

# Nine generators of the skein space of the 3–torus

ALESSIO CARREGA

We show that the skein vector space of the 3–torus is finitely generated. We show that it is generated by nine elements: the empty set, some simple closed curves representing the nonzero elements of the first homology group with coefficients in  $\mathbb{Z}_2$ , and a link consisting of two parallel copies of one of the previous nonempty knots.

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## 1 Introduction

An alternative approach to representation theory for *quantum invariants* is provided by *skein theory*. The word “skein” and the notion were introduced by Conway in 1970 for his model of the *Alexander polynomial*. This idea became quite useful after the work of Kauffman [10] which redefined the *Jones polynomial* in a very simple and combinatorial way passing through the *Kauffman bracket*. These combinatorial techniques allow us to reproduce all quantum invariants arising from the representations of  $U_q(\mathfrak{sl}_2)$  without any reference to representation theory. This also leads to many interesting and quite easy computations. This skein method was used by Blanchet, Habegger, Masbaum and Vogel [1], Kauffman and Lins [11] and Lickorish [12; 13; 15; 14] to reinterpret and extend some of the methods of representation theory.

The first notion in skein theory is that of a “*skein vector space*” (or *skein module*). These are vector spaces ( $R$ –modules) associated to oriented 3–manifolds, where the base field is equipped with a fixed invertible element  $A$ . These were introduced independently in 1988 by Turaev [24] and in 1991 by Przytycki [20]. We can think of them as an attempt to get an algebraic topology for knots: they can be seen as homology spaces obtained using isotopy classes instead of homotopy or homology classes. In fact, they are defined taking a vector space generated by subobjects (*framed links*) and then quotienting them by some relations. In this framework, the following questions arise naturally and are still open in general:

- Question 1.1**
- Are skein spaces (modules) computable?
  - How powerful are they to distinguish 3–manifolds and links?

- Do the vector spaces (modules) reflect the topology/geometry of the 3–manifolds (eg surfaces, geometric decomposition)?
- Does this theory have a functorial aspect? Can it be extended to a functor from a category of cobordisms to the category of vector spaces (modules) and linear maps?

Skein spaces (modules) can also be seen as deformations of the ring of the  $SL_2(\mathbb{C})$ –character variety of the 3–manifold; see Bullock [3]. Moreover, they are useful to generalize the Kauffman bracket, hence the Jones polynomial, to manifolds other than  $S^3$ . Thanks to Hoste and Przytycki [9], Przytycki [22] and (with different techniques) Costantino [4], now we can define the Kauffman bracket also in the connected sum  $\#_g(S^1 \times S^2)$  of  $g \geq 0$  copies of  $S^1 \times S^2$ .

Currently, there are only few 3–manifolds whose skein space (module) is known; see for instance Bullock [2], Hoste and Przytycki [7; 8; 9], Marché [16], Mroczkowski [18; 17], Mroczkowski and Dabkowski [19] and Przytycki [21; 22; 23]. Another natural question is:

**Question 1.2** Is the skein vector space of a closed oriented 3–manifold always finitely generated?

In this paper, we take as base field the set  $\mathbb{Q}(A)$  of all rational functions with rational coefficients and abstract variable  $A$ , and we note that every result in this work holds also for the field  $\mathbb{C}$  of complex numbers with  $A \in \mathbb{C}$  a nonzero number such that  $A^{2n} \neq 1$  for every  $n > 0$ .

**Theorem 1.3** *The skein space  $K(T^3)$  of the 3–torus  $T^3 = S^1 \times S^1 \times S^1$  is finitely generated.*

A set of nine generators is given by the empty set  $\emptyset$ , some simple closed curves representing the nonzero elements of the first homology group  $H_1(T^3; \mathbb{Z}_2) \cong (\mathbb{Z}_2)^3$  with coefficients in  $\mathbb{Z}_2$ , and a skein element  $\alpha$  that is equal to the link consisting of two parallel copies of any previous nonempty knots.

