Algebraic & Geometric Topology 17 (2017) 3621–3664

Slice implies mutant ribbon for odd 5–stranded pretzel knots

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A pretzel knot K is called *odd* if all its twist parameters are odd and *mutant ribbon* if it is mutant to a simple ribbon knot. We prove that the family of odd 5–stranded pretzel knots satisfies a weaker version of the slice-ribbon conjecture: all slice odd 5–stranded pretzel knots are *mutant ribbon*, meaning they are mutant to a ribbon knot. We do this in stages by first showing that 5–stranded pretzel knots having twist parameters with all the same sign or with exactly one parameter of a different sign have infinite order in the topological knot concordance group and thus in the smooth knot concordance group as well. Next, we show that any odd 5–stranded pretzel knot with zero pairs or with exactly one pair of canceling twist parameters is not slice.

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1 Introduction

A knot $K \subset S^3$ is *smoothly slice* if it bounds a smoothly embedded disk in the 4–ball. Similarly, a knot $K \subset S^3$ is said to be *topologically slice* if it bounds a locally flat embedded disk $D \subset B^4$, where D is a locally flat submanifold of B^4, if for every point $x \in D$ there exists a neighborhood $U \subset B^4$ of x such that the pair $(U, U \cap D)$ is homeomorphic to the pair $(\mathbb{R}^4, \mathbb{R}^2)$. The notions of smoothly slice and topologically slice knots can be used to define the smooth and topological knot concordance groups $\mathcal{C}$ and $\mathcal{T}$, respectively, under the operation of connected sum. These are widely studied groups for which the corresponding slice knot represents the identity element. For explicit information about the concordance relations, see Livingston [12]. Fine details of the group structure of $\mathcal{C}$ and $\mathcal{T}$ continue to elude mathematicians, but concordance order is one small way of gaining insights into these groups. The topic of determining smoothly slice knots and concordance order for knots within families of pretzel knots has also been studied with increasing frequency over the past 30 years and various results can be found in Greene and Jabuka [3], Lecuona [10], Miller [14], Herald, Kirk and Livingston [4] and Long [13]. This work will focus almost entirely on slice knots and concordance in the smooth case, except where "topological" is explicitly stated.

The slice-ribbon conjecture hypothesizes that if a knot is slice then it is also ribbon. Given that ribbon knots are easily seen to be slice, this is ultimately a conjecture about

Published: 4 October 2017 DOI: 10.2140/agt.2017.17.3621

the equivalence of the notions “slice” and “ribbon”. Previous work by Joshua Greene and Stanislav Jabuka in [3] on the slice-ribbon conjecture for odd 3–stranded pretzel knots and work by Ana Lecuona in [10] on even pretzel knots inspired this project. This paper studies sliceness and concordance order for odd 5–stranded pretzel knots.

A *k–stranded pretzel link*, denoted by $P(p_1, p_2, \dots, p_k)$, where the $p_i \in \mathbb{Z} - \{0\}$ are called the *twist parameters*, is a knot in two cases: when exactly one of the twist parameters is even, or when k is odd and all the twist parameters are odd. A pretzel knot will be called *even* in the former case and *odd* in the latter. A 0*–pair pretzel knot* is a pretzel knot for which there are no canceling pairs of twist parameters satisfying $p_i = -p_j$. A 1*–pair pretzel knot* is a pretzel knot for which there exists a canceling pair of twist parameters, but when the pair is removed from the k–tuple defining the knot, the resulting $(k-2)$–stranded knot is 0–pair. Generally, a t*–pair pretzel knot* is one for which removing a single canceling pair of twist parameters results in a $(t-1)$–pair pretzel knot with two fewer strands. With this definition, 5–stranded pretzel knots $P(a, b, c, d, e)$ can be 0–pair, 1–pair or 2–pair. See Figure 1.

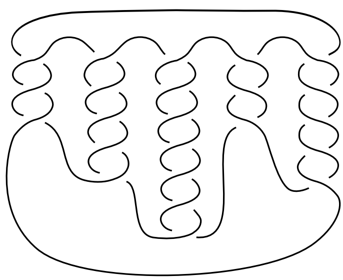


Figure 1: Pretzel knot $P(3, 5, 7, -3, -5)$

When proving statements about pretzel knots, it is often necessary to differentiate between the knots that contain twist parameters equal to ± 1 and those that do not. If for $K = P(p_1, \dots, p_k)$ there exists $i \in \{1, \dots, k\}$ such that $p_i = \pm 1$, then we say K is a *pretzel knot with single-twists*; otherwise, we say K is a *pretzel knot without single-twists*.

The classification of pretzel knots appears in Zieschang [20], a work that classifies the much larger class of Montesinos knots of which pretzel knots are a special case. The classification gives that two pretzel knots without single-twists are smoothly isotopic if their twist parameters differ by cyclic permutations, reflections, or compositions thereof. Two pretzel knots *with* single-twists are smoothly isotopic if their twist parameters differ by cyclic permutations, reflections and/or transpositions involving ± 1–twisted strands. Two k–stranded pretzel knots whose twist parameters are equal as *unordered*

k–tuples but not equal as *ordered* k–tuples are called *pretzel knot mutants*. This specific kind of mutation is the only type considered here, so "mutation" from this point on will always mean "pretzel knot mutation".

Mutation is a crucial topic for the problem of determining sliceness for k–stranded pretzel knots when $k \geq 4$ because many knot invariants used to obstruct sliceness are unable to detect pretzel knot mutants. In fact, any knot invariant based on the double branched cover of S^3 along the knot will fail to detect pretzel knot mutants; Bedient shows in [2] that any two pretzel knots defined by the same unordered k–tuple of twist parameters have the same double branched cover. Given a k–tuple $(p_1, \ldots, p_k)$ of twist parameters, $P\{p_1, \ldots, p_k\}$ will denote the set of pretzel knots having $\{p_1, \ldots, p_k\}$ as twist parameters, and also all mirrors of such knots.

Among pretzel knots is a subset of knots we will call *simple ribbon*. A *simple ribbon move* on a pretzel knot is the ribbon move shown in Figure 2, performed always on the topmost twist of two adjacent strands of K having canceling numbers of twists. We say a pretzel knot K is *simple ribbon* if there exists a sequence of simple ribbon moves that reduces K to a 1–stranded pretzel knot (if K is odd) or to a 2–stranded pretzel knot $P(a,b)$ where $a = -b-1$ (if K is even). A prerequisite for a pretzel knot to be simple ribbon is that if K is k–stranded, then K must be $\frac{1}{2}(k-1)$–pair. But, while all 1–pair 3–stranded pretzel knots are simple ribbon, not all 2–pair 5–stranded pretzel knots are simple ribbon. For example, the 2–pair knot $P(3,5,-3,-5,7)$ is not simple ribbon because no two adjacent strands have canceling numbers of twists. This phenomenon extends for all $k \geq 4$.

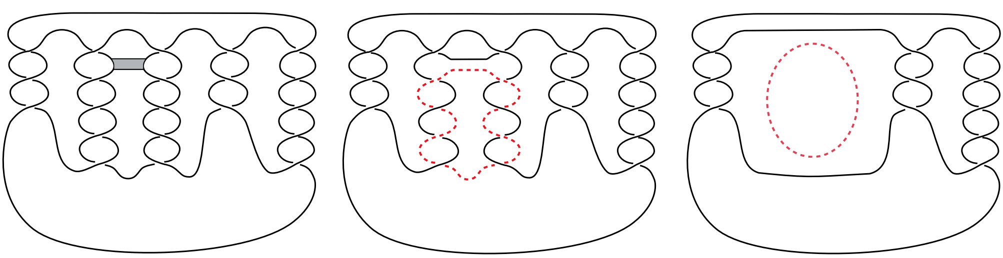


Figure 2: Simple ribbon move on pretzel knot $P(-3,-5,5,3,-5)$

The remainder of this paper is structured as follows: Section 2 presents our main results, the strongest of which is Corollary 2.5, a weak version of the slice-ribbon conjecture for generic odd 5–stranded pretzel knots. Section 3 gives foundational information on branched covers, framed links, weighted graphs and plumbings in the specific context of 4–dimensional topology. Section 4 describes a classical slice obstruction, the signature condition, and it gives the proof of Theorem 2.1. Section 5 gives details about Donaldson's diagonalization theorem and a resulting slice obstruction we call the lattice embedding condition. Section 6 addresses the final slice obstructions utilized in this work, d–invariants and the coset counting conditions. The proofs of the main results following Theorem 2.1 are given in Sections 7–10 and are organized by increasing number of slice obstructions needed to obtain the desired result.

Acknowledgements The author would like to thank her advisor Paul Melvin for his patience and guidance throughout this project. She is forever indebted to him for not only passing along his knowledge of this subject, but also for his invaluable edits of her work, both in style and content. She would also like to thank the referee for helpful suggestions and great attention to detail.

2 Results

As previously mentioned, this project was motivated by work of Greene and Jabuka in [3] on the slice-ribbon conjecture for odd 3–stranded pretzel knots and by work of Ana Lecuona in [10] on even pretzel knots. Lecuona writes down the following conjecture:

Pretzel ribbon conjecture (Lecuona) Let K be a pretzel knot whose twist parameters are all greater than 1 in absolute value. If K is ribbon, then K is simple ribbon.

For odd 3–stranded pretzel knots, the pretzel ribbon conjecture posits that the only ribbon knots are the simple ribbon knots, ie the 1–pairs. Similarly for odd 5–stranded pretzel knots, it says that the only ribbon knots are the simple ribbon knots which are 2–pairs *for which at least one of the canceling pairs is adjacent*. Greene and Jabuka show in [3] that odd 3–stranded pretzel knots satisfy both the pretzel ribbon conjecture and the slice-ribbon conjecture by proving that a knot of this type is slice if and only if it is 1–pair. This result, which proves the two aforementioned conjectures in a particular case, hints to the following possible strengthening of the slice-ribbon conjecture in the specific case of pretzel knots:

Pretzel slice-ribbon conjecture If K is a slice pretzel knot, then K is simple ribbon.

Of course, if the pretzel ribbon conjecture is true then the above is equivalent to the original version of the slice-ribbon conjecture. There is evidence that supports the pretzel slice-ribbon conjecture in the odd 5–stranded case. Herald, Kirk and Livingston [4] prove that $P(3,5,-3,-5,7)$ is *not* slice despite being mutant to the two simple ribbon knots $P(3,-3,5,-5,7)$ and $P(3,5,-5,-3,7)$. See Figure 2 for an illustration of the ribbon move on two adjacent strands in a 5–stranded pretzel knot whose twist numbers cancel.

This present work applies the techniques used by Jabuka and Greene on odd 3–stranded pretzel knots to odd 5–stranded pretzel knots, in the hope of showing that this new class of knots also satisfies the pretzel slice-ribbon conjecture as well. It should be noted that Greene and Jabuka went a step farther and proved that all nonslice odd 3–stranded pretzel knots have infinite order in $\mathcal{C}$. To obtain these results, they used three tools: the knot signature from classical knot theory, Donaldson's diagonalization theorem from gauge theory, and the d–invariant from Heegaard Floer theory.

The main results of this project are given below, accompanied by brief explanations as to where each of the above three tools comes into play. In Theorem 2.1 and its corollary, $\sigma(K)$ denotes the signature of K; s is the difference between the number of positive twist parameters and the number of negative twist parameters of K; $\widehat{e}$ is the orbifold Euler characteristic of K given by the sum of the reciprocals of the twist parameters; and sgn() is the function returning -1, 0, or $+1$ according to whether the input is negative, zero, or positive, respectively. The first result is about the class of odd pretzel knots:

Theorem 2.1 *If K is an odd pretzel knot, then $\sigma(K) = s - \operatorname{sgn}(\widehat{e})$. In particular, $\sigma(K) = 0$ if and only if $s = \operatorname{sgn}(\widehat{e})$.*

Corollary 2.2 *All odd pretzel knots with $s \neq \operatorname{sgn}(\widehat{e})$ have infinite order in the topological knot concordance group $\mathcal{T}$.*

The corollary follows from the fact that σ is a homomorphism from $\mathcal{T} \to \mathbb{Z}$, and it implies infinite order in the smooth knot concordance group $\mathcal{C}$ as well. It is a well-known fact that we call on later that if a knot K is slice, then $\sigma(K) = 0$. An implicit implication of Theorem 2.1 is that all odd pretzel knots for which $s \neq \pm 1$ are not slice, which is particularly easy to read off from the k–tuple defining the knot. For odd 5–stranded pretzel knots this tells us that if all or all but one of the twist parameters have the same sign, then K is not slice.

Powerful as the signature is as a concordance invariant, the signature alone is insufficient for determining sliceness in odd pretzel knots for which $s = \pm 1$. For example, the pretzel knot $K = P(-3, -5, -7, 9, 27)$ has vanishing signature, but the pretzel slice-ribbon conjecture gives us reason to think that K may not be slice. Such occurrences in the odd 3–stranded case prompted Jabuka and Greene to turn to an obstruction based on Donaldson's diagonalization theorem, which is ultimately phrased as a lattice embedding condition necessary for sliceness. This same obstruction was originally used by Paolo Lisca in [11] to classify slice knots within the family of 2–bridge knots.

The use of Donaldson's diagonalization theorem to define a "lattice embedding condition" for sliceness is based on the construction of a (potentially hypothetical) closed, definite 4–manifold X, created as follows: Assume K is a slice knot. Let Y be the double branched cover of S^3 along K; let W be the double branched cover of the 4–ball with branching set the slice disk for K, so that W is a rational homology 4–ball with $\partial W = Y$; let P be a canonical definite 4–dimensional plumbing[1] with $\partial P = Y$. Define $X = P \cup_Y (-W)$. The lattice embedding condition arises by applying the diagonalization theorem to X, for which it is necessary to verify that the intersection form on X, Q_X, can in fact be diagonalized over the integers. We do this in Section 5.

The lattice embedding condition for sliceness puts great restrictions on the possible k–tuples that can define a slice odd pretzel knot, so it enables us to conclude that all but a very select subset of such knots are not slice. Unfortunately, the knots that satisfy both the vanishing signature condition and the lattice embedding condition are not easily differentiated from the knots satisfying the signature condition but *not* the lattice embedding condition. For example, sliceness is obstructed for $P(-3,-17,27)$ and $P(-3,-7,-19,17,55)$ by the lattice embedding condition, but *not* obstructed for $P(-3,-17,29)$ and $P(-3,-7,-19,19,55)$.

For this reason Jabuka and Greene introduced a third slice obstruction based on the d–invariant from Heegaard Floer theory. It assumes the same construction used above involving K, Y, W, P, and X, but it boils down to a comparison of two different "counts" obtained by analysis on the homology long exact sequences of the pairs (X, W) and (P, Y). We refer to it here as "coset condition I". Combining the signature obstruction, the lattice embedding condition, and coset condition I, Jabuka and Greene were able to prove their full result. With these same tools, we obtain the following results for odd 5–stranded pretzel knots with signature zero:

Theorem 2.3 *If K is a 0–pair odd 5–stranded pretzel knot, then K is not slice.*

Coset condition I fails to obstruct sliceness in t–pair odd k–stranded pretzel knots K if $t \geq 1$, k is odd, and $\sigma(K) = 0$, so yet another tool is required to prove analogous results in the present case. When $k \geq 5$, the increased number of twist parameters introduces complexity not present when $k = 3$, requiring a more refined "count" than Jabuka and Greene needed when implementing the d–invariant obstruction. With just

[1] A canonical definite plumbing P is one for which the weights of the vertices in the corresponding plumbing graph are either all ≥ 2 or ≤ -2. It should be noted that not all knots have such plumbings, but that pretzel knots do.

a little bit of work we derive a stronger version of coset condition I and uncreatively call it coset condition II. Combining the signature obstruction, the lattice embedding condition, and coset condition II, we prove:

Theorem 2.4 *If K is a 1–pair odd 5–stranded pretzel knot without single-twists, then K is not slice.*

Theorem 2.4 avoids mention of odd 5–stranded pretzel knots *with* single-twists because they behave slightly differently from those without single-twists for the following reason: any strand with exactly one positive or negative half twist can be transposed with an adjacent strand through a *flype* as in Figure 3. Such a move preserves the smooth knot type thus, for example, $P(1,3,-5,1,-7)$ and $P(1,1,3,-5,-7)$ are not only mutants of one another but also members of the same smooth isotopy class.

