Algebraic & Geometric Topology 17 (2017) 3665–3701

Axioms for higher twisted torsion invariants of smooth bundles

CHRISTOPHER OHRT

This paper attempts to investigate the space of various characteristic classes for smooth manifold bundles with local system on the total space inducing a finite holonomy covering. These classes are known as twisted higher torsion classes. We will give a system of axioms that we require these cohomology classes to satisfy. Higher Franz–Reidemeister torsion and twisted versions of the higher Miller–Morita–Mumford classes will satisfy these axioms. We will show that the space of twisted torsion invariants is two-dimensional or one-dimensional depending on the torsion degree and is spanned by these two classes. The proof will greatly depend on results about the equivariant Hatcher constructions developed in Goodwillie, Igusa and Ohrt (2015).

19J10, 55R40; 57R80, 55R10

1 Introduction

Higher torsion invariants have been developed by J Wagoner, J R Klein, K Igusa, M Bismut, J Lott, W Dwyer, M Weiss, E B Williams, S Goette and many others; see Wagoner [12], Igusa [5], Igusa and Klein [9], Bismut and Lott [2], Dwyer, Weiss and Williams [3] and Bismut and Goette [1].

In [7], Igusa defined a higher torsion invariant of degree $2k$ to be a characteristic class $\tau(E) \in H^{4k}(B;\mathbb{R})$ of a smooth bundle $E \to B$ satisfying an additivity and a transfer axiom; see [7, Section 2]. He proved that the set of higher torsion invariants forms a two-dimensional vector space spanned by the higher Reidemeister torsion and the Miller–Morita–Mumford class.

But higher Reidemeister torsion or Igusa–Klein torsion can be defined in a more general way: it is a characteristic class $\tau^{\mathrm{IK}}(E,\rho) \in H^{2k}(B;\mathbb{R})$ for a smooth bundle with an unitary representation $\rho\colon \pi_1 E \to U(m)$ factorizing through a finite group; see for example Igusa [5]. For our purposes it will be better to look at finite complex local systems on E instead. After a choice of a base point, this corresponds to a representation of the fundamental group as can be found for example in T Szamuely's book [11, Corollary 2.6.2]. Regarding that, we will define a twisted higher torsion invariant in degree k to be a characteristic class $\tau(E;\mathcal{F}) \in H^{2k}(B;\mathbb{R})$ depending on

Published: 4 October 2017 DOI: 10.2140/agt.2017.17.3665

a finite local complex system $\mathcal{F}$ on E inducing a finite holonomy covering satisfying six axioms: the first two are versions of the original two axioms for nontwisted torsion invariants, which will respect the local system; the remaining four axioms will determine the dependence of the torsion class on the local system.

The goal of this paper is to show an analogous result to Igusa's on twisted torsion invariants. For this we will generalize Igusa's paper [7] step by step:

In Section 2, we will define twisted higher torsion invariants.

In Section 3, we will repeat why the Igusa–Klein torsion τ^{IK} satisfies the axioms, introduce a twisted version of the Miller–Morita–Mumford classes M^{2k} and show that these also satisfy the axioms. The MMM classes will be zero in degree $4l+2$. Then we will state our main theorem:

Theorem 1.1 (main theorem) *The space of higher twisted torsion invariants in degree $4l$ on bundles with simple fibers and base having a finite fundamental group is two-dimensional and spanned by the twisted MMM class and the twisted Igusa–Klein torsion, and one-dimensional in degree $4l+2$ and spanned by the Igusa–Klein torsion. In other words, for any twisted torsion invariant of even degree τ, there exist unique $a, b \in \mathbb{R}$ such that*

$$\tau = a\tau^{\mathrm{IK}} + bM,$$

and for every twisted torsion invariant τ of odd degree there exists a unique $a \in \mathbb{R}$ such that

$$\tau = a\tau^{\mathrm{IK}}.$$

The scalars a and b can be calculated as follows: For torsion in degree $4l$ we look at the universal line bundle $\lambda\colon ES^1 \to \mathbb{CP}^\infty$. Since the cohomology groups $H^{2k}(\mathbb{CP}^\infty;\mathbb{R})$ are one-dimensional, the torsion invariant of the associated S^1–bundle $S^1(\lambda)$ and the associated S^2–bundle $S^2(\lambda)$ over $\mathbb{CP}^\infty$ will determine the scalars a and b. In degree $4l+2$ we only have to calculate a by looking at a fiberwise quotient $S^1(\lambda)/(\mathbb{Z}/n)$ of the n–action on S^1. This admits a nontrivial finite complex local system and therefore has a nontrivial higher twisted torsion.

Before we prove the main theorem, we will extend a higher twisted torsion invariant to have values on bundles with vertical boundaries and then define a relative torsion for bundle pairs (see Section 4), which we will use to deconstruct any bundle into easier pieces and keep control over the torsion.

In Section 5, we will show that the main theorem holds on S^1–bundles. Then we will define the difference torsion to be

$$\tau^\delta := \tau - a\tau^{\mathrm{IK}} - bM,$$

and we will see that $\tau^\delta = 0$ for every sphere bundle, disc bundle and odd-dimensional lens space bundle. In Goodwillie, Igusa and Ohrt [4] we give an explicit base for the space of h–cobordism bundles of a lens space, and the calculations in Section 6 of this paper show that the difference torsion will be zero on these basis elements. From this crucial observation we can deduce that the difference torsion will be a fiber homotopy invariant, and in Section 7 we will show that this fiber homotopy invariant must be trivial if it is restricted to bundles with simple fiber and base having finite fundamental group.

Acknowledgements This paper is the product of my work with Kiyoshi Igusa during my stay at Brandeis University in the academic year 2011/2012, which led up to further work on the equivariant Hatcher constructions with Thomas Goodwillie and Kiyoshi Igusa in the academic year 2012/2013. I want to thank Kiyoshi for the great support, advice, and guidance he offered me. I also want to thank Ulrich Bunke from my home university in Regensburg, Germany, for the many comments and corrections he contributed.

2 Axioms and definitions

2.1 Preliminaries

Throughout the whole paper, let $F \hookrightarrow E \xrightarrow{p} B$ be a smooth fiber bundle, where E and B are compact smooth manifolds, p is a smooth submersion, and F is a compact orientable n–dimensional manifold with or without boundary. In the boundary case, there is a subbundle $\partial F \to \partial^v E \to B$ of E. We call $\partial^v E$ the vertical boundary of E. We assume that B is connected and that the action of $\pi_1 B$ on F preserves the orientation of F. We also assume that $\pi_1 B$ is finite, which immediately implies that the bundle E is unipotent (as required in [7]).

These are all similar assumptions to the ones for considering nontwisted higher torsion classes. Additionally to those, we assume that E comes equipped with a finite complex local system $\mathcal{F}$. By "finite" we mean that there exists a finite covering $\widetilde{E} \to E$ such that the pull-back of the local system is trivializable. These local systems are sometimes also called hermitian local coefficient systems because they induce a well defined hermitian inner product on each fiber. We will often call $\mathcal{F}$ just local coefficient system.

If $F \hookrightarrow E \to B$ is a smooth bundle we have the transfer map

$$\operatorname{tr}_B^E \colon H^*(E;\mathbb{Z}) \to H^*(B;\mathbb{Z}).$$

For an exact definition, one can consult [7, Section 2]. The most important property we will need is that we always have

$$\operatorname{tr}_E^B = (-1)^n \operatorname{tr}_B^E,$$

where n is the dimension of the fiber F. In particular, this implies $\mathrm{tr}_B^E = 0$ if $\dim F$ is odd and we only consider it on cohomology with real coefficients.

2.2 Higher twisted torsion invariants

We are ready to give the definition of a twisted higher torsion invariant. Most of the axioms were proposed by Igusa in [8, Section 4].

Definition 2.1 A higher twisted torsion invariant in degree $2k$ with $k \in \mathbb{N}$ is a rule τ_k, which assigns to any bundle $F \hookrightarrow E \to B$ with closed fiber F and local coefficient system $\mathcal{F}$ on E a cohomology class $\tau_k(E, \mathcal{F}) \in H^{2k}(B; \mathbb{R})$ subject to the axioms beneath. We will drop the degree out of the notation most of the time and just write τ.

Remark 2.2 We consider higher twisted torsion invariants as real cohomology classes (rather than rational ones) since our main example is Igusa–Klein torsion which can only be defined with real coefficients.

Axiom 1 (naturality) τ_k is a characteristic class in degree $2k$. That means for a map $f\colon B' \to B$ and a bundle $F \hookrightarrow E \to B$ with local coefficient system $\mathcal{F}$ on E we have

$$\tau_k(f^*(E), f^*\mathcal{F}) = f^*\tau(E, \mathcal{F}) \in H^{2k}(B'; \mathbb{R}),$$

where f^* denotes the pull-back along f.

Remark 2.3 The naturality axiom immediately implies triviality on trivial bundles $\tau_k(B \times F, \mathcal{F}) = 0$, if $\mathcal{F} = \mathbf{1}$ is the constant local system. Furthermore, if B is simply connected, a local system $\mathcal{F}$ on $B \times F$ will pull back from a local system $\mathcal{F}_F$ on F under the projection $B \times F \to F$. So if we view F as a trivial bundle over a point, naturality gives that $\tau(B \times F, \mathcal{F}) = 0$ for any local system $\mathcal{F}$ if B is simply connected.

If B is a space with finite fundamental group and $B \times F \to B$ is a trivial bundle with local system $\mathcal{F}$, we can look at the pull-back $\widetilde{B} \times F \to \widetilde{B}$ of $B \times F$ to the universal covering space $\pi\colon \widetilde{B} \to B$. By the previous paragraph we know that the twisted torsion of $\widetilde{B} \times F$ is trivial with respect to any finite local system and since the π is a finite covering the map $\pi^*\colon H^*(B; \mathbb{R}) \to H^*(B; \mathbb{R})$ is a monomorphism. By naturality we see that the torsion of a trivial bundle over a base with finite fundamental group is 0 with respect to any local system.

Let E_1 and E_2 be bundles over B with local coefficient systems $\mathcal{F}_1$ and $\mathcal{F}_2$, such that there is an isomorphism $\phi\colon \partial^v E_1 \to \partial^v E_2 \neq \varnothing$ and such that we have, for the restrictions of the local systems,

$$(\mathcal{F}_1)_{|\partial^v E_1} \cong \phi^*(\mathcal{F}_2)_{|\partial^v E}.$$

Then we can glue them together to a local coefficient system $\mathcal{F} := \mathcal{F}_1 \cup_\phi \mathcal{F}_2$ on $E_1 \cup_\phi E_2$.

Axiom 2 (geometric additivity) In the setting from above we have, for any twisted torsion invariant τ,

$$\tau(E_1 \cup_\phi E_2, \mathcal{F}) = \tfrac{1}{2}\big(\tau(DE_1, \mathcal{F}_1^l \cup_{\mathrm{id}} \mathcal{F}_1^r) + \tau(DE_2, \mathcal{F}_2^l \cup_{\mathrm{id}} \mathcal{F}_2^r)\big),$$

where DE_i denotes the fiberwise double $E_i^l \cup_{\mathrm{id}} E_i^r$ with a left copy E_i^l and a right copy E_i^r glued together along their isomorphic boundaries and the induced local coefficient system $\mathcal{F}_i^l \cup_{\mathrm{id}} \mathcal{F}_i^r$.

Now suppose again that $p\colon E \to B$ is a bundle with closed fiber F and local coefficient system $\mathcal{F}$ on E. Let $q\colon D \to E$ be an S^n–bundle which is isomorphic to the sphere bundle of a vector bundle. We get the local coefficient system $q^*\mathcal{F}$ on D by pulling back $\mathcal{F}$ along q.

Axiom 3 (geometric transfer) In the situation above, for a twisted torsion invariant τ, we have the following relation between the torsion class $\tau_B(D, q^*\mathcal{F}) \in H^{2k}(B;\mathbb{R})$ of D as a bundle over B and the torsion class $\tau_E(D, q^*\mathcal{F}) \in H^{2k}(E;\mathbb{R})$ of D as a bundle over E:

$$\tau_B(D, q^*\mathcal{F}) = \chi(S^n)\tau_B(E, \mathcal{F}) + \mathrm{tr}_B^E(\tau_E(D, q^*\mathcal{F})),$$

where χ denotes the Euler class, $\mathrm{tr}_B^E\colon H^{2k}(E;\mathbb{R}) \to H^{2k}(B;\mathbb{R})$ the transfer, and $\tau_E(D, q^*\mathcal{F})$ the twisted torsion class of D over E.

Remark 2.4 We have $\chi(S^n) = 2$ or 0 depending on whether n is even or odd.

Remark 2.5 If we take a twisted torsion class τ_{2k} with $k = 2l$ even, we will get a nontwisted torsion class in the sense of Igusa [7],

$$\tau_{\mathrm{nontw}}(E) := \tau(E, \mathbf{1}) \in H^{4k}(B;\mathbb{R}),$$

where $E \to B$ is a bundle and $\mathbf{1}$ the constant local system on E. We will denote this nontwisted torsion invariant simply by $\tau(E)$ without any local system in the argument.

