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Cosimplicial groups and spaces of homomorphisms

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Let G be a real linear algebraic group and L a finitely generated cosimplicial group. We prove that the space of homomorphisms $\mathrm{Hom}(L_n, G)$ has a homotopy stable decomposition for each $n \geq 1$. When G is a compact Lie group, we show that the decomposition is G–equivariant with respect to the induced action of conjugation by elements of G. In particular, under these hypotheses on G, we obtain stable decompositions for $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ and $\mathrm{Rep}(F_n/\Gamma_n^q, G)$, respectively, where F_n/Γ_n^q are the finitely generated free nilpotent groups of nilpotency class $q-1$.

The spaces $\mathrm{Hom}(L_n, G)$ assemble into a simplicial space $\mathrm{Hom}(L, G)$. When $G = U$ we show that its geometric realization $B(L, U)$, has a nonunital E_∞–ring space structure whenever $\mathrm{Hom}(L_0, U(m))$ is path connected for all $m \geq 1$.

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1 Introduction

Let G be a topological group and Γ a finitely generated group. The set of homomorphisms $\mathrm{Hom}(\Gamma, G)$ can be identified with the ordered tuples $(\rho(a_1), \dots, \rho(a_r))$ in G^r, where $\rho\colon \Gamma \to G$ is a homomorphism and $a_1, \dots, a_r$ is a generating set for Γ. Computing the homotopy type of $\mathrm{Hom}(\Gamma, G)$ has proven to be rather complicated. Nevertheless, there has been recognition of the stable homotopy type in several cases. When $G \subset \mathrm{GL}_n(\mathbb{C})$ is a closed subgroup, A Adem and F Cohen [2] gave a homotopy stable decomposition for $\mathrm{Hom}(\mathbb{Z}^n, G)$ as wedges of the quotient spaces $\mathrm{Hom}(\mathbb{Z}^k, G)/S_1(\mathbb{Z}^k, G)$ with $1 \leq k \leq n$. Here $S_1(\mathbb{Z}^k, G)$ stands for the k–tuples with at least one entry equal to the identity matrix I in G. For an arbitrary finitely generated abelian group π, Adem and J M Gómez [5] gave a similar stable decomposition for $\mathrm{Hom}(\pi, G)$, but in this case, G is a finite product of the compact Lie groups $\mathrm{SU}(r)$, $\mathrm{Sp}(k)$ and $U(m)$. We show that the homotopy stable decomposition in [2] (in general, a version of it) still holds if we replace the family $\{\mathbb{Z}^n\}_{n \geq 1}$ with a broader object, namely, a finitely generated cosimplicial group L, that is, for every $n \geq 0$, we have a finitely generated group L_n with its coface and codegeneracy homomorphisms. A specific example of L comes from the finitely generated free nilpotent groups F_n/Γ_n^q. Here F_n denotes the free group on n–generators and Γ_n^q

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is the q^{th} stage of its descending central series. We work under the assumption that G is a real linear algebraic group, and with it we can prove that the inclusions $S_{k+1}(F_n/\Gamma_n^q, G) \hookrightarrow S_k(F_n/\Gamma_n^q, G)$ are closed cofibrations for every $0 \le k \le n$, where $S_k(F_n/\Gamma_n^q, G)$ denotes the subspace of $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ with at least k entries equal to the identity element in G. This condition, as stated by Adem, Cohen and E Torres-Giese [4, page 102], says that real linear algebraic groups have *cofibrantly filtered elements*, and they show that this implies $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ splits after one suspension as wedges of $\mathrm{Hom}(F_k/\Gamma_k^q, G)/S_1(F_k/\Gamma_k^q, G)$, where $0 \le k \le n$. This decomposition was known before for some compact and connected Lie groups G (see for example Cohen and M Stafa [11, Remark 1, page 387 and Theorem 2.13, page 388]).

In a more general setting, let Δ stand for the category whose objects are the natural numbers and morphisms are order-preserving maps. If $L\colon \Delta \to \mathbf{Grp}$ is a finitely generated cosimplicial group and G is a topological group, we get the simplicial space $\mathrm{Hom}(L, G)\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$, where $\mathrm{Hom}(L, G)_n := \mathrm{Hom}(L_n, G)$. We give a homotopy stable decomposition for the n–simplices of $\mathrm{Hom}(L, G)$ as follows. Let X be a simplicial space. Define $S^t(X_n)$ as the subspace of X_n in which any element is in the image of the composition of at least t degeneracy maps. We say X is simplicially NDR when all pairs $(S^{t-1}(X_n), S^t(X_n))$ are neighborhood deformation retracts. It was proven by Adem, A Bahri, M Bendersky, Cohen, and S Gitler [1] that when X is simplicially NDR, each X_n is homotopy stable equivalent to wedges of $S^t(X_n)/S^{t+1}(X_n)$ with $0 \le t \le n$. When G is a real linear algebraic group, the simplicial space $X = \mathrm{Hom}(L, G)$ is simplicially NDR. We prove this by showing that the subspaces $S^t(X_n)$ are real affine subvarieties of X_n for all $0 \le t \le n$ and therefore can be simultaneously triangulated. Define $S_t(L_n, G) := S^t(\mathrm{Hom}(L_n, G))$.

Theorem 1.1 *Let G be a real linear algebraic group, and L a finitely generated cosimplicial group. For each n, there are natural homotopy equivalences*

$$\Theta(n)\colon \Sigma\,\mathrm{Hom}(L_n, G) \simeq \bigvee_{0 \le k \le n} \Sigma(S_k(L_n, G)/S_{k+1}(L_n, G)).$$

The free groups F_n assemble into a cosimplicial group, which we denote by F. In this case $\mathrm{Hom}(F, G)$ is NG, the nerve of G seen as a topological category with one object, which is also the underlying simplicial space of a model of the classifying space BG. For each n, we take quotients F_n/K_n by normal subgroups K_n that are compatible with coface and codegeneracy homomorphisms of F to get finitely generated cosimplicial groups, denoted by F/K. The induced simplicial spaces are more easily described since there is a simplicial inclusion $\mathrm{Hom}(F/K, G) \subset NG$. Fixing $q > 0$, the family of subgroups Γ_n^q arising from the descending central series of F_n is compatible with F, and induces the finitely generated cosimplicial group F/Γ^q. Adem, Cohen and Torres-Giese [4] conjectured that the closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ have cofibrantly filtered elements and thus the homotopy stable decomposition holds. Applying Theorem 1.1 to F/Γ^q allows us to prove the following version of the conjecture.

Corollary 1.2 *If G is a Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, then there are homotopy equivalences for the cosimplicial group F/Γ^q,*

$$\Sigma\,\mathrm{Hom}(F_n/\Gamma_n^q, G) \simeq \bigvee_{1\le k\le n} \Sigma\big(\bigvee^{\binom{n}{k}} \mathrm{Hom}(F_k/\Gamma_k^q, G)/S_1(F_k/\Gamma_k^q, G)\big)$$

for all n and q.

For any finitely generated cosimplicial group L, conjugation under elements of G gives $\mathrm{Hom}(L_n, G)$ a G–space structure. Moreover, if G is a real algebraic linear group, then it has a G–variety structure. The subspaces $S^t(\mathrm{Hom}(L_n, G))$ are subvarieties that are invariant under the action of G for all $0 \le t \le n$. Using techniques from D H Park and D Y Suh [21], when G is a compact Lie group, we show that $\mathrm{Hom}(L_n, G)$ has a G–CW–complex structure, where each $S^t(\mathrm{Hom}(L_n, G))$ is a G–subcomplex. This allows us to prove the equivariant version of the previous theorem. Let $\mathrm{Rep}(L_n, G)$ and $\bar{S}_t(L_n, G)$ denote the orbit spaces of $\mathrm{Hom}(L_n, G)$ and $S^t(\mathrm{Hom}(L_n, G))$, respectively.

Theorem 1.3 *Let G be a compact Lie group. Then, for each n, $\Theta(n)$ in Theorem 1.1 is a G–equivariant homotopy equivalence, and in particular we get homotopy equivalences*

$$\Sigma\,\mathrm{Rep}(L_n, G) \simeq \bigvee_{1\le k\le n} \Sigma(\bar{S}_k(L_n, G)/\bar{S}_{k+1}(L_n, G)).$$

Applying this to the cosimplicial group F/Γ^q as in Corollary 1.2, we obtain

$$\Sigma\,\mathrm{Rep}(F_n/\Gamma_n^q, G) \simeq \bigvee_{1\le k\le n} \Sigma\big(\bigvee^{\binom{n}{k}} \mathrm{Rep}(F_k/\Gamma_k^q, G)/\bar{S}_1(F_k/\Gamma_k^q, G)\big).$$

In the second part of this paper we study the geometric realization of $\mathrm{Hom}(L, G)$ for a finitely generated cosimplicial group, which we denote by $B(L, G)$. We show that the set of 1–cocycles of L, denoted by $Z^1(L)$, is in one-to-one correspondence with cosimplicial morphisms $F \to L$. With this we show that any 1–cocycle of L defines a principal G–bundle over $B(L, G)$.

When $G = U = \mathrm{colim}_m\, U(m)$ we show that $B(L, U)$ has an $\mathbb{I}$–rig structure, that is, if $\mathbb{I}$ stands for the category of finite sets and injections, the functor $B(L, U(\_))\colon \mathbb{I} \to \mathbf{Top}$ is symmetric monoidal with respect to both symmetric monoidal structures on $\mathbb{I}$. Using the machinery developed in [7] by Adem, Gómez, J Lind and U Tillman, we prove:

Theorem 1.4 *Let L be a finitely generated cosimplicial group and suppose that the space $\mathrm{Hom}(L_0, U(m))$ is path connected for all $m \ge 1$. Then $B(L, U)$ is a nonunital E_∞–ring space.*

This theorem is also true if we replace U by SU, Sp, SO or O.

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2 Homotopy stable decompositions

2.1 Spaces of homomorphisms

Let G be a topological group and Γ a finitely generated group. Any homomorphism $\rho\colon \Gamma \to G$ is uniquely determined by $(\rho(\gamma_1), \dots, \rho(\gamma_n)) \in G^n$ when $\gamma_1, \dots, \gamma_n \in \Gamma$ is a set of generators. On the other hand, if we fix a presentation of Γ, then an n–tuple $(g_1, \dots, g_n) \in G^n$ will induce an element in $\mathrm{Hom}(\Gamma, G)$ whenever $\{g_i\}_{i=1}^n$ satisfy the relations in the presentation of Γ. Thus, there is a one-to-one correspondence between the subset of such n–tuples in G^n and $\mathrm{Hom}(\Gamma, G)$. Topologize $\mathrm{Hom}(\Gamma, G)$ with the subspace topology on G^n.