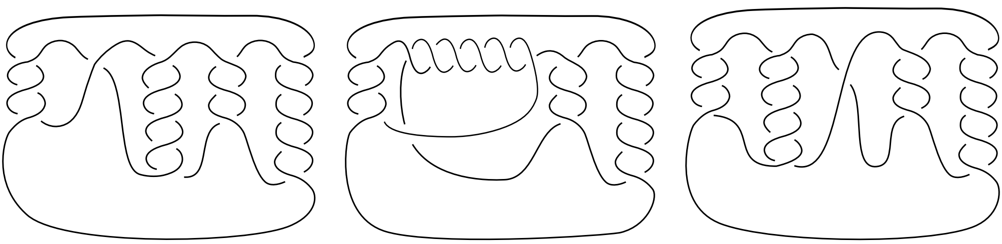


Figure 3: Transposition of a single-twist strand, turning $P(3,1,5,-3,-5)$ into $P(3,5,1,-3,-3)$

Furthermore, by flyping we can always "collect" all strands with ± 1 twists so that they occur in succession. This has the greatest impact on 1– and 2–pair pretzel knots for which at least one of the pairs is $\{-1,1\}$. If K is defined by $\{-1,1,b,c,d\}$, then K is not only concordant to $P(b,c,d)$ but also smoothly isotopic, regardless of the initial locations of 1 and -1 in the 5–tuple. It follows that every 2–pair odd 5–stranded pretzel knot containing the pair $\{-1,1\}$ is simple ribbon. To contrast, if $K\in P\{-a,a,b,c,d\}$ with $|a|\geq 3$, then K is smoothly concordant to $P(b,c,d)$ if and only if the pair $\{-a,a\}$ is adjacent; it is precisely this that leads to $P(3,5,-3,-5,7)$ and $P(3,-3,5,-5,7)$ having different smooth concordance order.

Theorems 2.3 and 2.4 together imply that odd 5–stranded pretzel knots without single-twists satisfy a weaker version of the slice-ribbon conjecture:

Corollary 2.5 *If K is a slice odd 5–stranded pretzel knot without single-twists, then K is mutant to a simple ribbon knot.*

For 2–pair, odd, 5–stranded pretzel knots (with or without single-twists) not containing the pair $\{-1,1\}$ and for 1–pairs with single-twists and pair $\{-a,a\} \neq \{-1,1\}$, the signature condition, the lattice embedding condition, and coset conditions I and II all fail to obstruct sliceness in the knots that are not simple ribbon because these slice obstructions, at their cores, obstruct the double branched covers of the knots from having certain properties. As previously mentioned, all mutants of a given pretzel knot share the same double branched cover and hence there is no hope of obstructing sliceness for a knot $K \in P\{a,b,c,d,e\}$ if any member of $P\{a,b,c,d,e\}$ is slice. Since 2–pair knots of the form $P(a,-a,b,-b,c,)$ and $P(a,b,-b,-a,c)$ are simple ribbon and therefore slice, we cannot use the aforementioned tools to say that $P(a,b,-a,-b,c)$ is not slice. Similarly, Remark 1.3 in [3] gives that the 1–pair knots with single-twists and pair $\{-a,a\} \neq \{-1,1\}$ of the form $P(a,-a,1,b,c)$ with $b+c=4$ are slice, and therefore again there is no way to distinguish between slice and suspected nonslice members of $P\{a,-a,1,b,c\}$.

In [4], Herald, Kirk and Livingston used twisted Alexander polynomials to show that the 2–pair knot $P(3,5,-3,-5,7)$ is not slice, despite being mutant to the simple ribbon knot $P(3,-3,5,-5,7)$. Twisted Alexander polynomials are able to distinguish mutants and, in fact, they can reveal when a knot is not *topologically* slice. The issue in using twisted Alexander polynomials to show that pretzel knots satisfy the slice-ribbon conjecture is that their computation relies on number-theoretic choices that often make it difficult to find general formulas for infinite families of knots. Of the examples computed for pretzel knots to date, there is only one infinite family of pretzel knots whose slice status has been determined using twisted Alexander polynomials. It is a subfamily of the 4–stranded family $K = P(2n,m,-2n \pm 1,-m)$, done by Allison Miller in [14].

3 Branched covers, framed links, weighted graphs and plumbings

Let $K = P(a_1,\dots,a_p,-b_1,\dots,-b_n)$ be an odd k–stranded pretzel knot with $k = p+n$ odd, $a_i, b_j > 0$, and let Y be the double branched cover of S^3 along K. As a 3–manifold, we will describe Y by two framed links L_0 and L_+ which differ by a sequence of moves in the Kirby calculus for framed links. The links L_0 and L_+ are equivalently represented by weighted star-shaped graphs Γ_0 and Γ_+, shown in Figure 4. In Γ_0 and Γ_+, each vertex v_i with weight $w(v_i)$ represents an unknot component K_i with framing $r_i = w(v_i)$; two components K_i and K_j link once in L_0 (resp. L_+) if the corresponding vertices v_i and v_j share an edge in Γ_0 (resp. Γ_+).

To obtain the double cover of S^3 branched along a pretzel knot, we use the following algorithm of Montesinos: start with an unknot in S^3 and the double cover of S^3

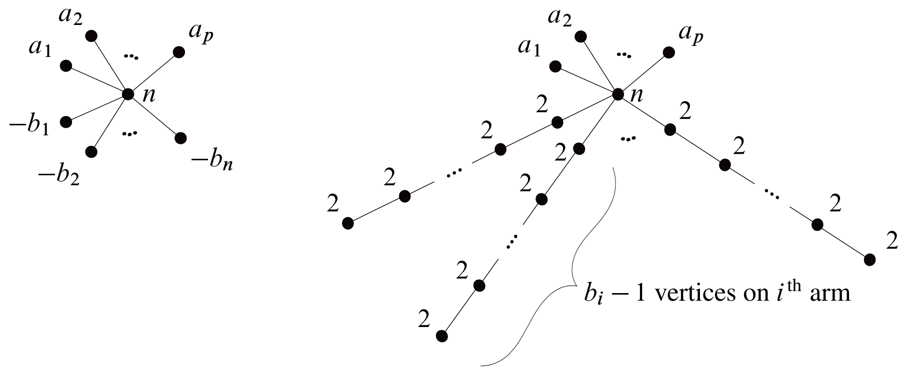


Figure 4: Weighted plumbing graphs Γ_{L_0} (left) and Γ_{L_+} (right)

branched along the unknot. Recall that the double cover of S^3 branched along the unknot is, again, S^3, which can be described as surgery on an unknot with 0–framing. The unknot can be turned into a $P(p_1, \dots, p_k)$ pretzel knot by replacing a 0–tangle in the unknot by a p_i–tangle for each i and replacing a single ∞–tangle by a 0–tangle, as shown in Figure 5 for a 5–stranded pretzel knot. Determining the double branched

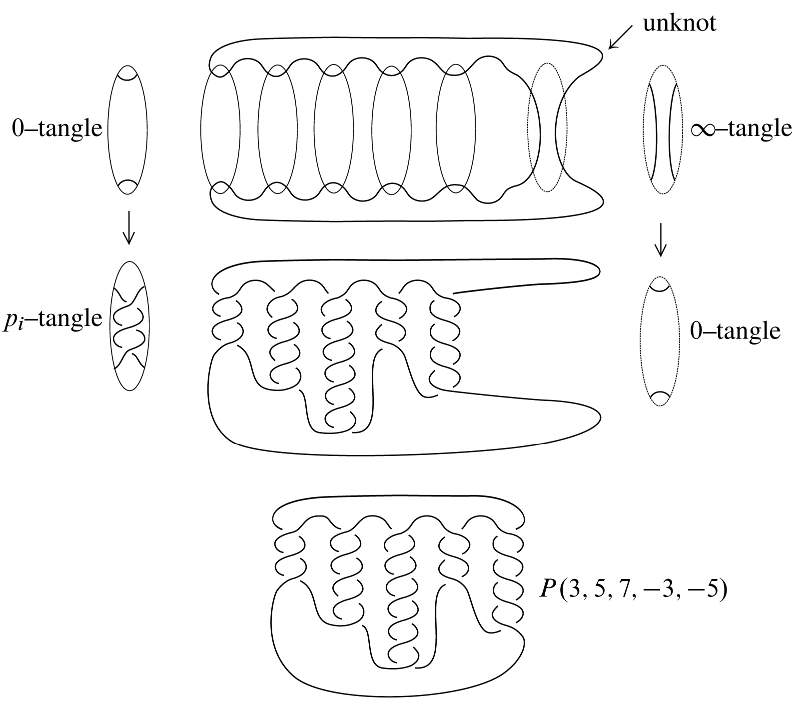


Figure 5: Obtaining a 5–stranded pretzel knot by replacing tangles

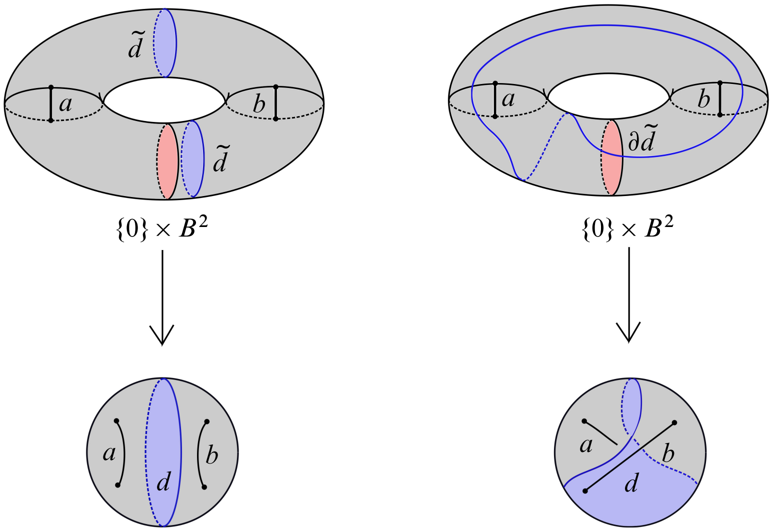
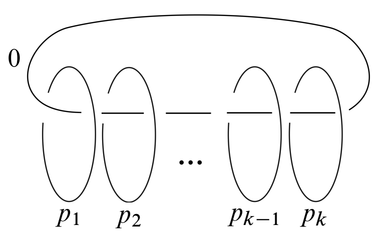


Figure 6: Double covers of B^3 branched along the ∞–tangle and over the $\frac{1}{2}$–tangle

cover of S^3 along $P(p_1, \dots, p_k)$ now amounts to accounting for the changes affected in the cover above by the tangle replacements in the knot below.

The double cover of B^3 branched along a p_i–tangle is a solid torus T_{p_i} parametrized by $S^1 \times B^2$ such that the lift $\widetilde{d}$ of the disk d in B^3 separating the two arcs of the tangle satisfies (i) $\partial\widetilde{d} \cap (\{0\} \times B^2) = 1$ and (ii) $\mathrm{lk}(\partial\widetilde{d}, S^1 \times \{0\}) = p_i$. See Figure 6. As such, the effect in the double branched cover of replacing a 0–tangle by a p_i–tangle is p_i–surgery on an unknot in S^3; the effect in the cover of replacing the ∞–tangle with the 0–tangle is 0–surgery on an unknot in S^3 that links once with each of the other unknots. Considering all surgeries as being done simultaneously, the double cover of S^3 branched along the pretzel knot $P(p_1, \dots, p_k)$ is thus obtained by surgery on S^3 described by this framed link:[2]



[2] From this construction, we see that any two pretzel knots defined by the same unordered k–tuple of twist parameters have the same double branched cover. As a result, any knot invariant based on the double branched cover of S^3 along the knot will fail to detect pretzel knot mutants.

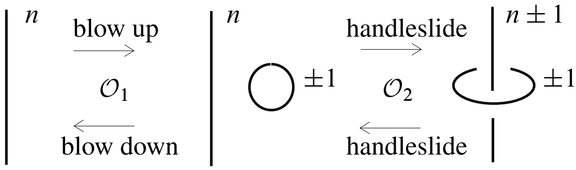


Figure 7: "Slam dunk" shortcut

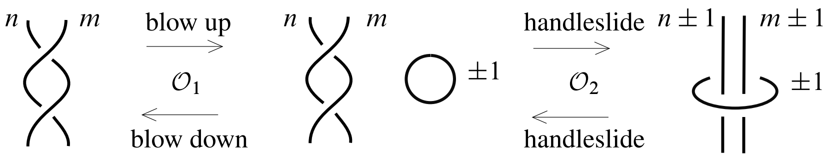


Figure 8: "Kirby twist" shortcut

To transform L_0 into L_+ via Kirby moves, we operate on the negatively framed components of L_0 using the shortcuts shown in Figures 7 and 8. The course of Kirby moves needed to change L_0 into L_+ is described in Figure 9. The accompanying

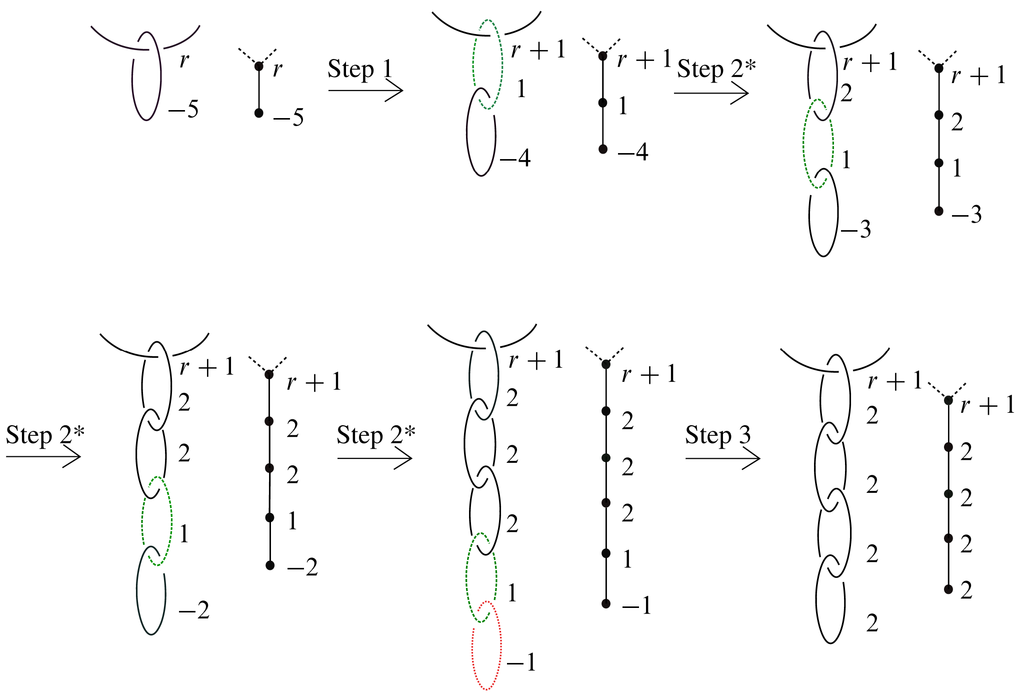


Figure 9: Kirby calculus sequence $L_0 \to L_+$. The asterisk on Step 2 indicates that it will be repeated until the bottommost component has framing -1.

weighted graphs are shown as well. If the prescribed sequence of moves is performed on a component with framing $-b_i$, then Step 2 is repeated $b_i - 3$ times and the original component is ultimately replaced by $b_i - 1$ new components, all of which are unknots with framing 2. In the corresponding weighted graphs, this translates into replacing a single arm of length one, whose lone vertex has weight $-b_i$, by an arm of length $b_i - 1$ containing all weight-2 vertices; the weight of the central vertex increases by 1 for each arm altered.

Any sequence of Kirby moves used to transform one framed link into another is mimicked in the corresponding graphs by adding (resp. deleting) vertices and edges to the graph at the expense of subtracting 1 (resp. adding 1) to the weights of the vertices sharing an edge with the added (resp. deleted) vertex.