Our main tool is the algebraic work of Frohman and Gelca [5]. The skein space (module) of a (thickened) surface has a natural algebra structure obtained by overlap of framed links. In their work, Frohman and Gelca gave a nice formula that describes the product in the skein space (algebra)  $K(T^2)$  of the 2–torus  $T^2 = S^1 \times S^1$ . A standard embedding of  $T^2$  in  $T^3$  makes this product commutative; hence we can get further relations from the formula of Frohman and Gelca.

A natural question is the following:

**Question 1.4** Is 9 the dimension of the skein vector space  $K(T^3)$  of the 3–torus?

After this paper was submitted, P Gilmer [6] answered this question positively.

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## 2 The result

### 2.1 Definition of skein module

Let  $M$  be an oriented 3-manifold,  $R$  a commutative ring with unit and  $A \in R$  an invertible element of  $R$ . Let  $V$  be the abstract free  $R$ -module generated by all framed links in  $M$  (considered up to isotopies) including the empty set  $\emptyset$ .

**Definition 2.1** The  $(R, A)$ -Kauffman bracket skein module of  $M$ , or the  $R$ -skein module, or simply the  $KBSM$ , sometimes indicated with  $KM(M; R, A)$ , is the quotient of  $V$  by all the possible skein relations:

$$\begin{aligned} \times &= A \smile + A^{-1} \frown, \\ L \sqcup \bigcirc &= (-A^2 - A^{-2})D, \\ \bigcirc &= (-A^2 - A^{-2})\emptyset. \end{aligned}$$

These are local relations where the framed links in an equation differ just in the pictured 3-ball that is equipped with a positive trivialization. An element of  $KM(M; R, A)$  is called a *skein* or a *skein element*. If  $M$  is the oriented  $I$ -bundle over a surface  $S$  (that is,  $M = S \times [-1, 1]$  if  $S$  is oriented), we simply write  $KM(S; R, A)$  and call it the *skein module* of  $S$ .

Let  $\mathbb{Q}(A)$  be field of all rational function with rational coefficients and abstract variable  $A$ . We set

$$K(M) := KM(M; \mathbb{Q}(A), A),$$

and we call it the *skein vector space*, or simply the *skein space*, of  $M$ .

**Remark 2.2** It is easy to verify that if we modify the framing of a component of a framed link, the skein changes by the multiplication of an integer power of  $-A^3$ :

$$\overbrace{\quad} = -A^3 \text{---}, \quad \overbrace{\quad} = -A^{-3} \text{---}.$$

### 2.2 The skein algebra of the 2-torus

**Definition 2.3** Let  $S$  be a surface; the skein module  $KM(S; R, A)$  has a natural structure of an  $R$ -algebra that is given by the linear extension of the multiplication

defined on framed links. Given two framed links  $L_1, L_2 \subset S \times [-1, 1]$ , the product  $L_1 \cdot L_2 \subset S \times [-1, 1]$  is obtained by putting  $L_1$  above  $L_2$ , so  $L_1 \cdot L_2 \cap S \times [0, 1] = L_1$  and  $L_1 \cdot L_2 \cap S \times [-1, 0] = L_2$ .

Consider the 2–torus  $T^2$  as the quotient of  $\mathbb{R}^2$  modulo the standard lattice of translations generated by  $(1, 0)$  and  $(0, 1)$ ; hence for any nonzero pair  $(p, q)$  of integers, we have the notion of  $(p, q)$ –*curve*: the simple closed curve in the 2–torus that is the quotient of the line passing trough  $(0, 0)$  and  $(p, q)$ .

