Algebraic & Geometric Topology 17 (2017) 3837–3852

Correction to the articles Homotopy theory of nonsymmetric operads, I–II

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We correct a mistake in Algebr. Geom. Topol. 11 (2011) 1541–1599 on the construction of push-outs along free morphisms of algebras over a nonsymmetric operad, and we fix the affected results from there and a follow-up article (Algebr. Geom. Topol. 14 (2014) 229–281).

18D50, 55U35; 18D10, 18D35, 18D20

Introduction

In [10, Section 8] we give a wrong construction of push-outs along free maps in the category of algebras over an operad. Contrary to what we intended and claimed in the introduction, it does not generalize Harper [6, Proposition 7.32], which is the correct construction. It does not even yield Schwede and Shipley's description [12] of push-outs along free maps in the category of monoids. As Donald Yau pointed out to us, the trivial ring only maps to itself (since it is characterized by the fact that $0 = 1$), but our construction yields $\bigotimes_{n\geq 1} Z^{\otimes n}$ for the coproduct of the trivial ring and the tensor algebra on Z. Here, we fix this mistake and its consequences in Muro [10; 11]. The main results of these papers, presented in their introductions, remain true as stated, modulo a modification in the nonsymmetric monoid axiom [10, Definition 9.1] and a strengthening in the hypotheses of [11, Theorem 1.13 and Corollary 1.14]. These changes do not affect the applications. Moreover, the results which are purely on operads, not on algebras, remain completely unaffected.

1 Push-out filtrations in symmetric monoidal categories

In this section we consider operads $\mathcal{O}$ (always nonsymmetric) and their algebras A in a bicomplete closed symmetric monoidal category $\mathscr{V}$ with tensor product $\otimes$ and tensor unit $\mathbb{I}$, as a preliminary step to the more general case in the following section.

Published: 4 October 2017 DOI: 10.2140/agt.2017.17.3837

We start with Harper's description of the $\mathcal{O}$–algebra push-out

$$
\begin{tikzcd}
\mathcal{F}_{\mathcal{O}}(Y) \arrow[r, "\mathcal{F}_{\mathcal{O}}(f)"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{push}"] & \mathcal{F}_{\mathcal{O}}(Z) \arrow[d, "g'"] \\
A \arrow[r, "f'"'] & B
\end{tikzcd}
\tag{1-1}
$$

Here $\mathcal{F}_{\mathcal{O}}$ is the free $\mathcal{O}$–algebra functor, $\mathcal{F}_{\mathcal{O}}(Y) = \coprod_{n\geq 0} \mathcal{O}(n) \otimes Y^{\otimes n}$, and we denote the adjoint of g by $\bar{g}\colon Y \to A$.

The *enveloping operad* $\mathcal{O}_A$ [5; 4; 3] is characterized by the fact that an operad map $\mathcal{O}_A \to \mathcal{P}$ is the same as an operad map $\mathcal{O} \to \mathcal{P}$ together with an $\mathcal{O}$–algebra map $A \to \mathcal{P}(0)$. Aritywise, $\mathcal{O}_A(t)$ is the (reflexive) coequalizer of the following diagram for $t \geq 0$ — compare [6, Proposition 7.28] —

$$
\begin{gathered}
\coprod_{s\geq 0} \mathcal{O}(s+t) \otimes \left(\frac{\Sigma_{s+t}}{\Sigma_s \times \Sigma_t} \cdot \mathcal{F}_{\mathcal{O}}(A)^{\otimes s} \otimes \mathbb{I}^{\otimes t} \right) \\
\downarrow\downarrow\uparrow \\
\coprod_{s\geq 0} \mathcal{O}(s+t) \otimes \left(\frac{\Sigma_{s+t}}{\Sigma_s \times \Sigma_t} \cdot A^{\otimes s} \otimes \mathbb{I}^{\otimes t} \right)
\end{gathered}
\tag{1-2}
$$

Here, given a permutation $\sigma \in \Sigma_n$ we write $\sigma \cdot X_1 \otimes \cdots \otimes X_n = X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)}$, given a subset $S \subset \Sigma_n$ we set $S \cdot X_1 \otimes \cdots \otimes X_n = \coprod_{\sigma \in S} \sigma \cdot X_1 \otimes \cdots \otimes X_n$, and $\Sigma_{s+t}/\Sigma_s \times \Sigma_t$ identifies with the set of (s,t)–shuffles. The two arrows pointing downwards are defined by the operad structure of $\mathcal{O}$ and the $\mathcal{O}$–algebra structure of A, respectively, and the arrow pointing upwards is given by the unit of $\mathcal{O}$. For $t = 0$, the previous formula reduces to the cotriple presentation of $\mathcal{O}_A(0) = A$.

Recall from [10, Section 4] that a map $f\colon Y \to Z$ in $\mathscr{V}$ is the same as a functor $f\colon \mathbf{2} \to \mathscr{V}$ from the poset $\mathbf{2} = \{0 < 1\}$. Given maps $f_i\colon Y_i \to Z_i$ in $\mathscr{V}$ for $1 \leq i \leq n$, their *push-out product* $f_1 \odot \cdots \odot f_n$ is the latching map of the functor

$$\mathbf{2}^n \xrightarrow{f_1 \otimes \cdots \otimes f_n} \mathscr{C}$$

at the final object $(1, \ldots, 1) \in \mathbf{2}^n$ [7, Definition 15.2.5].

The following lemma is a special case of [6, Proposition 7.32].

