

The closed-open string map for S^1 -invariant Lagrangians

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Given a monotone Lagrangian submanifold invariant under a loop of Hamiltonian diffeomorphisms, we compute a piece of the closed-open string map into the Hochschild cohomology of the Lagrangian which captures the homology class of the loop's orbit.

Our applications include split-generation and nonformality results for real Lagrangians in projective spaces and other toric varieties; a particularly basic example is that the equatorial circle on the 2-sphere carries a nonformal Fukaya A_∞ algebra in characteristic 2.

[53D37](#), [53D40](#), [57R58](#); [53D45](#)

1 Introduction

1.1 Overview of main results

Let X be a compact monotone symplectic manifold, $L \subset X$ a monotone Lagrangian submanifold, and \mathbb{K} a field. We assume that L satisfies the usual conditions making its Floer theory well defined over \mathbb{K} , namely, L has Maslov index at least 2 and is oriented and spin if $\text{char } \mathbb{K} \neq 2$. In this case, one can define a unital algebra over \mathbb{K} , the Floer cohomology $\text{HF}^*(L, L)$, which is invariant under Hamiltonian isotopies of L . A larger amount of information about L is captured by the Fukaya A_∞ algebra of L , and given this A_∞ algebra, one can build another associative unital algebra called the Hochschild cohomology $\text{HH}^*(L, L)$. There is the so-called (full) closed-open string map

$$\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(L, L),$$

which is a map of unital algebras, where $\text{QH}^*(X)$ is the (small) quantum cohomology of X . This map is of major importance in symplectic topology, particularly in light of Abouzaid's split-generation criterion (see Abouzaid [1], Sheridan [45] and Ritter and Smith [38]), one of whose versions in the case $\text{char } \mathbb{K} = 2$ says the following: if the closed-open map is *injective*, then L split-generates the w -summand $\mathcal{Fuk}(X)_w$ of the Fukaya category, where $w = w(L) \in \mathbb{K}$ is the so-called obstruction number of L .

(When $\text{char } \mathbb{K} \neq 2$, the hypothesis can be weakened to say that \mathcal{CO}^* is injective on a relevant eigensummand of $\text{QH}^*(X)$; we will recall this later.)

Split-generation of the Fukaya category $\mathcal{Fuk}(X)_w$ by a Lagrangian submanifold L is an algebraic phenomenon which has important geometric implications. For example, in this case, L must have nonempty intersection with any other monotone Lagrangian submanifold L' which is a nontrivial object in $\mathcal{Fuk}(X)_w$, namely such that $\text{HF}^*(L', L') \neq 0$ and $w(L') = w$. Another application, though not discussed here, is that split-generation results are used in proofs of homological mirror symmetry.

The present paper contributes new calculations of the closed-open map, motivated by the split-generation criterion and the general lack of explicit calculations known so far. (The closed-open map is defined by counting certain pseudoholomorphic disks with boundary on L , which makes it extremely hard to compute in general.)

There is a simplification of the full closed-open map, called the “zeroth-order” closed-open map, which is a unital algebra map

$$\mathcal{CO}^0: \text{QH}^*(X) \rightarrow \text{HF}^*(L, L).$$

It is the composition of \mathcal{CO}^* with the canonical projection $\text{HH}^*(L, L) \rightarrow \text{HF}^*(L, L)$, and if \mathcal{CO}^0 is injective, so is \mathcal{CO}^* (but not vice versa). Although \mathcal{CO}^0 generally carries less information than \mathcal{CO}^* , it is sometimes easier to compute. For example, we compute \mathcal{CO}^0 when L is the real locus of a complex toric Fano variety X ; see [Theorem 1.13](#). This map turns out to be noninjective in many cases, eg for $\mathbb{R}P^{2n+1} \subset \mathbb{C}P^{2n+1}$ over a characteristic-2 field. The aim of the present paper is to study the higher-order terms of the full closed-open map \mathcal{CO}^* , and to find examples when \mathcal{CO}^* is injective but \mathcal{CO}^0 is not.

Specifically, let us consider the following setting: a loop γ of Hamiltonian symplectomorphisms preserves a Lagrangian L setwise. Let $S(\gamma) \in \text{QH}^*(X)$ be the Seidel element of γ ; then from Charette and Cornea [\[11\]](#), one can see that

$$\mathcal{CO}^0(S(\gamma)) = 1_L,$$

where 1_L denotes the unit in $\text{HF}^*(L, L)$. Our main result, [Theorem 1.7](#), is a tool for distinguishing $\mathcal{CO}^*(S(\gamma))$ from the Hochschild cohomology unit in $\text{HH}^*(L, L)$; this way, it captures a nontrivial piece of the full closed-open map \mathcal{CO}^* not seen by \mathcal{CO}^0 . We apply [Theorem 1.7](#) to show that \mathcal{CO}^* is injective for some real Lagrangians in toric manifolds and also for monotone toric fibres which correspond to (non-Morse) A_2 -type critical points of the Landau–Ginzburg superpotential.

After this paper had appeared, Evans and Lekili [19] proved split-generation for all orientable real toric Lagrangians, and all monotone toric fibres in zero characteristic, by completely different methods. They make use of the fact that these are homogeneous Lagrangians (ie they are orbits of Hamiltonian group actions), while we only use the fact these Lagrangians are invariant under certain Hamiltonian loops.

We will now mention our examples regarding real Lagrangians and postpone all discussion of monotone toric fibres, along with an introductory part, to [Section 4](#).

Proposition 1.1 *Let \mathbb{K} be a field of characteristic 2 and $\mathbb{R}P^n$ the standard real Lagrangian in $\mathbb{C}P^n$. Then $\mathcal{CO}^*: \mathrm{QH}^*(\mathbb{C}P^n) \rightarrow \mathrm{HH}^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is injective for all n . In contrast, $\mathcal{CO}^0: \mathrm{QH}^*(\mathbb{C}P^n) \rightarrow \mathrm{HF}^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is injective if and only if n is even.*

Corollary 1.2 *Over a field of characteristic 2, $\mathbb{R}P^n$ split-generates $\mathcal{Fuk}(\mathbb{C}P^n)_0$.*

As hinted above, this corollary leads to a result on nondisplaceability of $\mathbb{R}P^n$ from other monotone Lagrangians which are Floer-theoretically nontrivial. This has been known due to Biran and Cornea [8, Corollary 8.1.2] and Entov and Polterovich [18]. Very recently, Konstantinov [29] showed that the Chiang Lagrangian in $\mathbb{C}P^3$ admits a higher-rank local system making it Floer-theoretically nontrivial over a characteristic-2 field; then he concludes via [Corollary 1.2](#) that the Chiang Lagrangian is nondisplaceable from $\mathbb{R}P^3$. It is possible that for Lagrangians with higher-rank local systems, a generalisation of [8] and [18] can be invoked instead of [Corollary 1.2](#), but we have not checked this.

We can extract another interesting consequence about projective spaces from our main computation of the closed-open map.

Proposition 1.3 *The Fukaya A_∞ algebra of the Lagrangian $\mathbb{R}P^{4n+1} \subset \mathbb{C}P^{4n+1}$ is not formal over a characteristic-2 field for any $n \geq 0$.*

Here formality means an existence of a quasi-isomorphism with the associative algebra $\mathrm{HF}^*(\mathbb{R}P^{4n+1}, \mathbb{R}P^{4n+1}) \cong \mathbb{K}[u]/(u^{4n+2} - 1)$ considered as an A_∞ algebra with trivial higher-order structure maps. In particular, the Fukaya A_∞ algebra of the equator $S^1 \subset S^2$ is not formal in characteristic 2; we devote a separate discussion to this fact in [Section 3](#), where we explicitly exhibit a nontrivial Massey product which provides an alternative proof of the nonformality. Below is another example of split-generation which we can prove using the same methods.