In addition to describing the double branched cover Y of S^3 along a pretzel knot, the framed links L_0 and L_+ and weighted graphs Γ_0 and Γ_+ define 4–dimensional plumbings P_0 and P_+ (respectively) bounded by Y. The plumbings P_0 and P_+ can be viewed as 4–dimensional handlebodies whose handle decompositions consist entirely of a single zero handle and a collection of 2–handles. For P_0, each component $C_i \subset L_0$ corresponds to a 4–dimensional 2–handle G_i that is attached to B^4 (the single 0–handle) by a map $f\colon \bigcup_i(\widetilde{G}_i) \to \partial B^4$, where $\widetilde{G}_i = S^1 \times B^2$ is the attaching region of G_i. The map f identifies $\widetilde{G}_i$ with a tubular neighborhood N_i of C_i, such that the core of $\widetilde{G}_i$ lies along C_i and a meridian of $\partial\widetilde{G}_i$ is identified with a curve in N_i that links n_i times with C_i. Plumbing P_+ is given similarly.

The effect on $\partial B^4 = S^3$ of attaching 2–handles to B^4 corresponds exactly to performing surgery on S^3 by viewing the identification of $\widetilde{G}_i$ with N_i instead as removing N_i and replacing it with the $\widetilde{G}_i$ according to the framing. Since P_0 and P_+ are described by L_0 and L_+ as plumbings/4–dimensional handlebodies and Y is described by L_0 and L_+ as surgery on S^3, it follows that $\partial P_0 = \partial P_+ = Y$.

The intersection form Q_{P_0} for P_0, represented as a matrix with basis equal to the set of classes represented by the zero-section of each plumbed disk bundle, is equal to the incidence matrix for Γ_0. Likewise, with an analogous choice of basis the intersection form of P_+ is equal to the incidence matrix for Γ_+. Knowing an exact sequence of Kirby moves between L_0 and L_+ allows one to compute the overall change in the signature from P_0 to P_+ by analyzing how the signature changes with each step.

At this point, it is worth detailing a labeling scheme for the vertices of Γ_0 and Γ_+ so that the bases for the incidence matrices are ordered consistently. Given the vertex labelings pictured in Figure 10, Γ_0 will have ordered basis $\{v_0, v_1, \dots, v_p, v_{p+1}, \dots, v_{p+n}\}$ and Γ_+ will have ordered basis $\{s_0, s_1, \dots, s_p, s_{1,1}, \dots, s_{1,r_1}, \dots, s_{n,1}, \dots, s_{n,r_n}\}$. Written succinctly, the basis for Γ_+ can be written $\{s_i, s_{j,r_j}\}$ where $0 \le i \le p$,

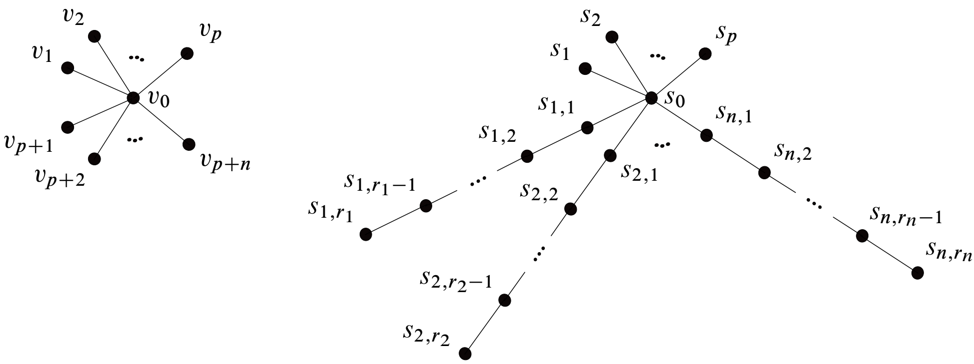


Figure 10: Vertex labeling scheme for Γ_0 (left) and Γ_+ (right)

$1 \leq j \leq n$, and r_j is equal to one less than the number of vertices in the j^{th} arm of Γ_+. It is with these ordered bases for Γ_0 and Γ_+ that the matrices for Q_{P_0} and Q_{P_+} are given later.

4 The signature condition and proof of Theorem 2.1

The *signature* of a symmetric matrix Q, denoted $\sigma(Q)$, is the difference between the number of positive diagonal entries and the number of negative diagonal entries of Q, after Q has been diagonalized over $\mathbb{R}$. The *signature of a knot* K is defined as $\sigma(K) = \sigma(V^T + V)$, where V is a Seifert matrix for K. Given a 4–manifold X with intersection form Q_X, the *signature of* X is defined as the signature of Q_X: $\sigma(X) = \sigma(Q_X)$. The signature is an abelian invariant based on the double branched cover of the knot, and therefore it cannot detect pretzel mutants.

The signature is a homomorphism $\sigma\colon \mathcal{T} \to \mathbb{Z}$, where $\mathcal{T}$ is the topological knot concordance group. Hence

(1) $\sigma(-K) = -\sigma(K)$, where $-K$ is the mirror of K, and

(2) $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$.

The signature is also invariant under mutation; see [8]. For pretzel knots, if we combine this fact with (1) we see that computation of the signature of $K = P(p_1, \dots, p_k)$ may be obtained using any knot in $P\{p_1, \dots, p_k\}$. Often, a specific K is chosen in order to simplify computations. Homomorphism property (2) implies that if $\sigma(K) > 0$, then the knot K will have infinite order in the topological (and therefore smooth) knot concordance group. A classical theorem (a proof of which can be found in [18]) states that *any slice knot has signature zero*.

With this result we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 Let $K = P(a_1, \dots, a_p, -b_1, \dots, -b_n)$ be an odd pretzel knot. For simplicity throughout this proof, we will make two notational simplifications: First, we will call a framed link and the 4–manifold it describes by the same name; in particular, we will use the framed link L_0 and the corresponding plumbing manifold P_0 from Section 3 in this proof but we will resort to using P_0 to describe both, letting context dictate whether we are referring to the framed link or the 4–manifold. This notational simplification will be implicit in all other framed links and 4–manifolds defined within this proof. The second notational simplification we will make is to write $\sigma(M)$ rather than $\sigma(Q_M)$ for the signature of a 4–manifold M; in fact, this is less of an idiosyncrasy in notation and more appropriately an understated definition of the signature of a 4–manifold as the signature of its intersection form.

Kauffman and Taylor prove in [7] that $\sigma(K) = \sigma(T)$, where T is the double branched cover of B^4 along any pushed-in Seifert surface of K. In [1], Akbulut and Kirby give an algorithm for computing the p–fold ($p \geq 2$) cyclic cover of B^4 branched along a pushed-in Seifert surface for a given knot, where the Seifert surface used is one that can be described as a disk with possibly twisted and possibly knotted bands attached. Such a Seifert surface F can be obtained for a pretzel knot by an isotopy of the "standard" Seifert surface, ie the Seifert surface obtained from Seifert's algorithm via the standard pretzel knot diagram. Using this particular Seifert surface F for K together with Akbulut and Kirby's algorithm for the double branched cover of B^4, we get a framed link that describes the handlebody structure of the 2–fold cover of B^4 branched along F pushed-in. We call this particular cover T. See Figure 11 for an example of F and T for the pretzel knot $P(3, 5, -5, -3, -5)$.

Rather than compute $\sigma(T) = \sigma(K)$ directly, we will instead show that $\sigma(T) = \sigma(T \# (S^2 \tilde{\times} S^2)) = \sigma(P_0)$, where P_0 is the plumbing manifold from Section 3. By choosing the basis for Q_{P_0} to be the set of spheres obtained from the cores of the attaching 2–handles together with hemispheres in B^4 (alternatively, the spheres are the 0–sections of the disk bundles used to create P_0), Q_{P_0} is given by the incidence matrix of the plumbing graph Γ_0 from Section 3. A straightforward diagonalization of Q_{P_0} shows that

$$Q_{P_0} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a_1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & a_p & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -b_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & -b_n \end{bmatrix} \sim \begin{bmatrix} -\hat{e} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & a_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b_1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & -b_n \end{bmatrix}.$$

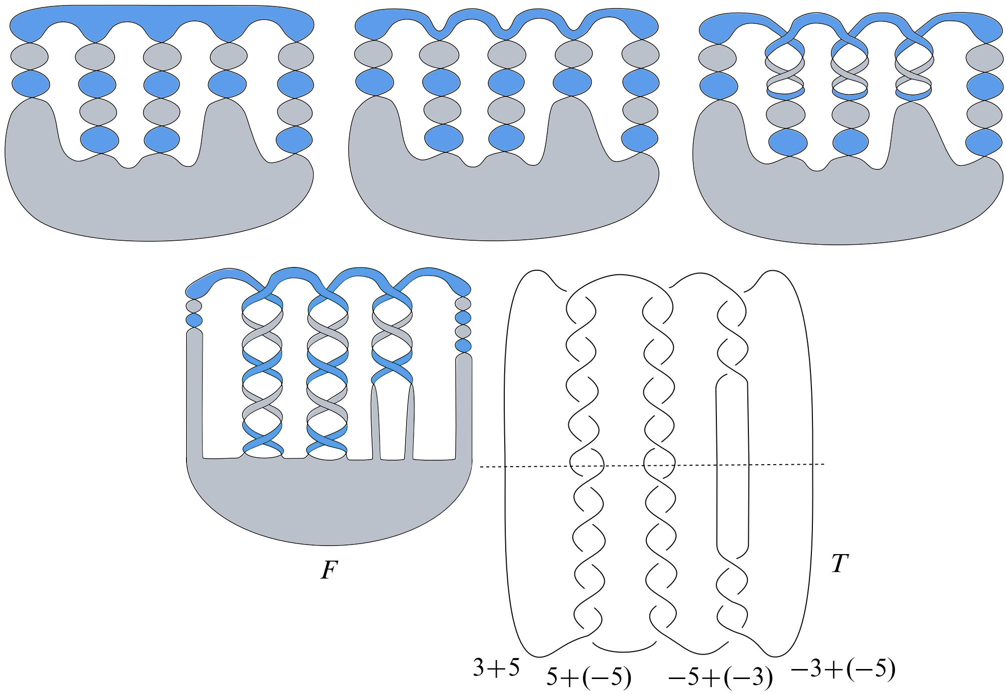


Figure 11: The Seifert surface F and the double cover T of B^4 branched along F pushed-in for the pretzel knot $P(3,5,-5,-3,-5)$ according to Akbulut and Kirby's algorithm.

Therefore, $\sigma(P_0) = s - \operatorname{sgn}(\hat{e})$.

Now, P_0 can be seen to be equivalent to $T\#(S^2 \widetilde{\times} S^2)$ by performing handle slides: for each adjacent pair of nonzero framed handles $((h_i,\alpha_i),(h_{i+1},\alpha_{i+1}))$ of P_0, where h_i is the i^{th} nonzero-framed handle[3] and a_i its framing, we slide h_i over h_{i+1}. The result is a new pair of adjacent handles $((\widetilde{h}_i,\alpha_i+\alpha_{i+1}),(h_{i+1},\alpha_{i+1}))$ that link α_{i+1} times. This is performed for the pairs of handles $((h_i,a_i),(h_{i+1},a_{i+1}))$ for $1 \le i \le p-1$, for the pair $((h_p,a_p),(h_{p+1},b_1))$, and lastly for the pairs $((h_{p+j},b_j),(h_{p+j+1},b_{j+1}))$ for $1 \le j \le n-1$. The result of performing these handle slides is a framed link diagram of T linked with a Hopf link, where one component of the Hopf link has framing 0 and the other has odd framing. This entire process is summarized in Figure 12.

By Lemma 4.4 in [9], a Hopf link with such framings is equivalent to a Hopf link H with a single 0–framed component and a single 1–framed component, which gives a description of the twisted sphere bundle $S^2 \widetilde{\times} S^2$. By Lemma 4.5 in [9], H can be

[3] Handles ordered from left to right in the standard diagram of P_0.

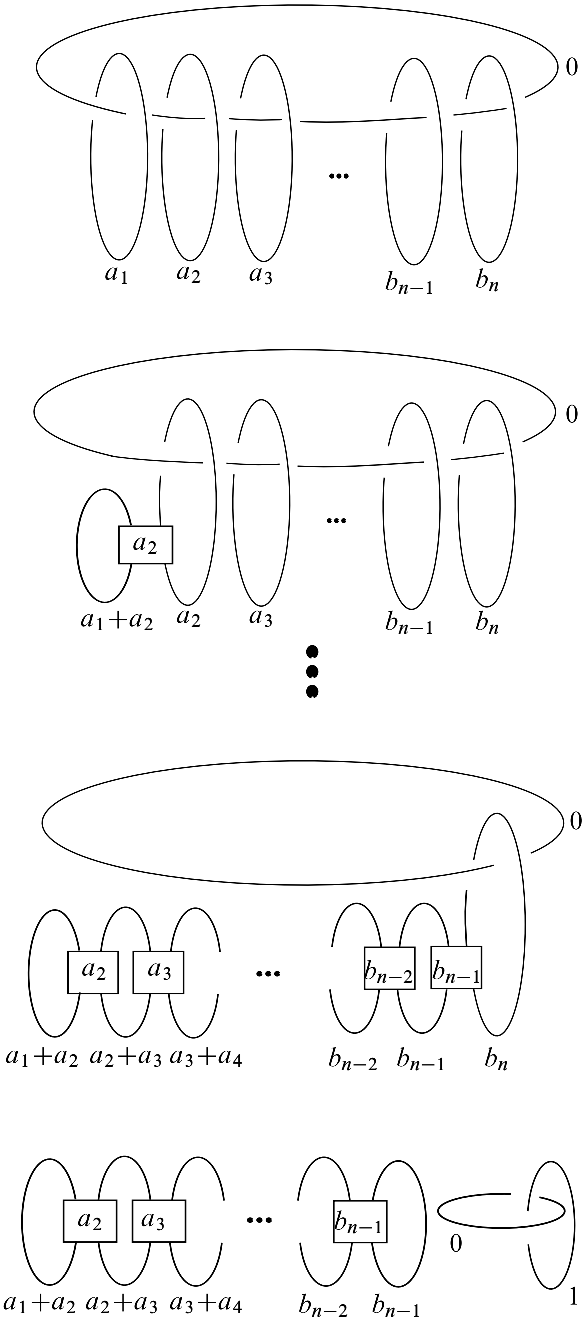


Figure 12: The sequence of handle slides showing that P_0 is equivalent to $T\,\#\,(S^2 \widetilde{\times} S^2)$ as 4–manifolds

unlinked from T through handle slides to yield $T \sqcup H$, which describes the 4–manifold $T\,\#\,(S^2 \mathbin{\tilde{\times}} S^2)$. Given that we moved from P_0 to $T\,\#\,(S^2 \mathbin{\tilde{\times}} S^2)$ exclusively through handle slides (no blow-ups or blow-downs), P_0 and $T\,\#\,(S^2 \mathbin{\tilde{\times}} S^2)$ are the same 4–manifold and thus $\sigma(P_0) = \sigma(T\,\#\,(S^2 \mathbin{\tilde{\times}} S^2))$.

To finish, recall that the signature is additive under connected sums and $\sigma(S^2 \mathbin{\tilde{\times}} S^2) = 0$. Hence

$$\begin{aligned}\sigma(K) = \sigma(T) = \sigma(T) + 0 &= \sigma(T) + \sigma(S^2 \mathbin{\tilde{\times}} S^2)\\ &= \sigma(T\,\#\,(S^2 \mathbin{\tilde{\times}} S^2))\\ &= \sigma(P_0)\\ &= s - \operatorname{sgn}(\hat{e}).\end{aligned}$$

□

Let $a, b, c, d, e \geq 3$. As mentioned in Section 2, Theorem 2.1 shows nonsliceness for all odd 5–stranded pretzel knots K in $P\{a, b, c, d, e\}$ and in $P\{a, b, c, d, -e\}$, since s fails to equal ± 1. But, it also shows nonsliceness for all K in $P\{a, b, c, -d, -e\}$ for which $1/a + 1/b + 1/c > 1/d + 1/e$. For example, $K = P(5, 5, 5, -3, -3)$ has nonvanishing signature by Theorem 2.1 and is therefore not slice.