Lemma 1.1 *The map f' in* (1-1) *is the transfinite composition of a sequence*

$$A = B_0 \xrightarrow{\varphi_1} B_1 \to \cdots \to B_{t-1} \xrightarrow{\varphi_t} B_t \to \cdots$$

in $\mathscr{V}$ such that the morphism φ_t for $t \geq 1$ is given by the push-out square

$$\begin{tikzcd} \bullet \arrow[r, "\mathcal{O}_A(t)\otimes f^{\odot t}"] \arrow[d, "\psi_t"'] \arrow[dr, phantom, "\text{push}"] & \bullet \arrow[d, "\bar{\psi}_t"] \\ B_{t-1} \arrow[r, "\varphi_t"'] & B_t \end{tikzcd}$$

where the attaching map ψ_t is defined by the following maps for $1 \leq i \leq t$:

$$\mathcal{O}_A(t) \otimes Z^{\otimes(i-1)} \otimes Y \otimes Z^{\otimes(t-i)} \to \mathcal{O}_A(t) \otimes Z^{\otimes(i-1)} \otimes A \otimes Z^{\otimes(t-i)} \xrightarrow{\circ_i} \mathcal{O}_A(t-1) \otimes Z^{\otimes(t-1)} \to B_{t-1},$$

where the first map is defined by $\bar{g}\colon Y \to A$ and the last is $\bar{\psi}_{t-1}$ if $t > 1$ and the identity if $t = 1$.

This lemma is also the arity-0 part of the following one. Observe that the enveloping operad $\mathcal{O}_A$ is functorial on A in the obvious way. Moreover, it is a functor of the pair $(\mathcal{O}, A)$ regarded as a object in the Grothendieck construction of the categories of algebras over all operads.

Lemma 1.2 *If we have an $\mathcal{O}$–algebra push-out* (1-1), *$\mathcal{O}_{f'}\colon \mathcal{O}_A \to \mathcal{O}_B$ is the transfinite composition of a sequence of maps in the category of sequences*

$$\mathcal{O}_A = \mathcal{O}_{B,0} \xrightarrow{\Phi_1} \mathcal{O}_{B,1} \to \cdots \to \mathcal{O}_{B,t-1} \xrightarrow{\Phi_t} \mathcal{O}_{B,t} \to \cdots$$

such that $\Phi_t(n)$ for $t \geq 1$ and $n \geq 0$ is given by the push-out square

$$\begin{tikzcd} \bullet \arrow[r, "\mathcal{O}_A(t+n)\otimes((\Sigma_{t+n}/(\Sigma_t\times\Sigma_n))\cdot f^{\odot t}\otimes \mathbb{I}^{\otimes n})"] \arrow[d, "\Psi_t(n)"'] \arrow[dr, phantom, "\text{push}"] & \bullet \arrow[d, "\bar{\Psi}_t(n)"] \\ \mathcal{O}_{B,t-1}(n) \arrow[r, "\Phi_t(n)"'] & \mathcal{O}_{B,t}(n) \end{tikzcd}$$

where the attaching map $\Psi_t(n)$ is defined, as in Lemma 1.1, from $\bar{g}$, the composition laws $\circ_i\colon \mathcal{O}_A(t+n) \otimes A \to \mathcal{O}_A(t+n-1)$, and also $\bar{\Psi}_{t-1}(n)$ if $t > 1$.

The universal property of $\mathcal{O}_B$ allows us to obtain it as the push-out in the category of operads

(1-3)
$$\begin{tikzcd} \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\tilde{g}"'] \arrow[dr, phantom, "\text{push}"] & \mathcal{F}(Z) \arrow[d] \\ \mathcal{O}_A \arrow[r] & \mathcal{O}_B \end{tikzcd}$$

Here $\mathcal{F}$ is the free operad functor, f is regarded as a map of sequences concentrated in arity 0, and $\widetilde{g}$ is the adjoint of $\overline{g}\colon Y \to A \subset \mathcal{O}_A$. Lemma 1.2 follows from the description of operad pushouts in [10, Section 5].

Assume now that we have a push-out square in the category of operads

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{push}"] & \mathcal{F}(V) \arrow[d, "g'"] \\ \mathcal{O} \arrow[r, "f'"'] & \mathcal{P} \end{tikzcd} \tag{1-4}$$

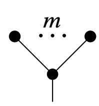
and A is a $\mathcal{P}$–algebra. The universal properties of $\mathcal{P}$ and $\mathcal{P}_A$ show that we have a similar push-out

$$\begin{tikzcd} \mathcal{F}(U)_A \arrow[r, "\mathcal{F}(f)_A"] \arrow[d, "g_A"'] \arrow[dr, phantom, "\text{push}"] & \mathcal{F}(V)_A \arrow[d, "g'_A"] \\ \mathcal{O}_A \arrow[r, "f'_A"'] & \mathcal{P}_A \end{tikzcd} \tag{1-5}$$

The enveloping operad $\mathcal{F}(V)_A$ of an algebra over a free operad admits a description similar to the free operad $\mathcal{F}(V)$; compare [1, Section 3; 10, Section 5]. Namely,

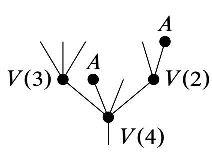
$$\mathcal{F}(V)_A(n) = \coprod_T \bigotimes_{v \in I(T)} \overline{V}(\widetilde{v}). \tag{1-6}$$

Here $\overline{V}$ is the sequence with $\overline{V}(0) = A$ and $\overline{V}(m) = V(m)$ for $m > 0$, T runs over all (isomorphism classes of) trees (planted, planar and with leaves) with n leaves [10, Section 3] which do not contain any *forbidden configuration* for $m \geq 1$,

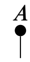


and $I(T)$ is the set of inner vertices of T. Each coproduct factor in (1-6) is usually depicted by labeling each inner vertex v of T with $\overline{V}(\widetilde{v})$, where $\widetilde{v}$ is the number of edges adjacent to v from above, eg

(1-7)



The reason for the forbidden configuration is that we must take into account the $\mathcal{F}(V)$–algebra structure maps $V(m) \otimes A^{\otimes m} \to A$. The operad structure on $\mathcal{F}(V)_A$ is defined by formal tree grafting, applying (repeatedly) if necessary the previous structure maps whenever a forbidden configuration appears, collapsing it to



Hence, the push-out (1-5) admits a filtration description analogous to (1-4). The *level* of a vertex v of T is the number of edges in the shortest path to the root. We say that v is *even* if it has even level. *Odd* vertices are defined similarly. The sets of even and odd inner vertices in T are denoted by $I^e(T)$ and $I^o(T)$, respectively.