Since according to Igusa's definition there are no higher torsion invariants in degree $4l + 2 = 2k$, we also need the following axiom:

Axiom 4 (triviality) For a twisted torsion invariant in degree $4l + 2$, we have, for every bundle $E \to B$ and the constant local system $\mathbf{1}$ on E,

$$\tau(E, \mathbf{1}) = 0 \in H^{4l+2}(B;\mathbb{R}).$$

These axioms so far were only modifications of the axioms for nontwisted torsion invariants. We also need some axioms concerning the local system $\mathcal{F}$ on E:

Axiom 5 (additivity for coefficients) If $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ for local systems $\mathcal{F}_i$ on E, with $E \to B$ a bundle, then we have, for every twisted torsion invariant τ,

$$\tau(E, \mathcal{F}) = \sum_i \tau(E, \mathcal{F}_i).$$

Axiom 6 (transfer/induction for coefficients) If $\widetilde{E} \to B$ and $E \to B$ are bundles and $\pi\colon \widetilde{E} \to E$ is a finite fiberwise covering, then we have, for every local system $\mathcal{F}$ on $\widetilde{E}$,

$$\tau(\widetilde{E}, \mathcal{F}) = \tau(E, \pi_* \mathcal{F}),$$

where π_* denotes the push-down operator for local systems.

Remark 2.6 Igusa [8, Section 4.7] proposed this axiom originally in the following form, which corresponds to our formulation:

If G is a group that acts freely and fiberwise on $E \to B$, H is a subgroup of G, and V is a unitary representation of H, then the torsions of the orbit bundles E/G, $E/H \to B$ are related by

$$\tau(E/G, \operatorname{Ind}_H^G V) = \tau(E/H, V).$$

Lastly we need a continuity axiom. It roughly states that if we fix a bundle $E \to B$ then the values of a twisted torsion invariant on E depend continuously on the different local systems $\mathcal{F}$ we might choose. More explicitly we can look at the universal linear S^1–bundle $S^\infty \to \mathbb{CP}^\infty$. If we identify the quotient $\mathbb{Q}/\mathbb{Z}$ with the roots of unity in $\mathbb{C}$ we get a local system $\mathcal{F}_\zeta$ on $S^\infty/(\mathbb{Z}/n)$ for every $\zeta \in \mathbb{Q}/\mathbb{Z}$ of degree n. We can use this and a fixed torsion invariant τ to define a map

$$f_\tau\colon \mathbb{Q}/\mathbb{Z} \to H^{2k}(\mathbb{CP}^\infty, \mathbb{R}) \cong \mathbb{R}$$

given by $f_\tau(\zeta) := \tau(S^\infty/(\mathbb{Z}/n), \mathcal{F}_\zeta)$. Details will provided in Section 5 where we need to use the following axiom:

Axiom 7 (continuity) The map $f_\tau\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ is continuous.

3 Statement of main theorem

3.1 Examples of higher twisted torsion invariants

Our main example of higher twisted torsion is the higher Franz–Reidemeister torsion or Igusa–Klein torsion

$$\tau_k^{\mathrm{IK}}(E, \partial_0 E, \mathcal{F}) \in H^{2k}(B; \mathbb{R}),$$

which is defined for any unipotent bundle pair $(F, \partial_0 F) \to (E, \partial_0 E) \to B$ with $\partial_0 E \subseteq \partial^v E$ and local system $\mathcal{F}$ on E; for details, see [5].

Igusa proved the following result:

Theorem 3.1 [7, Theorem 9.4; 5, Theorem 2.4.7 and Theorem 2.7.1] *The Igusa–Klein torsion invariants are higher twisted torsion invariants for bundles with closed fibers.*

Besides this torsion, we also have the Miller–Morita–Mumford classes in degree $4l$ with $l \in \mathbb{N}$

$$M^{2l}(E) := \mathrm{tr}_B^E((2l!)\,\mathrm{ch}_{4l}(T^v E)),$$

where $\mathrm{ch}_{4l}(T^v E) = \frac{1}{2}\,\mathrm{ch}_{4l}(T^v E \otimes \mathbb{C})$. We will consider this to be a real characteristic class. Igusa also showed that this class is a higher nontwisted torsion invariant [7, Proposition 9.1]. To make it a higher twisted torsion invariant we simply define, for an $m-$dimensional local system $\mathcal{F}$ on E,

$$M^{2l}(E, \mathcal{F}) := m M^{2l}(E) \in H^{4l}(B; \mathbb{R}).$$

Furthermore we set

$$M^{2l+1}(E, \mathcal{F}) := 0,$$

since there is no nontwisted torsion in degree $2k = 2(2l+1)$, and the twisted MMM torsion always induces nontrivial nontwisted torsion. Knowing that the MMM class is a nontwisted torsion invariant (and therefore fulfills the first three axioms) it is now easy to see:

Theorem 3.2 *The twisted MMM class is a higher twisted torsion invariant.*

We also know that for any bundle $F \to E \to B$ with closed l–dimensional fiber F, twice the transfer map tr_B^E is rationally trivial, if l is odd. Therefore we get:

Proposition 3.3 $M^k(E, \mathcal{F}) = 0$ *for closed odd-dimensional fiber* F.

3.2 The space of twisted torsion invariants

We are moving on to the space of higher twisted torsion invariants in degree $2k$. We begin with an elementary observation:

Lemma 3.4 *For each* k, *the set of all twisted torsion invariants of degree* $2k$ *is a vector space over* $\mathbb{R}$.

Of course, the same statement holds for the set of nontwisted higher torsion invariants. Igusa proved for the space of nontwisted higher torsion invariants:

Theorem 3.5 [7, Theorem 4.4] *For any l the space of higher nontwisted torsion invariants in degree $4l$ is two-dimensional and spanned by the nontwisted MMM class M^{4l} and the nontwisted Igusa–Klein torsion τ_{2l}^{IK}. In other words, for any nontwisted torsion invariant τ there exist unique $a, b \in \mathbb{R}$ such that*

$$\tau = a\tau^{\mathrm{IK}} + bM.$$

Now, let $\mathrm{Top}_{\mathrm{fin}}$ be the full subcategory of Top of topological spaces with finite fundamental group and $\mathrm{Top}_{\mathrm{sim}}$ the full subcategory of simple topological spaces. A space F is called simple if the fundamental group $\pi_1 F$ acts trivially on the higher homotopy groups $\pi_* F$. If we restrict a twisted torsion invariant to bundles with fibers in $\mathrm{Top}_{\mathrm{sim}}$ and base in $\mathrm{Top}_{\mathrm{fin}}$ we get the main theorem:

Theorem 3.6 (main theorem) *In the setting above, the space of higher twisted torsion invariants in degree $2k$ on bundles with simple fibers and base having finite fundamental group is two-dimensional and spanned by the twisted MMM class and the twisted Igusa–Klein torsion, if k is even, and one-dimensional and spanned by the Igusa–Klein torsion, if k is odd. In other words, for any twisted torsion invariant τ of degree $4l$, there exist unique $a, b \in \mathbb{R}$ such that*

$$\tau = a\tau^{\mathrm{IK}} + bM,$$

and for every twisted torsion invariant τ of odd degree $4l + 2$ there exists a unique $a \in \mathbb{R}$ such that

$$\tau = a\tau^{\mathrm{IK}}.$$

Remark 3.7 If k is even, we get a nontwisted torsion invariant from the twisted one by always inserting the trivial representation. Then the numbers a and b used in both theorems above will be the same.

The proof of the main theorem is developed in Sections 4 to 7. In the very technical Section 4 we will introduce relative torsion of bundles with vertical boundary and we will turn the geometric additivity axiom into two eye-pleasing formulas that will allow us to dissect the fiber F into easier pieces meeting along a common vertical boundary. Section 5 is devoted to investigating the higher twisted torsion of linear S^1–bundles. Concretely, we show that the continuity, geometric additivity, and geometric transfer axioms together imply that the space of twisted torsion invariants restricted to only the universal bundle $S^1 \to S^\infty \to \mathbb{CP}^\infty$ is one-dimensional. This together with the results in the untwisted case implies that the difference torsion $\tau^\delta := \tau - a\tau^{\mathrm{IK}} - bM$ is trivial on all linear disc and sphere bundles. The goal of Section 6 is to use this to show that τ^δ is a fiber homotopy invariant which will follow from τ^δ being trivial on any lens space bundle. The proof of this last assertion relies on the twisted Hatcher example we defined in [4]. Armed with the fiber homotopy invariance we then proceed in Section 7 to use homotopical tools to replace any fiber bundle $E \to B$ with another one that has homologically trivial fibers and the same difference torsion as E and prove triviality on those.

3.3 The scalars a and b

Before we get to the proof of the main theorem let us assume for now that it is true. This section aims to explain how given a torsion invariant one can calculate the scalars of the equation $\tau = a\tau^{\mathrm{IK}} + bM$. We need to distinguish between τ having degree $2k = 4l$ or $2k = 4l + 2$.

3.3.1 In degree $2k = 4l$ First we first look at a twisted torsion invariant in degree $2k = 4l$. In this case the scalars must be the same as the ones we get for the corresponding nontwisted torsion. To determine them we follow Igusa's approach [7, Section 4.2] and look at the universal $S^1 \cong U(1) \cong \mathrm{SO}(2)$–bundle λ over $\mathbb{CP}^\infty \cong \mathrm{BU}(1)$. Furthermore, let $S^1(\lambda)$ be the associated circle bundle with λ and $S^2(\lambda)$ the S^2–bundle associated with $S^1(\lambda)$ (by fiberwise suspension of $S^1(\lambda)$). Since the cohomology ring of $\mathbb{CP}^\infty$ is a polynomial algebra generated by $c_1(\lambda)$, the cohomology group $H^{2k}(\mathbb{CP}^\infty;\mathbb{R}) \cong \mathbb{R}$ is generated by $\mathrm{ch}_{2k}(\lambda) = c_1^k/k!$.

From this, we immediately get scalars $s_1, s_2 \in \mathbb{R}$ for any twisted torsion invariant in degree $2k = 4l$ with

$$\tau(S^1(\lambda)) = s_1 \,\mathrm{ch}_{2k}(\lambda) \quad \text{and} \quad \tau(S^2(\lambda)) = s_2 \,\mathrm{ch}_{2k}(\lambda).$$

Furthermore we have the following two propositions:

Proposition 3.8 [5, Chapter 2.7] *We get*

$$\tau_{2l}^{\mathrm{IK}}(S^n(\lambda)) = (-1)^{l+n}\zeta(2l+1)\,\mathrm{ch}_{4l}(\lambda).$$

Proposition 3.9 [7, Proposition 9.2] $M_k(S^2(\lambda)) = 2k!\,\mathrm{ch}_{2k}(\lambda)!.$

Now we are taking into account that the MMM class is trivial on odd-dimensional fibers, and therefore we get that $\tau(S^1(\lambda)) = a\tau^{\mathrm{IK}}(S^1(\lambda))$. From this we get

$$a = s_1/((-1)^{1+l}\zeta(2l+1)).$$

Looking at the $S^2(\lambda)$ case, we have

$$s_2 = a(-1)^l\zeta(2l+1) + b2k! = -s_1 + b2k!$$

and therefore

$$b = \frac{s_1 + s_2}{2k!}.$$

3.3.2 In degree $2k = 4l + 2$ Now let the degree be $2k = 4l + 2$. In this case τ does not define a nontrivial nontwisted torsion invariant. On the other hand we also just need to determine a since the MMM class vanishes in this degree.

Furthermore, we cannot use the standard universal bundle for linear S^1–bundles $ES^1 \to BS^1$, since ES^1 is contractible and therefore will not admit a nonconstant local system. But we can replace it by a very similar construction. First, recall that ES^1 can be constructed as follows: Take $S^1 \subseteq \mathbb{C}$ and $S^{2N-1} \subseteq \mathbb{C}^N$. Then we have a fibration $S^1 \hookrightarrow S^{2N-1} \to \mathbb{CP}^{N-1}$. Taking the direct limit of this will yield an S^1–bundle with total space S^∞, which is contractible and therefore the universal S^1–principal bundle $S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$.

We can look at a $\mathbb{Z}/n$–action on S^1 given by multiplication with the primitive n^{th} root of unity $e^{2\pi i/n}$. This will give rise to a fiberwise $\mathbb{Z}/n$–action on the bundle $S^1 \hookrightarrow S^{2N-1} \to \mathbb{CP}^{N-1}$. The action of $\mathbb{Z}/n$ on S^{2N-1} is by construction the same as the one being taken to get a lens space L_n^{2N-1} as quotient out of S^{2N-1}. Therefore taking the fiberwise quotient under the given $\mathbb{Z}/n$–action gives a bundle (since $S^1/n \cong S^1$)

$$S^1 \hookrightarrow L_n^{2N-1} \to \mathbb{CP}^{N-1},$$

which yields in the limit to

$$S^1 \hookrightarrow L_n^\infty \to \mathbb{CP}^\infty.$$

We will refer to this bundle as $S^1(\lambda)/n$, since it has the S^1–bundle associated with the universal line bundle as its n–fold covering. The n–fold Galois covering $S^{2N-1} \to L_n^{2N-1}$ gives a bundle $S^{2N-1} \times \mathbb{C} \to L_n^{2N-1}$ where a fixed generator of $\mathbb{Z}/n$ acts on $\mathbb{C}$ by multiplication with an n^{th} root of unity ζ_n. Using this we can make the following important definition.

Definition 3.10 In the setting above, the nonconstant local system $\mathcal{F}_{\zeta_n}$ on L_n^{2N-1} is defined to be the nonconstant local system of the sections of the bundle $S^{2N-1} \times \mathbb{C} \to L_n^{2N-1}$. The nonconstant local system $\mathcal{F}_{\zeta_n}$ on L_n^∞ is defined as the direct limit of these local systems on L_n^{2N-1}.