Lemma 2.1 *Let $\varphi\colon \Gamma \to \Gamma'$ be a homomorphism of finitely generated groups. If G is a topological group, then $\varphi^*\colon \mathrm{Hom}(\Gamma', G) \to \mathrm{Hom}(\Gamma, G)$ is continuous.*

Proof Suppose $\Gamma = \langle a_1, \dots, a_r \mid R\rangle$ and $\Gamma' = \langle b_1, \dots, b_m \mid R'\rangle$. Recall that the induced map $\varphi^*\colon \mathrm{Hom}(\Gamma', G) \to \mathrm{Hom}(\Gamma, G)$ is given by

$$(\rho(b_1), \dots, \rho(b_m)) \mapsto \big(\rho(\varphi(a_1)), \dots, \rho(\varphi(a_r))\big)$$

for $\rho\colon \Gamma' \to G$. For any i, $\varphi(a_i) = b_{i_1}^{n_{i_1}} \cdots b_{i_{q_i}}^{n_{i_{q_i}}}$. By fixing one presentation for each $\varphi(a_i)$ we get that φ^* is given by

$$\begin{aligned}(\rho(b_1), \dots, \rho(b_m)) &\mapsto (\rho(b_{1_1}^{n_{1_1}} \cdots b_{1_{q_1}}^{n_{1_{q_1}}}), \dots, \rho(b_{r_1}^{n_{r_1}} \cdots b_{r_{q_r}}^{n_{r_{q_r}}})) \\ &= (\rho(b_{1_1})^{n_{1_1}} \cdots \rho(b_{1_{q_1}})^{n_{1_{q_1}}}, \dots, \rho(b_{r_1})^{n_{r_1}} \cdots \rho(b_{r_{q_r}})^{n_{r_{q_r}}}).\end{aligned}$$

Therefore φ^* is the restriction of the map $G^m \to G^r$ given by

$$(g_1, \dots, g_m) \mapsto (g_{1_1}^{n_{1_1}} \cdots g_{1_{q_1}}^{n_{1_{q_1}}}, \dots, g_{r_1}^{n_{r_1}} \cdots g_{r_{q_r}}^{n_{r_{q_r}}}),$$

which is continuous. □

In particular, this lemma tells us that given any two presentations of Γ, we get an isomorphism $\varphi\colon \Gamma \to \Gamma$ and hence a homeomorphism φ^* between the induced spaces of homomorphisms. Therefore the topology on the space of homomorphisms does not depend on the choice of presentations.

Recall that an affine variety is the zero locus in k^n of a family of polynomials on n variables over a field k. Throughout this paper we will focus only on $k = \mathbb{R}$. An affine variety that has a group structure with group operations given by polynomial maps, ie maps $f = (f_1, \dots, f_n)$, where each f_i is a polynomial, is called a linear algebraic group. For example, consider any matrix group. It is easy to check that matrix multiplication is in fact a polynomial map. For the inverse operation of matrices, it is easier to think of matrix groups as subgroups of $\mathrm{SL}(n, \mathbb{R})$. Any matrix A in $\mathrm{SL}(n, \mathbb{R})$ satisfies $A^{-1} = C^t$, the transpose of the cofactor matrix C of A. Since the cofactor matrix is described only in terms of minors of A, the map $A \mapsto C^t$ is a polynomial map. In fact, this is the general example, since it can be shown that any linear algebraic group is isomorphic to a group of matrices (see for example [15, page 63]).

Lemma 2.2 *Let G be a linear algebraic group; then for any finitely generated group Γ, $\mathrm{Hom}(\Gamma, G)$ is an affine variety. Moreover, if φ is a homomorphism of finitely generated groups, then φ^* is a polynomial map.*

Proof Suppose Γ is generated by $\gamma_1, \gamma_2, \dots, \gamma_r$ and has a presentation $\{p_\alpha\}_{\alpha \in \Lambda}$. Each p_α is of the form $\gamma_{k_1}^{n_1} \cdots \gamma_{k_q}^{n_q} = e$ with $n_j \in \mathbb{Z}$ and $\gamma_{k_l} \in \{\gamma_1, \dots, \gamma_r\}$ for all $1 \le j \le q$. For any homomorphism $\rho\colon \Gamma \to G$ and any such relation p_α, we have

$$\rho(p_\alpha) = \rho(\gamma_{i_1}^{n_1} \cdots \gamma_{i_q}^{n_q}) = \rho(\gamma_{i_1})^{n_1} \cdots \rho(\gamma_{i_q})^{n_q} = I,$$

the identity matrix in G. Since products and inverses in G are given in terms of polynomials, this sets up a family of polynomial relations $\{y_{\alpha,i,j}\}_{\alpha,i,j}$, where each $y_{\alpha,i,j}$ is induced by $\rho(p_\alpha)_{i,j} = \delta_{ij}$, the (i, j)–entry of the matrix equality $\rho(p_\alpha) = I$. These relations do not depend on ρ, only on p_α, in the sense that any r–tuple $(g_1, \dots, g_r) \in G$ satisfying $\{y_{\alpha,i,j}\}_{\alpha,i,j}$, ie

$$(g_{i_1}^{n_1} \cdots g_{i_q}^{n_q})_{i,j} = \delta_{ij}$$

for all $\alpha \in \Lambda$ and $1 \le i, j \le n$, is an element of $\mathrm{Hom}(\Gamma, G)$. Adding the polynomial relations $\{y_{\alpha,i,j}\}_{\alpha \in \Lambda}$ to the ones describing G^r defines $\mathrm{Hom}(\Gamma, G)$ as an affine variety.

For the second part, recall from the proof of Lemma 2.1, that φ^* is defined in terms of products and inverses of matrices and thus is a polynomial map. □

Similarly, this lemma tells us that the affine variety structure on $\mathrm{Hom}(\Gamma, G)$ does not depend on the presentation of Γ. Indeed, any isomorphism of groups will induce an isomorphism of affine varieties.

2.2 Triangulation of semialgebraic sets

Definition 2.3 A real semialgebraic set is a finite union of subsets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) > 0 \text{ and } g_j(x) = 0 \text{ for all } i \text{ and } j\},$$

where $f_i(x)$ and $g_j(x)$ are a finite number of polynomials with real coefficients.

Using Hilbert's basis theorem, all affine varieties over $\mathbb{R}$ are real semialgebraic sets. Indeed, the zero locus ideal of an affine variety will be finitely generated and thus the affine variety can be carved out by finitely many polynomials.

What makes semialgebraic sets more interesting is that images of semialgebraic sets in $\mathbb{R}^n$ under a polynomial map $\mathbb{R}^n \to \mathbb{R}^m$ are semialgebraic sets in $\mathbb{R}^m$ (see [14, page 167]), as opposed to affine varieties and regular maps.

Let M and N be semialgebraic subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. A continuous map $f\colon M \to N$ is said to be semialgebraic if its graph is a semialgebraic set in $\mathbb{R}^m \times \mathbb{R}^n$. The next result is proven in [14, page 170].

Proposition 2.4 *Given a finite system of bounded semialgebraic sets M_i in $\mathbb{R}^n$, there is a simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $k\colon |K| \to \bigcup \overline{M}_i$, where each M_i is a finite union of sets $k(\operatorname{int}|\sigma|)$ with $\sigma \in K$.*

Remark 2.5 Proposition 2.4 can be stated without the boundedness condition and the details can be found in [21, Theorem 2.12], where they add the hypothesis that $\bigcup M_i$ is closed in $\mathbb{R}^n$.

In the next sections, we will be using this last result in its full power, but a first application is that any affine variety Z can be triangulated, that is, there exists a simplicial complex K and a homeomorphism $|K| \cong Z$. With Lemma 2.2 and Proposition 2.4 we prove the following.

Corollary 2.6 *Let Γ be a finitely generated group and G a real linear algebraic group. Then $\operatorname{Hom}(\Gamma, G)$ is a triangulated space.*

2.3 Simplicial spaces and homotopy stable decompositions

Let Δ be the category of finite sets $[n] = \{0, 1, \dots, n\}$ with morphisms order preserving maps $f\colon [n] \to [m]$. It can be shown that all morphisms in this category are generated by composition of maps, denoted by $\boldsymbol{d}^i\colon [n-1] \to [n]$ and $\boldsymbol{s}^i\colon [n+1] \to [n]$ for $0 \le i \le n$. These maps are determined by the relations

$$\begin{aligned} \boldsymbol{d}^j\boldsymbol{d}^i &= \boldsymbol{d}^i\boldsymbol{d}^{j-1} && \text{if } i<j,\\ \boldsymbol{s}^j\boldsymbol{s}^i &= \boldsymbol{s}^{i-1}\boldsymbol{s}^j && \text{if } i>j,\\ \boldsymbol{s}^j\boldsymbol{d}^i &= \begin{cases} \boldsymbol{d}^i\boldsymbol{s}^{j-1} & \text{if } i<j,\\ \mathrm{Id} & \text{if } i=j \text{ or } i=j+1,\\ \boldsymbol{d}^{i-1}\boldsymbol{s}^j & \text{if } i>j+1, \end{cases} \end{aligned}$$

which are called cosimplicial identities. For any category $\boldsymbol{C}$, let $\boldsymbol{C}^{\mathrm{op}}$ denote its opposite category. A functor

$$X\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$$

is called a simplicial space. Here **Top** stands for k–spaces, ie topological spaces where each compactly closed subset is closed. We write $X_n := X([n])$ and the maps $d_i = X(\boldsymbol{d}^i)$ and $s_i = X(\boldsymbol{s}^i)$ are called face and degeneracy maps, respectively.

Fix n. Define $S^0(X_n) = X_n$ and, for $0 < t \le n$,

$$S^t(X_n) = \bigcup_{J_{n,t}} s_{i_1} \circ \cdots \circ s_{i_t}(X_{n-t}),$$

where $s_{i_j}\colon X_{n-j} \to X_{n-j+1}$ is a degeneracy map, $1 \le i_1 < \cdots < i_t \le n$ is a sequence of t numbers between 1 and n, and $J_{n,t}$ stands for all possible sequences. This defines a decreasing filtration of X_n,

$$S^n(X_n) \subset S^{n-1}(X_n) \subset \cdots \subset S^0(X_n) = X_n.$$

For each n there is a homotopy decomposition of ΣX_n in terms of the quotient spaces $S^k(X_n)/S^{k+1}(X_n)$ with $k \le n$. To do this we need the following.