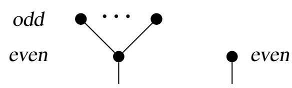
Lemma 1.3 *Given an operad push-out* (1-4) *and a $\mathcal{P}$–algebra A, $f'_A\colon \mathcal{O}_A \to \mathcal{P}_A$ is the transfinite composition of a sequence of maps of sequences*

$$\mathcal{O}_A = \mathcal{P}_{A,0} \xrightarrow{\Phi_1} \mathcal{P}_{A,1} \to \cdots \to \mathcal{P}_{A,t-1} \xrightarrow{\Phi_t} \mathcal{P}_{A,t} \to \cdots$$

such that $\Phi_t(n)$ for $t \geq 1$ and $n \geq 0$ is given by the push-out square

$$\begin{tikzcd}
\bullet \arrow[rr, "\coprod_T \bigodot_{v\in I^e(T)} f(\tilde{v})\otimes\bigotimes_{w\in I^o(T)} \mathcal{O}_A(\tilde{w})"] \arrow[d, "\Psi_t(n)"'] \arrow[drr, phantom, "\text{push}"] & & \bullet \arrow[d, "\overline{\Psi}_t(n)"] \\
\mathcal{P}_{A,t-1}(n) \arrow[rr, "\Phi_t(n)"'] & & \mathcal{P}_{A,t}(n)
\end{tikzcd}$$

where T runs over the isomorphism classes of trees with n leaves concentrated in even levels and t inner even vertices not containing



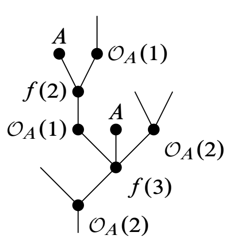
The attaching map $\Psi_t(n)$ is defined by the maps from

$$U(\tilde{u}) \otimes \bigotimes_{v\in I^e(T)\setminus\{u\}} V(\tilde{v}) \otimes \bigotimes_{w\in I^o(T)} \mathcal{O}_A(\tilde{w}), \quad u \in I^e(T),$$

defined by the composite $U \to \mathcal{O} \to \mathcal{O}_A$, composition in $\mathcal{O}_A$, the structure maps $V(m) \otimes A^{\otimes m} \to A$, and the previous $\overline{\Psi}_s(n)$, $s < t$.

Each factor of the coproduct of maps in the statement of the previous lemma is depicted by labeling each even inner vertex v of T with $f(\tilde{v})$, and each odd inner vertex w with $\mathcal{O}_A(\tilde{w})$, eg

(1-8)



The proof of Lemma 1.3 is a slight variation of the explicit construction of the push-out (1-4) given in [10, Section 5]. We believe that this result is new in the literature.

2 Push-out filtrations in nonsymmetric settings

We now turn to our general setting, where operads $\mathcal{O}$ still live in $\mathscr{V}$ but their algebras A live in a bicomplete biclosed monoidal category $\mathscr{C}$ (possibly nonsymmetric) endowed with a strong monoidal left adjoint $z\colon \mathscr{V} \to \mathscr{C}$ which is central, meaning that it is equipped with coherent isomorphisms $z(X) \otimes Y \cong Y \otimes z(X)$. Objects in $\mathscr{V}$ have "underlying" objects in $\mathscr{C}$ via z. We will often drop z from notation. Here we indicate how the three previous lemmas extend to this context.

Enveloping operads do not make sense in this setting since they should live in $\mathscr{C}$, but the definition of operad requires a symmetric tensor product. We must instead consider (always nonsymmetric) *functor-operads* $F = \{F(n)\}_{n \geq 0}$ in $\mathscr{C}$ [9], also known as *multitensors* [2]. They consist of a sequence of functors $F(n)\colon \mathscr{C}^n \to \mathscr{C}$ equipped with composition and unit natural transformations

$$\begin{aligned} \circ_i\colon F(p)(\overset{i-1}{\ldots}, F(q), \overset{p-i}{\ldots}) &\to F(p+q-1), \quad 1 \leq i \leq p,\ q \geq 0, \\ u\colon \mathrm{id}_{\mathscr{C}} &\to F(1), \end{aligned}$$

satisfying relations similar to operads. The values $\mathcal{O}_A(t)(X_1, \ldots, X_t)$ of the *enveloping functor-operad* $\mathcal{O}_A$ are defined by replacing $\mathbb{I}^{\otimes t}$ with $X_1 \otimes \cdots \otimes X_t$ in (1-2). Again, $\mathcal{O}_A(0)$, which is a functor from the discrete category on one object $\mathscr{C}^0$, identifies with A. Enveloping functor-operads satisfy the same functoriality properties as enveloping operads do when $\mathscr{C} = \mathscr{V}$.

Consider the $\mathcal{O}$–algebra push-out (1-1), now in our current setting. As above, we regard maps in $\mathscr{C}$ as functors $\mathbf{2} \to \mathscr{C}$. We now present our first amended statement, where the numbering refers to the cited paper.

Lemma 8.1 [10] *The map f' in* (1-1) *is the transfinite composition of a sequence*

$$A = B_0 \xrightarrow{\varphi_1} B_1 \to \cdots \to B_{t-1} \xrightarrow{\varphi_t} B_t \to \cdots$$

in $\mathscr{C}$ such that the morphism φ_t for $t \geq 1$ is given by the push-out square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\psi_t"'] \arrow[dr, phantom, "push"] & \bullet \arrow[d, "\bar{\psi}_t"] \\ B_{t-1} \arrow[r, "\varphi_t"'] & B_t \end{tikzcd}$$

where the top map is the latching map of $\mathcal{O}_A(t)(f,\dots,f)$ at the final object and the attaching map ψ_t is given by, for $1 \le i \le t$,

$$\mathcal{O}_A(t)(Z,\overset{i-1}{\dots},Z,Y,Z,\overset{t-i}{\dots},Z)\to \mathcal{O}_A(t)(Z,\overset{i-1}{\dots},Z,A,Z,\overset{t-i}{\dots},Z)$$
$$\xrightarrow{\circ_i}\mathcal{O}_A(t-1)(Z,\overset{t-1}{\dots},Z)\to B_{t-1}$$

where the first map is defined by $\bar{g}\colon Y\to A$ and the last is $\bar{\psi}_{t-1}$ if $t>1$ and the identity if $t=1$.