Again, we can use the fact that the cohomology of $\mathbb{CP}^\infty$ is a group ring and that $H^{2k}(\mathbb{CP}^\infty; \mathbb{R})$ will be spanned by $\operatorname{ch}_{2k}(\lambda)$ and therefore

$$\tau(S^1(\lambda)/n, \mathcal{F}_{\zeta_n}) = s_1 \operatorname{ch}_{2k}(\lambda).$$

Furthermore we have again the following result from Igusa [6]:

Proposition 3.11 *For the Igusa–Klein torsion we have*

$$\tau^{\mathrm{IK}}(S^1(\lambda)/n, \mathcal{F}_{\zeta_n}) = -n^k L_{k+1}(\zeta_n)\,\mathrm{ch}_{2k}(\lambda),$$

where L_{k+1} denotes the polylogarithm

$$L_{k+1}(\zeta) := \mathrm{Re}\biggl(\frac{1}{i^k}\sum_{l=1}^{\infty}\frac{\zeta_n^l}{n^{k+1}}\biggr).$$

Putting this together we get

$$a = -s_1/(n^k L_{k+1}(\zeta)).$$

We will prove later that a is independent of the choice of the local system.

4 Extension of higher twisted torsion

Now we present some easy consequences of the geometric additivity and transfer axioms. More precisely, we introduce twisted torsion and calculations thereof for bundles with vertical boundary and bundle pairs. This is completely parallel to the corresponding Section 5 in [7] and all the proofs can be translated word-by-word and will be skipped. While the material is very technical the formulas to keep in mind are Lemma 4.2 and Example 4.7.

First we define the higher twisted torsion on bundles with vertical boundary:

Definition 4.1 (higher twisted torsion for bundles with vertical boundary) Suppose $F \hookrightarrow E \to B$ is a bundle with vertical boundary $\partial^v E \to B$ and local coefficient system $\mathcal{F}$ on E and τ is a higher twisted torsion invariant. Then the twisted torsion of the bundle with boundary is defined by

$$\tau(E,\mathcal{F}) := \tfrac{1}{2}\bigl(\tau(DE, \mathcal{F}^l \cup_{\mathrm{id}} \mathcal{F}^r) + \tau(\partial^v E, \mathcal{F}_{|\partial^v E})\bigr),$$

where $DE := E^l \cup_{\mathrm{id}} E^r$ denotes the fiberwise double as before.

Building onto two lemmas one can prove the following formula (compare to [7], Proposition 5.4):

Lemma 4.2 (additivity in the boundary case) *Suppose E is a bundle over B and (E_1, ∂_0) and (E_2, ∂_0) are bundle pairs such that $E_1, E_2 \subseteq E$, $\partial_0 E_1 = \partial_0 E_2 = E_1 \cap E_2$ and $E = E_1 \cup E_2$. Let $\mathcal{F}$ be a local system on E and $\mathcal{F}_1 := \mathcal{F}_{|E_1}$ and $\mathcal{F}_2 := \mathcal{F}_{|E_2}$. Then*

$$\tau(E_1 \cup E_2, \mathcal{F}) = \tau(E_1, \mathcal{F}_1) + \tau(E_2, \mathcal{F}_2) - \tau(E_1 \cap E_2, \mathcal{F}_{|E_1 \cap E_2}).$$

Furthermore, we get the transfer formula (compare [7], Proposition 5.5):

Lemma 4.3 (transfer in the boundary case) *Let $X \to D \xrightarrow{q} E$ be an oriented disc or sphere bundle over a bundle $F \to E \to B$ with local coefficient system $\mathcal{F}$ on E. As for the transfer axiom this pulls up to a local coefficient system $q^*\mathcal{F}$ on D and we get*

$$\tau_B(D, q^*\mathcal{F}) = \chi(X)\tau(E, \mathcal{F}) + \mathrm{tr}_B^E(\tau_E(D), q^*\mathcal{F}).$$

Now we turn to bundle pairs.

Definition 4.4 A pair of bundles $(F, \partial_0) \to (E, \partial_0) \to B$ is called a bundle pair if the vertical boundary $\partial^v E$ is the union $\partial^v E = \partial_0 E \cup \partial_1 E$ of two subbundles which meet along their common boundary $\partial_0 E \cap \partial_1 E = \partial^v \partial_0 E = \partial^v \partial_1 E$.

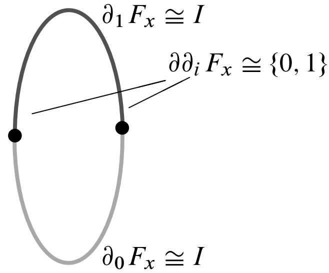


Figure 1: The fiber over x of a bundle pair with fiber $F \cong D^2$

Definition 4.5 (relative torsion) For a bundle pair $(F, \partial_0) \to (E, \partial_0) \to B$ with local coefficient system $\mathcal{F}$ on E we define the relative torsion to be

$$\tau(E, \partial_0, \mathcal{F}) := \tau(E, \mathcal{F}) - \tau(\partial_0 E, \mathcal{F}_{|\partial_0 E}).$$

We get the following proposition (compare [7], Proposition 5.7):

Proposition 4.6 (relative additivity) *Suppose $E \to B$ is a smooth bundle with local coefficient system $\mathcal{F}$, which can be written as the union of two subbundles $E = E_1 \cup E_2$, which meet along a subbundle of their respective vertical boundaries $E_1 \cap E_2 = \partial_0 E_2 \subseteq \partial^v E_1$. Let $\partial^v E_1 = \partial_0 E_1 \cup \partial_1 E_1$ be a decomposition of $\partial^v E_1$, so that $\partial_0 E_2 \subseteq \partial_1 E_1$ and $(E_i, \partial_0) \to B$ for $i = 1, 2$ are smooth bundle pairs. Then $(E, \partial_0 E) \to B$ is a smooth bundle pair and*

$$\tau(E_1 \cup E_2, \partial_0 E_1, \mathcal{F}) = \tau(E_1, \partial_0, \mathcal{F}_{|E_1}) + \tau(E_2, \partial_0, \mathcal{F}_{|E_2}).$$

Example 4.7 The example to keep in mind here are h–cobordism bundles. That is bundle pairs $B \times M \subset E \to B$ such that the fibers are h–cobordisms of M with 0–end the fiber of the trivial bundle. Often we have two h–cobordism bundles $B\times M \subset E \to B$ and $B\times M' \subset E' \to B$ and an inclusion into the 1–end $B\times M' \hookrightarrow E$. This is exactly the situation in which we want to apply the relative additivity, and we get, with an appropriate local system $\mathcal{F}$,

$$\tau(E\cup E', B\times M, \mathcal{F}) = \tau(E, B\times M, \mathcal{F}) + \tau(E', B\times M', \mathcal{F}),$$

where we regard $E\cup E'$ as the h–cobordism bundle obtained by "gluing E'" on top of E.

To state the transfer axiom in the relative case, we need the relative transfer:

$$\mathrm{tr}_B^{(E,\partial_0)}\colon H^*(E;\mathbb{R}) \to H^*(B;\mathbb{R}),$$

which is also introduced in [7, Section 5].

Proposition 4.8 (relative transfer; compare [7, Proposition 5.9]) *Let $(F,\partial_0) \to (E,\partial_0) \to B$ and $(X,\partial_0) \to (D,\partial_0) \xrightarrow{q} E$ be bundle pairs with local system $\mathcal{F}$ on E, so that the second bundle is an oriented linear S^n or D^n bundle with $\partial_0 X = S^{n-1}, D^{n-1}$ or $\varnothing$. Then*

$$\tau_B(D, \partial_0 D \cup q^{-1}\partial_0 E, q^*\mathcal{F}) = \chi(X,\partial_0)\tau(E,\partial_0,\mathcal{F}) + \mathrm{tr}_B^{E,\partial_0}(\tau_E(D,\partial_0,q^*\mathcal{F})).$$

Remark 4.9 Note that we do not have a result analogous to the product formula [7, Corollary 5.10]. However, we still have the following corollary.

Corollary 4.10 (stability theorem) *If $(E,\partial_0)\to B$ is a smooth bundle pair with local system $\mathcal{F}$ on E, then so is $(E\times D^n, \partial_0 E\times D^n)$ and the relative torsion is the same:*

$$\tau(E\times D^n, \partial_0 E\times D^n, \mathcal{F}\times \mathbf{1}) = \tau(E,\partial_0,\mathcal{F}),$$

where $\mathcal{F}\times\mathbf{1}$ is the local system constant on D^n.

5 Higher twisted torsion of sphere bundles

The goal of this section is to calculate the higher twisted torsion of linear S^1–bundles only using the axioms. Before we can do this we will discuss why we can always restrict our calculations to finite cyclic local systems on bundles with simply connected base.

5.1 Reduction of the representation

In the following we will simplify the local systems:

Proposition 5.1 *To prove the main theorem, Theorem 3.6, it is enough to only consider bundles with simply connected base instead of base having finite fundamental group. We can also restrict to only considering local systems (on the fiber) that induce n–fold holonomy covers with transition group $\mathbb{Z}/n$ instead of just finite local systems.*

Remark 5.2 Let $F \hookrightarrow E \to B$ be a fiber bundle and $\mathcal{F}$ a finite local system on E. This corresponds to its holonomy cover $\widetilde{E} \to E$ with finite transition group G and representation $\rho\colon G \to U(m)$. On the other hand every finite covering $\widetilde{E} \xrightarrow{G} E$ with representation $\rho\colon G \to U(m)$ gives us a local system $\mathcal{F}_\rho$ as the sections of the bundle $\widetilde{E} \times_G \mathbb{C}^m \to E$ where G acts on $\mathbb{C}^m$ via ρ. This construction is a one-to-one correspondence. Now let $H \subseteq G$ be a subgroup. From the covering $\widetilde{E} \xrightarrow{G} E$ we get coverings $\pi_H\colon \widetilde{E}/H \to E$ and $\widetilde{E} \xrightarrow{H} \widetilde{E}/H$. Suppose we have a representation $\rho_H\colon H \to U(m)$ and thereby get a local system $\mathcal{F}_{\rho_H}$ on $\widetilde{E}/H$. Then we can either form the induced representation $\operatorname{Ind}_H^G(\rho_H)\colon G \to U(m)$ and its corresponding local system $\mathcal{F}_{\operatorname{Ind}_H^G(\rho_H)}$ on E or the local system $\pi_*\mathcal{F}_{\rho_H}$ on E given by the push-down of the local system $\mathcal{F}_\rho$. It follows from an easy calculation that

$$\mathcal{F}_{\operatorname{Ind}_H^G(\rho_H)} = \pi_*\mathcal{F}_{\rho_H}.$$

Proof of the proposition Let $\mathcal{F}$ be again a local system on E corresponding to a finite covering $\widetilde{E} \xrightarrow{G} E$ with representation $\rho\colon G \to U(m)$. Let $\mathcal{H} = \{H_i\}$ be the finite set of cyclic subgroups H_i of G. By Artin's induction theorem, we can write the character of ρ rationally as linear combination of characters of one-dimensional representations. Since we are working over $\mathbb{C}$, we therefore can write ρ rationally as a linear combination of one-dimensional representations $\lambda_i\colon H_i \to U(1)$ and inductions thereof. Concretely we have

$$n\rho \cong \bigoplus_i n_i \operatorname{Ind}_{H_i}^G(\lambda_i) \quad \text{with } n, n_i \in \mathbb{Z}.$$

Let τ be a twisted torsion invariant and $\pi_i\colon \widetilde{E}/H \to E$ be a covering. Then we have, using the transfer of coefficient axiom and the calculation above,

$$\begin{aligned} n\tau(E,\mathcal{F}) &= \sum_i n_i \tau(E, \mathcal{F}_{\operatorname{Ind}_{H_i}^G(\lambda_i)}) = \sum_i n_i \tau(E, \pi_*\mathcal{F}_{\lambda_i}) \\ &= \sum_i n_i \tau(\widetilde{E}/H_i, \mathcal{F}_{\lambda_i}) \in H^{2k}(B;\mathbb{R}). \end{aligned}$$

Therefore it suffices for the rest of the paper to work with local systems with n–fold holonomy covers with cyclic transition group $\mathbb{Z}/n$.

Now let $F \to E \to B$ be a bundle with local system $\mathcal{F}$ on E and the base B having a finite fundamental group. We have the universal covering $q\colon \widetilde{B} \to B$ and pulling back

E along q gives a bundle $\widetilde{E} := q^*E \to \widetilde{B}$ with local system $\widetilde{\mathcal{F}} := q^*\mathcal{F}$. Naturality implies

$$\tau(\widetilde{E}, \widetilde{\mathcal{F}}) = q^*\tau(E, \mathcal{F}) \in H^{2k}(\widetilde{B}; \mathbb{R}).$$

Furthermore we know that $q^*\colon H^{2k}(B;\mathbb{R}) \to H^{2k}(\widetilde{B};\mathbb{R})$ is injective. By this construction it suffices to prove the main theorem only on bundles with simply connected base. □

5.2 Twisted torsion for S^1–bundles

We want to show the following theorem:

Theorem 5.3 *For every S^1–bundle $S^1 \hookrightarrow E \to B$ with B simply connected and local system $\mathcal{F}$ on E with $\mathbb{Z}/n$–fold holonomy cover $\widetilde{E}_n \to E$ every twisted torsion invariant τ is given by*

$$\tau(E, \mathcal{F}) = a\tau^{\mathrm{IK}}(E, \mathcal{F}),$$

where a is the scalar defined earlier.

We will follow an approach Igusa introduced in [8, Section 4]. Since $B\mathrm{Diff}(S^1) \simeq B\mathrm{SO}(2)$ it suffices to look at linear S^1–bundles. These pull back from the universal S^1–bundle $S^1(\lambda)$ given by $S^1 \hookrightarrow S^\infty \to \mathbb{CP}^\infty$.