Proposition 1.4 *Let \mathbb{K} be a field of characteristic 2, $X = \text{Bl}_{\mathbb{C}P^1} \mathbb{C}P^9$ the blowup of $\mathbb{C}P^9$ along a complex line which intersects $\mathbb{R}P^9$ in a circle, and $L \subset X$ the blowup of $\mathbb{R}P^9$ along that circle. Then \mathcal{CO}^* : $\text{QH}^*(X) \rightarrow \text{HH}^*(L, L)$ is injective, although \mathcal{CO}^0 : $\text{QH}^*(X) \rightarrow \text{HF}^*(L, L)$ is not. Consequently, L split-generates $\mathcal{Fuk}(X)_0$.*

(The manifold $\text{Bl}_{\mathbb{C}P^1} \mathbb{C}P^9$ is the first instance among $\text{Bl}_{\mathbb{C}P^k} \mathbb{C}P^n$ for which L is monotone of Maslov index at least 2 and such that \mathcal{CO}^0 is not injective; the last requirement makes the use of our general results essential in this example.) In general, it is known that the real Lagrangian in a toric Fano variety is not displaceable from the monotone toric fibre; this was proved by Alston and Amorim [5]. Proposition 1.4 implies a much stronger nondisplaceability result like the one which has been known for $\mathbb{R}P^n \subset \mathbb{C}P^n$.

Corollary 1.5 *Let \mathbb{K} and $L \subset X$ be as in Proposition 1.4, and $L' \subset X$ any other monotone Lagrangian, perhaps equipped with a local system $\pi_1(L') \rightarrow \mathbb{K}^\times$, with minimal Maslov number at least 2 and such that $\text{HF}^*(L', L') \neq 0$. If $w(L') \neq 0$, we also assume the technical Hypothesis 3.9, which is expected to hold following Ganatra [22]. Then $L \cap L' \neq \emptyset$.*

Here $\text{HF}^*(L', L')$ denotes the Floer cohomology of L' with respect to the local system ρ , so a better notation would be $\text{HF}^*((L', \rho), (L', \rho))$. For brevity, we decided to omit ρ from our notation of Floer and Hochschild cohomologies throughout the article when it is clear that a Lagrangian is equipped with such a local system. The point of allowing local systems in Corollary 1.5 is to introduce more freedom in achieving the nonvanishing of $\text{HF}^*(L', L')$.

Note that Corollary 1.5 does not require that the obstruction number of L' matches the one of L , namely zero. If $w(L') \neq 0$, we can pass to $X \times X$ noticing that $w(L' \times L') = 2w(L') = 0$ and similarly $w(L \times L) = 0$, so we have well-defined Floer theory between the two product Lagrangians. This trick was observed by Abreu and Macarini [3] and has also been used in Alston and Amorim [5]. So it suffices to show that $L \times L$ split-generates $\mathcal{Fuk}(X \times X)_0$; this follows from Proposition 1.4 by the general expectation that the condition of the Abouzaid's split-generation criterion is "preserved" under Künneth isomorphisms. As we explain later, this general expectation is contingent upon a certain commutative diagram which we formulate as Hypothesis 3.9, and which is largely substantiated by Ganatra [22]; see also Abouzaid and Smith [2] and Amorim [6].

As in the case of $\mathbb{R}P^n$, we also prove a nonformality statement.

Proposition 1.6 *The Fukaya A_∞ algebra of the Lagrangian $\text{Bl}_{\mathbb{R}P^1} \mathbb{R}P^9 \subset \text{Bl}_{\mathbb{C}P^1} \mathbb{C}P^9$ from Proposition 1.4 is not formal over a characteristic-2 field.*

Although we cannot prove that \mathcal{CO}^* is injective for the real locus of an arbitrary toric Fano variety, we are able to do this in a slightly wider range of examples, which we postpone to Section 3. We will prove Proposition 1.1 and Corollary 1.2 at the end of the introduction, and the remaining statements from above will be proved in Section 3. Now we state the main theorem; the new pieces of notation are explained right after the statement.

Theorem 1.7 *Let X be a compact monotone symplectic manifold and $L \subset X$ a monotone Lagrangian submanifold of Maslov index at least 2, possibly equipped with a local system $\rho: H_1(L) \rightarrow \mathbb{K}^\times$. If $\text{char } \mathbb{K} \neq 2$, assume L is oriented and spin.*

Let $\gamma = \{\gamma_t\}_{t \in S^1}$ be a loop of Hamiltonian symplectomorphisms of X , and denote by $S(\gamma) \in \text{QH}^(X)$ the corresponding Seidel element. Suppose the loop γ preserves L setwise, that is, $\gamma_t(L) = L$. Denote by $l \in H_1(L)$ the homology class of an orbit $\{\gamma_t(q)\}_{t \in S^1}$, $q \in L$. Finally, assume $\text{HF}^*(L, L) \neq 0$.*

- (a) *Then $\mathcal{CO}^0(S(\gamma)) = (-1)^{\epsilon(l)} \cdot \rho(l) \cdot 1_L$, where $1_L \in \text{HF}^*(L, L)$ is the unit.*
- (b) *Suppose there exists no $a \in \text{HF}^*(L, L)$ such that*
- (*) *$\mu^2(a, \Phi(y)) + \mu^2(\Phi(y), a) = (-1)^{\epsilon(l)} \rho(l) \cdot \langle y, l \rangle \cdot 1_L$ for each $y \in H^1(L)$.*

Then $\mathcal{CO}^(S(\gamma)) \in \text{HH}^*(L, L)$ is linearly independent from the Hochschild cohomology unit.*

- (c) *More generally, suppose $Q \in \text{QH}^*(X)$ and there exists no $a \in \text{HF}^*(L, L)$ such that*
- (**) *$\mu^2(a, \Phi(y)) + \mu^2(\Phi(y), a)$*

$$= (-1)^{\epsilon(l)} \rho(l) \cdot \langle y, l \rangle \cdot \mathcal{CO}^0(Q) \quad \text{for each } y \in H^1(L).$$

Then $\mathcal{CO}^(S(\gamma) * Q)$ and $\mathcal{CO}^*(Q)$ are linearly independent in the Hochschild cohomology $\text{HH}^*(L, L)$.*

Here μ^2 is the product on $\text{HF}^*(L, L)$, $\langle -, - \rangle$ is the pairing $H^1(L) \otimes H_1(L) \rightarrow \mathbb{K}$, and $S(\gamma) * Q$ is the quantum product of the two elements. Next,

$$\Phi: H^1(L) \rightarrow \text{HF}^*(L, L)$$

is the PSS map of Albers [4], which is canonical and well defined if $\text{HF}^*(L, L) \neq 0$. Its well-definedness in a setting closer to ours was studied by eg Biran and Cornea [9],

and later we discuss it in more detail. Note that Φ is not necessarily injective, although in our applications, where $\mathrm{HF}^*(L, L) \cong H^*(L)$, it will be. Finally, in the theorem, we have allowed L to carry an arbitrary local system, which modifies the Fukaya A_∞ structure of L by counting the same punctured holomorphic disks as in the case without a local system with coefficients which are the values of ρ on the boundary loops of such disks. The algebras $\mathrm{HF}^*(L, L)$ and $\mathrm{HH}^*(L, L)$ get modified accordingly, although their dependence on ρ is not reflected by our notation, as mentioned earlier. We allow nontrivial local systems in view of our application to toric fibres and will only need the trivial local system $\rho \equiv 1$ for applications to real Lagrangians.

