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Gorenstein duality for real spectra

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Following Hu and Kriz, we study the C_2–spectra $BP\mathbb{R}\langle n\rangle$ and $E\mathbb{R}(n)$ that refine the usual truncated Brown–Peterson and the Johnson–Wilson spectra. In particular, we show that they satisfy Gorenstein duality with a representation grading shift and identify their Anderson duals. We also compute the associated local cohomology spectral sequence in the cases $n = 1$ and 2.

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1 Introduction

1A Background

Philosophy For us, *real spectrum* is a loose term for a C_2–spectrum built from the C_2–spectrum $M\mathbb{R}$ of real bordism, considered by Araki [2], Araki and Murayama [3], Landweber [22], and Hu and Kriz [18]. The present article shows that bringing together real spectra and Gorenstein duality reveals rich and interesting structures.

It is part of our philosophy that theorems about real spectra can often be shown in the same style as theorems for the underlying complex oriented spectra, although the details might be more difficult, and groups needed to be graded over the real representation ring $RO(C_2)$ (indicated by $\star$) rather than over the integers (indicated by $*$). This extends a well known phenomenon: complex orientability of equivariant spectra makes it easy to reduce questions to integer gradings, and we show that even in the absence of complex orientability, good behaviour of coefficients can be seen by grading with representations.

Bordism with reality In studying these spectra, the real regular representation $\rho = \mathbb{R}C_2$ plays a special role. We write σ for the sign representation on $\mathbb{R}$, so $\rho = 1+\sigma$. One of the crucial features of $M\mathbb{R}$ is that it is *strongly even* in the sense of Meier and Hill [27], ie

(1) the restriction functor $\pi^{C_2}_{k\rho} M\mathbb{R} \to \pi_{2k} MU$ is an isomorphism for all $k \in \mathbb{Z}$, and

(2) the groups $\pi^{C_2}_{k\rho-1} M\mathbb{R}$ are zero for all $k \in \mathbb{Z}$.

We now localize at 2, and (with two exceptions) all spectra and abelian groups will henceforth be 2–local. The Quillen idempotent has a C_2–equivariant refinement, and this defines the C_2–spectrum $BP\mathbb{R}$ as a summand of $M\mathbb{R}_{(2)}$. The isomorphism (1) allows us to lift the usual v_i to classes $\bar{v}_i \in \pi^{C_2}_{(2^i-1)\rho} BP\mathbb{R}$. The real spectra we are interested in are quotients of $BP\mathbb{R}$ by sequences of $\bar{v}_i$ and localizations thereof. For example, we can follow [18] and Hu [17] and define

$$BP\mathbb{R}\langle n\rangle = BP\mathbb{R}/(\bar{v}_{n+1}, \bar{v}_{n+2}, \dots) \quad \text{and} \quad E\mathbb{R}(n) = BP\mathbb{R}\langle n\rangle[\bar{v}_n^{-1}].$$

These spectra are still strongly even, as we will show. Apart from the extensive literature on K-theory with reality (eg Atiyah [4], Dugger [8] and Bruner and Greenlees [7]), real spectra have been studied by Hu and Kriz, in a series of papers by Kitchloo and Wilson (see eg [21] for one of the latest instalments), by Banerjee [5], by Ricka [28] and by Lorman [24]. A crucial point is that a morphism between strongly even C_2–spectra is an equivalence if it is an equivalence of underlying spectra [27, Lemma 3.4].

We are interested in two dualities for real spectra: Anderson duality and Gorenstein duality. These are closely related (see Greenlees and Stojanoska [13]) but apply to different classes of spectra.

Anderson duality The Anderson dual $\mathbb{Z}^X$ of a spectrum X is an integral version of its Brown–Comenetz dual (in accordance with our general principle, $\mathbb{Z}$ denotes the 2–local integers). The homotopy groups of the Anderson dual lie in a short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\pi_{-*-1}X, \mathbb{Z}) \to \pi_*(\mathbb{Z}^X) \to \mathrm{Hom}_{\mathbb{Z}}(\pi_{-*}X, \mathbb{Z}) \to 0. \tag{1.1}$$

One reason to be interested in the computation of Anderson duals is that they show up in universal coefficient sequences; see Anderson [1] or Section 3B. The situation is nicest for spectra that are Anderson self-dual in the sense that $\mathbb{Z}^X$ is a suspension of X. Many important spectra like KU, KO, Tmf (see Stojanoska [31]) or $Tmf_1(3)$ are indeed Anderson self-dual. These examples are all unbounded as the sequence (1.1) nearly forces them to be.

Anderson duality also works C_2–equivariantly as first explored in [28]; the only change in the above short exact sequence is that equivariant homotopy groups are used. The C_2–spectra $K\mathbb{R}$ (see Heard and Stojanoska [14]) and $Tmf_1(3)$ [27] are also C_2–equivariantly Anderson self-dual, at least if we allow suspensions by *representation spheres*.

One simpler example is essential background: if $\underline{\mathbb{Z}}$ denotes the constant Mackey functor (ie with restriction being the identity and induction being multiplication by 2) then the Anderson dual of its Eilenberg–Mac Lane spectrum is the Eilenberg–Mac Lane spectrum for the dual Mackey functor $\underline{\mathbb{Z}}^* = \mathrm{Hom}_{\mathbb{Z}}(\underline{\mathbb{Z}}, \mathbb{Z})$ (ie with restriction being multiplication by 2 and induction being the identity). It is then easy to check that in fact $H(\underline{\mathbb{Z}}^*) \simeq \Sigma^{2(1-\sigma)} H\underline{\mathbb{Z}}$. (From one point of view this is the fact that $\mathbb{R}P^1 = S(2\sigma)/C_2$ is equivalent to the circle). The dualities we find are in a sense all dependent on this one.

Gorenstein duality By contrast with Anderson self-duality, many connective ring spectra are Gorenstein in the sense of Dwyer, Greenlees and Iyengar [9]. We sketch the theory here, explaining it more fully in Sections 6 and 7.

The starting point is a connective commutative ring C_2–spectrum R, whose 0^{th} homotopy Mackey functor is constant at $\mathbb{Z}$:

$$\underline{\pi}_0^{C_2}(R) \cong \underline{\mathbb{Z}}.$$

This gives us a map $R \to H\underline{\mathbb{Z}}$ of commutative ring spectra by killing homotopy groups. We say that R is *Gorenstein* of shift $a \in RO(C_2)$ if there is an equivalence of R–modules

$$\mathrm{Hom}_R(H\underline{\mathbb{Z}}, R) \simeq \Sigma^a H\underline{\mathbb{Z}}.$$

We are interested in the duality this often entails. Note that the Anderson dual $\mathbb{Z}^R$ obviously has the Matlis lifting property

$$\mathrm{Hom}_R(H\underline{\mathbb{Z}}, \mathbb{Z}^R) \simeq H\underline{\mathbb{Z}}^*,$$

where $\mathbb{Z}^* = \mathrm{Hom}_{\mathbb{Z}}(\underline{\mathbb{Z}}, \mathbb{Z})$ as above. Thus if R is Gorenstein, in view of the equivalence $H(\underline{\mathbb{Z}}^*) \simeq \Sigma^{2(1-\sigma)} H\underline{\mathbb{Z}}$, we have equivalences

$$\begin{aligned}\mathrm{Hom}_R(H\underline{\mathbb{Z}}, \mathrm{Cell}_{H\underline{\mathbb{Z}}} R) &\simeq \mathrm{Hom}_R(H\underline{\mathbb{Z}}, R)\\ &\simeq \Sigma^a H\underline{\mathbb{Z}}\\ &\simeq \Sigma^{a-2(1-\sigma)} H(\underline{\mathbb{Z}}^*)\\ &\simeq \mathrm{Hom}_R(H\underline{\mathbb{Z}}, \Sigma^{a-2(1-\sigma)} \mathbb{Z}^R).\end{aligned}$$

Here, $\mathrm{Cell}_{H\underline{\mathbb{Z}}}$ denotes the $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellularization as in Section 2B. We would like to remove the $\mathrm{Hom}_R(H\underline{\mathbb{Z}}, \cdot)$ from the composite equivalence above.

Definition 1.2 We say that R has *Gorenstein duality* of shift b if we have an equivalence of R–modules

$$\mathrm{Cell}_{H\underline{\mathbb{Z}}} R \simeq \Sigma^b \mathbb{Z}^R.$$

As in the nonequivariant setting, the passage from Gorenstein to Gorenstein duality requires showing that the above composite equivalence is compatible with the right action of $\mathcal{E} = \mathrm{Hom}_R(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}})$. This turns out to be considerably more delicate than the nonequivariant counterpart because connectivity is harder to control; but if one can lift the R–equivalence to an $\mathcal{E}$–equivalence, the conclusion is that if R is Gorenstein of shift a, then it has Gorenstein duality of shift $b = a - 2(1-\sigma)$.

Local cohomology The duality statement becomes more interesting when the cellularization can be constructed algebraically. For any finitely generated ideal J of the $RO(C_2)$–graded coefficient ring $R^{C_2}_\star$, we may form the stable Koszul complex $\Gamma_J R$, which only depends on the radical of J. In our examples, this applies to the augmentation ideal $J = \ker(R^{C_2}_\star \to H\underline{\mathbb{Z}}^{C_2}_\star)$, which may be radically generated by finitely many elements $\bar{v}_i$ in degrees which are multiples of ρ. Adapting the usual proof to the real context, Proposition 3.8 shows that $\Gamma_J R \to R$ is an $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellularization:

$$\mathrm{Cell}_{H\underline{\mathbb{Z}}} R \simeq \Gamma_J R.$$

The $RO(C_2)$–graded homotopy groups of $\Gamma_J R$ can be computed using a spectral sequence involving local cohomology.

Conclusion In favourable cases, the Gorenstein condition on a ring spectrum R implies Gorenstein duality for R; this in turn establishes a strong duality property on the $RO(C_2)$–graded coefficient ring, expressed using local cohomology.

1B Results

Our main theorems establish Gorenstein duality for a large family of real spectra. We present in this introduction the particular cases of $BP\mathbb{R}\langle n\rangle$ and $E\mathbb{R}(n)$, deferring the more general theorem to Section 5. Let again σ denote the nontrivial representation of C_2 on the real line and $\rho = 1+\sigma$ the real regular representation. Furthermore, set $D_n = 2^{n+1}-n-2$ so that $D_n\rho = |\bar{v}_1|+\cdots+|\bar{v}_n|$. Other terms in the statement will be explained in Section 3.

Theorem 1.3 *For each $n\geq 1$, the C_2–spectrum $BP\mathbb{R}\langle n\rangle$ is Gorenstein of shift $-D_n\rho-n$, and has Gorenstein duality of shift $-D_n\rho-n-2(1-\sigma)$. This means that*

$$\mathbb{Z}_{(2)}^{BP\mathbb{R}\langle n\rangle} \simeq \Sigma^{D_n\rho+n+2(1-\sigma)}\Gamma_{\bar{J}_n} BP\mathbb{R}\langle n\rangle,$$

where $\bar{J}_n = (\bar{v}_1,\dots,\bar{v}_n)$. This induces a local cohomology spectral sequence

$$H^*_{\bar{J}_n}(BP\mathbb{R}\langle n\rangle^{C_2}_\star) \implies \pi^{C_2}_\star(\Sigma^{-D_n\rho-n-2(1-\sigma)}\mathbb{Z}_{(2)}^{BP\mathbb{R}\langle n\rangle}).$$

Theorem 1.4 *For each $n\geq 1$, the C_2–spectrum $E\mathbb{R}(n)$ has Gorenstein duality of shift $-D_n\rho-(n-1)-2(1-\sigma)$. This means that*

$$\begin{aligned}\mathbb{Z}_{(2)}^{E\mathbb{R}(n)} &\simeq \Sigma^{D_n\rho+(n-1)+2(1-\sigma)}\Gamma_{\bar{J}_{n-1}} E\mathbb{R}(n)\\ &\simeq \Sigma^{(n+2)(2^{2n+1}-2^{n+2})+n+3}\Gamma_{J_{n-1}} E\mathbb{R}(n)\end{aligned}$$

for $J_{n-1} = \bar{J}_{n-1}\cap \pi^{C_2}_* E\mathbb{R}(n)$. This induces likewise a local cohomology spectral sequence.

We note that this has implications for the C_2–fixed point spectrum $(BP\mathbb{R}\langle n\rangle)^{C_2} = BPR\langle n\rangle$. The graded ring

$$\pi_*(BPR\langle n\rangle) = \pi^{C_2}_*(BP\mathbb{R}\langle n\rangle)$$

is the integer part of the $RO(C_2)$–graded coefficient ring $\pi^{C_2}_\star(BP\mathbb{R}\langle n\rangle)$. However, since the ideal $\bar{J}_n$ is not generated in integer degrees, the statement for $BPR\langle n\rangle$ is usually rather complicated, and one of our main messages is that working with the equivariant spectra gives more insight. On the other hand, $ER(n) = E\mathbb{R}(n)^{C_2}$ has integral Gorenstein duality because one can use the additional periodicity to move the representation suspension and the ideal $\bar{J}_n$ to integral degrees.

We will discuss the general result in more detail later, but the two first cases are about familiar ring spectra.

Example 1.5 (see Sections 6 and 11) For $n=1$, connective K-theory with reality $k\mathbb{R}$ is 2–locally a form of $BP\mathbb{R}\langle 1\rangle$. For this example, we can work without 2–localization, so that $\mathbb{Z}$ means the integers. Our first theorem states that $k\mathbb{R}$ is Gorenstein of shift $-\rho-1=-2-\sigma$ and that it has Gorenstein duality of shift $-4+\sigma$. This just means that

$$\mathbb{Z}^{k\mathbb{R}} \simeq \Sigma^{4-\sigma}\operatorname{fib}(k\mathbb{R}\to K\mathbb{R}).$$

The local cohomology spectral sequence collapses to a short exact sequence associated to the fibre sequence just mentioned. We will see in Section 11 that the sequence is not split, even as abelian groups.

Theorem 1.4 recovers the main result of [14], ie that $\mathbb{Z}^{K\mathbb{R}} \simeq \Sigma^4 K\mathbb{R}$, which also implies $\mathbb{Z}^{KO}\simeq \Sigma^4 KO$. It is a special feature of the case $n=1$ that we also get a nice duality statement for the fixed points in the connective case. Indeed, by considering the $RO(C_2)$–graded homotopy groups of $k\mathbb{R}$, one sees [7, Corollary 3.4.2] that

$$(k\mathbb{R}\otimes S^{-\sigma})^{C_2}\simeq \Sigma^1 ko.$$

This implies that connective ko has untwisted Gorenstein duality of shift -5, ie that

$$\mathbb{Z}^{ko}\simeq \Sigma^5\operatorname{fib}(ko\to KO).$$

This admits a closely related nonequivariant proof, combining the fact that ku is Gorenstein (clear from coefficients) and the fact that complexification $ko\to ku$ is relatively Gorenstein (connective version of Wood's theorem [7, Lemma 4.1.2]).

Example 1.6 (see Examples 4.13 and 5.12 or Lemma 7.1 and Corollary 7.5) The 2–localization of the C_2–spectrum $tmf_1(3)$ is a form of $BP\mathbb{R}\langle 2\rangle$, and the theorem is closely related to results in [27]. It states that $tmf_1(3)$ is Gorenstein of shift $-4\rho-2=-6-4\sigma$ and has Gorenstein duality of shift $-8-2\sigma$. We show in Section 13 that there are nontrivial differentials in the local cohomology spectral sequence.

Passing to fixed points, we obtain the 2–local equivalence

$$BPR\langle 2\rangle = (BP\mathbb{R}\langle 2\rangle)^{C_2} = tmf_0(3).$$

By contrast with the $n=1$ case, as observed in [27], $tmf_0(3)$ does not have untwisted Gorenstein duality of any integer degree.

A variant of Theorem 1.4 also computes the C_2–equivariant Anderson dual of $TMF_1(3)$, and the computation of the Anderson dual of $Tmf_1(3)$ from [27] follows as well.

The results apply to $tmf_1(3)$ and $TMF_1(3)$ themselves (ie with just 3 inverted, and not all other odd primes).

Our main theorem also recovers the main result of [28] about the Anderson self-duality of integral real Morava K-theory.

1C Guide to the reader

While the basic structure of this paper is easily visible from the table of contents, we want to comment on a few features.

The precise statements of our main results can be found in Section 5. We will give two different proofs of them. One (Part III) might be called "the hands-on approach" which is elementary and explicit, and one (Part II) uses Gorenstein techniques inspired by commutative algebra. The intricacy and dependence on specific calculations in the explicit approach make the conceptual approach valuable. The subtlety of the structural requirements of the conceptual approach make the reassurance of the explicit approach valuable. The results from the latter approach are also a bit more general: In Part III, we prove a version of Gorenstein duality for a quite general class of quotients of $BP\mathbb{R}$, but we treat only $BP\mathbb{R}\langle n\rangle$ itself in Part II.

While the Gorenstein approach only relies on the knowledge of the homotopy groups of $H\underline{\mathbb{Z}}$ and the reduction theorem Corollary 4.7, we need detailed information about the homotopy groups of quotients of $BP\mathbb{R}$ for the hands-on approach. In the appendix, we give a streamlined version of the computation of $\pi_\star^{C_2} BP\mathbb{R}$ (which appeared first in [18]). In Section 4, we give a rather self-contained account of the homotopy groups of $BP\mathbb{R}\langle n\rangle$ and of other quotients of $BP\mathbb{R}$, which can also be read independently of the rest of the paper. While some of this is rather technical, most of the time we just have to use Corollary 4.6 whose statement (though not proof, perhaps) is easy to understand.

We give separate arguments for the computation of the Anderson dual of $k\mathbb{R}$ so that this easier case might illustrate the more complicated arguments of our more general theorems. Thus, if the reader is only interested in $k\mathbb{R}$, he or she might ignore most of this paper. More precisely, under this assumption one might proceed as follows: First one looks at Section 11B for a quick reminder on $\pi_\star^{C_2} k\mathbb{R}$, then one skims through Sections 2 and 3 to pick up the relevant definitions, and then one proceeds directly to Section 6 or Section 8 to get the proof of the main result in the case of $k\mathbb{R}$. Afterwards, one may look at the pictures and computations in the rest of Section 11 to see what happens behind the scenes of Gorenstein duality.

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Part I Preliminaries and results

2 Basics and conventions about C_2–spectra

2A Basics and conventions

We will work in the homotopy category of genuine G–spectra (ie stable for suspensions by S^V for any finite dimensional representation V) for $G = C_2$, the group of order 2. We will denote by $\otimes$ the derived smash product of spectra.

We may combine the equivariant and nonequivariant homotopy groups of a C_2–spectrum into a Mackey functor, which we denote by $\underline{\pi}_*^{C_2} X$ and denote C_2–equivariant and underlying homotopy groups correspondingly by $\pi_*^{C_2} X$ and $\pi_*^e X$. For an abelian group A, we write $\underline{A}$ for the constant Mackey functor (ie restriction maps are the identity), and $\underline{A}^*$ for its dual (ie induction maps are the identity). We write HM for the Eilenberg–Mac Lane spectrum associated to a Mackey functor M.

Another C_2–spectrum of interest to us is $k\mathbb{R}$, the C_2–equivariant connective cover of Atiyah's K-theory with reality [4]. The term "real spectra" derives from this example. The examples of real bordism and the other C_2–spectra derived from it will be discussed in Section 4.

We will usually grade our homotopy groups by the real representation ring $RO(C_2)$, and we write $M_\star$ for $RO(C_2)$–graded groups. In addition to the real sign representation σ and the regular representation ρ, the virtual representation $\delta = 1 - \sigma$ is also significant. Examples of $RO(C_2)$–graded homotopy classes are the geometric Euler classes $a_V \colon S^0 \to S^V$; in particular, $a = a_\sigma$ will play a central role. In addition to a, we will also often have a class $u = u_{2\sigma}$ of degree 2δ.

We often want to be able to discuss gradings by certain subsets of $RO(C_2)$. To start with, we often want to refer to gradings by multiples of the regular representation (where we write $M_{*\rho}$), but we also need to discuss gradings of the form $k\rho - 1$. For this, we use the notation

$$*\rho- = \{k\rho \mid k \in \mathbb{Z}\} \cup \{k\rho - 1 \mid k \in \mathbb{Z}\}.$$

Following [27], we call an $RO(C_2)$–graded object M *even* if $M_{k\rho-1} = 0$ for all k. An $RO(C_2)$–graded Mackey functor is *strongly even* if it is even and all the Mackey functors in gradings $k\rho$ are constant. We call a C_2–spectrum (strongly) even if its homotopy groups are (strongly) even.

If X is a strongly even C_2–spectrum and $x \in \pi_{2k} X$, we denote by $\overline{x}$ its counterpart in $\pi_{k\rho}^{C_2} X$. If we want to stress that we consider a certain spectrum as a C_2–spectrum, we will also sometimes indicate this by a bar; for example, we may write $\overline{tmf_1(3)}$ if we want to stress the C_2–structure of $tmf_1(3)$.

2B Cellularity

In a general triangulated category, it is conventional to say M is K*–cellular* if M is in the localizing subcategory generated by K (or equivalently by all integer suspensions of K). A reference in the case of spectra is [9, Section 4.1]. We say that a C_2–spectrum M is K-$\mathbb{R}$*–cellular* (for a C_2–spectrum K) if it is in the localizing subcategory generated by the suspensions $S^{k\rho} \otimes K$ for all integers k. We note that this is the same as the localizing subcategory generated by integer suspensions of K and $(C_2)_+ \otimes K$ because of the cofibre sequence

$$(C_2)_+ \to S^0 \to S^\sigma.$$

We say that a map $N \to M$ is a K-$\mathbb{R}$*–cellularization* if N is K-$\mathbb{R}$–cellular and the induced map

$$\mathrm{Hom}(K, N) \to \mathrm{Hom}(K, M)$$

is an equivalence of C_2–spectra. The K-$\mathbb{R}$–cellularization is clearly unique up to equivalence.

We note that cellularity and $\mathbb{R}$–cellularity are definitely different. For example, $(C_2)_+$ is not S^0–cellular, but it is S^0-$\mathbb{R}$–cellular.

In this article, we will only use $\mathbb{R}$–cellularity.

2C The slice filtration

Recall from [16, Section 4.1] or [15] that the *slice cells* are the C_2–spectra of the form

- $S^{k\rho}$ of dimension $2k$,
- $S^{k\rho-1}$ of dimension $2k-1$, and
- $S^k \otimes (C_2)_+$ of dimension k.

A C_2–spectrum X is $\leq k$ if for every slice cell W of dimension $\geq k+1$, the mapping space $\Omega^\infty \mathrm{Hom}_{\mathbb{S}}(W, X)$ is equivariantly contractible. As explained in [16, Section 4.2], this leads to the definition of $X \to P^k X$, which is the universal map into a C_2–spectrum that is $\leq k$. The fibre of

$$X \to P^k X$$

is denoted by $P_{k+1}X$. The *k-slice* $P^k_k X$ is defined as the fibre of

$$P^k X \to P^{k-1} X,$$

or equivalently, as the cofibre of the map $P_{k+1}X \to P_k X$. We have the following two useful propositions:

Proposition 2.1 [15, Corollary 2.12, Theorem 2.18] *If X is an even C_2–spectrum, then $P^{2k-1}_{2k-1} X = 0$ for all $k \in \mathbb{Z}$.*

Proposition 2.2 [15, Corollary 2.16, Theorem 2.18] *If X is a C_2–spectrum such that the restriction map in $\underline{\pi}^{C_2}_{k\rho}$ is injective, then $P^{2k}_{2k} X$ is equivalent to the Eilenberg–Mac Lane spectrum $\underline{\pi}^{C_2}_{k\rho} X$.*

This allows us to give a characterization of an Eilenberg–Mac Lane spectrum based on regular representation degrees.

Corollary 2.3 *Any even C_2–spectrum X with*

$$\underline{\pi}^{C_2}_{k\rho}(X) = \begin{cases} \underline{A} & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for an abelian group A, is equivalent to $H\underline{A}$.

Proof By the last two propositions, we have

$$P^k_k X \simeq \begin{cases} H\underline{A} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By convergence of the slice spectral sequence [16, Theorem 4.42], the result follows. □

3 Anderson duality, Koszul complexes and Gorenstein duality

3A Duality for abelian groups

It is convenient to establish some conventions for abelian groups to start with, so as to fix notation.

Pontrjagin duality is defined for all graded abelian groups A by

$$A^\vee = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}).$$

Similarly, the rational dual is defined by

$$A^{\vee\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}).$$

Since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective abelian groups these two dualities are homotopy invariant and pass to the derived category. Finally the Anderson dual A^* is defined by applying $\mathrm{Hom}_{\mathbb{Z}}(A, \cdot)$ to the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

so we have a triangle

$$A^* \to A^{\vee\mathbb{Q}} \to A^{\vee}.$$

If M is a free abelian group, then the Anderson dual is simply calculated by

$$M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

(since M is free, the Hom need not be derived).

If M is a graded abelian group which is an $\mathbb{F}_2$–vector space then up to suspension the Anderson dual is the vector space dual:

$$M^{\vee} = \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_2) \simeq \Sigma^{-1} M^*$$

(since vector spaces are free, Hom need not be derived).

3B Anderson duality

Anderson duality is the attempt to topologically realize the algebraic duality from the last subsection. It appears that it was invented by Anderson (only published in mimeographed notes [1]) and Kainen [19], with similar ideas by Brown and Comenetz [6]. For brevity and consistency, we will only use the term Anderson duality instead of Anderson–Kainen duality or Anderson–Brown–Comenetz duality throughout. We will work in the category of C_2–spectra, where Anderson duality was first explored by Ricka in [28].

Let I be an injective abelian group. Then we let $I^{\mathbb{S}}$ denote the C_2–spectrum representing the functor

$$X \mapsto \mathrm{Hom}(\pi_*^{C_2} X, I).$$

For an arbitrary C_2–spectrum, we define I^X as the function spectrum $F(X, I^{\mathbb{S}})$. For a general abelian group A, we choose an injective resolution

$$A \to I \to J$$

and define A^X as the fibre of the map $I^X \to J^X$. For example, we get a fibre sequence

$$\mathbb{Z}^X \to \mathbb{Q}^X \to (\mathbb{Q}/\mathbb{Z})^X.$$

In general, we get a short exact sequence of homotopy groups

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\pi^{C_2}_{-k-1}(X), A) \to \pi^{C_2}_k(A^X) \to \mathrm{Hom}(\pi^{C_2}_{-k}(X), A) \to 0.$$

The analogous exact sequence is true for $RO(C_2)$–graded Mackey functor valued homotopy groups by replacing X by $(C_2/H)_+ \wedge \Sigma^V X$. Our most common choices will be $A = \mathbb{Z}$ and $A = \mathbb{Z}_{(2)}$.

From time to time we use the following property of Anderson duality: If R is a strictly commutative C_2–ring spectrum and M an R–module, then $\mathrm{Hom}_R(M, A^R) \simeq A^M$ as R–modules as can easily be seen by adjunction.

One of the reasons to consider Anderson duality is that it provides universal coefficient sequences. In the C_2–equivariant world, this takes the following form [28, Proposition 3.11]:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(E^{C_2}_{\alpha-1}(X), A) \to (A^E)^{C_2}_\alpha(X) \to \mathrm{Hom}_{\mathbb{Z}}(E^{C_2}_\alpha(X), A) \to 0,$$

where E and X are C_2–spectra, $\alpha \in RO(C_2)$ and A is an abelian group.

Our first computation is the Anderson dual of the Eilenberg–Mac Lane spectrum of the constant Mackey functor $\underline{\mathbb{Z}}$.