5 Donaldson's diagonalization theorem and the lattice embedding condition

Donaldson's diagonalization theorem constitutes a small piece of the larger topic of Yang–Mills gauge theory. It remains one of the most significant results in 4–manifold topology, and it has useful applications in many other areas of low-dimensional topology. Donaldson's diagonalization theorem can be used to obstruct knot sliceness and it is with this goal in mind that we call on it here. Recall that a closed, oriented 4–manifold X has a unimodular intersection form[4]

$$Q_X\colon H_2(X)/\operatorname{Tor} \otimes H_2(X)/\operatorname{Tor} \to \mathbb{Z},$$

and that Q_X is *definite* if $|\sigma(Q_X)| = \operatorname{rk}(Q_X)$. Then:

Theorem (Donaldson 1987) *Let X be a smooth, closed, oriented, 4–manifold with positive definite intersection form Q_X. Then Q_X is equivalent over the integers to the standard diagonal form, so in some base,*

$$Q_X(u_1, u_2, \dots, u_r) = u_1^2 + u_2^2 + \dots + u_r^2.$$

[4] Here, Tor denotes the torsion part of $H_2(X)$.

Remark Donaldson's diagonalization theorem was originally phrased for Q_X negative definite, making all the u_i^2 terms negative. Also, Q_X being definite and diagonalizable means that the pair $(\mathbb{Z}^{b_2(X)}, Q_X)$ can be viewed geometrically as a lattice that is isomorphic to $\mathbb{Z}^{b_2(X)}$ with the standard dot product.

Donaldson's diagonalization theorem is used to obstruct sliceness of a knot in the following way: Assume the knot $K \subset S^3$ is slice and that Y is the 2–fold branched cover of S^3 along K. Let P be a canonical definite 4–dimensional plumbing manifold satisfying $\partial P = Y$, and let W be the double branched cover of B^4 along a slicing disk for K. Since K is a knot, Y is a rational homology 3–sphere. Furthermore, W is a rational homology 4–ball with $\partial W = Y$, which follows from the more general fact that the double branched cover of a $\mathbb{Z}/2\mathbb{Z}$–homology ball branched along a codimension-2 $\mathbb{Z}/2\mathbb{Z}$–homology ball is again a $\mathbb{Z}/2\mathbb{Z}$–homology ball. For a proof of this, see [6, Lemma 17.2]. A new 4–manifold X is formed by gluing P and W together along their common boundary Y in the usual, orientation-preserving way. This new manifold X will be compact, smooth, oriented, and have definite intersection form, and thus the diagonalization theorem applies. This gives that $(\mathbb{Z}^{b_2(X)}, Q_X)$ is lattice isomorphic to $(\mathbb{Z}^{b_2(X)}, \mathrm{Id})$, the standard n–dimensional integer lattice.

The Mayer–Vietoris sequence involving $X = P \cup_Y (-W)$ with rational coefficients shows that $H_2(P)$ includes into $H_2(X)$, and therefore $(\mathbb{Z}^{b_2(P)}, Q_P)$ must embed into $(\mathbb{Z}^{b_2(X)}, Q_X)$ as a sublattice of full rank. Algebraically, $(\mathbb{Z}^{b_2(P)}, Q_P)$ embeds into $(\mathbb{Z}^{b_2(X)}, \mathrm{Id})$ if there exists an injection $\alpha\colon \mathbb{Z}^{b_2(P)} \to \mathbb{Z}^{b_2(X)}$ such that $Q_P(a,b) = \mathrm{Id}(\alpha(a), \alpha(b))$ [3]. If this embedding does not exist then the conclusion is that X, as constructed, does not exist. The only assumption made in this construction was that K is slice; therefore the contradiction implies this cannot be the case. Thus, the existence of an embedding α of the lattice $(\mathbb{Z}^{b_2(P)}, Q_P)$ into $(\mathbb{Z}^{b_2(X)}, Q_X)$ is a necessary condition for the knot K to be slice, which is precisely the obstruction to sliceness utilized in [11] and [3]. We call this the lattice embedding condition.

In practice, showing the embedding α exists amounts to writing down a matrix A for α that satisfies $A^T A = Q_P$. This requires a choice basis for $H_2(P)$ and for $H_2(X)/\operatorname{Tor}$. The basis $\{s_i\}$ chosen for $H_2(P)/\operatorname{Tor}$ is the set of classes represented by the zero-sections in the disk bundles used to create P; the basis $\{e_i\}$ for $H_2(X)/\operatorname{Tor}$ is chosen to be one that makes Q_X diagonal by Donaldson's theorem. As such, each column of A corresponds to one of those 2–spheres in P whose intersection information is recorded by the plumbing graph of P. That is, the columns of A must have standard dot products consistent with the information given by the plumbing graph for P.

In an attempt to use Donaldson's diagonalization theorem to obstruct sliceness of an odd pretzel knot K, we refer back to the Section 3 and take $P = P_+$, which has plumbing graph Γ_+ and intersection form Q_{P_+}, with matrix equal to the incidence matrix for Γ_+ with respect to the above bases. By the signature obstruction to sliceness, we need only consider odd pretzel knots K for which $\sigma(K) = 0$. In order to utilize the positive definite version of Donaldson's diagonalization theorem, we need to prove that Q_X is positive definite for $X = P_+ \cup_Y (-W)$. This is done with the help of the following lemma:

Lemma 5.1 *If K is an odd k–stranded pretzel knot with k odd and $\sigma(K) = 0$, then either $Q_{P_+}(K)$ or $Q_{P_+}(-K)$ is positive definite.*

Proof From Theorem 2.1, we know that $\hat{e} \neq 0$ for pretzel knots K with $\sigma(K) = 0$. Then, Theorem 5.2 in [15] tells us that Q_{P_+} is either positive definite or negative definite, according to whether $\hat{e} > 0$ or $\hat{e} < 0$, respectively. Taking the mirror $-K$ of a knot K will change Q_{P_+} from positive definite to negative definite, or vice versa. Thus after mirroring if necessary, it is always possible to choose K so that Q_{P_+} is positive definite. □

With our eye on applying the diagonalization theorem to X and the help of Lemma 5.1, we argue that Q_X is also positive definite for $X = P_+ \cup_Y (-W)$. Consider the following portion of the Mayer–Vietoris sequence for X with rational coefficients:

$$0 \longrightarrow \mathbb{Q}^n \oplus 0 \xrightarrow{i_*} H_2(X;\mathbb{Q}) \longrightarrow 0.$$

The map i_* is an isomorphism, which implies every element $x \in H_2(X)$ is a $\mathbb{Q}$–linear combination of basis elements $\{s_i\}$ for $H_2(P_+)$ and torsion elements of $H_2(W)$. Bilinearity of Q_X and positive-definiteness of Q_{P_+} yield that Q_X is positive definite. Thus, we are free to utilize the previously described construction using Donaldson's diagonalization theorem, with $P = P_+$, to obtain the embedding criterion for sliceness on odd, 5–stranded pretzel knots.

In all the results that follow, we use Theorem 2.1 to immediately reduce to considering only those odd, 5–stranded pretzel knots of the form $P(-a,-b,-c,d,e)$ for which $\operatorname{sgn}(\hat{e}) = -1$. We use $P(-a,-b,-c,d,e)$ rather than its mirror in order to use the positive definite formulation of Donaldson's theorem. As stated in the explanation of the lattice embedding condition, we wish to write down a matrix A satisfying $A^T A = Q_{P_+}$. This condition can be phrased as a collection of conditions on the column vectors of A:

Embedding conditions For a slice odd 5–stranded pretzel knot K of the form $P(-a,-b,-c,d,e)$, there exist vectors $v_i, v_{j,r} \in \mathbb{Z}^m$, with $m = a+b+c$, satisfying

$$
\begin{array}{llll}
(0) & v_0 \cdot v_0 = 3, & (1) & v_1 \cdot v_0 = 1, \\
(2) & v_2 \cdot v_0 = 1, & (3) & v_1 \cdot v_2 = 0, \\
(4) & v_1 \cdot v_1 = d, & (5) & v_2 \cdot v_2 = e, \\
(6) & v_{j,1} \cdot v_0 = 1, & (7) & v_{j,r} \cdot v_{j,r} = 2, \\
(8) & v_{j,r} \cdot v_{j,r\pm1} = 1 & \text{for } r \geq 2, & \\
(9) & v_{j,r} \cdot v_* = 0 & \text{for } r \geq 2 \text{ and all vectors } v_* \neq v_{j,r\pm1}. &
\end{array}
$$

The embedding conditions impose severe restrictions on the form each v_i and v_{j,r_j} can take. Condition (0) for example, implies that v_0 must have exactly three entries equal to ±1 and zeros otherwise; similarly, condition (7) implies that each vector v_{j,r_j} must have exactly two entries equal to ±1 and zeros otherwise. It can be verified using conditions (0)–(7) that up to a change of basis, A will have the following form, with $\alpha, \beta, \gamma, x, y, z \in \mathbb{Z}$:

$$
\left[\begin{array}{ccc|cccc|cccc|cccc}
v_0 & v_1 & v_2 & v_{a,1} & v_{a,2} & \cdots & v_{a,a-1} & v_{b,1} & v_{b,2} & \cdots & v_{b,b-1} & v_{c,1} & v_{c,2} & \cdots & v_{c,c-1} \\
\hline
1 & \alpha & x & 1 & 0 & \cdots & 0 & & & & & & & & \\
0 & \alpha & x & -1 & -1 & & 0 & & & & & & & & \\
0 & \alpha & x & 0 & 1 & & 0 & & & & & & & & \\
0 & \alpha & x & 0 & 0 & \cdots & 0 & & & 0 & & & & 0 & \\
 & \vdots & & \vdots & & \ddots & \vdots & & & & & & & & \\
0 & \alpha & x & 0 & 0 & \cdots & -1 & & & & & & & & \\
0 & \alpha & x & 0 & 0 & \cdots & 1 & & & & & & & & \\
\hline
1 & \beta & y & & & & & 1 & 0 & \cdots & 0 & & & & \\
0 & \beta & y & & & & & -1 & -1 & & 0 & & & & \\
0 & \beta & y & & & & & 0 & 1 & & 0 & & & & \\
0 & \beta & y & & & 0 & & 0 & 0 & \cdots & 0 & & & 0 & \\
 & \vdots & & & & & & \vdots & & \ddots & \vdots & & & & \\
0 & \beta & y & & & & & 0 & 0 & \cdots & -1 & & & & \\
0 & \beta & y & & & & & 0 & 0 & \cdots & 1 & & & & \\
\hline
1 & \gamma & z & & & & & & & & & 1 & 0 & \cdots & 0 \\
0 & \gamma & z & & & & & & & & & -1 & -1 & & 0 \\
0 & \gamma & z & & & & & & & & & 0 & 1 & & 0 \\
0 & \gamma & z & & & 0 & & & & 0 & & 0 & 0 & \cdots & 0 \\
 & \vdots & & & & & & & & & & \vdots & & \ddots & \vdots \\
0 & \gamma & z & & & & & & & & & 0 & 0 & \cdots & -1 \\
0 & \gamma & z & & & & & & & & & 0 & 0 & \cdots & 1
\end{array}\right]
$$

Having A in this explicit form allows us to put restrictions on the unordered 5–tuples $\{a, b, c, d, e\}$ ensuring the embedding conditions are satisfied. For fixed a, b, and c, we enumerate the embedding conditions in terms of the entries of the column vectors of A, which reduces the problem of finding the desired embedding to the problem of finding integers α, β, γ, x, y, z that satisfy the following system of nonlinear equations. Each new condition is numbered to correspond to the original embedding condition that implies it. In references by number, no distinction is made between the original and updated conditions since the updated conditions are direct implications of the originals.

(Updated) embedding conditions For a slice odd 5–stranded pretzel knot K of the form $P(-a,-b,-c,d,e)$, there exist integers α, β, γ, x, y, z satisfying

(1) $\alpha+\beta+\gamma=1$,

(2) $x+y+z=1$,

(3) $a\alpha x+b\beta y+c\gamma z=0$,

(4) $a\alpha^2+b\beta^2+c\gamma^2=d$,

(5) $ax^2+by^2+cz^2=e$.

In fact, these updated embedding conditions are exactly the contents of Theorem 4.1.6 in [13], so a more detailed account of these facts can be found there.[5]

6 The d–invariants and the coset conditions

Peter Ozsváth and Zoltán Szabó defined the d–invariant $d(Y,\mathfrak{s})\in\mathbb{Q}$ in the setting of Heegaard Floer homology for a rational homology 3–sphere Y equipped with a Spin^c structure $\mathfrak{s}$. While the d–invariant has an important function as a correction term for the grading in Heegaard Floer homology, it is significant in 4–manifold topology because it is a Spin^c rational homology bordism invariant. As stated in [16], if $(Y_1,\mathfrak{s}_1)$ and $(Y_2,\mathfrak{s}_2)$ are two pairs such that Y_i is a rational homology 3–sphere and $\mathfrak{s}_i$ is a Spin^c structure on Y_i, then if there exists a connected, oriented, smooth cobordism W from Y_1 to Y_2 with $H_i(W;\mathbb{Q})=0$ for $i=1,2$ which can be endowed with a Spin^c structure $\mathfrak{t}$ whose restriction to Y_i is $\mathfrak{s}_i$, then $d(Y_1,\mathfrak{s}_1)=d(Y_2,\mathfrak{s}_2)$. The proof of this highly nontrivial fact is given in Proposition 9.9 of [16], and it has the following corollary:

[5] Warning: Long's approach to the problem of sliceness in 5–stranded pretzel knots uses a negative definite convention rather than the positive definite convention of this paper.

Corollary 6.1 (Ozsváth and Szabó) *Let Y be a rational homology 3–sphere with Spin^c structure $\mathfrak{s}$, and let W be a rational homology 4–ball with $\partial W = Y$ and Spin^c structure $\mathfrak{t}$. If $\mathfrak{s}$ can be extended over W so that $\mathfrak{t}|_Y = \mathfrak{s}$, then $d(Y, \mathfrak{s}) = 0$.*

In general $d(Y, \mathfrak{s})$ may be hard to compute, but in [17] Ozsváth and Szabó give a formula for $d(Y, \mathfrak{s})$ when Y is the boundary of a 4–dimensional plumbing manifold P. Their formula holds in more generality than the version presented below, but the formula is stated here in the special case relevant to the present situation of determining sliceness of pretzel knots. Throughout this section, we refer to K, Y, P, W, and X as defined in Section 5. To remind the reader of these definitions: K is assumed to be a slice odd pretzel knot; Y is the double branched cover of S^3 along K; W is the double branched cover of B^4 along a fixed slice disk for K with $\partial W = Y$; $P = P_+$ is a positive definite plumbing manifold with $\partial P = Y$; and $X = P \cup_Y (-W)$ is a closed positive definite manifold. Under these assumptions, W is a rational homology 4–ball and Y is a rational homology 3–sphere. To state the aforementioned formula easily and to give a more geometric flavor to the material that follows, we first discuss an identification of $\mathrm{Spin}^c(Y)$ with $H_1(Y)$.

If Y is a 3–manifold such that $H_1(Y)$ is odd torsion, then there is a natural identification of $\mathrm{Spin}^c(Y)$ with $H_1(Y)$. In our current work Y is the double branched cover of S^3 along a knot K and a bit of straightforward algebraic topology reveals that $H_1(Y)$ is always odd torsion in this case. The first step in the identification shows a one-to-one correspondence between $\mathrm{Spin}^c(Y)$ and $\mathrm{vect}(Y)$, the set of Euler structures on Y. An Euler structure on a smooth closed connected oriented 3–manifold Y is an equivalence class of nonsingular tangent vector fields on Y, where two vector fields u and v on Y are deemed equivalent if u and v are homotopic as nonsingular vector fields outside of some closed 3–dimensional ball. This particular identification of $\mathrm{Spin}^c(Y)$ with $\mathrm{vect}(Y)$ is due to Vladimir Turaev and constitutes Lemma 1.4 in [19], so the reader is directed there for details. The salient feature of this step is that it allows us to view a Spin^c structure on Y as a vector field over Y under some notion of equivalence.