To complete the statement of [Theorem 1.7](#), we need to explain the appearance of the sign $(-1)^{\epsilon(l)} = \pm 1$. By a spin Lagrangian, we always mean a Lagrangian with a fixed spin structure (rather than admitting one). We have two natural trivialisations of TL over the loop $\gamma_t(q) \subset L$: the one induced from a fixed basis of T_qL by the Hamiltonian loop γ and the one determined by the spin structure on L . We put $\epsilon(l)$ to be 0 if the two trivialisations agree and 1 otherwise.

Outline of proof We mentioned earlier that part (a) of [Theorem 1.7](#) is an easy consequence of the paper by Charette and Cornea [\[11\]](#). The proof of parts (b) and (c) also starts by using a result from that paper, and then the main step is an explicit computation of

$$\mathcal{CO}^1(S(\gamma))|_{\mathrm{CF}^1(L, L)}: \mathrm{CF}^1(L, L) \rightarrow \mathrm{CF}^0(L, L)$$

on the cochain level, which turns out to be dual to taking the γ -orbit of a point up to the factor $(-1)^{\epsilon(l)}\rho(l)$: this is [Proposition 2.8](#). The final step is to check whether the computed nontrivial piece of the Hochschild cocycle $\mathcal{CO}^*(S(\gamma))$ survives to cohomology; this is controlled by equations [\(*\)](#) and [\(**\)](#).

Remark 1.8 In our examples, we will never encounter a nontrivial sign $(-1)^{\epsilon(l)}$: for real Lagrangians, we shall be working over characteristic 2, and for toric fibres with the standard spin structure, this sign is easily seen to be $+1$. The examples when the sign $(-1)^{\epsilon(l)}$ is negative have been found by J Smith [\[47, Remark 5.3.2\]](#); they occur for $\mathrm{PSU}(N-1)$ -homogeneous Lagrangians. We are grateful to him for pointing out the presence of this sign in general, which was missed in the previous versions of the paper.

1.2 The split-generation criterion

We will now briefly discuss the split-generation criterion in more detail, particularly because we wish to pay attention to both cases: $\mathrm{char} \mathbb{K} = 2$ and $\mathrm{char} \mathbb{K} \neq 2$. We continue

to denote by $L \subset X$ a monotone Lagrangian submanifold with minimal Maslov number at least 2, which is oriented and spin if $\text{char } \mathbb{K} \neq 2$. If $\text{char } \mathbb{K} = 2$, we allow L to be nonorientable. Consider the quantum multiplication by the first Chern class as an endomorphism of quantum cohomology, $- * c_1(X): \text{QH}^*(X) \rightarrow \text{QH}^*(X)$. If \mathbb{K} is algebraically closed, we have an algebra decomposition $\text{QH}^*(X) = \bigoplus_w \text{QH}^*(X)_w$, where $\text{QH}^*(X)_w$ is the generalised w -eigenspace of $- * c_1(X)$ for $w \in \mathbb{K}$.

Recall that $w(L) \in \mathbb{K}$ denotes the obstruction number of L , ie the count of Maslov index 2 disks with boundary on L . By an observation of Auroux, Kontsevich and Seidel, $\mathcal{CO}^0(2c_1) = 2w(L) \cdot 1_L$, which in $\text{char } \mathbb{K} \neq 2$ implies that $\mathcal{CO}^0(c_1) = w(L) \cdot 1_L$; see eg [45]. Now suppose that $\text{char } \mathbb{K} = 2$ and $c_1(X)$ lies in the image of $H^2(X, L; \mathbb{K}) \rightarrow H^2(X; \mathbb{K})$, which is true if L is orientable (because the Maslov class goes to twice the Chern class under $H^2(X, L; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$, and the Maslov class of an orientable manifold is integrally divisible by two). In this case, the same argument shows again that $\mathcal{CO}^0(c_1) = w(L) \cdot 1_L$. This way one deduces the following lemma, which is well known but usually stated only for $\text{char } \mathbb{K} \neq 2$.

Lemma 1.9 *For \mathbb{K} of any characteristic, if L is orientable, then $\mathcal{CO}^0: \text{QH}^*(X) \rightarrow \text{HF}^*(L, L)$ vanishes on all summands except maybe $\text{QH}^*(X)_{w(L)}$. \square*

(If $w(L)$ is not an eigenvalue of $- * c_1(X)$, then \mathcal{CO}^0 vanishes altogether, and it follows that $\text{HF}^*(L, L) = 0$. Recall that L is required to be monotone.) The same vanishing statement is expected to hold for the full map \mathcal{CO}^* . Keeping this vanishing in mind, we see that the “naive” version of the split-generation criterion stated in the introduction, that $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(L, L)$ is injective, can only be useful when $\text{char } \mathbb{K} = 2$ and L is nonorientable. In other cases, it must be replaced by a more practical criterion which does not ignore the eigenvalue decomposition; we will now state both versions of the criterion. Let $\mathcal{Fuk}(X)_w$ denote the Fukaya category whose objects are monotone Lagrangians in X with minimal Maslov number at least 2, oriented and spin if $\text{char } \mathbb{K} \neq 2$, and whose obstruction number equals $w \in \mathbb{K}$.

Theorem 1.10 *Let $L_1, \dots, L_n \subset X$ be Lagrangians which are objects of $\mathcal{Fuk}(X)_w$, and $\mathcal{G} \subset \mathcal{Fuk}(X)_w$ the full subcategory generated by L_1, \dots, L_n . Then \mathcal{G} split-generates $\mathcal{Fuk}(X)_w$ if either of the two following statements hold:*

- (a) $\text{char } \mathbb{K} \neq 2$ and $\mathcal{CO}^*|_{\text{QH}^*(X)_w}: \text{QH}^*(X)_w \rightarrow \text{HH}^*(\mathcal{G})$ is injective.
- (b) \mathbb{K} is arbitrary and $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(\mathcal{G})$ is injective. \square

In the monotone case, this theorem is due to Ritter and Smith [38] and Sheridan [45]. It is more common to only state part (a), but it is easy to check the same proof works for part (b) as well. (In part (a), we could also allow $\text{char } \mathbb{K} = 2$, if L is orientable.) [Theorem 1.10](#) is most easily applied when $\text{QH}^*(X)_w$ is 1-dimensional: because \mathcal{CO}^* is unital, it automatically becomes injective. We apply this theorem in more complicated cases. Before we proceed, let us mention one easy corollary of split-generation. We say that L_1, \dots, L_n split-generate the Fukaya category when \mathcal{G} does.

Lemma 1.11 *If Lagrangians $L_1, \dots, L_n \subset X$ split-generate $\mathcal{Fuk}(X)_w$, and $L \subset X$ is another Lagrangian that is an object of $\mathcal{Fuk}(X)_w$ with $\text{HF}^*(L, L) \neq 0$, then L has nonempty intersection and nonzero Floer cohomology with one of the L_i . \square*

1.3 \mathcal{CO}^0 for real toric Lagrangians

In this subsection, we state a theorem that computes \mathcal{CO}^0 for real Lagrangians in toric manifolds. Using this, it is easy to identify the cases when \mathcal{CO}^0 is injective (and the split-generation follows immediately), and the cases when \mathcal{CO}^0 is not injective, and therefore, a further study of \mathcal{CO}^* is required to establish the split-generation. Our subsequent goal is to apply the main result, [Theorem 1.7](#), to some examples of the latter type.