Lemma 3.1 *The Anderson dual of the Eilenberg–Mac Lane spectrum $H\underline{\mathbb{Z}}$ (as an $\mathbb{S}$–module) is given by the following, where $\delta = 1 - \sigma$:*

$$\mathbb{Z}^{H\underline{\mathbb{Z}}} \simeq H\underline{\mathbb{Z}}^* \simeq \Sigma^{2\delta} H\underline{\mathbb{Z}}.$$

Proof The first equivalence follows from the isomorphisms

$$\underline{\pi}^{C_2}_*(\mathbb{Z}^{H\underline{\mathbb{Z}}}) \cong \mathrm{Hom}_{\mathbb{Z}}(\underline{\pi}^{C_2}_{-*} H\underline{\mathbb{Z}}, \mathbb{Z}) \cong \underline{\mathbb{Z}}^*.$$

Since

$$\pi^{C_2}_*(S^{2-2\sigma} \otimes H\underline{\mathbb{Z}}) = H^*_{C_2}(S^{2\sigma-2}; \underline{\mathbb{Z}}) = H^*(S^{2\sigma-2}/C_2; \mathbb{Z}),$$

and $S^{2\sigma} = S^0 * S(2\sigma)$ is the unreduced suspension of $S(2\sigma)$, the second equivalence is a calculation of the cohomology of $\mathbb{R}P^1$. □

Remark 3.2 This proof shows that if C_2 is replaced by a cyclic group of any order, we still have

$$\mathbb{Z}^{H\underline{\mathbb{Z}}} = H\underline{\mathbb{Z}}^* \simeq \Sigma^\lambda H\underline{\mathbb{Z}},$$

where $\lambda = \epsilon - \alpha$ (with ϵ the trivial one dimensional complex representation and α a faithful one dimensional representation).

Anderson duality works, of course, also for nonequivariant spectra. We learnt the following proposition comparing the equivariant and nonequivariant version in a conversation with Nicolas Ricka.

Proposition 3.3 *Let A be an abelian group. We have $(A^X)^{C_2} \simeq A^{(X^{C_2})}$ for every C_2–spectrum X.*

Proof Let $\inf_e^{C_2} Y$ denote the inflation of a spectrum Y to a C_2–spectrum with "trivial action", ie the left derived functor of first regarding it as a naive C_2–spectrum with trivial action and then changing the universe. This is the (derived) left adjoint for the fixed point functor [25, Proposition 3.4].

Let I be an injective abelian group. Then there is for every spectrum Y a natural isomorphism

$$\begin{aligned}
[Y, (I^X)^{C_2}] &\cong [\inf_e^{C_2} Y, I^X]^{C_2} \\
&\cong \mathrm{Hom}(\pi_0^{C_2}(\inf_e^{C_2} Y \otimes X), I) \\
&\cong \mathrm{Hom}(\pi_0(Y \otimes X^{C_2}), I) \\
&\cong [Y, I^{(X^{C_2})}].
\end{aligned}$$

Here, we use that fixed points commute with filtered homotopy colimits and cofibre sequences and therefore also with smashing with a spectrum with trivial action. Thus, there is a canonical isomorphism in the homotopy category of spectra between $I^{(X^{C_2})}$ and $(I^X)^{C_2}$ that is also functorial in I (by Yoneda). For a general abelian group A, we can write $A^{(X^{C_2})}$ as the fibre of $(I^0)^{X^{C_2}} \to (I^1)^{X^{C_2}}$ (and similarly for the other side) for an injective resolution $0 \to A \to I^0 \to I^1$. Thus, we obtain a (possibly noncanonical) equivalence between $A^{(X^{C_2})}$ and $(A^X)^{C_2}$. □

Remark 3.4 An analogous result holds, of course, for every finite group G.

3C Koszul complexes and derived power torsion

Let R be a nonequivariantly E_∞ C_2–ring spectrum and M an R–module. In this section, we will recall two versions of stable Koszul complexes. Among their merits is that they provide models for cellularization or $\mathbb{R}$–cellularization in cases of interest for us. A basic reference for the material in this section is [11].

As classically, the r–power torsion in a module N can be defined as the kernel of $N \to N[1/r]$, we define the *derived J–power torsion* of M with respect to an ideal

$J = (x_1, \dots, x_n) \subseteq \pi_\star^{C_2}(R)$ as

$$\Gamma_J M = \mathrm{fib}\Big(R \to R\Big[\frac{1}{x_1}\Big]\Big) \otimes_R \cdots \otimes_R \mathrm{fib}\Big(R \to R\Big[\frac{1}{x_n}\Big]\Big) \otimes_R M.$$

This is also sometimes called the *stable Koszul complex*, also denoted by $K(x_1, \dots, x_n)$. As shown in [11, Section 3], this only depends on the ideal J and not on the chosen generators. As algebraically, the derived functors of J–power torsion are the local cohomology groups, we might expect a spectral sequence computing the homotopy groups of $\Gamma_J M$ in terms of local cohomology. As in [11, Section 3], this takes the form

$$H_J^s(\pi_{\star+V}^{C_2} M) \implies \pi_{V-s}^{C_2}(\Gamma_J M). \tag{3.5}$$

Our second version of the Koszul complex can be defined in the one-generator case as

$$\kappa_R(x) = \underrightarrow{\mathrm{holim}}\, \Sigma^{(1-l)|x|} R/x^l$$

for $x \in \pi_\star^{C_2}(R)$. Here, the map $R/x^l \to \Sigma^{-|x|} R/x^{l+1}$ is induced by the diagram of cofibre sequences:

$$\begin{tikzcd}
\Sigma^{|x^l|} R \arrow[r, "x^l"] \arrow[d, "="] & R \arrow[r] \arrow[d, "x"] & R/x^l \arrow[d, dashed] \\
\Sigma^{|x^l|} R \arrow[r, "x^{l+1}"] & \Sigma^{-|x|} R \arrow[r] & \Sigma^{-|x|} R/x^{l+1}
\end{tikzcd}$$

More generally, we can make, for a sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ in $\pi_\star^{C_2}(R)$, the definition

$$\begin{aligned}
\kappa_R(\boldsymbol{x}; M) &:= \kappa_R(x_1) \otimes_R \cdots \otimes_R \kappa_R(x_n) \otimes_R M \\
&\simeq \underrightarrow{\mathrm{holim}}\, \Sigma^{-((l_1-1)+\cdots+(l_n-1))|x|} M/(x_1^{l_1}, \dots, x_n^{l_n}).
\end{aligned}$$

The spectrum $\kappa_R(x)$ comes with an obvious filtration by $\Sigma^{(1-l)|x|} R/x^l$ with filtration quotients $\Sigma^{-l|x|} R/x$. We can smash these filtrations together to obtain a filtration of $\kappa_R(\boldsymbol{x})$ with filtration quotients wedges of summands of the form

$$\Sigma^{-l_1|x_1|-\cdots-l_n|x_n|} R/(x_1, \dots, x_n);$$

see [32, Definition 1.3.11, Proposition 12] or [33, Remark 2.8, Lemma 2.12]. Using the following lemma, we obtain also a corresponding filtration on $\Gamma_J R$.

Lemma 3.6 *For $\boldsymbol{x}$ as above, we have*

$$\kappa_R(\boldsymbol{x}) \simeq \Sigma^{|x_1|+\cdots+|x_n|+n} \Gamma_J R.$$

Proof See [11, Lemma 3.6]. □

We can also define $\kappa_R(\boldsymbol{x}; M)$ (and likewise the other versions of Koszul complexes) for an infinite sequence of x_i by just taking the filtered homotopy colimit over all finite subsequences. Usually Lemma 3.6 breaks down in the infinite case.

Remark 3.7 The homotopy colimit defining $\kappa_R(\boldsymbol{x}; M)$ has a directed cofinal subsystem, both in the finite and in the infinite case. Indeed, the colimit ranges over all sequences $(l_1, l_2, \dots)$ with only finitely many entries nonzero. For the directed subsystem, we start with $(0, 0, \dots)$ and raise in the n^{th} step the first n entries by 1. Directed homotopy colimit are well known to be weak colimits in the homotopy category of R–modules, ie every system of compatible maps induces a (possibly nonunique) map from the homotopy colimit [26, Section 3.1; 29, Section II.5].

One of the reasons for introducing $\Gamma_J M$ is that it provides a model for the $\mathbb{R}$–cellularization of M with respect to $R/J = (R/x_1) \otimes_R \cdots \otimes_R (R/x_n)$ in the sense of Section 2B.

Proposition 3.8 *Suppose that $x_1, \dots, x_n \in \pi_{*\rho}^{C_2} R$, and set $J = (x_1, \dots, x_n)$. Then $\Gamma_J M \to M$ is an $\mathbb{R}$–cellularization with respect to R/J in the (triangulated) category of R–modules.*

Proof Clearly, $\kappa_R(x_1, \dots, x_n; M)$ is $\mathbb{R}$–cellular with respect to M/J; furthermore M/J is R/J-$\mathbb{R}$–cellular as clearly M is R–cellular. To finish the proof, we have to show that

$$\mathrm{Hom}_R(R/J, \Gamma_J M) \to \mathrm{Hom}_R(R/J, M)$$

is an equivalence. Note that $\Gamma_J M = \Gamma_{x_n}(\Gamma_{(x_1,\dots,x_{n-1})} M)$. Thus, it suffices by induction to show that

$$\mathrm{Hom}_R(A/x, \Gamma_x B) \to \mathrm{Hom}_R(A/x, B)$$

is an equivalence for all R–modules A, B. This is equivalent to

$$\mathrm{Hom}_R(A/x, B[x^{-1}]) = 0,$$

which is true as multiplication by x induces an equivalence

$$\mathrm{Hom}_R(A, B[x^{-1}]) \xrightarrow{x^*} \mathrm{Hom}_R(\Sigma^{|x|} A, B[x^{-1}]). \qquad \square$$

Corollary 3.9 *Let M be a connective R–module and A an abelian group. Then the Anderson dual A^M is $\mathbb{R}$–cellular with respect to R/J for every ideal $J \subset \pi_\star^{C_2}$ finitely generated in degrees $a + b\sigma$ with $a \geq 1$ and $a + b \geq 1$.*

Proof By the last proposition, we have to show that $\Gamma_J A^M \simeq A^M$. For this, it suffices to show that $A^M[x^{-1}]$ is contractible for every generator x of J. As M is connective, we know that $\pi_{a+b\sigma} M = 0$ if $a < 0$ and $a+b < 0$ (this follows, for example, using the cofibre sequence $(C_2)_+ \to S^0 \to S^\sigma$). Thus, $\pi_{a+b\sigma} A^M = 0$ if $a > 0$ and $a+b > 0$. The result follows. $\square$

4 Real bordism and the spectra $BP\mathbb{R}\langle n\rangle$

4A Basics and homotopy fixed points

The C_2–spectrum $M\mathbb{R}$ of *real bordism* was originally defined by Araki and Landweber. In the naive model of C_2–spectra, where a C_2–spectrum is just given as a sequence (X_n) of pointed C_2–spaces together with maps

$$\Sigma^\rho X_n \to X_{n+1},$$

it is just given by the Thom spaces $M\mathbb{R}_n = BU(n)^{\gamma_n}$ with complex conjugation as C_2–action. Defining it as a strictly commutative C_2–orthogonal spectrum requires more care and was done in [30, Example 2.14] and [16, Section B.12]. An important fact is that the geometric fixed points of $M\mathbb{R}$ are equivalent to MO (first proven in [3] and reproven in [16, Proposition B.253]).

As shown in [2] and [18, Theorem 2.33], there is a splitting

$$M\mathbb{R}_{(2)} \simeq \bigoplus_i \Sigma^{m_i\rho} BP\mathbb{R},$$

where the underlying spectrum of $BP\mathbb{R}$ agrees with BP. This splitting corresponds on geometric fixed points to the splitting

$$MO \simeq \bigoplus_i \Sigma^{m_i} H\mathbb{F}_2.$$

As shown in [18] (see also the appendix), the restriction map

$$\pi^{C_2}_{*\rho} BP\mathbb{R} \to \pi_{2*} BP$$

is an isomorphism. Choose now arbitrary indecomposables $v_i \in \pi_{2(2^i-1)} BP$ and denote their lifts to $\pi^{C_2}_{(2^i-1)\rho} BP\mathbb{R}$ and their images in $\pi^{C_2}_{(2^i-1)\rho} M\mathbb{R}$ by $\bar{v}_i$. We denote by $BP\mathbb{R}\langle n\rangle$ the quotient

$$BP\mathbb{R}/(\bar{v}_{n+1}, \bar{v}_{n+2}, \dots)$$

in the homotopy category of $M\mathbb{R}$–modules. At least a priori, this depends on the choice of v_i.

We want to understand the homotopy groups of $BP\mathbb{R}\langle n\rangle$. This was first done by Hu in [17] (beware though that Theorem 2.2 is not correct as stated there) and partially redone in [20]. For the convenience of the reader, we will give the computation again. Note that our proofs are similar but not identical to the ones in the literature. The main difference is that we do not use ascending induction and prior knowledge of $H\mathbb{Z}$ to compute $\Phi^{C_2}BP\mathbb{R}\langle n\rangle$, but precise knowledge about $\pi_\star^{C_2}BP\mathbb{R}$; this is not simpler than the original approach, but gives extra information about other quotients of $BP\mathbb{R}$, which we will need later. We recommend that the reader looks at the appendix for a complete understanding of the results that follow.

We will use the Tate square [12] and consider the following diagram in which the rows are cofibre sequences:

$$\begin{tikzcd} BP\mathbb{R}\langle n\rangle\otimes(EC_2)_+ \arrow[r] \arrow[d, "\simeq"] & BP\mathbb{R}\langle n\rangle \arrow[r] \arrow[d] & BP\mathbb{R}\langle n\rangle\otimes\widetilde{E}C_2 \arrow[r] \arrow[d] & \Sigma BP\mathbb{R}\langle n\rangle\otimes(EC_2)_+ \arrow[d] \\ BP\mathbb{R}\langle n\rangle^{(EC_2)_+}\otimes(EC_2)_+ \arrow[r] & BP\mathbb{R}\langle n\rangle^{(EC_2)_+} \arrow[r] & BP\mathbb{R}\langle n\rangle^{(EC_2)_+}\otimes\widetilde{E}C_2 \arrow[r] & \Sigma BP\mathbb{R}\langle n\rangle\otimes(EC_2)_+ \end{tikzcd}$$

After taking fixed points, this becomes the diagram:

$$\begin{tikzcd} BP\mathbb{R}\langle n\rangle_{hC_2} \arrow[r] \arrow[d, "="] & BP\mathbb{R}\langle n\rangle^{C_2} \arrow[r] \arrow[d] & BP\mathbb{R}\langle n\rangle^{\Phi C_2} \arrow[r] \arrow[d] & \Sigma BP\mathbb{R}\langle n\rangle_{hC_2} \arrow[d] \\ BP\mathbb{R}\langle n\rangle_{hC_2} \arrow[r] & BP\mathbb{R}\langle n\rangle^{hC_2} \arrow[r] & BP\mathbb{R}\langle n\rangle^{tC_2} \arrow[r] & \Sigma BP\mathbb{R}\langle n\rangle_{hC_2} \end{tikzcd}$$

First, we compute the homotopy groups of the homotopy fixed points. For this, we use the $RO(C_2)$–graded homotopy fixed point spectral sequence, described for example in [27, Section 2.3].

Proposition 4.1 *The $RO(C_2)$–graded homotopy fixed point spectral sequence*

$$E_2 = H^*(C_2;\pi_\star^e BP\mathbb{R}\langle n\rangle) \cong \mathbb{Z}_{(2)}[\bar v_1,\dots,\bar v_n,u^{\pm1},a]/2a \implies \pi_\star^{C_2}(BP\mathbb{R}\langle n\rangle^{(EC_2)_+})$$

has differentials generated by $d_{2^{i+1}-1}(u^{2^{i-1}}) = a^{2^{i+1}-1}\bar v_i$ for $i\le n$ and $E_{2^{n+1}} = E_\infty$.

Proof The description of $E_{2^{n+1}}$ is entirely analogous to the proof of A.2, using that $a^{2^{i+1}-1}\bar v_i = 0$ in $\pi_\star^{C_2}BP\mathbb{R}\langle n\rangle^{(EC_2)_+}$. Now we need to show that there are no further differentials: As every element in filtration f is divisible by a^f in $E^{2^{n+1}}$, the existence of a nonzero d_m (with $m\ge 2^{n+1}$) implies the existence of a nonzero d_m with source in the 0–line. Moreover, a nonzero d_m of some element $u^l\bar v$ (for $\bar v$ a polynomial in the $\bar v_i$) on the 0–line implies a nonzero d_m on u^l as $\bar v$ is a permanent cycle (in the image from $BP\mathbb{R}$). The image of such a differential must be of the form $a^m u^{l'} \bar{v}'$, where $\bar{v}'$ is a polynomial in $\bar{v}_1, \dots, \bar{v}_n$. As $a^m \bar{v}_i = 0$ for $1 \le i \le n$ in $E^{2^{n+1}}$, the polynomial $\bar{v}'$ must be constant. Counting degrees, we see that

$$(2l-1) - 2l\sigma = |u^l| - 1 = |a^m u^{l'}| = 2l' - (2l'+m)\sigma,$$

and thus $m = 2l - 2l' = 1$. This is clearly a contradiction. □

Corollary 4.2 *We have*

$$\pi_\star^{C_2}\big(BP\mathbb{R}\langle n\rangle^{(EC_2)_+} \otimes \widetilde{E}C_2\big) \cong \mathbb{F}_2[u^{\pm 2^n}, a^{\pm 1}].$$

In particular, we get $\pi_* BP\mathbb{R}\langle n\rangle^{tC_2} \cong \mathbb{F}_2[x^{\pm 1}]$, *where* $x = u^{2^n} a^{-2^{n+1}}$ *and* $|x| = 2^{n+1}$. *These are understood to be additive isomorphisms.*

Proof Recall that

$$\pi_\star^{C_2}\big(BP\mathbb{R}\langle n\rangle^{(EC_2)_+} \otimes \widetilde{E}C_2\big) = \pi_\star^{C_2}\big(BP\mathbb{R}\langle n\rangle^{(EC_2)_+}\big)[a^{-1}].$$

as $S^{\infty\sigma}$ is a model of $\widetilde{E}C_2$. The result follows as all $\bar{v}_i$ are a–power torsion, but $u^{2^n m}$ is not. □

4B The homotopy groups of $BP\mathbb{R}\langle n\rangle$

Computing the homotopy groups of the fixed points is more delicate than the computation of the homotopy fixed points. We first have to use our detailed knowledge about the homotopy groups of $BP\mathbb{R}$. Given a sequence $\underline{l} = (l_1, \dots)$, we denote by $BP\mathbb{R}/\underline{\bar{v}}^{\underline{l}}$ the spectrum $BP\mathbb{R}/(\bar{v}_{i_1}^{l_{i_1}}, \bar{v}_{i_2}^{l_{i_2}}, \dots)$, where i_j runs over all indices such that $l_{i_j} \neq 0$. Similarly $BP\mathbb{R}/\bar{v}_i^j$ is understood to be $BP\mathbb{R}$ if $j = 0$. We use the analogous convention when we have algebraic quotients of homotopy groups.

We recommend the reader skips the proof of the following result for first reading, as the technical detail is not particularly illuminating.

Proposition 4.3 *Let* $k \ge 1$ *and let* $\underline{l} = (l_1, l_2, \dots)$ *be a sequence of nonnegative integers with* $l_k = 0$. *Then the element* $\bar{v}_k$ *acts injectively on* $(\pi_{*\rho-c}^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$ *if* $0 \le c \le 2^{k+1}+1$, *with a splitting on the level of* $\mathbb{Z}_{(2)}$*–modules.*

Proof Recall from the appendix that $\pi_\star^{C_2} BP\mathbb{R}$ is isomorphic to the subalgebra of

$$P/(2a, \bar{v}_i a^{2^{i+1}-1})$$

(where i runs over all positive integers) generated by $\bar{v}_i(j) = u^{2^i j} \bar{v}_i$ (with $i, j \in \mathbb{Z}$ and $i \ge 0$) and a, where $P = \mathbb{Z}_{(2)}[a, \bar{v}_i, u^{\pm 1}]$. The degrees of the elements are $|a| = 1 - \rho$ and

$$|\bar{v}_i(j)| = (2^i - 1)\rho + 2^i j(4 - 2\rho) = (2^i - 1 - 2^{i+1} j)\rho + 2^{i+2} j.$$

We add the relations $\bar{v}_i^{l_i} = 0$ if $l_i \neq 0$.

We will first show that the ideal of $\bar{v}_k$–torsion elements in $(\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$ is contained in the ideal generated by $a^{2^{k+1}-1}$ and $\bar{v}_s^{l_s-1}\bar{v}_s(j)$ for s with $l_s \neq 0$ and j divisible by 2^{k-s} if $s < k$. Indeed, because the ideal $(2a, \bar{v}_i a^{2^{i+1}-1}, \underline{\bar{v}}^{\underline{l}}) \subset P$ is generated by monomials, a polynomial in P defines a $\bar{v}_k$–torsion element in $(\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$ if and only if each of its monomials define $\bar{v}_k$–torsion elements. A monomial x_P in P can only define a nonzero $\bar{v}_k$–torsion element in $(\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$ if it is divisible by $a^{2^{k+1}-1}$ or $\bar{v}_s^{l_s}$. In the latter case, x_P is of the form $\bar{v}\bar{v}_s^{l_s}u^m$, where $\bar{v}$ is a polynomial in the $\bar{v}_i$. This is divisible by $\bar{v}_s^{l_s}$ in $\pi_\star^{C_2} BP\mathbb{R}$ if and only if m is divisible by 2^i for some $\bar{v}_i$ in $\bar{v}$. Thus, x_P defines a nonzero element x in $(\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$ such that $\bar{v}_k x$ defines 0 only if $2^k | m$, which corresponds to the condition above.

Let x be a nonzero $\bar{v}_k$–torsion element in $(\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$, represented by a monomial in P. First assume that x is divisible by a^n with $n \geq 2^{k+1}-1$, but not by a^{n+1}. Then, x is not divisible by any $\bar{v}_i(j)$ with $i \leq k$ as $a^n\bar{v}_i(j) = 0$. We demand that x is in degree $*\rho - c$ with $c \geq 0$; in particular, $x \neq a^n$. Let $\bar{v}_i(j)$ a divisor of x with minimal i. Thus, the degree of x must be of the form $*\rho + 2^{i+2}m + n$. We know that $n \leq 2^{i+1}-2$. The largest negative value the non-ρ-part can take is $-2^{i+2} + 2^{i+1} - 2 = -2^{i+1} - 2$. The statement about injectivity follows in this case as $i > k$.

Now assume that x is a $\bar{v}_k$–torsion element not divisible by a^n for $n \geq 2^{k+1}-1$. Then x must be of the form $\bar{v}_s^{l_s-1}\bar{v}_s(j)$, where j is divisible by 2^{k-s} if $s < k$. Observe that

$$\bar{v}_s^{l_s-1}\bar{v}_s(j)\bar{v}_t(m) = \bar{v}_s^{l_s}\bar{v}_t(2^{s-t}j + m) = 0 \in (\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}}$$

for $t < s$, so y is not divisible by any $\bar{v}_t(m)$ for $t < s$. Likewise observe that if $s \leq t \leq k$, then

$$\bar{v}_s^{l_s-1}\bar{v}_s(j)\bar{v}_t(m) = \bar{v}_s^{l_s}\bar{v}_t(m + 2^{k-t}j') = 0 \in (\pi_\star^{C_2} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}},$$

where $j = 2^{k-s}j'$. Thus, y is also not divisible by any $\bar{v}_t(m)$ with $s \leq t \leq k$. As $|\bar{v}_s(j)| = *\rho + d$, where d is divisible by 2^{k+2}, and the same is true for $|\bar{v}_t(j)|$ with $t > k$, we see that if $|x|$ is of the form $*\rho - c$ with $c \geq 0$, then we have

$$c \geq 2^{k+2} - (2^{k+1} - 2) = 2^{k+1} + 2.$$

The statement about injectivity follows also in this case.

We still have to show the split injectivity. If $\bar{v}_k y = 2z$, but y is not divisible by 2, then y must be of the form $2\bar{v}u^{2^k j}$ in P, where $\bar{v}$ is a polynomial in the $\bar{v}_i$. Thus, $|y| = 2^{k+2}j + *\rho$, so we are fine in degree $*\rho - c$ for $0 \leq c \leq 2^{k+1} + 1 \leq 2^{k+2} - 1$. □

Remark 4.4 The exact bounds in the preceding proposition are not very important. The only important part for later inductive arguments is that the bound grows with k.

Corollary 4.5 *Let $\underline{l} = (l_1, l_2, \dots)$ be a sequence with only finitely many nonzero entries, and let j be the smallest index such that $l_j \neq 0$. Then the map*

$$(\pi^{C_2}_{*\rho-c} BP\mathbb{R})/\underline{\bar{v}}^{\underline{l}} \to \pi^{C_2}_{*\rho-c}(BP\mathbb{R}/\underline{\bar{v}}^{\underline{l}})$$

is an isomorphism for $0 \le c \le 2^{j+1}$.

Proof We use induction on the number n of nonzero indices in $\underline{l}$. If $n = 0$ (and $j = \infty$), the statement is clear.

For the step, define $\underline{l}'$ to be the sequence obtained from $\underline{l}$ by setting $l_j = 0$. Consider the short exact sequence

$$0 \to \big(\pi^{C_2}_{*\rho-c}(B/\underline{\bar{v}}^{\underline{l}'})\big)/\bar{v}_j^{l_j} \to \pi^{C_2}_{*\rho-c}(B/\underline{\bar{v}}^{\underline{l}}) \to \big\{\pi^{C_2}_{*\rho-(c+1)}(B/\underline{\bar{v}}^{\underline{l}'})\big\}_{\bar{v}_j^{l_j}} \to 0.$$

Here, the notation $\{X\}_z$ denotes the elements in X killed by z.

Assume $c \le 2^{j+1}$. By the induction hypothesis, $\pi^{C_2}_{*\rho-c}(B/\underline{\bar{v}}^{\underline{l}'}) \cong (\pi^{C_2}_{*\rho-c} B)/\underline{\bar{v}}^{\underline{l}'}$ as $c \le 2^{j+2}$, so $(\pi^{C_2}_{*\rho-c}(B/\underline{\bar{v}}^{\underline{l}'}))/\bar{v}_j^{l_j} \cong (\pi^{C_2}_{*\rho-c} B)/\underline{\bar{v}}^{\underline{l}}$. Furthermore,

$$\big\{\pi^{C_2}_{*\rho-(c+1)}(B/\underline{\bar{v}}^{\underline{l}'})\big\}_{\bar{v}_j^{l_j}} \cong \big\{(\pi^{C_2}_{*\rho-(c+1)} B)/\underline{\bar{v}}^{\underline{l}'}\big\}_{\bar{v}_j^{l_j}} \cong 0,$$

as follows from $c+1 \le 2^{j+2}$ and $c+1 \le 2^{j+1}+1$ by the induction hypothesis and Proposition 4.3. Thus $(\pi^{C_2}_{*\rho-c} B)/\underline{\bar{v}}^{\underline{l}} \to \pi^{C_2}_{*\rho-c}(B/\underline{\bar{v}}^{\underline{l}})$ is an isomorphism. □

The following corollary is crucial:

Corollary 4.6 *Let $I \subset \mathbb{Z}_{(2)}[\bar{v}_1, \dots]$ be an ideal generated by powers of the $\bar{v}_i$. Then $BP\mathbb{R}/I$ is strongly even.*

Proof As being strongly even is a property closed under filtered homotopy colimits, we are reduced to the case that I is finitely generated. By the last corollary, it suffices to show that $BP\mathbb{R}$ itself is strongly even. That the Mackey functor $\underline{\pi}^{C_2}_{*\rho}(BP\mathbb{R})$ is constant is clear from Theorem A.4.

Assume that x is a nonzero element in $\pi^{C_2}_{*\rho-1} BP\mathbb{R}$. We can assume that x is represented by $a^k u^l \bar{v}$ in the E_2–term of the homotopy fixed point spectral sequence for $BP\mathbb{R}$, where $\bar{v}$ is a monomial in the $\bar{v}_i$ (with $\bar{v}_0 = 2$), $k \ge 0$ and $l \in \mathbb{Z}$. The element x is in degree $k + 4l + *\rho$. Let $j \ge 0$ be the minimal number such that $\bar{v}_j | \bar{v}$. Then $2^j | l$ and $k \le 2^{j+1} - 2$. This is clearly in contradiction with $k + 4l = -1$. □

We recover the C_2–case of the reduction theorem of [18, Proposition 4.9] and [16, Theorem 6.5].