The same proof as in the case $\mathscr{C}=\mathscr{V}$ [6, Proposition 7.32] works here, mutatis mutandis. The lemma actually extends to enveloping functor-operads. Given a functor $F\colon \mathscr{C}^n\to\mathscr{C}$ and a permutation $\sigma\in\Sigma_n$, we let $F\sigma\cdot(X_1,\dots,X_n)=F(X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(n)})$, and given a subset $S\subset\Sigma_n$ we set $FS\cdot(X_1,\dots,X_n)=\coprod_{\sigma\in S}F\sigma\cdot(X_1,\dots,X_n)$.

Lemma 2.1 *If we have an $\mathcal{O}$–algebra push-out* (1-1)*, $\mathcal{O}_{f'}$ is the transfinite composition of a sequence of natural transformations between sequences of functors*

$$\mathcal{O}_A=\mathcal{O}_{B,0}\xrightarrow{\Phi_1}\mathcal{O}_{B,1}\to\cdots\to\mathcal{O}_{B,t-1}\xrightarrow{\Phi_t}\mathcal{O}_{B,t}\to\cdots$$

such that, pointwise, $\Phi_t(n)(X_1,\dots,X_n)$ for $t\ge1$ and $n\ge0$ is given by the push-out

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[d, "\Psi_t(n)(X_1,\dots,X_n)"'] \arrow[drr, phantom, "push"] & & \bullet \arrow[d, "\overline{\Psi}_t(n)(X_1,\dots,X_n)"] \\
\mathcal{O}_{B,t-1}(n)(X_1,\dots,X_n) \arrow[rr, "\Phi_t(n)(X_1,\dots,X_n)"'] & & \mathcal{O}_{B,t}(n)(X_1,\dots,X_n)
\end{tikzcd}$$

where the top map is the latching map of

$$\mathcal{O}_A(t+n)(\Sigma_{t+n}/(\Sigma_t\times\Sigma_n))\cdot(f,\overset{t}{\dots},f,X_1,\dots,X_n)$$

at the final object and the attaching map $\Psi_t(n)(X_1,\dots,X_n)$ is defined, as in Lemma 8.1 above, from $\overline{g}$, the composition laws

$$\circ_i\colon \mathcal{O}_A(t+n)(\dots,A,\dots)\to\mathcal{O}_A(t+n-1),$$

and also $\overline{\Psi}_{t-1}(n)(X_1,\dots,X_n)$ if $t>1$.

This lemma can be proved by fitting Lemma 8.1 above into the coequalizer definition of $\mathcal{O}_B$.

Given a sequence V in $\mathscr{V}$ we identify the object $V(n)$ with the functor $\mathscr{C}^n\to\mathscr{C}\colon (X_1,\dots,X_n)\mapsto V(n)\otimes X_1\otimes\cdots\otimes X_n$. In this way, a sequence in $\mathscr{V}$ can be regarded as a sequence of functors. We similarly identify a map of sequences in $\mathscr{V}$ with the obvious natural transformations. An operad in $\mathscr{V}$ yields a functor-operad in $\mathscr{C}$ through this assignment, and the natural operad map $\mathcal{O}\to\mathcal{O}_A$ becomes a functor-operad map when A is in $\mathscr{C}$.

Lemma 2.2 *Given an operad push-out* (1-4) *and a $\mathcal{P}$–algebra A, the map $f'_A\colon \mathcal{O}_A \to \mathcal{P}_A$ is the transfinite composition of a sequence of natural transformations between sequences of functors*

$$\mathcal{O}_A = \mathcal{P}_{A,0} \xrightarrow{\Phi_1} \mathcal{P}_{A,1} \to \cdots \to \mathcal{P}_{A,t-1} \xrightarrow{\Phi_t} \mathcal{P}_{A,t} \to \cdots$$

such that $\Phi_t(n)(X_1,\dots,X_n)$ for $t \geq 1$ and $n \geq 0$ is given by the push-out square

$$\begin{tikzcd}
\bullet \arrow[rr, "\coprod_T \widetilde{\Phi}_t(T)(X_1,\dots,X_n)"] \arrow[d, "\Psi_t(n)(X_1,\dots,X_n)"'] \arrow[drr, phantom, "push"] & & \bullet \arrow[d, "\overline{\Psi}_t(n)(X_1,\dots,X_n)"] \\
\mathcal{P}_{A,t-1}(n)(X_1,\dots,X_n) \arrow[rr, "\Phi_t(n)(X_1,\dots,X_n)"'] & & \mathcal{P}_{A,t}(n)(X_1,\dots,X_n)
\end{tikzcd}$$

where T runs over the same set of trees as in Lemma 1.3, $\widetilde{\Phi}_t(T)(X_1,\dots,X_n)$ is the latching map at the final object of the functor $\mathbf{2}^t \to \mathscr{C}$ obtained by composing horizontally the natural transformations $f(\widetilde{v})$ for $v \in I^e(T)$ and the functors $\mathcal{O}_A(\widetilde{w})$ for $w \in I^o(T)$ according to the structure of the tree T (see eg (1-8)), *evaluating at $X_1,\dots,X_n$ in the slots indicated by the leaves, and the attaching map $\Psi_t(n)(X_1,\dots,X_n)$ is defined by the composite $U \to \mathcal{O} \to \mathcal{O}_A$, composition in $\mathcal{O}_A$, the structure maps $V(m) \otimes A^{\otimes m} \to A$, and the previous $\overline{\Psi}_s(n)(X_1,\dots,X_n)$ for $s < t$.*

For the proof of this lemma, we can fit the filtration for the bottom map in (1-4) constructed in [10, Section 5] into the coequalizer definition of $\mathcal{P}_A$.