**Definition 2.4** Let  $p$  and  $q$  be two coprime integers; hence  $(p, q) \neq (0, 0)$ . We denote by  $(p, q)_T$  the  $(p, q)$ –curve in the 2–torus  $T^2$  equipped with the blackboard framing. Given a framed knot  $\gamma$  in an oriented 3–manifold  $M$  and an integer  $n \geq 0$ , we denote by  $T_n(\gamma)$  the skein element defined by induction as follows:

$$\begin{aligned} T_0(\gamma) &:= 2 \cdot \emptyset, \\ T_1(\gamma) &:= \gamma, \\ T_{n+1}(\gamma) &:= \gamma \cdot T_n(\gamma) - T_{n-1}(\gamma), \end{aligned}$$

where  $\gamma \cdot T_n(\gamma)$  is the skein element obtained adding a copy of  $\gamma$  to all the framed links that compose the skein  $T_n(\gamma)$ . For  $p, q \in \mathbb{Z}$  such that  $(p, q) \neq (0, 0)$ , we denote by  $(p, q)_T$  the skein element

$$(p, q)_T := T_{\text{MCD}(p,q)} \left( \left( \frac{p}{\text{MCD}(p,q)}, \frac{q}{\text{MCD}(p,q)} \right)_T \right),$$

where  $\text{MCD}(p, q)$  is the maximum common divisor of  $p$  and  $q$ . Finally, we set

$$(0, 0)_T := 2 \cdot \emptyset.$$

It is easy to show that the set of all the skein elements  $(p, q)_T$  with  $p, q \in \mathbb{Z}$  generates  $\text{KM}(T^2; R, A)$  as  $R$ –module.

This is not the standard way to color framed links in a skein module. The colorings  $\text{JW}_n(\gamma)$ ,  $n \geq 0$ , with the Jones–Wenzl projectors are defined in the same way as  $T_n(\gamma)$ , but at the 0–level we have  $\text{JW}_0(\gamma) = \emptyset$ .

**Theorem 2.5** (Frohman and Gelca [5]) *For any  $p, q, r, s \in \mathbb{Z}$ , the following holds in the skein module  $\text{KM}(T^2; R, A)$  of the 2–torus  $T^2$ :*

$$(p, q)_T \cdot (r, s)_T = A \begin{vmatrix} p & q \\ r & s \end{vmatrix} (p+r, q+s)_T + A^{-1} \begin{vmatrix} p & q \\ r & s \end{vmatrix} (p-r, q-s)_T,$$

where  $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$  is the determinant  $ps - qr$ . □

### 2.3 The abelianization

**Definition 2.6** Let  $B$  be a  $R$ -algebra for a commutative ring with unity  $R$ . We denote by  $C(B)$  the  $R$ -module defined as the quotient

$$C(B) := \frac{B}{[B, B]},$$

where  $[B, B]$  is the submodule of  $B$  generated by all the elements of the form  $ab - ba$  for  $a, b \in B$ . We call  $C(B)$  the *abelianization* of  $B$ .

**Remark 2.7** Usually in noncommutative algebra, the *abelianization* is the  $R$ -algebra defined as the quotient of  $B$  modulo the subalgebra (submodule and ideal) generated by all the elements of the form  $ab - ba$ . In our definition, the denominator is just a submodule and we only get an  $R$ -module. We use the word “abelianization” anyway.

Now we work with  $C(K(T^2))$ , and we still use  $(p, q)_T$  and  $(p, q)_T \cdot (r, s)_T$  to denote the class of  $(p, q)_T \in K(T^2)$  and  $(p, q)_T \cdot (r, s)_T \in K(T^2)$  in  $C(K(T^2))$ .

**Lemma 2.8** Let  $(p, q)$  be a pair of integers different from  $(0, 0)$ . Then in the abelianization  $C(K(T^2))$  of the skein algebra  $K(T^2)$  of the 2-torus  $T^2$ , we have

$$(p, q)_T = \begin{cases} (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1 \text{ and } q \in 2\mathbb{Z}, \\ (0, 1)_T & \text{if } p \in 2\mathbb{Z} \text{ and } q \in 2\mathbb{Z} + 1, \\ (1, 1)_T & \text{if } p, q \in 2\mathbb{Z} + 1, \\ (2, 0)_T & \text{if } p, q \in 2\mathbb{Z}. \end{cases}$$

Hence  $C(K(T^2))$  is generated as a  $\mathbb{Q}(A)$ -vector space by the empty set  $\emptyset$ , the framed knots  $(1, 0)_T$ ,  $(0, 1)_T$ ,  $(1, 1)_T$ , and a framed link consisting of two parallel copies of  $(1, 0)_T$ .