Let X be a (smooth, compact) toric Fano variety with minimal Chern number at least 2, ie $\langle c_1(X), H_2(X; \mathbb{Z}) \rangle = N\mathbb{Z}$ with $N \geq 2$. As a toric manifold, X has a canonical antiholomorphic involution $\tau: X \rightarrow X$. Its fixed locus is the so-called real Lagrangian $L \subset X$ which is smooth (see Duistermaat [16, page 419]), monotone, and whose minimal Maslov number equals the minimal Chern number of X (see Haug [23]). When speaking of such real Lagrangians, we will always be working over a field \mathbb{K} of characteristic 2. In particular, there is the Frobenius map

$$\mathcal{F}: \text{QH}^*(X) \rightarrow \text{QH}^{2*}(X), \quad \mathcal{F}(x) = x^2.$$

Because $\text{char } \mathbb{K} = 2$, we have that \mathcal{F} is a map of unital algebras. We have reflected in our notation that \mathcal{F} multiplies the $\mathbb{Z}/2N$ -grading by two. A classical theorem of Duistermaat [16] constructs, again in $\text{char } \mathbb{K} = 2$, the isomorphisms $H^i(L) \cong H^{2i}(X)$. We can package these isomorphisms into a single isomorphism of unital algebras,

$$\mathcal{D}: H^{2*}(X) \xrightarrow{\cong} H^*(L).$$

Let us now recall a recent theorem of Haug [23].

Theorem 1.12 *If $\text{char } \mathbb{K} = 2$, then $\text{HF}^*(L, L) \cong H^*(L)$ as vector spaces. Using the identification coming from a specific perfect Morse function from [23], and also identifying $\text{QH}^*(X) \cong H^*(X)$, the same map*

$$\mathcal{D}: \text{QH}^{2*}(X) \xrightarrow{\cong} \text{HF}^*(L, L)$$

is again an isomorphism of unital algebras. □

It turns out that it is possible to completely compute \mathcal{CO}^0 for real toric Lagrangians. This rather quickly follows by combining the works of Charette and Cornea [11], Hyvrier [25], and McDuff and Tolman [32]; we explain this theorem in Section 3.

Theorem 1.13 *The diagram below commutes:*

$$\begin{array}{ccc} \text{QH}^*(X) & \xrightarrow{\mathcal{F}} & \text{QH}^{2*}(X) \\ & \searrow \mathcal{CO}^0 & \downarrow \mathcal{D} \cong \\ & & \text{HF}^*(L, L) \end{array}$$

In particular, \mathcal{CO}^0 is injective if and only if \mathcal{F} is injective.

1.4 Split-generation for the real projective space

We conclude the introduction by proving Proposition 1.1 and Corollary 1.2. The crucial idea is that when n is odd, the kernel of $\mathcal{CO}^0: \text{QH}^*(\mathbb{C}P^n) \rightarrow \text{HF}^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is the ideal generated by the Seidel element of a nontrivial Hamiltonian loop preserving $\mathbb{R}P^n$; this allows us to apply Theorem 1.7 and get new information about \mathcal{CO}^* . Recall that $\text{QH}^*(X) \cong \mathbb{K}[x]/(x^{n+1} - 1)$ and $w(\mathbb{R}P^n) = 0$ because the minimal Maslov number of $\mathbb{R}P^n$ equals $n + 1$ (when $n = 1$, we still have $w(S^1) = 0$ for $S^1 \subset S^2$).

Proof of Proposition 1.1 If n is even, the Frobenius map on $\text{QH}^*(\mathbb{C}P^n)$ is injective, so by Theorem 1.13, $\mathcal{CO}^0: \text{QH}^*(\mathbb{C}P^n) \rightarrow \text{HF}^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is injective, and hence so is \mathcal{CO}^* .

Now suppose n is odd, and let $n = 2p - 1$. Given $\text{char } \mathbb{K} = 2$, we have $\text{QH}^*(\mathbb{C}P^n) \cong \mathbb{K}[x]/(x^p + 1)^2$, so $\ker \mathcal{F} = \ker \mathcal{CO}^0$ is the ideal generated by $x^p + 1$. Consider the Hamiltonian loop γ on $\mathbb{C}P^n$ which in homogeneous coordinates $(z_1 : \dots : z_{2p})$ is the

rotation $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ for $t \in [0, \pi]$, applied simultaneously to the pairs $(z_1, z_2), \dots, (z_{2p-1}, z_{2p})$. Note that t runs to π , not 2π . This loop is Hamiltonian isotopic to the loop

$$(z_0 : \dots : z_{2p-1}) \mapsto (e^{2it} z_0 : z_1 : \dots : e^{2it} z_{2p-1} : z_{2p}), \quad t \in [0, \pi],$$

so $S(\gamma) = x^p$; see [32]. The loop γ clearly preserves the real Lagrangian $\mathbb{R}P^n \subset \mathbb{C}P^n$, and its orbit l is a generator of $H_1(\mathbb{R}P^n) \cong \mathbb{K}$. Taking $y \in H^1(\mathbb{R}P^n)$ to be the generator, we get $\langle y, l \rangle = 1$, and the right-hand side of equation (**) from Theorem 1.7 equals $\mathcal{CO}^0(Q)$. On the other hand, the product on $\text{HF}^*(\mathbb{R}P^n, \mathbb{R}P^n)$ is commutative by Theorem 1.12, so the left-hand side of (**) necessarily vanishes. We conclude that the hypothesis of Theorem 1.7(c) is satisfied for any $Q \notin \ker \mathcal{CO}^0$.

Let us prove that $\mathcal{CO}^*(P) \neq 0$ for each nonzero $P \in \text{QH}^*(\mathbb{C}P^n)$. If $\mathcal{CO}^0(P) \neq 0$, we are done, so it suffices to suppose that $\mathcal{CO}^0(P) = 0$. This means that

$$P = (x^p + 1) * Q = (S(\gamma) + 1) * Q$$

for some $Q \in \text{QH}^*(\mathbb{C}P^n)$. Note that if $Q \in \ker \mathcal{CO}^0 = \ker \mathcal{F}$, then $P \in (\ker \mathcal{F})^2 = \{0\}$. So if $P \neq 0$, then $\mathcal{CO}^0(Q) \neq 0$, and thus $\mathcal{CO}^*(P) \neq 0$ by Theorem 1.7(c) and the observation earlier in this proof. □

Remark 1.14 When n is even, $c_1(\mathbb{C}P^n)$ is invertible in $\text{QH}^*(\mathbb{C}P^n)$, so the 0–eigenspace $\text{QH}^*(\mathbb{C}P^n)_0$ is trivial; but L is nonorientable, so this does not contradict Lemma 1.9. On the other hand, when n is odd, L is orientable, but $c_1(\mathbb{C}P^n)$ vanishes in $\text{char } \mathbb{K} = 2$, so the whole $\text{QH}^*(\mathbb{C}P^n)$ is its 0–eigenspace; this is also consistent with Lemma 1.9.

Proof of Corollary 1.2 This follows from Proposition 1.4 and Theorem 1.10(b). □

The same trick of finding a real Hamiltonian loop whose Seidel element generates $\ker \mathcal{CO}^0$ works for some other toric manifolds which have “extra symmetry” in addition to the toric action, like a Hamiltonian action of $\text{SU}(2)^{\dim_{\mathbb{C}} X/2}$ which was essentially used above. As already mentioned, we will provide more explicit examples in Section 3.

Acknowledgements The author is most grateful to his supervisor Ivan Smith for many useful comments as well as constant care and enthusiasm. Yankı Lekili and Jack Smith have provided valuable feedback and pointed out two inaccuracies in the previous versions of this paper. The paper has also benefited from discussions with Mohammed Abouzaid, Lino Amorim, François Charette, Georgios Dimitroglou Rizell, Jonny Evans,

Alexander Ritter, Paul Seidel, Nick Sheridan and Renato Vianna. We are grateful to the referee for helpful suggestions on improving the exposition.

The author was funded by the Cambridge Commonwealth, European and International Trust, and acknowledges travel funds from King's College, Cambridge. The work was partially supported by RFBR grant 15-01-06302.