Let $E \to B$ be an S^1–bundle with local system $\mathcal{F}$ on E inducing a finite holonomy covering. At first we look at the following n–fold holonomy Galois covering:

$$\begin{array}{ccc}
S^1 & \xrightarrow{\ n\ } & S^1 \\
\downarrow & & \downarrow \\
\widetilde{E}_n & \xrightarrow{\ n\ } & E \\
\downarrow & & \downarrow \\
B & = & B
\end{array}$$

Now $\widetilde{E}_n$ is again a linear S^1–bundle with fiberwise $\mathbb{Z}/n$–action. This will pull back equivariantly from the universal S^1–bundle $S^1(\lambda)$ given by $S^\infty \to \mathbb{CP}^\infty$, which also admits an $\mathbb{Z}/n$–action. Therefore E will pull back from the quotient $S^1(\lambda)/(\mathbb{Z}/n)$. Also the local system $\mathcal{F}$ on E will pull back from the local system $\mathcal{F}_{\zeta_n}$ on S^∞ for some n^{th} root of unity ζ_n. We defined this earlier (Definition 3.10) to be given by the bundle $S^1(\lambda) \times \mathbb{C} \to S^1(\lambda)/n$ where the action on $\mathbb{C}$ is given by multiplication by ζ_n. So because of naturality it is enough to show:

Theorem 5.4 *For all n and ζ_n,*

$$\tau(S^1(\lambda)/n, \mathcal{F}_{\zeta_n}) = a\tau^{\mathrm{IK}}(S^1(\lambda)/n, \mathcal{F}_{\zeta_n}) \in H^{2k}(\mathbb{CP}^\infty; \mathbb{R}).$$

First we prove two important lemmas already introduced in [8] (Lemmas 4.11 and 4.12). These will isolate certain properties of $\tau(S^1(\lambda)/n, \mathcal{F}_\zeta)$ thought of as a function of ζ.

Lemma 5.5 *Suppose we have a bundle $E \to B$ and a free fiberwise nm–action on E, where $n, m \in \mathbb{N}$. Then we have, for any twisted torsion invariant and n^{th} root of unity ζ_n,*

$$\tau(E/n, \mathcal{F}_{\zeta_n^m}) = \sum_{\xi^m=1} \tau(E/(nm), \mathcal{F}_{\xi\zeta_n}),$$

where the local systems $\mathcal{F}_{\zeta_n}$ on E/n are given by the construction above.

Proof Denote the projection by $\pi\colon E/n \to E/(nm)$. We get

$$\pi_*\mathcal{F}_{\zeta_n^m} = \bigoplus_{\xi^m=1} \mathcal{F}_{\xi\zeta_n}.$$

Now we can use the transfer of coefficients and the additivity axiom to get

$$\tau(E/n, \mathcal{F}_{\zeta_n^m}) = \tau(E/(nm), \pi_*\mathcal{F}_{\zeta_n^m}) = \sum_{\xi^m=1} \tau(E/(nm), \mathcal{F}_{\xi\zeta_n}).$$ □

Lemma 5.6 *For every linear S^1–bundle $E \to B$ and any n^{th} root of unity ζ_n, we have, for every twisted torsion class in degree $2k$,*

$$\tau(E/(nm), \mathcal{F}_{\zeta_n}) = m^k \tau(E/n, \mathcal{F}_{\zeta_n}).$$

Proof Again we look at the universal circle bundle $S^1(\lambda)$, and by the naturality axiom it is enough to show the lemma only on $E = S^1(\lambda)$. We have that $S^1(\lambda)/m$ is again a circle bundle over $\mathbb{CP}^\infty$ and therefore classified by a map

$$f_m\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty.$$

In degree 2 we can see (by looking at circle bundles over spheres S^2) that this map is multiplication by m on H^2. Then it follows that f_m^* is multiplication by m^k on $H^{2k}(\mathbb{CP}^\infty; \mathbb{R})$. The classifying maps for $S^1(\lambda)/nm$ and $S^1(\lambda)/n$ are related by

$$f_{mn} = f_n \circ f_m.$$

The lemma now follows from naturality. □

Now let $f\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$ a function. It is said to satisfy the Kubert identity if

$$f(x) = m^{s-1} \sum_{k=0}^{m-1} f\left(\frac{x+k}{m}\right)$$

for fixed s and all integers m and all $x \in \mathbb{Q}/\mathbb{Z}$. Identifying $\mathbb{Q}/\mathbb{Z}$ with the roots of unity in $\mathbb{C}$ (by $x \mapsto e^{2\pi i x}$), we can write $f(x) = L(e^{2\pi i x})$ and the Kubert identity becomes

$$L(\zeta^m) = m^{s-1} \sum_{\xi^m=1} L(\zeta\xi).$$

The following result can be proved by considering Fourier coefficients:

Theorem 5.7 (Milnor 1983 [10, Section 3, Theorem 1]) *Let $\mathbb{Q}/\mathbb{Z}$ have the quotient topology. The space of continuous functions $f\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$ satisfying the Kubert identity is two-dimensional and splits into two one-dimensional spaces, the first of which contains all the functions with $L(\zeta) = L(\overline{\zeta})$ and the second, the ones with $L(\zeta) = -L(\overline{\zeta})$.*

Remark 5.8 Milnor states this theorem for continuous functions $\mathbb{R}/\mathbb{Z} \to \mathbb{C}$ rather than $\mathbb{Q}/\mathbb{Z} \to \mathbb{C}$, but since $\mathbb{Q} \subset \mathbb{R}$ is dense this does not impact the statement. Furthermore the original theorem is formulated without identifying $\mathbb{R}/\mathbb{Z}$ with the unit sphere in $\mathbb{C}$. The two one-dimensional subspaces are then formed by the functions satisfying $f(x) = -f(1-x)$ and $f(x) = f(1-x)$, which correspond exactly to the equations $L(\zeta) = \pm L(\overline{\zeta})$ on the unit sphere.

Proof of Theorem 5.4 To any higher twisted torsion invariant τ we get, for any n^{th} root of unity, a coefficient $s_1(\tau,\zeta)$ defined by

$$\tau(S^1(\lambda)/n, \mathcal{F}_\zeta) = s_1(\tau,\zeta)\,\mathrm{ch}_{2k}(\lambda) \in H^{2k}(\mathbb{CP}^\infty;\mathbb{R}) \cong \mathbb{R}.$$

Identifying $\mathbb{Q}/\mathbb{Z}$ with the roots of unity, we get a function $f_\tau\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ defined by

$$f_\tau(\zeta) := \frac{1}{n^k} s_1(\tau,\zeta),$$

where $\zeta^n = 1$. This is well defined, since by the previous lemma we have

$$\tau(S^1(\lambda)/(nm), \mathcal{F}_\zeta) = m^k \tau(S^1(\lambda)/n, \mathcal{F}_\zeta),$$

so $f_\tau(\zeta)$ is by construction independent from the choice of n with $\zeta^n = 1$.

Our goal is to show that this satisfies the Kubert identity and then to use Milnor's result to prove our theorem. But for this, f_τ needs to be continuous, a fact which we cannot prove, but must assume. Therefore we need the following last axiom:

Axiom 7 (continuity) For any twisted torsion invariant, the function $f_\tau\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{R}$ constructed above is continuous.

As explained earlier in the paper, this axiom basically states that for a fixed bundle $E \to B$ the twisted torsion depends continuously on the local system $\mathcal{F}$ on E.

Continuation of the proof Now we calculate for $\zeta \in \mathbb{Q}/\mathbb{Z}$ with $\zeta^n = 1$ using the two lemmas from above:

$$\begin{aligned} f_\tau(\zeta^m)\,\mathrm{ch}_{2k}(\lambda) &= \frac{1}{n^k}\,\tau(S^1(\lambda)/nm, \mathcal{F}_{\zeta^m}) \\ &= \frac{1}{n^k}\sum_{\xi^m=1} \tau(S^1(\lambda)/nm, \mathcal{F}_{\xi\zeta}) \\ &= m^k \sum_{\xi^m=1} f_\tau(\xi\zeta)\,\mathrm{ch}_{2k}(\lambda). \end{aligned}$$

So f_τ satisfies the Kubert identity (with $s = k+1$) for any τ.

We note that the change of representation from ζ to $\overline{\zeta}$ represents a change of orientation in the fiber. Therefore, it corresponds to a map $g\colon \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, giving $g_*\colon \pi_1 S^1 \to \pi_1 S^1$ as multiplication by -1. Using that $\pi_1 S^1(\lambda)/n \cong \mathbb{Z}/n$, we get the following commutative diagram relating the exact sequence of the homotopy groups of the fibration $S^1 \hookrightarrow S^1(\lambda)/n \to \mathbb{CP}^\infty$ to itself under g_*:

$$\begin{array}{ccccccc} \pi_2\mathbb{CP}^\infty & \xrightarrow{\cdot n} & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle -1} & & \downarrow{\scriptstyle g_*} & & \\ \pi_2\mathbb{CP}^\infty & \xrightarrow{\cdot n} & \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/n & \longrightarrow & 0 \end{array}$$

From this one can see that $g_*\colon \pi_2\mathbb{CP}^\infty \to \pi_2\mathbb{CP}^\infty$ is multiplication by -1. Since $\mathbb{CP}^\infty$ is simply connected, g_* is also multiplication by -1 in homology of degree 2. Since $\mathbb{CP}^\infty$ is an Eilenberg–Mac Lane space, g^* must be multiplication by -1 on degree-2 cohomology and thus multiplication by $(-1)^k$ on degree-$2k$ cohomology.

This yields

$$f_\tau(\zeta) = (-1)^k f_\tau(\overline{\zeta})$$

for any τ with degree $2k$. So f_τ is in one specific one-dimensional subspace of the space of functions satisfying the Kubert identity for any torsion invariant τ of degree $2k$, and therefore we have, for an arbitrary torsion invariant τ and the Igusa–Klein torsion τ^{IK},

$$f_\tau = a f_{\tau^{\mathrm{IK}}}$$

for a certain $a \in \mathbb{R}$. This translates to

$$\tau(S^1(\lambda)/n, \mathcal{F}_\zeta) = a\tau^{\mathrm{IK}}(S^1(\lambda)/n, \mathcal{F}_\zeta)$$

for any root of unity ζ and proves the theorem. □

Remark 5.9 This also shows that the scalar a that we calculated earlier by choosing an arbitrary local system is well-defined and does not depend on this choice.

Remark 5.10 It is an unproven conjecture by Milnor [10] that any function satisfying the Kubert identity is already continuous. If this conjecture was proven we could drop the continuity axiom.

6 The difference torsion

Given a twisted torsion invariant τ, we can now form the twisted difference torsion

$$\tau^\delta := \tau - a\tau^{\mathrm{IK}} - bM,$$

where the scalars a and b are the ones from Theorem 3.6 (and b is 0 if the torsion has degree $4l+2$). Clearly, τ^δ is a twisted torsion invariant.

Our goal in this section and the next is to show $\tau^\delta(E,\mathcal{F}) = 0$ for every bundle $E \to B$ with every local coefficient system $\mathcal{F}$ on E and base B having finite fundamental group. In this section we will show that τ^δ is a fiber homotopy invariant. Here is a sketch of our approach: Given two fiber bundles $E \to B$ and $E' \to B'$ with appropriate local system $\mathcal{F}$ and fiber homotopy equivalence $g\colon E \to E'$ (which we can without restriction assume to be an embedding) we can view $E' \backslash g(E)$ as a bundle with fibers h–cobordisms (this is not necessarily an h–cobordism bundle) the torsion of which is exactly the difference of the torsions of E and E'. (This is done in the proof of Theorem 6.12.) To show that the difference torsion of bundles with h–cobordisms as fibers is trivial in Lemma 6.11 we embed one end of the bundle in a trivial lens space bundle (we have to use lens spaces instead of discs or spheres to preserve a nontrivial first homotopy group) making it a bundle fiber homotopy equivalent to a trivial lens space bundle. Now we use the fiber homotopy from such a bundle to the trivial lens space bundle to get an h–cobordism bundle of a lens space (done in Lemma 6.10). Finally, in our paper [4] we essentially classified all h–cobordism bundles of a lens space and showed that their Igusa–Klein torsion can be calculated only using the axioms. So their difference torsion is zero.

6.1 Lens spaces

Any cyclic group $\mathbb{Z}/n$ acts on the complex numbers $\mathbb{C}$ by rotation. For the rest of the paper, we will pick a generator $1 \in \mathbb{Z}/n$ and have it act by multiplication with $e^{2\pi i/n}$ on $\mathbb{C}$. Then we get a componentwise action on the odd-dimensional sphere $S^{2N+1} \subset \mathbb{C}^{N+2}$.

Definition 6.1 The odd-dimensional lens space L_n^{2N+1} is defined to be the quotient $S^{2N+1}/(\mathbb{Z}/n)$ by the action defined above.

It is well known that the CW–structure on L_n^{2N+1} has a cell in every dimension and its associated chain complex is given by

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

In particular we see that L_n^{2N+1} is rationally spherical with fundamental group $\mathbb{Z}/n$. Recall that we are interested in twisted torsion invariants and thereby require our manifolds to have nontrivial fundamental group, so the odd-dimensional lens spaces will play the role of finite-dimensional spheres in some sense.

In Section 7 we will also need spaces with nontrivial fundamental group that are rationally contractible to provide a twisted analogue of the infinite-dimensional sphere $S^\infty \simeq *$ or large-dimensional discs D^N. The even-dimensional lens spaces are exactly going to fulfill this condition:

Definition 6.2 The even-dimensional lens space $L_n^{2N} \subset L_n^{2N+1}$ is obtained from the odd-dimensional one by omitting the top cell in the CW decomposition described above.