**Proof** By Theorem 2.5, for every  $p, q \in \mathbb{Z}$ , we have

$$\begin{aligned} A^{-q}(p + 2, q)_T + A^q(p, q)_T &= (p + 1, q)_T \cdot (1, 0)_T \\ &= (1, 0)_T \cdot (p + 1, q)_T \\ &= A^q(p + 2, q)_T + A^{-q}(-p, -q)_T. \end{aligned}$$

Since  $(p, q)_T = (-p, -q)_T$ , we have  $(A^q - A^{-q})(p, q)_T = (A^q - A^{-q})(p + 2, q)_T$ . Hence if  $q \neq 0$ , we get  $(p, q)_T = (p + 2, q)_T$  (here we use the fact that the base ring is a field and that  $A^{2n} \neq 1$  for every  $n > 0$ ). Thus

$$(p, q)_T = \begin{cases} (0, q)_T & \text{if } p \in 2\mathbb{Z} \text{ and } q \neq 0, \\ (1, q)_T & \text{if } p \in 2\mathbb{Z} + 1 \text{ and } q \neq 0. \end{cases}$$

Analogously, by using  $(0, 1)_T$  instead of  $(1, 0)_T$  for  $p \neq 0$ , we get

$$(p, q)_T = \begin{cases} (p, 0)_T & \text{if } q \in 2\mathbb{Z} \text{ and } q \neq 0, \\ (p, 1)_T & \text{if } q \in 2\mathbb{Z} + 1 \text{ and } q \neq 0. \end{cases}$$

Therefore, if  $p, q \in 2\mathbb{Z} + 1$ , we have  $(p, q)_T = (1, 1)_T$ . If  $p \neq 0$ , we get

$$(p, 0)_T = (p, 2)_T = \begin{cases} (0, 2)_T & \text{if } p \in 2\mathbb{Z}, \\ (1, 2)_T = (1, 0)_T & \text{if } p \in 2\mathbb{Z} + 1. \end{cases}$$

In the same way for  $q \neq 0$ , we get

$$(0, q)_T = (2, q)_T = \begin{cases} (2, 0)_T & \text{if } p \in 2\mathbb{Z}, \\ (2, 1)_T = (0, 1)_T & \text{if } p \in 2\mathbb{Z} + 1. \end{cases}$$

In particular, we have

$$(2, 0)_T = (2, 2)_T = (2, -2)_T = (0, 2)_T = (p, q)_T \quad \text{for } (p, q) \neq (0, 0), p, q \in 2\mathbb{Z}. \quad \square$$

## 2.4 The $(p, q, r)$ -type curves

As for the 2-torus  $T^2$ , we look at the 3-torus  $T^3$  as the quotient of  $\mathbb{R}^3$  modulo the standard lattice of translations generated by  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ .

**Definition 2.9** Let  $(p, q, r)$  be a triple of coprime integers; that means we have  $\text{MCD}(p, q, r) = 1$ , where  $\text{MCD}(p, q, r)$  is the maximum common divisor of  $p$ ,  $q$  and  $r$ , and in particular, we have  $(p, q, r) \neq (0, 0, 0)$ . The  $(p, q, r)$ -curve is the simple closed curve in the 3-torus that is the quotient (under the standard lattice) of the line passing through  $(0, 0, 0)$  and  $(p, q, r)$ . We denote by  $[p, q, r]$  the  $(p, q, r)$ -curve equipped with the framing that is the collar of the curve in the quotient of any plane containing  $(0, 0, 0)$  and  $(p, q, r)$ . The framing does not depend on the choice of the plane.