Let $A \subset Z$ be topological spaces. Recall that (Z, A) is an NDR pair if there exist continuous functions

$$h\colon Z \times [0,1] \to Z, \quad u\colon Z \to [0,1]$$

such that the following conditions are satisfied:

1. $A = u^{-1}(0)$.
2. $h(z, 0) = z$ for all $z \in Z$.
3. $h(a, t) = a$ for all $a \in A$ and all $t \in [0,1]$.
4. $h(z, 1) \in A$ for all $z \in u^{-1}([0,1))$.

Examples of NDR pairs are pairs consisting of CW–complexes and subcomplexes. Indeed, if Z is a CW–complex and $A \subset Z$ a subcomplex, then the inclusion $A \hookrightarrow Z$ is a cofibration, which is equivalent to a retraction $X \times I$ to $A \times I \cup X \times \{0\}$ relative to $A \times \{0\}$.

When X is a simplicial space, we call X *simplicially* NDR if $(S^{t-1}(X_n), S^t(X_n))$ is an NDR pair for every n and $t \geq 1$. The following result can be found in [1, Theorem 1.6].

Proposition 2.7 *Let X be a simplicial space, and suppose X is simplicially* NDR. *Then for every $n \geq 0$ there is a natural homotopy equivalence*

$$\Theta(n)\colon \Sigma X_n \simeq \bigvee_{0\leq k\leq n} \Sigma(S^k(X_n)/S^{k+1}(X_n)).$$

For each n, the map $\Theta(n)$ is natural with respect to morphisms of simplicial spaces, that is, natural transformations $X \to Y$.

2.4 Cosimplicial groups, 1–cocycles and $\mathrm{Hom}(L, G)$

Definition 2.8 Let **Grp** denote the category of groups. A functor $L\colon \Delta \to \mathbf{Grp}$ is called a *cosimplicial group*. The homomorphisms $d^i = L(\boldsymbol{d}^i)$ and $s^i = L(\boldsymbol{s}^i)$ are called coface and codegeneracy homomorphisms, respectively. We say that L is a *finitely generated cosimplicial group* if each L_n is finitely generated.

There are two canonical finitely generated cosimplicial groups that arise from finitely generated free groups.

Definition 2.9 Define $F\colon \Delta \to \mathbf{Grp}$ as follows: set $F_0 = \{e\}$ and for $n \geq 1$ let $F_n = \langle a_1, \dots, a_n\rangle$, the free group on n generators. The coface homomorphisms $d^i\colon F_{n-1} \to F_n$ are given on the generators by

$$\begin{aligned} d^0(a_j) &= a_{j+1}, \\ d^i(a_j) &= \begin{cases} a_j & \text{if } j < i, \\ a_j a_{j+1} & \text{if } j = i, \\ a_{j+1} & \text{if } j > i, \end{cases} \qquad \text{for } 1 \leq i \leq n-1, \\ d^n(a_j) &= a_j; \end{aligned}$$

and the codegeneracy homomorphisms $s^i\colon F_{n+1} \to F_n$ by

$$s^i(a_j) = \begin{cases} a_j & \text{if } j \leq i, \\ e & \text{if } j = i+1, \\ a_{j-1} & \text{if } j > i+1, \end{cases}$$

for $0 \leq i \leq n$.

Definition 2.10 Define $\overline{F}\colon \Delta \to \mathbf{Grp}$ as $\overline{F}_n := \langle a_0, \dots, a_n \rangle$ for any $n \geq 0$; coface and codegeneracy homomorphisms $\overline{d}^i\colon \overline{F}_{n-1} \to \overline{F}_n$ and $\overline{s}^i\colon \overline{F}_{n+1} \to \overline{F}_n$, respectively, are given on the generators by

$$\overline{d}^i(a_j) = \begin{cases} a_j & \text{if } j < i, \\ a_{j+1} & \text{if } j \geq i, \end{cases} \qquad \text{and} \qquad \overline{s}^i(a_j) = \begin{cases} a_j & \text{if } j \leq i, \\ a_{j-1} & \text{if } j > i, \end{cases}$$

for all $0 \leq i \leq n$.

Definition 2.11 We will say that a family of normal subgroups $K_n \subset F_n$ is *compatible* with F if $d^i(K_{n-1}) \subset K_n$ and $s^i(K_{n+1}) \subset K_n$ for all n and all i. Similarly we define *compatible* families of $\overline{F}$.

Given $\{K_n\}_{n \geq 0}$ a compatible family with F, we get induced homomorphisms

$$\begin{array}{ccc} F_{n-1} & \xrightarrow{d^i} & F_n \\ \downarrow & & \downarrow \\ F_{n-1}/K_{n-1} & \xrightarrow{d^i} & F_n/K_n \end{array} \qquad \begin{array}{ccc} F_{n+1} & \xrightarrow{s^i} & F_n \\ \downarrow & & \downarrow \\ F_{n+1}/K_{n+1} & \xrightarrow{s^i} & F_n/K_n \end{array}$$

Define

$$F/K\colon \Delta \to \mathbf{Grp}$$

as $(F/K)_n = F_n/K_n$ with coface and codegeneracy maps the quotient homomorphisms d^i and s^i, respectively. This way F/K is a finitely generated cosimplicial group. Similarly, with a compatible family $\{\overline{K}_n\}_{n \geq 0}$ of $\overline{F}$, we can define $\overline{F}/\overline{K}\colon \Delta \to \mathbf{Grp}$.

Example 2.12 We describe two families of finitely generated cosimplicial groups that can be constructed using F/K and $\overline{F}/\overline{K}$ through the commutator subgroup.

- Let A be a group, define inductively $\Gamma^1(A) = A$ and $\Gamma^{q+1}(A) = [\Gamma^q(A), A]$ for $q > 1$. The descending central series of A is

$$\Gamma^q(A) \trianglelefteq \cdots \trianglelefteq \Gamma^2(A) \trianglelefteq \Gamma^1(A) = A.$$

Given a homomorphism of groups $\phi\colon A \to B$, we have $\phi[a, a'] = [\phi(a), \phi(a')]$ for all $a, a' \in A$, so that

$$\phi(\Gamma^q(A)) \subset \Gamma^q(B).$$

Taking $A = F_n$, and writing $\Gamma^q_n := \Gamma^q(F_n)$, we have that the family of normal subgroups $\{\Gamma^q_n\}_{n \geq 0}$ is compatible with d_i and s_i. Thus we can define F/Γ^q as $(F/\Gamma^q)_n = F_n/\Gamma^q_n$ for all q and $1 \leq i \leq n$. In particular, for $q = 2$, we obtain $F_n/\Gamma^2_n = \mathbb{Z}^n$ for all $n \geq 0$.

- Another example using the commutator is the derived series of a group A,

$$A^{(q)} \trianglelefteq \cdots \trianglelefteq A^{(1)} \trianglelefteq A^{(0)} = A,$$

where $A^{(i+1)} = [A^{(i)}, A^{(i)}]$. Again, $\phi(A^{(q)}) \subset B^{(q)}$ for any homomorphism $\phi\colon A \to B$. Thus $F/F^{(q)}$, where $(F/F^{(q)})_n = F_n/F_n^{(q)}$ defines a finitely generated cosimplicial group.

Similarly, $F_{n+1}^{(q)}, \Gamma_{n+1}^q \subset \overline{F}_n$ define compatible families of $\overline{F}$ and we obtain the finitely generated cosimplicial groups $\overline{F}/\Gamma_{*+1}^q$ and $\overline{F}/F_{*+1}^{(q)}$.

Example 2.13 Here is one example of a cosimplicial group that does not come from a compatible family. Let $L_0 = \Sigma_2 = \langle \tau \rangle$ and $L_1 = \Sigma_3 = \langle \sigma_1, \sigma_2 \rangle$ and define coface homomorphisms

$$L_0 \xrightarrow{d^i} L_1 \quad \text{for } i = 0, 1$$

by $d^0(\tau) = \sigma_2$ and $d^1(\tau) = \sigma_1$. The codegeneracy homomorphism $s_0\colon L_1 \to L_0$ is given by $s^0(\sigma_1) = s^0(\sigma_2) = \tau$. This defines a 1–truncated cosimplicial group, which we denote by $\Sigma_{2,3}$, that is, a functor $\Sigma_{2,3}\colon \Delta_{\leq 1} \to \mathbf{Grp}$. Here $\Delta_{\leq 1}$ stands for the full subcategory of Δ with objects [0] and [1]. We can extend $\Sigma_{2,3}$ to Δ by using its left Kan extension.

For our purposes we describe the second stage of this extension: We have that

$$L_2 = \langle a, b, c \mid a^2 = b^2 = c^2,\ aba = bab,\ aca = cac,\ bcb = cbc \rangle,$$

coface homomorphisms

$$L_1 \xrightarrow{d^i} L_2 \quad \text{for } i = 0, 1, 2$$

are given by

$$d^0(\sigma_1) = a, \quad d^1(\sigma_1) = c, \quad d^2(\sigma_1) = c,$$
$$d^0(\sigma_2) = b, \quad d^1(\sigma_2) = b, \quad d^2(\sigma_2) = a,$$

and codegeneracy homomorphisms

$$L_1 \xleftarrow{s^i} L_2 \quad \text{for } i = 0, 1$$

by

$$s^0(a) = s^0(c) = \sigma_1, \quad s^1(a) = s^1(b) = \sigma_2$$
$$s^0(b) = \sigma_2, \qquad s^1(c) = \sigma_1.$$

Remark 2.14 The symmetric groups Σ_n can not be assembled all together as a cosimplicial group. This is because there are no surjective homomorphisms $\Sigma_n \to \Sigma_{n-1}$ for $n \geq 5$ to use as codegeneracy homomorphisms. Indeed, given a homomorphism

$\varphi\colon \Sigma_n \to \Sigma_{n-1}$, $\ker\varphi$ is a normal subgroup of Σ_n, that is, A_n or Σ_n. Thus the image of φ is either the identity element or a subgroup of order 2.

We describe another method of constructing new cosimplicial groups that arise from a given one. To do this, we recall a concept that was originally introduced in [10, page 284] to define cohomotopy groups (and pointed sets) for a cosimplicial group.

Definition 2.15 Let L be a cosimplicial group. The elements b in L_1 satisfying

(1) $$d^2(b)d^0(b) = d^1(b)$$

are called 1–cocycles of L. The set of 1–cocycles is denoted by $Z^1(L)$.