Assuming Turaev's identification of $\mathrm{Spin}^c(Y)$ with $\mathrm{vect}(Y)$, the second step is to identify $\mathrm{vect}(Y)$ with $H_1(Y)$. Start by fixing a trivialization τ of the tangent bundle TY. Since $H_1(Y)$ has only odd torsion, τ is unique off a 3–ball up to homotopy. Let $[Y, S^2]$ denote the space of smooth maps from Y to S^2 up to homotopy. The identification of $\mathrm{vect}(Y)$ with $H_1(Y)$ will be done via a composition $\mathrm{vect}(Y) \to [Y, S^2] \to H_1(Y)$.

For each equivalence class of nonvanishing vector fields on Y, we choose a representative vector field $\mathcal{V}$. By a straight line homotopy, we can assume that each vector in $\mathcal{V}$ is a unit vector, where the length is measured according to the trivialization τ. For each point $p \in Y$, the tangent space T_pY at p is isomorphic to R^3 and thus provides a way to give Euclidean coordinates to $v_p \in \mathcal{V}$, the vector based at p. Let (x_p, y_p, z_p) be the Euclidean coordinates for v_p obtained from τ. With this we define a smooth map $\phi_{\mathcal{V}}\colon Y \to S^2$ by sending $p \in Y$ to the vector $(x_p, y_p, z_p) \in S^2$. The map $g\colon \mathrm{vect}(Y) \to [Y, S^2]$ is then defined as $g([\mathcal{V}]) = \phi_{\mathcal{V}}$.

For the second map, recall that any map $\phi \in [Y, S^2]$ has the property that the preimage $\phi^{-1}(z)$ of a regular value $z \in S^2$ will be a submanifold of Y of codimension 2, namely, a curve γ. From this fact, define $h\colon [Y, S^2] \to H_1(Y)$ by $h(\phi) = [\gamma]$, where γ is the preimage of any regular value of ϕ. The Pontryagin–Thom construction shows that this map is well defined. It follows that Spin^c is identified with $H_1(Y)$ via Turaev's identification of Spin^c with $\mathrm{vect}(Y)$ followed by the composition $h \circ g \circ f$.

A second topic necessary to discuss before stating the d–invariant formula is that of characteristic elements of $H_2(X)/\mathrm{Tor}$, $H_2(P)$, and $H_2(P, Y)$. These definitions involve intersection numbers, and in all cases we will abbreviate the intersection number of two elements a, b in $H_2(X)/\mathrm{Tor}$, $H_2(P)$, or $H_2(P, Y)$ by $a \cdot b$ and let the definition of $a \cdot b$ be given by context. As before, Q_X and Q_P are the intersection forms on X and P, respectively. The map Q_P^{-1} is the relative intersection form on (P, Y) given as the inverse of Q_P over $\mathbb{Q}$. We define:

- $a \cdot b = Q_X(a, b)$ if $a, b \in H_2(X)/\mathrm{Tor}$.
- $a \cdot b = Q_P(a, b)$ if $a, b \in H_2(P)$.
- $a \cdot b = Q_P(x, b)$ if $a \in H_2(P, Y)$ and $b \in H_2(P)$, where $x = Q_P^{-1}(a) \in H_2(P)$.
- $a \cdot b = Q_P(x, y)$ if $a, b \in H_2(P, Y)$, where again $x = Q_P^{-1}(a) \in H_2(P)$ and $y = Q_P^{-1}(b) \in H_2(P)$.

By choosing bases for $H_2(X)/\mathrm{Tor}$, $H_2(P)$, and $H_2(P, Y)$, homology classes in these groups can be represented by column vectors and the intersection forms Q_X and Q_P can be represented by matrices. We choose bases as follows: the basis $\{e_i\}$ for $H_2(X)/\mathrm{Tor}$ is the one that makes Q_X diagonal by Donaldson's theorem; the basis $\{s_i\}$ for $H_2(P)$ is the set of homology classes represented by the zero-sections of the disk bundles used to create P; lastly, the basis $\{d_i\}$ for $H_2(P, Y)$ is the set of classes represented by single fiber disks in each of the disk bundles of P. Note that the fiber disks $\{d_i\}$ are the Hom-duals of the $\{s_i\}$.

With fixed bases the above intersection numbers can be computed using column vector representatives for homology classes and the matrix representatives for Q_X and Q_P. As matrices with the above bases, recall that Q_P is equal to the incidence matrix of the weighted graph representing P and Q_X is equal to the identity matrix of rank $b_2(X)$. By an abuse of notation, we use Q_P to denote both the intersection form for P and its matrix representative in this case. This allows us to write and compute the above intersection numbers in terms of column vectors a, b as follows:

- If $a, b \in H_2(X)/\operatorname{Tor}$, then $a \cdot b = a^T b$.
- If $a, b \in H_2(P)$, then $a \cdot b = a^T Q_P b$.
- If $a \in H_2(P, Y)$ and $b \in H_2(P)$, then $a \cdot b = x^T Q_P b$, where $x = Q_P^{-1}(a) \in H_2(P)$. This simplifies to $a \cdot b = a^T b$.
- If $a, b \in H_2(P, Y)$, then $a \cdot b = Q_P(x, y)$, where again $x = Q_P^{-1}(a) \in H_2(P)$ and $y = Q_P^{-1}(b) \in H_2(P)$. This simplifies to $a \cdot b = a^T Q_P^{-1} b$.

Now, we say that an absolute class $w \in H_2(X)/\operatorname{Tor}$ is a *characteristic class of* X if $w \cdot x \equiv x \cdot x \pmod 2$, for all $x \in H_2(X)/\operatorname{Tor}$; we say a characteristic class w is *minimal* if $w \cdot w \le z \cdot z$ for all characteristic classes z. Characteristic and minimal characteristic elements of $H_2(P)$ are defined similarly. A relative class $w \in H_2(P, Y)$ is *characteristic in* X *with respect to* $\mathfrak{s}$, where $\mathfrak{s}$ is regarded as an element of $H_1(Y)$, if $\partial w = \mathfrak{s}$ and $w \cdot u \equiv u \cdot u \pmod 2$, for all $u \in H_2(P)$. The set of characteristic elements in $H_2(P, Y)$ relative to $\mathfrak{s}$ is denoted by $\operatorname{Char}_{\mathfrak{s}}(P)$, which makes an appearance in the formula below.

We are now ready to state Ozsváth and Szabó's formula for $d(Y, \mathfrak{s})$ in the case that Y bounds a certain type of 4–dimensional plumbing:

Theorem 6.2 (Ozsváth and Szabó) *Let P be a 4–dimensional plumbing with positive definite intersection form Q_P, such that the weighted graph of P has at most two vertices whose weights are less than their valences. Then under the identification* $\operatorname{Spin}^c(Y) \to H_1(Y)$,

$$d(Y, \mathfrak{s}) = \min_{w \in \operatorname{Char}_{\mathfrak{s}}(P)} \frac{w \cdot w - \sigma(P)}{4}. \tag{1}$$

In [3], Greene and Jabuka use Theorem 6.2 and Corollary 6.1 to give an obstruction to sliceness for odd pretzel knots through some analysis of the cohomology long exact sequences of the pairs (P, Y) and (X, W). Here, we derive their results in terms of homology and obtain the following commutative diagram at the top of the next page. In the diagram the horizontal maps arise from the long exact sequences of the pairs (P, Y) and (X, W); the vertical maps r and γ are induced by inclusions; β is an isomorphism due to excision; and q is the usual quotient map.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(P) & \xrightarrow{\lambda} & H_2(P,Y) & \xrightarrow{\partial} & H_1(Y) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle r} & & {\scriptstyle \beta}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
\cdots & \longrightarrow & H_2(X) & \xrightarrow{g} & H_2(X,W) & \longrightarrow & H_1(W) & \longrightarrow & H_1(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & \nearrow{\scriptstyle \mu} & & & & & & & \\
\cdots & \longrightarrow & H_2(X)/\operatorname{Tor} & & & & & & & &
\end{array}
$$

Because $H_2(P)$ is free and r is a homomorphism, the image of r lies entirely in the free part of $H_2(X)$. Let $\alpha = qr$ and $\alpha^* = \beta^{-1}\mu$; this allows us to use the first isomorphism theorem to eliminate $H_2(X)$ from the diagram. By commutativity, λ can be seen to have the factorization $\lambda = \alpha^*\alpha$, converting the previous diagram into:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(P) & \xrightarrow{\lambda} & H_2(P,Y) & \xrightarrow{\partial} & H_1(Y) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle \alpha}\downarrow & \nearrow{\scriptstyle \alpha^*} & {\scriptstyle \beta}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
\cdots & \longrightarrow & H_2(X)/\operatorname{Tor} & \xrightarrow{\mu} & H_2(X,W) & \xrightarrow{\nu} & H_1(W) & \longrightarrow & H_1(X) & \longrightarrow & 0
\end{array}
$$

To use this diagram in conjunction with the lattice embedding condition, it is advantageous to work with matrix representatives of the maps α, α^*, and λ. We choose the bases for $H_2(P)$, $H_2(P,Y)$, and $H_2(X)/\operatorname{Tor}$ as before, we let A be the matrix representative for the map α (induced by the embedding of P into X), and we let A^* be the matrix for α^*.

The columns of A express the basis elements $\{e_i\}$ of $H_2(X)/\operatorname{Tor}$ in terms of the basis disks $\{d_i\}$ for $H_2(P,Y)$. Consequently, the *rows* of A^T express the spheres $\{s_i\}$ in terms of the $\{e_i\}$, which implies that the ij^{th} entry in A^TA gives the intersection number between the spheres s_i and s_j. Thus A^TA is the matrix of the intersection form Q_P of P with respect to the basis $\{s_i\}$.

Recall that each basis element d_i of $H_2(P,Y)$ is the Hom-dual of the basis element s_i of $H_2(P)$, and therefore $\lambda(s_i) = \sum_j (s_i \cdot s_j) d_j$. This implies that with respect to the chosen bases, λ (as a linear map from $H_2(P)$ to $H_2(P,Y)$) is represented by the same matrix as is Q_P (regarded as a bilinear map from $H_2(P) \times H_2(P)$ to $\mathbb{Z}$). Namely, λ is also represented by A^TA. Given that $\lambda = \alpha^*\alpha$, it follows that $A^*A = A^TA$ as matrices. Since Q_P is invertible over $\mathbb{Q}$, so is A; whence $A^* = A^T$. By reinstating the abuse in notation whereby we use Q_P to denote both the intersection form on P

and its matrix representative in this case, we let the matrix Q_P represent λ with respect to the chosen bases.

Dropping the less relevant maps, the previous commutative diagram becomes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_2(P) & \xrightarrow{Q_P} & H_2(P,Y) & \longrightarrow & H_1(Y) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle A} & \nearrow{\scriptstyle A^T} & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
\cdots & \longrightarrow & H_2(X)/\operatorname{Tor} & \longrightarrow & H_2(X,W) & \longrightarrow & H_1(W) & \longrightarrow & H_1(X) & \longrightarrow & 0
\end{array}$$

We use this to restate and reprove — in a homological setting — Greene and Jabuka's d–invariant obstruction to sliceness in odd pretzel knots:

Theorem 6.3 (Greene and Jabuka) *Let K be a slice odd pretzel knot with Y, W, $P = P_+$, and X as in the above commutative diagram. Then every coset of* $\operatorname{coker}(\alpha)$ *contains a minimal characteristic class of $H_2(X)/\operatorname{Tor}$.*

Proof Under the assumption that K is slice, it follows that K satisfies the embedding conditions and $\sigma(P) = \operatorname{rk}(Q_P) = \operatorname{rk}(Q_X) = b_2(X) := m$. It also follows from Corollary 6.1 that $d(Y,\mathfrak{s}) = 0$ for every $\mathfrak{s}$ that extends over W. In general, the Spin^c structures on a rational homology 3–sphere Y that extend over a rational homology 4–ball W are identified with precisely those elements in $H_1(Y)$ that bound relative homology classes in $H_2(W,Y)$. As such, they are in one-to-one correspondence with the elements of $\ker(\gamma)$, where $\gamma\colon H_1(Y) \to H_1(W)$ is induced by inclusion.

Theorem 6.2 applies to Y since the plumbing graph of P will have exactly one vertex whose weight is less than its valence, namely, the central node. Ozsváth and Szabó's formula

$$d(Y,\mathfrak{s}) = \min_{w \in \operatorname{Char}_{\mathfrak{s}}(P)} \frac{w\cdot w - \sigma(P)}{4}$$

implies that $d(Y,\mathfrak{s}) = 0$ if and only if there exists $w \in \operatorname{Char}_{\mathfrak{s}}(P)$ such that $w \cdot w = m$. A straightforward diagram chase shows that for each $w \in \operatorname{Char}_{\mathfrak{s}}(P)$ there exists an element $x \in H_2(X)/\operatorname{Tor}$ such that $\alpha^*(x) = w$. In addition, x is characteristic in X and $x \cdot x = w \cdot w$, so in general the characteristic classes of P relative to $\mathfrak{s}$ correspond to absolute characteristic classes of X with equal intersection number. This fact, which is verified below, allows us to compute $w \cdot w$, which appears in formula (1), by using $x \cdot x$ instead.

Fix the bases for $H_2(P)$, $H_2(P,Y)$, and $H_2(X)/\operatorname{Tor}$ as before, and let $A = (a_{ij})$ again be the matrix representative of α with respect to these bases. Let $w \in \operatorname{Char}_{\mathfrak{s}}(P)$

and $x = (x_1, \dots, x_m) \in H_2(X)/\operatorname{Tor}$ such that $\alpha^*(x) = w$. Recall that α^* is represented by the matrix A^T with respect to these bases. To show that x is characteristic in X, it suffices to show that $x \cdot e_j \equiv e_j \cdot e_j \pmod 2$, for all basis elements e_j. Since $e_j \cdot e_j = 1$, one need only show that $x \cdot e_j$ — that is, the j^{th} component of x — is odd for all j. Stated differently, it must be shown that every component of x is odd.

By definition of $\operatorname{Char}_{\mathfrak{s}}(P)$, $w \cdot u \equiv u \cdot u \pmod 2$ for all $u \in H_2(P)$, in particular for u equal to a basis element s_j for $H_2(P)$: $w \cdot s_j \equiv s_j \cdot s_j \pmod 2$. Observe that for all j,

$$w \cdot s_j = A^T x \cdot s_j = x^T A s_j = \sum_i x_i a_{ij},$$

$$s_j \cdot s_j = (Q_P)_{jj} = (A^T A)_{jj} = \sum_i a_{ij} a_{ij} \equiv \sum_i a_{ij} \pmod 2.$$

Hence, $\sum_i x_i a_{ij} \equiv \sum_i a_{ij} \pmod 2$ for all j. Letting $x_i \equiv 1 \pmod 2$ yields a solution to this equation, which in fact is the unique solution since A is invertible modulo 2. Given that this holds for all j, it has thus been shown that x has all odd entries and is therefore characteristic in X. Furthermore, since $w \in H_2(P, Y)$, it follows from above that $w \cdot w = w^T Q_P^{-1} w$. Making the substitutions $Q_P = A^T A$ and $w = A^T x$ shows that $w \cdot w = x \cdot x$.

In addition, the diagram chase from before shows that $\ker(\gamma) \cong \operatorname{coker}(\alpha)$. Combining this with the preceding information implies that $d(Y, \mathfrak{s}) = 0$ for all $\mathfrak{s} \in \ker(\gamma)$ with corresponding $k \in \operatorname{coker}(\alpha)$ if and only if there exists $w \in \operatorname{Char}_{\mathfrak{s}}(P)$ and $x \in \operatorname{Char}(X)$ such that $w = A^T x$, $x \cdot x = m$, and $x + \operatorname{im}(\alpha) = k$. Clearly, $x \cdot x = m$ only if $x_i = \pm 1$ for all i, which implies that x is a *minimal* characteristic class of X. Hence, K slice implies that every element of $\operatorname{coker}(\alpha)$, ie every coset of $\operatorname{im}(\alpha)$, contains a minimal characteristic class of X. □

Theorem 6.3 gives a necessary condition for sliceness for odd 5–stranded pretzel knots that can be rephrased in a simpler, more geometric way by analyzing the quotient $\operatorname{coker}(\alpha) = (H_2(X)/\operatorname{Tor})/\operatorname{im}(\alpha)$. We will reduce the problem of finding minimal characteristic vectors in each coset of $\operatorname{im}(\alpha)$ to a more visualizable problem of finding lattice points in $\mathbb{Z}^2$ with certain properties.