2 Proof of Theorem 1.7

Let X be a monotone symplectic manifold and $w \in \mathbb{K}$. We recall that the objects in the monotone Fukaya category $\mathcal{Fuk}(X)_w$ are monotone Lagrangian submanifolds $L \subset X$ with minimal Maslov number at least 2, oriented and spin if $\text{char } \mathbb{K} \neq 2$, equipped with local systems $\rho: \pi_1(L) \rightarrow \mathbb{K}^\times$, whose count of Maslov index 2 disks (weighted using ρ) equals w . We will use the definition of the Fukaya category based on achieving transversality by explicit Hamiltonian perturbations of the pseudoholomorphic equation. This setup was developed by Seidel [40] for exact manifolds and carries over to monotone ones; see [38; 45]. There is a notion of bounding cochains from [20], generalising the notion of a local system, and all results are expected carry over to them as well.

2.1 A theorem of Charette and Cornea

Suppose $\gamma = \{\gamma_t\}_{t \in S^1}$ is a loop of Hamiltonian symplectomorphisms on X . As explained by Seidel in [40, Section 10c], the loop γ gives rise to a natural transformation γ^\sharp from the identity functor on $\mathcal{Fuk}(X)_w$ to itself. Any such natural transformation is a cocycle of the Hochschild cochain complex $\text{CC}^*(\mathcal{Fuk}(X)_w)$ [40, Section 1d]. Denote the corresponding Hochschild cohomology class by

$$[\gamma^\sharp] \in \text{HH}^*(\mathcal{Fuk}(X)_w).$$

We denote, as earlier, the closed-open map by $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(\mathcal{Fuk}(X)_w)$ and the Seidel element by $S(\gamma) \in \text{QH}^*(X)$. The following theorem was proved by Charette and Cornea [11].

Theorem 2.1 *If we take for $\mathcal{Fuk}(X)_w$ the Fukaya category of Lagrangians with trivial local systems only, then $\mathcal{CO}^*(S(\gamma)) = [\gamma^\sharp]$. □*

Let us now restrict to a single Lagrangian L which is preserved by the Hamiltonian loop γ and denote by $l \in H_1(L)$ the homology class of an orbit of γ on L . Let $\text{CC}^*(L, L)$ denote the Hochschild cochain complex of the A_∞ algebra $\text{CF}^*(L, L)$,

and let $\mathrm{HH}^*(L, L)$ be its Hochschild cohomology. (The definition of Hochschild cohomology will be reviewed later in this section.) We will now need to recall the proof of [Theorem 2.1](#) for several reasons: first, we wish to see how [Theorem 2.1](#) gets modified in the presence of a local system on L ; second, we shall see the appearance of a sign “hidden” in γ^\sharp ; third and most importantly, we will recall the definition of the moduli spaces computing γ^\sharp in the process. Eventually, for later use, we need a form of [Theorem 2.1](#) expressed by formula (2-2) below, which takes the local system and the sign into account.

Pick some Floer datum $\{H_s, J_s\}_{s \in [0,1]}$ and perturbation data defining an A_∞ structure on Floer’s complex $\mathrm{CF}^*(L, L)$ [40]. Recall that the maps

$$\mathcal{CO}^k(S(\gamma)): \mathrm{CF}^*(L, L)^{\otimes k} \rightarrow \mathrm{CF}^*(L, L)$$

count 0–dimensional moduli space of disks satisfying a perturbed pseudoholomorphic equation (with appropriately chosen perturbation data) with $k + 1$ boundary punctures (k inputs and one output) and one interior marked point. These disks satisfy the Lagrangian boundary condition L , and their interior marked point is constrained to a cycle dual to $S(\gamma)$; see [Figure 1](#), top left (in this figure, we abbreviate the datum $\{H_s, J_s\}$ simply to H). A disk u is counted with coefficient $\pm \rho(\partial u)$, where the sign \pm comes from the orientation on the moduli space, and $\rho(\partial u) \in \mathbb{K}^\times$ is the monodromy of the local system. The collection of maps $\mathcal{CO}^*(S(\gamma)) := \{\mathcal{CO}^k\}_{k \geq 0}$ is a Hochschild cochain in $\mathrm{CC}^*(L, L)$ if all perturbation data are chosen consistently with gluing.

The argument of Charette and Cornea starts by passing to a more convenient definition of the closed-open map in which \mathcal{CO}^k counts holomorphic disks with $k + 1$ boundary punctures and one interior puncture (instead of a marked point). We can view the neighbourhood of the interior puncture as a semi-infinite cylinder; then the pseudo-holomorphic equation restricts on this semi-infinite cylinder to a Hamiltonian Floer equation with some Floer datum $\{F_t, J_t\}_{t \in S^1}$. We input the PSS image of $S(\gamma)$ to the interior puncture (see [Figure 1](#), top middle) given as a linear combination of some Hamiltonian orbits z (in the figure, we abbreviate the datum $\{F_t, J_t\}$ simply to F_t).

The PSS image of $S(\gamma)$ counts configurations shown in the upper part of [Figure 1](#), top middle, consisting of disks with one output puncture (say, asymptotic to an orbit y), and a cylinder counting continuation maps from $(\gamma_t^{-1})^*y$ seen as an orbit of Floer’s complex with datum pulled back by the loop γ_t^{-1} [39, Lemmas 2.3 and 4.1] to another orbit z of the original Floer’s complex with datum $\{F_t, J_t\}$. Let us glue the z –orbits

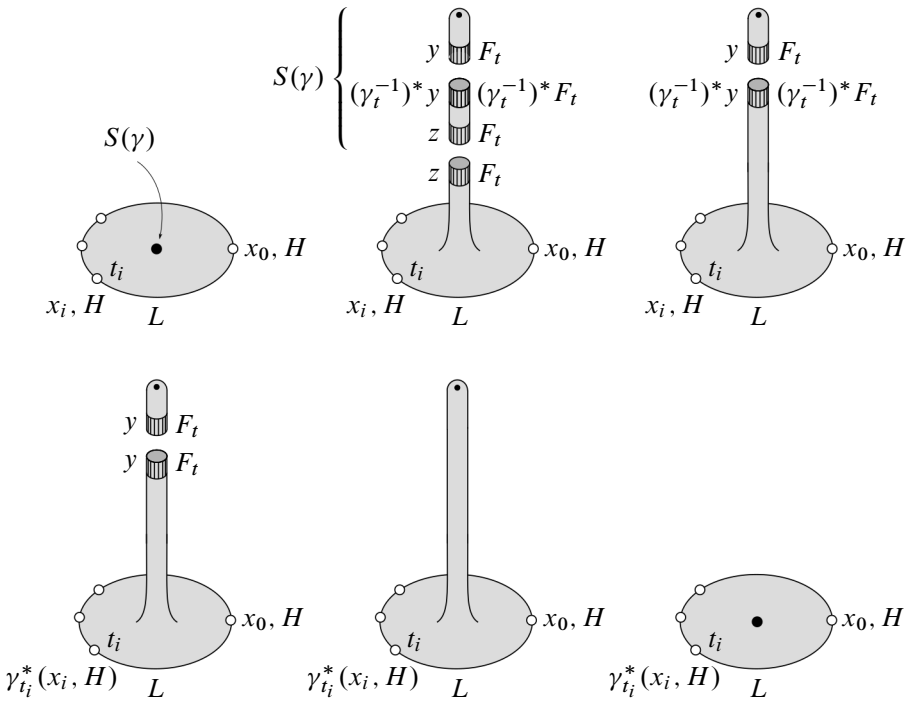


Figure 1: A computation of $CO^*(S(\gamma))$ by Charette and Cornea

together, passing to [Figure 1](#), top right, and then substitute each lower punctured pseudoholomorphic disk u in [Figure 1](#), top right, by \tilde{u} defined as

$$(2-1) \quad \tilde{u}(re^{2\pi it}) = \gamma_t^{-1} \circ u(re^{2\pi it}),$$

assuming that the interior puncture is located at $0 \in \mathbb{C}$ and the output puncture at $1 \in \mathbb{C}$. Let us look at the effect of this substitution.