If b is a 1–cocycle, then applying s^0 to (1), we obtain $s^0d^2(b) = e$ and, using the cosimplicial identities, $d^1s^0(b) = e$, which implies $b \in \ker s^0$. Define inductively $b_n \in L_n$ as $b_{n+1} = d^{n+1}(b_n)$, where $b_1 := b$. These elements will satisfy

(2) $$d^2(b_n)d^0(b_n) = d^1(b_n),$$

(3) $$b_n \in \ker s^0$$

for all $n \geq 1$. Given a 1–cocycle b, we build a new cosimplicial group.

Construction of L^b Define $L^b\colon \Delta \to \mathbf{Grp}$ as follows. For each $n \geq 0$, $L^b_n := \overline{F}_0 * L_n$ with codegeneracy homomorphisms $s^i_b := \mathrm{Id} * s^i$, $i \geq 0$. The coface homomorphisms are $d^i_b := \mathrm{Id} * d^i$ for $i > 0$. To define d^0_b consider the homomorphism $k_n\colon \overline{F}_0 \to \overline{F}_0 * L_n$ given by $k_n(a_0) = a_0 b_n$ for all $n \geq 0$, then $d^0_b := k_n * d^0$. There is a canonical inclusion

$$\iota_b\colon L \hookrightarrow L^b$$

induced by the inclusions $L_n \hookrightarrow \overline{F}_0 * L_n$.

Example 2.16 • When $b = e$, $L^e = \overline{F}_0 * L$, where $\overline{F}_0$ represents the constant cosimplicial group with value $\overline{F}_0$.

• Consider the finitely generated free cosimplicial group F. The codegeneracy homomorphism $s^0\colon F_1 \to F_0$ is the constant map and thus $\ker s^0 = F_1 = \langle a_1 \rangle$. Also

$$d^1(a_1) = a_1 a_2 = d^2(a_1)d^0(a_1),$$

and hence $a_1 \in Z^1(F)$. Note that any other power of a_1 will fail to satisfy the cocycle condition (1), that is $Z^1(F) = \{e, a_1\}$. Let $F^+ = F^{a_1}$. We denote the canonical inclusion by $\iota_+\colon F \hookrightarrow F^+$. A similar argument shows that $Z^1(F/\Gamma^q) = \{e, a_1\}$ for $q > 2$. We also let $(F/\Gamma^q)^{a_1} = F/\Gamma^{q+}$ and $\iota_+\colon F/\Gamma^q \hookrightarrow F/\Gamma^{q+}$.

- Consider F/Γ^2. As in the previous example, $\ker s_0 = \langle a_1 \rangle$, but since $F_2/\Gamma^2_2 = \mathbb{Z}^2$ all powers of a_1 will satisfy the cocycle condition, that is, $Z^1(F/\Gamma^2) = \mathbb{Z}$. Thus for each positive $m \in \mathbb{Z}$ we get nonisomorphic cosimplicial groups $(F/\Gamma^2)^m$ and inclusions $\iota_m\colon F/\Gamma^2 \hookrightarrow (F/\Gamma^2)^m$. When $m = 1$, we write $(F/\Gamma^2)^1 = F/\Gamma^{2^+}$.

- Consider $\Sigma_{2,3}$, defined in Example 2.13. The product $\sigma_1\sigma_2 \in (\Sigma_{2,3})_1$ satisfies

$$d^2(\sigma_1\sigma_2)d^0(\sigma_1\sigma_2) = caab = cb = d^1(\sigma_1\sigma_2)$$

and thus $\sigma_1\sigma_2$ is a 1–cocycle and we get the cosimplicial group $\Sigma_{2,3}^{\sigma_1\sigma_2}$.

Now we turn our attention to spaces of homomorphisms. For any topological group G, its underlying group structure defines the functor

$$\mathrm{Hom}_{\mathbf{Grp}}(\_, G)\colon \mathbf{Grp}^{\mathrm{op}} \to \mathbf{Set}.$$

If L is a cosimplicial group, the composition of functors $\mathrm{Hom}_{\mathbf{Grp}}(\_, G)L$, which we denote by $\mathrm{Hom}(L, G)$, defines a simplicial set. Whenever L is finitely generated, for each n we can topologize $\mathrm{Hom}(L_n, G)$ in a way that the induced face and degeneracy maps are continuous. Therefore we get the simplicial space $\mathrm{Hom}(L, G)\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$. We list some known simplicial spaces:

- $\mathrm{Hom}(F, G) = NG$, the nerve of G as a category with one object.
- $\mathrm{Hom}(F^+, G) = (EG)_*$, Steenrod's model for the total space of the universal principal G–bundle $p\colon EG \to BG$, where p is induced by the simplicial map $\iota_+^*\colon \mathrm{Hom}(F^+, G) \to \mathrm{Hom}(F, G)$.
- $\mathrm{Hom}(F/\Gamma^q, G) = (B(q, G))_*$, the underlying simplicial space of the classifying space $B(q, G)$ defined in [4], [6] (for $q = 2$) and [7].
- $\mathrm{Hom}(F/\Gamma^{q^+}, G) = (E(q, G))_*$, the underlying simplicial space of the total space of the universal bundle $p\colon E(q, G) \to B(q, G)$ also defined in [4], [6] (for $q = 2$) and [7]. Again, p is induced by $\iota_+^*\colon \mathrm{Hom}(F/\Gamma^{q^+}, G) \to \mathrm{Hom}(F/\Gamma^q, G)$.
- $\mathrm{Hom}(\overline{F}, G) = N\overline{G}$, the nerve of the category $\overline{G}$ that has G as space of objects and a unique morphism between any two objects.

Remark 2.17 Consider the morphism of cosimplicial groups $\gamma\colon F \to \overline{F}$ given on generators by $\gamma^n(a_i) = a_{i-1}a_i^{-1}$, where $\gamma^n\colon F_n \to \overline{F}_n$. Let G be a topological group. The induced map $\gamma\colon N\overline{G} \to NG$ is the underlying simplicial map of Segal's fat geometric realization model for the universal G–bundle. A detailed version of this can be seen in [13, page 66].

2.5 Homotopy stable decomposition of $\mathrm{Hom}(L_n, G)$

Lemma 2.18 *Let G be a linear algebraic group and L a finitely generated cosimplicial group. Let $s^i\colon L_{n+1} \to L_n$ be a codegeneracy map. Then, the image of $s_i := (s^i)^*\colon \mathrm{Hom}(L_n, G) \to \mathrm{Hom}(L_{n+1}, G)$ is a subvariety for all $0 \le i \le n$.*

Proof Suppose $L_{n+1} = \langle a_1, \dots, a_r\rangle$. The homomorphism $s^i\colon L_{n+1} \to L_n$ is surjective, so $L_n \cong L_{n+1}/\ker s^i$. We can describe $\ker s^i = \langle \{b_\alpha\}_{\alpha\in\Lambda}\rangle$, where each b_α is a fixed product of powers of generators a_k. Let $\rho\colon L_{n+1} \to G$ be a homomorphism. Then $(\rho(a_1), \dots, \rho(a_r))$ is in $s_i(\mathrm{Hom}(L_n, G))$ if and only if $\rho(b_\alpha) = I$ for all $\alpha \in \Lambda$. That is, these r–tuples in $\mathrm{Hom}(L_{n+1}, G)$ are determined by the polynomial equations $\{\rho(b_\alpha) = I\}_{\alpha\in\Lambda}$ and hence they build up an affine variety. □

For a cosimplicial group L, write $S_t(L_k, G) := S^t(\mathrm{Hom}(L_k, G))$.

Theorem 2.19 *Let G be a real algebraic linear group, and L a finitely generated cosimplicial group. Then for each n we have homotopy equivalences*

$$\Theta(n)\colon \Sigma\,\mathrm{Hom}(L_n, G) \simeq \bigvee\nolimits_{0\le k\le n} \Sigma(S_k(L_n, G)/S_{k+1}(L_n, G)).$$

Proof Fix n. Using Proposition 2.7, we only need to show that

$$(S_{t-1}(L_n, G), S_t(L_n, G))$$

is a strong NDR pair for all $0 \le t \le n$. By Lemma 2.18, each $s_j(\mathrm{Hom}(L_k, G))$ is an affine variety for all $0 \le j, k \le n$. Then, for all $t \ge 1$, the finite union

$$S_t(L_n, G) = \bigcup_{J_{n,t}} s_{i_1} \circ \cdots \circ s_{i_t}(\mathrm{Hom}(L_{n-t}, G))$$

is also an affine variety. Consider the natural filtration

$$S_n(L_n, G) \subset S_{n-1}(L_n, G) \subset \cdots \subset S_0(L_n, G) = \mathrm{Hom}(L_n, G).$$

The union $\bigcup_t S_t(L_n, G) = \mathrm{Hom}(L_n, G)$ is an affine variety, and therefore is a closed subspace of some euclidean space. By Remark 2.5, $\mathrm{Hom}(L_n, G)$ can be triangulated in a way that each $S_t(L_n, G)$ is a finite union of interiors of simplices. Since $S_t(L_n, G)$ are closed subspaces, it follows that under the triangulation they are subcomplexes. This way the inclusions $S_t(L_n, G) \subset S_{t-1}(L_n, G)$ are cofibrations and hence NDR pairs. Therefore $\mathrm{Hom}(L, G)$ is simplicially NDR. □

Lemma 2.20 *Let G be a topological group and consider the cosimplicial group F/Γ^q. Then*

$$S_k(F_n/\Gamma^q_n, G)/S_{k+1}(F_n/\Gamma^q_n, G) \cong \bigvee^{\binom{n}{k}} \mathrm{Hom}(F_k/\Gamma^q_k, G)/S_1(F_k/\Gamma^q_k, G)$$

for all $1 \le k \le n$.