3 Corrected results

We will sometimes restrict to the following class of operads with homotopically well-behaved enveloping (functor-)operads.

Definition 3.1 Suppose that the tensor unit of $\mathscr{V}$ is cofibrant. An operad $\mathcal{O}$ is *excellent* if the functor $A \mapsto \mathcal{O}_A$ takes an $\mathcal{O}$–algebra A with underlying cofibrant object to a cofibrant sequence, and a weak equivalence between $\mathcal{O}$–algebras with underlying cofibrant objects to a weak equivalence of sequences.

The meaning is clear in case $\mathscr{C} = \mathscr{V}$. In the general case we must consider sequences of functors $\mathscr{C}^n \to \mathscr{C}$ for $n \geq 0$ rather than objects in $\mathscr{V}$. Homotopical notions in this more general context will be defined below. When the tensor unit is not cofibrant, the previous definition makes sense but it is not useful. We will also deal with this more general case below.

An operad which is not excellent, with $\mathscr{C} = \mathscr{V}$ the category of chain complexes over a commutative ring, is the operad whose algebras are nonunital DG–algebras A with

$A^3 = 0$. This operad, which has an underlying cofibrant sequence, can be used to construct examples showing the necessity of the excellence assumption in several statements below.

3.2 Corrections to statements

Note that [10, Lemma 8.1] has already been amended in the previous section. We do not repeat it here.

Proposition 9.2(2) [10] *Consider the push-out diagram* (1-1) *in* $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$.

(2) *Suppose that* f *is a cofibration in* $\mathscr{C}$ *and either* A *is cofibrant in* $\mathscr{C}$ *and* $\mathcal{O}$ *is excellent or* A *is cofibrant as an* $\mathcal{O}$*–algebra in* $\mathscr{C}$ *and* $\mathcal{O}(n)$ *is cofibrant in* $\mathscr{V}$ *for* $n \geq 0$. *Then* $f'\colon A \to B$ *is a cofibration in* $\mathscr{C}$.

We will modify not [10, Proposition 9.2(1)] but the definition of the nonsymmetric monoid axiom, so that the statement will be tautologically true by Lemma 8.1 above.

Definition 9.1 [10] The *monoid axiom* in the $\mathscr{V}$–algebra $\mathscr{C}$ says that relative K'–cell complexes are weak equivalences, where K' is the class of morphisms

$$K' = \Big\{ f \otimes X,\ X \otimes f,\ \text{latching map of } \mathcal{O}_A(t)(f, \dots, f) \text{ at the final object} \ \Big| \\ X \text{ is an object in } \mathscr{C},\ f \text{ is a trivial cofibration in } \mathscr{C},\ \mathcal{O} \text{ is an operad in } \mathscr{V}, \\ A \text{ is an } \mathcal{O}\text{–algebra in } \mathscr{C},\ t \geq 1 \Big\}.$$

This axiom is equivalent to Schwede and Shipley [12, Definition 3.3] if $\mathscr{C} = \mathscr{V}$.

The following two modifications are forced by the previous amendments.

(6-2) [11] Replace this equation with the latching map of $\mathcal{O}_A(t)(f, \dots, f)$ at the final object.

Definition 2.3(3) [11] Replace with the new [10, Definition 9.1] above.

Now, in [11, **Theorems 1.13, 8.1 and D.13, Corollaries 1.14 and 8.2 and Propositions 8.3 and D.14**], we must assume in addition that the operad $\mathcal{O}$ is excellent.

The most general of these results is [11, Theorem D.13], which follows from Proposition 3.4.3 and, if the tensor unit is not cofibrant, the remarks in Section 3.7 below.

Note that [11, Lemmas 6.6 and D.1] are not useful any more, since the map [11, (6-2)] plays no role after the corrections.

Corollary D.2 [11] *Suppose that $\mathscr{C}$ satisfies the strong unit axiom and either A is pseudocofibrant in $\mathscr{C}$ and $\mathcal{O}$ is excellent, or A is pseudocofibrant as an $\mathcal{O}$–algebra in $\mathscr{C}$ and $\mathcal{O}(n)$ is cofibrant in $\mathscr{C}$ for $n \geq 0$. Then any cofibration $\phi\colon A \rightarrowtail B$ in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$ is also a cofibration in $\mathscr{C}$.*

3.3 Correct statements needing new proofs

The nonsymmetric monoidal category $\mathrm{Graph}_S(\mathscr{V})$ of $\mathscr{V}$–graphs with object set S [10, Definition 10.1] still satisfies the amended nonsymmetric monoid axiom.

Proof of [10, Proposition 10.3] It is easy to check (using the symmetry of $\mathscr{V}$) that the latching map of $\mathcal{O}_A(t)(f,\dots,f)$ at the final object is componentwise a coproduct of maps, each of which is the tensor product of a single object in $\mathscr{V}$ with a push-out product of components of f, which are trivial cofibrations in $\mathscr{V}$. Such a push-out product is again a trivial cofibration by the push-out product axiom. Hence, any K'–cell complex is componentwise a K–cell complex in the sense of [10, Definition 6.1], and therefore a weak equivalence by the monoid axiom for $\mathscr{V}$. □

The modifications made to [10, Proposition 9.2] have no impact on [10, Lemma 9.4 and Corollary 9.5], however [10, Lemma 9.6 and Theorem 1.3; 11, Theorems 6.7 and D.4] require a new proof. They follow from the arity-0 part of Proposition 3.4.2 and, if the tensor unit is not cofibrant, the remarks in 3.7 below. The modification in [11, (6-2)] forces us to give new proofs of [11, Propositions 7.3 and D.6]. They follow from the arity-0 part of Proposition 3.4.6 and Section 3.7.

3.4 Auxiliary results

We need the following results to prove the amended statements and to fix proofs of correct statements affected by the amendments.