Corollary 4.7 *There is an equivalence* $BP\mathbb{R}/(\bar{v}_1, \bar{v}_2, \dots) \simeq H\underline{\mathbb{Z}}_{(2)}$.

Proof This follows directly from the last corollary and Corollary 2.3. □

Corollary 4.8 *Let* $I \subset \mathbb{Z}_{(2)}[\bar{v}_1, \dots]$ *be an ideal generated by powers of the* $\bar{v}_i$. *Then*

$$\pi^{C_2}_{*\rho+1} BP\mathbb{R}/I \cong \mathbb{F}_2\{a\} \otimes \mathbb{Z}_{(2)}[\bar{v}_1, \bar{v}_2, \dots]/I.$$

Proof As $BP\mathbb{R}/I$ is strongly even, this follows from [27, Lemma 2.15]. □

This allows us to compute $\pi^{C_2}_\star BP\mathbb{R}\langle n\rangle$.

Proposition 4.9 *The spectrum* $BP\mathbb{R}\langle n\rangle$ *is the connective cover of its Borel completion* $BP\mathbb{R}\langle n\rangle^{(EC_2)_+}$. *The cofibre* C *of* $BP\mathbb{R}\langle n\rangle \to BP\mathbb{R}\langle n\rangle^{(EC_2)_+}$ *has homotopy groups*

$$\pi^{C_2}_\star C \cong \mathbb{F}_2[a^{\pm 1}, u^{-2^n}]u^{-2^n},$$

with the naming of the elements indicating the map $\pi^{C_2}_\star BP\mathbb{R}\langle n\rangle^{(EC_2)_+} \to \pi^{C_2}_\star C$.

Proof This is clear on underlying homotopy groups. Thus, we have only to show that $BP\mathbb{R}\langle n\rangle^{C_2} \to BP\mathbb{R}\langle n\rangle^{hC_2}$ is a connective cover. For that purpose, it is enough to show that $BP\mathbb{R}\langle n\rangle^{\Phi C_2}$ is connective and that the fibre of $BP\mathbb{R}\langle n\rangle^{\Phi C_2} \to BP\mathbb{R}\langle n\rangle^{tC_2}$ has homotopy groups only in negative degrees.

We have $BP\mathbb{R}\langle n\rangle^{\Phi C_2} \simeq BP\mathbb{R}^{\Phi C_2}/(\bar{v}_{n+1}, \dots)$. As noted in the proof of Proposition A.1, the image of $\bar{v}_i$ in $M\mathbb{R}^{\phi C_2}$ is 0. As the quotient $BP\mathbb{R}^{\Phi C_2}/(\bar{v}_{n+1}, \dots)$ can be taken in the category of $M\mathbb{R}^{\Phi C_2}$–modules, we are only quotienting out by 0. It follows easily that $(BP\mathbb{R}/(\bar{v}_{n+1}, \dots, \bar{v}_{n+m}))^{\Phi C_2}$ has in the homotopy groups an $\mathbb{F}_2$ in all degrees of the form $\sum_{i=n+1}^{n+m} \varepsilon_i(|v_i|+1) = \sum_{i=n+1}^{n+m} \varepsilon_i 2^i$ with $\varepsilon_i = 0$ or 1. As geometric fixed point commute with homotopy colimits, we see that $\pi_* BP\mathbb{R}\langle n\rangle^{\Phi C_2} \cong \mathbb{F}_2[y]$ (additively) with $|y| = 2^{n+1}$. It remains to show that y^k maps nonzero to $\pi_* BP\mathbb{R}\langle n\rangle^{tC_2}$ (and hence maps to x^k).

It is enough to show that $a^{-|y^k|-1}y^k$ maps nonzero to $\pi^{C_2}_\star \Sigma BP\mathbb{R}\langle n\rangle \otimes (EC_2)_+$ in the sequence coming from the Tate square, ie that $a^{-|y^k|-1}y^k$ is not in the image from (the fixed points of) $BP\mathbb{R}\langle n\rangle$. But $a^{-|y^k|-1}y^k$ is in degree $(|y^k|+1)\rho - 1$ and $\pi^{C_2}_{(|y^k|+1)\rho-1} BP\mathbb{R}\langle n\rangle = 0$ by Corollary 4.6. □

Let us describe the homotopy groups of $BP\mathbb{R}\langle n\rangle$ in more detail. We set $\bar{v}_0 = 2$ for convenience. Denote by BB (for *basic block*) the $\mathbb{Z}_{(2)}[a, \bar{v}_1, \dots, \bar{v}_n]/2a$–submodule of

$$\mathbb{Z}_{(2)}[\bar{v}_1, \dots, \bar{v}_n]/(a^{2^{k+1}-1}\bar{v}_k)_{0 \le k \le n}$$

generated by 1 and by $\bar{v}_k(m) = u^{2^k m}\bar{v}_k$ for $0 \le k < n$ and $0 < m < 2^{n-k}$. By Proposition 4.1, we see that

$$\pi_\star^{C_2} BP\mathbb{R}\langle n\rangle^{(EC_2)_+} \cong BB[U^{\pm 1}]$$

with $U = u^{2^n}$. Note that this isomorphism is not claimed to be multiplicative; in general, $BP\mathbb{R}\langle n\rangle$ is not even known to have any kind of (homotopy unital) multiplication.

Define BB' to be the kernel of the map $BB \to \mathbb{F}_2[a]$ given by sending all $\bar{v}_k$ and $\bar{v}_k(m)$ to zero. Set $NB = \Sigma^{\sigma-1}\mathbb{F}_2[a]^\vee \oplus BB'$, where NB stands for *negative block*. Then it is easy to see from the last proposition that

$$\pi_\star^{C_2} BP\mathbb{R}\langle n\rangle \cong BB[U] \oplus U^{-1}NB[U^{-1}],$$

where this isomorphism is again only meant additively. We will be a little bit more explicit about the homotopy groups of $BP\mathbb{R}\langle n\rangle$ in the cases $n = 1$ and 2 in Part IV.

4C Forms of $BP\mathbb{R}\langle n\rangle$

Our next goal is to identify certain spectra as forms of $BP\mathbb{R}\langle n\rangle$. We take the following definition from [27]:

Definition 4.10 Let E be an even 2–local commutative and associative C_2–ring spectrum up to homotopy. By [27, Lemma 3.3], E has a real orientation, and after choosing one, we obtain a formal group law on $\pi_{*\rho}^{C_2} E$. The 2–typification of this formal group law defines a map $\pi_{2*}^e BP \cong \pi_{*\rho}^{C_2} BP\mathbb{R} \to \pi_{*\rho}^{C_2} E$. We call E a *form of* $BP\mathbb{R}\langle n\rangle$ if the map

$$\underline{\mathbb{Z}_{(2)}[\bar{v}_1, \dots, \bar{v}_n]} \subset \underline{\pi}_{*\rho} BP\mathbb{R} \to \underline{\pi}_{*\rho} E$$

is an isomorphism of constant Mackey functors.

This depends neither on the choice of $\bar{v}_i$ nor on the chosen real orientation, as can be seen using that $\bar{v}_i$ is well defined modulo $(2, \bar{v}_1, \dots, \bar{v}_{i-1})$.

Equivalently, one can say that E is a form of $BP\mathbb{R}\langle n\rangle$ if and only if E is strongly even and its underlying spectrum is a form of $BP\langle n\rangle$. We want to show that every form of $BP\mathbb{R}\langle n\rangle$ is also of the form $BP\mathbb{R}/(\bar{v}_{n+1}, \bar{v}_{n+2}, \dots)$ for some choice of elements $\bar{v}_i$. For this, we need the following lemma from [27, Lemma 3.4]:

Lemma 4.11 *Let $f\colon E \to F$ be a map of C_2–spectra. Assume that f induces isomorphisms*

$$\pi_{k\rho}^{C_2} E \to \pi_{k\rho}^{C_2} E \quad \text{and} \quad \pi_k E \to \pi_k F$$

for all $k \in \mathbb{Z}$. Assume furthermore that $\pi_{k\rho-1}^{C_2} E \to \pi_{k\rho-1}^{C_2} F$ is an injection for all $k \in \mathbb{Z}$ (for example, if $\pi_{k\rho-1}^{C_2} E = 0$). Then f is an equivalence of C_2–spectra.

Proposition 4.12 *Let E be a form of $BP\mathbb{R}\langle n\rangle$. Then one can choose indecomposables $\bar{v}_i \in \pi^{C_2}_{(2^i-1)\rho} BP\mathbb{R}$ for $i \geq n+1$ such that $E \simeq BP\mathbb{R}/(\bar{v}_{n+1}, \bar{v}_{n+2}, \dots)$.*

Proof First choose any system of $\bar{v}_i$. Also choose a real orientation $f\colon BP\mathbb{R} \to E$ and denote $f(\bar{v}_i)$ by x_i. Define a multiplicative section

$$s\colon \pi^{C_2}_{*\rho} E \to \pi^{C_2}_{*\rho} BP\mathbb{R}$$

by $s(x_i) = \bar{v}_i$ for $1 \leq i \leq n$.

Now define a new system of $\bar{v}_i$ by

$$\bar{v}_i^{\text{new}} = \bar{v}_i - s(f_*(\bar{v}_i))$$

for $i \geq n+1$. As these agree with $\bar{v}_i$ mod $(\bar{v}_1, \dots, \bar{v}_n)$, they are still indecomposable. Furthermore, the $\bar{v}_i^{\text{new}}$ are for $i \geq n+1$ clearly in the kernel of f_*. Thus, we obtain a map $BP\mathbb{R}\langle n\rangle/(\bar{v}^{\text{new}}_{n+1}, \bar{v}^{\text{new}}_{n+2}, \dots) \to E$ that is an isomorphism on $\pi^{C_2}_{*\rho}$. By Corollary 4.6, the source is strongly even. By Lemma 4.11, the map is an equivalence. □

Examples 4.13 We consider real versions of the classical examples ku and $tmf_1(3)$.

(1) The connective real K-theory spectrum $k\mathbb{R}_{(2)}$ is a form of $BP\mathbb{R}\langle 1\rangle$. Indeed, the underlying spectrum $ku_{(2)}$ is well known to be a form of $BP\langle 1\rangle$ and $k\mathbb{R}_{(2)}$ is also strongly even (as can be seen by the results from [7, Section 3.7.D] or from the computation in Section 11).

(2) Define $\overline{tmf_1(3)}$ as the equivariant connective cover of the spectrum $\overline{Tmf_1(3)}$, ie $Tmf_1(3)$ with the algebro-geometrically defined C_2–action (see [27, Section 4.1] for details). As shown in [27, Corollary 4.17], $\overline{tmf_1(3)}_{(2)}$ is a form of $BP\mathbb{R}\langle 2\rangle$. By Proposition 4.12, we can construct $\overline{tmf_1(3)}_{(2)}$ by killing a sequence $\bar{v}_2, \bar{v}_3, \dots$ in $BP\mathbb{R}$. This construction is used in [23] to define a C_2–equivariant version of $tmf_1(3)_{(2)}$. In particular, we see (using the discussion before Proposition 4.23 in [27]) that $\overline{TMF_1(3)}_{(2)}$ (with the algebro-geometrically defined C_2–action) agrees with the $\mathbb{TMF}_1(3)_{(2)}$ of [23].

5 Results and consequences

In this section, we want to discuss our main results in more detail than in the introduction and we will also derive some consequences and give some examples. Recall to that purpose the notation from Sections 3C and 4A. Furthermore, we will implicitly localize everything at 2, so $\mathbb{Z}$ means $\mathbb{Z}_{(2)}$, etc. Our main theorem is the following:

Theorem 5.1 *Let $(m_1, m_2, \dots)$ be a sequence of nonnegative integers with only finitely many entries bigger than 1, and let M be the quotient $BP\mathbb{R}/(\bar{v}_1^{m_1}, \bar{v}_2^{m_2}, \dots)$, where we only quotient by the positive powers of $\bar{v}_i$. Denote by $\underline{\bar{v}}$ the sequence of $\bar{v}_i$ in $\pi_\star^{C_2} M\mathbb{R}$ such that $m_i = 0$, by $|\underline{\bar{v}}|$ the sum of their degrees and by m' the sum of all $(m_i - 1)|\bar{v}_i|$ for $m_i > 1$. Then*

$$\mathbb{Z}^M \simeq \Sigma^{-m'+4-2\rho} \kappa_{M\mathbb{R}}(\underline{\bar{v}}; M).$$

The most important case is where $m_{n+1} = m_{n+2} = \cdots = 1$, so

$$M = BP\mathbb{R}\langle n\rangle/(\bar{v}_1^{m_1}, \dots, \bar{v}_n^{m_n}).$$

If k is the number of elements in $\underline{\bar{v}}$, we also get

$$\mathbb{Z}^M \simeq \Sigma^{-m'+k+|\underline{\bar{v}}|+4-2\rho} \Gamma_{\underline{\bar{v}}} M,$$

where we view M as an $M\mathbb{R}$–module.

The first form will be proved as Theorem 10.1 and the second follows from it using Lemma 3.6. The second form also follows from Corollary 7.5 (using that $\Gamma_{\underline{\bar{v}}}$ preserves cofibre sequences to pass to quotients of $BP\mathbb{R}\langle n\rangle$).

Example 5.2 $\mathbb{Z}^{BP\mathbb{R}\langle n\rangle} \simeq \Sigma^{n+D_n\rho+4-2\rho} \Gamma_{(\bar{v}_1,\dots,\bar{v}_n)} BP\mathbb{R}\langle n\rangle$ for $D_n = |v_1| + \cdots + |v_n|$. This says $BP\mathbb{R}\langle n\rangle$ has Gorenstein duality with respect to $H\underline{\mathbb{Z}} \simeq BP\mathbb{R}\langle n\rangle/(\bar{v}_1, \dots, \bar{v}_n)$. (The last equivalence follows from Corollary 4.7.)

Example 5.3 Set $k\mathbb{R}(n) = BP\mathbb{R}\langle n\rangle/(\bar{v}_1, \dots, \bar{v}_{n-1})$ to be connective integral real Morava K-theory and $K\mathbb{R}(n) = k\mathbb{R}(n)[\bar{v}_n^{-1}]$ its periodic version. Then

$$\begin{aligned} \mathbb{Z}^{k\mathbb{R}(n)} &\simeq \Sigma^{1+|\bar{v}_n|+4-2\rho} \Gamma_{\bar{v}_n} k\mathbb{R}(n) \\ &\simeq \Sigma^{(2^n-3)\rho+4} \operatorname{cof}(k\mathbb{R}(n) \to K\mathbb{R}(n)). \end{aligned}$$

This includes for $n = 1$ the case of usual (2–local) connective real K-theory.

Example 5.4 To have a slightly stranger example, take $M = BP\mathbb{R}\langle 3\rangle/(\bar{v}_1^4, \bar{v}_3^2)$. Then

$$\mathbb{Z}^M \simeq \Sigma^{5-9\rho} \Gamma_{\bar{v}_2} M.$$

So far, we have only talked about *quotients* of $BP\mathbb{R}$. This does not include important real spectra like the real Johnson–Wilson theories $E\mathbb{R}(n) = BP\mathbb{R}\langle n\rangle[\bar{v}_n^{-1}]$ or the (integral) real Morava K-theories $K\mathbb{R}(n)$. For this, we have to study the behaviour of our constructions under localizations.

Let M be an $RO(C_2)$–graded $\mathbb{Z}[v]$–module, where v has some degree $|v| \in RO(C_2)$. We say that M has *bounded v–divisibility* if for every degree $a + b\sigma$, there is a k such that

$$v^k\colon M_{a+b\sigma-|v^k|} \to M_{a+b\sigma}$$

is zero. We will also apply the concept to modules that are just $\mathbb{Z}|v|$–graded.

Lemma 5.5 *The class of $RO(C_2)$–graded $\mathbb{Z}[v]$–modules of bounded v–divisibility is closed under submodules, quotients and extensions.*

Proof This is clear for submodules and quotients. Let

$$0 \to K \to M \to N \to 0$$

be a short exact sequence of $\mathbb{Z}[v]$–modules where K and N are of bounded v–divisibility. For a given degree $\alpha \in RO(C_2)$, we know that there is a k such that v^k maps trivially into K_α. Furthermore, there is an n such that v^n maps trivially into $N_{\alpha-k|v|}$. Thus, multiplication by v^{n+k} is the zero map $M_{\alpha-(k+n)|v|} \to M_\alpha$. □

Let M be an $M\mathbb{R}$–module. We say that M is of *bounded $\bar{v}_n$–divisibility* if both $\pi^{C_2}_\star M$ and $\pi^e_* M$ are of bounded $\bar{v}_n$–divisibility. This is, for example, true if M is connective.

Lemma 5.6 *We have the following two properties of $\bar{v}_n$–divisibility.*

(1) *Being of bounded $\bar{v}_n$–divisibility is closed under cofibres and suspensions.*
(2) *An $M\mathbb{R}$–module M is of bounded $\bar{v}_n$–divisibility if and only if $\pi^{C_2}_{*\rho} M$ and $\pi^e_* M$ are of bounded $\bar{v}_n$–divisibility.*

Proof Both statements follow from the last lemma. For the second item, we additionally use the exact sequence

$$\pi^e_{a+b+1} M \to \pi^{C_2}_{a+(b+1)\sigma} M \to \pi^{C_2}_{a+b\sigma} M \to \pi^e_{a+b} M$$

induced by the cofibre sequence

$$(C_2)_+ \to S^0 \to S^\sigma. \qquad \square$$

Lemma 5.7 *If M has bounded $\bar{v}_n$–divisibility, then there is a natural equivalence*

$$M[\bar{v}_n^{-1}] \simeq \Sigma \operatorname{holim}_{\leftarrow}(\cdots \to \Sigma^{|\bar{v}_n|} \Gamma_{\bar{v}_n} M \xrightarrow{\bar{v}_n} \Gamma_{\bar{v}_n} M)$$

of $M\mathbb{R}$–modules.

Proof We apply the endofunctor $H\colon N \mapsto \operatorname{holim}_{\leftarrow}(\cdots \to \Sigma^{|\bar{v}_n|}N \xrightarrow{\bar{v}_n} N)$ of $M\mathbb{R}$–modules to the cofibre sequence

$$\Gamma_{\bar{v}_n} M \to M \to M[\bar{v}_n^{-1}].$$

Clearly $H(M[\bar{v}_n^{-1}]) \simeq M[\bar{v}_n^{-1}]$. Thus, we just have to show that $H(M) \simeq 0$. This follows by the $\lim^1$–sequence and bounded $\bar{v}_n$–divisibility. □

Lemma 5.8 *Let B be a quotient of $BP\mathbb{R}$ by powers of the $\bar{v}_i$. Then $B[\bar{v}^{-1}]$ has bounded $\bar{v}_n$–divisibility if $\bar{v}$ is a product of $\bar{v}_i$ not containing $\bar{v}_n$. Hence, the same is also true for the stable Koszul complex $\Gamma_{\underline{\bar{v}}} B$, where $\underline{\bar{v}}$ is a sequence of $\bar{v}_i$ not containing $\bar{v}_n$.*

Proof By Lemma 5.6, it is enough to check the first statement on $\pi_{*\rho}^{C_2}$ and on π_*^e. On the latter, it is clear and the former is isomorphic to it by Corollary 4.6. For the second statement, we use that $\Gamma_{\underline{\bar{v}}} B$ is the fibre of $B \to \check{C}(\underline{\bar{v}}; B)$, where $\check{C}(\underline{\bar{v}}; B)$ has a filtration with subquotients $M\mathbb{R}$–modules of the form $\Sigma^{?} B[x^{-1}]$ for some $x \in \pi_\star^{C_2} M\mathbb{R}$ [11, Lemma 3.7]. Thus, the second statement follows from Lemma 5.6. □

Theorem 5.9 *Let the notation be as in Theorem 5.1, and assume for simplicity that only finitely many m_i are zero and that $m_n = 0$. Then*

$$\mathbb{Z}^{M[\bar{v}_n^{-1}]} \simeq \Sigma^{-m'+|\underline{\bar{v}}|+(k-1)+4-2\rho}\Gamma_{\underline{\bar{v}}\setminus \bar{v}_n} M.$$

Here $\underline{\bar{v}} \setminus \bar{v}_n$ denotes the sequence of all $\bar{v}_i$ such that $m_i = 0$ and $i \neq n$.

Proof The preceding lemmas imply the following chain of equivalences:

$$\begin{aligned}
\mathbb{Z}^{M[\bar{v}_n^{-1}]} &\simeq \mathbb{Z}^{\operatorname{holim}_{\rightarrow}(M \xrightarrow{\bar{v}_n} \Sigma^{-|\bar{v}_n|}M \xrightarrow{\bar{v}_n} \cdots)} \\
&\simeq \operatorname{holim}_{\leftarrow}(\cdots \xrightarrow{\bar{v}_n} \mathbb{Z}^M) \\
&\simeq \Sigma^{-m'+|\underline{\bar{v}}|+k+4-2\rho}\operatorname{holim}_{\leftarrow}(\cdots \xrightarrow{\bar{v}_n} \Gamma_{\underline{\bar{v}}} M) \\
&\simeq \Sigma^{-m'+|\underline{\bar{v}}|+k+4-2\rho}\operatorname{holim}_{\leftarrow}(\cdots \xrightarrow{\bar{v}_n} \Gamma_{\bar{v}_n}(\Gamma_{\underline{\bar{v}}\setminus \bar{v}_n} M)) \\
&\simeq \Sigma^{-m'+|\underline{\bar{v}}|+(k-1)+4-2\rho}(\Gamma_{\underline{\bar{v}}\setminus \bar{v}_n} M)[\bar{v}_n^{-1}] \\
&\simeq \Sigma^{-m'+|\underline{\bar{v}}|+(k-1)+4-2\rho}\Gamma_{\underline{\bar{v}}\setminus \bar{v}_n}(M[\bar{v}_n^{-1}]).
\end{aligned}$$
□

Example 5.10 We recover the following result by Ricka [28]:

$$\mathbb{Z}^{K\mathbb{R}(n)} \simeq \Sigma^{4-2\rho} K\mathbb{R}(n).$$

Here, $K\mathbb{R}(n)$ denotes integral Morava K-theory $E\mathbb{R}(n)/(\bar{v}_1, \ldots, \bar{v}_{n-1})$.

Example 5.11 In the following, we will use the fact that there are invertible classes $x, \bar{v}_n \in \pi_\star^{C_2} E\mathbb{R}(n)$ of degree $-2^{2n+1}+2^{n+2}-\rho$ and $(2^n-1)\rho$, respectively, where $x = \bar{v}_n^{1-2^n} u^{2^n(1-2^{n-1})}$:

$$\begin{aligned}\mathbb{Z}^{E\mathbb{R}(n)} &\simeq \Sigma^{D_{n-1}\rho+(n-1)+4-2\rho}\Gamma_{(\bar{v}_1,\dots,\bar{v}_{n-1})}E\mathbb{R}(n)\\ &\simeq \Sigma^{-(n+2)\rho+(n+3)}\Gamma_{(\bar{v}_1,\dots,\bar{v}_{n-1})}E\mathbb{R}(n)\\ &\simeq \Sigma^{(n+2)(2^{2n+1}-2^{n+2})+n+3}\Gamma_{(\bar{v}_1,\dots,\bar{v}_{n-1})}E\mathbb{R}(n).\end{aligned}$$

This says that $E\mathbb{R}(n)$ has Gorenstein duality with respect to $E\mathbb{R}(n)/(\bar{v}_1,\dots,\bar{v}_{n-1}) = K\mathbb{R}(n)$. Note that we can replace the ideal $(\bar{v}_1,\dots,\bar{v}_{n-1})$ by an ideal generated in integral degrees, namely $(\bar{v}_1 x,\dots,\bar{v}_{n-1}x^{2^{n-1}-1})$.

Example 5.12 Recall from [27] the spectra $tmf_1(3)$, $Tmf_1(3)$ and $TMF_1(3)$ and the corresponding C_2–spectra $\overline{tmf_1(3)}$, $\overline{Tmf_1(3)}$ and $\overline{TMF_1(3)}$. Recall that we have $\pi_* tmf_1(3) = \mathbb{Z}[a_1, a_3]$, where a_1 and a_3 can be identified with the images of the Hazewinkel generators v_1 and v_2, and that $\overline{tmf_1(3)}$ is a form of $BP\mathbb{R}\langle 2\rangle$ (as already discussed in Examples 4.13). This gives the Anderson dual of $\overline{tmf_1(3)}$. Tweaking the last theorem a little bit allows us also to show that

$$\mathbb{Z}^{\overline{TMF_1(3)}} \simeq \Sigma^{5+2\rho}\Gamma_{\bar{v}_1}\overline{TMF_1(3)}.$$

We can also recover one of the main results of [27], namely that

$$\mathbb{Z}^{\overline{Tmf_1(3)}} \simeq \Sigma^{5+2\rho}\overline{Tmf_1(3)}.$$

Indeed, $Tmf_1(3)$ is by [27, Section 4.3] the cofibre of the map

$$\Gamma_{\bar{v}_1,\bar{v}_2}\overline{tmf_1(3)} \to \overline{tmf_1(3)}.$$

As the source is equivalent to $\Sigma^{-6-2\rho}\mathbb{Z}^{\overline{tmf_1(3)}}$, applying Anderson duality shows that $\mathbb{Z}^{\overline{Tmf_1(3)}}$ is the fibre of

$$\Sigma^{6+2\rho}\overline{tmf_1(3)} \to \Sigma^{6+2\rho}\Gamma_{\bar{v}_1,\bar{v}_2}\overline{tmf_1(3)}.$$

This is equivalent to $\Sigma^{5+2\rho}\overline{Tmf_1(3)}$. This example does not require 2–localization, only that 3 is inverted.

Remark 5.13 By Proposition 3.3, all the results in this section have direct implications for the Anderson duals of the fixed point spectra. These are easiest to understand in the case of $ER(n) = (E\mathbb{R}(n))^{C_2}$, where we get

$$\mathbb{Z}^{ER(n)} \simeq \Sigma^{(n+2)(2^{2n+1}-2^{n+2})+n+3}\Gamma_{(\bar{v}_1 x,\dots,\bar{v}_{n-1}x^{2^n-1})}ER(n).$$

Part II The Gorenstein approach

In this part, we explain the Gorenstein approach to prove Gorenstein duality, first for $k\mathbb{R}$ and then for $BP\mathbb{R}\langle n\rangle$.

6 Connective K-theory with reality

The present section considers K-theory with reality, which is more familiar than $BP\mathbb{R}\langle n\rangle$ for general n, and no 2–localization is necessary. The arguments are especially simple, firstly because $k\mathbb{R}$ is a commutative ring spectrum, and secondly because we only need to consider principal ideals. Simple as the argument is, we see in Section 11 that the consequences for coefficient rings are interesting.

6A Gorenstein condition and Matlis lift

It is well known that there is a cofibre sequence

$$\Sigma^{\epsilon} ku \xrightarrow{v} ku \to H\mathbb{Z}.$$

If one knows the coefficient ring $ku_* = \mathbb{Z}[v]$, this is easy to construct, since we can identify ku/v as the Eilenberg–Mac Lane spectrum from its homotopy groups.

There is a version with reality [8]. Indeed, we may construct the cofibre sequence

$$\Sigma^{\rho} k\mathbb{R} \xrightarrow{\bar{v}} k\mathbb{R} \to H\underline{\mathbb{Z}},$$

where $k\mathbb{R}/\bar{v}$ is identified using Corollary 2.3

Since the Dugger sequence is self dual we immediately deduce that $k\mathbb{R}$ is Gorenstein.

Lemma 6.1 $$\mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, k\mathbb{R}) = \Sigma^{-\rho-1} H\underline{\mathbb{Z}},$$

and $k\mathbb{R} \to H\underline{\mathbb{Z}}$ is Gorenstein.

Proof Apply $\mathrm{Hom}_{k\mathbb{R}}(\cdot\,, k\mathbb{R})$ to the Dugger sequence. □

To actually get Gorenstein duality we need to construct a Matlis lift (adapted from [9, Section 6]), which is a counterpart in topology of the injective hull of the residue field.