It follows immediately that L_n^{2N} is rationally acyclic with fundamental group $\mathbb{Z}/n$. We choose a universal covering $\widetilde{L_n^{2N}} \to L_n^{2N}$. This comes equipped with a $(2N-1)$–connected map $\tilde{i}\colon \widetilde{L_n^{2N}} \to S^{2N+1}$.

Lastly, note that there is a chain of inclusions

$$\cdots \subset L_n^{2N-2} \subset L_n^{2N-1} \subset L_n^{2N} \subset L_n^{2N+1} \subset \cdots.$$

6.2 Lens space bundles

Following the outline above, we first want to show that the difference torsion is zero on every linear odd-dimensional lens space bundle $L_n^{2N+1} \hookrightarrow E_n^{2N+1} \to B$ with local coefficient system $\mathcal{F}$ on E_n^{2N+1}. We already know from the base case that the difference torsion is zero on every S^1–bundle. Furthermore, if we take an S^l–bundle with $l > 1$ or disc bundle, we know that the fundamental group of the fiber is trivial and it therefore admits no nonconstant local system. So the twisted difference torsion on these bundles is always given by the nontwisted difference torsion. But the nontwisted difference torsion is zero everywhere as Igusa showed in [7]. From this we get the following lemma:

Lemma 6.3 *For the difference torsion τ^δ associated with any higher twisted torsion invariant, we have*

$$\tau^\delta(E, \mathcal{F}) = 0$$

for any disc or sphere bundle $E \to B$ with local system $\mathcal{F}$ on E.

At first we will prove:

Lemma 6.4 *The difference torsion is* 0 *on any linear odd-dimensional lens space bundle* $L_n^{2N+1} \hookrightarrow E_n^{2N+1} \to B$*. By linear we mean that it is covered by a linear sphere bundle* $S^{2N+1} \hookrightarrow \widetilde{E}^{2N+1} \to B$*.*

Remark 6.5 The corresponding statement [7, Lemma 7.3] only deals with linear disc bundles, the proof of which follows swiftly from the product formula for relative torsion. Unfortunately, there is no twisted product formula, so our proof is slightly more difficult.

Proof The covering sphere bundle $\widetilde{E}^{2N+1}$ is a subbundle of an $(N+1)$–dimensional complex vector bundle. By the splitting principle, it suffices to look at the direct sum of $N+1$ complex line bundles. The sphere bundle will become the fiberwise join of the circle bundles associated with the line bundles:

$$S^1 * \cdots * S^1 \hookrightarrow \widetilde{E}_1^1 * \cdots * \widetilde{E}_{N+1}^1 \to B.$$

Now we have

$$\begin{aligned} L_n^{2N+1} &\cong (S^{2N-1} * S^1)/n \\ &= (S^{2N-1} \times D^2)/n \cup_{(S^{2N-1} \times S^1)/n} (D^{2N} \times S^1)/n. \end{aligned}$$

Fiberwise, this gives us

$$E_n^{2N+1} = H_n^{2N-1} \cup H_n^1,$$

where $H_n^{2N-1} \to B$ is an $(S^{2N-1} \times D^2)/n$–bundle and $H_n^1 \to B$ is a $(D^{2N} \times S^1)/n$–bundle, both meeting along their common vertical boundary, which is given by an $(S^{2N-1} \times S^1)/n$–bundle G_n. The $\mathbb{Z}/n$–action is given by the simultaneous action on each component of the products. While the $\mathbb{Z}/n$–action on any disc is not free, the simultaneous action will guarantee that it is free on the product. We can restrict every local coefficient system $\mathcal{F}$ on E_n^{2N+1} to H_n^{2N-1}, H_n^1 and G_n and use the additivity axiom.

Now we will continue the proof by induction. We know that the difference torsion is 0 on every $L_n^1 \cong S^1$–bundle. Let us then assume that the difference torsion is 0 on any linear L_n^{2N-1}–bundle with any representation of the fundamental group. Given a linear L_n^{2N+1}–bundle $E_n^{2N+1} \to B$ with local coefficient system $\mathcal{F}$, the construction above yield, by Lemma 4.2,

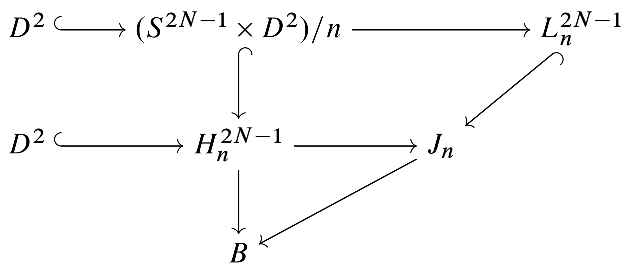
$$\tau^\delta(E_n^{2N+1}, \mathcal{F}) = \tau^\delta(H_n^{2N-1}, \mathcal{F}_{|H_n^{2N-1}}) + \tau^\delta(H_n^1, \mathcal{F}_{|H_n^1}) - \tau^\delta(G_n, \mathcal{F}_{|G_n}).$$

We have nontrivial fibrations

$$D^2 \hookrightarrow (S^{2N-1} \times D^2)/n \to L_n^{2N-1},$$

$$D^{2N} \hookrightarrow (D^{2N} \times S^1)/n \to L_n^1,$$
$$S^1 \to (S^{2N-1} \times S^1)/n \to L_n^{2N-1}.$$

The first of these splits the bundle H_n^{2N-1} in the following manner:



where $J_n \to B$ is an L_n^{2N-1}–bundle and $H_n^{2N-1} \to J_n$ is a D^2–bundle. Since D^2 is contractible, we get a local system $\mathcal{F}_J$ on J_n the pull-back of which to H_n^{2N-1} is isomorphic to $\mathcal{F}_{|H_n^{2N-1}}$. Now we can use the geometric transfer and the fact that we already determined the difference torsion to be 0 on L_n^{2N-1}–bundles and D^2–bundles to show

$$\tau^\delta(H_n^{2N-1}, \mathcal{F}_{|H_n^{2N-1}}) = \chi(D^2)\tau(J_n, \mathcal{F}_J) + \mathrm{tr}_B^{J_n}(\tau_{J_n}(H_n^{2N-1}, \mathcal{F}_{|H_n^{2N-1}})) = 0.$$

A similar argument holds for H_n^1 and G_n, and this completes the proof. □

6.3 Difference torsion as a fiber homotopy invariant

In this section, we will prove that the difference torsion τ^δ is a fiber homotopy invariant. By this we mean that for any two bundles $F_1 \hookrightarrow E_1 \to B$ and $F_2 \hookrightarrow E_2 \to B$ and fiber homotopy equivalence $f \colon E_1 \to E_2$ with local coefficient systems $\mathcal{F}_2$ on E_2 and $f^*\mathcal{F}_2 \cong \mathcal{F}_1$ on E_1, we have

$$\tau^\delta(E_1, \mathcal{F}_1) = \tau^\delta(E_2, \mathcal{F}_2) \in H^{2k}(B; \mathbb{R}).$$

This section will greatly rely on the construction of the equivariant Hatcher examples from [4]. We will especially use some techniques involving h–cobordism bundles, for a basic depiction of which the reader is also referred to [4, Section 1].

First we show the following lemmas:

Lemma 6.6 *For any linear disc bundle $D \xrightarrow{q} E$ and any bundle pair $(E, \partial_0) \to B$ with local coefficient system $\mathcal{F}$ we have*

$$\tau_B^\delta(D, \partial_0, q^*\mathcal{F}) = \tau_B^\delta(E, \partial_0, \mathcal{F}),$$

where we pull the system up to D and $\partial_0 D = q^{-1}\partial_0 E$ as usual.

Proof By geometric transfer (Proposition 4.8) we have

$$\tau_B^\delta(D, \partial_0, q^*\mathcal{F}) = \tau^\delta(E, \partial_0, \mathcal{F}) + \mathrm{tr}_B^E(\tau_E^\delta(D, q^*\mathcal{F}))$$

and $\tau_E^\delta(D, q^*\mathcal{F}) = 0$ because D is a disc bundle over E. □

Remark 6.7 The same statement still holds in the nonrelative case.

We will now need to prove three subsequent lemmas before we can prove the fiber homotopy invariance.

Remark 6.8 As before (Proposition 5.1) it is enough to look at local systems that induce holonomy covers with cyclic transformation group. So we will always assume that.

Lemma 6.9 *Let B be a space with finite fundamental group. Then for sufficiently large integers N the difference torsion τ^δ is zero on any h–cobordism bundle of L_n^{2N-1} over B for a given n.*

Proof Since we can assume that B is simply connected, all local systems on an h–cobordism bundle of $L_n^{2N-1} \times D^M$ inducing an n–fold cyclic holonomy are isomorphic to the local systems of the form $\mathcal{F}_\zeta$, where ζ is an n^{th} root of unity. We will now fix such a ζ.

We will follow Igusa [7, Lemma 7.11] closely in his discussion of the untwisted version of this crucial proof. By the stability of higher torsion (Corollary 4.10) we can view the difference torsion as a map

$$\tau^\delta(\_\_, \mathcal{F}_\zeta)\colon [B, B\mathcal{P}(L_n^{2N-1})] = [B, B(\mathrm{colim}_M\, \mathcal{C}(L_n^{2N-1} \times D^M))] \to H^*(B; \mathbb{R})$$

sending an h–cobordism bundle $h \to B$ to $\tau^\delta(h, \mathcal{F}_\zeta)$. Here $\mathcal{C}(M)$ is the concordance space and $\mathcal{P}(M)$ is the stable concordance space; for details see [4, Section 1]. We can give the set $[B, B\mathcal{P}(L_n^{2N-1})]$ a group structure by the fiberwise gluing together of the h–cobordisms as explained in [7]. From the additivity properties of higher twisted torsion (in particular Example 4.7) it follows that $\tau^\delta(\_\_, \mathcal{F}_\zeta)$ is a group homomorphism. So it is enough to give rational generators of $[B, B\mathcal{P}(L_n^{2N-1})]$ and show that the difference torsion is zero on these generators.

For N large enough ($N \gg \dim B$) we have

$$[B, B\mathcal{P}(L_n^{2N-1})] = [B, \mathcal{H}(L_n^{2N-1})] \cong [B, \mathcal{H}(B\mathbb{Z}/n)],$$

where $\mathcal{H}$ denotes the classifying space of h–cobordism bundles. In [4, Section 3.2], we define the twisted Hatcher maps $\Delta^i\colon G_n/U \to \mathcal{H}(B\mathbb{Z}/n)$ and the main theorem thereof uses those to show that the space $\mathbb{Q}\otimes[B,\mathcal{H}(B\mathbb{Z}/n)]$ is spanned by various Hatcher constructions (also defined in [4]) of one nontrivial vector bundle ξ over B with fiber a homotopically trivial sphere bundle. The calculations in [4, Section 4.1] only rely on the axioms and ensure that the difference torsion of these Hatcher constructions is zero. This is because the nontrivial Igusa–Klein torsion of those bundles arises from the torsion of sphere bundles and linear lens space bundles. □

Lemma 6.10 *Let N be an sufficiently large integer and $E\to B$ a bundle with local system $\mathcal{F}$ on E inducing an n–fold cyclic holonomy covering. Then we have $\tau^\delta(E,\mathcal{F})=0$ if there is a fiber homotopy equivalence:*

$$\begin{array}{ccc} E & \xrightarrow{\sim} & L_n^{2N-1}\times B \\ \downarrow & & \downarrow \\ B & \xrightarrow{=} & B \end{array}$$

Proof Denote the fiber homotopy equivalence $H\colon E\to L_n^{2N-1}\times B$. We can take the product of $L_n^{2N-1}\times B$ with a large-dimensional disc D^M and make H into an embedding

$$\overline{H}\colon E\overset{\sim}{\hookrightarrow} D^M\times L_n^{2N-1}\times B.$$

Then we can take a tubular neighborhood of $\overline{H}(E)\subseteq D^M\times L_n^{2N-1}\times B$ to get a codimension-0 embedding of an M'–dimensional disc bundle $D(E)$ over E

$$G\colon D(E)\overset{\sim}{\hookrightarrow} D^M\times L_n^{2N-1}\times B.$$

Then $(D^M\times L_n^{2N-1}\times B)\backslash G(D^\circ(E))$ is an h–cobordism bundle of $L_n^{2N-1}\times S^{M-1}$ over B and by Lemma 4.2 its difference torsion is given by

$$\begin{aligned}\tau^\delta((D^M\times L_n^{2N-1}\times B)\backslash G(D^\circ(E)),\mathcal{F})+\tau^\delta(D(E),\mathcal{F})-\tau^\delta(S(E),\mathcal{F})\\ =\tau^\delta(D^M\times L_n^{2N-1}\times B,\mathcal{F})\\ =0,\end{aligned}$$

since the last bundle is trivial. $S(E)$ denotes the sphere bundle given as the vertical boundary of $D(E)$. We can use the transfer axiom to show that

$$\tau^\delta(D(E),\mathcal{F})=\tau^\delta(E,\mathcal{F}),$$

and, given that M' is even,

$$\tau^\delta(S(E),\mathcal{F})=\chi(S^{M'-1})\tau^\delta_B(E,\mathcal{F})+\operatorname{tr}^E_B\,\tau^\delta_E(S(E),\mathcal{F})=0,$$

because the difference torsion is zero on any disc and sphere bundles. Therefore it suffices to show that the difference torsion is zero on any h–cobordism bundle of $L_n^{2N-1} \times S^{M-1}$ over B for arbitrarily large N. Such a bundle can easily be reduced to an h–cobordism bundle of L_n^{2N-1} without changing its torsion: Let $H \to B$ be an h–cobordism bundle of $L_n^{2N-1} \times S^{M-1}$. We can embed $S^{M-1} \times I$ as a tubular neighborhood of S^{M-1} into D^M and thereby get

$$H \supseteq L_n^{2N-1} \times S^{M-1} \times B \hookrightarrow L_n^{2N-1} \times D^M \times 1 \times B \subseteq L_n^{2N-1} \times D^M \times I \times B,$$

and we can define the h–cobordism bundle of $L_n^{2N-1} \times D^M$ (and thereby of L_n^{2N-1} by stability)

$$H' := H \cup_{L_n^{2N-1} \times S^{M-1} \times B} L_n^{2N-1} \times D^M \times I \times B.$$

Intuitively, we get H' by gluing the h–cobordism bundle H of $L_n^{2N-1} \times S^{M-1}$ on top of a trivial h–cobordism bundle of $L_n^{2N-1} \times D^M$ along the inclusion $S^{M-1} \hookrightarrow D^M$.