Proof Let $1 \leq i_1 < \cdots < i_{n-k} \leq n$. Consider the projections

$$P_{i_1,\ldots,i_m}\colon G^n \to G^{n-k}$$

given by $(x_1,\ldots,x_n) \mapsto (x_{i_1},\ldots,x_{i_{n-k}})$. We claim that the image of $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ under this projection lies on $\mathrm{Hom}(F_{n-k}/\Gamma_{n-k}^q, G)$. Indeed, each projection $P_{i_1,\ldots,i_m}$ is induced by the homomorphism $\varphi\colon F_{n-k} \to F_n$ given on generators by $a_j = a_{i_j}$. Since $\varphi(\Gamma_{n-k}^q) \subset \Gamma_n^q$ we get the homomorphism $\overline{\varphi}\colon F_{n-k}/\Gamma_{n-k}^q \to F_n/\Gamma_n^q$ which proves our claim. Assemble the restrictions of $P_{i_1,\ldots,i_m}$ to $\mathrm{Hom}(F_n/\Gamma_n^q, G)$ so that we build up a continuous map

$$\eta_n\colon \mathrm{Hom}(F_n/\Gamma_n^q, G) \to \prod_{J_{n,k}} \mathrm{Hom}(F_{n-k}/\Gamma_{n-k}^q, G)$$

given by

$$(x_1,\ldots,x_n) \mapsto \{P_{i_1,\ldots,i_m}(x_1,\ldots,x_n)\}_{(i_1,\ldots,i_{n-k})\in J_{n,k}},$$

where $J_{n,k}$ runs over all possible sequences $1 \leq i_1 < \cdots < i_{n-k} \leq n$ of length $n-k$. Since all sequences $(i_1,\ldots,i_{n-k}) \in J_{n,k}$ are disjoint, the restriction

$$\eta_n|_k\colon S_k(F_n/\Gamma_n^q, G) \to \bigvee\nolimits_{J_{n,k}} \mathrm{Hom}(F_{n-k}/\Gamma_{n-k}^q, G)/S_1(F_k/\Gamma_k^q, G)$$

has a continuous inverse $\bigvee_{J_{n,k}} s_{j_1} \circ \cdots \circ s_{j_k}$, where $1 \leq j_1 < \cdots < j_k \leq n$ and $\{j_1,\ldots,j_k\} \cap \{i_1,\ldots,i_{n-k}\} = \varnothing$. Therefore $\eta_n|_k$ is a homeomorphism for every k. Finally, note that $S_{k+1}(F_n/K_n, G)$ is mapped to $\bigvee S_1(F_{n-k}/\Gamma_{n-k}^q, G)$. Taking quotients we get the desired homeomorphism. □

The next corollary was first conjectured in [4, page 12] for closed subgroups of $\mathrm{GL}_n(\mathbb{C})$. Since any real linear algebraic group is Zariski closed we have the following version of the conjecture, which follows from Theorem 2.19 and Lemma 2.20.

Corollary 2.21 *If G is a Zariski closed subgroup of $\mathrm{GL}_n(\mathbb{C})$, then there are homotopy equivalences for the cosimplicial group F/Γ^q,*

$$\Theta(n)\colon \Sigma\,\mathrm{Hom}(F_n/\Gamma_n^q, G) \simeq \bigvee\nolimits_{1\leq k\leq n} \Sigma\Big(\bigvee\nolimits^{\binom{n}{k}} \mathrm{Hom}(F_k/\Gamma_k^q, G)/S_1(F_k/\Gamma_k^q, G)\Big)$$

for all n and q.

Example 2.22 Let $G = \mathrm{SU}(2)$ and consider F/Γ^q.

- The case $q = 2$ ($F_n/\Gamma_n^2 = \mathbb{Z}^n$) has been largely studied (for example [3; 9; 12]). In this example we follow [3, pages 482–484]. First a few preliminaries. Let T be the maximal torus of G that consists of all diagonal matrices $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \bar\lambda\end{smallmatrix}\right)$ with $\lambda \in S^1$ and $W = N(T)/T = \{[w], e\}$ its Weyl group, where $w = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$. The group W acts on T

via $[w]\cdot t=\overline{wtw^{-1}}=t^{-1}$ and using left translation on G/T we get a diagonal action on $G/T\times T^n$. Let $\mathfrak{t}\cong i\mathbb{R}$ be the Lie algebra of T with the induced action of W. There is an equivariant homeomorphism $\mathfrak{t}\to T-\{I\}$ so that $G/T\times_W(T-\{I\})^n\cong G/T\times_W\mathfrak{t}^n$. The quotient map $G/T\to(G/T)/W\cong\mathbb{RP}^2$ is a principal W–bundle and we can take the associated vector bundle $p_n\colon G/T\times_W\mathfrak{t}^n\to\mathbb{RP}^2$. Let λ_2 be the canonical vector bundle over $\mathbb{RP}^2$; then we can identify p_n with $n\lambda_2$, the Whitney sum of n copies of λ_2. The pieces in the homotopy stable decomposition before suspending are

$$\mathrm{Hom}(\mathbb{Z}^n,\mathrm{SU}(2))/S_1(\mathbb{Z}^n,\mathrm{SU}(2))\cong\begin{cases}S^3 & \text{if } n=1,\\ (\mathbb{RP}^2)^{n\lambda_2}/s_n(\mathbb{RP}^2) & \text{if } n\geq 2,\end{cases}$$

where $(\mathbb{RP})^{n\lambda_2}$ is the associated Thom space of $n\lambda_2$ and s_n is its zero section. Therefore,

$$\Sigma\,\mathrm{Hom}(\mathbb{Z}^n,\mathrm{SU}(2))\simeq\Sigma\bigvee_n S^3\vee\bigvee_{2\leq k\leq n}\Sigma\big(\bigvee^{\binom{n}{k}}(\mathbb{RP}^2)^{k\lambda_2}/s_k(\mathbb{RP}^2)\big).$$

- Let $q=3$. An n–tuple $(g_1,\dots,g_n)$ lies in $\mathrm{Hom}(F_n/\Gamma_n^3,\mathrm{SU}(2))$ if and only if $[[g_i,g_j],g_k]=I$ for all $1\leq i,j,k\leq n$, ie the commutators $[g_i,g_j]$ are central in the subgroup generated by $g_1,\dots,g_n$. We claim that the center of every nonabelian subgroup of $\mathrm{SU}(2)$ sits inside $\{\pm I\}$. Suppose a and b are two elements in $\mathrm{SU}(2)$ such that $[a,b]\neq I$. Then, the cyclic groups $\langle a\rangle$ and $\langle b\rangle$ are contained in different tori, say T_1 and T_2, respectively. Since the center of $\langle a,b\rangle$ is abelian it must lie in the intersection $T_1\cap T_2$. These two circles can only intersect at $\{\pm I\}$, which proves our claim. Therefore the central elements $[g_i,g_j]$ are in $\{\pm I\}$ for all $1\leq i,j\leq n$. Consider

$$B_n(\mathrm{SU}(2),\{\pm I\})=\big\{(g_1,\dots,g_n)\in\mathrm{SU}(2)^n\mid[g_i,g_j]\in\{\pm I\}\big\},$$

the space of almost commuting tuples in $\mathrm{SU}(2)$. By the previous observation,

$$\mathrm{Hom}(F_n/\Gamma_n^3,\mathrm{SU}(2))=B_n(\mathrm{SU}(2),\{\pm I\}).$$

In [3, pages 485–486], they show that

$$B_n(\mathrm{SU}(2),\{\pm I\})/S_1(\mathrm{SU}(2),\{\pm I\})\\ \cong\mathrm{Hom}(\mathbb{Z}^n,\mathrm{SU}(2))/S_1(\mathbb{Z}^n,\mathrm{SU}(2))\vee\bigvee_{K(n)}\mathrm{PU}(2)_+,$$

where $K(1)=0$ and $K(n)=\frac{7^n}{24}-\frac{3^n}{8}+\frac{1}{12}$ for $n\geq 2$. Here $S_1(\mathrm{SU}(2),\{\pm I\})$ are the n–tuples in $B_n(\mathrm{SU}(2),\{\pm I\})$ with at least one coordinate equal to I. Since $\mathrm{PU}(2)\cong\mathbb{RP}^3$ and $S_1(\mathrm{SU}(2),\{\pm I\})=S_1(F_n/\Gamma_n^3,\mathrm{SU}(2))$ we conclude that

$$\Sigma\,\mathrm{Hom}(F_n/\Gamma_n^3,\mathrm{SU}(2))\\ \simeq\Sigma\bigvee_n S^3\vee\bigvee_{2\leq k\leq n}\Sigma\big(\bigvee^{\binom{n}{k}}(\mathbb{RP}^2)^{k\lambda_2}/s_k(\mathbb{RP}^2)\vee\bigvee_{K(k)}\mathbb{RP}^3_+\big).$$

Remark 2.23 For $q \geq 4$ we can find nilpotent subgroups of $\mathrm{SU}(2)$ of class q. Indeed, if ξ_n is a representative in $\mathrm{SU}(2)$ of a primitive n^{th} root of unity, then the subgroup generated by the set $\{\xi_{2q}, w\}$ with w as above, is of nilpotency class q. With this we can show that the spaces $\mathrm{Hom}(F_n/\Gamma_n^q, \mathrm{SU}(2))$ for $q \geq 4$ have more connected components than $\mathrm{Hom}(F_n/\Gamma_n^3, \mathrm{SU}(2))$. See [8] for more details.

2.6 Equivariant homotopy stable decomposition of Hom(L_n, G)

Let G and H be topological groups and $f\colon G \to H$ a continuous homomorphism. If L is a finitely generated cosimplicial group then for each n, we have the commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}(L_n, G) & \xrightarrow{d_i} & \mathrm{Hom}(L_{n-1}, G) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Hom}(L_n, H) & \xrightarrow{d_i} & \mathrm{Hom}(L_{n-1}, H) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}(L_n, G) & \xrightarrow{s_i} & \mathrm{Hom}(L_{n+1}, G) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Hom}(L_n, H) & \xrightarrow{s_i} & \mathrm{Hom}(L_{n+1}, H) \end{array}$$

for all $n \geq 0$ and all $0 \leq i \leq n$, so that f_* is a simplicial map. Conjugation by elements of G defines a homomorphism $G \to G$, so that $\mathrm{Hom}(L_n, G)$ is a G–space and each $S_t(L_n, G)$ is a G–subspace.

Definition 2.24 Let M be a G–space. We say that M has a G–CW–structure if there exists a pair (X, ξ) such that X is a G–CW–complex and $\xi\colon X \to M$ is a G–equivariant homeomorphism.

We want to show that for all n, $\mathrm{Hom}(L_n, G)$ has a G–CW–complex structure for which $S_n(L_n, G) \subset S_{n-1}(L_n, G) \subset \cdots \subset \mathrm{Hom}(L_n, G)$ are G–subcomplexes. To show this we slightly generalize some results in [21].