Definition 6.2 If M is an $H\underline{\mathbb{Z}}$–module, we say that a $k\mathbb{R}$–module $\widetilde{M}$ is a *Matlis lift* of M if $\widetilde{M}$ is $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellular, and

$$\mathrm{Hom}_{k\mathbb{R}}(T, \widetilde{M}) \simeq \mathrm{Hom}_{H\underline{\mathbb{Z}}}(T, M)$$

for all $H\underline{\mathbb{Z}}$–modules T.

The Anderson dual provides one such example.

Lemma 6.3 *The $k\mathbb{R}$–module $\Sigma^{-2(1-\sigma)}\underline{\mathbb{Z}}^{k\mathbb{R}}$ is a Matlis lift of $H\underline{\mathbb{Z}}$. Indeed,*

(i) *$\underline{\mathbb{Z}}^{k\mathbb{R}}$ is $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellular, and*

(ii) *there is an equivalence*

$$\Sigma^{2\delta} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^* = \mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, \underline{\mathbb{Z}}^{k\mathbb{R}}),$$

where $\delta = 1-\sigma$.

Proof One could prove the first part from the slice tower, but it also follows directly from Corollary 3.9.

The second statement is immediate from Lemma 3.1. □

6B Gorenstein duality

We next want to move on to Gorenstein duality, so we write

$$\mathcal{E} = \mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}}).$$

Combining Lemmas 6.1 and 6.3, we have

$$\mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, k\mathbb{R}) \simeq \Sigma^{-\rho-1} H\underline{\mathbb{Z}} \simeq \mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, \Sigma^{-4+\sigma}\underline{\mathbb{Z}}^{k\mathbb{R}}). \tag{6.4}$$

We now want to remove the $\mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, \cdot)$ from this equivalence.

Lemma 6.5 (effective constructibility) *The evaluation map*

$$\mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, M) \otimes_{\mathcal{E}} H\underline{\mathbb{Z}} \to M$$

is an $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellularization for every left $k\mathbb{R}$–module M.

Proof Since the domain is clearly $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellular, it is enough to show the map is an equivalence for all cellular modules M.

This is clear for $M = H\underline{\mathbb{Z}}$. The class of M for which the statement is true is closed under (i) triangles, (ii) coproducts (since $H\underline{\mathbb{Z}}$ is small) and (iii) suspensions by representations. This gives all $\mathbb{R}$–cellular modules. □

Local cohomology gives an alternative approach to cellularization. Recall that we define the $\bar{v}$–power torsion of a $k\mathbb{R}$–module M by the fibre sequence

$$\Gamma_{\bar{v}} M \to M \to M[1/\bar{v}].$$

The following lemma is a special case of Proposition 3.8.

Lemma 6.6 *The map*

$$\Gamma_{\overline{v}} M \to M$$

is an $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellularization.

It remains to check that the two $\mathcal{E}$–actions on $H\underline{\mathbb{Z}}$ coincide.

Lemma 6.7 *There is a unique right $\mathcal{E}$–module structure on $H\underline{\mathbb{Z}}$.*

Proof Suppose that $H\underline{\mathbb{Z}}'$ is a right $\mathcal{E}$–module whose underlying C_2–spectrum is equivalent to the Eilenberg–Mac Lane spectrum $H\underline{\mathbb{Z}}$. We first claim that $H\underline{\mathbb{Z}}'$ can be constructed as an $\mathcal{E}$–module with cells in degrees $k\rho$ for $k \leq 0$:

$$H\underline{\mathbb{Z}}' \simeq_{\mathcal{E}} S^0_{\mathcal{E}} \cup e^{-\rho}_{\mathcal{E}} \cup e^{-2\rho}_{\mathcal{E}} \cup \cdots .$$

Once that is proved, we argue as follows. If $H\underline{\mathbb{Z}}''$ is another right $\mathcal{E}$–module with underlying C_2–spectrum $H\underline{\mathbb{Z}}$, we may construct a map $H\underline{\mathbb{Z}}' \to H\underline{\mathbb{Z}}''$ skeleton by skeleton in the usual way. We start with the $\mathcal{E}$–module map $\mathcal{E} = (H\underline{\mathbb{Z}}')^{(0)} \to H\underline{\mathbb{Z}}'$ giving the unit, and successively extend the map over the cells of $H\underline{\mathbb{Z}}'$. At each stage the obstruction to the existence of an extension over $(H\underline{\mathbb{Z}}')^{-k\rho}$ lies in $\pi^{C_2}_{-k\rho-1}(H\underline{\mathbb{Z}}'')$. These groups are zero. We end with a map which is an isomorphism on 0^{th} homotopy Mackey functors and therefore an equivalence.

For the cell-structure, it is enough to show that for every right $\mathcal{E}$–module $H\underline{\mathbb{Z}}'$ of the homotopy type of the Eilenberg–Mac Lane spectrum $H\underline{\mathbb{Z}}$, there is a map $\mathcal{E} \to H\underline{\mathbb{Z}}'$ of right $\mathcal{E}$–modules whose fibre has the homotopy type of $\Sigma^{-\rho-1}H\underline{\mathbb{Z}}$. Indeed, suppose we have already constructed a right $\mathcal{E}$–module $(H\underline{\mathbb{Z}}')^{(n)}$ with an $\mathcal{E}$–map to $H\underline{\mathbb{Z}}'$ with fibre of the homotopy type $\Sigma^{-(n+1)\rho-1}H\underline{\mathbb{Z}}$. Then it is easy to see that the cofibre $(H\underline{\mathbb{Z}}')^{(n+1)}$ of the map $\Sigma^{-(n+1)\rho-1}\mathcal{E} \to \Sigma^{-(n+1)\rho-1}H\underline{\mathbb{Z}} \to (H\underline{\mathbb{Z}}')^{(n)}$ has the analogous property. Taking the homotopy colimit, we get a map $\underrightarrow{\mathrm{holim}}(H\underline{\mathbb{Z}}')^{(n)} \to H\underline{\mathbb{Z}}'$ with fibre $\underrightarrow{\mathrm{holim}}\Sigma^{-(n+1)\rho-1}H\underline{\mathbb{Z}}$, which is clearly zero (eg by Lemma 4.11 and the fact that $H\underline{\mathbb{Z}}$ is even; we refer to [28, Section 3.4] for a table of $\pi^{C_2}_{\star}H\underline{\mathbb{Z}}$).

We choose the map $f\colon \mathcal{E} \to H\underline{\mathbb{Z}}'$ representing $1 \in \pi^{C_2}_0 H\underline{\mathbb{Z}}'$ and call the fibre F. We want to show that f agrees with the canonical map $\mathcal{E} \to H\underline{\mathbb{Z}}$ on homotopy groups of the form $\pi^{C_2}_{k-\sigma}$ for $k \in \mathbb{Z}$. Indeed, the only nonzero class in $H\underline{\mathbb{Z}}'$ in these degrees is $a \in \pi^{C_2}_{-\sigma}H\underline{\mathbb{Z}}'$, which has to be hit by $a \in \pi^{C_2}_{-\sigma}\mathcal{E}$ as it comes from the sphere. Thus, $\pi^{C_2}_{k-\sigma}F \cong \pi^{C_2}_{k-\sigma}\Sigma^{-1-\rho}H\underline{\mathbb{Z}}$ for all k and hence $F \simeq \Sigma^{-1-\rho}H\underline{\mathbb{Z}}$ as C_2–spectra, as we needed to show. □

From this the required statement follows.

Corollary 6.8 (Gorenstein duality) *There is an equivalence of $k\mathbb{R}$–modules*

$$\Gamma_{\overline{v}} k\mathbb{R} \simeq \Sigma^{-4+\sigma}\mathbb{Z}^{k\mathbb{R}}.$$

Proof By (6.4) and Lemma 6.7, we know that

$$\mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, k\mathbb{R}) \otimes_{\mathcal{E}} H\underline{\mathbb{Z}} \simeq \mathrm{Hom}_{k\mathbb{R}}(H\underline{\mathbb{Z}}, \Sigma^{-4+\sigma}\mathbb{Z}^{k\mathbb{R}}) \otimes_{\mathcal{E}} H\underline{\mathbb{Z}}.$$

By Lemma 6.5, the two sides are the cellularizations of $k\mathbb{R}$ and $\Sigma^{-4+\sigma}\mathbb{Z}^{k\mathbb{R}}$ respectively. By Lemmas 6.6 and 6.3, the former is $\Gamma_{\overline{v}} k\mathbb{R}$ and the latter is $\Sigma^{-4+\sigma}\mathbb{Z}^{k\mathbb{R}}$ itself. □

The implications of this equivalence for the coefficient ring are investigated in Section 11.

7 $BP\langle n\rangle$ with reality

We now turn to the case of $BP\mathbb{R}\langle n\rangle$ for a general n. The counterpart of the argument of Section 6 is a little simpler when $BP\mathbb{R}\langle n\rangle$ is a commutative ring spectrum. For $n=1$ and $n=2$, the spectra $k\mathbb{R}$, and $tmf_1(3)$, are both known to be a commutative ring spectra, and their 2–localizations give $BP\mathbb{R}\langle n\rangle$ when $n=1$ and $n=2$ respectively. However for higher n it is not known that $BP\mathbb{R}\langle n\rangle$ is a commutative ring spectrum. This is a significant technical issue, but one that is familiar when working with nonequivariant BP–related theories since BP is not known to be a commutative ring. The established method for getting around this is to use the fact that BP and $BP\langle n\rangle$ are modules over the commutative ring MU. We will adopt precisely the same method by working with $M\mathbb{R}$–modules. The only real complication is that we are forced to work with spectra whose homotopy groups are bigger than we might like, but if we focus on the relevant part, it causes no real difficulties.

7A Gorenstein condition and Matlis lift

As mentioned in the introduction of this section, we will work in the setting of $M\mathbb{R}$–modules. More precisely, we will always (implicitly) localize at 2 and set $S = M\mathbb{R}_{(2)}$. As discussed in Section 4A, we can define S–modules $BP\mathbb{R}\langle n\rangle$, once we have chosen a sequence of $\overline{v}_i$ (for example, the Hazewinkel or Araki generators).

The ideal

$$\overline{J}_n = (\overline{v}_1, \dots, \overline{v}_n)$$

plays a prominent role, and we will abuse notation by writing

$$S/\overline{J}_n := \mathrm{cof}(S \xrightarrow{\overline{v}_1} S) \otimes_S \mathrm{cof}(S \xrightarrow{\overline{v}_2} S) \otimes_S \cdots \otimes_S \mathrm{cof}(S \xrightarrow{\overline{v}_n} S),$$

and then

$$M/\overline{J}_n := M \otimes_S S/\overline{J}_n.$$

In particular,

$$BP\mathbb{R}\langle n\rangle/\bar{J}_n = BP\mathbb{R}\langle n\rangle/\bar{v}_n/\bar{v}_{n-1}/\cdots/\bar{v}_1 \simeq H\underline{\mathbb{Z}}$$

by the C_2–case of the reduction theorem, here proved as Corollary 4.7.

If $BP\mathbb{R}\langle n\rangle$ is a ring spectrum,

$$\mathrm{Hom}_{BP\mathbb{R}\langle n\rangle}(H\underline{\mathbb{Z}}, M) = \mathrm{Hom}_{BP\mathbb{R}\langle n\rangle}(BP\mathbb{R}\langle n\rangle \otimes_S S/\bar{J}_n, M) = \mathrm{Hom}_S(S/\bar{J}_n, M).$$

The right-hand side gives a way for us to express the fact that certain $BP\mathbb{R}\langle n\rangle$–modules (such as $BP\mathbb{R}\langle n\rangle$ and $\mathbb{Z}^{BP\mathbb{R}\langle n\rangle}$) are Matlis lifts, using only module structures over S.

Applying this when $M = BP\mathbb{R}\langle n\rangle$, we obtain the Gorenstein condition.

Lemma 7.1 *The map $BP\mathbb{R}\langle n\rangle \to H\underline{\mathbb{Z}}$ is Gorenstein of shift $-D_n\rho - n$ in the sense that*

$$\mathrm{Hom}_S(S/\bar{J}_n, BP\mathbb{R}\langle n\rangle) \simeq \Sigma^{-D_n\rho-n} H\underline{\mathbb{Z}},$$

where

$$D_n\rho = |\bar{v}_n| + |\bar{v}_{n-1}| + \cdots + |\bar{v}_1| = [2^{n+1} - n - 2]\rho.$$

Proof Since each of the maps $\bar{v}_i\colon \Sigma^{|\bar{v}_i|} S \to S$ is self-dual, for any S–module M, we have

$$\mathrm{Hom}_S(S/\bar{J}_n, M) \simeq \Sigma^{-D_m\rho-n} S/\bar{J}_n \otimes_S M. \qquad \square$$

Applying this when $M = \mathbb{Z}^{BP\mathbb{R}\langle n\rangle}$, we obtain the Anderson Matlis lift.

Lemma 7.2 *The Anderson dual of $BP\mathbb{R}\langle n\rangle$ is a Matlis lift of $H\underline{\mathbb{Z}}^*$ in the sense that*

(i) *$\mathbb{Z}^{BP\mathbb{R}\langle n\rangle}$ is $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellular, and*

(ii) *there is an equivalence*

$$\Sigma^{2-2\sigma} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}}^* \simeq \mathrm{Hom}_S(S/\bar{J}_n, \mathbb{Z}^{BP\mathbb{R}\langle n\rangle}).$$

Proof One could prove the first part from the slice tower, but it also follows directly from Corollary 3.9.

For the second statement, observe that

$$\mathrm{Hom}_S(S/\bar{J}_n, \mathbb{Z}^{BP\mathbb{R}\langle n\rangle}) \simeq \mathrm{Hom}_S(S/\bar{J}_n \otimes_S BP\mathbb{R}\langle n\rangle, \mathbb{Z}^S) \simeq \mathbb{Z}^{H\underline{\mathbb{Z}}}.$$

Thus, Lemma 3.1 implies the statement. $\square$

7B Gorenstein duality

Throughout this section, we will write $R = BP\mathbb{R}\langle n\rangle$ for brevity. Combining Lemmas 7.1 and 7.2, we have an equivalence of S–modules

$$\mathrm{Hom}_S(S/\bar{J}_n, R) \simeq \Sigma^{-D_n\rho - n} H\underline{\mathbb{Z}} \simeq \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{-(D_n+n+2)-(D_n-2)\sigma}\mathbb{Z}^R).$$

We want to remove the $\mathrm{Hom}_S(S/\bar{J}_n, \cdot)$ from this equivalence. The endomorphism ring

$$\widetilde{\mathcal{E}}_n = \mathrm{Hom}_S(S/\bar{J}_n, S/\bar{J}_n)$$

of the small S–module $S/\bar{J}_n$, replaces $\mathcal{E}_n = \mathrm{Hom}_R(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}})$ from the case that $R = BP\mathbb{R}\langle n\rangle$ is a ring spectrum. We note that

$$\widetilde{\mathcal{E}}_n \otimes_S R = \mathrm{Hom}_S(S/\bar{J}_n, S/\bar{J}_n) \otimes_S R \simeq \mathrm{Hom}_S(S/\bar{J}_n, S/\bar{J}_n) \otimes_S R).$$

If $R = BP\mathbb{R}\langle n\rangle$ were a commutative ring, then this would be a ring equivalent to $\mathrm{Hom}_R(H\underline{\mathbb{Z}}, H\underline{\mathbb{Z}})$.

In any case, the following is proved exactly like Lemma 6.5.

Lemma 7.3 (effective constructibility) *The evaluation map*

$$\mathrm{Hom}_S(S/\bar{J}_n, M) \otimes_{\widetilde{\mathcal{E}}_n} S/\bar{J}_n \to M$$

is an $S/\bar{J}_n$-$\mathbb{R}$–cellularization. □

Of course local cohomology gives an alternative approach to cellularization. Recall that we define

$$\Gamma_{\bar{J}_n} M = \Gamma_{\bar{v}_1} S \otimes_S \Gamma_{\bar{v}_2} S \otimes_S \cdots \otimes_S \Gamma_{\bar{v}_n} S \otimes_S M.$$

Then Proposition 3.8 gives the following lemma.

Lemma 7.4
$$\Gamma_{\bar{J}_n} M \to M$$
is an $H\underline{\mathbb{Z}}$-$\mathbb{R}$–cellularization.

It remains to check that the two $\widetilde{\mathcal{E}}_n$ actions on $H\underline{\mathbb{Z}}$ coincide. For $k\mathbb{R}$ (ie $n = 1$), we showed there was a unique right $\mathcal{E}_n$–module structure on $H\underline{\mathbb{Z}}$. This may be true for $\widetilde{\mathcal{E}}_n$–module structures, but we will instead just prove in the next subsection that the two particular $\widetilde{\mathcal{E}}_n$–modules that arose from the left and right-hand ends of the first display of this subsection are equivalent.

The required Gorenstein duality statement follows. Its implications for the coefficient ring for $n = 2$ are investigated explicitly in Section 13.

Corollary 7.5 (Gorenstein duality) *There is an equivalence of $M\mathbb{R}$–modules*

$$\Gamma_{\bar{J}_n} R \simeq \Sigma^{-(D_n+n+2)-(D_n-2)\sigma} \mathbb{Z}^R$$

with $R = BP\mathbb{R}\langle n\rangle$.

Proof We will argue in Section 7C that the equivalence

$$\mathrm{Hom}_S(S/\bar{J}_n, R) \simeq \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{-D_n\rho-n-2\delta}\mathbb{Z}^R),$$

is in fact an equivalence of right modules over $\widetilde{\mathcal{E}}_n$. By Lemma 7.3, we see that R and $\Sigma^{-(D_n+n+2)-(D_n-2)\sigma}\mathbb{Z}^R$ have equivalent $S/\bar{J}_n$ cellularizations. We have seen above that the cellularization of R is $\Gamma_{\bar{J}_n} BP\mathbb{R}\langle n\rangle$ and that $\Sigma^{-D_n\rho-n-2\delta}\mathbb{Z}^R$ itself is cellular. □

7C The equivalence of induced and coinduced Matlis lifts of $H\underline{\mathbb{Z}}$

For brevity, we will still write $R = BP\mathbb{R}\langle n\rangle$, and note that we have a map $S = M\mathbb{R} \to BP\mathbb{R}\langle n\rangle = R$. The two S–modules that concern us are of a very special sort, one looks as if it is obtained from an S–module by "extension of scalars from S to R" and one looks as if it is obtained by "coextension of scalars from S to R".

Lemma 7.6 *We have equivalences of right $\widetilde{\mathcal{E}}_n$–modules*

$$\mathrm{Hom}_S(S/\bar{J}_n, R) \simeq \mathrm{Hom}_S(S/\bar{J}_n, S) \otimes_S R,$$
$$\mathrm{Hom}_S(S/\bar{J}_n, \mathbb{Z}^R) = \mathrm{Hom}_S(R, \mathrm{Hom}_S(S/\bar{J}_n, \mathbb{Z}^S)).$$

Proof The first equivalence is immediate from the smallness of $S/\bar{J}_n$.

The second equivalence is a consequence of the following equivalence of S–modules:

$$\mathbb{Z}^R \simeq \mathrm{Hom}_S(R, \mathbb{Z}^S).$$ □

Suspending the equivalences from Lemma 7.6 so that we are comparing two $\widetilde{\mathcal{E}}_n$–modules equivalent to $H\underline{\mathbb{Z}}$ (see Lemma 7.2), we have

$$Y_1 = \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{D_n\rho+n} R) \simeq \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{D_n\rho+n} S) \otimes_S R = X_1 \otimes_S R$$

and

$$Y_2 = \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{2\delta}\mathbb{Z}^R) \simeq \mathrm{Hom}_S(S, \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{2\delta}\mathbb{Z}^S)) = \mathrm{Hom}_S(R, X_2).$$

In Section 7D, we will construct an $\widetilde{\mathcal{E}}_n$–map $\alpha : X_1 \to Y_2$ and then argue in Section 7E that this extends along $X_1 = X_1 \otimes_S S \to X_1 \otimes_S R = Y_1$ to give a map $\tilde{\alpha}$: $Y_1 \to Y_2$

which is easily seen to be an equivalence: it is clearly a $*\rho-$ isomorphism and hence an equivalence by Lemma 4.11.

To see our strategy, note that the extension problem

$$\begin{tikzcd} X_1 \arrow[r, "\alpha"] \arrow[d] & \mathrm{Hom}_S(S/\bar{J}_n, \mathrm{Hom}_S(R, \mathbb{Z}^S)) \\ X_1 \otimes_S R \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd}$$

in the category of $\widetilde{\mathcal{E}}_n$–modules is equivalent to the extension problem

$$\begin{tikzcd} X_1 \otimes_{\widetilde{\mathcal{E}}_n} S/\bar{J}_n \otimes_S R \arrow[r, "\alpha'"] \arrow[d] & \mathbb{Z}^S \\ X_1 \otimes_{\widetilde{\mathcal{E}}_n} S/\bar{J}_n \otimes_S R \otimes_S R \arrow[ur, dashed, "\tilde{\alpha}'"'] & \end{tikzcd}$$

in the category of S–modules. The point is that by the defining property of the Anderson dual, this latter extension problem can be tackled by looking in $\pi_0^{C_2}$. The 0^{th} homotopy groups of the spectra on the left are easily calculated from the known ring $\pi_\star^{C_2}(H\underline{\mathbb{Z}})$.

7D Construction of the map α

We construct the map α using a similar method as in the proof of Lemma 6.7.

Lemma 7.7 *There is a map*

$$\alpha\colon X_1 \to Y_2$$

of right $\widetilde{\mathcal{E}}_n$–modules that takes the image of $1 \in \pi_0^{C_2}(S)$ to a generator of $\pi_0^{C_2}(H\underline{\mathbb{Z}}) = \mathbb{Z}$.

Proof First we claim that X_1 has a $\widetilde{\mathcal{E}}_n$–cell structures with one 0–cell and other cells in dimensions which are negative multiples of ρ. More precisely, there is a filtration

$$\widetilde{\mathcal{E}}_n \simeq X_1^{[0]} \to X_1^{[1]} \to X_1^{[2]} \to \cdots \to X_1$$

such that $X_1 \simeq \underrightarrow{\mathrm{holim}}_d X_1^{[d]}$, and there are cofibre sequences

$$X_1^{[d-1]} \to X_1^{[d]} \to \bigvee \Sigma^{-d\rho} \widetilde{\mathcal{E}}_n.$$

By definition, $X_1 = \mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{D_n\rho+n} S)$. By Proposition 3.8 and Lemma 3.6, this is equivalent to

$$\mathrm{Hom}_S(S/\bar{J}_n, \Sigma^{D_n\rho+n} \Gamma_{\bar{J}_n} S) \simeq \mathrm{Hom}_S(S/\bar{J}_n, \kappa_S(\bar{v}_1, \ldots, \bar{v}_n))$$

because $\Gamma_{\bar{J}_n} S \to S$ is an $S/\bar{J}_n$-$\mathbb{R}$–cellularization. The usual construction of the stable Koszul complex from the unstable Koszul complex recalled in Section 3C, shows that

$$\kappa_S(\bar{v}_1, \dots, \bar{v}_n)$$

has a filtration with subquotients sums of $(-k\rho)$–fold suspensions of $S/\bar{J}_n$. This induces a corresponding filtration on X_1.

As in Lemma 6.7 we may construct α by obstruction theory. Indeed, we start by choosing a map $\widetilde{\mathcal{E}}_n = X_1^{[0]} \to Y_2^{[0]}$ taking the unit to a generator. At the d^{th} stage we have a problem:

$$\begin{tikzcd} X_1^{[d-1]} \arrow[r] \arrow[d] & Y_2 \\ X_1^{[d]} \arrow[ur, dashed] & \end{tikzcd}$$

The obstruction to extension is in a finite product of groups

$$[\Sigma^{-d\rho-1}\widetilde{\mathcal{E}}_n, Y_2]^{\widetilde{\mathcal{E}}_n} = \pi^{C_2}_{-d\rho-1}(H\underline{\mathbb{Z}}) = 0,$$

where the vanishing is from the known value of $\pi^{C_2}_\star(H\underline{\mathbb{Z}})$. $\square$

7E The map $\widetilde{\alpha}$

Referring to the second extension problem diagram above, we note $S/\bar{J}_n \otimes_S R \simeq H\underline{\mathbb{Z}}$ as S–modules. Thus, we have to solve the lifting problem

$$\begin{tikzcd} X_1 \otimes_{\widetilde{\mathcal{E}}_n} H\underline{\mathbb{Z}} \otimes_S S \arrow[r, "\alpha'"] \arrow[d, "1\otimes 1\otimes\pi"'] & \mathbb{Z}^S \\ X_1 \otimes_{\widetilde{\mathcal{E}}_n} H\underline{\mathbb{Z}} \otimes_S R \arrow[ur, dashed, "\widetilde{\alpha}'"'] & \end{tikzcd}$$

where $H\underline{\mathbb{Z}}$ is equipped with some $\widetilde{\mathcal{E}}_n$–module structure. Denote the upper left corner by T. The map $T \to T \otimes_S R$ is a split inclusion on underlying MU–modules. Indeed,

$$T \simeq X_1 \otimes_{\widetilde{\mathcal{E}}_n} S/\bar{J}_n \otimes_S R,$$

and the map $R \to R \otimes_S R$ is a split inclusion on underlying spectra because $BP\langle n\rangle$ has the structure of a homotopy unital MU–algebra [10, Theorem V.2.6].

By the definition of Anderson duals, we have a diagram of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\pi_{-1}^{C_2}(T \otimes_S R), \mathbb{Z}) \arrow[r] \arrow[d] & {[T \otimes_S R, \mathbb{Z}^S]^S} \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\pi_0^{C_2}(T \otimes_S R), \mathbb{Z}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\pi_{-1}^{C_2}(T), \mathbb{Z}) \arrow[r] & {[T, \mathbb{Z}^S]^S} \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\pi_0^{C_2}(T), \mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

We want to show that the maps $\pi_k^{C_2} T \to \pi_k^{C_2} T \otimes_S R$ are split injections for $k = 0, -1$, which solves the problem. For the computation of $\pi_*^{C_2} T$, recall from the last section that X_1 has a filtration starting with $X_1^{[x]} = \widetilde{\mathcal{E}}_n$ and with subquotients sums of terms of the form $\Sigma^{-d\rho}\widetilde{\mathcal{E}}_n$. Thus, T obtains a filtration starting with $T^{[1]} = H\underline{\mathbb{Z}}$ and with subquotients sums of terms of the form $\Sigma^{-d\rho} H\underline{\mathbb{Z}}$. The map $H\underline{\mathbb{Z}} = T^{[1]} \to T$ clearly induces isomorphisms on $\underline{\pi}_k^{C_2}$ for $k = 0, -1$ by the known homotopy groups of $H\underline{\mathbb{Z}}$; see eg [28, Section 3.4] for a table. Thus, $\underline{\pi}_{-1}^{C_2} T = 0$ and $\underline{\pi}_0^{C_2} T = \underline{\mathbb{Z}}$.

If we have a map $\underline{\mathbb{Z}} \to M$ from the constant Mackey functor, it is a split injection on (C_2/C_2) if it is one on (C_2/e). But we have already seen above that on underlying spectra $T \to T \otimes_S R$ is a split inclusion. Thus, we have shown that $\pi_k^{C_2} T \to \pi_k^{C_2}(T \otimes_S R)$ is split injective, which provides the map $\widetilde{\alpha}'$.

Part III The hands-on approach

In this part, we give a different way to compute the Anderson dual of $BP\mathbb{R}\langle n\rangle$ by first computing the Anderson dual of $BP\mathbb{R}$ itself. Again, we will first do the case of $k\mathbb{R}$.

8 The case of $k\mathbb{R}$ again

To illustrate our strategy, we give an alternative calculation of the Anderson dual of $k\mathbb{R}$. This can also be deduced from our main theorem below, but it might be helpful to see the proof in this simpler case first. General references for the $RO(C_2)$–graded homotopy groups of $k\mathbb{R}$ are [7, Section 3.7] or Section 11B.

We want to show the following proposition:

Proposition 8.1 *There is an equivalence* $\kappa_{k\mathbb{R}}(\bar{v}) \to \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}$.