We calculate, using the relative additivity properties of higher torsion (Example 4.7), for any local system $\mathcal{F}$ in L_n^{2N-1} extended naturally to H, H' and $L_n^{2N-1} \times D^M \times I \times B$,

$$\begin{aligned}\tau^\delta(H', \mathcal{F}) &= \tau^\delta(H', L_n \times D^M \times 0 \times B, \mathcal{F}) \\ &= \tau^\delta(H, L_n^{2N-1} \times S^{M-1} \times B, \mathcal{F}) \\ &\qquad\qquad + \tau^\delta(L_n^{2N-1} \times D^m \times I \times B, L_n^{2N-1} \times D^m \times 0 \times B, \mathcal{F}) \\ &= \tau^\delta(H, \mathcal{F}).\end{aligned}$$

With this construction on h–cobordism bundles the proof now follows from the previous lemma. □

Lemma 6.11 *The difference torsion τ^δ is 0 on any bundle pair $(E, \partial_0) \to B$ the fibers (F, ∂_0) of which are h-cobordisms and have a local system $\mathcal{F}$ inducing a cyclic n–fold holonomy covering.*

Proof This proof can be translated directly from the proof of Lemma 8.3 in [7] by replacing the high-dimensional discs D^N with high-dimensional lens spaces L_n^{2N}. □

Theorem 6.12 *The difference torsion τ^δ is a fiber homotopy invariant of smooth bundle pairs with local systems.*

Proof Same as for Theorem 8.4 in [7]. □

Remark 6.13 Since τ^δ is a fiber homotopy equivalence, it is well defined on any fibration $(Z, C) \to B$ with fiber (X, A) and local system $\mathcal{F}$ on X which is smoothable in the sense that it is fiber homotopy equivalent to a smooth bundle pair (E, ∂_0) with compact manifold fiber (F, ∂_0).

7 Triviality of the difference torsion

Using the fiber homotopy invariance of the difference torsion we will first show that we can replace any bundle $E \to B$ with another one with the same torsion and real acyclic fiber. Then we will show that the difference torsion (or more general any torsion invariant that is fiber homotopy invariant) must be zero on any bundle with acyclic fibers.

7.1 Lens space suspensions

As outlined above, our first goal is to eliminate the real homology groups of the fiber F of a bundle $E \to B$. We will use the fact that the stable homotopy groups of F are rationally equivalent to the rational homology groups. This means that sufficiently large k and for an element $\alpha \in H_{m+k}(\Sigma^k F; \mathbb{R}) \cong H_m(F; \mathbb{R})$ there is a map $S^{m+k} \to \Sigma^k F$ representing α as an element of $\pi_{m+k}(\Sigma^k F) \otimes \mathbb{R}$. We then can glue in an $(m+k+1)$–cell along this map to effectively kill off the element α and continue inductively. Unfortunately, this naive construction has a big problem for us: even just one suspension destroys the first homotopy group of F leaving us with only the trivial local system which is not very interesting (or helpful). So we need an alternative suspension construction that shifts up the rational homology groups, exhibits the isomorphism to stable homotopy groups after sufficiently many suspensions and preserves the first homotopy group. We achieve all of this by suspending via a push-out along two high-dimensional (even) lens spaces rather than discs.

Let us recall that the usual suspension ΣF is defined by the (homotopy) push-out:

$$\begin{array}{ccc} F & \longrightarrow & D^N \\ \downarrow & & \downarrow \\ D^N & \longrightarrow & \Sigma F \end{array}$$

Since D^N is contractible, we know that $\pi_1 \Sigma F = 0$, and therefore this construction cannot give us a nonconstant local system on ΣF. Now we make the following definition:

Definition 7.1 (lens space suspension) Let F be a topological space with local system $\mathcal{F}$ on F inducing an n–fold holonomy cover $\widetilde{F} \to F$ with finite cyclic transition group. The cover gives us a mapping $F \to L_n^{2N}$ for a large $N \in \mathbb{N}$ (because $L_n^\infty \cong K(\mathbb{Z}/n, 1)$). Using this map, we can define the lens space suspension $\Sigma_n F$ as the homotopy push-out:

$$\begin{array}{ccc} F & \longrightarrow & L_n^{2N} \\ \downarrow & & \downarrow \\ L_n^{2N} & \longrightarrow & \Sigma_n(N) F \end{array}$$

Remark 7.2 We will drop N from the notation and consider it to be very large.

We have the earlier introduced local systems $\mathcal{F}_\zeta$ on L_n^{2N} for an n^{th} root of unity ζ. By choosing the map $i\colon F \to L_n^{2N}$ properly, we can assume $\mathcal{F} = i^*\mathcal{F}_{e^{2\pi i/n}}$. So we get a local system $\Sigma\mathcal{F} = \mathcal{F}_{e^{2\pi i/n}} \cup_{\mathcal{F}} \mathcal{F}_{e^{2\pi i/n}}$ on $\Sigma_n F$. From this we get the holonomy covering $\widetilde{\Sigma_n F} \xrightarrow{n} \Sigma_n F$; but we also have the holonomy covering $\widetilde{F} \xrightarrow{n} F$. These two covering spaces are related by the following lemma:

Lemma 7.3 *In the setting above, we have*

$$\pi_i\, \widetilde{\Sigma_n F} \cong \pi_i\, \Sigma\widetilde{F}$$

in low degrees i (smaller than $2N$).

Proof Let $\Sigma(N)\widetilde{F}$ be the suspension of $\widetilde{F}$ along S^{2N} (instead of S^∞). This forms an n–fold covering $\Sigma(N)\widetilde{F} \to \Sigma_n(N)F$, which must be homotopy equivalent to the universal covering of $\widetilde{\Sigma_n(N)F} \to \Sigma_n(N)F$. □

For the usual suspension, it is well known that $H_{k+1}(\Sigma F;\mathbb{R}) \cong H_k(F;\mathbb{R})$ for all $k \geq 1$. For the lens space suspension this becomes:

Lemma 7.4 *For every topological space F with local system inducing an n–fold holonomy covering, we have, for $k \geq 1$,*

$$H_{k+1}(\Sigma_n F;\mathbb{R}) \cong H_k(F;\mathbb{R}).$$

Proof Using the Mayer–Vietoris sequence for the defining push-out of the lens space suspension, we get:

$$\cdots \to H_{k+1}(L_n^{2N};\mathbb{R}) \oplus H_{k+1}(L_n^{2N};\mathbb{R}) \to H_{k+1}(\Sigma_n F;\mathbb{R}) \\ \to H_k(F;\mathbb{R}) \to H_k(L_n^{2N};\mathbb{R}) \oplus H_k(L_n^{2N};\mathbb{R}) \to \cdots$$

The fact that L_n^{2N} is rationally homologically trivial now yields the desired isomorphism. □

Furthermore, we know for the usual suspension that $\pi_m^S(F) \otimes \mathbb{R} \cong \overline{H}_m(F;\mathbb{R})$, where $\pi_m^S(F) := \pi_m(\operatorname{colim}_k \Omega^k \Sigma^k F)$ denotes the stabilized homotopy group. This becomes:

Lemma 7.5 *If $k \in \mathbb{N}$ is large enough, and F is a space with local system inducing an n–fold holonomy covering, we have an isomorphism*

$$\pi_{m+k}(\Sigma_n^k F) \otimes \mathbb{R} \cong \overline{H}_{m+k}(\Sigma^k \widetilde{F};\mathbb{R})$$

for $m + k < N$.

Proof We get the n–fold holonomy covering $\widetilde{F} \to F$. Using Lemma 7.3 several times, we get in low degrees i

$$\pi_i \widetilde{\Sigma_n^k F} \cong \pi_i \Sigma(\widetilde{\Sigma_n^{k-1} F}) \cong \cdots \cong \pi_i \Sigma^k \widetilde{F}.$$

Thus we have, for $N > m+k > 1$,

$$\begin{aligned}\pi_{m+k}(\Sigma_n^k F) \otimes \mathbb{R} &\cong \pi_{m+k}(\Sigma^k \widetilde{F}) \otimes \mathbb{R} \\ &\cong \pi_m^S(\widetilde{F}) \otimes \mathbb{R} \quad \text{for } k \text{ large} \\ &\cong \overline{H}_m(\widetilde{F};\mathbb{R}) \\ &\cong \overline{H}_{m+k}(\Sigma^k \widetilde{F};\mathbb{R}).\end{aligned}$$

□

Remark 7.6 Although we require k to be large in the last lemma, it does not depend on N at all, meaning that we can still choose N to be much larger than k.

We will need the following definition and proposition:

Definition 7.7 A topological space F is called simple if $\pi_1 F$ is abelian and acts trivially on every $\pi_i F$ for $i \geq 2$.

Proposition 7.8 *Let F be a path connected, simple space and $\widetilde{F} \xrightarrow{n} F$ an n–fold Galois covering. Then the transition group $\mathbb{Z}/n$ will act trivially on $H_*(\widetilde{F};\mathbb{R})$*

Proof Let $\{F^l\}$ be the Postnikov tower for F; that is a sequence of spaces with $\lim_l F^l \cong F$ and $\pi_i F^l \cong \pi_i F$ for $0 \leq i \leq l$ and $\pi_i F^l \cong 0$ for $i > l$. Since we have $\pi_1 F^l \cong \pi_1 F$ for every $l > 0$, we have n–fold coverings $\widetilde{F}^l \xrightarrow{n} F^l$. We will prove by induction that $\mathbb{Z}/n$ acts trivially on $H_*(\widetilde{F}^l;\mathbb{R})$. The sequence $\{\widetilde{F}^l\}$ will clearly provide a Postnikov tower for $\widetilde{F}$, and since the real homology of the stages of a Postnikov tower stabilizes in every degree, this will prove the proposition.

To start the induction we look at $F^1 \simeq K(\pi_1 F, 1)$, which will only have the first homotopy group $\pi_1 F^1 \cong \pi_1 F$. The covering $\widetilde{F} \xrightarrow{n} F$ gives a map $\alpha\colon \pi_1 F \to \mathbb{Z}/n$. Using this, we see that the covering $\widetilde{F}^1 \xrightarrow{n} F^1$ will be an Eilenberg–Mac Lane space:

$$\widetilde{F}^1 \simeq K(\ker\alpha, 1).$$

The group $\mathbb{Z}/n$ acts trivially on $\ker\alpha \subseteq \pi_1 F$ because $\pi_1 F$ is abelian, and therefore $\mathbb{Z}/n$ acts trivially on $\widetilde{F}^1 \simeq K(\ker\alpha, 1)$ and $H_*(\widetilde{F}^1;\mathbb{R})$. This starts the induction.

Now assume that $\mathbb{Z}/n$ acts trivially on $H_*(\widetilde{F}^{l-1};\mathbb{R})$ with $l > 1$. We have the fibration

$$K(\pi_l \widetilde{F}, l) \to \widetilde{F}^l \to \widetilde{F}^{l-1}.$$

Since we know $\pi_l F \cong \pi_l \widetilde{F}$, the group $\mathbb{Z}/n$ will act trivially on $\pi_l \widetilde{F}$ and thereby also trivially on $K(\pi_l \widetilde{F}, l)$ and $H_*(K(\pi_l \widetilde{F}, l); \mathbb{R})$. By induction assumption it must also act trivially on

$$H_i(\widetilde{F}^{l-1}; H_k(K(\pi_l(\widetilde{F}), l); \mathbb{R})),$$

and thereby it acts trivially on the whole Leray–Serre spectral sequence for the fibration $K(\pi_l \widetilde{F}, l) \to \widetilde{F}^l \to \widetilde{F}^{l-1}$. From this it follows that $\mathbb{Z}/n$ acts unipotently on $H_*(\widetilde{F}_l; \mathbb{R})$, and since $\mathbb{R}[\mathbb{Z}/n]$ is semisimple, this includes that $\mathbb{Z}/n$ acts trivially on $H_*(\widetilde{F}; \mathbb{R})$. □

From this we get the following important corollary.

Corollary 7.9 *If F is a simple topological space with local system inducing an n–fold holonomy covering $\widetilde{F} \xrightarrow{n} F$, then we have*

$$H_l(\Sigma(N)^k \widetilde{F}; \mathbb{R}) \cong H_l(\Sigma_n^k F; \mathbb{R})$$

for all $l < 2N$. (Recall that $\Sigma(N)\widetilde{F}$ is the suspension of $\widetilde{F}$ along S^{2N} instead of S^∞.)