First, $[\partial u] = [\partial \tilde{u}] + l \in H_1(L)$, so the count of configurations in [Figure 1](#), top right (before substitution) is equal to the count of configurations in [Figure 1](#), bottom left (after substitution) multiplied by $\rho(l)$.

Second, \tilde{u} satisfies the same boundary condition L because $\gamma_t L = L$, but the perturbation data defining the pseudoholomorphic equation get pulled back accordingly. In particular, the Lagrangian Floer datum $\{H_s, J_s\}_{s \in [0,1]}$ and the asymptotic chord at a strip-like end corresponding to the boundary puncture at $t_i \in S^1$ get pulled back by γ_{t_i} .

Third, (2-1) gives an abstract bijection $u \mapsto \tilde{u}$ between the respective zero-dimensional moduli spaces, but we should discuss how this bijection behaves with respect to the

signs attached to u and \tilde{u} by the orientations on the moduli spaces. Assume for simplicity that the Hamiltonian perturbation is small enough so that we can canonically deform ∂u and $\partial \tilde{u}$ to loops inside L . Take the trivialisation of $TL|_{\partial u}$ defined by the spin structure on L and push it forward by γ to a trivialisation of $TL|_{\partial \tilde{u}}$. Obviously, (2-1) preserves the signs computed using these trivialisations; we remark that this trivialisation of $TL|_{\partial \tilde{u}}$ is used in the general definition of γ^\sharp . In our specific case, \tilde{u} is again a curve with boundary on L (rather than with a moving Lagrangian boundary condition), and we wish to consider a different orientation scheme for \tilde{u} , namely the usual scheme for orienting moduli spaces of curves with boundary on L using the given spin structure. We shall be using this orientation scheme from now on, and we observe that it uses the trivialisation of $TL|_{\partial \tilde{u}}$ coming from the spin structure, which may be different from the pushforward trivialisation of $TL|_{\partial \tilde{u}}$ mentioned before. We denote the sign difference between the two orientations for \tilde{u} by $(-1)^\epsilon = \pm 1$; this sign equals $+1$ if and only if the two trivialisations of $TL|_{\partial \tilde{u}}$ from above are homotopic. (Ultimately, we are going to use the fact that for our choice of orientation scheme, the moduli space of constant unconstrained disks is positively oriented; this may not be true for the γ pushforward orientation scheme.)

Remark 2.2 In general, the number ϵ does not necessarily equal $\epsilon(l)$ from the introduction. This equality holds when $\partial \tilde{u}$ is contractible, as easily seen from the definitions.

Fourth, near the interior puncture, \tilde{u} satisfies the Hamiltonian Floer equation with original datum $\{F_t, J_t\}$ at the interior puncture, and is asymptotic orbit y . So we can glue the y -orbits, passing to Figure 1, bottom middle, and Figure 1, bottom right, is another drawing of the same domain we got after gluing, namely the disk with $k + 1$ boundary punctures and one interior unconstrained marked point fixed at $0 \in \mathbb{C}$. Let us explain the presence of the marked point $0 \in \mathbb{C}$: it is carried over from a marked point on the upper disk in Figure 1, top middle, where the interior marked point serves to stabilise the domain; such a marked point is present in the definition of the Seidel element.

Summing up,

$$(2-2) \quad \mathcal{CO}^k(S(\gamma))(x_1 \otimes \cdots \otimes x_k) = (-1)^\epsilon \cdot \rho(l) \cdot \sum \# \mathcal{M}^\gamma(x_1, \dots, x_k; x_0) \cdot x_0,$$

where $\mathcal{M}^\gamma(x_1, \dots, x_k; x_0)$ is the 0-dimensional moduli space of disks shown in Figure 1, bottom right, which satisfy the inhomogeneous pseudoholomorphic equation defined by domain- and modulus-dependent perturbation data in the sense of [40] such that:

- The disks carry the unconstrained interior marked point fixed at $t = 0$, the output boundary puncture fixed at $t_0 = 1$, and k free input boundary punctures at $t_i \in S^1, i = 1, \dots, k$.
- On a strip-like end corresponding to a boundary puncture $t_i \in S^1$, perturbation data restrict to the Floer datum which is the γ_{t_i} -pullback of the original Floer datum $\{H_s, J_s\}_{s \in [0,1]}$, and the asymptotic chord for this strip must be the γ_{t_i} -pullback of the asymptotic chord x_i of the original Floer datum.
- The data must be consistent with gluing strip-like ends at $t_i \in S^1$ to strip-like ends of punctured pseudoholomorphic disks carrying the γ_{t_i} -pullbacks of the perturbation data defining the A_∞ structure on $\text{CF}^*(L, L)$.
- We use the standard orientation scheme for curves with boundary on L to orient the \mathcal{M}^γ , and the sign $(-1)^\epsilon$ was explained above.

The counts $\#\mathcal{M}^\gamma$ are signed and weighted by ρ ; the third condition guarantees that $\mathcal{CO}^*(S(\gamma))$ is a Hochschild cocycle. Formula (2-2) coincides with the formula from [40, Section 10c] defining the natural transformation $[\gamma^\#]$ up to $(-1)^\epsilon \rho(l)$, and we have clarified this difference.

Remark 2.3 The fixed interior marked point at $t = 0$ and the fixed boundary marked point at $t_0 = 1$ make sure our disks have no automorphisms, so the values $t_i \in S^1$ of the other boundary punctures are uniquely defined.

Before proceeding, note that we are already able to compute $\mathcal{CO}^0(S(\gamma))$.

Corollary 2.4 If $\{\gamma_t\}_{t \in S^1}$ is a Hamiltonian loop such that $\gamma_t(L) = L$, then on the chain level,

$$\mathcal{CO}^0(S(\gamma)) = (-1)^{\epsilon(l)} \cdot \rho(l) \cdot 1_L \in \text{HF}^0(L, L).$$

(Here 1_L is a chain-level representative of the cohomology unit; see (2-3) below.)

Proof When $k = 0$, the moduli space in formula (2-2) is exactly the moduli space defining the cohomological unit in $\text{CF}^*(L, L)$; see eg [45, Section 2.4]. The equality $\epsilon = \epsilon(l)$ holds by Remark 2.2: for a small Hamiltonian, the curves computing the unit are close to being constant and therefore have contractible boundary. □

Proof of Theorem 1.7(a) This is the homology-level version of Corollary 2.4. □

2.2 The PSS maps in degree one

Our goal will be to compute a “topological piece” of $\mathcal{CO}^1(S(\gamma))$. This subsection introduces some background required for the computation: in particular, we recall that there is a canonical map $\Phi: H^1(L) \rightarrow \text{HF}^*(L, L)$ which was used in the statement of [Theorem 1.7](#). This is the Lagrangian PSS map of Albers [4], and the fact it is canonical was discussed, for instance, by Biran and Cornea [9, Proposition 4.5.1(ii)] in the context of Lagrangian quantum cohomology.

First, recall that once the Floer datum is fixed, the complex $\text{CF}^*(L, L)$ acquires the Morse \mathbb{Z} -grading. This grading is not preserved by the Floer differential or the A_∞ structure maps, but is still very useful. Assume that the Hamiltonian perturbation, as part of the Floer datum, is chosen to have a unique minimum x_0 on L , which means that $\text{CF}^0(L, L)$ is one-dimensional and generated by x_0 . We denote by

$$(2-3) \quad 1_L \in \text{CF}^0(L, L)$$

the chain-level cohomological unit defined in [45, Section 2.4], which is proportional to x_0 . Now pick a metric and a Morse–Smale function f on L with a single minimum; together they define the Morse complex which we denote by $C^*(L)$. Consider the “Maslov index 0” versions of the PSS maps, which are linear (but *not* chain) maps

$$(2-4) \quad \Psi: \text{CF}^*(L, L) \rightarrow C^*(L), \quad \Phi: C^*(L) \rightarrow \text{CF}^*(L, L).$$

These maps are defined as in the paper of Albers [4], with the difference that Φ and Ψ count configurations with Maslov index 0 disks only. For example, the map Ψ counts configurations consisting of a Maslov index 0 pseudoholomorphic disk with boundary on L and one input boundary puncture, followed by a semi-infinite gradient trajectory of f which outputs an element of $C^*(L)$. Similarly, Φ counts configurations in which a semi-infinite gradient trajectory is followed by a Maslov index 0 disk with an output boundary puncture. The maps Ψ and Φ preserve \mathbb{Z} -gradings on the two complexes.