Since $H_2(X)/\operatorname{Tor} \cong \mathbb{Z}^m$, it follows that $\operatorname{coker}(\alpha) \cong \mathbb{Z}^m/\operatorname{im}(\alpha)$. Given that the image of α with the chosen bases is equal to the span of the columns of A, $\operatorname{coker}(\alpha)$ is isomorphic to the quotient of $\mathbb{Z}^m$ by the columns of A. Let $\mathcal{U} = \{v_{j,r_j}\}$ be the set of column vectors of A with standard dot product, where $1 \le j \le n$ and $1 \le r_j \le j-1$. Then the columns of A, as vectors, are given by $\{v_0, v_1, v_2, \mathcal{U}\}$.

Define $B\colon \mathbb{Z}^m \to \mathbb{Z}^2$ by

$$(x_1,\dots,x_a,y_1,\dots,y_b,z_1,\dots,z_c)^T \mapsto \left(\sum_{i=1}^{a} x_i - \sum_{k=1}^{c} z_k, \sum_{j=1}^{b} y_j - \sum_{k=1}^{c} z_k\right)^T.$$

It is straightforward to see that $\ker(B) = \langle v_0, \mathcal{U}\rangle$ and that B is onto, so by the first isomorphism theorem $\mathbb{Z}^2 \cong \mathbb{Z}^m/\langle v_0,\mathcal{U}\rangle$. It follows that

$$\operatorname{coker}(\alpha) \cong \mathbb{Z}^m/\langle v_0, v_1, v_2, \mathcal{U}\rangle \cong \mathbb{Z}^2/\langle B(v_1), B(v_2)\rangle.$$

Let $\overline{v}_1 := B(v_1)$ and $\overline{v}_2 := B(v_2)$. Using the above isomorphisms, the slice condition in Theorem 6.3 can now be rephrased to say that every coset in $\mathbb{Z}^2/\langle \overline{v}_1, \overline{v}_2\rangle$ must have a representative in $B(\{\pm 1\}^m)$. Thus, $\mathbb{Z}^2/\langle \overline{v}_1, \overline{v}_2\rangle$ and $B(\{\pm 1\}^m)$ are analyzed:

$$\begin{aligned}
\overline{v}_1 &= B((\alpha,\dots,\alpha,\beta,\dots,\beta,\gamma,\dots,\gamma))^T = (a\alpha - c\gamma, b\beta - c\gamma)^T,\\
\overline{v}_2 &= B((x,\dots,x,y,\dots,y,z,\dots,z))^T \;= (ax-cz, by-cz)^T,\\
B(\{\pm 1\}^m) &= \{(q-s, r-s) \mid -a-c \le q-s \le a+c \text{ and } -b-c \le r-s \le b+c\}.
\end{aligned}$$

The vectors $\{\overline{v}_1, \overline{v}_2\}$ define a fundamental domain $R \subset \mathbb{R}^2$. Let $\mathcal{R}$ be the set of lattice points in R that represent unique cosets of $\operatorname{im}(\alpha)$ coming from $\operatorname{coker}(\alpha)$. Note that $\mathcal{R} \subset \mathbb{Z}^2$ and $\mathcal{R} \subset R$. Also, let $\mathcal{H} := B(\{\pm 1\}^m)$. Then $\mathcal{H}$ is a collection of lattice points (x, y) satisfying $-a-c \le x \le a+c$ and $-b-c \le y \le b+c$. Since a, b, and c are odd and positive, every element of $\mathcal{H}$ is an element of $2\mathbb{Z}^2$ and collectively these points lie in a hexagonal region $H \subset \mathbb{R}^2$. Hence, $\mathcal{H} \subset 2\mathbb{Z}^2$ and $\mathcal{H} \subset H$.

Greene and Jabuka observed that Theorem 6.3, which gives the slice condition that every element of $\operatorname{coker}(\alpha)$ contains a minimal characteristic vector of the form $(\{\pm 1\}^n)$, is equivalent to the condition that every lattice point of $\mathcal{R}$ can be translated onto a lattice point of $\mathcal{H}$ by a linear combination of $\overline{v}_1$ and $\overline{v}_2$. Hence, a knot K cannot be slice if there exists an element of $\operatorname{coker}(\alpha)$ that does not contain a minimal characteristic vector of X by Theorem 6.3. The correspondence between cosets of $\operatorname{im}(\alpha)$, minimal characteristic vectors of X, and lattice points implies that K is not slice if there exists a lattice point in $\mathcal{R}$ that can not be translated onto a lattice point in $\mathcal{H}$ by a linear combination of $\overline{v}_1$ and $\overline{v}_2$. By the definition of $\mathcal{R}$, every element of $\mathcal{R}$ represents a distinct coset in the quotient $\mathbb{Z}^2/\langle \overline{v}_1, \overline{v}_2\rangle$, and thus if there are more lattice points in $\mathcal{R}$ than there are in $\mathcal{H}$ for a knot K, then K is not slice. This proves the following:

Coset condition I *If $P(a,b,c,d,e)$ is a slice odd 5–stranded pretzel knot, then $|\mathcal{R}| \le |\mathcal{H}|$.*

It is possible, however, for many points in $\mathcal{H}$ to belong to the same coset in $\mathbb{Z}^2/\langle \overline{v}_1, \overline{v}_2\rangle$. Let $\overline{\mathcal{H}} := \mathcal{H}/\langle \overline{v}_1, \overline{v}_2\rangle$, so that $|\overline{\mathcal{H}}|$ is the number of $\langle \overline{v}_1, \overline{v}_2\rangle$–cosets in $\mathcal{H}$. With this observation and Theorem 6.3 and the above observation, K is not slice if $|\mathcal{R}| > |\overline{\mathcal{H}}|$. This condition is a refinement of coset condition I, which the author unimaginatively calls coset condition II:

Coset condition II *If $P(a,b,c,d,e)$ is a slice odd 5–stranded pretzel knot, then $|\mathcal{R}| \le |\overline{\mathcal{H}}|$.*

7 Proof of Theorem 2.3

Due to the slightly different nature of pretzel knots with single-twists versus those without, the proof of Theorem 2.3 is divided according to this distinction. A technical lemma, Lemma 7.1, is given first and then it is shown that all 0–pair odd 5–stranded pretzel knots *without* single-twists are not slice. Lemma 9.1 refines Lemma 7.1 and is then used to show that all 0–pair odd 5–stranded pretzel knots *with* single-twists are not slice.

Recall from Section 4 that a knot K is slice if and only if its mirror $-K$ is slice. To make the computations in the proof easier, the knot $K = P(-a,-b,-c,d,e)$ in $P\{a,b,c,-d,-e\}$ will be used rather than its mirror $P(a,b,c,-d,-e)$. Lemma 7.1, which is given next, states the conditions on α, β, γ, x, y, and z under which $P(-a,-b,-c,d,e)$ will be a 0–pair pretzel knot. Without loss of generality, assume throughout that $a \le b \le c$.

Lemma 7.1 *If $K \in P\{a,b,c,-d,-e\}$ is 0–pair and satisfies the embedding conditions, then at most one of α, β, γ, x, y, z is zero. Furthermore, if the set $\{\alpha,\beta,\gamma,x,y,z\}$ contains 0, then $d \ge 4a+b$ and $e \ge a+b+c$; otherwise, both d and e are greater than or equal to $a+b+c$.*

Proof Choose $K = P(-a,-b,-c,d,e)$. First it will be shown that if any two of α, β, γ are zero or if any two of x, y, z are zero, then K is not 0–pair. By the symmetry of the embedding conditions on $\{\alpha,\beta,\gamma\}$ and $\{x,y,z\}$, it suffices to prove this only for $\{\alpha,\beta,\gamma\}$. Suppose two of the parameters α, β, γ are zero. Then the third parameter is equal to 1 by embedding condition (1), and thus $d \in \{a,b,c\}$ by embedding condition (4). Consequently, K has at least one pair of canceling twist parameters, a contradiction. It follows that at most one of α, β, γ is zero and at most one of x, y, z is zero. The remainder of the proof consists in showing the stronger statement that the sets $\{\alpha,\beta,\gamma\}$ and $\{x,y,z\}$ cannot *both* contain 0.

Without loss of generality, suppose $\alpha = 0$ and $\beta \ne \gamma \ne 0$. It will be shown that if any of x, y, z is zero, then either K is not 0–pair or there is a contradiction to $x,y,z \in \mathbb{Z}$.

With the assumptions on α, β, and γ, embedding conditions (1) and (4) immediately yield $d = b\beta^2 + c\gamma^2 \leq 4b + c \leq 4a + b$, and embedding condition (3) implies

$$b\beta y = -c\gamma z. \tag{2}$$

From (2), if either one of y or z is 0, then so is the other. Thus, K is not 0–pair by the first paragraph of the proof and therefore y and z are both nonzero.

If $x = 0$, embedding condition (2) implies that $z = 1 - y$. Substituting this into (2) and solving for y yields

$$y = \frac{c\gamma}{c\gamma - b\beta}. \tag{3}$$

Since $\alpha = 0$, embedding condition (1) implies that $\beta + \gamma = 1$. If either one of β or γ is equal to 1, then the other vanishes. This violates the assumption that $\beta \neq \gamma \neq 0$; therefore $\beta, \gamma \notin \{0, 1\}$. Note that β and γ always have different signs. If $\gamma \geq 2$, then $\beta \leq -1$ and thus $-b\beta > 0$. Thus, (3) takes on the form

$$y = \frac{p}{p+q}, \tag{4}$$

where $p, q \in \mathbb{Z}^+$. Thus y cannot be an integer, contradicting the embedding conditions. If instead $\gamma \leq -1$, then $\beta \geq 2$. In this case, one can take (2) and solve for z instead of y, yielding

$$z = \frac{b\beta}{b\beta - c\gamma}. \tag{5}$$

By the same argument given for $\gamma \geq 2$, if $\beta \geq 2$ it follows that z cannot be an integer and the embedding conditions are again contradicted. Thus if $\alpha = 0$, each of x, y, z must be nonzero and $e = ax^2 + by^2 + cz^2 \geq a + b + c$ by embedding condition (5).

If $\beta = 0$, the proof follows similarly with $d = a\alpha^2 + c\gamma^2 \geq 4a + c \geq 4a + b$; if $\gamma = 0$, then again the proof follows similarly with $d = a\alpha^2 + b\beta^2 \geq 4a + b$. In all three cases, $e = ax^2 + by^2 + cz^2 \geq a + b + c$. Lastly, if none of α, β, γ, x, y, z is zero, then embedding conditions (4) and (5) imply that $d = a\alpha^2 + b\beta^2 + c\gamma^2 \geq a + b + c$ and $e = ax^2 + by^2 + cz^2 \geq a + b + c$, since $c \geq b \geq a \geq 1$. □

8 Proof of Theorem 2.3 without single-twists

The proof of Theorem 2.3 will now proceed by showing that if K is a 0–pair odd 5–stranded pretzel knot without single-twists, then coset condition I is violated. Assume K is slice. It follows that K satisfies the signature condition and the lattice embedding condition. Furthermore, it may also be assumed that $K = P(-a,-b,-c,d,e)$ with $a \le b \le c$. Let $\mathcal{R}$ and $\mathcal{H}$ be as in coset condition I.

The fact that K is 0–pair implies that $d \ge 4a+b$ and $e \ge a+b+c$ by Lemma 7.1. Given that $\ker(\gamma) \cong \operatorname{coker}(\alpha)$ (where γ and α here refer to the maps in Section 6) and $|\ker(\gamma)| = \sqrt{|H_1(Y)|} = \sqrt{|\det(K)|}$, it follows that $|\mathcal{R}| = |\operatorname{coker}(\alpha)| = \sqrt{|\det(K)|}$. Theorem 1.4 in [5] gives the following formula for the determinant of odd pretzel knots $P(p_1,\dots,p_k)$:

$$\det(K) = \sum_{i=1}^{k} p_1 \cdots p_{i-1} \widehat{p}_i p_{i+1} \cdots p_k.$$

Using this with the above choice of K, one gets

$$\det(K) = -abcd - abce + abde + acde + bcde.$$

To compute $|\overline{\mathcal{H}}|$, a direct computation shows that the closed hexagonal region H in which $\mathcal{H}$ is contained is a region in $\mathbb{R}^2$ defined by the $2(a+c) \times 2(b+c)$ rectangle centered at the origin, minus the lower-right and upper-left half-square triangular regions with side lengths $2c$. See Figure 13. The set $\mathcal{H}$ contains all lattice points in $2\mathbb{Z}^2$ in the interior of H and on the boundary of H. These can be counted in many different ways but are counted here by observing that there are $a+b+1$ even lattice points in the perpendicular boundary components of H lying in the third quadrant of $\mathbb{R}^2$, and there are $c+1$ copies of this L-shaped collection of even lattice points repeated throughout H; furthermore, H includes a $a \times b$ rectangle of even lattice points. Hence

$$|\mathcal{H}| = (a+b+1)(c+1) + ab = ab + ac + bc + a + b + c + 1.$$

To violate coset condition I, it will be argued that $|\mathcal{R}|^2 > |\mathcal{H}|^2$ using the facts that

(1) $3 \le a \le b \le c$,

(2) $d \ge 4a+b$ and $e \ge a+b+c$ or $d, e \ge a+b+c$, and

(3) $ab > a+b+\frac{1}{2}$ for $a, b \ge 3$.