Recall here that $\bar{v} \in \pi_\rho^{C_2} k\mathbb{R}$ is the Bott element for real K-theory, and

$$\kappa_{k\mathbb{R}}(\bar{v}) = \operatorname*{hocolim}_n \Sigma^{-(n-1)\rho} k\mathbb{R}/\bar{v}^n.$$

Our idea is simple: to get a map from the homotopy colimit, we have just to give maps

$$\Sigma^{-(n-1)\rho} k\mathbb{R}/\bar{v}^n \to \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}$$

that are compatible in the homotopy category (see Remark 3.7). We will show in the next lemma that these maps are essentially unique: the Mackey functor of homotopy classes of $k\mathbb{R}$–linear maps $\Sigma^{-(n-1)\rho}k\mathbb{R}/\bar{v}^n \to \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}$ is isomorphic to $\underline{\mathbb{Z}}$ and the precomposition with the map $\Sigma^{-(n-1)\rho}k\mathbb{R}/\bar{v}^n \to \Sigma^{-n\rho}k\mathbb{R}/\bar{v}^{n+1}$ induces the identity on $\underline{\mathbb{Z}}$. Choosing the C_2–equivariant map $\kappa_{k\mathbb{R}}(\bar{v}) \to \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}$ that corresponds to $1 \in \mathbb{Z}$ for every n induces an equivalence on underlying homotopy groups. By Lemma 4.11, the result follows as soon as we have established that $\kappa_{k\mathbb{R}}(\bar{v})$ is strongly even and that the Mackey functor $\underline{\pi}_{*\rho}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}$ is constant. These two facts will also be shown in the following lemma, finishing the proof of the proposition.

Lemma 8.2 *For a $\underline{\mathbb{Z}}[\bar{v}]$ module M, denote by $\{M\}_{\bar{v}^n}$ its $\bar{v}^n$–torsion. Then we have:*

(1) *$k\mathbb{R}/\bar{v}^n$ is strongly even, and hence the same is true for $\kappa_{k\mathbb{R}}(\bar{v})$.*

(2) *$\underline{\pi}^{C_2}_{n\rho}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}} \cong \underline{\pi}^{C_2}_{(n-2)\rho+4}\mathbb{Z}^{k\mathbb{R}}$ is constant for all $n \in \mathbb{Z}$.*

(3) *$\underline{[\Sigma^{-(n-1)\rho}k\mathbb{R}/\bar{v}^n, \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}]}^{C_2}_{k\mathbb{R}} \cong \{\underline{\pi}^{C_2}_{-(n-1)\rho}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}\}_{\bar{v}^n} \cong \underline{\mathbb{Z}}$.*

Proof The first part follows as

$$\underline{\pi}^{C_2}_{k\rho-i}(k\mathbb{R}/\bar{v}^n) = \underline{\pi}^{C_2}_{k\rho-i}(k\mathbb{R})/\bar{v}^n$$

for $i = 0, 1$ because $\pi^{C_2}_{k\rho-i}k\mathbb{R} = 0$ for $i = 1, 2$.

For the second part, consider the short exact sequence

$$0 \to \operatorname{Ext}(\underline{\pi}^{C_2}_{k\rho-5}k\mathbb{R}, \mathbb{Z}) \to \underline{\pi}^{C_2}_{-k\rho+4}\mathbb{Z}^{k\mathbb{R}} \to \operatorname{Hom}(\underline{\pi}^{C_2}_{k\rho-4}k\mathbb{R}, \mathbb{Z}) \to 0.$$

We have $\underline{\pi}^{C_2}_{k\rho-5}k\mathbb{R} = 0$ for all $k \in \mathbb{Z}$. For $k < 2$, the Mackey functor $\underline{\pi}^{C_2}_{k\rho-4}k\mathbb{R}$ vanishes as well and for $k \geq 2$, we have $\underline{\pi}^{C_2}_{k\rho-4}k\mathbb{R} \cong \underline{\mathbb{Z}}^*$, generated by v^{k-2} and $2\bar{v}^{k-2}u$. Thus,

$$\underline{\pi}^{C_2}_{-k\rho+4}\mathbb{Z}^{k\mathbb{R}} \cong \begin{cases} 0 & \text{if } k < 2, \\ \underline{\mathbb{Z}} & \text{if } k \leq 2. \end{cases}$$

This shows part (2). As multiplication by $\bar{v}^n$ does not hit $\underline{\pi}^{C_2}_{(n+1)\rho-4}k\mathbb{R}$, the whole Mackey functor $\underline{\pi}^{C_2}_{-(n+1)\rho+4}\mathbb{Z}^{k\mathbb{R}}$ is $\bar{v}^n$–torsion. This gives the second isomorphism of the third part.

For the remaining isomorphism, note that the cofibre sequence

$$\Sigma^{\rho}k\mathbb{R} \xrightarrow{\bar{v}^n} \Sigma^{-(n-1)\rho}k\mathbb{R} \to \Sigma^{-(n-1)\rho}k\mathbb{R}/\bar{v}^n \to \Sigma^{\rho+1}k\mathbb{R}$$

induces a short exact sequence

$$0 \to (\underline{\pi}^{C_2}_{\rho+1}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}})/\bar{v}_n \to \underline{[\Sigma^{-(n-1)\rho}k\mathbb{R}/\bar{v}^n, \Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}]}^{C_2}_{k\mathbb{R}} \to \{\underline{\pi}^{C_2}_{-(n-1)\rho}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}}\}_{\bar{v}^n} \to 0.$$

We have $\underline{\pi}^{C_2}_{\rho+1}\Sigma^{2\rho-4}\mathbb{Z}^{k\mathbb{R}} \cong \underline{\pi}^{C_2}_{5-\rho}\mathbb{Z}^{k\mathbb{R}}$, which sits in a short exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(\underline{\pi}^{C_2}_{\rho-6}k\mathbb{R}, \mathbb{Z}) \to \underline{\pi}^{C_2}_{5-\rho}\mathbb{Z}^{k\mathbb{R}} \to \mathrm{Hom}_{\mathbb{Z}}(\underline{\pi}^{C_2}_{\rho-5}k\mathbb{R}, \mathbb{Z}) \to 0.$$

But because of connectivity, $\underline{\pi}^{C_2}_{\rho-c}k\mathbb{R} = 0$ for $c \geq 3$. □

9 Duality for $BP\mathbb{R}$

We will use throughout the abbreviation $B = BP\mathbb{R}$ and will furthermore implicitly localize everything at 2, so $\mathbb{Z} = \mathbb{Z}_{(2)}$ etc, and all Hom and Ext groups are over $\mathbb{Z} = \mathbb{Z}_{(2)}$ unless marked otherwise. Denote by $\underline{\bar{v}}$ a sequence of indecomposable elements $\bar{v}_i \in \pi^{C_2}_{(2^i-1)\rho}B$. The aim of this section is to show that $\Sigma^{2\rho-4}\mathbb{Z}^B \simeq \kappa_{M\mathbb{R}}(\underline{\bar{v}}; B)$.

Recall that $\kappa_{M\mathbb{R}}(\underline{\bar{v}}; B)$ is defined as follows: Given a sequence $\underline{l} = (l_1, l_2, \dots)$ with $l_i \geq 0$, we denote by $B/\underline{\bar{v}}^{\underline{l}}$ the spectrum $B/(\bar{v}_{i_1}^{l_{i_1}}, \bar{v}_{i_2}^{l_{i_2}}, \dots)$, where i_j runs over all indices such that $l_{i_j} > 0$. Set

$$|\underline{l}| = l_1|\bar{v}_1| + l_2|\bar{v}_2| + \cdots .$$

Then

$$\kappa_{M\mathbb{R}}(\underline{\bar{v}}; B) = \operatorname*{hocolim}_{\underline{l}} \Sigma^{-|\underline{l}-\underline{1}|}B/\underline{\bar{v}}^{\underline{l}},$$

where $\underline{l}$ runs over all sequences such that all but finitely many l_i are zero, and $\underline{1}$ denotes the constant sequence of ones. Furthermore, the i^{th} entry of $\underline{l}-\underline{1}$ is defined to be the maximum of 0 and $l_i - 1$.

Thus, to get a map $\kappa_{M\mathbb{R}}(\underline{\bar{v}}; B) \to \Sigma^{2\rho-4}\mathbb{Z}^B$, we have to understand the homotopy classes of maps $B/\underline{\bar{v}}^{\underline{l}} \to \Sigma^{2\rho-4}\mathbb{Z}^B$. This will be the content of the next subsection.

9A Preparation

Recall the Mackey functor $\underline{\mathbb{Z}}^*$ defined by

$$\underline{\mathbb{Z}}^*(C_2/C_2) \cong \underline{\mathbb{Z}}^*(C_2/e) \cong \mathbb{Z}$$

with transfer equalling 1 while restriction is multiplication by 2.

Lemma 9.1 *As $\mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]$–modules, we have the following isomorphisms:*

(1) $\underline{\pi}^{C_2}_{*\rho-4}B \cong \underline{\mathbb{Z}}^* \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]$, *where $\underline{\mathbb{Z}}^*$ is generated by 1 on underlying and by $2u^{-1}$ on C_2–equivariant homotopy groups.*

(2) $\underline{\pi}^{C_2}_{*\rho-5}B = 0$.

(3) $\pi^{C_2}_{*\rho-6}B \cong \mathbb{F}_2\{a^2\bar{v}_1(-1)\} \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]$.

Proof By Theorem A.4, the groups $\pi^{C_2}_{*\rho-c}B$ are additively generated by nonzero elements of the form $x = a^l\bar{v}$ with $\bar{v}$ a monomial in the $\bar{v}_i(j)$. Let $\bar{v}_i(j)$ be the one occurring with minimal i, where j is chosen such that $\bar{v} = \bar{v}_i(j)\bar{v}'$ with $\bar{v}'$ a monomial in the $\bar{v}_k$ (this is possible by the third relation in Theorem A.4). Then $|x| = *\rho + j2^{i+2} + l$ and $0 \le l < 2^{i+1} - 1$.

For $c = 4$, this implies $j = -1$, $i = 0$ and $l = 0$. Thus, x is of the form $\bar{v}_0(-1)\bar{v}'$. As the restriction of $\bar{v}_0(-1)$ to $\pi^e_0 B$ equals 2, the result follows.

For $c = 5$, we must have $l \ge 2^{i+2} - 5$, which implies $l \ge 2^{i+1} - 1$ or $i = 0$; in the latter case l must be zero, which is not possible.

For $c = 6$, we must have $l = -j2^{i+2} - 6$, which implies $l \ge 2^{i+1} - 1$ or $i \le 1$ and $j = -1$. As $i = 0$ is again not possible, $x = a^2\bar{v}_1(-1)\bar{v}'$ with $\bar{v}' \in \pi^{C_2}_{*\rho}$. □

Lemma 9.2 *For a sequence $\underline{l} = (l_1, l_2, \dots)$, the map*

$$\underline{\pi}^{C_2}_{*\rho+4}\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{l}}} \to \mathrm{Hom}(\underline{\pi}^{C_2}_{-*\rho-4}B/\underline{\bar{v}}^{\underline{l}}, \mathbb{Z}) \cong \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} (\mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]/\underline{\bar{v}}^{\underline{l}})^*$$

is an isomorphism, where $\mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots], \mathbb{Z})$ (so that the gradings become nonpositive). Here, the second map is the dual of the map

$$\underline{\mathbb{Z}}^* \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]/\underline{\bar{v}}^{\underline{l}} \to \underline{\pi}^{C_2}_{-*\rho-4}B/\underline{\bar{v}}^{\underline{l}}$$

sending $1 \in \underline{\mathbb{Z}}^*(C_2/C_2)$ to the image of u^{-1} under the map $B \to B/\underline{\bar{v}}^{\underline{l}}$ and sending $1 \in \underline{\mathbb{Z}}^*(C_2/e)$ to 1.

Proof We have a short exact sequence

$$0 \to \mathrm{Ext}(\underline{\pi}^{C_2}_{-*\rho-5}B/\underline{\bar{v}}^{\underline{l}}, \mathbb{Z}) \to \underline{\pi}^{C_2}_{*\rho-4}\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{l}}} \to \mathrm{Hom}(\underline{\pi}^{C_2}_{-*\rho-4}B/\underline{\bar{v}}^{\underline{l}}, \mathbb{Z}) \to 0.$$

If $l_1 = 0$, then Corollary 4.5 and Lemma 9.1 directly imply the statement. If $l_1 \neq 0$, Corollary 4.5 only allows us to identify the homotopy Mackey functor in degree $-*\rho-4$, but not the one in degree $-*\rho-5$. We give a separate argument in this case.

If $l_1 \neq 0$, consider the sequence $\underline{l}' = (0, l_2, l_3, \dots)$ and the corresponding cofibre sequence

$$\Sigma^{l_1\rho}B/\underline{\bar{v}}^{\underline{l}'} \xrightarrow{\bar{v}_1^{l_1}} B/\underline{\bar{v}}^{\underline{l}'} \to B/\underline{\bar{v}}^{\underline{l}} \to \Sigma^{l_1\rho+1}B/\underline{\bar{v}}^{\underline{l}'}.$$

This induces a short exact sequence

$$0 \to (\underline{\pi}^{C_2}_{*\rho-5}B/\underline{\bar{v}}^{\underline{l}'})/\bar{v}_1^{l_1} \to \underline{\pi}^{C_2}_{*\rho-5}B/\underline{\bar{v}}^{\underline{l}} \to \{\underline{\pi}^{C_2}_{*\rho-6}B/\underline{\bar{v}}^{\underline{l}'}\}_{\bar{v}_1^{l_1}} \to 0.$$

Here the last term denotes the Mackey subfunctor of $\underline{\pi}^{C_2}_{*\rho-6}B/\underline{\bar{v}}^{\underline{l}'}$ killed by $\bar{v}_1^{l_1}$. By Corollary 4.5 and Lemma 9.1, we see that $\underline{\pi}^{C_2}_{*\rho-5}B/\underline{\bar{v}}^{\underline{l}} = 0$. □

As $B = BP\mathbb{R}$ is not known to have an E_∞–structure, we have to work with $M\mathbb{R}$–linear maps instead, for which the following lemma is useful:

Lemma 9.3 *The map*

$$\mathbb{Z}^B \simeq \mathrm{Hom}_{M\mathbb{R}}(M\mathbb{R}, \mathbb{Z}^B) \to \mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^B)$$

is an equivalence.

Proof Let $e\colon M\mathbb{R} \to M\mathbb{R}$ be the Quillen–Araki idempotent. Recall that

$$B = \mathrm{hocolim}(M\mathbb{R} \xrightarrow{e} M\mathbb{R} \xrightarrow{e} \cdots).$$

Thus,

$$\mathbb{Z}^B \simeq \underleftarrow{\mathrm{holim}}(\cdots \xrightarrow{e^*} \mathbb{Z}^{M\mathbb{R}} \xrightarrow{e^*} \mathbb{Z}^{M\mathbb{R}}).$$

Hence,

$$\mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^B) \simeq \underleftarrow{\mathrm{holim}}\big(\cdots \xrightarrow{e^*} \mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^{M\mathbb{R}}) \xrightarrow{e^*} \mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^{M\mathbb{R}})\big).$$

As every $\mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^{M\mathbb{R}})$ is equivalent to a holim over

$$\mathrm{Hom}_{M\mathbb{R}}(M\mathbb{R}, \mathbb{Z}^{M\mathbb{R}}) \simeq \mathbb{Z}^{M\mathbb{R}}$$

connected by e^*, we get that $\mathrm{Hom}_{M\mathbb{R}}(B, \mathbb{Z}^B)$ is the homotopy limit $\mathrm{holim}_{\mathbb{Z}^- \times \mathbb{Z}^-} \mathbb{Z}^{M\mathbb{R}}$, where $\mathbb{Z}^-$ denotes the poset of negative numbers and all connecting maps are e^*. This is equivalent to the homotopy limit indexed over the diagonal, which in turn is equivalent to the homotopy limit indexed over a vertical. □

Recall that we want to show that $X = \Sigma^{2\rho-4}\mathbb{Z}^B$ is equivalent to $\kappa_{M\mathbb{R}}(\bar{\underline{v}}, B)$. The reason for the choice of suspension is essentially (as before) that $H\underline{\mathbb{Z}} \simeq \Sigma^{2\rho-4} H\underline{\mathbb{Z}}^*$.

Proposition 9.4 *For a sequence $\underline{l} = (l_1, l_2, \dots)$, we have an isomorphism*

$$\underline{[\Sigma^{*\rho} B/\bar{\underline{v}}^{\underline{l}}, X]^{C_2}_{M\mathbb{R}}} \cong \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} (\mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]/\bar{\underline{v}}^{\underline{l}})^*,$$

natural with respect to the maps $B/\bar{\underline{v}}^{\underline{l}} \to \Sigma^{-|\underline{l}'-\underline{l}|\rho} B/\bar{\underline{v}}^{\underline{l}'}$ in the defining homotopy colimit for $\kappa_{M\mathbb{R}}(\bar{\underline{v}}; B)$ for $\underline{l}' = (l_1', l_2', \dots)$ a sequence with $l_i' \ge l_i$ for all $i \ge 1$.

Proof The last lemma implies that we also have

$$\mathbb{Z}^{B/\bar{\underline{v}}^{\underline{l}}} \simeq \mathrm{Hom}_{M\mathbb{R}}(B/\bar{\underline{v}}^{\underline{l}}, \mathbb{Z}^B)$$

as the functors $\mathbb{Z}^?$ and $\mathrm{Hom}_{M\mathbb{R}}(?, \mathbb{Z}^B)$ behave the same way with respect to cofibre sequences and (filtered) homotopy colimits. Then we just have to apply Lemma 9.2. □

9B The theorem

We first describe the homotopy groups of $X = \Sigma^{2\rho-4}\mathbb{Z}^B$ with $B = BP\mathbb{R}$ as before. By Lemma 9.2, we get

$$\underline{\pi}^{C_2}_{*\rho} X \cong \mathrm{Hom}(\underline{\pi}^{C_2}_{(*+2)\rho-4} B, \mathbb{Z}) \cong \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{v}_1, \bar{v}_2, \dots]^*.$$

Let $\underline{l}$ be a sequence with only finitely many nonzero entries. By Proposition 9.4, the element $(\underline{\bar{v}}^{\underline{l}-\underline{1}})^*$ induces a corresponding $M\mathbb{R}$–linear map $\Sigma^{-|\underline{l}-\underline{1}|} B/\underline{\bar{v}}^{\underline{l}} \to X$, which is unique up to homotopy. By this uniqueness, these maps are also compatible for comparable $\underline{l}$. By Remark 3.7, this induces a map

$$\kappa_{M\mathbb{R}}(\underline{\bar{v}}, B) = \operatorname*{hocolim}_{\underline{l}} (\Sigma^{-|\underline{l}-\underline{1}|} B/\underline{\bar{v}}^{\underline{l}}) \xrightarrow{h} X,$$

where $\underline{l}$ ranges over all sequences where only finitely many l_i are nonzero.

Theorem 9.5 *The map $h\colon \kappa_{M\mathbb{R}}(\underline{\bar{v}}; B) \to X$ is an equivalence of C_2–spectra.*

Proof By Corollary 4.6, we get on $\underline{\pi}_{*\rho}$–level

$$\operatorname*{colim}_{\underline{l}} \Sigma^{-|\underline{l}-\underline{1}|} \underline{\mathbb{Z}}[\bar{v}_1, \bar{v}_2, \dots]/(\bar{v}_1^{l_1}, \dots) \to \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}[\bar{v}_1, \dots]^*,$$

which is an isomorphism. The odd underlying homotopy groups of both sides are zero. To apply Lemma 4.11, it is left to show that $\pi^{C_2}_{k\rho-1}\kappa_{M\mathbb{R}}(\underline{\bar{v}}; B) = 0$ for all $k \in \mathbb{Z}$. Again by Corollary 4.6, it is even true that $\pi^{C_2}_{k\rho-1}(B/\underline{\bar{v}}^{\underline{l}})$ is zero for all $k \in \mathbb{Z}$ and all sequences $\underline{l}$. □

10 Duality for regular quotients

The goal of this section is to prove our main result Theorem 5.1:

Theorem 10.1 *Let $(m_1, m_2, \dots)$ be a sequence of nonnegative integers with only finitely many entries bigger than 1. Denote by $\underline{\bar{v}}'$ the sequence of $\bar{v}_i$ in $\pi^{C_2}_\star M\mathbb{R}$ such that $m_i = 0$ and by m' the sum of all $(m_i - 1)|\bar{v}_i|$ for $m_i > 1$. Then there is an equivalence*

$$\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{m}}} \simeq \Sigma^{-m'+4-2\rho} \kappa_{M\mathbb{R}}(\underline{\bar{v}}'; B/\underline{\bar{v}}^{\underline{m}}).$$

Here and for the rest of the section we will implicitly localize everything at 2 again. Before we prove the theorem, we need some preparation.

Lemma 10.2 *Let $\underline{m} = (m_1, \dots)$ be a sequence of nonnegative integers with a finite number n of nonzero entries. Then*

$$\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{m}}} \simeq \Sigma^{-|\underline{m}|-n}(\mathbb{Z}^B)/\underline{\bar{v}}^{\underline{m}}.$$

Proof Let Y be an arbitrary (C_2–)spectrum and $\Sigma^{|v|}Y \xrightarrow{v} Y \to Y/v$ be a cofibre sequence. Then we have an induced cofibre sequence

$$\mathbb{Z}^{Y/v} \to \mathbb{Z}^{Y} \xrightarrow{v} \Sigma^{-|v|}\mathbb{Z}^{Y} \to \Sigma\mathbb{Z}^{Y/v} \simeq \Sigma^{-|v|}(\mathbb{Z}^{Y})/v.$$

Thus, $\mathbb{Z}^{Y/v} \simeq \Sigma^{-|v|-1}(\mathbb{Z}^{Y})/v$. The claim follows by induction. □

Lemma 10.3 *The element $\bar{v}_i^{3k}$ acts trivially on $B/\bar{v}_i^k$ for every $i \geq 1$ and $k \geq 1$.*

Proof By the commutativity of the diagram

$$\begin{tikzcd} \Sigma^{k|\bar{v}_i|}B \arrow[r, "q"] \arrow[d, "\bar{v}_i^k"] & \Sigma^{k|\bar{v}_i|}B/\bar{v}_i^k \arrow[d, "\bar{v}_i^k"] \\ B \arrow[r] & B/\bar{v}_i^k \end{tikzcd}$$

we see that the composite $\bar{v}_i^k q$ is zero, and so the $\bar{v}_i^k$ on the right factors over an $M\mathbb{R}$–linear map $\Sigma^{2k|\bar{v}_i|+1}B \to B/\bar{v}_i^k$. As $[\Sigma^{2k|\bar{v}_i|+1}B, B/\bar{v}_i^k]_{M\mathbb{R}}$ is a retract of $[\Sigma^{2k|\bar{v}_i|+1}M\mathbb{R}, B/\bar{v}_i^k]_{M\mathbb{R}} \cong \pi^{C_2}_{2k|\bar{v}_i|+1}B/\bar{v}_i^k$, we just have to show that $\bar{v}_i^{2k}x = 0$ for every $x \in \pi_{2k|\bar{v}_i|+1}B/\bar{v}_i^k$.

We have a short exact sequence

$$0 \to (\pi^{C_2}_\star B)/\bar{v}_i^k \to \pi^{C_2}_\star(B/\bar{v}_i^k) \to \{\pi^{C_2}_{\star-k|\bar{v}_i|-1}B\}_{\bar{v}_i^k} \to 0.$$

As $\bar{v}_i^k x$ clearly maps to zero, it is the image of a $y \in (\pi^{C_2}_\star B)/\bar{v}_i^k$. But $\bar{v}_i^k y = 0$. □

Lemma 10.4 *We have*

$$B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_j^m \simeq B/(\bar{v}_i^l, \bar{v}_j^m).$$

Furthermore, there is an equivalence

$$\operatorname*{hocolim}_l \Sigma^{-(l-1)|\bar{v}_i|}B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_i^m \simeq \Sigma^{|\bar{v}_i|+1}B/\bar{v}_i^m$$

of $M\mathbb{R}$–modules if $m \geq 1$.

Proof We have

$$B \otimes_{M\mathbb{R}} B \simeq \operatorname{hocolim}(B \xrightarrow{e} B \xrightarrow{e} \cdots) \simeq B,$$

where e denotes again the Quillen–Araki idempotent, and thus also

$$B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_j^m \simeq B/(\bar{v}_i^l, \bar{v}_j^m).$$

Thus, the maps in the homotopy colimit in the lemma are induced by the following diagram of cofibre sequences:

$$\begin{tikzcd}
\Sigma^{|\bar{v}_i|} B/\bar{v}_i^m \arrow[r, "\bar{v}_i^l"] \arrow[d, "\mathrm{id}"] & \Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^m \arrow[r] \arrow[d, "\bar{v}_i"] & \Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_i^m \arrow[d] \\
\Sigma^{|\bar{v}_i|} B/\bar{v}_i^m \arrow[r, "\bar{v}_i^{l+1}"] & \Sigma^{-l|\bar{v}_i|} B/\bar{v}_i^m \arrow[r] & \Sigma^{-l|\bar{v}_i|} B/\bar{v}_i^{l+1} \otimes_{M\mathbb{R}} B/\bar{v}_i^m
\end{tikzcd}$$

We can assume that the homotopy colimit only runs over $l \geq 3m$ so that, by the last lemma, the two cofibre sequences split, and we get

$$\Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_i^m \simeq \Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^m \oplus \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m.$$

The corresponding map

$$\Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^m \oplus \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m \to \Sigma^{-l|\bar{v}_i|} B/\bar{v}_i^m \oplus \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m$$

induces multiplication by $\bar{v}_i$ on the first summand, the identity on the second plus possibly a map from the second summand to the first.

Using this decomposition, it is easy to show that

$$\operatorname*{hocolim}_l \Sigma^{-(l-1)|\bar{v}_i|} B/\bar{v}_i^l \otimes_{M\mathbb{R}} B/\bar{v}_i^m \to \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m$$

(defined by the projection on the second summand for $l \geq 3m$) is an equivalence. Indeed, on homotopy groups the map is clearly surjective. And if

$$(x, y) \in \pi_\star^{C_2} \Sigma^{-l|\bar{v}_i|} B/\bar{v}_i^m \oplus \pi_\star^{C_2} \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m$$

maps to $0 \in \pi_\star^{C_2} \Sigma^{|\bar{v}_i|+1} B/\bar{v}_i^m$, then $y = 0$ and $(x, 0)$ represents 0 in the colimit because $\bar{v}_i$ acts nilpotently. □

Proof of Theorem 10.1 As in the theorem, let $\underline{\bar{v}}'$ be the sequence of $\bar{v}_i$ such that $m_i = 0$ and also denote by $\underline{\bar{v}}'' = (\bar{v}_{i_1}, \bar{v}_{i_2}, \dots)$ the sequence of $\bar{v}_i$ such that $m_i \neq 0$.

We begin with the case that $\underline{m}$ has only finitely many nonzero entries (say n). By Lemma 10.2, we see that

$$\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{m}}} \simeq \Sigma^{-|\underline{m}|-n}(\mathbb{Z}^B)/\underline{\bar{v}}^{\underline{m}}.$$

Combining this with Theorem 9.5, we obtain

$$\begin{aligned}
\mathbb{Z}^{B/\underline{\bar{v}}^{\underline{m}}} &\simeq \Sigma^{-|\underline{m}|-n+4-2\rho} \kappa_{M\mathbb{R}}(\underline{\bar{v}}, B)/\underline{\bar{v}}^{\underline{m}} \\
&\simeq \Sigma^{-|\underline{m}|-n+4-2\rho} \kappa_{M\mathbb{R}}(\underline{\bar{v}}', \kappa_{M\mathbb{R}}(\underline{\bar{v}}'', B/\underline{\bar{v}}^{\underline{m}})).
\end{aligned}$$

Thus, we have to show that $\kappa_{M\mathbb{R}}(\underline{\bar{v}}'', B/\underline{\bar{v}}^{\underline{m}}) \simeq \Sigma^{|\bar{v}_{i_1}|+\cdots+|\bar{v}_{i_n}|+n} B/\underline{\bar{v}}^{\underline{m}}$.