We continue using the techniques of the previous section, so we require G to be a real linear algebraic group. It is known that any compact Lie group has a unique algebraic group structure (see [19, page 247]). Assuming G is a compact Lie group, every representation space of G has finite orbit types (see [20]), so when M is an algebraic G–variety, the equivariant algebraic embedding theorem [21, Proposition 3.2] implies that M has finite orbit types. Also, this theorem guarantees the existence of a G–invariant algebraic map $f\colon M \to \mathbb{R}^d$ for some d such that the induced map $\bar{f}\colon M/G \to f(M)$ is a homeomorphism and $f(M)$ is a closed semialgebraic set in $\mathbb{R}^d$ [21, Lemma 3.4]. If $\tau\colon |K| \to M/G$ is a triangulation, we say that τ is compatible with a family of subsets $\{D_i\}$ of M if $\pi(D_i)$ is a union of some $\tau(\mathrm{int}\,|\sigma|)$, where $\sigma \in K$ and $\pi\colon M \to M/G$ is the quotient map.

Proposition 2.25 *Let G be a compact Lie group, M_0 an algebraic G–variety and $\{M_j\}_{j=1}^n$ a finite system of G–subvarieties of M_0. Then there exists a semialgebraic triangulation $\tau\colon |K| \to M/G$ compatible with the collection*

$$\{M_{j\,(H)} \mid H \text{ is a subgroup of } G\}_{j=0}^n,$$

where $M_{j\,(H)} = \{x \in M_j \mid G_x = gHg^{-1} \text{ for some } g \in G\}$.

Proof Let $H_1, \dots, H_s \subset G$ be the orbit types of G on M and $f\colon M \to \mathbb{R}^d$ as above. By [21, Lemma 3.3], all $M_{j\,(H_i)}$ are semialgebraic sets, and therefore all $f(M_{j\,(H_i)})$ are also semialgebraic. Since i and j vary on finite sets, we can use Remark 2.5 and obtain a semialgebraic triangulation

$$\lambda\colon |K| \to f(M) = \bigcup_{ij} f(M_{j\,(H_i)})$$

such that each $f(M_{j\,(H_i)})$ is a finite union of some $\lambda(\operatorname{int}|\sigma|)$, where $\sigma \in K$. Take $\tau = \bar{f}^{-1} \circ \lambda$. □

Proposition 2.26 *Let G be a compact Lie group. Let M_0 be an algebraic G–variety and $\{M_j\}_{j=1}^k$ a finite system of G–subvarieties. Then M_0 has a G–CW–complex structure such that each M_j is a G–subcomplex of M.*

Proof Let $\tau\colon |K| \to M/G$ be as in Proposition 2.25 and $\pi\colon M \to M/G$ the orbit map. Let K' be a barycentric subdivision of K, which guarantees that, for any simplex Δ^n of K', $\pi^{-1}(\tau(\Delta^n - \Delta^{n-1})) \subset M_{j\,(H_n)}$ for some $H_n \subset G$ and $0 \le j \le k$. Since $\tau|\colon \pi^{-1}(\tau(\Delta^n))/G \to \Delta^n$ is a homeomorphism and the orbit type of $\pi^{-1}(\tau(\Delta^n - \Delta^{n-1}))$ is constant, by [16, Lemma 4.4] there exists a continuous section $s\colon \tau(\Delta^n) \to M_j$ such that $s \circ \tau(\Delta^n - \Delta^{n-1})$ has a constant isotropy subgroup H_n. Consequently there is an equivariant homeomorphism

$$\pi^{-1}\tau(\Delta^n - \Delta^{n-1}) \cong G/H_n \times (\Delta^n - \Delta^{n-1}).$$

Collecting G–cells $Gs \circ \tau(\Delta^n)$ for all simplices of K' we get a G–CW–structure over all M_j for $0 \le j \le k$. □

For a finitely generated cosimplicial group L, let $\operatorname{Rep}(L_n, G) := \operatorname{Hom}(L_n, G)/G$ and $\bar{S}_t(L_n, G) := S_t(L_n, G)/G$.

Theorem 2.27 *Let G be a compact Lie group and L a finitely generated cosimplicial group. Then, for each n, $\Theta(n)$ from Theorem 2.19 is a G–equivariant homotopy equivalence, and in particular we get homotopy equivalences*

$$\Sigma \operatorname{Rep}(L_n, G) \simeq \bigvee\nolimits_{1 \le k \le n} \Sigma(\bar{S}_k(L_n, G)/\bar{S}_{k+1}(L_n, G)).$$

Proof Assume $G \subset \mathrm{GL}_N(\mathbb{R})$. Under conjugation by elements of G, $\mathrm{Hom}(L_n, G)$ is an affine G–variety and by Lemma 2.18 the subspaces $S_j(L_n, G)$ are G–subvarieties for all $1 \le j \le n$. Hence, by Proposition 2.26, $\mathrm{Hom}(L_n, G)$ can be given a G–CW–complex structure, where each $S_j(L_n, G)$ is a G–subcomplex. Similarly, the quotient $S_k(L_n, G)/S_{k+1}(L_n, G)$ has a G–CW–complex structure.

To prove that the map $\Theta(n)$ is a G–equivariant homotopy equivalence, first recall that conjugation by elements of G defines a simplicial action on $\mathrm{Hom}(L, G)$, and by the naturality of each $\Theta(n)$, the G–equivariance follows. Let $H \subset G$ be a closed subgroup. The fixed point spaces $\mathrm{Hom}(L_n, G)^H$ and $S_k(L_n, G)^H$ inherit a CW–complex structure, so that $\mathrm{Hom}(L, G)^H$ is simplicially NDR. By Proposition 2.7, we have homotopy equivalences

$$\Theta(n, H)\colon \Sigma(\mathrm{Hom}(L_n, G)^H) \to \bigvee_{0 \le k \le n} \Sigma(S_k(L_n, G)^H / S_{k+1}(L_n, G)^H)$$

for each $n \ge 1$. The fixed points map $\Theta(n)^H$ agrees by naturality with $\Theta(n, H)$ and thus is a homotopy equivalence. The result now follows from the equivariant Whitehead theorem. □

Corollary 2.28 *Let G be a compact Lie group. Then the homotopy equivalences in Corollary 2.21 are G–equivariant homotopy equivalences, and in particular we get*

$$\Sigma\,\mathrm{Rep}(F_n/\Gamma_n^q, G) \simeq \bigvee_{1 \le k \le n} \Sigma\Big(\bigvee^{\binom{n}{k}} \mathrm{Rep}(F_k/\Gamma_k^q, G)/\bar{S}_1(F_k/\Gamma_k^q, G)\Big).$$

Example 2.29 Let $G = \mathrm{SU}(2)$ and $L = F/\Gamma^q$.

- For $q = 2$, it was proven in [3, page 484] that

$$\mathrm{Rep}(\mathbb{Z}^n, \mathrm{SU}(2))/\bar{S}_1(\mathbb{Z}^n, \mathrm{SU}(2)) \simeq T^{\wedge n}/W = S^n/\Sigma_2,$$

where the action of the generating element on Σ_2 is given by

$$(x_0, x_1, \dots, x_n) \mapsto (x_0, -x_1, \dots, -x_n)$$

for any $(x_0, x_1, \dots, x_n)$. Identifying $S^n = \Sigma S^{n-1}$, we can see the orbit space S^n/Σ_2 as first taking antipodes, and then suspending, that is, $S^n/\Sigma_2 \cong \Sigma\,\mathbb{RP}^{n-1}$. Thus

$$\Sigma\,\mathrm{Rep}(\mathbb{Z}^n, \mathrm{SU}(2)) \simeq \bigvee_{1 \le k \le n} \Sigma\Big(\bigvee^{\binom{n}{k}} \Sigma\,\mathbb{RP}^{k-1}\Big).$$

- Let $q = 3$. We have shown that $\mathrm{Rep}(F_n/\Gamma_n^3, \mathrm{SU}(2)) = B_n(\mathrm{SU}(2), \{\pm I\})/G$ and using the description of these spaces given in [3, p. 486], the stable pieces are

$$\mathrm{Rep}(F_n/\Gamma_n^3, \mathrm{SU}(2))/\bar{S}_1(F_n/\Gamma_n^3, \mathrm{SU}(2)) \simeq \Big(\bigvee_{K(n)} S^0\Big) \vee \Sigma\,\mathbb{RP}^{n-1},$$

where $K(n)$ is as in Example 2.22. Therefore

$$\Sigma\,\mathrm{Rep}(F_n/\Gamma_n^3, \mathrm{SU}(2)) \simeq \bigvee_{1 \le k \le n} \Sigma\Big(\bigvee^{\binom{n}{k}} \Big(\bigvee_{K(k)} S^0\Big) \vee \Sigma\,\mathbb{RP}^{k-1}\Big).$$

3 Homotopy properties of $B(L,G)$

3.1 Geometric realization of Hom(L,G)

Definition 3.1 Let L be a finitely generated cosimplicial group and G a topological group. Let

$$B(L,G) := |\mathrm{Hom}(L,G)|.$$

For the cosimplicial groups F/Γ^q we get that $B(F/\Gamma^q, G) = B(q,G)$, the classifying space for G–bundles of transitional nilpotency class less than q. A natural question is whether or not the space $B(L,G)$ is a classifying space for a specific class of G–bundles.

Lemma 3.2 *Let L be a cosimplicial group. The 1–cocycles of L are in one-to-one correspondence with cosimplicial morphisms $F \to L$.*

Proof Suppose b is a 1–cocycle. Any generator $a_j \in F_n$ is in the image of $a_1 \in F_1$ under composition of coface homomorphisms, eg $a_j = (d^0)^{j-1}(d^2)^{n-j}(a_1)$ for all $j \geq 1$. Define $h^n\colon F_n \to L_n$ as $h^n(a_j) = (d^0)^{j-1}(d^2)^{n-j}(b)$. To show that h is cosimplicial, consider the diagrams

$$\begin{array}{ccc} F_{n-1} & \xrightarrow{h^{n-1}} & L_{n-1} \\ {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ F_n & \xrightarrow{h^n} & L_n \end{array} \qquad \begin{array}{ccc} F_{n+1} & \xrightarrow{h^{n+1}} & L_{n+1} \\ {\scriptstyle s^i}\downarrow & & \downarrow{\scriptstyle s^i} \\ F_n & \xrightarrow{h^n} & L_n \end{array}$$

We prove the case of coface homomorphisms. Let $a_j \in F_{n-1}$. On one side we get

$$h_n d^i(a_j) = \begin{cases} (d^0)^{j-1}(d^2)^{n-j}(b) & \text{if } j < i, \\ (d^0)^{j-1}(d^2)^{n-j}(b)(d^0)^j(d^2)^{n-j-1}(b) & \text{if } j = i, \\ (d^0)^j(d^2)^{n-j-1}(b) & \text{if } j > i, \end{cases}$$

and applying the cosimplicial identity $d^k d^l = d^l d^{k-1}$, where $k > l$, we obtain

$$d^i h^{n-1}(a_j) = \begin{cases} (d^0)^{j-1}(d^{i-j+1})(d^2)^{n-j-1}(b) & \text{if } j < i, \\ (d^0)^{j-1}(d^1)(d^2)^{n-j-1}(b) & \text{if } j = i, \\ (d^0)^j(d^2)^{n-j-1}(b) & \text{if } j > i. \end{cases}$$

We need to analyze two cases:

- $j < i$ implies that $i - j + 1 \geq 2$, thus $d^{i-j+1}(d^2)^{n-j-1} = (d^2)^{n-j}$.
- $j = i$. Then the equality follows from (2) applied to $(d^2)^{n-j-1}(b) = b_{n-j}$.