Proposition 3.4.1 *Let $\mathcal{O}$ be an operad with underlying cofibrant sequence. For any cofibration with cofibrant source $f'\colon A \to B$ in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$, the map $\mathcal{O}_{f'}\colon \mathcal{O}_A \to \mathcal{O}_B$ is a cofibration of sequences. In particular, $\mathcal{O}_A$ is a cofibrant sequence for any cofibrant A in $\mathrm{Alg}_{\mathscr{C}}(\mathcal{O})$.*

Proposition 3.4.2 *Let $\phi\colon \mathcal{O} \xrightarrow{\sim} \mathcal{P}$ be a weak equivalence in $\mathrm{Op}(\mathscr{V})$. Assume that the objects $\mathcal{O}(n)$ and $\mathcal{P}(n)$ are cofibrant in $\mathscr{V}$ for all $n \geq 0$. Given a cofibrant $\mathcal{O}$–algebra A in $\mathscr{C}$, the map $\phi_{\eta_A}\colon \mathcal{O}_A \to \mathcal{P}_{\phi_* A}$ induced by ϕ and by the unit $\eta_A\colon A \to \phi^*\phi_* A$ of the change of operad adjunction $\phi_* \dashv \phi^*$ [10, (1)] is a weak equivalence of sequences.*

Proposition 3.4.3 *If $\mathcal{O}$ is an excellent operad in $\mathscr{V}$ and*

$$\begin{tikzcd} A \arrow[r, tail, "\psi"] \arrow[d, "\varphi"', "\sim"] \arrow[dr, phantom, "push"] & B \arrow[d, "\varphi'"] \\ C \arrow[r, tail, "\psi'"'] & C \cup_A B \end{tikzcd}$$

is a push-out of $\mathcal{O}$–algebras in $\mathscr{C}$ such that the underlying objects of A and C are cofibrant, then φ' is a weak equivalence.

The goal of the two following results is to exhibit a huge class of excellent operads. They are not strictly required to correct the results in [10; 11], but they are essential for applications. We believe that these results, which imply homotopy invariance of enveloping (functor-)operads, are new in the literature in this generality. Similar results for chain complexes have been obtained in [4, Section 17.4].

Proposition 3.4.4 *The initial operad $\mathtt{Ass}^{\mathscr{V}}$, and $\mathtt{uAss}^{\mathscr{V}}$, are excellent.*

Proposition 3.4.5 *If $f'\colon \mathcal{O} \rightarrowtail \mathcal{P}$ is a cofibration in $\mathrm{Op}(\mathscr{V})$ and $\mathcal{O}$ is an excellent operad such that $\mathcal{O}(n)$ is cofibrant for all $n \geq 0$, then so is $\mathcal{P}$.*

In the following result, in addition to our standing context $(\mathscr{V}, \mathscr{C})$ we have another one $(\mathscr{W}, \mathscr{D})$ satisfying the same formal properties. Both of them are related by Quillen pairs, $F\colon \mathscr{V} \rightleftarrows \mathscr{W} :G$ and $\bar{F}\colon \mathscr{C} \rightleftarrows \mathscr{D} :\bar{G}$, with colax monoidal left adjoints F and $\bar{F}$, equipped with a coherent natural map $\tau(X)\colon \bar{F}z(X) \to zF(X)$ which is a weak equivalence for X cofibrant [11, Section 7]. They give rise to a functor between operad categories $F^{\mathrm{oper}}\colon \mathrm{Op}(\mathscr{V}) \to \mathrm{Op}(\mathscr{W})$ and, for each operad $\mathcal{O}$ in $\mathscr{V}$, a functor between algebra categories $\bar{F}_{\mathcal{O}}\colon \mathrm{Alg}_{\mathscr{C}}(\mathcal{O}) \to \mathrm{Alg}_{\mathscr{D}}(F^{\mathrm{oper}}(\mathcal{O}))$. These functors are left adjoint to the obvious functors defined by the lax monoidal functors G and $\bar{G}$. In particular, we obtain a map of sequences $\chi_{\mathcal{O}}\colon F(\mathcal{O}) \to F^{\mathrm{oper}}(\mathcal{O})$ and natural transformations $\chi_{\mathcal{O},A}(n)\colon \bar{F}\mathcal{O}_A(n) \to F^{\mathrm{oper}}(\mathcal{O})_{\bar{F}_{\mathcal{O}}(A)}(n)\bar{F}^{\times n}$ between functors $\mathscr{C}^n \to \mathscr{D}$ for $n \geq 0$ for any $\mathcal{O}$–algebra A.

Proposition 3.4.6 *If $\bar{F} \dashv \bar{G}$ is a weak monoidal Quillen adjunction, $\mathscr{V}$ and $\mathscr{W}$ have cofibrant tensor units, $\mathcal{O}$ is a cofibrant operad in $\mathscr{V}$ and A is a cofibrant $\mathcal{O}$–algebra in $\mathscr{C}$, then $\chi_{\mathcal{O},A}(n)$ is a weak equivalence in $\mathscr{D}$ when evaluated at n cofibrant objects in $\mathscr{C}$ for $n \geq 0$.*

3.5 Proofs for $\mathscr{C} = \mathscr{V}$ and cofibrant tensor units

In this special case our results admit easier proofs which do not need the sophisticated homotopical notions for functors introduced below.

Proof of Proposition 9.2(2) By Lemma 1.1 and the usual inductive transfinite composition and retract argument, this boils down to proving that $\mathcal{O}_A(t)\otimes f^{\odot t}$ is a cofibration for all $t\geq 1$. If the sequence $\mathcal{O}_A$ is cofibrant, this follows from the push-out product axiom. This sequence is cofibrant under the first set of hypotheses, by excellence. The second case is the arity-0 part of Proposition 3.4.1. □

Proof of Proposition 3.4.1 By Lemma 1.2, the usual transfinite composition and retract argument, and the push-out product axiom, it suffices to notice that the map $\mathcal{O}_A(t+n)\otimes f^{\odot t}$ for $t\geq 1$ and $n\geq 0$ is a cofibration provided f is a cofibration and $\mathcal{O}_A$ is a cofibrant sequence, and the sequence $\mathcal{O}_A$ is cofibrant if $A=\mathcal{O}(0)$ is the initial $\mathcal{O}$–algebra, since $\mathcal{O}_{\mathcal{O}(0)}=\mathcal{O}$. □