Proof Since F is simple, the group $\mathbb{Z}/n$ will act trivially on $H_*(\widetilde{F}; \mathbb{R})$. It is well known that this implies

$$H_*(F; \mathbb{R}) \cong H_*(\widetilde{F}; \mathbb{R}).$$

The inclusion $S^{2N} \to S^\infty$ gives a map $\Sigma(N)\widetilde{F} \to \Sigma\widetilde{F}$ that is evidently $2N$–connected. By using Lemma 7.4 we see

$$H_*(\Sigma(N)^k \widetilde{F}; \mathbb{R}) \cong H_{*-k}(\widetilde{F}; \mathbb{R}) \cong H_{*-k}(F; \mathbb{R}) \cong H_*(\Sigma_n^k F; \mathbb{R})$$

up to degree $2N$. □

Now we are turning back to bundles. For a fiber bundle $F \hookrightarrow E \to B$ with local system $\mathcal{F}$ on F inducing a finite cyclic n–fold holonomy covering, we get a fiberwise map $E \to B \times L_n^{2N}$ and can use this to define the fiberwise lens space suspension as the (homotopy) push-out:

$$\begin{array}{ccc} E & \longrightarrow & B \times L_n^{2N} \\ \downarrow & & \downarrow \\ B \times L_n^{2N} & \longrightarrow & \Sigma_{n,B} E \end{array}$$

It is easy to see that $\Sigma_{n,B} E \to B$ is a bundle with fiber $\Sigma_n F$ and as before we get a local system $\Sigma\mathcal{F}$ on $\Sigma_{n,B} E$. We have the following lemma analogous to [7, Lemma 8.7]:

Lemma 7.10 *The bundle $\Sigma_{n,B} E$ is smoothable (ie fiber homotopy equivalent to a smooth bundle) if E is smoothable, and we have*

$$\tau^\delta(E, \mathcal{F}) = -\tau^\delta(\Sigma_{n,B} E, \Sigma\mathcal{F}).$$

7.2 Reducing the homology of the fiber

We now attempt to make the fiber of a bundle $F \hookrightarrow E \to B$ with a local system on F, simply connected base B, and simple fiber F rationally homologically trivial without changing the difference torsion. This is the general strategy: Assume that m is the largest integer such that $H_m(F;\mathbb{R})$ is nontrivial. Picking an element $\alpha \in H_l(F;\mathbb{R})$, Lemma 7.12 asserts that we can find a representative of α

$$B \times L_n^{m+k} \to \Sigma^k_{n,B} E,$$

and then Lemma 7.11 uses this to create a new fiber bundle $E_1 \to B$ with the same difference torsion as E and the fiber F_1 having overall one dimension lower homology than F. Then Lemma 7.13 puts everything together inductively. The basic ideas reflect what has been done by Igusa in [7], yet the fact that we need to preserve the representation of a fundamental group poses some challenges. In the following, let N always be a sufficiently large integer.

Lemma 7.11 *Suppose $F \hookrightarrow E \to B$ is a fibration with local system $\mathcal{F}$ on F inducing a finite cyclic n–fold holonomy covering. Let $m \in \mathbb{N}$ denote the largest integer for which $\overline{H}_m(F;\mathbb{R}) \neq 0$. Suppose that we have $H_l(F;\mathbb{R}) \cong H_l(\widetilde{F};\mathbb{R})$ for $0 < l < m + \dim B$. Suppose further that m is odd and let α be a map*

$$\alpha\colon B \times L_n^m \to E$$

with the following properties: on each fiber we have $\alpha^*\mathcal{F} \cong \mathcal{F}_\zeta$ for some n^{th} root of unity ζ and $\alpha_*\colon \overline{H}_m(L_n^m;\mathbb{R}) \to \overline{H}_m(F;\mathbb{R})$ is nontrivial. Then if we look at the bundle

$$E_1 = E \cup_{B \times L_n^m} B \times L_n^{2N}$$

with fiber F_1 with local system $\mathcal{F}_1 := \mathcal{F} \cup_{\mathcal{F}_\zeta} \mathcal{F}_\zeta$ and corresponding covering $\widetilde{F}_1 \xrightarrow{n} F_1$, we have

$$\dim_{\mathbb{R}} H_*(F_1;\mathbb{R}) < \dim_{\mathbb{R}} H_*(F;\mathbb{R})$$

and

$$H_l(F_1;\mathbb{R}) \cong H_l(\widetilde{F}_1;\mathbb{R}) \quad \text{for } 0 < l < m + \dim B.$$

Proof Assume that we have a map $\alpha\colon B \times L_n^m \to E$ such that the induced map

$$\alpha_*\colon \overline{H}_m(L_n^m;\mathbb{R}) \to \overline{H}_m(F;\mathbb{R})$$

is nontrivial. Note this implies that the integer m is odd. Then the homology of the fiber $F_1 = F \cup_{L_n^m} L_n^{2N}$ will be given by the Mayer–Vietoris sequence as (where $i < 2N$)

$$H_i(L_N^m;\mathbb{R}) \xrightarrow{\alpha_*} H_i(F;\mathbb{R}) \oplus 0 \to H_i(F_1;\mathbb{R}) \to 0$$

and therefore we have $\dim_{\mathbb{R}} H_*(F_1;\mathbb{R}) < \dim_{\mathbb{R}} H_*(F;\mathbb{R})$. This also shows that $H_i(F_1;\mathbb{R}) \cong H_i(F;\mathbb{R})$ for $i \neq m$.

To show that this F_1 will satisfy the second property, we can use a similar sequence and show $H_i(\widetilde{F}_1;\mathbb{R}) \cong H_i(\widetilde{F};\mathbb{R})$. □

Lemma 7.12 *Suppose $F \hookrightarrow E \to B$ is a fibration with simply connected base B and local system $\mathcal{F}$ on F inducing a finite cyclic n–fold holonomy covering. As before let $m \in \mathbb{N}$ denote the largest integer for which $\overline{H}_m(F;\mathbb{R}) \neq 0$ and suppose that we have $H_l(F;\mathbb{R}) \cong H_l(\widetilde{F};\mathbb{R})$ for $0 < l < m + \dim B$. Then there exists an integer $k \in \mathbb{N}$ and a map*

$$\alpha\colon B \times L_n^{m+k} \to \Sigma_{n,B}^k E$$

such that $\alpha^* \Sigma^k \mathcal{F} \cong \mathcal{F}_\zeta$ for some n^{th} root of unity ζ and $\alpha_*\colon \overline{H}_{m+k}(L_n^{m+k};\mathbb{R}) \to \overline{H}_{m+k}(\Sigma_n^k F;\mathbb{R})$ is nontrivial.

Proof Note that in the following, m and n are fixed, already determined integers, whereas k is an sufficiently large integer bounded by the sufficiently large integer N. Furthermore $m+k$ must be odd, such that L_n^{m+k} has a nonvanishing rational homology group in degree $m+k$, but we can choose k in such a way that this is satisfied.

Such a map α will correspond to a section s of the bundle

$$\operatorname{Map}(L_n^{m+k}, \Sigma_n^k F) \hookrightarrow \operatorname{Map}_B(B \times L_n^{m+k}, \Sigma_{n,B}^k E) \to B,$$

which is a homologically nontrivial map in each fiber. In this context the notation $\operatorname{Map}_B(B \times L_n^{m+k}, \Sigma_{n,b}^E)$ will always mean the space of fiberwise maps between $B \times L_n^{m+k}$ and $\Sigma_{n,B}^k E$. We will construct this section using obstruction theory. Let B_l denote the l–skeleton of B. Firstly, we will give $s_1\colon B_1 \to \operatorname{Map}_B(B \times L_n^{m+k}, \Sigma_{n,B}^k E)$. By the choice of m we have a nonzero element

$$\tilde{\gamma} \in \overline{H}_{m+k}(\Sigma(N)^k \widetilde{F};\mathbb{R}) \cong \overline{H}_{m+k}(\Sigma_n^k F;\mathbb{R}) \cong \overline{H}_m(F;\mathbb{R}).$$

Since the reduced homology is isomorphic to rationalized stabilized homotopy, we can view $\tilde{\gamma}$ as an element of $\pi_{m+k}(\Sigma(N)^k \widetilde{F}) \otimes \mathbb{R}$, if k is large enough. Now choose a representative $\tilde{\alpha}_1\colon S^{m+k} \to \Sigma(N)^k \widetilde{F}$ of $\tilde{\gamma}$. The map $\tilde{\alpha}_1$ will clearly be nontrivial on homology.

Our goal is now to modify $\tilde{\alpha}_1$ to $\tilde{\alpha}\colon S^{m+k} \to \Sigma(N)^k \widetilde{F}$ such that it covers an $\alpha\colon L_n^{m+k} \to \Sigma_n^k F$. Since $H_{m+k}(\Sigma(N)^k \widetilde{F};\mathbb{R}) \cong H_{m+k}(\Sigma_n^k F;\mathbb{R})$, the map α will be nontrivial on homology. Furthermore the covering will ensure $\alpha^* \mathcal{F} \cong \mathcal{F}_\zeta$ for some n^{th} root of unity ζ. To begin, we have from the last lens space suspension an inclusion

$$i\colon L_n^{m+k} \hookrightarrow \Sigma_n^k F$$

trivial on homology. This will be covered by a homologically trivial equivariant inclusion

$$\tilde{i}\colon S^{m+k} \hookrightarrow \widetilde{\Sigma_n^k F}.$$

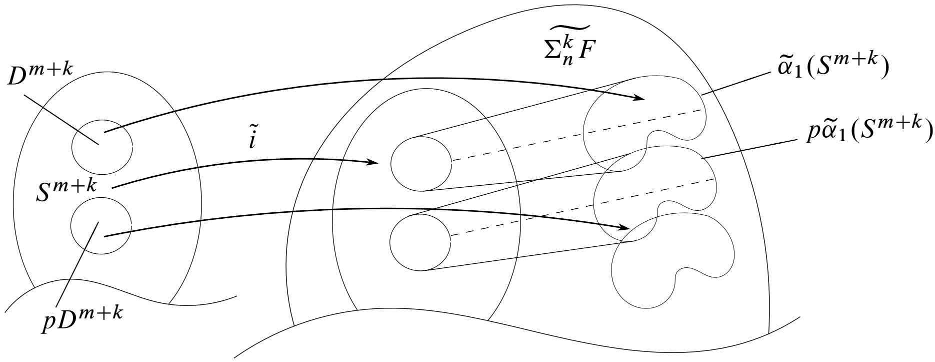


Figure 2: Modifying the inclusion $\tilde{i}\colon S^{m+k} \hookrightarrow \widetilde{\Sigma_n^k F}$

The idea now is to take a small disc D^{m+k} in $S^{m+k} \subseteq \widetilde{\Sigma_n^k F}$ and connect it to the image $\widetilde{\alpha}_1(S^{m+k})$. Then we can map S^{m+k} to this new image instead and this map will be nontrivial on homology because $\widetilde{\alpha}_1$ is nontrivial on homology. To make it equivariant we do the same construction equivariantly to every disc $p^i D^{m+k}$ in the orbit of D^{m+k} under the $\mathbb{Z}/n$ action on S^{m+k}. Here $p \in \mathbb{Z}/n$ denotes a generator. This is illustrated in Figure 2.

The formal construction is the following: Choose a small disc $D^{m+k} \subseteq S^{m+k}$. By doing this in a slightly bigger disc, we can modify the inclusion such that it factorizes

$$D^{m+k} \to * \hookrightarrow \widetilde{\Sigma_n^k F}.$$

Using $D^{m+k}/\partial D^{m+k} \simeq S^{m+k}$, we can glue in $\widetilde{\alpha}_1$ and modify the inclusion again so that it factorizes

$$D^{m+k} \xrightarrow{\widetilde{\alpha}_1} \widetilde{\Sigma_n^k F}.$$

Now let $p \in \mathbb{Z}/n$ be a generator. If we make D^{m+k} small enough, it will not intersect with any of the $p^i D^{m+k} \subseteq S^{m+k}$ for $0 < i < n$. Doing the same construction to every $p^i D^{m+k}$ using $p^i \widetilde{\alpha}_1$, we can modify the inclusion to a map

$$\widetilde{\alpha}\colon S^{m+k} \to \widetilde{\Sigma_n^k F},$$

which will clearly be n–equivariant and thus cover a map

$$\alpha\colon L_n^{m+k} \to \Sigma_n F.$$

The corresponding rationalized homotopy class of $\widetilde{\alpha}$ in $\pi_{m+k}(\Sigma_n^k F) \otimes \mathbb{R}$ is given by

$$[\widetilde{\alpha}] = [\widetilde{\alpha}_1] + p[\widetilde{\alpha}_1] + \cdots + p^{n-1}[\widetilde{\alpha}_1] = n[\widetilde{\alpha}_1] \neq 0,$$

since $\pi_1 F$ acts trivially on

$$\pi_{m+k}\widetilde{\Sigma_n^k F}\otimes\mathbb{R}\cong\pi_{m+k}(\Sigma(N)^k\widetilde{F})\otimes\mathbb{R}\cong H_{m+k}(\Sigma(N)^k\widetilde{F};\mathbb{R})$$
$$\cong H_m(\widetilde{F};\mathbb{R})\cong H_m(F;\mathbb{R})$$

(otherwise the map $H^m(F;\mathbb{R})\hookrightarrow H^m(\widetilde{F};\mathbb{R})$ would not be an isomorphism and thereby $H_m(F;\mathbb{R})$ would not be isomorphic to $H_m(\widetilde{F};\mathbb{R})$ either). So α will be nontrivial in rational homology.