**Definition 2.10** An embedding  $e: T^2 \rightarrow T^3$  of the 2-torus in the 3-torus is *standard* if it is the quotient (under the standard lattice) of a plane in  $\mathbb{R}^3$  that is the image of the plane generated by  $(1, 0, 0)$  and  $(0, 1, 0)$  under a linear map defined by a matrix of  $\text{SL}_3(\mathbb{Z})$  (a  $3 \times 3$  matrix with integer entries and determinant 1).

**Remark 2.11** There are infinitely many standard embeddings, even up to isotopies. A standard embedding of  $T^2$  in  $T^3$  is the quotient under the standard lattice of the plane generated by two columns of a matrix of  $\text{SL}_3(\mathbb{Z})$ .

**Lemma 2.12** Let  $(p, q, r)$  be a triple of coprime integers. Then the skein element  $[p, q, r] \in K(T^3)$  is equal to  $[x, y, z]$ , where  $x, y, z \in \{0, 1\}$  and they have respectively the same parities as  $p$ ,  $q$  and  $r$ .

**Proof** Every embedding  $e: T^2 \rightarrow T^3$  of the 2-torus defines a linear map between the skein spaces

$$e_*: K(T^2) \rightarrow K(T^3).$$

The map  $e_*$  factorizes with the quotient map  $K(T^2) \rightarrow C(K(T^2))$ . In fact, we can slide the framed links in  $e(T^2 \times [-1, 1])$  from above to below, getting  $e_*(L_1 \cdot L_2) = e_*(L_2 \cdot L_1)$  for every two framed links,  $L_1$  and  $L_2$ , in  $T^2 \times [-1, 1]$ . As said in Remark 2.11, a standard embedding  $e: T^2 \rightarrow T^3$  corresponds to the plane generated by two columns  $(p_1, q_1, r_1), (p_2, q_2, r_2) \in \mathbb{Z}^3$  of a matrix in  $SL_3(\mathbb{Z})$ . In this correspondence,  $e_*((a, b)_T) = [ap_1 + bp_2, aq_1 + bq_2, ar_1 + br_2]$  for every coprime  $a, b \in \mathbb{Z}$ . Therefore, by Lemma 2.8, we get

$$\begin{aligned} [a'p_1 + b'p_2, a'q_1 + b'q_2, a'r_1 + b'r_2] &= e_*((a', b')_T) \\ &= e_*((a, b)_T) \\ &= [ap_1 + bp_2, aq_1 + bq_2, ar_1 + br_2] \end{aligned}$$

for every two pairs  $(a, b), (a', b') \in \mathbb{Z}^2$  of coprime integers such that  $a + a', b + b' \in 2\mathbb{Z}$ .

Let  $(p, q, r)$  be a triple of coprime integers. By permuting  $p, q$  and  $r$ , we get either  $(p, q, r) = (1, 0, 0)$  or  $p, q \neq 0$ . Consider the case where  $p, q \neq 0$ . Let  $d$  be the maximum common divisor of  $p$  and  $q$ , and let  $\lambda, \mu \in \mathbb{Z}$  such that  $\lambda p + \mu q = d$ . The following matrix belongs in  $SL_3(\mathbb{Z})$ :

$$M_1 := \begin{pmatrix} p/d & -\mu & 0 \\ q/d & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let  $v_1^{(1)}$  and  $v_3^{(1)}$  be the first and the third columns of  $M_1$ . We have  $(p, q, r) = dv_1^{(1)} + rv_3^{(1)}$ . Hence

$$[p, q, r] = \begin{cases} [\frac{p}{d}, \frac{q}{d}, 0] & \text{if } d \in 2\mathbb{Z} + 1 \text{ and } r \in 2\mathbb{Z}, \\ [0, 0, 1] & \text{if } d \in 2\mathbb{Z} \text{ and } r \in 2\mathbb{Z} + 1, \\ [\frac{p}{d}, \frac{q}{d}, 1] & \text{if } d, r \in 2\mathbb{Z} + 1. \end{cases}$$

The integers  $p, q, r$  cannot be all even because they are coprime; hence  $d$  and  $r$  cannot be both even. Therefore, we just need to study the cases where  $r \in \{0, 1\}$ .