By [13, Theorem 2.0.3], 0–pair odd 5–stranded pretzel knots $P(-a,-b,-c,d,e)$ are not slice if $d, e \ge a+b+c$, thus that case is omitted here. Hence, we assume $d \ge 4a+b$ and $e \ge a+b+c$. First consider $|\mathcal{H}|^2$:

$$|\mathcal{H}|^2 = (ab+ac+bc+a+b+c+1)^2 = L+M+N+S,$$

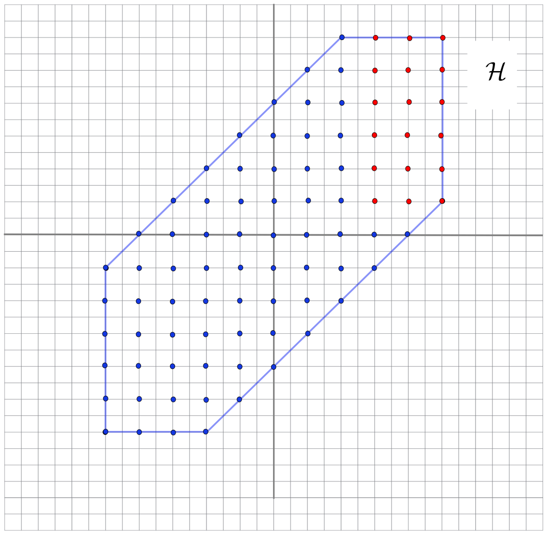


Figure 13: $\mathcal{H}$ for $P(-3,-7,-19,d,e)$

where

$$\begin{aligned} L &= a^2b^2+a^2c^2+b^2c^2+2(a^2bc+ab^2c+abc^2), \\ M &= 2(a^2b+a^2c+ab^2+b^2c), \\ N &= 2c^2\left(a+b+\tfrac{1}{2}\right)+6abc+4(ab+ac)+3bc, \\ S &= a^2+b^2+bc+2(a+b+c)+1. \end{aligned}$$

It will be shown that $|\mathcal{R}|^2 > |\mathcal{H}|^2$ by proving equivalently that

$$|\mathcal{R}|^2 - L - M - N > |\mathcal{H}|^2 - L - M - N.$$

Consider $|\mathcal{R}|^2$:

$$\begin{aligned} |\mathcal{R}|^2 &= |\det(K)| = |-abcd-abce+abde+acde+bcde| \\ &= abd(e-c)+bce(d-a)+acde \\ &\geq 5a^2b^2+4a^2c^2+b^2c^2+8a^2bc+5ab^2c+4abc^2+4a^3(b+c)+b^3(a+c) \\ &=: E_3, \end{aligned}$$

where the inequality follows from making the substitutions $d \geq 4a+b$ and $e \geq a+b+c$. Thus

$$|\mathcal{R}|^2 - L \geq E_3 - L.$$

Next we have

$$\begin{aligned} E_3 - L &= 4a(a^2b+a^2c)+b(ab^2+b^2c)+4a^2b^2+3a^2c^2+6a^2bc+3ab^2c+2abc^2 \\ &\geq 12(a^2b+a^2c)+3(ab^2+b^2c)+4a^2b^2+3a^2c^2+6a^2bc+3ab^2c+2abc^2 \\ &=: E_2, \end{aligned}$$

where the inequality comes from the fact that $4a \geq 12$ since $a \geq 3$. Therefore $|\mathcal{R}|^2 - L \geq E_3 - L > E_2$, and so

$$|\mathcal{R}|^2 - L - M \geq E_3 - L - M > E_2 - M.$$

Next we have

$$\begin{aligned} E_2 - M &= 10(a^2b+a^2c)+(ab^2+b^2c)+4a^2b^2+3a^2c^2+6a^2bc+3ab^2c+2abc^2 \\ &> 2c^2\left(a+b+\tfrac{1}{2}\right)+6abc+4(ab+ac)+3bc+4a^2b^2+3a^2c^2 \\ &\qquad\qquad +3ab^2c+6(a^2b+a^2c)+ab^2 \\ &=: E_1, \end{aligned}$$

where the inequality comes from the following four facts, obtained from the assumption that $c \geq b \geq a \geq 3$:

- $2abc^2 = 2c^2(ab) > 2c^2\left(a+b+\frac{1}{2}\right)$,
- $b^2c > 3bc$,
- $6a^2bc > 6abc$,
- $10(a^2b+a^2c) = 6(a^2b+a^2c)+4(a^2b+a^2c) > 6(a^2b+a^2c)+4(ab+ac)$.

This shows that $E_2 - M > E_1$, so it follows that

$$|\mathcal{R}|^2 - L - M - N \geq E_3 - L - M - N > E_2 - M - N > E_1 - N.$$

Now observe that

$$\begin{aligned} E_1 - N &= 4a^2b^2+3a^2c^2+3ab^2c+6(a^2b+a^2c)+ab^2 \\ &= 6a^2b+ab^2+3a^2c^2+(4a^2b^2+3ab^2c+6a^2c) \\ &> a^2+b^2+bc+2(a+b+c)+1 \\ &= S = |\mathcal{H}|^2 - L - M - N, \end{aligned}$$

where the inequality comes from the following six facts, again obtained from the assumption that $c \geq b \geq a \geq 3$:

$$6a^2b > a^2, \quad ab^2 > b, \quad 3ab^2c > bc, \quad 3a^2c^2 > 2a, \quad 4a^2b^2 > 2b, \quad 6a^2c > 2c+1.$$

Combining everything, one sees that

$$|\mathcal{R}|^2 - L - M - N > |\mathcal{H}|^2 - L - M - N,$$

which implies that $|\mathcal{R}|^2 > |\mathcal{H}|^2$, as desired. This completes the proof that all 0–pair odd 5–stranded pretzel knots without single-twists are not slice.

9 Proof of Theorem 2.3 with single-twists

Next, the knots with single-twists are addressed. As before, assume all knots in question are slice and therefore satisfy the signature condition, the lattice embedding condition, and both coset conditions. Just a bit of thought reveals that, possibly after mirroring, the signature condition yields only three cases to consider for 0–pair odd 5–stranded pretzel knots with single-twists. Note that $d, e \geq a$ in all cases due to embedding conditions (1), (2), (4), and (5). For $K \in P\{-a, -b, -c, d, e\}$, the cases are

(1) $a = b = c = 1$ and $d, e \geq 3$,

(2) $a = b = 1$ and $c, d, e \geq 3$,

(3) $a = 1$ and $b, c, d, e \geq 3$.

Since the lattice embedding conditions hold, there exist α, β, γ, x, y, $z \in \mathbb{Z}$ satisfying the system of equations given in Section 5. Thus, this proof for nonsliceness of 0–pair pretzel knots with single-twists has the same starting point as the previous proof for nonsliceness of 0–pair knots without single-twists. Lemma 7.1 still applies here for all three cases of 0–pair pretzel knots with single-twists. To obstruct sliceness for 0–pair knots $P(-a, -b, -c, d, e)$ with single twists, however, it is necessary to get more precise lower bounds on d and e than are obtained in Lemma 9.1.

Lemma 9.1 *If $K \in P\{-a, -b, -c, d, e\}$ is 0–pair and d is equal to its lower bound (either $d = 4a + b$ or $d = a + b + c$), then $e \geq 4a + 4b + c$.*

Proof First, suppose $d = 4a + b$ and $e = a + b + c$. By the embedding conditions, it follows that $\alpha = 2, \beta = -1$, and $\gamma = 0$, and $|x| = |y| = |z| = 1$. Embedding condition (3) says $a\alpha x + b\beta y + c\gamma z = 0$, which reduces to $\pm 2a = b$ after substitutions. But, b is odd so this is a contradiction. If instead one supposes that $e = a + b + c$, then by the embedding conditions, $|\alpha| = |\beta| = |\gamma| = |x| = |y| = |z| = 1$. After substitutions, embedding condition (3) becomes $c = \pm a \pm b$, which is again a contradiction since all three of a, b, c are odd.

Hence when $d = 4a + b$ or $d = a + b + c$, we have $e \neq a + b + c$. In words, both d and e cannot simultaneously achieve their lower bounds as given in Lemma 7.1. It follows that at least one of $|x|$, $|y|$, or $|z|$ must be ≥ 2. But, in fact, it will be shown presently that at least two of $|x|$, $|y|$, and $|z|$ must be ≥ 2. If $x = 2$, embedding condition (2) implies that $y + z = -1$; if $x = -2$, then $y + z = 3$. In both cases, it is impossible for both $|y| = 1$ and $|z| = 1$, and therefore $|y| \geq 2$ or $|z| \geq 2$. By the symmetry in x, y, z of embedding condition (2), similar results follow if $y = \pm 2$ or if $z = \pm 2$. Hence, at least two of $|x|$, $|y|$, or $|z|$ must be ≥ 2.

The choices of $|x|, |y|, |z|$ that satisfy the above discovery and that minimize e are $|x| = |y| = 2$ and $|z| = 1$, which yields $e = ax^2 + by^2 + cz^2 = 4a + 4b + c$. Thus, if d is equal to a lower bound then $e \geq 4a + 4b + c$. □

The proof of Theorem 2.3 will now proceed. The goal in each of the following cases is to arrive at a contradiction to coset condition I by showing that $|\mathcal{R}|^2 > |\mathcal{H}|^2$.

Case 1 $K \in P\{-a, -b, -c, d, e\}$ with $a = b = c = 1$.

By Lemma 7.1, $d \geq 4a + b = 5$ or $d \geq a + b + c = 3$. Assume $d = 3$. By Lemma 7.1, it follows that $e \geq 4a + 4b + c = 9$. Then

$$\begin{aligned}|\mathcal{R}|^2 &= |\det(K)| = |-abcd - abce + abde + acde + bcde| \\ &= d(e-1) + e(d-1) + de \geq 69 \\ &> 49 = (ab + ac + bc + a + b + c + 1)^2 \\ &= |\mathcal{H}|^2,\end{aligned}$$

as desired.

Case 2 $K \in P\{-a, -b, -c, d, e\}$ with $a = b = 1$ and $c \geq 3$.

By Lemma 7.1, $d \geq 4a + b = 5$ or $d \geq a + b + c = 2 + c$. But, $c \geq 3$ so in any case we have $d \geq 5$ and thus $e \geq 4a + 4b + c = 8 + c$ by Lemma 9.1. Then

$$\begin{aligned}|\mathcal{R}|^2 &= |\det(K)| = |-abcd - abce + abde + acde + bcde| \\ &= d(e-c) + ce(d-1) + cde \\ &\geq 5(8 + c - c) + c(8 + c)(5 - 1) + 5c(8 + c) = 9c^2 + 72c + 40 \\ &> 9c^2 + 24c + 16 = (3c + 4)^2 \\ &= (ab + ac + bc + a + b + c + 1)^2 \\ &= |\mathcal{H}|^2,\end{aligned}$$

as desired.

Case 3 $K \in P\{-a, -b, -c, d, e\}$ with $a = 1$ and $b, c \geq 3$.

By Lemma 7.1, $d \geq 4a + b = 4 + b$ or $d \geq a + b + c = 1 + b + c$. Assuming that $d \geq b + 4$ accounts for both situations. By Lemma 9.1, $e \geq 4a + 4b + c = 4 + 4b + c$.

Then

$$\begin{aligned}|\mathcal{R}|^2 &= |\det(K)| = |-abcd-abce+abde+acde+bcde|\\ &= bd(e-c)+bce(d-1)+cde\\ &\geq b(b+4)(4+4b+c-c)+bc(4+4b+c)(b+4-1)+c(b+4)(4+4b+c)\\ &= 4b^3c+b^2c^2+4b^3+20b^2c+4bc^2+20b^2+32bc+4c^2+16b+16c.\end{aligned}$$

Also

$$\begin{aligned}|\mathcal{H}|^2 &= (ab+ac+bc+a+b+c+1)^2\\ &= b^2c^2+4b^2c+4bc^2+4b^2+12bc+4c^2+8b+8c+4.\end{aligned}$$

Let $L = b^2c^2+4b^2c+4bc^2+4b^2+12bc+4c^2+8b+8c$. Then

$$|\mathcal{R}|^2-|\mathcal{H}|^2 = 4b^3c+16b^2c+20bc+8c+4b^3+16b^2+8b-4>0$$

Thus, $|\mathcal{R}|^2 > |\mathcal{H}|^2$. This concludes the proof that 0–pair odd pretzel knots with single-twists are not slice, and therefore all 0–pair odd 5–stranded pretzel knots are not slice. □

10 Proof of Theorem 2.4

Theorem 2.4 asserts that if K is a 1–pair odd 5–stranded pretzel knot without single-twists, then K is not slice. This will be shown by proving that coset condition II is violated for the knots in question. It suffices to consider only the 1–pair pretzel knots $P(a,b,c,d,e)$ for which the signature vanishes and both the lattice embedding condition and coset condition I are satisfied. Let $a,b,c,d,e>0$ such that $a\leq b\leq c$, and assume that $K=P(-a,-b,-c,d,e)$ throughout. Let Y, $P=P_+$, W, X, and the embedding map $\alpha\colon H_2(P)\to H_2(X)/\operatorname{Tor}$ be as usual.

Theorem 6.3 gives that if K is slice, then every coset of $\operatorname{im}(\alpha)$ coming from $\operatorname{coker}(\alpha)$ has a coset representative in the set $\{\pm 1\}^m$, where $m=a+b+c$. Let v_1 and v_2 be the second and third columns (respectively) in the matrix A of α with respect to the bases chosen in Sections 5 and 6; lastly, let B be the map outlined in Section 6.

Recall from the coset conditions the sets $\mathcal{R}$ and $\mathcal{H}$ associated with A. The set $\mathcal{R}$ consists of the integer lattice points in a fundamental region $R\subset\mathbb{R}^2$ defined by $\overline{v}_1$ and $\overline{v}_2$ that correspond to unique cosets of $\operatorname{im}(\alpha)$; $\mathcal{H}$ is the set of lattice points $(x,y)\in 2\mathbb{R}^2$ such that $-a-c\leq x\leq a+c$ and $-b-c\leq y\leq b+c$, lying in a hexagonal region $H\subset\mathbb{R}^2$. The argument now reduces to determining $|\mathcal{R}|$ and $|\overline{\mathcal{H}}|$, where $\overline{\mathcal{H}}=\mathcal{H}/\langle\overline{v}_1,\overline{v}_2\rangle$. An important note is that the computation of $|\mathcal{R}|$ will be done differently here from how it is done in Chapter 7.

Observe that $\mathcal{R}$ consists of all lattice points in the interior of R as well as all lattice points on the boundary of R modulo $\overline{v}_1$ and $\overline{v}_2$. The action of modding out the boundary of R by $\overline{v}_1$ and $\overline{v}_2$ has the effect of removing half of all boundary lattice points, plus one more. The extra lattice point that must be removed is, without loss of generality, the one at the tip of $\overline{v}_1$ that gets identified with $(0,0)$. Hence, if i is the number of interior lattice points of R and b is the total number of lattice points lying on the boundary, then

$$|\mathcal{R}| = i + \frac{b}{2} - 1.$$

In a lucky turn of events, Pick's theorem equates the right hand side of this expression with the area of R. In it's general form, Pick's theorem states that the area A of any polygon P in $\mathbb{R}^2$ with vertices at integer lattice points is given by

$$A(P) = i + \frac{b}{2} - 1,$$

where i is the number of integer lattice points in the interior of the polygon and b is the number of integer lattices points lying on the boundary of the polygon. Thus,

$$|\mathcal{R}| = i + \frac{b}{2} - 1 = A(R).$$

Given that R is a parallelogram in $\mathbb{R}^2$ defined by $\overline{v}_1, \overline{v}_2 \in \mathbb{Z}^2$, its area $A(R)$ — and thus $|\mathcal{R}|$ — is equal to the absolute value of the determinant of the 2×2 matrix whose column vectors are $\overline{v}_1$ and $\overline{v}_2$:

$$|\mathcal{R}| = A(R) = \begin{vmatrix} a\alpha - c\gamma & ax - cz \\ b\beta - c\gamma & by - cz \end{vmatrix}.$$

Also, recall from Section 8 that

$$|\mathcal{H}| = (a+b+1)(c+1) + ab = ab + ac + bc + a + b + c + 1.$$

In obstructing sliceness for 1–pair pretzel knots (still under the assumption $a \le b \le c$), three cases must be considered: (1) when the pair is $\{a, -a\}$, (2) when the pair is $\{b, -b\}$, and (3) when the pair is $\{c, -c\}$. By assumption, the twist parameters in all three cases satisfy the embedding criterion.

Case 1 $K \in P\{-a, -b, -c, a, e\}$ with $e \notin \{b, c\}$.

When the twist parameters contain the pair $\{a, -a\}$, we obtain $\alpha = 1$, $\beta = \gamma = x = 0$, and that y and z are nonzero. This yields $\overline{v}_1 = (a, 0)$ and $\overline{v}_2 = (-cz, by - cz)$, hence

$$|\mathcal{R}| = \begin{vmatrix} a & -cz \\ 0 & by - cz \end{vmatrix} = a|by - cz|.$$

As $y \to \infty$, it follows that $z \to -\infty$ by embedding condition (2) which says that $1 = x + y + z = 0 + y + z$; thus $|\mathcal{R}| \to \infty$. Similarly, as $y \to -\infty$ (and $z \to \infty$), $|\mathcal{R}| \to \infty$. For this reason, $|\mathcal{R}|$ is minimized when y and z are both small in absolute value, ie when $\overline{v}_2$ is short. Given that $b < c$, we have $\overline{v}_2$ shortest when $y = 2$ and $z = -1$. In this case

$$|\mathcal{R}| = a|2b + c| = 2ab + ac. \tag{6}$$

An upper bound for $|\overline{\mathcal{H}}|$ will now be computed. Due to the shape and dimensions of $\mathcal{H}$, we can see that many of the lattice points of $\mathcal{H}$ lie in the same $\langle \overline{v}_1, \overline{v}_2 \rangle$–coset because any two lattice points in $\mathcal{H}$ that differ by multiple of $\overline{v}_1 = (a, 0)$ will be identified. Furthermore, modding out by $\overline{v}_2$ would only result in more identification among the lattice points of $\mathcal{H}$. Hence

$$|\overline{\mathcal{H}}| = |\mathcal{H}/\langle \overline{v}_1, \overline{v}_2 \rangle \le \mathcal{H}/\langle \overline{v}_1 \rangle.$$

Note: Figures 14–19 show the lattice points in $2\mathbb{Z}^2$, that is, each grid square is 2×2.