With this we can define $s_0\colon B_0\simeq *\to\mathrm{Map}_B(B\times L_n^{m+k},\Sigma_{n,B}^k E)$ nontrivial in the homology of the fiber. Since B is simply connected, this section, defined over a point of B, can be extended to a section $s_1\colon B_1\to\mathrm{Map}_B(B\times L_n^{m+k},\Sigma_{n,B}^k E)$.

Let us now continue inductively. Suppose we already have a section $s_l\colon B_l\to\mathrm{Map}_B(B\times L_n^{m+k},\Sigma_{n,B}^k E)$ with $1\le l<\dim B$. By restriction, we will get sections

$$s_{l,i}\colon B_l\to\mathrm{Map}_B(B\times L_n^i,\Sigma_{n,B}^k E).$$

Let us first extend $s_{l,1}$ to $s_{l+1,1}\colon B_{l+1}\to\mathrm{Map}_B(B\times L_n^1,\Sigma_n^k E)$: This depends on the obstruction class

$$\theta(s_l,1)\in H^{l+1}(B,B_l;\pi_l(\mathrm{Map}(L_n^1,\Sigma_n^k F)))\cong H^{l+1}(B,B_l;\pi_{l+1}(\Sigma_n^k F)),$$

because $L_n^1\simeq S^1$. So $\theta(s_{l,1})$ is rationally trivial, if k is large enough (larger than $l+1$). This is enough to extend $s_{l,1}$ as Igusa showed in the nontwisted version [7, Lemma 8.9] of this lemma.

We now want to extend $s_{l+1,1}$ to $s_{l+1,2}$ relative to $s_{l,2}$. For this we look at the cofibration sequence

$$L_n^1\hookrightarrow L_n^2\to S^2,$$

which gives us the fibration sequence

$$\Omega^2(\Sigma_n^k F)\hookrightarrow\mathrm{Map}_B(B\times L_n^2,\Sigma_{n,B}^k E)\to\mathrm{Map}_B(B\times L_n^1,\Sigma_{n,B}^k E).$$

From this we get the commutative diagram

$$\begin{array}{ccc}
 & & \Omega^2(\Sigma_n^k F)\\
 & & \big\downarrow\\
B_l & \xrightarrow{\;s_{l,2}\;} & \mathrm{Map}_B(B\times L_n^2,\Sigma_{n,B}^k E)\\
\big\downarrow & \overset{s_{l+1,2}}{\nearrow} & \big\downarrow\\
B_{l+1} & \xrightarrow{\;s_{l+1,1}\;} & \mathrm{Map}_B(B\times L_n^1,\Sigma_{n,B}^k E)
\end{array}$$

where the right column is a fibration sequence. Consequently the extension from $s_{l+1,1}$ to $s_{l+1,2}$ depends on the obstruction class

$$\theta(s_{l,1}) \in H^{l+1}(B, B_l; \pi_l(\Omega^2(\Sigma_n^k F))) \cong H^{l+1}(B, B_l; \pi_{l+2}(\Sigma_n^k F)),$$

which is, again, rationally trivial for large k.

Now assume that we have already constructed $s_{l+1,i}$ with $i \in \mathbb{N}$ even. Next, look at the cofibration

$$L_n^i \hookrightarrow L_n^{i+2} \to M(\mathbb{Z}_n, i),$$

where

$$M(\mathbb{Z}_n, i) := \mathrm{cof}(S^i \xrightarrow{n} S^i)$$

is the Moore space. Directly from the definition of the Moore space, we get that $\pi_l(\mathrm{Map}(M(\mathbb{Z}_n, i), X))$ is finite for any space X. Using the fibration

$$\mathrm{Map}(M(\mathbb{Z}_n, i), \Sigma_n^k F) \hookrightarrow \mathrm{Map}_B(B \times L_n^{i+2}, \Sigma_{n,B}^k E) \to \mathrm{Map}_B(B \times L_{n,B}^i \Sigma_n^k E),$$

the commutative diagram

$$\begin{array}{ccc} & & \mathrm{Map}(M(\mathbb{Z}_n, i), \Sigma_n^k F) \\ & & \downarrow \\ B_l & \xrightarrow{s_{l,i+2}} & \mathrm{Map}_B(B \times L_n^{i+2}, \Sigma_{n,B}^k E) \\ \downarrow & \overset{s_{l+1,i+2}}{\nearrow} & \downarrow \\ B_{l+1} & \xrightarrow{s_{l+1,i}} & \mathrm{Map}_B(B \times L_n^i, \Sigma_{n,B}^k E) \end{array}$$

tells us that extending $s_{l+1,i}$ to $s_{l+1,i+2}$ depends on the obstruction class

$$\theta(s_{l+1,i}) \in H^{l+1}(B, B_l; \pi_l(\mathrm{Map}(M(\mathbb{Z}_n, i), \Sigma_n^k F))),$$

which is rationally trivial.

Using this inductively, we get $s_{l+1,k+m-1}$. To extend this to $s_{l+1,k+m} = s_{l+1}$, we use again the cofibration sequence

$$L_n^{k+m-1} \hookrightarrow L_N^{k+m} \to S^{k+m},$$

the induced fibration sequence

$$\Omega^{k+m}(\Sigma_n^k F) \hookrightarrow \mathrm{Map}_B(B \times L_n^{k+m}, \Sigma_{n,B}^k E) \to \mathrm{Map}_B(B \times L_n^{k+m-1}, \Sigma_{n,B}^k E)$$

and the commutative diagram

$$\begin{array}{ccc} & & \Omega^{k+m}(\Sigma^k_n F) \\ & & \big\downarrow \\ B_l & \xrightarrow{s_{l,k+m}} & \mathrm{Map}_B(B \times L_n^{k+m}, \Sigma^k_{n,B} E) \\ \big\downarrow & \overset{s_{l+1,k+m}}{\nearrow} & \big\downarrow \\ B_{l+1} & \xrightarrow{s_{l+1,k+m-1}} & \mathrm{Map}_B(B \times L_n^{k+m-1}, \Sigma^k_{n,B} E) \end{array}$$

making the obstruction class

$$\theta(s_{l+1,k+m-1}) \in H^{l+1}(B, B_l; \pi_{k+m+l}(\Sigma^k_n F)).$$

However, if k is large enough, we have

$$\begin{aligned} \pi_{k+m+l}(\Sigma^k_n F) \otimes \mathbb{R} &\cong \pi_{k+m+l}(\Sigma(N)^k \widetilde{F}) \otimes \mathbb{R} \\ &\cong \overline{H}_{k+m+l}(\Sigma(N)^k \widetilde{F}; \mathbb{R}) \\ &\cong H_{k+m+l}(\Sigma^k_n F; \mathbb{R}) \\ &\cong H_{m+l}(F; \mathbb{R}) \cong 0 \end{aligned}$$

by assumption because $m + l < m + \dim B$. This guarantees that we can extend $s_{l+1,k+m-1}$ to s_{l+1} and completes the proof. □

Lemma 7.13 *Let $F \hookrightarrow E \to B$ be a fibration with simply connected base B and local system $\mathcal{F}$ on F inducing a finite cyclic n–fold holonomy covering. Suppose further that F is simple. Then there exists a bundle $F' \hookrightarrow E' \to B$ with local coefficient system $\mathcal{F}'$ on F', where F' is rationally homologically trivial such that*

$$\tau^\delta(E, \mathcal{F}) = \pm\tau^\delta(E', \mathcal{F}').$$

Proof Let again m be the largest integer such that $H_m(F; \mathbb{R})$ is nontrivial. Since F is simple we get $H_*(F; \mathbb{R}) \cong H_*(\widetilde{F}; \mathbb{R})$ by Corollary 7.9, and we can use Lemma 7.12 to get, for an integer k, a bundle map

$$\alpha\colon B \times L_n^{m+k} \to \Sigma^k_{n,B} E$$

nontrivial on the $(m+k)^{\text{th}}$ homology. By Lemma 7.3 the n–fold covering of $\Sigma^k_n F$ is given in low degrees by $\Sigma(N)^k \widetilde{F}$. Since both Σ^k_n and $\Sigma(N)^k$ only shift rational homology up by k degrees we have

$$H_l(\Sigma^k_n F; \mathbb{R}) \cong H_l(\Sigma(N)^k \widetilde{F}; R) \cong H_l(\widetilde{\Sigma^k_n F}; \mathbb{R})$$

for all $0 < l < m + k + \dim B$. Furthermore the highest nontrivial homology group of $\Sigma_n^k F$ is in degree $m + k$ and we also have

$$\dim_{\mathbb{R}} H_*(\Sigma_n^k F; \mathbb{R}) = \dim_{\mathbb{R}} H_*(F; \mathbb{R}).$$

Now we can apply the construction of Lemma 7.11 to get a bundle $F_1 \hookrightarrow E_1 \to B$ such that

$$\dim_{\mathbb{R}} H_*(F_1, \mathbb{R}) < \dim_{\mathbb{R}} H_*(\Sigma_n^k F; \mathbb{R}) = \dim_{\mathbb{R}} H_*(F; \mathbb{R}).$$

By definition of E_1, and since the torsion of trivial bundles is zero, we get with additivity and Lemma 7.10

$$\tau^\delta(E_1, \mathcal{F}_1) = \tau^\delta(\Sigma_{n,B}^k E, \Sigma^k \mathcal{F}) = (-1)^k \tau^\delta(E, \mathcal{F}).$$

Since Lemma 7.11 guarantees that $H_l(F_1; \mathbb{R}) \cong H_l(\widetilde{F}_1; \mathbb{R})$ for $0 < l < m+k+\dim B$ we now can repeat this process and decrease the dimension of the rational homology until we will get the bundle $F' \hookrightarrow E' \to B$ with local system $\mathcal{F}'$ on F such that

$$\tau^\delta(E, \mathcal{F}) = \pm\tau^\delta(E', \mathcal{F}')$$

and F' is rationally homologically trivial. □

We are now finally in the position to prove the main theorem. As a consequence of Lemma 7.13 it suffices to only determine τ^δ on bundles with rationally trivial fiber, so we conclude with the following lemma.

Lemma 7.14 *We have $\tau^\delta(Z, \mathcal{F}) = 0$ for any torsion invariant, smoothable bundle $X \hookrightarrow Z \to B$ with $\overline{H}_*(X; \mathbb{R}) = 0$, simply connected base B and local system $\mathcal{F}$ inducing an n–fold holonomy covering.*

Proof This is completely analogous to the proof of Lemma 8.11 in [7]. We will only explain the main points. We replace the bundle by a manifold bundle $M \hookrightarrow E \to B$ and its universal covering $\widetilde{M} \to \widetilde{E} \to B$. Choosing a section of $E \to B$ gives disc bundles $D \subset E \to B$ and $\widetilde{D} \subset \widetilde{E} \to B$. Now there is a universal torsion class

$$\tau^\delta \in H^{2k}(B\mathrm{Diff}_n(\widetilde{M} \text{ rel } \widetilde{D}); \mathbb{R}),$$

where $B\mathrm{Diff}_n(\widetilde{M} \text{ rel } \widetilde{D})$ is the classifying space of $\mathbb{Z}/n$–equivariant diffeomorphisms of $\widetilde{M}$ relative to $\widetilde{D}$.

In the original paper [7], one only has to consider $B\mathrm{Diff}(M \text{ rel } D)$, but luckily in the end we are only interested in maps that leave a certain base point fixed and we have

$$B\mathrm{Diff}_{0,n}(\widetilde{M} \text{ rel } \widetilde{D}) \cong B\mathrm{Diff}_0(M \text{ rel } D),$$

where the subscript 0 indicates the identity component. From here on the proof is parallel to the proof in [7]. □

References

[1] **J-M Bismut**, **S Goette**, *Families torsion and Morse functions*, Astérisque 275, Soc. Math. France, Paris (2001) MR

[2] **J-M Bismut**, **J Lott**, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. 8 (1995) 291–363 MR

[3] **W Dwyer**, **M Weiss**, **B Williams**, *A parametrized index theorem for the algebraic K–theory Euler class*, Acta Math. 190 (2003) 1–104 MR

[4] **T Goodwillie**, **K Igusa**, **C Ohrt**, *An equivariant version of Hatcher's G/O construction*, J. Topol. 8 (2015) 675–690 MR

[5] **K Igusa**, *Higher Franz–Reidemeister torsion*, AMS/IP Studies in Advanced Mathematics 31, Amer. Math. Soc., Providence, RI (2002) MR

[6] **K Igusa**, *Higher complex torsion and the framing principle*, Mem. Amer. Math. Soc. 835, Amer. Math. Soc., Providence, RI (2005) MR

[7] **K Igusa**, *Axioms for higher torsion invariants of smooth bundles*, J. Topol. 1 (2008) 159–186 MR

[8] **K Igusa**, *Outline of higher Igusa–Klein torsion*, lecture notes (2009) Available at `http://people.brandeis.edu/~igusa/Papers/AIM0911.pdf`

[9] **K Igusa**, **J Klein**, *The Borel regulator map on pictures, II: An example from Morse theory*, K–Theory 7 (1993) 225–267 MR

[10] **J Milnor**, *On polylogarithms, Hurwitz zeta functions, and the Kubert identities*, Enseign. Math. 29 (1983) 281–322 MR

[11] **T Szamuely**, *Galois groups and fundamental groups*, Cambridge Studies in Advanced Mathematics 117, Cambridge Univ. Press (2009) MR

[12] **J B Wagoner**, *Diffeomorphisms, K_2, and analytic torsion*, from "Algebraic and geometric topology, 1" (R J Milgram, editor), Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 23–33 MR

Department of Mathematics, University of California
Los Angeles, CA, United States

Current address: *Department of Mathematics, Stanford University*
Stanford, CA, United States

cohrt@stanford.edu

Received: 28 September 2016 Revised: 24 March 2017