By Lemma 10.4, we have an equivalence

$$(B/\underline{\bar{v}}^{\underline{m}})/(\bar{v}_{i_1}^{l_{i_1}},\dots,\bar{v}_{i_n}^{l_{i_n}}) \simeq (B/\bar{v}_1^{l_{i_1}} \otimes_{M\mathbb{R}} B/\bar{v}_1^{m_{i_1}}) \otimes_{M\mathbb{R}} \dots \otimes_{M\mathbb{R}} (B/\bar{v}_n^{l_{i_n}} \otimes_{M\mathbb{R}} B/\bar{v}_n^{m_{i_n}}).$$

If we let now the homotopy colimit run over the sequences $(l_{i_1},\dots,l_{i_n})$, we can do it separately for each tensor factor. Hence, we obtain again by Lemma 10.4 an equivalence

$$\kappa_{M\mathbb{R}}(\underline{\bar{v}}'', B/\underline{\bar{v}}^{\underline{m}}) \simeq \Sigma^{|\bar{v}_{i_1}|+\dots+|\bar{v}_{i_n}|+n} B/\underline{\bar{v}}^{\underline{m}}.$$

Thus, we have shown the theorem when $\underline{m}$ has only finitely many nonzero entries.

We prove the case that $\underline{m}$ has possibly infinitely many nonzero entries by a colimit argument. Define $\underline{m}_{\le k}$ to be the sequence obtained from $\underline{m}$ by setting $m_{k+1}, m_{k+2}, \dots$ to zero. Then $B/\underline{m} \simeq \operatorname{hocolim}_k B/\underline{m}_{\le k}$ and thus $\mathbb{Z}^{B/\underline{m}} \simeq \operatorname{holim}_k \mathbb{Z}^{B/\underline{m}_{\le k}}$. Denote by $\underline{\bar{v}}'_{\le k}$ the sequence of $\bar{v}_i$ such that $m_i = 0$ or $i > k$ and by m'_k the quantity $|\underline{m}_{\le k} - \underline{1}|$; note that $m'_k = m'$ for k large.

We have to show that the map

$$h\colon \Sigma^{-m'}\kappa_{M\mathbb{R}}(\underline{\bar{v}}', B/\underline{\bar{v}}^{\underline{m}}) \to \operatorname*{holim}_k \Sigma^{-m'_k}\kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\le k}, B/\underline{\bar{v}}^{\underline{m}_{\le k}})$$

is an equivalence. This map is defined as follows: We know that

$$\kappa_{M\mathbb{R}}(\underline{\bar{v}}', B/\underline{\bar{v}}^{\underline{m}}) \simeq \operatorname*{hocolim}_k \kappa_{M\mathbb{R}}(\underline{\bar{v}}', B/\underline{\bar{v}}^{\underline{m}_{\le k}}).$$

Using this, we get a map induced from the maps

$$\kappa_{M\mathbb{R}}(\underline{\bar{v}}', B/\underline{\bar{v}}^{\underline{m}_{\le k}}) \to \kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\le k}, B/\underline{\bar{v}}^{\underline{m}_{\le k}})$$

for k large.

By Corollary 4.6, we can describe what happens on $\pi_{*\rho}^{C_2}$: The left-hand side has as $\mathbb{Z}$–basis monomials of the form $\underline{\bar{v}}^{\underline{n}}$ with only finitely many n_i nonzero, $n_i \le 0$ and $n_i \ge -m_i + 1$ if $m_i \ne 0$. Likewise,

$$\pi_{*\rho}^{C_2}\big(\Sigma^{m'_k}\kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\le k}, B/\underline{\bar{v}}^{\underline{m}_{\le k}})\big)$$

has as $\mathbb{Z}$–basis monomials of the form $\underline{\bar{v}}^{\underline{n}}$ with only finitely many n_i nonzero, $n_i \le 0$ and $n_i \ge -m_i + 1$ if $m_i \ne 0$ and $i \le k$. The maps in the homotopy limit induce the obvious inclusion maps. Thus, clearly the map

$$\pi_{*\rho}^{C_2}\big(\Sigma^{m'}\kappa_{M\mathbb{R}}(\underline{\bar{v}}', B/\underline{\bar{v}}^{\underline{m}})\big) \to \lim_k \pi_{*\rho}^{C_2}\big(\Sigma^{m'_k}\kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\le k}, B/\underline{\bar{v}}^{\underline{m}_{\le k}})\big)$$

is an isomorphism.

It remains to show $\lim_k^1 \pi_{*\rho+1}^{C_2}\big(\Sigma^{m'_k}\kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\le k}, B/\underline{\bar{v}}^{\underline{m}_{\le k}})\big)$ vanishes. By Corollary 4.8, every term has as $\mathbb{F}_2$–basis monomials of the form $a\underline{\bar{v}}^{\underline{n}}$ with only finitely many n_i nonzero, $n_i \le 0$ and $n_i \ge -m_i + 1$ if $m_i \ne 0$ and $i \le k$. The system becomes stationary in every degree, more precisely if $* > -2^{k+1}$. Thus, the $\lim^1$–term vanishes.

A similar $\lim^1$–argument also shows that the odd underlying homotopy groups of $\operatorname{holim}_k \Sigma^{-m'_k} \kappa_{M\mathbb{R}}(\underline{\bar{v}}'_{\leq k}, B/\underline{\bar{v}}^{m_{\leq k}})$ vanish.

As the source of h is strongly even by Corollary 4.6 and by the arguments we just gave the morphism h induces an isomorphism on $\underline{\pi}^{C_2}_{*\rho}$ and on (odd) underlying homotopy groups, Lemma 4.11 implies that h is an equivalence. □

Part IV Local cohomology computations

In Part IV, we will describe the local cohomology spectral sequence in some detail, and use it to understand the structure of the $H\underline{\mathbb{Z}}$–cellularization of $BP\mathbb{R}\langle n\rangle$. The calculation is not difficult, but on the other hand it is quite hard to follow because it is made up of a large number of easy calculations which interact a little, and because one needs to find a helpful way to follow the $RO(C_2)$–graded calculations.

In contrast, the case of $k\mathbb{R}$ is simple enough to be explained fully without further scaffolding, and it introduces many of the structures that we will want to highlight. Since it may also be of wider interest than the general case of $BP\mathbb{R}\langle n\rangle$ we devote Section 11 to it before returning to the general case in Section 12. Section 13 will then give a more detailed account in the interesting case $n=2$.

Let us also recall some notation used throughout this part. As in the rest of the paper we work 2–locally, except when speaking about $k\mathbb{R}$ or $tmf_1(3)$ when fewer primes need be inverted. We often write $\delta = 1-\sigma \in RO(C_2)$. We also recall the duality conventions from Section 3A; in particular, for an $\mathbb{F}_2$–vector space $V^\vee$ equals the dual vector space $\operatorname{Hom}_{\mathbb{F}_2}(V, \mathbb{F}_2)$ and for a torsion-free $\mathbb{Z}$–module M, we set $M^* = \operatorname{Hom}(M, \mathbb{Z})$.

If R is a C_2–spectrum, we will use the notation $R^{C_2}_\star$ for its $RO(C_2)$–graded homotopy groups. We will also write $R^{hC_2}_\star = \pi^{C_2}_\star(R^{(EC_2)_+})$ and similarly for geometric fixed points and the Tate construction.

11 The local cohomology spectral sequence for $k\mathbb{R}$

This section focuses entirely on the classical case of $k\mathbb{R}$, where there are already a number of features of interest. This gives a chance to introduce some of the structures we will use for the general case.

11A The local cohomology spectral sequence

Gorenstein duality for $k\mathbb{R}$ (Corollary 6.8) has interesting implications for the coefficient ring, both computationally and structurally. Writing $\star$ for $RO(C_2)$–grading as usual, the local cohomology spectral sequence [11, Section 3] takes the following form.

Proposition 11.1 *There is a spectral sequence of $k\mathbb{R}^{C_2}_\star$–modules*

$$H^*_{(\bar{v})}(k\mathbb{R}^{C_2}_\star) \implies \Sigma^{-4+\sigma}\pi^{C_2}_\star(\mathbb{Z}^{k\mathbb{R}}).$$

The homotopy of the Anderson dual in an arbitrary degree $\alpha \in RO(C_2)$ lies in an exact sequence

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(k\mathbb{R}^{C_2}_{-\alpha-1}, \mathbb{Z}) \to \pi^{C_2}_\alpha(\mathbb{Z}^{k\mathbb{R}}) \to \mathrm{Hom}_{\mathbb{Z}}(k\mathbb{R}^{C_2}_{-\alpha}, \mathbb{Z}) \to 0.$$

Since local cohomology is entirely in cohomological degrees 0 and 1, the spectral sequence collapses to a short exact sequence

$$0 \to \Sigma^{-1} H^1_{(\bar{v})}(k\mathbb{R}^{C_2}_\star) \to \Sigma^{-4+\sigma}\pi^{C_2}_\star(\mathbb{Z}^{k\mathbb{R}}) \to H^0_{(\bar{v})}(k\mathbb{R}^{C_2}_\star) \to 0.$$

This sequence is not split, even as abelian groups.

One should not view Proposition 11.1 as an algebraic formality: it embodies the fact that $k\mathbb{R}^{C_2}_\star$ is a very special ring. To illustrate this, we recall the calculation of $k\mathbb{R}^{C_2}_\star$ in Section 11B. In Section 11C, we calculate its local cohomology, and how the Gorenstein duality isomorphism with the known homotopy of the Anderson dual works.

11B The ring $k\mathbb{R}^{C_2}_\star$

One may easily calculate $k\mathbb{R}^{C_2}_\star$. This has already been done in [7], but we sketch a slightly different method. We will first calculate $k\mathbb{R}^{hC_2}_\star$ and then use the Tate square [12].

In the homotopy fixed point spectral sequence

$$\mathbb{Z}[\bar{v}, a, u^{\pm 1}]/2a \implies k\mathbb{R}^{hC_2}_\star,$$

all differentials are generated by $d_3(u) = \bar{v}a^3$. Indeed, this differential is forced by $\eta^4 = 0$ and there is no room for further ones. It follows that $U = u^2$ is an infinite cycle, and so the whole ring is U–periodic:

$$k\mathbb{R}^{hC_2}_\star = BB[U, U^{-1}],$$

where BB is a certain "basic block". This basic block is a sum

$$BB = BR \oplus (2u)\cdot\mathbb{Z}[\bar{v}]$$

as BR–modules, where

$$BR = \mathbb{Z}[\bar{v}, a]/(2a, \bar{v}a^3).$$

It is worth illustrating BB in the plane (with $BB_{a+b\sigma}$ placed at the point (a,b)); see Figure 1. The squares and circles represent copies of $\mathbb{Z}$, and the dots represent

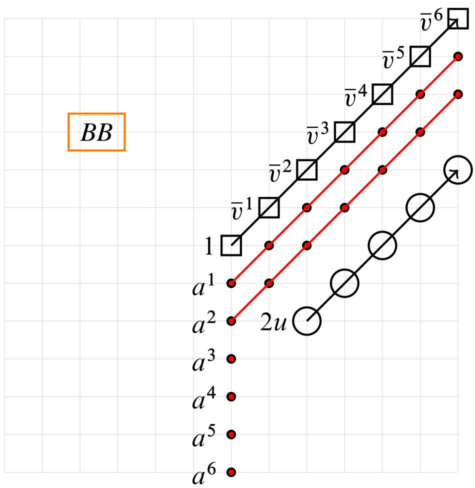

Figure 1: The basic block *BB*

copies of $\mathbb{F}_2$. The left-hand vertical column consists of 1 (at the origin, $(0,0)$) and the powers of a, but the feature to concentrate on is the diagonal lines representing $\mathbb{Z}[\overline{v}]$–submodules. These are either copies of $\mathbb{Z}[\overline{v}]$ or of $\mathbb{F}_2[\overline{v}]$ or simply copies of $\mathbb{F}_2$.

Proceeding with the calculation, we may invert a to find the homotopy of the Tate spectrum $k\mathbb{R}^t = F(E(C_2)_+, k\mathbb{R}) \wedge S^{\infty\sigma}$:

$$k\mathbb{R}^{tC_2}_\star = \mathbb{F}_2[a, a^{-1}][U, U^{-1}].$$

One also sees that the homotopy of the geometric fixed points (the equivariant homotopy of $k\mathbb{R}^\Phi = k\mathbb{R} \wedge S^{\infty\sigma}$) is

$$k\mathbb{R}^{\Phi C_2}_\star = \mathbb{F}_2[a, a^{-1}][U]$$

using the following lemma:

Lemma 11.2 *Let X be a C_2–spectrum which is nonequivariantly connective and such that $X^{C_2} \to X^{hC_2}$ is a connective cover. Then $X^{\Phi C_2} \to X^{tC_2}$ is a connective cover as well.*

Proof This follows from the diagram of long exact sequences

$$\begin{tikzcd}
\pi_k X_{hC_2} \arrow[r] \arrow[d] & \pi_k X^{C_2} \arrow[r] \arrow[d] & \pi_k X^{\Phi C_2} \arrow[r] \arrow[d] & \pi_{k-1} X_{hC_2} \arrow[r] \arrow[d] & \pi_{k-1} X^{C_2} \arrow[d] \\
\pi_k X_{hC_2} \arrow[r] & \pi_k X^{hC_2} \arrow[r] & \pi_k X^{tC_2} \arrow[r] & \pi_{k-1} X_{hC_2} \arrow[r] & \pi_{k-1} X^{hC_2}
\end{tikzcd}$$

the fact that X_{hC_2} is connective, and the five lemma. □

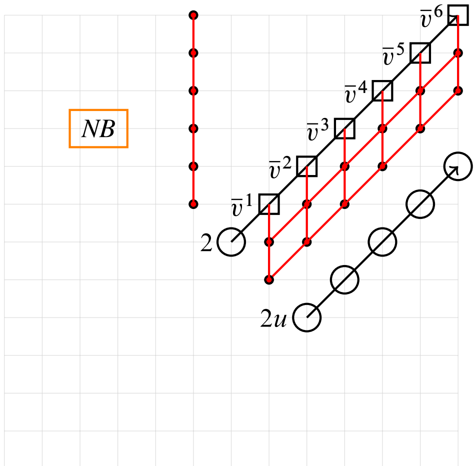

Figure 2: The negative block *NB*

Now the Tate square

$$\begin{tikzcd} k\mathbb{R} \arrow[r] \arrow[d] & k\mathbb{R} \wedge S^{\infty\sigma} \arrow[d] \\ k\mathbb{R}^{(EC_2)_+} \arrow[r] & k\mathbb{R}^{(EC_2)_+} \wedge S^{\infty\sigma} \end{tikzcd}$$

gives $k\mathbb{R}^{C_2}_{\star}$.

It is convenient to observe that the two rows are of the form $M \to M[1/a]$, so the fibre is $\Gamma_a M$. Since the two rows have equivalent fibres, we calculate the homotopy of the second and obtain

$$k\mathbb{R}^{\star}_{hC_2} = NB[U, U^{-1}],$$

where NB is quickly calculated as the (a)–local cohomology $H^*_{(a)}(BB)$ (and named NB for "negative block"). The element a acts vertically and we can immediately read off the answer: the tower $\mathbb{Z}[a]/(2a)$ gives some H^1, and the rest is a–power torsion:

$$NB = BB' \oplus \Sigma^{-\delta}\mathbb{F}_2[a]^\vee,$$

where $BB' \subset BB$ is the BR–submodule generated by 2, $\bar{v}$ and $2u$ (informally, we may say that BB' omits from BB all monomials a^k for $k \geq 1$ and the generator 1). Note that NB is placed so that its element 2 is in degree 0 for ease of comparison to BB; all occurrences of NB in $k\mathbb{R}^{C_2}_{\star}$ involve nontrivial suspensions.

Again, it is helpful to display the negative block; see Figure 2. This differs from BB in that the powers of a have been deleted, and replaced by a new left-hand column $\Sigma^{-\delta}\mathbb{F}_2[a]^\vee$. The other new feature is that the copy of $\mathbb{Z}[\bar{v}]$ generated by 1 has been replaced by the kernel $(2,\bar{v})$ of $\mathbb{Z}[\bar{v}] \to \mathbb{F}_2$, as indicated by the circle at the origin, labelled by its generator 2.

The Tate square then lets us read off

$$k\mathbb{R}^{C_2}_\star = \bigoplus_{k\le -1} NB\cdot\{U^k\} \oplus \bigoplus_{k\ge 0} BB\cdot\{U^k\} = (U^{-1}\cdot NB[U^{-1}]) \oplus BB[U].$$

The $\mathbb{Z}[U]$ module structure is given by letting U act in the obvious way on the NB and BB parts, and by the maps

$$NB \to BB' \to BB$$

in passage from the U^{-1} factor of NB to the U^0 factor of BB.

Perhaps it is helpful to note that with the exception of the towers $U^{-k}\Sigma^{-\delta}\mathbb{F}_2[a]^\vee$, we have a subring of $BB[U,U^{-1}]$, which consists of blocks $BB\cdot U^i$ for $i\ge 0$ and blocks $BB'\cdot U^i$ for $i<0$.

11C Local cohomology

Recall that we are calculating local cohomology with respect to the principal ideal $(\bar{v})$ so that we only need to consider $k\mathbb{R}^{C_2}_\star$ as a $\mathbb{Z}[\bar{v}]$–module. As such it is a sum of suspensions of the blocks BB and NB, so we just need to calculate the local cohomology of these.

More significantly, $\mathbb{Z}[\bar{v}]$ is graded over multiples of the regular representation, so local cohomology calculations may be performed on one diagonal at a time (ie we fix n and consider gradings $n+*\rho$). The only modules that occur are

$$\mathbb{Z}[\bar{v}], \quad \mathbb{F}_2[\bar{v}], \quad \mathbb{F}_2 \quad \text{and the ideal } (2,\bar{v}) \subseteq \mathbb{Z}[\bar{v}],$$

each of which has local cohomology that is very easily calculated.

Lemma 11.3 *The local cohomology of the basic block BB is as follows:*

$$H^0_{(\bar{v})}(BB) = a^3\mathbb{F}_2[a],$$
$$H^1_{(\bar{v})}(BB) = \Sigma^{-\rho}\mathbb{Z}[\bar{v}]^* \oplus \Sigma^{-\rho+2\delta}\mathbb{Z}[\bar{v}]^* \oplus \Sigma^{-\rho-\sigma}\mathbb{F}_2[\bar{v}]^\vee \oplus \Sigma^{-\rho-2\sigma}\mathbb{F}_2[\bar{v}]^\vee.$$

Proof The local cohomology is the cohomology of the complex

$$BB \to BB[1/\bar{v}].$$

It is clear that

$$BB[1/\bar{v}] = \mathbb{Z}[\bar{v},\bar{v}^{-1}] \oplus u\cdot\mathbb{Z}[\bar{v},\bar{v}^{-1}] \oplus a\cdot\mathbb{F}_2[\bar{v},\bar{v}^{-1}] \oplus a^2\cdot\mathbb{F}_2[\bar{v},\bar{v}^{-1}]. \qquad \square$$

Turning to NB, we recall that $NB = BB' \oplus \Sigma^{-\delta}\mathbb{F}_2[a]^\vee$, and we have a short exact sequence

$$0 \to BB' \to BB \to \mathbb{F}_2[a] \to 0.$$

The local cohomology is thus easily deduced from that of BB.

Lemma 11.4 *The local cohomology of the negative block NB is as follows:*

$$H^0_{(\bar{v})}(NB) = \Sigma^{-\delta}\mathbb{F}_2[a]^\vee,$$
$$H^1_{(\bar{v})}(NB) = \Sigma^{-\rho}\mathbb{Z}[\bar{v}]^* \oplus \mathbb{F}_2 \oplus \Sigma^{-\rho+2\delta}\mathbb{Z}[\bar{v}]^* \oplus \Sigma^{-\sigma}\mathbb{F}_2[\bar{v}]^\vee \oplus \Sigma^{-2\sigma}\mathbb{F}_2[\bar{v}]^\vee.$$

More properly, the $\mathbb{Z}[\bar{v}]$–module structure of the sum of the first two terms is

$$\Sigma^{-\rho}\mathbb{Z}[\bar{v}]^* \oplus \mathbb{F}_2 \cong \mathbb{Z}[\bar{v}]^*/(2\cdot 1^*).$$

Proof The local cohomology is the cohomology of the complex

$$NB \to NB[1/\bar{v}].$$

It is clear that $NB[1/\bar{v}] = BB[1/\bar{v}]$, which makes the part coming from the 2–torsion clear. For the $\mathbb{Z}$–torsion free part, it is helpful to consider the exact sequence

$$0 \to (2, \bar{v}) \to \mathbb{Z}[\bar{v}] \to \mathbb{F}_2 \to 0$$

and then consider the long exact sequence in local cohomology. □

Immediately from the defining cofibre sequence $\Gamma_{\bar{v}}k\mathbb{R} \to k\mathbb{R} \to k\mathbb{R}[1/\bar{v}]$ we see that there is a short exact sequence

$$0 \to H^1_{(\bar{v})}(\Sigma^{-1}k\mathbb{R}^{C_2}_\star) \to \pi^{C_2}_\star(\Gamma_{(\bar{v})}k\mathbb{R}) \to H^0_{(\bar{v})}(k\mathbb{R}^{C_2}_\star) \to 0.$$

This gives $\pi^{C_2}_\star(\Gamma_{(\bar{v})}k\mathbb{R})$ up to extension. The Gorenstein duality isomorphism can be used to resolve the remaining extension issues, and the answer is recorded in the proposition below.

The diagram Figure 3 should help the reader interpret the statement and proof of the calculation of the homotopy of $\Gamma_{(\bar{v})}k\mathbb{R}$. We have omitted dots, circles and boxes except at the ends of diagonals or where an additional generator is required. The vertical lines denote multiplication by a and the dashed vertical line is an exotic multiplication by a that is not visible on the level of local cohomology. The green diamond does not denote a class, but marks the point one has to reflect (nontorsion classes) at to see Anderson duality. Torsion classes are shifted by -1 after reflection (ie shifted one step horizontally to the left).

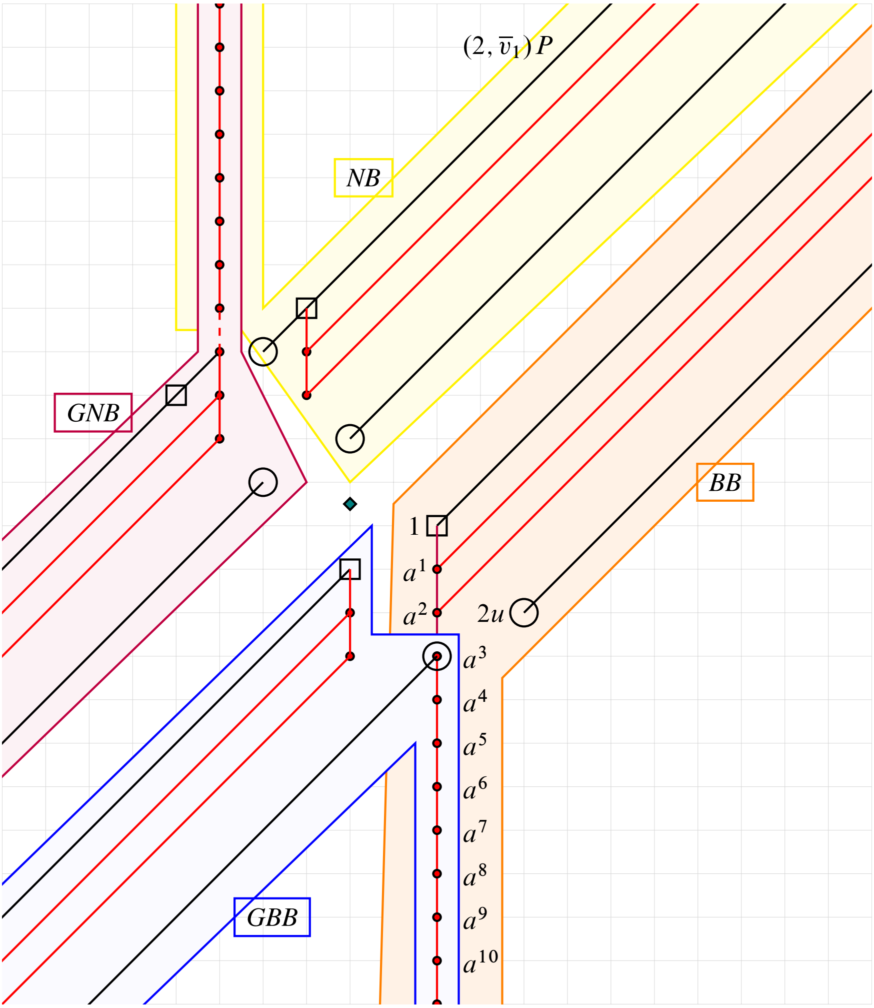

Figure 3: Gorenstein duality for $k\mathbb{R}$

Proposition 11.5 *The homotopy of the derived $\overline{v}$–power torsion is given by*

$$\pi_\star^{C_2}(\Gamma_{(\overline{v})}k\mathbb{R}) \cong (U^{-1}\cdot GNB[U^{-1}]) \oplus GBB[U],$$

where GBB and GNB are based on the local cohomology of BB and NB respectively, and described as follows. We have

$$GBB = \Sigma^{-2-\sigma}\big[\mathbb{Z}[\overline{v}]^* \oplus a\cdot\mathbb{F}_2[\overline{v}]^\vee \oplus a^2\cdot\mathbb{F}_2[\overline{v}]^\vee \oplus u\cdot N\big],$$

where N (with top in degree 0) is given by an exact sequence nonsplit in degree 0:

$$0 \to \mathbb{Z}[\overline{v}]^* \to N \to \mathbb{F}_2[a] \to 0.$$

Similarly,

$$GNB = \Sigma^{-1}\big[\mathbb{Z}[\bar{v}]^*/(2\cdot(1^*)) \oplus a\cdot\mathbb{F}_2[\bar{v}]^\vee \oplus a^2\cdot\mathbb{F}_2[\bar{v}]^\vee \oplus \Sigma^{1-3\sigma}\mathbb{Z}[\bar{v}]^* \oplus \Sigma^{\sigma}\mathbb{F}_2[a]^\vee\big],$$

where the action of a is as suggested by the sum decomposition except that multiplication by a is nontrivial wherever possible (ie when one dot is vertically above another, or where a box is vertically above a dot).

Proof We first note that the contributions from the different blocks do not interact. Indeed, the only time that different blocks give contributions in the same degree come from the $\mathbb{F}_2[a]$ towers of BB: one class in that degree is $\bar{v}$–divisible (and not killed by $\bar{v}$) and the other class is annihilated by $\bar{v}$. We may therefore consider the blocks entirely separately.

The block GBB comes from the local cohomology of BB and therefore lives in a short exact sequence

$$0 \to H^1_{(\bar{v})}(\Sigma^{-1}BB) \to GBB \to H^0_{(\bar{v})}(BB) \to 0.$$

The block GNB comes from the local cohomology of NB and therefore lives in a short exact sequence

$$0 \to H^1_{(\bar{v})}(\Sigma^{-1}NB) \to GNB \to H^0_{(\bar{v})}(NB) \to 0.$$

Most questions about module structure over $BB[U]$ are resolved by degree, but there are two which remain. These can be resolved by Gorenstein duality (Corollary 6.8) and the known module structure in $\mathbb{Z}^{k\mathbb{R}}$.

In GBB, the additive extension in $\pi^{C_2}_{-3\sigma}$ is nontrivial:

$$\pi^{C_2}_{-3\sigma}(\Gamma_{(\bar{v})}k\mathbb{R}) \cong \mathbb{Z}.$$

Also the multiplication by a

$$\mathbb{F}_2 \cong GNB_{-1+\sigma} \to GNB_{-1} \cong \mathbb{F}_2$$

is nonzero (where $GNB_{-1+\sigma}$ corresponds to $\pi^{C_2}_{-5+5\sigma}(\Gamma_{(\bar{v})}k\mathbb{R})$ in the U^{-1}–shift). $\square$

Remark 11.6 It is striking that the duality relates the top BB to the bottom NB (ie Anderson duality takes the part of $\Gamma_{\bar{v}}k\mathbb{R}$ coming from the local cohomology of BB to NB), and it takes the bottom NB to the top BB (ie Anderson duality takes the part of $\Gamma_{\bar{v}}k\mathbb{R}$ coming from the local cohomology of NB to BB). Indeed, as commented after Lemma 11.2, since $NB = \Gamma_{(a)}BB$, we have

$$\Sigma^{2+\sigma}\Gamma_{(\bar{v})}BB \simeq (\Gamma_{(a)}BB)^* \quad \text{and} \quad \Gamma_{(\bar{v},a)}BB \simeq \Sigma^{-2-\sigma}BB^*,$$

with the second stating that BB is Gorenstein of shift $-2-\sigma$ for the ideal $(a,\bar{v})$.