Commutativity for the codegeneracy homomorphisms is similar, but using the cosimplicial identity $s^k d^l = d^l s^{k-1}$ with $k > l$ and condition (3) above. Hence h is uniquely determined by b. Given a morphism $F \to L$, the element b is given by the image of $a_1 \in F_1$. □

Proposition 3.3 *Let L be a cosimplicial group and $h_b\colon F \to L$ be the morphism defined on $F_1 \to L_1$ as $a_1 \mapsto b$. Then the diagram*

$$\begin{array}{ccc} F & \xhookrightarrow{\iota_+} & F^+ \\ {\scriptstyle h_b}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}*h_b} \\ L & \xhookrightarrow{\iota_b} & L^b \end{array}$$

is a pushout of cosimplicial groups.

Proof Suppose $f\colon F^+ \to K$ and $g\colon L \to K$ are morphisms such that $f \circ \iota = g \circ h_b$. Define $h\colon L^b \to K$ on each $L^b_n = \overline{F}_0 * L_n$ as $h^n(a_0) = f^n(a_0)$ (here f^n is evaluated on $a_0 \in \overline{F}_0 * F_n$) and $h^n(x) = g^n(x)$ for any $x \in L_n$. To check that h is in fact a cosimplicial homomorphism, by construction of L^b and h, we just need to verify commutativity with coface maps at level $i = 0$. Consider

$$\begin{array}{ccc} L^b_{n-1} & \xrightarrow{h^{n-1}} & K_{n-1} \\ {\scriptstyle d^0_b}\downarrow & & \downarrow{\scriptstyle d^0} \\ L^b_n & \xrightarrow{h^n} & K_n \end{array}$$

We only need to see what happens at $a_0 \in L^b_{n-1}$:

$$d^0 h^{n-1}(a_0) = d^0 f^{n-1}(a_0) = f^n d^0_+(a_0) = f^n(a_0 a_1) = f^n(a_0) f^n(a_1),$$
$$h^n d^0_+(a_0) = h^n(a_0 b_n) = f^n(a_0) g^n(b_n).$$

Let $b = b_1$. By hypothesis, $g^1(b_1) = f^1(a_1)$. Since

$$g^n(b_n) = (d^2)^{n-1} g^1(b_1) \quad \text{and} \quad f^n(a_1) = (d^2)^{n-1} f^1(a_1),$$

the desired equality holds. □

Corollary 3.4 *Let G be a well-based topological group and L a finitely generated cosimplicial group. Using the notation above, suppose $h_b\colon F \to L$ is a morphism. Then the inclusion $\iota_b\colon L \hookrightarrow L^b$ defines a principal G–bundle $|\iota_b^*|\colon B(L^b, G) \to B(L, G)$.*

Proof From the pushout diagram in Proposition 3.3, and applying the functors $\mathrm{Hom}(\_, G)$ and geometric realization, we obtain the pullback diagram

$$\begin{array}{ccc} B(L^b, G) & \longrightarrow & EG \\ {\scriptstyle |\iota_b^*|}\downarrow & & \downarrow \\ B(L, G) & \xrightarrow{|h_b^*|} & BG \end{array}$$

and hence $|\iota_b^*|$ is a principal G–bundle. □

Example 3.5 We have seen that there is only one nonconstant homomorphism $h_{a_1} = \mathrm{Id}\colon F \to F$. For $q > 2$ it can be shown that the same is true for $L = F/\Gamma^q$, where $h_{a_1}\colon F \to F/\Gamma^q$ at each n is the quotient homomorphism. The corresponding $B((F/\Gamma^q)^+, G)$ is the space $E(q, G)$ defined in [4, page 94], and $|h_{a_1}^*|\colon B(q, G) \to BG$ is the inclusion. The bundle $E(q, G) \to B(q, G)$ classifies transitionally nilpotent bundles of class less than q (see [7, Section 5]). The case $q = 2$ is more interesting since $Z^1(F/\Gamma^2) = \mathbb{Z}$. For $m = 1$ we obtain $B((F/\Gamma^2)^+, G) = E(2, G)$ and $E(2, G) \to B(2, G)$ classifies transitionally commutative bundles (see [6, Section 2]). Since multiplication by -1 induces a cosimplicial automorphism of F/Γ^2, all constructions are equivalent for $m = -1$. Now let $m > 1$. The bundle $B((F/\Gamma^2)^m, G) \to B(2, G)$ will classify G–bundles whose transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ factor through

$$\begin{array}{ccc} U_\alpha \cap U_\beta & \xrightarrow{\rho_{\alpha,\beta}} & G \\ & {\scriptstyle g_{\alpha\beta}}\searrow & \downarrow{\scriptstyle m} \\ & & G \end{array}$$

where the $\rho_{\alpha\beta}$ are transitionally commutative and m denotes taking the m^{th} power of elements in G.

3.2 Relation between commutative $\mathbb{I}$–monoids and infinite loop spaces

In this section we recall briefly the notion of $\mathbb{I}$–monoid and how it is related to infinite loop spaces. This is more widely covered in [7]. Our goal is to use this machinery to show that, for a finitely generated cosimplicial group L, $B(L, U) = \mathrm{colim}_n\, B(L, U(n))$ is a nonunital E_∞–ring space when $\mathrm{Hom}(L_0, U)$ is path connected.

Let $\mathbb{I}$ stand for the category whose objects are the sets $[0] = \varnothing$ and $[n] = \{1, \dots, n\}$ for each $n \geq 1$, and morphisms are injective functions. Any morphism $j\colon [n] \to [m]$ in $\mathbb{I}$ can be factored as a canonical inclusion $[n] \hookrightarrow [m]$ and a permutation $\sigma \in \Sigma_m$. This category is symmetric monoidal under two different operations. One is concatenation $[n]\sqcup[m]=[n+m]$ with symmetry morphism the permutation $\tau_{m,n}\in\Sigma_{n+m}$ defined as

$$\tau_{m,n}(i)=\begin{cases} n+i & \text{if } i\le m,\\ i-m & \text{if } i>m,\end{cases}$$

and identity object $[0]$. The second operation is the Cartesian product $[m]\times[n]=[mn]$ with $\tau^{\times}_{m,n}\in\Sigma_{mn}$ given by

$$\tau^{\times}_{m,n}((i-1)n+j)=(j-1)m+i,$$

where $1\le i\le m$ and $1\le j\le n$. In this case the identity object is $[1]$. Cartesian product is distributive under concatenation (both left and right).

Definition 3.6 An $\mathbb{I}$*–space* is a functor $X\colon \mathbb{I}\to\mathbf{Top}$. This functor is determined by the following:

1. A family of spaces $\{X[n]\}_{n\ge 0}$, where each $X[n]$ is a Σ_n–space;
2. Σ_n–equivariant structural maps $j_n\colon X[n]\to X[n+1]$ (here we consider $X[n+1]$ is a Σ_n–space under the restriction of the Σ_{n+1}–action to the canonical inclusion $\Sigma_n\hookrightarrow\Sigma_{n+1}$) with the property: for any $j\colon [n]\to[m]$ and any $\sigma,\sigma'\in\Sigma_m$ whose restrictions in Σ_n are equal, we have $\sigma\cdot x=\sigma'\cdot x\in X(j)(X[n])$.

We say that an $\mathbb{I}$–space X is a *commutative* $\mathbb{I}$*–monoid* if it is a symmetric monoidal functor $X\colon (\mathbb{I},\sqcup,[0])\to(\mathbf{Top},\times,\{\mathrm{pt}\})$. Additionally, we say that X is a *commutative* $\mathbb{I}$*–rig* if X is also symmetric monoidal with respect to $(\mathbb{I},\times,[1])$. For the latter definition we also require X to preserve distributivity.

Definition 3.7 Let $\boldsymbol{C}$ be a small category and $Y\colon \boldsymbol{C}\to\mathbf{Top}$ a functor. Denote by $\boldsymbol{C}\ltimes Y$ the category of elements of Y, that is, objects are pairs (c,x) consisting of an object c of $\boldsymbol{C}$ and a point $x\in Y(c)$. A morphism in $\boldsymbol{C}\ltimes Y$ from (c,x) to (c',x') is a morphism $\alpha\colon c\to c'$ in $\boldsymbol{C}$ satisfying the equation $Y(\alpha)(x)=x'$.

Given $Y\colon \boldsymbol{C}\to\mathbf{Top}$, with the notation above, if we consider $\boldsymbol{C}\ltimes Y$ as a topological category whose space of objects and space of morphisms are

$$\bigsqcup_{c\in\mathrm{obj}(\boldsymbol{C})} Y(c)\quad\text{and}\quad\bigsqcup_{f\in\mathrm{mor}(\boldsymbol{C})} Y(f),$$

respectively, then we have that the homotopy colimit of Y is the classifying space $B(\boldsymbol{C}\ltimes Y)=\mathrm{hocolim}_{\boldsymbol{C}}Y$, that is, the realization of the nerve of the category $\boldsymbol{C}\ltimes Y$.

Let X denote a commutative $\mathbb{I}$–monoid. The category of elements $\mathbb{I}\ltimes X$ is a permutative category, that is, a symmetric monoidal category where associativity and unit relations hold strictly. According to [18], the classifying space of a permutative category has an E_∞–space structure, and so we get that $\mathrm{hocolim}_{\mathbb{I}} X$ has an E_∞–space structure. Here we think of an E_∞–space as a space with an operation that is associative and commutative up to a system of coherent homotopies. Thus, the group completion $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$ is an infinite loop space. If X is a commutative $\mathbb{I}$–rig, then $\mathbb{I} \ltimes X$ is a bipermutative category and its classifying space is an E_∞–ring space (as explained in [7]), that is, an E_∞–space with an operation that is associative and commutative (up to coherent homotopy) and distributive (up to coherent homotopy) over the E_∞–space operation.