Proof of Proposition 3.4.2 This follows from the proof of [3, Proposition 5.7], but we here give an argument which extends to our general case. By the aforementioned inductive argument and Lemma 1.2, it is enough to check that the statement holds for $A=\mathcal{O}(0)$ the initial $\mathcal{O}$–algebra and that, assuming the result true for A, the map ϕ_{η_A} induces a weak equivalence of cofibrations $\phi_{\eta_A}(t+n)\otimes f^{\odot t}\colon \mathcal{O}_A(t+n)\otimes f^{\odot t}\to \mathcal{P}_{\phi_*A}(t+n)\otimes f^{\odot t}$, with cofibrant source and target, for f a cofibration as in (1-1). For $A=\mathcal{O}(0)$, $\phi_{\eta_{\mathcal{O}(0)}}=\phi\colon \mathcal{O}\to\mathcal{P}$, which is a weak equivalence by hypothesis. For any cofibrant $\mathcal{O}$–algebra A, $\mathcal{O}_A$ and $\mathcal{P}_{\phi_*A}$ are cofibrant sequences by Proposition 3.4.1. We can assume that $f\colon Y\rightarrowtail Z$ has cofibrant source, replacing it with its push-out $A\rightarrowtail Z\cup_Y A$ along $\bar{g}$ if necessary. Hence, by the push-out product axiom, $\phi_{\eta_A}(t+n)\otimes f^{\odot t}$ is indeed a weak equivalence between cofibrations with cofibrant source and target. □

Proof of Proposition 3.4.3 By the previous inductive argument, we can assume that $\psi=f'$ in (1-1) with f a cofibration. We can also suppose as in the proof of Proposition 3.4.2 that the source of f is cofibrant. By Lemma 1.1, it suffices to notice that

$$\mathcal{O}_\varphi(t)\otimes f^{\odot t}\colon \mathcal{O}_A(t)\otimes f^{\odot t}\to \mathcal{O}_C(t)\otimes f^{\odot t}$$

is a weak equivalence between cofibrations with cofibrant source. Here we use excellence and the push-out product axiom. □

Proof of Proposition 3.4.4 This follows from the fact that $\mathtt{uAss}^{\mathscr{V}}_A(n)=A^{\otimes(n+1)}$ for $n\geq 0$, $\mathtt{Ass}^{\mathscr{V}}_A(0)=A$ and $\mathtt{Ass}^{\mathscr{V}}_A(n)=(A\amalg\mathbb{I})^{\otimes(n+1)}$ for $n\geq 1$, and, for $\mathcal{O}$ the initial operad, $\mathcal{O}_A(0)=A$, $\mathcal{O}_A(1)=\mathbb{I}$ and $\mathcal{O}_A(n)=\varnothing$ for $n\geq 2$. □

Proof of Proposition 3.4.5 As in previous proofs, we can assume that f' fits into a push-out square (1-3) with f a cofibration between cofibrant sequences. Let A be an

$\mathcal{O}$–algebra with underlying cofibrant object and $\varphi\colon A\to B$ a weak equivalence between such $\mathcal{O}$–algebras. By Lemma 1.3, it is enough to notice that, a push-out product of maps $f(n)$ tensored with objects $\mathcal{O}_A(n)$ for $n\geq 0$ is a cofibration between cofibrant objects. Moreover, if we replace $\mathcal{O}_A(n)$ with $\mathcal{O}_\varphi(n)$ we get a weak equivalence between these cofibrations. We are using here the push-out product axiom and the excellence assumption. □

Proof of Proposition 3.4.6 Under the standing hypotheses of this subsection, $\mathscr{C}=\mathscr{V}$, $\mathscr{D}=\mathscr{W}$, and $\chi_{\mathcal{O},A}\colon F(\mathcal{O}_A)\to F^{\mathrm{oper}}(\mathcal{O})_{F_{\mathcal{O}}(A)}$ is a map of sequences in $\mathscr{W}$. If A is the initial $\mathcal{O}$–algebra then $\chi_{\mathcal{O},A}=\chi_{\mathcal{O}}$, so the statement follows from [11, Proposition 4.2]. For general cofibrant $\mathcal{O}$–algebras, using the inductive argument and Lemma 1.2, it suffices to notice that, if the result holds for A and f is a cofibration between cofibrant objects in $\mathscr{V}$, then the map $F(\mathcal{O}_A(t+n)\otimes f^{\odot t})\to F^{\mathrm{oper}}(\mathcal{O})_{F_{\mathcal{O}}(A)}(t+n)\otimes F(f)^{\odot t}$ for $t\geq 1$ and $n\geq 0$ induced by the comultiplication of F and $\chi_{\mathcal{O},A}$ is a weak equivalence between cofibrations with cofibrant source (and target). Here we are using the push-out product axiom and the cofibrancy results in Proposition 3.4.1 and [11, Corollary 3.8 and Lemma 4.3]. □

3.6 Proofs for $\mathscr{C}\neq\mathscr{V}$ and cofibrant tensor units

The Reedy model structure [7, Section 15.3] on the category of diagrams indexed by $\mathbf{2}^n$ can be generalized as follows.

Proposition 3.6.1 *If $\mathscr{M}$ is a model category and $S\subset\{1,\dots,n\}$, there is a model structure $\mathscr{M}^{\mathbf{2}^n}_S$ on the diagram category $\mathscr{M}^{\mathbf{2}^n}$ such that a map $\tau\colon F\to G$ is*

- *a fibration if $\tau(x_1,\dots,x_n)\colon F(x_1,\dots,x_n)\to G(x_1,\dots,x_n)$ is a fibration in $\mathscr{M}$ for all $(x_1,\dots,x_n)$ in $\mathbf{2}^n$,*
- *a weak equivalence if $\tau(x_1,\dots,x_n)$ is a weak equivalence in $\mathscr{M}$ for all $(x_1,\dots,x_n)$ in $\mathbf{2}^n$ with $x_i=0$ if $i\in S$, and*
- *a cofibration if the relative latching map of τ at any $(x_1,\dots,x_n)$ in $\mathbf{2}^n$ is a cofibration, and moreover a trivial cofibration if $x_i=1$ for some $i\in S$.*

Note that $\mathscr{M}^{\mathbf{2}^n}_S$ is a right Bousfield localization of the Reedy model structure $\mathscr{M}^{\mathbf{2}^n}_\varnothing$. Cofibrant diagrams take cofibrant values and have cofibrant latching objects. Moreover, any weak equivalence between cofibrant functors induces weak equivalences between latching objects.