Let $d_0: \text{CF}^*(L, L) \rightarrow \text{CF}^{*+1}(L, L)$ be the “Morse” part of the Floer differential counting the contribution of Maslov index 0 disks; see Oh [34]. Let $d_{\text{Morse}}: C^*(L) \rightarrow C^{*+1}(L)$ be the usual Morse differential. The lemma below is a version of [4, Theorem 4.11].

Lemma 2.5 *Φ and Ψ are chain maps with respect to d_0 and d_{Morse} , and are cohomology inverses of each other. \square*

Lemma 2.6 *Suppose $\text{HF}^*(L, L) \neq 0$. If $y \in C^1(L)$ is a Morse cocycle (resp. coboundary) then $\Phi(y)$ is a Floer cocycle (resp. coboundary).*

Proof This follows from the fact that the image and the kernel of d_0 and the full Floer differential d coincide on $\text{CF}^1(L, L)$ if $\text{HF}^*(L, L) \neq 0$ by Oh's decomposition of the Floer differential [34]. \square

Consequently, if $\text{HF}^*(L, L) \neq 0$, we get a map

$$\Phi: H^1(L) \rightarrow \text{HF}^*(L, L).$$

For Ψ , we have a weaker lemma using [34] (this lemma is not true for coboundaries instead of cocycles).

Lemma 2.7 *If $y \in \text{CF}^*(L, L)$ is a Floer cocycle, then $\Psi(y) \in C^*(L)$ is a Morse cocycle.* \square

By Lemmas 2.6 and 2.7, given $\text{HF}^*(L, L) \neq 0$, we have the following induced maps which we denote by the same symbols Ψ and Φ , abusing notation:

$$(2-5) \quad \Psi: \text{CF}^*(L, L) \rightarrow H^*(L), \quad \Phi: H^*(L) \rightarrow \text{HF}^*(L, L).$$

In particular, Theorem 1.7 in Section 1 refers to this cohomological version of the map Φ . We remind that Ψ does not necessarily descend to a map from $\text{HF}^*(L, L)$.

2.3 Computing the topological part of $\mathcal{CO}^1(S(\gamma))$

We continue to use the above conventions and definitions, namely we use the \mathbb{Z} -grading on $\text{CF}^*(L, L)$, the maps Φ and Ψ , and the choice of a Hamiltonian perturbation on L with a unique minimum x_0 . From now on, we assume $\text{HF}^*(L, L) \neq 0$. Recall that $\mathcal{CO}^*(S(\gamma))$ is determined via formula (2-2) by the moduli spaces $\mathcal{M}^\nu(x_1, \dots, x_k; x_0)$. The connected components of $\mathcal{M}^\nu(x_1, \dots, x_k; x_0)$ corresponding to disks of Maslov index μ have dimension

$$|x_0| + k + \mu - \sum_{i=1}^k |x_i|,$$

where $|x_i|$ are the \mathbb{Z} -gradings of the $x_i \in \text{CF}^*(L, L)$. Consequently,

$$\mathcal{CO}^k(S(\gamma)): \text{CF}^*(L, L)^{\otimes k} \rightarrow \text{CF}^*(L, L)$$

is a sum of maps of degrees

$$-k - mN_L, \quad m \geq 0,$$

where N_L is the minimal Maslov number of L . In particular, the restriction of $\mathcal{CO}^1(S(\gamma))$ to $\text{CF}^1(L, L)$ is of pure degree -1 ; that is, its image lands in $\text{CF}^0(L, L)$:

$$\mathcal{CO}^1(S(\gamma))|_{\text{CF}^1(L, L)}: \text{CF}^1(L, L) \rightarrow \text{CF}^0(L, L).$$

Moreover, this map is determined by the moduli space consisting of Maslov index 0 disks only and can be computed in purely topological terms. This is the main technical computation which we now perform; recall that $l \in H_1(L)$ is the homology class of an orbit of γ in L , and the sign $(-1)^{\epsilon(l)}$ was defined in [Section 1](#).

Proposition 2.8 *Suppose $\text{HF}^*(L, L) \neq 0$. If $x \in \text{CF}^1(L, L)$ is a Floer cocycle, then on the chain level,*

$$\mathcal{CO}^1(S(\gamma))(x) = (-1)^{\epsilon(l)} \cdot \rho(l) \cdot \langle \Psi(x), l \rangle \cdot 1_L.$$

(Here $\Psi(x) \in H^1(L)$ is from [\(2-5\)](#), $\langle -, - \rangle$ denotes the pairing $H^1(L) \otimes H_1(L) \rightarrow \mathbb{K}$, we consider \mathcal{CO}^1 on the chain level, and 1_L is a chain-level cohomology unit as in [\(2-3\)](#).)

Proof All disks with boundary on L we consider in this proof are assumed to have Maslov index 0. We identify the domains of all disks that appear in the proof with the unit disk in \mathbb{C} , and their boundaries are identified with the unit circle $S^1 \subset \mathbb{C}$. In the subsequent figures, punctured marked points will be drawn by circles filled white, and unpunctured marked points by circles filled black. According to formula [\(2-2\)](#), for a generator $x \in \text{CF}^1(L, L)$ we have

$$\mathcal{CO}^1(S(\gamma))(x) = (-1)^\epsilon \cdot \rho(l) \cdot \#\mathcal{M}^\gamma(x; x_0),$$

where $\mathcal{M}^\gamma(x; x_0)$ consists of (perturbed pseudoholomorphic) Maslov index 0 disks whose domains are shown in [Figure 2](#), left.

Step 1 (perturbation data producing bubbles with unpunctured points) Recall that the domains appearing in the moduli space $\mathcal{M}^\gamma(x; x_0)$ are disks with the interior marked point 0 and boundary punctures 1 and t , where $t \in S^1 \setminus \{1\}$. For further use, we will choose perturbation data defining $\mathcal{M}^\gamma(x; x_0)$ whose bubbling behaviour as $t \rightarrow 1$ differs from the standard one. Usually, the perturbation data would be chosen so as to be compatible, as $t \rightarrow 1$, with the gluing G_1 shown in [Figure 2](#), where the bubble meets the principal disk along a puncture, meaning that near this puncture it satisfies a Floer equation and shares an asymptotic Hamiltonian chord with the corresponding puncture

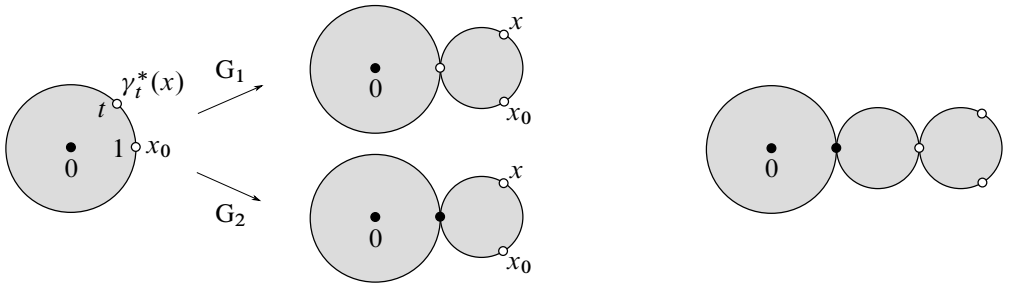


Figure 2: Two types of gluings for $\mathcal{M}^\gamma(x; x_0)$ and a way to interpolate between the glued perturbation data

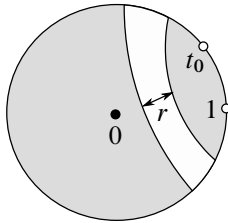


Figure 3: Collision of two boundary marked points seen as stretching the unshaded strip with parameter $r \rightarrow +\infty$

of the principal disk. On the other hand, we will use perturbation data consistent with gluing G_2 shown in Figure 2, where the bubble is attached to the principal disk by an unpunctured marked point. Near the unpunctured marked point, the disks satisfy a holomorphic equation with no Hamiltonian term.