Consider the subcategory of $\mathbb{I}$ consisting of the same set of objects and all isomorphisms. We denote it by $\mathbb{P}$. The (bi)permutative structure on $\mathbb{I} \ltimes X$ restricts to $\mathbb{P} \ltimes X$, so that $\mathrm{hocolim}_{\mathbb{P}} X$ is also an E_∞–space (E_∞–ring space) and its group completion $\Omega B(\mathrm{hocolim}_{\mathbb{P}} X)$ is an infinite loop space (E_∞–ring space). The maps $X[n] \to *$ induce a map of (bi)permutative categories $\mathbb{P} \ltimes X \to \mathbb{P} \ltimes *$ and therefore a map of infinite loop spaces (E_∞–ring spaces)

$$\rho^X \colon \Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \to \Omega B(\mathrm{hocolim}_{\mathbb{P}} *).$$

It follows that the homotopy fiber $\mathrm{hofib}\, \rho^X$ is an infinite loop space (nonunital E_∞–ring space). Let $X_\infty := \mathrm{hocolim}_{\mathbb{N}} X$, where $\mathbb{N}$ denotes the subcategory of $\mathbb{I}$ with same set of objects and as arrows the canonical inclusions, and X_∞^+ its Quillen plus construction applied with respect to the maximal perfect subgroup of $\pi_1(X_\infty)$. The following proposition is proved in [7, Theorem 3.1].

Proposition 3.8 *Let $X \colon \mathbb{I} \to$ **Top** be a commutative $\mathbb{I}$–monoid. Assume that*

- *the action of Σ_∞ on $H_*(X_\infty)$ is trivial;*
- *the inclusions induce natural isomorphisms $\pi_0(X[n]) \simeq \pi_0(X_\infty)$ of finitely generated abelian groups with multiplication compatible with the Pontrjagin product and in the center of the homology Pontrjagin ring;*
- *the commutator subgroup of $\pi_1(X_\infty)$ is perfect (for each component) and X_∞^+ is abelian.*

Then $\mathrm{hofib}\, \rho^X \simeq X_\infty^+$*, and in particular X_∞^+ is an infinite loop space.*

Note that the last two conditions of the previous Proposition are satisfied when each $X[n]$ is connected and X_∞ is abelian. Under these hypothesis X_∞ has an infinite loop space structure.

3.3 Nonunital E_∞–ring space structure of $B(L, U)$

Our first example and application of the machinery described in the previous section is showing the classical result

$$\operatorname*{colim}_m U(m) = U$$

has an infinite loop space structure. This will allow us to prove nonunital E_∞–ring space structures on our spaces of interest.

First, we show that $U(\_)$ is a commutative $\mathbb{I}$–rig. Recall that $\Sigma_m \subset U(m)$ as permutation matrices, so that $U(m)$ has a Σ_m action. Consider the inclusions

$$i_m\colon U(m) \to U(m+1), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix},$$

which are continuous and preserve group structure. The maps i_m restrict to the canonical inclusions $\Sigma_m \hookrightarrow \Sigma_{m+1}$, therefore

$$i_m(\sigma \cdot A) = i_m(\sigma) i_m(A) i_m(\sigma)^{-1} = \sigma \cdot i_m(A),$$

where $\sigma \in \Sigma_m$, $A \in U(m)$ and on the right-hand side $\sigma \in \Sigma_{m+1}$. Now let $\sigma, \sigma' \in \Sigma_r$ with $m < r$ and suppose both restrictions to the subset $\{1, \dots, m\}$ determine equal permutations in Σ_m. Let $i = i_r \circ i_{r-1} \circ \cdots \circ i_m$. Then, for $A \in U(m)$,

$$\sigma \cdot i(A) = \begin{pmatrix} (\sigma|_m) A (\sigma|_m^{-1}) & 0 \\ 0 & I_{r-m} \end{pmatrix} = \begin{pmatrix} (\sigma'|_m) A (\sigma'|_m^{-1}) & 0 \\ 0 & I_{r-m} \end{pmatrix} = \sigma' \cdot i(A).$$

Therefore $U(\_)\colon \mathbb{I} \to \mathbf{Top}$ is a functor. This $\mathbb{I}$–space has a commutative $\mathbb{I}$–rig structure as follows. Let $\oplus_{m,n}\colon U(m) \times U(n) \to U(m+n)$ denote the block sum of matrices, which is a group homomorphism. The (m,n) shuffle map $U(n+m) \to U(n+m)$ is given by $A \mapsto \tau_{m,n} \cdot A$. We have the commutative diagram

$$\begin{array}{ccc} U(m) \times U(n) & \xrightarrow{\oplus_{m,n}} & U(m+n) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau_{m,n}} \\ U(n) \times U(m) & \xrightarrow{\oplus_{n,m}} & U(m+n) \end{array}$$

where $\tau(A, B) = (B, A)$. Therefore $U(\_)$ is a commutative $\mathbb{I}$–monoid. The other monoidal structure is given by $\otimes_{m,n}\colon U(m) \times U(n) \to U(mn)$ the tensor product of matrices. Indeed, by definition $\tau^{\times}_{m,n} \cdot \otimes_{m,n}(A, B) = \otimes_{n,m} \tau(A, B)$, where $A \in U(m)$ and $B \in U(n)$. Since the image $\oplus_{m,n}(U(m) \times U(n))$ correspond to direct sum, then associativity, left and right distributivity over $\otimes_{m,n}$ hold.

Now we check the conditions of Proposition 3.8: the action of Σ_m on $U(m)$ is homologically trivial since conjugation action on $U(m)$ is trivial up to homotopy, $U(m)$ being path connected. The inclusions i_m are cellular and hence $U(\_)_\infty \simeq U$ and since U is an H–space under block sum of matrices, it is abelian. Therefore $U(\_)_\infty \simeq U$ is an infinite loop space (nonunital E_∞–ring space).

Lemma 3.9 *Let L be a finitely generated cosimplicial group and G, H real algebraic linear groups. Let $p_1\colon G\times H\to G$ and $p_2\colon G\times H\to H$ be the projections. Then*

$$B(L,p_1)\times B(L,p_2)\colon B(L,G\times H)\to B(L,G)\times B(L,H)$$

is a natural homeomorphism.

Proof Since $G\times H$ is a direct product, p_1 and p_2 are continuous homomorphism and therefore

$$p=(p_1)_*\times(p_2)_*\colon \mathrm{Hom}(L,G\times H)\to\mathrm{Hom}(L,G)\times\mathrm{Hom}(L,H)$$

is a simplicial map. Its easy to check that in fact p is a simplicial isomorphism. Both G and H being real algebraic imply that $\mathrm{Hom}(L_n,G)$ and $\mathrm{Hom}(L_n,H)$ have a CW–complex structure, and therefore are k–spaces. By [17, Theorem 11.5], the composition

$$B(L,G\times H)\xrightarrow{|p|}\mathrm{Hom}(L,G)\times\mathrm{Hom}(L,H)|\xrightarrow{|\pi_1|\times|\pi_2|}B(L,G)\times B(L,H)$$

is a natural homeomorphism, where $|\pi_1\circ p|\times|\pi_2\circ p|=B(L,p_1)\times B(L,p_2)$. □

Proposition 3.10 *Let L be a finitely generated cosimplicial group; then $B(L,U(\_))$ is a commutative $\mathbb{I}$–rig.*

Proof Consider the $\mathbb{I}$–rig $U(\_)$. Both the structural maps i_m and the action by elements of Σ_m are continuous group homomorphisms and hence $B(L,U(\_))=B(L,\_)U(\_)$ is an $\mathbb{I}$–space. Also, block sum of matrices and tensor product are topological group morphisms, so that with Lemma 3.9 we can define

$$\mu_{m,n}=B(L,\oplus_{m,n})\circ(B(L,p_1)\times B(L,p_2))^{-1},$$
$$\pi_{m,n}=B(L,\otimes_{m,n})\circ(B(L,p_1)\times B(L,p_2))^{-1},$$

where $p_1\colon U(m)\times U(n)\to U(m)$ and $p_2\colon U(m)\times U(n)\to U(n)$ are the projections. Let $p_1'\colon U(n)\times U(m)\to U(n)$ and $p_2'\colon U(n)\times U(m)\to U(m)$ also denote projections. Notice that

$$\tau\circ B(L,p_2')\times B(L,p_1')=B(L,p_1)\times B(L,p_2)\circ B(L,\tau)$$

(where τ, as before, is the symmetry morphism in **Top**). This implies that all properties satisfied by $\oplus_{m,n}$ and $\otimes_{m,n}$ will be preserved by $\mu_{m,n}$ and $\pi_{m,n}$. □

Theorem 3.11 *Let L be a finitely generated cosimplicial group and suppose that the space $\mathrm{Hom}(L_0,U(m))$ is path connected for all $m\geq 1$. Then $B(L,U)$ is a nonunital E_∞–ring space.*

Proof By Proposition 3.10, $B(L, U(\_))$ is a commutative $\mathbb{I}$–rig. It remains to check the conditions of Proposition 3.8. Note that the conjugation action of Σ_n is homologically trivial since it factors through conjugation action on $U(m)$. Since all $\mathrm{Hom}(L_0, U(m))$ are path connected, $|\mathrm{Hom}(L, U(m))| = B(L, U(m))$ is path connected for all $m \geq 1$. The colimit $B(L, U)$ is also an H–space under block sum of matrices, and therefore abelian. □

Example 3.12 The property $\pi_0\big(\mathrm{Hom}(L_0, U(m))\big) = 0$ for all $m \geq 1$ is satisfied by the following cosimplicial groups:

- $L = F/\Gamma^q$ and $L = F/F^{(q)}$ since $L_0 = \{e\}$ in both cases.
- $L = \overline{F}/\Gamma^q$ and $L = \overline{F}/F^{(q)}$ since $\mathrm{Hom}(L_0, U(m)) = U(m)$ in both cases.
- Consider $\Sigma_{2,3}$, and the cosimplicial morphism $h_{\sigma_1\sigma_2}\colon F \to \Sigma_{2,3}$. The image $h_{\sigma_1\sigma_2}(F)$ defines a cosimplicial subgroup of $\Sigma_{2,3}$, such that $h_{\sigma_1\sigma_2}(F)_0 = \{e\}$.

Remark 3.13 The results in this section also apply to the groups SU and Sp. For SO and O the proofs are not exactly similar, but still true. The arguments used in [7, Theorem 4.1] also apply in our case.

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