Given model categories $\mathscr{M}$ and $\mathscr{N}$, we introduce some naive homotopical notions for functors of several variables between them. They rely on the previous model structures, hence many facts from ordinary model categories extend to these big functor categories.

Definition 3.6.2 A natural transformation $\tau\colon F \to G$ between functors $\mathscr{N}^n \to \mathscr{M}$ is a *weak equivalence*, *fibration* or *cofibration* if, given cofibrations between cofibrant objects $g_1, \dots, g_n$ in $\mathscr{N}$, $\tau(g_1, \dots, g_n)$ has that property in $\mathscr{M}_S^{\mathbf{2}^n}$ for any $S \subset \{1, \dots, n\}$ such that g_i is a trivial cofibration if $i \in S$. These notions extend aritywise to sequences of functors $F(n)\colon \mathscr{N}^n \to \mathscr{M}$ for $n \geq 0$.

Weak equivalences and fibrations can be just characterized pointwise on cofibrant objects. The condition on cofibrations is stronger. For $n = 0$ we recover the original notions in $\mathscr{M}$. Cofibrant functors preserve cofibrant objects and weak equivalences between them.

When $\mathscr{M} = \mathscr{N}$, we can horizontally compose functors of several variables $\mathscr{M}^n \to \mathscr{M}$, ie $F(\dots, G, \dots)$, and natural transformations between them. Weak equivalences are preserved by horizontal composition if source and target are cofibrant. Cofibrant functors are also preserved, provided we compose at a colimit-preserving slot. All slots in enveloping functor-operads preserve colimits.

The meaning of Definition 3.1 is now clear in the general case for cofibrant tensor units. The proofs in the previous subsection extend straightforwardly, using Lemmas 8.1 above, 2.1 and 2.2 instead of Lemmas 1.1, 1.2 and 1.3, respectively.

3.7 Noncofibrant tensor units

In the proofs of Section 3.5 we have used that the tensor unit is cofibrant at some places. This hypothesis can be relaxed using the theory of *pseudocofibrant* and $\mathbb{I}$*–cofibrant* objects developed in [11, Appendices A and B] provided our monoidal model categories satisfy the *strong unit axiom* and all left Quillen functors satisfy the *pseudocofibrant* and $\mathbb{I}$*–cofibrant axioms*. These will be standing assumptions. We recently learned that pseudocofibrant objects were previously introduced in [8], where they are called semicofibrant.

For $\mathscr{C} = \mathscr{V}$, Definition 3.1 must be modified replacing cofibrancy with pseudocofibrancy. Proposition 3.4.4 holds with the same proof. Essentially the same proofs work for Propositions 3.4.2 and 3.4.3 if we only demand underlying pseudocofibrant objects. Moreover, if we only make pseudocofibrancy hypotheses in Propositions 3.4.1 and 3.4.5, we obtain pseudocofibrant outcomes and honest cofibrations between them. Proposition 3.4.6 holds without $\mathbb{I}$ being cofibrant under our standing assumptions (using [11, Corollary C.3 and Lemma B.14] in the proof). The proof of Corollary D.2 is similar to the proof of Proposition 9.2(2) above.

For $\mathscr{C} \neq \mathscr{V}$, we need new and modified homotopical notions in functor categories. *Pseudocofibrant* and $\mathbb{I}$*–cofibrant* objects F in diagram categories $\mathscr{M}_S^{\mathbf{2}^n}$ are defined by the existence of a cofibration $X \rightarrowtail F$ from a constant diagram on an object X satisfying the corresponding property in $\mathscr{M}$ (which must be monoidal). In Definition 3.6.2, we allow the sources of the g_i to be pseudocofibrant.

The operads in Proposition 3.4.4 are excellent since

$$\mathtt{uAss}_A^{\mathscr{V}}(n)(X_1,\dots,X_n) = A \otimes \bigotimes_{i=1}^{n}(X_i \otimes A) \quad \text{for } n \geq 0,$$

$\mathtt{Ass}_A^{\mathscr{V}}(n)(X_1,\dots,X_n) = (A \amalg \mathbb{I}) \otimes \bigotimes_{i=1}^{n}(X_i \otimes (A \amalg \mathbb{I}))$ for $n \geq 1$ and, for $\mathcal{O}$ the initial operad, $\mathcal{O}_A(1)(X_1) = X_1$ and $\mathcal{O}_A(n)(X_1,\dots,X_n) = \varnothing$ for $n \geq 2$. In Proposition 3.4.1, if we only demand that $\mathcal{O}$ is pseudocofibrant in $\mathscr{C}$ we obtain as outcomes functor-operads with underlying pseudocofibrant sequences and cofibrations between them. In Proposition 3.4.2, the map of sequences in $\mathscr{C}$ underlying ϕ must be a weak equivalence between pseudocofibrant objects. Propositions 3.4.3 and 3.4.5 are true when the underlying objects are pseudocofibrant in $\mathscr{C}$. The analog of Proposition 9.2(2) for noncofibrant $\mathbb{I}$ is Corollary D.2 above.

For the proof of Proposition 3.4.6 without cofibrant tensor units, we must modify again the homotopical notions in Definition 3.6.2, allowing the sources of the g_i to be just ($\mathbb{I}$–)cofibrant. The natural transformations $\chi_{\mathcal{O},A}(n)$ are weak equivalences in this sense, ie when evaluated at ($\mathbb{I}$–)cofibrant objects.

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Received: 24 July 2015 Revised: 8 March 2017