, \pi_{S_1\times\{0\}}(c)) \\ &+ d_{C(S\times\{0\})}(\pi_{S_1\times\{0\}}(c), f(\pi_{S_3\times\{1\}}(d))) \\ &+ d_{C(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\ &+ d_{C(S\times\{0\})}(f(b\times\{1\}), f(\gamma\times\{1\})) \end{aligned}$$

$$
\begin{aligned}
&\le 1 + d_{\mathcal{C}(S_1\times\{0\})}(a\times\{0\}, \pi_{S_1\times\{0\}}(c)) \\
&\quad + d_{\mathcal{C}(S_1\times\{0\})}(\pi_{S_1\times\{0\}}(c), f(\pi_{S_3\times\{1\}}(d))) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(b\times\{1\}), f(\gamma\times\{1\})) \\
&\le 1+2+1+ d_{\mathcal{C}(S\times\{0\})}(f(\pi_{S_3\times\{1\}}(d)), f(b\times\{1\})) \\
&\quad + d_{\mathcal{C}(S\times\{0\})}(f(b\times\{1\}), f(\gamma\times\{1\})) \\
&\le 4 + d_{\mathcal{C}(S\times\{1\})}(\pi_{S_3\times\{1\}}(d), b\times\{1\}) \\
&\quad + d_{\mathcal{C}(S\times\{1\})}(b\times\{1\}, \gamma\times\{1\}) \\
&\le 4 + d_{\mathcal{C}(S_3\times\{1\})}(\pi_{S_3\times\{1\}}(d), b\times\{1\}) \\
&\quad + d_{\mathcal{C}(S_3\times\{1\})}(b\times\{1\}, \gamma\times\{1\}) \\
&\le 7.
\end{aligned}
$$

This completes the proof of Claim 3.9. □

By Claim 3.9, the translation length of f is at most 7. This contradicts the assumption on f and completes the proof of Proposition 3.1. □

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