If  $r = 0$ , we consider the trivial embedding of  $T^2$  in  $T^3$ . The corresponding matrix of  $SL_3(\mathbb{Z})$  is the identity. We have  $(\frac{p}{d}, \frac{q}{d}, 0) = \frac{p}{d}(1, 0, 0) + \frac{q}{d}(0, 1, 0)$ ; hence

$$[p, q, 0] = [\frac{p}{d}, \frac{q}{d}, 0] = \begin{cases} [1, 0, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z} + 1 \text{ and } \frac{q}{d} \in 2\mathbb{Z}, \\ [0, 1, 0] & \text{if } \frac{p}{d} \in 2\mathbb{Z} \text{ and } \frac{q}{d} \in 2\mathbb{Z} + 1, \\ [1, 1, 0] & \text{if } \frac{p}{d}, \frac{q}{d} \in 2\mathbb{Z} + 1. \end{cases}$$

If  $r = 1$ , we take the matrix of  $SL_3(\mathbb{Z})$

$$M_2 := \begin{pmatrix} 0 & 0 & 1 \\ q & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let  $v_1^{(2)}$  and  $v_3^{(2)}$  be the first and the third columns of  $M_2$ . We have  $(p, q, 1) = pv_3^{(2)} + v_1^{(2)}$ ; hence

$$[p, q, 1] = \begin{cases} [1, q, 1] & \text{if } p \in 2\mathbb{Z} + 1, \\ [0, q, 1] & \text{if } p \in 2\mathbb{Z}. \end{cases}$$

By permuting  $p, q$  and  $r$ , we reduce the case  $(p, q, r) = (0, q, 1)$  to the case  $p, q \neq 0, r = 0$  that we studied before.

It remains to consider the case  $p = r = 1$ . We consider the matrix of  $SL_3(\mathbb{Z})$

$$M_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Let  $v_1^{(3)}$  and  $v_2^{(3)}$  be the first and the second columns of  $M_3$ . We have  $(1, q, 1) = v_1^{(3)} + qv_2^{(3)}$ . Hence

$$[1, q, 1] = \begin{cases} [1, 0, 1] & \text{if } q \in 2\mathbb{Z}, \\ [1, 1, 1] & \text{if } q \in 2\mathbb{Z} + 1. \end{cases} \quad \square$$

**Lemma 2.13** *The intersection of any two different standardly embedded 2-tori in  $T^3$  contains a  $(p, q, r)$ -type curve.*

**Proof** Let  $T_1$  and  $T_2$  be two standardly embedded tori in the 3-torus, and let  $\pi_1$  and  $\pi_2$  be two planes in  $\mathbb{R}^3$  whose projections under the standard lattice are respectively  $T_1$  and  $T_2$ . The intersection  $T_1 \cap T_2$  contains the projection of  $\pi_1 \cap \pi_2$ . We just need to prove that in  $\pi_1 \cap \pi_2$ , there is a point  $(p, q, r) \neq (0, 0, 0)$  with integer coordinates  $p, q, r \in \mathbb{Z}$ . Every plane defining a standardly embedded torus is generated by two vectors with integer coordinates, and hence it is described by an equation  $ax + by + cz = 0$  with integer coefficients  $a, b, c \in \mathbb{Z}$ . Applying a linear map described by a matrix of  $SL_3(\mathbb{Z})$ , we can suppose that  $\pi_1$  is the trivial plane  $\{z = 0\}$ . Let  $a, b, c \in \mathbb{Z}$  such that  $\pi_2 = \{ax + by + cz = 0\}$ . The vector  $(-b, a, 0)$  is nonzero and lies on  $\pi_1 \cap \pi_2$ . □