From Figure 14, we see that each of the $b + c + 1$ rows in $\mathcal{H}$ has a distinct $\langle \overline{v}_1, \overline{v}_2 \rangle$–cosets. The result of eliminating repeated representatives from each coset to obtain $\overline{\mathcal{H}}$

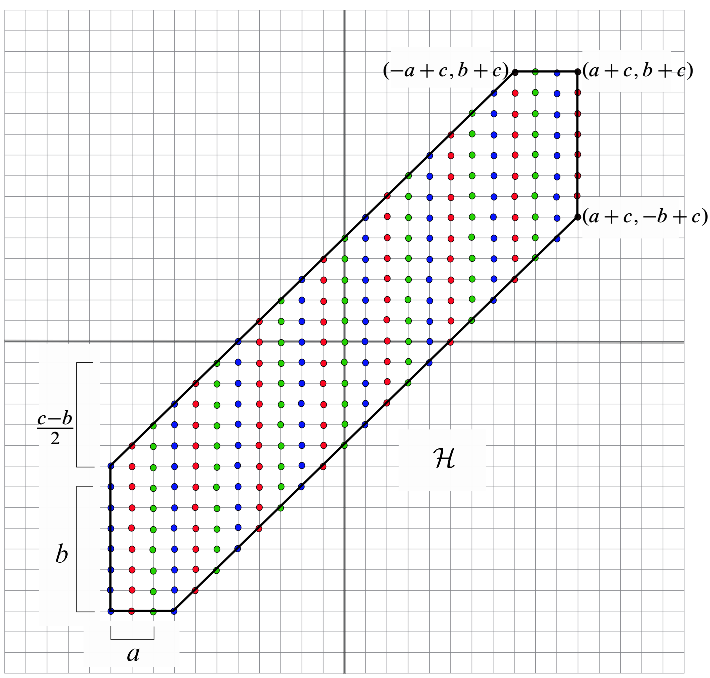


Figure 14: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3, -7, -19, 3, 47)$. Points of the same color *with the same y–coordinate* represent the same $\mathcal{R}$–coset.

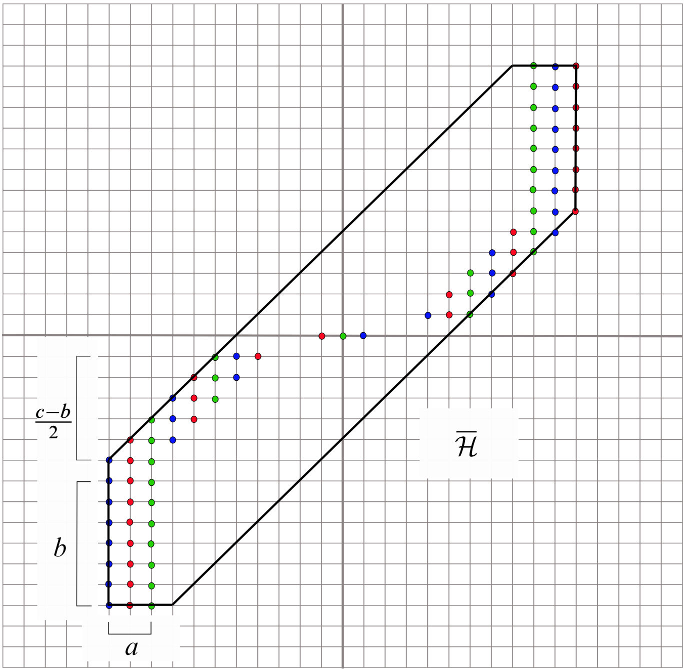


Figure 15: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3,-7,-19,3,47)$, repeat representatives removed

is shown in Figure 15. Thus, an upper bound for $|\overline{\mathcal{H}}|$ is given by

$$|\overline{\mathcal{H}}| \le \mathcal{H}/\langle \overline{v}_1 \rangle = a(b+c+1) = ab+ac+a.$$

Comparing this to (6), the desired result is achieved:

$$|\overline{\mathcal{H}}| \le ab+ac+a < 2ab+ac = |\mathcal{R}|,$$

since $a \le b \le c$. Hence, an odd 5–stranded pretzel knot $K \in P\{-a,-b,-c,a,d\}$, with $a,b,c,d \ge 3$, is not slice by coset condition I.

Case 2 $K \in P\{-a,-b,-c,b,d\}$ with $e \notin \{a,c\}$.

When the twist parameters contain the pair $\{b,-b\}$, we obtain $\beta = 1$, $\alpha = \gamma = y = 0$, and that x and z are nonzero. With this, $\overline{v}_1 = (0,b)$ and $\overline{v}_2 = (-cz, by-cz)$, hence

$$|\mathcal{R}| = \begin{vmatrix} 0 & ax-cz \\ b & -cz \end{vmatrix} = b|ax-cz|.$$

Following the logic of Case 1, it suffices to show that $|\mathcal{R}| > |\overline{\mathcal{H}}|$ when the length of $\overline{v}_2$

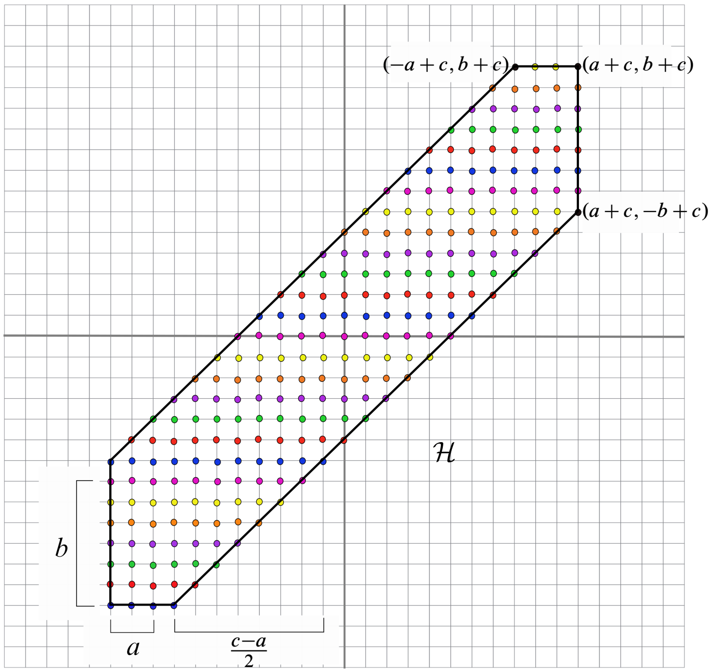


Figure 16: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3,-7,-19,7,31)$. Points of the same color *with the same x–coordinate* represent the same $\mathcal{R}$–coset.

is minimized. Since $a < c$, we have $\overline{v}_2$ shortest when $x = 2$ and $z = -1$, so

(7) $$|\mathcal{R}| = b|2a+c| = 2ab + bc.$$

The upper bound for $|\overline{\mathcal{H}}|$ is computed for Case 2 in a similar manner as for Case 1, the only difference being that $\overline{v}_1 = (0, b)$, and thus lattice points in $\mathcal{H}/\langle \overline{v}_1 \rangle$ are in the same coset when they differ by multiple of $(0, b)$ (vertical translations), as seen in Figure 16. Each of the $a+c+1$ columns in $\mathcal{H}$ always has b distinct $\mathcal{R}$–cosets. The result of eliminating repeated representatives from each coset to obtain $\overline{\mathcal{H}}$ is shown in Figure 17. Thus, an upper bound for $|\overline{\mathcal{H}}|$ is given by

$$|\overline{\mathcal{H}}| \leq \mathcal{H}/\langle \overline{v}_1 \rangle = b(a+c+1) = ab + bc + b.$$

Comparing this to (7), again the desired result is achieved:

$$|\overline{\mathcal{H}}| \leq ab + bc + b < 2ab + bc = |\mathcal{R}|,$$

since $b \geq a \geq 3$. Hence, 5–stranded pretzel knots $K \in P\{-a, -b, -c, b, d)\}$, with $a, b, c, d \geq 3$, are not slice.

Case 3 $K \in P\{-a, -b, -c, c, d\}$ with $e \notin \{a, b\}$.

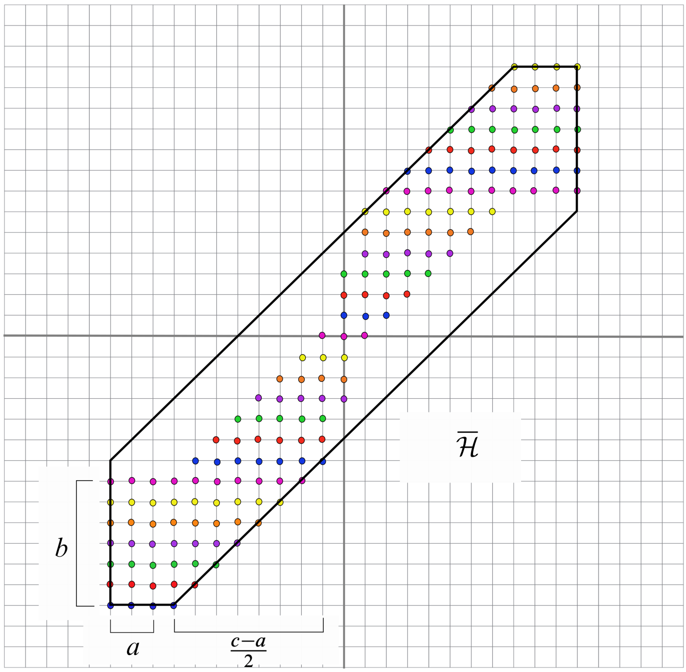


Figure 17: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3,-7,19,7,31)$, repeat representatives removed.

The final case of 5–stranded odd 1–pair pretzel knots has $\{c,-c\}$ as the pair in the twist parameters. Unlike for the 1–pair cases where the pair of canceling twist parameters is $\{a,-a\}$ or $\{b,-b\}$, the case with $\{c,-c\}$ does not necessarily imply that $\gamma=1$, $\alpha=\beta=z=0$, with x and y nonzero. Since $c \geq b \geq a$, it is possible that both α and β are nonzero and embedding condition (4) is satisfied by $c=a\alpha^2+b\beta^2$. In this case, however, the proof of Lemma 9.1 shows we would have $c \geq 4a+b$ and $e=ax^2+by^2+cz^2 \geq a+b+c$, which implies that $P(-a,-b,-c,c,d)$ is not slice by the proof of Theorem 2.3.

Hence, the only case that need be considered is the case in which $\gamma=1$, $\alpha=\beta=z=0$, with x and y nonzero. Under these conditions, $\overline{v}_1=(-c,-c)$ and $\overline{v}_2=(ax,by)$, and therefore

$$|\mathcal{R}| = \begin{vmatrix} -c & ax \\ -c & by \end{vmatrix} = c\,|ax-by|.$$

Again by following the logic from Case 1 and Case 2, it suffices to show that $|\mathcal{R}|>|\overline{\mathcal{H}}|$ when $\overline{v}_2$ is at its shortest. Since $a<b$, the length of $\overline{v}_2$ is minimized when $x=2$ and

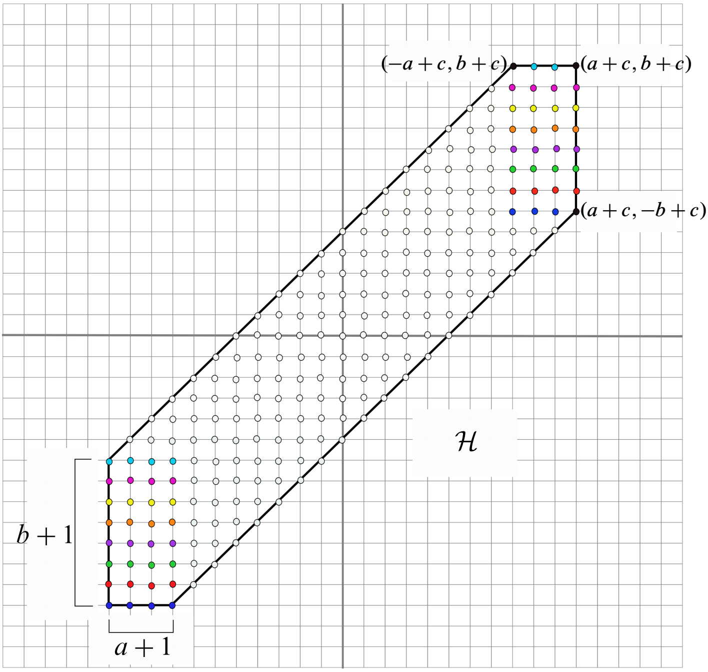


Figure 18: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3,-7,-19,19,55)$. Each white point represents a distinct $\mathcal{R}$–coset; colored points *lying along the same* 45*–degree diagonal* represent the same $\mathcal{R}$–coset.

$y=-1$. In this case,

$$|\mathcal{R}| = c\,|2a+b| = 2ac+bc. \tag{8}$$

The computation of an upper bound for $|\overline{\mathcal{H}}|$ in Case 3 is similar to those in Cases 1 and 2. Namely, it is computed by identifying lattice points in $\mathcal{H}$ via multiples of $\overline{v}_1=(-c,-c)$ (45–degree diagonal translations). The computations are also done as before using the well-understood region $\mathcal{H}$, however it is more efficient now to subtract off the number of repeat $\langle \overline{v}_1\rangle$–coset representatives from $|\mathcal{H}|$, rather than count the cosets directly as in Cases 1 and 2. Figure 18 indicates that

$$\begin{aligned}|\overline{\mathcal{H}}| &\le |\mathcal{H}|-(a+1)(b+1)\\ &= ab+ac+bc+a+b+c+1-(ab+a+b+1)\\ &= ac+bc+c.\end{aligned}$$

Since $c\ge a\ge 3$, comparing this result with (8) gives the result

$$|\overline{\mathcal{H}}| \le ac+bc+c < 2ac+bc = |\mathcal{R}|.$$

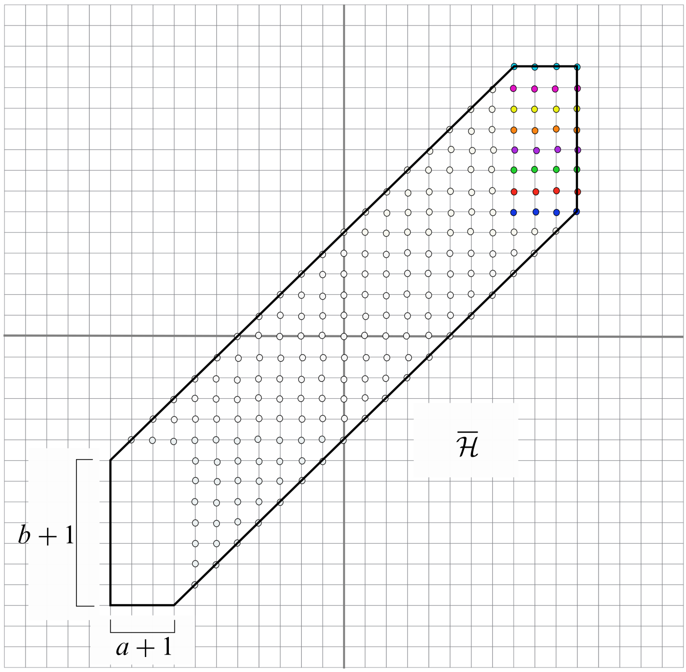


Figure 19: $\mathcal{H}$ and its $\mathcal{R}$–cosets for $P(-3,-7,19,19,55)$, repeat representatives removed.

Refer to Figure 19 for $\overline{\mathcal{H}}$ in this case. Thus, 5–stranded odd pretzel knots of the form $P(-a,-b,-c,c,d)$, with $a,b,c,d \geq 3$, are not slice. The proof is complete. □

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Received: 21 September 2016 Revised: 7 February 2017