By extension, Anderson duality takes the part of $\Gamma_{\bar{v}} k\mathbb{R}$ coming from the local cohomology of all copies of BB to all copies of NB and vice versa. This might suggest separating $k\mathbb{R}$ into a part with homotopy $BB[U]$, giving a cofibre sequence

$$\langle BB[U]\rangle \to k\mathbb{R} \to \langle U^{-1}NB[U^{-1}]\rangle,$$

where the angle brackets refer to a spectrum with the indicated homotopy. However one may see that there is no C_2–spectrum with homotopy the Mackey functor corresponding to $BB[U]$ (considering the $b\sigma$ and $(b+1)\sigma$ rows one sees that the nonequivariant homotopy of the spectrum would be zero up to about degree $2b$; taking all rows together it would have to be nonequivariantly contractible and hence a–periodic). Similarly, there is no spectrum with homotopy $U^{-1}NB[U^{-1}]$, so these dualities are purely algebraic.

12 The local cohomology spectral sequence for $BP\mathbb{R}\langle n\rangle$

Gorenstein duality for $BP\mathbb{R}\langle n\rangle$ (Example 5.2) has interesting implications for the coefficient ring, both computationally and structurally. Writing $\star$ for $RO(C_2)$–grading as usual, the local cohomology spectral sequence [11, Section 3] takes the form described in the following proposition. We now revert to our standard assumption of working 2–locally, so $\mathbb{Z}$ means the 2–local integers.

Proposition 12.1 *There is a spectral sequence of $BP\mathbb{R}\langle n\rangle^{C_2}_\star$–modules*

$$H^*_{\bar{J}_n}(BP\mathbb{R}\langle n\rangle^{C_2}_\star) \implies \Sigma^{-(D_n+n+2)-(D_n-2)\sigma}\pi^{C_2}_\star(\mathbb{Z}^{BP\mathbb{R}\langle n\rangle})$$

for $\bar{J}_n = (\bar{v}_1,\dots,\bar{v}_n)$. The homotopy of the Anderson dual in an arbitrary degree $\alpha\in RO(C_2)$ is easily calculated:

$$0\to \mathrm{Ext}_{\mathbb{Z}}(BP\mathbb{R}\langle n\rangle^{C_2}_{-\alpha-1},\mathbb{Z}) \to \pi^{C_2}_\alpha \mathbb{Z}^{BP\mathbb{R}\langle n\rangle} \to \mathrm{Hom}_{\mathbb{Z}}(BP\mathbb{R}\langle n\rangle^{C_2}_{-\alpha},\mathbb{Z})\to 0.$$

For $n\geq 2$, the local cohomology spectral sequence has some nontrivial differentials.

One should not view Proposition 12.1 as an algebraic formality: it embodies the fact that $BP\mathbb{R}\langle n\rangle^{C_2}_\star$ is a very special ring.

In the present section, we will discuss the implications of this for the coefficient ring for general n. The perspective is a bit distant so the reader is encouraged to refer back to $k\mathbb{R}$ (ie the case $n=1$) in Section 11 to anchor the generalities.

However the case $n=1$ is too simple to show some of what happens, so we will also illustrate the case $tmf_1(3)$ (ie the case $n=2$) in Section 13.

12A Reduction to diagonals

For brevity, we write $R_\star = BP\mathbb{R}\langle n\rangle_\star^{C_2}$. Because the ideal $\overline{J}_n = (\overline{v}_1, \dots, \overline{v}_n)$ is generated by elements whose degrees are a multiple of ρ, we can do $\overline{J}_n$–local cohomology calculations over the subring $R_{*\rho}$ of elements in degrees which are multiples of ρ.

Thus, for an $R_\star$–module $M_\star$ we have a direct sum decomposition

$$M_\star = \bigoplus_d M_{d+*\rho}$$

as $R_{*\rho}$–modules, where we refer to the gradings $d + *\rho$ as the *d–diagonal*. Hence, we also have

$$H^i_{\overline{J}_n}(M_\star) = \bigoplus_d H^i_{\overline{J}_n}(M_{d+*\rho}).$$

(We have abused notation by also writing $\overline{J}_n$ for the ideal of $R_{*\rho}$ generated by $\overline{v}_1, \dots, \overline{v}_n$.)

12B The general shape of $BP\mathbb{R}\langle n\rangle_\star^{C_2}$

By the description at the end of Section 4B, we have an isomorphism

$$R_\star = U^{-1}\cdot NB[U^{-1}] \oplus BB[U]$$

with BB and NB as described there. It is easy to see that BB and NB decompose as $R_{*\rho}$–modules into modules of a certain form we will describe now. We will implicitly 2–localize everywhere.

The modules BB and NB decompose into are

$$P = R_{*\rho} = \mathbb{Z}[\overline{v}_1, \dots, \overline{v}_n] \quad \text{and} \quad \overline{P}_s = P/(\overline{v}_0, \dots, \overline{v}_s) = \mathbb{F}_2[\overline{v}_{s+1}, \dots, \overline{v}_n]$$

for $s \geq 0$ and the ideals expressed by the exact sequences

$$0 \to (2, \overline{v}_1, \dots, \overline{v}_t) \to P \to \overline{P}_t \to 0 \quad \text{or} \quad 0 \to (\overline{v}_{s+1}, \dots, \overline{v}_t) \to \overline{P}_s \to \overline{P}_t \to 0$$

with $s \geq 0$.

Their local cohomology is easily calculated. In the first two cases, the modules only have local cohomology in a single degree:

$$H^*_{\overline{J}_n}(P) = H^n_{\overline{J}_n}(P) = P^*(-D_n\rho),$$
$$H^*_{\overline{J}_n}(\overline{P}_s) = H^{n-s}_{\overline{J}_n}(\overline{P}_s) = \overline{P}_s^\vee((D_s - D_n)\rho).$$

The top nonzero degree of P^* is zero, so $1^* \in P^*(-D_n\rho)$ is in degree $-D_n\rho = -|\overline{v}_1| - \cdots - |\overline{v}_n|$. We alert the reader to the fact that star is used in two ways:

occasionally in H^* to mean cohomological grading and rather frequently here in P^* to mean the $\mathbb{Z}$–dual of P.

Now we turn to the ideal $(\overline{v}_{s+1}, \dots, \overline{v}_t)$. If $t = s+1$ the ideal is principal and $(\overline{v}_{s+1}) \cong \overline{P}_s((s+1)\rho)$; thus we get a single local cohomology group

$$H^{n-s}_{\overline{J}_n}((\overline{v}_{s+1})\overline{P}_s) = \overline{P}_s^{\vee}((D_s - D_n + s + 1)\rho)$$

as can be seen from the long exact sequence of local cohomology.

Otherwise we get two local cohomology groups

$$H^{n-s}_{\overline{J}_n}((\overline{v}_{s+1}, \dots, \overline{v}_t)\overline{P}_s) = \overline{P}_s^{\vee}((D_n - D_s)\rho)$$

and

$$H^{n-t+1}_{\overline{J}_n}((\overline{v}_{s+1}, \dots, \overline{v}_t)\overline{P}_s) = \overline{P}_t^{\vee}((D_n - D_t)\rho).$$

The case of $(2, \overline{v}_1, \dots, \overline{v}_t)$ is similar but with an extra case. The case $t = 0$ is easy since then $(2) \cong P$ so the local cohomology is all in cohomological degree n where it is $P^*(-D_n\rho)$. If $t = 1$ we again get a single local cohomology group

$$H^{n}_{\overline{J}_n}((2, \overline{v}_1)P) = P^*(-D_n\rho) \oplus \overline{P}_1^{\vee}((D_1 - D_n)\rho).$$

Otherwise we get two local cohomology groups

$$H^{n}_{\overline{J}_n}((2, \dots, \overline{v}_t)P) = P^*(-D_n\rho) \quad \text{and} \quad H^{n-t+1}_{\overline{J}_n}((2, \dots, \overline{v}_t)P) = \overline{P}_t^{\vee}((D_t - D_n)\rho).$$

12C The special case $n = 1$

The best way to make the patterns apparent is to look at the simplest cases. In this section, we begin with $k\mathbb{R}^{C_2}_{\star}$ as treated in Section 11 above, and we encourage the reader to relate the calculations here to the diagrams in Section 11. In that case,

$$P = k\mathbb{R}^{C_2}_{*\rho} = \mathbb{Z}[\overline{v}_1], \quad \overline{P}_0 = \mathbb{F}_2[\overline{v}_1] \quad \text{and} \quad \overline{P}_1 = \mathbb{F}_2.$$

Table 1 (left) displayes BB by d–diagonal. The position of the modules along the d–diagonal can be inferred from the label at the top of the column. Thus the first column has generators in degree $-d\sigma$, and the second column similarly, but in the column of u (namely the 2–column). Noting that u is on the 4–diagonal, the d^{th} row has generators in $|u| - (d-4)\sigma = 2 - (d-2)\sigma$. For example, along the 4–diagonal we have $a^4\overline{P}_1 \oplus (2u)P$.

Taking local cohomology, and shifting $H^s_{\overline{J}_n}$ down by s (as in the local cohomology spectral sequence), we have Table 1 (right). Note that shifting down by s both lowers d by s and adds a shift by $-s\rho$. For example, considering the 3–diagonal of this table,

	BB	
d	1	u
0	P	
1	$\overline{P}_0$	
2	$\overline{P}_0$	
3	$\overline{P}_1$	
4	$\overline{P}_1$	$(2)P$
5	$\overline{P}_1$	
6	$\overline{P}_1$	
7	$\overline{P}_1$	
8	$\overline{P}_1$	

	$H^*_{(\overline{v}_1)}(BB)$	
d	1	u
-1	$P^*(-2\rho)$	
0	$\overline{P}_0^\vee(-2\rho)$	
1	$\overline{P}_0^\vee(-2\rho)$	
2		
3	$\overline{P}_1$	$P^*(-2\rho)$
4	$\overline{P}_1$	
5	$\overline{P}_1$	
6	$\overline{P}_1$	
7	$\overline{P}_1$	
8	$\overline{P}_1$	

Table 1: BB (left) and the local cohomology (right) by d-diagonal for $n=1$. The H^1–groups are coloured brown.

the $\overline{P}_1$ comes directly from the 3–diagonal of BB, whilst the $P^*(-2\rho)$ comes from the $(2)P$ on the 4–diagonal of BB; the local cohomology is $P^*(-\rho)$, but its diagonal is shifted by -1 since it is a first local cohomology, and because it is by reference to the 2–column the shift is $-\rho$. The top of this module is calculated by reference to the column of $|u|$ (ie the 2–column), and has top in degree $2-(3-2)\sigma-2\rho=-3\sigma$.

We saw in Section 11 that the two modules on the 3–diagonal give a nontrivial additive extension (in degree -3σ) after running the spectral sequence.

12D The special case $n = 2$

Continuing our effort to make patterns visible, we consider $tmf_1(3)^{C_2}_{\star}$ in this subsection (ie the case $n=2$). With $\mathbb{Z}$ denoting the integers with 3 inverted here, this has

$$P=tmf_1(3)^{C_2}_{*\rho}=\mathbb{Z}[\overline{v}_1,\overline{v}_2],\quad \overline{P}_0=\mathbb{F}_2[\overline{v}_1,\overline{v}_2],\quad \overline{P}_1=\mathbb{F}_2[\overline{v}_2]\quad\text{and}\quad \overline{P}_2=\mathbb{F}_2.$$

See Table 2. Once again, the column labelled u^i is the $2i^{\text{th}}$ column, and shifts along the diagonal have as reference point where this column meets the relevant diagonal.

We take local cohomology, again remembering that $H^s_{\overline{J}_n}$ is shifted down by s, which changes the diagonal by s. For example, on the 7–diagonal, $\overline{P}_2$ comes from the 7–diagonal in BB, whereas the $\overline{P}_0^\vee(-5\rho)$ comes from the 2^{nd} local cohomology of the entry $(\overline{v}_1)\overline{P}_0$ on the 9–diagonal; the local cohomology of $\overline{P}_0$ is $\overline{P}_0^\vee(-4\rho)$, this is shifted by a further -2ρ from the change of diagonal, and $+\rho$ because of the $\overline{v}_1$.

	BB			
d	1	u	u^2	u^3
0	P			
1	$\overline{P}_0$			
2	$\overline{P}_0$			
3	$\overline{P}_1$			
4	$\overline{P}_1$	(2)		
5	$\overline{P}_1$			
6	$\overline{P}_1$			
7	$\overline{P}_2$			
8	$\overline{P}_2$		$(2,\overline{v}_1)P$	
9	$\overline{P}_2$		$(\overline{v}_1)\overline{P}_0$	
10	$\overline{P}_2$		$(\overline{v}_1)\overline{P}_0$	
11	$\overline{P}_2$			
12	$\overline{P}_2$			(2)
13	$\overline{P}_2$			

	$H^*_{(\overline{v}_1,\overline{v}_2)}(BB)$			
d	1	u	u^2	u^3
-2	$P^*(-6\rho)$			
-1	$\overline{P}_0^\vee(-6\rho)$			
0	$\overline{P}_0^\vee(-6\rho)$			
1				
2	$\overline{P}_1^\vee(-4\rho)$	$P^*(-6\rho)$		
3	$\overline{P}_1^\vee(-4\rho)$			
4	$\overline{P}_1^\vee(-4\rho)$			
5	$\overline{P}_1^\vee(-4\rho)$			
6			$\overline{P}_1^\vee(-5\rho)\oplus P^*(-6\rho)$	
7	$\overline{P}_2$		$\overline{P}_0^\vee(-5\rho)$	
8	$\overline{P}_2$		$\overline{P}_0^\vee(-5\rho)$	
9	$\overline{P}_2$			
10	$\overline{P}_2$			$P^*(-6\rho)$
11	$\overline{P}_2$			
12	$\overline{P}_2$			
13	$\overline{P}_2$			

Table 2: BB (left) and the local cohomology (right) by d-diagonal for $n=2$. The H^1–groups are coloured in brown and the H^2–groups in teal.

We will see below that there are nontrivial extensions on the 2– and 10–diagonals, and that there are differentials in the local cohomology spectral sequence from the 7–, 8– and 9–diagonals (differentials go from the d–diagonal to the $(d-1)$–diagonal).

12E Moving from the basic block *BB* to the negative block *NB*

Moving from BB to NB only affects the 0 column, where in each case M is replaced by $\ker(M\to\mathbb{F}_2)=(2)M$. In effect, this replaces $\overline{P}_n$ by 0. It also adds on a new (-1)–column of $\overline{P}_n=\mathbb{F}_2$ going up from the σ row. We resist the temptation to display a table for NB explicitly, but note that $NB=\Gamma_{(a)}BB$ as for $k\mathbb{R}$.

12F Gorenstein duality

With the above data in mind, we may consider the d–diagonal BB_d, where the lowest value of d is 0 and the highest is $N=4(2^n-1)$. If we ignore the difference between BB and NB (which is at most $\mathbb{F}_2$ in any degree) we find approximately that BB_d has a relationship to BB_{N-d}, namely something like an equality

$$H^n_{J_n}(BB_d)^*=BB_{N-d}.$$

There are various ways in which this is inaccurate and needs to be modified. Firstly, if the local cohomology of BB_d is entirely in cohomological degree $n-\epsilon$ with $\epsilon \neq 0$, there will be a shift of ϵ (if it is in several degrees there is a further complication). Secondly, Anderson duality introduces a shift of one diagonal if applied to torsion modules. Thirdly, we have seen that there may be extensions between these local cohomology groups, sometimes removing $\mathbb{Z}$–torsion. Finally, there may be differentials.

In fact, all of these effects are "small" in the sense that the growth rate along a diagonal is bounded by a polynomial of degree $n-1$. Encouraged by this, if we ignore all of these effects, we see that BB is a Gorenstein module in the sense that the reverse-graded version is equivalent to the dual of its local cohomology:

$$H^n_{\bar{J}_n}(BB)^* = \mathrm{rev}(BB).$$

This is rather as if there is a cofibre sequence

$$S \to BP\mathbb{R}\langle n\rangle \to Q$$

where S is Gorenstein and Q is a Poincaré duality algebra of formal dimension $N = 2(1-\sigma)(2^n-1)$.

13 The local cohomology spectral sequence for $tmf_1(3)$

We examine the local cohomology spectral sequence and Gorenstein duality in more detail for $tmf_1(3)$. Actually, our calculations are equally valid for all forms of $BP\mathbb{R}\langle 2\rangle$, but we prefer the more evocative name $tmf_1(3)$ of the most prominent example. More of the general features are visible for $tmf_1(3)$ than for $k\mathbb{R}$.

As usual we will implicitly localize everywhere at 2 (although for $tmf_1(3)$ itself it would actually suffice to just invert 3).

13A The local cohomology spectral sequence

We make explicit the implications for the coefficient ring, both computationally and structurally. Writing $\star$ for $RO(C_2)$–grading as usual, the spectral sequence takes the following form.

Proposition 13.1 *There is a spectral sequence of $tmf_1(3)^{C_2}_\star$–modules*

$$H^*_{\bar{J}_n}(tmf_1(3)^{C_2}_\star) \implies \Sigma^{-8-2\sigma}\pi^{C_2}_\star(\mathbb{Z}^{tmf_1(3)}).$$

The homotopy of the Anderson dual is easily calculated:

$$0 \to \mathrm{Ext}_{\mathbb{Z}}(tmf_1(3)^{C_2}_{-\alpha-1}, \mathbb{Z}) \to \pi^{C_2}_\alpha \mathbb{Z}^{tmf_1(3)} \to \mathrm{Hom}_{\mathbb{Z}}(tmf_1(3)^{C_2}_{-\alpha}, \mathbb{Z}) \to 0.$$

The local cohomology spectral sequence has some nontrivial differentials.

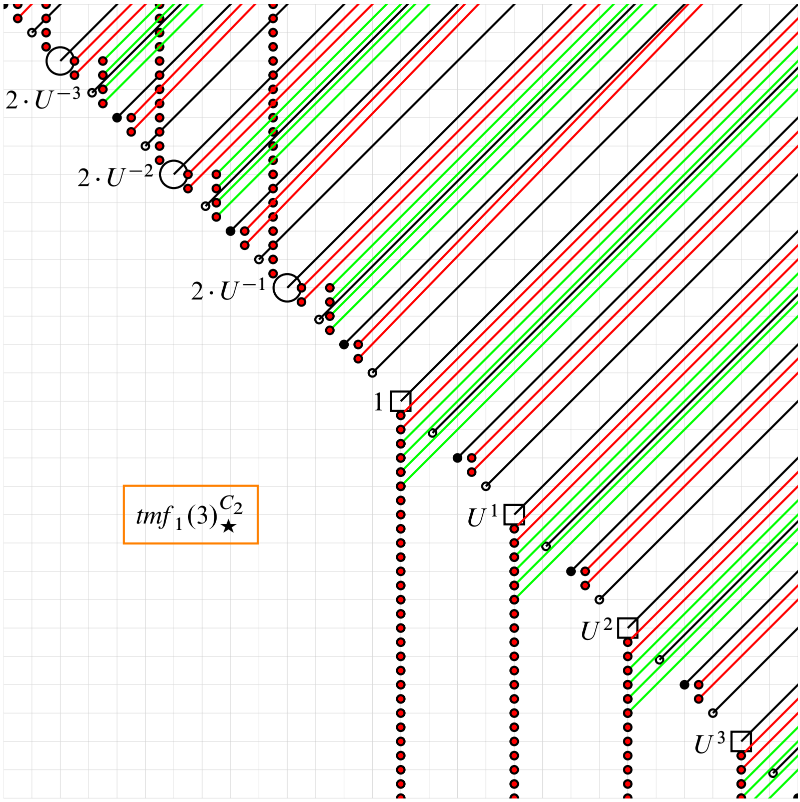

Figure 4: The homotopy of $tmf_1(3)$

13B The ring $tmf_1(3)^{C_2}_\bigstar$

The ring $tmf_1(3)^{C_2}_\bigstar$ is approximately calculated in [27] and more precisely described as

$$BB[U] \oplus U^{-1}NB[U^{-1}]$$

as at the end of Section 4B with $n = 2$. We already tabulated BB in Section 12D, but we want also want to display a bigger chart of $\pi^{C_2}_\bigstar tmf_1(3)$ as Figure 4 to give the reader a feeling of how the blocks piece together.

A black diagonal line means a copy of P when it starts in a box, a copy of $(2)P$ when it starts in a small circle, a copy of $(2, \overline{v}_1)P$ when it starts in a dot and a copy of $(2, \overline{v}_1, \overline{v}_2)$ when it starts in a big circle. In Figure 4, a red diagonal line means a copy of $\overline{P}_0$ and a green diagonal line a copy of $\overline{P}_1$. A red dot is a copy of $\mathbb{F}_2 = \overline{P}_2$.

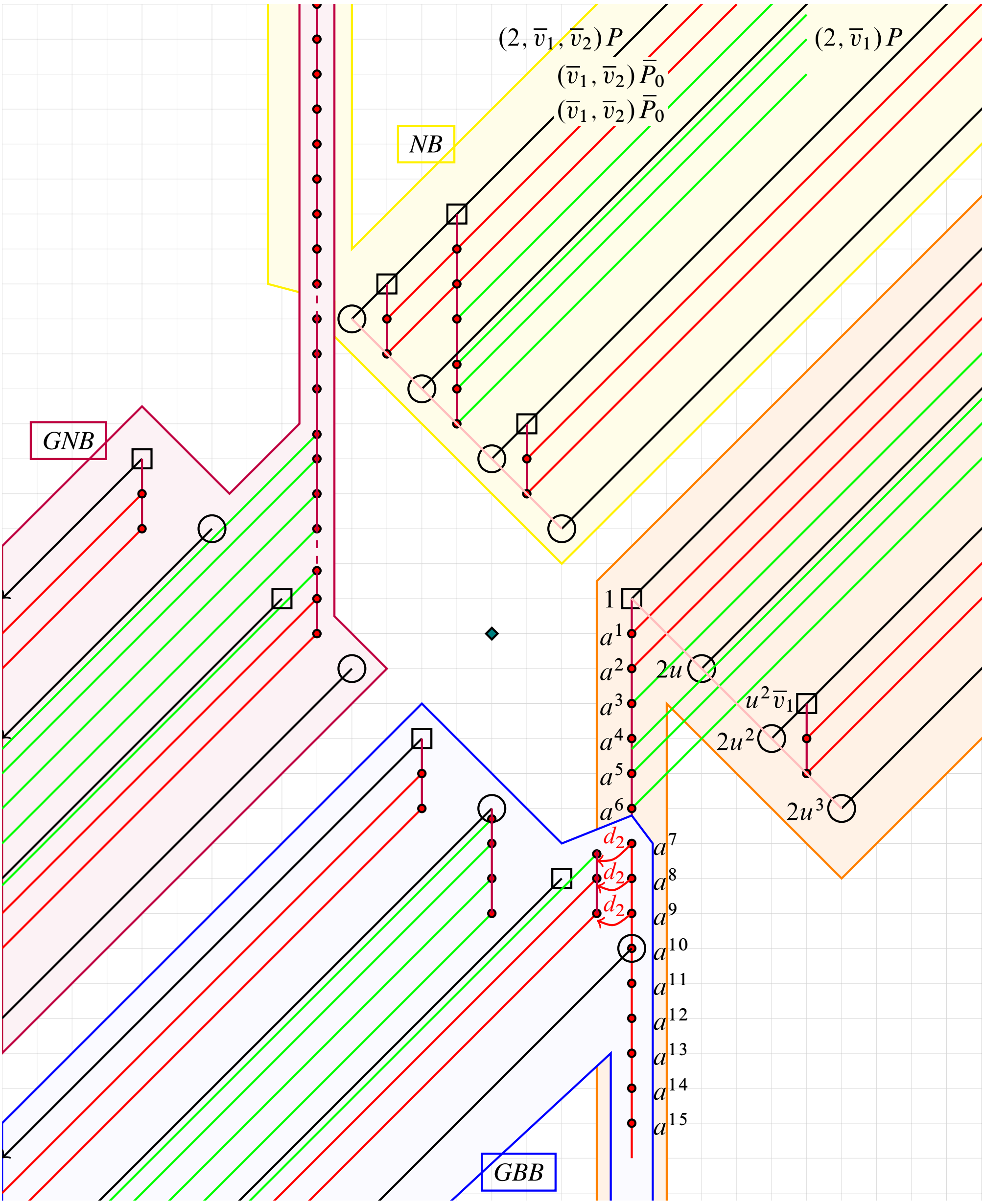

Figure 5: Gorenstein duality for $tmf_1(3)$

13C Local cohomology

We are calculating local cohomology with respect to the ideal $\bar{J}_2 = (\overline{v}_1, \overline{v}_2)$ so that we only need to consider $tmf_1(3)^{C_2}_{\star}$ as a $\mathbb{Z}[\overline{v}_1, \overline{v}_2]$–module. As such it is a sum of suspensions of the blocks BB and NB, so we just need to calculate the local cohomology of these. This was described in Section 12 above. Here we will simply describe the extensions and the behaviour of the local cohomology spectral sequence.

The basis of this discussion are the tables of BB and GBB from Section 12D together with the analogues for NB and GNB. Although these are organized by diagonal, Figure 5 displaying BB, GBB, $U^{-1}NB$ and $U^{-1}GNB$ may help visualize the way the modules are distributed along each diagonal. The vertical lines denote multiplication by a and the dashed vertical line is an exotic multiplication by a that is not visible on the level of local cohomology. The green diamond does not denote a class, but marks the point one has to reflect (nontorsion classes) at to see Anderson duality. Torsion classes are shifted after reflection by -1 (ie one step horizontally to the left).

The strategy is to take the known subquotients from the local cohomology calculation, and resolve the extension problems using Gorenstein duality.

Proposition 13.2 *We have an isomorphism*

$$\pi_\star^{C_2}\Gamma_{\bar{J}_2} tmf_1(3) \cong GBB[U] \oplus U^{-1}GNB[U^{-1}],$$

where GBB and GNB are described in the following. We will simultaneously describe what differentials and extensions in the local cohomology spectral sequence caused the passage from $H^*_{\bar{J}_2}(BB)$ and $H^*_{\bar{J}_2}(NB)$ to GBB and GNB respectively.

(i) *The $\mathbb{Z}[\bar{v}_1, \bar{v}_2]$–modules along the diagonals in GBB are as in Table 3 (left). There are three nontrivial differentials*

$$d_2\colon H^0_{\bar{J}_2}(BB) \to H^2_{\bar{J}_2}(BB)$$

from the groups at $-7\sigma, -8\sigma, -9\sigma$ to the groups at $-7\sigma-1, -8\sigma-1, -9\sigma-1$, which have affected the values on the 6–, 7–, 8– and 9–diagonals in Table 3 (left).