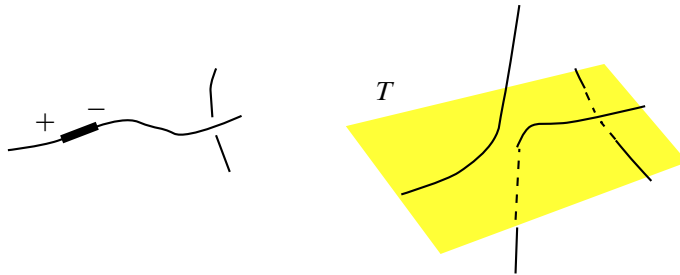


Figure 1: Diagrams of framed links in  $T^3$ . The plane is a part of the standardly embedded torus  $T \subset T^3$  where the links project. If we look at the framed links in  $T^3$  as framed tangles in  $T \times [-1, 1]$ , the two strands that get out vertically from the plane end in the boundary points  $(x, 1)$  and  $(x, -1)$  for some  $x \in T$ .

### 2.5 Diagrams

Framed links in  $T^3$  can be represented by diagrams in the 2-torus  $T^2$ . These diagrams are like the usual link diagrams but with further oriented signs on the edges; see Figure 1 (left). Fix a standardly embedded 2-torus  $T$  in  $T^3$ . After a cut along a parallel copy  $T'$  of  $T$ , the 3-torus becomes diffeomorphic to  $T \times [-1, 1]$ , and framed links in  $T^3$  correspond to framed tangles of  $T \times [-1, 1]$ . These diagrams are generic projections on  $T$  of the framed tangles in  $T \times [-1, 1]$  via the natural projection  $(x, t) \mapsto x$ . The further signs on the diagrams represent the intersection of the framed links with the boundary  $T \times \{-1, 1\}$ . In other words, they represent the passages of the links along the  $(p, q, r)$ -type curve that, in the Euclidean metric, is orthogonal to  $T$ ; see Figure 1 (right). If  $T$  is the trivial torus  $S^1 \times S^1 \times \{x\}$ , the further signs represent the passages through the third  $S^1$ -factor. We use the proper notion of blackboard framing.

### 2.6 Generators for the 3-torus

The following is the main theorem proved in this paper. We use all the previous lemmas to get a set of nine generators of  $K(T^3)$ .

**Theorem 2.14** *The skein space  $K(T^3)$  of the 3-torus  $T^3$  is generated by the empty set  $\emptyset$ ,  $[1, 0, 0]$ ,  $[0, 1, 0]$ ,  $[0, 0, 1]$ ,  $[1, 1, 0]$ ,  $[1, 0, 1]$ ,  $[0, 1, 1]$ ,  $[1, 1, 1]$  and a skein  $\alpha$  that is equal to the framed link consisting of two parallel copies of any  $(p, q, r)$ -type curve.*

**Proof** Let  $T$  be the trivial embedded 2-torus: the one containing the  $(p, q, r)$ -type curves with  $r = 0$ . Use  $T$  to project the framed links and make diagrams. By using the first skein relation on these diagrams, we can see that  $K(T^3)$  is generated by the framed links described by diagrams on  $T$  without crossings. These diagrams are unions of simple closed curves on  $T$  equipped with some signs as the one with +

and  $-$  in Figure 1. These simple closed curves are either parallel to a  $(p, q)$ -curve or homotopically trivial. The framed links described by these diagrams lie in the standardly embedded tori that are the projections (under the standard lattice) of the planes generated by  $(0, 0, 1)$  and  $(p, q, 0)$  for some  $p$  and  $q$ . Therefore,  $K(T^3)$  is generated by the images of  $K(T^2)$  under the linear maps induced by the standard embeddings of  $T^2$  in  $T^3$ .

As said in the proof of Lemma 2.12, the linear map  $e_*$  induced by any standard embedding  $e: T^2 \rightarrow T^3$  factorizes with the quotient map  $K(T^2) \rightarrow C(K(T^2))$ . Lemma 2.8 applied to the standard embedding  $e$  shows that the image  $e_*(K(T^2))$  is generated by  $\emptyset$ , three  $(p, q, r)$ -type curves lying on  $e(T^2)$ , and the skein  $\alpha_e$  that is equal to the framed link consisting of two parallel copies of any  $(p, q, r)$ -type curve lying on  $e(T^2)$ .