The extensions

$$0 \to P^* \to [(2, \bar{v}_1)P]^* \to \mathbb{F}_2[\bar{v}_2]^\vee \to 0$$

on the 2–diagonal and the 6–diagonal are Anderson dual to the defining short exact sequence

$$0 \to (2, \bar{v}_1)P \to P \to \mathbb{F}_2[\bar{v}_2] \to 0$$

in the following sense: The Anderson dual of the latter exact sequence is a triangle

$$\mathbb{F}_2[\bar{v}_2]^* \to P^* \to [(2, \bar{v}_1)P]^* \to \Sigma\mathbb{F}_2[\bar{v}_2]^* \cong \mathbb{F}_2[\bar{v}_2]^\vee$$

which induces (on homology) the extensions above. The extension

$$0 \to P^* \to [(2, \bar{v}_1, \bar{v}_2)P]^* \to \mathbb{F}_2 \to 0$$

	GBB			*GNB*	
d	module	top degree	d	module	top degree
			$-k \leq -3$	$\mathbb{F}_2$	$-1-k\sigma$
-2	P^*	$-6-4\sigma$	-2	$P^*, \mathbb{F}_2$	$-6-4\sigma, -1+\sigma$
-1	$\bar{P}_0^\vee$	$-6-5\sigma$	-1	$\bar{P}_0^\vee, \mathbb{F}_2$	$-6-5\sigma, -1+0\sigma$
0	$\bar{P}_0^\vee$	$-6-6\sigma$	0	$\bar{P}_0^\vee, \mathbb{F}_2$	$-6-6\sigma, -1-\sigma$
1	0		1	$\mathbb{F}_2$	$-1-2\sigma$
2	$[(2, \bar{v}_1) P]^*$	$-4-6\sigma$	2	$P^*, \bar{P}_1^\vee$	$-4-6\sigma, -1-3\sigma$
3	$\bar{P}_1^\vee$	$-4-7\sigma$	3	$\bar{P}_1^\vee$	$-1-4\sigma$
4	$\bar{P}_1^\vee$	$-4-8\sigma$	4	$\bar{P}_1^\vee$	$-1-5\sigma$
5	$\bar{P}_1^\vee$	$-4-9\sigma$	5	$\bar{P}_1^\vee$	$-1-6\sigma$
6	$[(2, \bar{v}_1) P]^*$	$-2-8\sigma$	6	$[(2, \bar{v}_1) P]^*$	$-1-7\sigma$
7	$(\bar{v}_1, \bar{v}_2) \bar{P}_0$	$-2-9\sigma$	7	$\bar{P}_0^\vee$	$-1-8\sigma$
8	$(\bar{v}_1, \bar{v}_2) \bar{P}_0$	$-2-10\sigma$	8	$\bar{P}_0^\vee$	$-1-9\sigma$
9	0		9	0	
10	$[(2, \bar{v}_1, \bar{v}_2) P]^*$	$0-10\sigma$	10	P^*	$0-10\sigma$
$10+k \geq 11$	$\mathbb{F}_2$	$0-(10+k)\sigma$			

Table 3: $\mathbb{Z}[\bar{v}_1, \bar{v}_2]$–modules as described in Proposition 13.2

on the 10*–diagonal is Anderson dual to the short exact sequence*

$$0 \to (2, \bar{v}_1, \bar{v}_2) P \to P \to \mathbb{F}_2 \to 0.$$

(ii) *The* $\mathbb{Z}[\bar{v}_1, \bar{v}_2]$*–modules along the diagonals in* GNB *are as in* Table 3 (*right*) (*take the direct sum of the two entries for the* (-2)–, (-1)–, 0– *and* 2*–diagonals*). *The extension*

$$0 \to P^* \to [(2, \bar{v}_1) P]^* \to \mathbb{F}_2[v_2]^\vee \to 0$$

on the 6*–diagonal is Anderson dual to the short exact sequence*

$$0 \to (2, \bar{v}_1) P \to P \to \mathbb{F}_2[v_2] \to 0.$$

Proof We first note that the contributions from the different blocks do not interact. Indeed, the only time that different blocks give contributions in the same degree comes from the $\mathbb{F}_2[a]$ towers of BB, and one class in that degree is divisible by $\bar{v}_1$ or $\bar{v}_2$ and not killed by both $\bar{v}_1$ and $\bar{v}_2$. We may therefore consider the blocks entirely separately.

The block GBB comes from the local cohomology of BB in the sense that there is a spectral sequence

$$H^*_{J_2}(BB) \implies GBB.$$

d	1	u^2	u^4	u^6
	$H^*_{J_2}(NB)$			
-2	$P^*(-6\rho)$			
-1	$\overline{P}_0^\vee(-6\rho)\oplus\overline{P}_2$			
0	$\overline{P}_0^\vee(-6\rho)\oplus\overline{P}_2$			
1	$\overline{P}_2$			
2	$\overline{P}_1^\vee(-4\rho)$	$P^*(-6\rho)$		
3	$\overline{P}_1^\vee(-4\rho)$			
4	$\overline{P}_1^\vee(-4\rho)$			
5	$\overline{P}_1^\vee(-4\rho)$			
6			$\overline{P}_1^\vee(-5\rho)\oplus P^*(-6\rho)$	
7			$\overline{P}_0^\vee(-5\rho)$	
8			$\overline{P}_0^\vee(-5\rho)$	
9				
10				$P^*(-6\rho)$
11				
12				
13				

Table 4: Local cohomology for $n=2$ from the proof of Proposition 13.2. Again, the H^1–groups are coloured in brown and the H^2–groups in teal.

Thus there is a filtration

$$GBB = GBB^0 \supseteq GBB^1 \supseteq GBB^2 \supseteq GBB^3 = 0$$

with

$$0 \to GBB^0/GBB^1 \to H^0_{J_2}(BB) \xrightarrow{d_2} \Sigma^{-1}H^2_{J_2}(BB) \to \Sigma^1 GBB^2 \to 0$$

and

$$GBB^1/GBB^2 \cong \Sigma^{-1}H^1_{J_2}(BB).$$

The block GNB comes from the local cohomology of NB in a precisely analogous way.

Most questions about module structure over $BB[U]$ are resolved by degree. The remaining issues are resolved by using Gorenstein duality.

Referring to the table for $H^*_{J_2}(BB)$ in Section 12D, the first potential extension is on the 2–diagonal. Using Gorenstein duality to compare with $NB_{\delta=8}$ we see that the actual extension on the 2–diagonal of GBB is

$$0 \to P^* \to [(2,\bar{v}_1)P]^* \to \overline{P}_1^\vee \to 0,$$

δ	δ' s.t. $H^*_{\bar{J}_2}(BB_\delta)^* \sim NB_{\delta'}$	δ	δ' s.t. $H^*_{\bar{J}_2}(NB_\delta)^* \sim BB_{\delta'}$
0	12	0	12
1	10	1	10
2	9	2	9
3	8	3	8
4	$8, 6$	4	$8, 6$
5	5	5	5
6	4	6	4
7	2	7	.
8	$4, 3$	8	4
9	2	9	2
10	$1, 0$	10	1
11	0	11	.
12	0	12	0

Table 5: Diagonal contributions from Remark 13.3(i)

where we have shifted the modules so they all have top degree 0. There is an additive extension on the 10–diagonal by reference to the Anderson dual. Finally the three nonzero d_2 differentials from $-1-k\sigma$ for $k=7, 8$ and 9 are necessary for connectivity (this removes the need to discuss the possible extensions on the 7– and 8–diagonals).

The situation is rather similar for GNB. We will not explicitly display NB since the only effect (apart from the addition of $\mathbb{F}_2[a]^\vee$) is on the first column, where a module is replaced by the kernel of a surjection to $\mathbb{F}_2$. It is perhaps worth displaying $H^2_{\bar{J}_2}(NB)$, where we leave out the big $\mathbb{F}_2[a]^\vee$–tower in $H^0_{\bar{J}_2}NB$. See Table 4. In this case, all extensions are split, except for the one on the 6–diagonal and there are no differentials. The a multiplications in the $\mathbb{F}_2[a]^\vee$ tower are clear from Gorenstein duality and the a–tower $\mathbb{F}_2[a]$ in BB. □

Remark 13.3 (i) In Table 5, we summarize the way a diagonal BB_δ contributes to NB as in

$$H^*_{\bar{J}_2}(BB_\delta)^* \sim NB_{\delta'}$$

as sketched in Section 12F. Because most of the modules are 2–torsion the most common pairing is between δ and $11-\delta$ rather than between δ and $12-\delta$ as happens for the main U–power diagonals.

(ii) We also note as before that since $NB = \Gamma_{(a)}BB$, we have

$$\Sigma^{6+4\sigma}\Gamma_{(\bar{v}_1,\bar{v}_2)}BB \sim (\Gamma_{(a)}BB)^*$$

(where we have written $\sim$ rather than $\simeq$ in recognition of the differentials) and

$$\Sigma^{6+4\sigma}\Gamma_{(\bar{v}_1,\bar{v}_2,a)}BB \simeq BB^*,$$

with the second stating that BB is Gorenstein of shift $-6-4\sigma$ for the ideal $(\bar{v}_1,\bar{v}_2,a)$.

Appendix: The computation of $\pi_\star^{C_2}BP\mathbb{R}$

Our main goal in this appendix is to compute the homotopy fixed point spectral sequence for $BP\mathbb{R}$ and hence for $M\mathbb{R}$. All the results in this appendix and the essential idea of the argument for Proposition A.2 are contained in [18] (see especially Formula 4.16). We just rearranged their arguments and added some details. Our argument for the multiplicative extensions might be considered new though. We have strived for elementary and short proofs though they retain some computational complexity. We hope this is helpful for the reader to understand this crucial computation. Note that even before Hu and Kriz, the computation of $\pi_\star^{C_2}BP\mathbb{R}$ was announced in [3].

We will work throughout 2–locally. As before, we denote by ρ the regular real C_2–representation and by σ the sign representation. We need a few facts, first proven by Araki:

(1) If E is a real-oriented spectrum, then $E_{C_2}^\star(\mathbb{C}P^\infty) \cong E_{C_2}^\star[\![u]\!]$ with $|u| = -\rho$ and $E_{C_2}^\star(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E_{C_2}^\star[\![1\otimes u, u\otimes 1]\!]$. This induces a formal group law on $\pi_{*\rho}^{C_2}E$ and the forgetful map $\pi_{*\rho}^{C_2}E \to \pi_{2*}^{e}E$ maps it to the usual formal group law from the complex orientation of E. [18, Theorem 2.10]

(2) Thus, we get a ring map $\pi_{2*}^{e}MU \to \pi_{*\rho}^{C_2}M\mathbb{R}$ from the Lazard ring such that $\pi_{2*}^{e}MU$ is a retract of $\pi_{*\rho}^{C_2}M\mathbb{R}$. For every class in $x \in \pi_{2*}MU$, we have thus a corresponding class $\bar{x} \in \pi_{*\rho}^{C_2}M\mathbb{R}$. [18, Proposition 2.27]

(3) There is a splitting $M\mathbb{R}_{(2)} \simeq \bigoplus_{m_i}\Sigma^{m_i\rho}BP\mathbb{R}$, where the underlying spectrum of $BP\mathbb{R}$ agrees with BP. This splitting corresponds on geometric fixed points to the splitting $MO \simeq \bigoplus_{m_i}\Sigma^{m_i}H\mathbb{F}_2$. [18, Theorem 2.33]

Define $a\colon S^0 \to S^\sigma$ as before to be the inclusion of the points 0 and ∞; we will denote the image of a in $\pi_\star M\mathbb{R}$ and $\pi_\star BP\mathbb{R}$ by the same symbol. The class a has degree $-\sigma = 1-\rho$.

Proposition A.1 *We have $a^{2^{n+1}-1}\bar{v}_n = 0$ in $\pi_\star^{C_2}M\mathbb{R}$.*

Proof We have a fibre sequence

$$(EC_2)_+ \otimes M\mathbb{R} \to M\mathbb{R} \to \widetilde{E}C_2 \otimes M\mathbb{R}.$$

First, we claim that the image of $\bar{v}_n$ under $M\mathbb{R} \to \widetilde{E}C_2 \otimes M\mathbb{R}$ is zero. Indeed, as a is invertible on $\widetilde{E}C_2 \otimes M\mathbb{R}$, the formal group law on $\pi_{*\rho}^{C_2}(\widetilde{E}C_2 \otimes M\mathbb{R})$ agrees with that on $\pi_*^{C_2}(\widetilde{E}C_2 \otimes M\mathbb{R}) = \pi_* MO$, which is additive. Therefore, the map

$$MU_{2*} \to \pi_{*\rho}^{C_2} M\mathbb{R} \to \pi_{*\rho}^{C_2} \widetilde{E}C_2 \otimes M\mathbb{R}$$

sends all v_n to zero. Thus, $\bar{v}_n$ and hence also $a^{2^{n+1}-1}\bar{v}_n$ are in the image of the map

$$(EC_2)_+ \otimes M\mathbb{R} \to M\mathbb{R}.$$

Observe that

$$|a^{2^{n+1}-1}\bar{v}_n| = -(2^{n+1}-1)\sigma + (2^n-1)(1+\sigma) = 2^n - 1 - 2^n\sigma.$$

We claim that $\pi_{2^n-1-2^n\sigma}^{C_2}((EC_2)_+ \otimes M\mathbb{R})$ is zero. Indeed, we have

$$\pi_{2^n-1-2^n\sigma}^{C_2}((EC_2)_+ \otimes M\mathbb{R}) \cong \pi_{2^n-1}(\Sigma^{2^n\sigma} M\mathbb{R})_{hC_2}.$$

This can be computed by the homotopy orbit spectral sequence

$$H_p(C_2; \pi_q \Sigma^{2^n\sigma} M\mathbb{R}) \implies \pi_{p+q}(\Sigma^{2^n\sigma} M\mathbb{R})_{hC_2}.$$

But $\pi_q \Sigma^{2^n\sigma} M\mathbb{R} = 0$ for $q < 2^n$, so $\pi_{2^n-1}(\Sigma^{2^n\sigma} M\mathbb{R})_{hC_2} = 0$. Thus, we see that $a^{2^{n+1}-1}\bar{v}_n = 0$ in $\pi_\star^{C_2} M\mathbb{R}$. □

For a C_2–spectrum X, the $RO(C_2)$ graded homotopy fixed point spectral sequence is defined by combining the homotopy fixed point spectral sequences

$$E_2^{p,q}(r) = H^q(C_2, \pi_{p+q}(X \wedge S^{-r\sigma})) \implies \pi_p^{C_2}((X \wedge S^{-r\sigma})^{hC_2}) \cong \pi_{p+r\sigma}^{C_2}(X^{(EC_2)_+})$$

into a single spectral sequence with differential

$$d_n\colon E_n^{p,q}(r) \to E_n^{p-1,q+n}(r).$$

Note that we use an Adams grading convention here. We will often call $p + r\sigma$ the *degree* of an element.

The $RO(C_2)$–graded homotopy fixed point spectral sequence (HFPSS) for $BP\mathbb{R}$ has E_2–term

$$\mathbb{Z}_{(2)}[a, u^{\pm 1}, \bar{v}_1, \bar{v}_2, \ldots]/2a$$

with

$$|a| = (-\sigma, 1), \quad |u| = (2 - 2\sigma, 0) \quad \text{and} \quad |\bar{v}_i| = ((2^i - 1)\rho, 0).$$

This can be seen, for example, by the identification with the Bockstein spectral sequence for a discussed in [27, Lemma 4.8]. As $BP\mathbb{R}$ is a retract of $M\mathbb{R}_{(2)}$, it has the structure of a (homotopy) ring spectrum and thus the $RO(C_2)$–graded homotopy fixed point spectral sequence is multiplicative by [27, Section 2.3].

By the discussion above, a and the $\bar{v}_i$ are permanent cycles. As $a^{2^{n+1}-1}\bar{v}_n$ is zero, it must be hit by a differential. This is the crucial ingredient for the following central proposition. It is fully formal in the sense that we do not need any other input in addition to the things we already discussed; we argue just with the form of the spectral sequence. We will set $\bar{v}_0 = 2$ for convenience.

Proposition A.2 *In the HFPSS for $BP\mathbb{R}$, we have $E_{2^n} = E_{2^{n+1}-1}$, and it is the subalgebra of*

$$E_2/(a^3\bar{v}_1, \dots, a^{2^n-1}\bar{v}_{n-1})$$

generated by a, $u^{\pm 2^{n-1}}$, the $\bar{v}_i$ for $i \geq 0$ and by the $\bar{v}_i u^{2^i j}$ for $i < n-1$ and $j \in \mathbb{Z}$.

Proof We prove it by induction. It is obviously true for $n = 1$ by the checkerboard phenomenon; indeed, for all generators of the E_2–term in degree $(a + b\rho, q)$ we have $a + q$ even.

Now assume it to be true for a given n. First, we will show that $d_{2^{n+1}-1}(u^{2^{n-1}}) = a^{2^{n+1}-1}\bar{v}_n$. Indeed, as $a^{2^{n+1}-1}\bar{v}_n$ is nonzero in $E_{2^{n+1}-1}$, it must be hit by a $d_{2^{n+1}-1}$. Its source x is in the zero-line in degree $2^{n+1} - 2^n\rho$. As the zero-line in E_2 is generated by u of degree $4 - 2\rho$ and by the $\bar{v}_i$ in regular representation degrees, we see that the exponent of u in x must be 2^{n-1}, so there is no room for further $\bar{v}_i$. Thus, $d_{2^{n+1}-1}(u^{2^{n-1}}) = a^{2^{n+1}-1}\bar{v}_n$.

Next, we want to show that $d_q(\bar{v}_i u^{2^i j}) = 0$ for $2^{n+1} - 1 \leq q < 2^{n+2} - 1$ and $i < n$. Write $d_q(\bar{v}_i u^{2^i j}) = a^q x$. The degree of x is

$$(2^i - 1)\rho + 2^i j(4 - 2\rho) - q(1 - \rho) - 1 = (2^{i+2} j - q - 1) + (2^i - 2^{i+1} j + q - 1)\rho.$$

Thus, $x = u^{2^i j - (q+1)/4}\bar{v}$, where $\bar{v}$ is a polynomial in the $\bar{v}_\nu$. The degree of $\bar{v}$ is $(2^i - 2 + \frac{1}{2}(q + 1))\rho$. As $\frac{1}{2}(q + 1) < 2^{n+1}$, we have

$$|\bar{v}| < |\bar{v}_{n+1}^2| < |\bar{v}_r|$$

for $r \geq n + 2$. Thus, no monomial in $\bar{v}$ is divisible by $\bar{v}_{n+1}^2$ or $\bar{v}_r$. Assume that $|\bar{v}| = |\bar{v}_{n+1}|$. Then $\frac{1}{2}(q + 1) = 2^{n+1} - 1 + 2 - 2^i = 2^{n+1} - 2^i + 1$, which is odd; but then $\frac{1}{4}(q + 1) \notin \mathbb{Z}$, which is a contradiction. Thus, every monomial in $\bar{v}$ is divisible by some $\bar{v}_k$ for some $k \leq n$ as $\bar{v} \neq 1$ for degree reasons. But $a^q\bar{v}_k = 0$ in E_q. Thus, also $a^q x = 0$ in E_q.

Similarly, write $d_q(u^{2^n}) = a^q x$ for $2^{n+1} - 1 \leq q < 2^{n+2} - 1$ and assume that this is nonzero. The degree of x is

$$2^n(4 - 2\rho) - q(1 - \rho) - 1 = (2^{n+2} - q - 1) + (q - 2^{n+1})\rho.$$

Thus, we can write x in E_2 as $u^{2^n-(q+1)/4}\bar{v}$, where $\bar{v}$ is a polynomial in the $\bar{v}_\nu$. The degree of $\bar{v}$ is $\frac{1}{2}(q-1) < 2^{n+1}-1$. Thus, no monomial in $\bar{v}$ can be divisible by $\bar{v}_r$ for $r \geq n+1$. Thus, every monomial in $\bar{v}$ is divisible by some $\bar{v}_k$ for some $k \leq n$ as $\bar{v} \neq 1$ for degree reasons. But $a^q\bar{v}_k = 0$ in E_q. Thus, $d_q(u^{2^n}) = 0$.

By the Leibniz rule, this implies the proposition. □

Before we solve the multiplicative extension issues, we need a technical lemma.

Lemma A.3 *Assume that there is an element $a^k u^l \bar{v} \neq 0$ above the zero line in the E_∞–term of the $RO(C_2)$–graded HFPSS for $BP\mathbb{R}$ with $\bar{v}$ a monomial in the $\bar{v}_\nu$ and in the same degree as $\bar{v}_i\bar{v}_m u^{2^m j}$. Let p be the minimal index such that $\bar{v}_p$ divides $\bar{v}$ (which we will show to exist). Then $i > p+m$.*

Proof The degree of $\bar{v}_i\bar{v}_m u^{2^m j}$ is

$$2^m j(4-2\rho) + (2^i - 1 + 2^m - 1)\rho = 2^{m+2} j + (2^i + 2^m - 2^{m+1} j - 2)\rho.$$

Let $a^k u^l \bar{v} \neq 0$ be an element in E_∞ in this degree with $\bar{v}$ a monomial in the $\bar{v}_\nu$ of degree $n\rho$ and assume that $k > 0$. (In the following, we will use the notation $\|\bar{v}_p\| = |\bar{v}_p|/\rho$ so that $\|\bar{v}\| = n$.) We get

$$4l + k = 2^{m+2} j,$$
$$n - 2l - k = 2^i + 2^m - 2^{m+1} j - 2.$$

This implies $n = 2^i + 2^m - 2 + \frac{1}{2}k$. We see that $n \neq 0$. Let p be the minimal index such that $\bar{v}_p|\bar{v}$. Then $2^p|l$ and we set $c = l/2^p$. Then $k = 2^{m+2} j - 2^{p+2} c$. Due to the relation $a^{2^{p+1}-1}\bar{v}_p = 0$, we have $k \leq 2^{p+1} - 2$ and thus $m+2 \leq p$ (as else $2^{p+1}|k$ and thus $k \geq 2^{p+1}$). In particular, 2^{m+1} divides $\frac{1}{2}k$. Now observe that $n \geq \|\bar{v}_p\| = 2^p - 1$, so

$$2^i + 2^m - 1 \geq 2^p - \tfrac{1}{2}k.$$

As $k \leq 2^{p+1} - 2$, the right-hand side is positive; as it is also divisible by 2^{m+1} it is thus it is at least 2^{m+1}. We see that $i \geq m+1$. Thus $n \equiv 2^m - 2 \mod 2^{m+1}$. As $\|\bar{v}_q\| \equiv -1 \mod 2^{m+1}$ for $q \geq p > m+1$, we see that the total exponent of $\bar{v}$ (ie the degree of $\bar{v}$ as a monomial in the $\bar{v}_\nu$) must be $\equiv 2^m + 2 \mod 2^{m+1}$. In particular, $n \geq \|\bar{v}_p\|(2^m + 2) = (2^p - 1)(2^m + 2)$. Thus,

$$\tfrac{1}{2}k = n - 2^i - 2^m + 2 \geq 2^{p+m} - 2^i + (2^{p+1} - 2^{m+1}).$$

If $p + m \geq i$, then the right-hand side is at least 2^p, which would be a contradiction. Thus $i > p+m$. □

Now, we are ready to prove the main result of the appendix. Note that [18, Theorem 4.11] gives a different relation than our last one; our relation implies their relation, but not vice versa. Note also that our arguments for the multiplicative relations are completely algebraic (using the form of the spectral sequence), while [18] uses additionally a C_2–equivariant Adams spectral sequence.

Theorem A.4 *The ring $\pi_\star^{C_2} BP\mathbb{R}$ is isomorphic to the E_∞–term of the homotopy fixed point spectral sequence above, ie to the subalgebra of*

$$\mathbb{Z}_{(2)}[a, \bar{v}_i, u^{\pm 1}]/(2a, \bar{v}_i a^{2^{i+1}-1})$$

(where i runs over all positive integers) generated by $\bar{v}_m(n) = u^{2^m n}\bar{v}_m$ (with $m, n \in \mathbb{Z}$ and $m \geq 0$) and a with $\bar{v}_0 = 2$. Consequently, it is the quotient R of the ring

$$\mathbb{Z}_{(2)}[a,\ \bar{v}_m(n) \mid m \geq 0,\ n \in \mathbb{Z}]$$

by the relations

$$\begin{aligned} \bar{v}_0(0) &= 2, \\ a^{2^{m+1}-1}\bar{v}_m(n) &= 0, \\ \bar{v}_i(j)\bar{v}_m(n) &= \bar{v}_i\bar{v}_m(2^{i-m}j + n) \quad \text{for } i \geq m, \end{aligned}$$

with $\bar{v}_i = \bar{v}_i(0)$. Here, $|a| = 1 - \rho$ and $|\bar{v}_m(n)| = 2^{m+2}n + (2^m - 1 - 2^{m+1}n)\rho$.

Proof It suffices to show that the expression above computes the homotopy fixed points $\pi_\star^{C_2} BP\mathbb{R}^{(EC_2)_+}$. Indeed, Proposition A.2 implies that $(a^{-1}BP\mathbb{R}^{(EC_2)_+})^{C_2} \simeq H\mathbb{F}_2$, so the map $BP\mathbb{R}^{\Phi C_2} \to BP\mathbb{R}^{tC_2}$ is an equivalence and hence also $BP\mathbb{R} \to BP\mathbb{R}^{(EC_2)_+}$ by the Tate square.

Set $\bar{v}_0(0) = 2$. By Proposition A.2, the classes $u^{2^m n}\bar{v}_m$ are permanent cycles in the HFPSS; choose element $\bar{v}_m(n) \in \pi_\star^{C_2} BP\mathbb{R}^{(EC_2)_+}$ representing them. Again by Proposition A.2, the $\bar{v}_m(n)$ generate together with a the E_∞–term of the HFPSS. Thus, we get a surjective map $R \to E_\infty$. The third relation defining R allows to define a normal form: Every monomial in the $\bar{v}_i(j)$ equals in R an element of the form $\bar{v}\,\bar{v}_m(k)$, where $\bar{v}$ is a monomial in the $\bar{v}_i$ and m was the smallest index of all $\bar{v}_i(j)$. Thus, two monomials in the $\bar{v}_i(j)$ are equal in R if they are equal in E_∞; hence, the map $R \to E_\infty$ is also injective.

We now check that the relations are also satisfied in $\pi_\star^{C_2} BP\mathbb{R}^{(EC_2)_+}$. This is clear or was already shown for the first two relations. Let now i be the least number such that $m \leq i$ and

$$\bar{v}_i(j)\bar{v}_m(n) \neq \bar{v}_i\bar{v}_m(2^{i-m}j + n)$$

for some j, m, n if such an i exists. The difference must be detected by a class $a^k u^l \bar{v}$, where $\bar{v}$ is a polynomial in the $\bar{v}_\nu$. Let p the minimal index such that every monomial in $\bar{v}$ is divisible by a $\bar{v}_r$ with $r \le p$. From Lemma A.3, we know that $p \le i-1$ (and in particular $i \ge 1$). Thus,

$$\bar{v}_i(j)\bar{v}_m(n)\bar{v}_{i-1} \neq \bar{v}_i\bar{v}_m(2^{i-m}j+n)\bar{v}_{i-1}$$

as their difference is detected by a nonzero class $a^k u^l \bar{v}\bar{v}_{i-1}$ (indeed, this could only be zero if $k \ge 2^i - 1$, but $k < 2^{p+1} - 1$). By the minimality of i, we have

$$\bar{v}_m(2^{i-m}j+n)\bar{v}_{i-1} = \bar{v}_{i-1}(2j)\bar{v}_m(n).$$

In addition, $\bar{v}_i\bar{v}_{i-1}(2j) = \bar{v}_i(j)\bar{v}_{i-1}$ because there is no element of higher filtration in the same degree as $\bar{v}_{i-1}\bar{v}_i(j)$ by Lemma A.3. The last two equations combine to the chain of equalities

$$\begin{aligned}\bar{v}_i(j)\bar{v}_m(n)\bar{v}_{i-1} &= \bar{v}_i\bar{v}_{i-1}(2j)\bar{v}_m(n)\\ &= \bar{v}_i\bar{v}_m(2^{i-m}j+n)\bar{v}_{i-1}.\end{aligned}$$

This is a contradiction to the inequality above. Thus,

$$\bar{v}_i(j)\bar{v}_m(n) = \bar{v}_i\bar{v}_m(2^{i-m}j+n)$$

is always true for $i \ge m$. □

Remark A.5 We remark that all the work above for the multiplicative extensions was actually necessary. For example, we get from the homotopy fixed point spectral sequence only that $\bar{v}_5\bar{v}_1(1) - \bar{v}_5(1)\bar{v}_1(-15)$ has filtration at least 1. But there are indeed classes in this degree of higher filtration, for example, $a^8\bar{v}_3^3\bar{v}_4$.

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