Let  $e_1, e_2: T^2 \rightarrow T^3$  be two standard embeddings. By Lemma 2.13,  $e_1(T^2) \cap e_2(T^2)$  contains a  $(p, q, r)$ -type curve  $\gamma$ ; hence  $\alpha_{e_1}$  and  $\alpha_{e_2}$  coincide with the framed link that is two parallel copies of  $\gamma$ . Therefore, the skein element  $\alpha_e$  does not depend on the embedding  $e$ .

We conclude by using Lemma 2.12, which says that the skein of any  $(p, q, r)$ -type curve is equal to the one of a standard representative of a nonzero element of the first homology group  $H_1(T^3; \mathbb{Z}_2)$  with coefficient in  $\mathbb{Z}_2$ , namely a  $(p, q, r)$ -type curve with  $p, q, r \in \{0, 1\}$ . □

**Remark 2.15** Theorem 2.14, Lemma 2.8 and Lemma 2.12 work for every base pair  $(R, A)$  such that  $A^{2n} - 1$  is an invertible element of  $R$  for any  $n > 0$ . In particular, they work for  $(\mathbb{C}, A)$ , where  $A^{2n} \neq 1$  for any  $n > 0$ . Unfortunately, we do not know what happens with the base pair  $(\mathbb{C}, \pm 1)$ , which is the one used for the connection with the  $SL_2(\mathbb{C})$ -character variety [3]. In fact, in Lemma 2.8, we would get just trivial equalities if  $A = \pm 1$ .

### 2.7 Linear independence

Here we talk about the linear independence of our generators of  $K(T^2)$ . The following proposition shows a direct sum decomposition of  $K(T^3)$ .

**Proposition 2.16** *The skein space  $K(T^3)$  is the direct sum of eight subspaces,*

$$K(T^3) = V_0 \oplus V_1 \oplus \dots \oplus V_7,$$

such that

- (1)  $V_0$  is generated by the empty set  $\emptyset$  and the skein  $\alpha$  (see Theorem 2.14);
- (2) every  $(p, q, r)$ -type curve generates a  $V_j$  with  $j > 0$ , and every  $V_j$  with  $j > 0$  is generated by one such curve.

**Proof** The skein relations relate framed links in the same  $\mathbb{Z}_2$ –homology class. Hence for every oriented 3–manifold  $M$ , we have a direct sum decomposition

$$\text{KM}(M; R, A) = \bigoplus_{h \in H_1(M; \mathbb{Z}_2)} V_h,$$

where  $V_h$  is generated by the framed links whose  $\mathbb{Z}_2$ –homology class is  $h$ . The statement follows by this observation and the fact that if  $[p, q, r]$  and  $[p', q', r']$  represent the same  $\mathbb{Z}_2$ –homology class, then  $[p, q, r] = [p', q', r'] \in K(T^3)$ .  $\square$

**Remark 2.17** Given a triple of integers  $(x, y, z) \neq (0, 0, 0)$  such that  $x, y, z \in \{0, 1\}$ , we can easily find an orientation-preserving diffeomorphism of the 3–torus  $T^3$  sending  $[x, y, z]$  to  $[1, 0, 0]$ . Hence if the skein of one such curve  $[x, y, z]$  is null, then also all the other skein elements of such curves are null. Therefore, by Proposition 2.16, the possible dimensions of the skein space  $K(T^3)$  are 0, 1, 2, 7, 8 and 9.

After the submission of this paper, P Gilmer [6] showed that the skein of the  $(1, 0, 0)$ –curve is not null and that the empty set and the skein  $\alpha$  are linear independent. This answers Question 1.4 in the affirmative by proving that the set of nine generators is actually a basis for the skein space.

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*Dipartimento di Matematica “Tonelli”, Università di Pisa  
Pisa, Italy*

carrega@mail.dm.unipi.it

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