Let us explain how to define both types of data more explicitly. The domain in Figure 2, left, with free parameter t close to 1, is biholomorphic to the domain shown in Figure 3 whose boundary marked points are fixed at 1 and some $t_0 \in S^1$, upon which a stretching procedure along the unshaded strip is performed. This stretching procedure changes the complex structure on the disk by identifying the strip with $[0, 1] \times [0, 1]$, removing it, and gluing back the longer strip $[0, 1] \times [0, r]$. The parameter $r \in [1, +\infty)$ is free and replaces the free parameter t , so that tending $r \rightarrow +\infty$ replaces the collision of two marked points $t \rightarrow 1$.

In order to get perturbation data which are consistent with the usual bubbling, as shown by G_1 in Figure 2, one requires the perturbed pseudoholomorphic equation to coincide, on the strip $[0, 1] \times [0, r]$, with the usual Floer equation defining the Floer differential, which uses a Hamiltonian perturbation translation-invariant in the direction of $[0, r]$. In order to get perturbation data producing the bubbling pattern shown by G_2 in Figure 2,

we simply put an unperturbed pseudoholomorphic equation on the strip $[0, 1] \times [0, r]$ without using a Hamiltonian perturbation at all.

Both ways of defining perturbation data are subject to appropriate gluing and compactness theorems, which precisely say that as we tend $r \rightarrow \infty$, the solutions bubble in one of the two corresponding ways shown in [Figure 2](#). The standard choice is used, for example, to prove that $[\gamma^\#]$ (obtained from the counts of various \mathcal{M}^γ) is a Hochschild cocycle in the Fukaya A_∞ algebra of L defined using Seidel’s setup with Hamiltonian perturbations. The other choice will be more convenient for our computations. Note that the two different types of perturbation data give the same count $\#\mathcal{M}^\gamma(x; x_0)$: this is proved by interpolating between them using the two-parametric space of perturbation data obtained from gluing together the disks in [Figure 2](#), right, with different length parameters. Recall that all disks in $\mathcal{M}^\gamma(x; x_0)$ have Maslov index 0 as the Morse index $|x| = 1$; therefore, no unnecessary bubbling occurs. (Since we do not want to compute the moduli spaces \mathcal{M}^γ other than the $\mathcal{M}^\gamma(x; x_0)$ for $|x| = 1$, we do not have to worry about extending our unusual type of perturbation data to the other moduli spaces.)

In addition, we will assume that the Hamiltonian perturbation vanishes over the principal disk in [Figure 2](#), bottom, making this disk J -holomorphic and hence constant, because the disk has Maslov index 0. Such configurations can be made consistent with gluing: for this, one just needs to make the Hamiltonian perturbation vanish over the shaded subdomain in [Figure 3](#), for all t close to 1. Note that regularity can be achieved by perturbing the pseudoholomorphic equation over the subdomain to the right of the unshaded strip in [Figure 3](#).

Step 2 (a one-dimensional cobordism from $\mathcal{M}^\gamma(x; x_0)$) In what follows, we will use (and assume familiarity with) the theory of holomorphic pearly trees developed by Sheridan in his Morse–Bott definition of the Fukaya category [\[45\]](#); see also the earlier work of Cornea and Lalonde [\[15\]](#). Sheridan performs the analysis based on extending Seidel’s setup of Fukaya categories from [\[40\]](#). Although [\[45\]](#) considers exact Lagrangians instead of monotone ones, all the analysis works equally well in the nonexact case if we only consider disks of Maslov index 0, because here unpunctured disk bubbles cannot occur just like in the exact case. Techniques for dealing with holomorphic pearly trees (or “clusters”) with disks of arbitrary Maslov index have appeared in [\[15; 10\]](#), but we will not actually need to appeal to them.

We will now define a family of domains depending on two parameters $s \in [0, 2\pi]$ and $l \in [-1, +\infty]$. When $s \notin \{0, 2\pi\}$, the domains are shown in [Figure 4](#), where $t = e^{is}$;

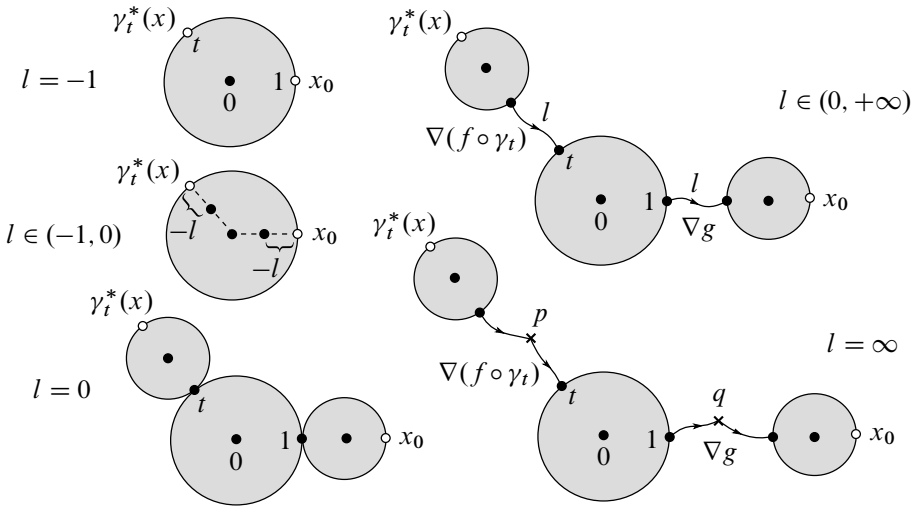


Figure 4: The domains for $s \in (0, 2\pi)$ and $l \in [-1, \infty]$, where $t = e^{is}$

we discuss the case $s \in \{0, 2\pi\}$ later. When $l = -1$, the domain is the disk from the definition of $\mathcal{M}^\gamma(x; x_0)$. When $l \in (-1, 0)$, the domain is the same disk (called principal) with two additional interior marked points whose position is determined by the parameter l : the first point lies on the line segment $[0, t]$, the second one lies on the line segment $[0, 1]$, and both points have distance $1 + l$ from 0. When $l = 0$, the domain consists of the principal disk with marked points $0, 1, t$, and two bubble disks attached to the principal disk at points 1 and t . The first bubble disk has marked points $0, 1$ and a boundary puncture at -1 ; the second one has marked points at $0, -1$ and a boundary puncture at 1 . When $0 < l < \infty$, the domain contains the same three disks, now disjoint from each other, plus two line segments of length l connecting the bubble disks to the principal one along the boundary marked points at which the disks used to be attached to each other. (So far, the length is just a formal parameter associated with the domain, but soon it will become the length of the flowline corresponding to the segment.) When $l = \infty$, we replace each line segment by two rays $[0, +\infty) \sqcup (-\infty, 0]$.

When $s = 0$ or $s = 2\pi$, the domains obtain extra bubbles as those discussed above in the definition of $\mathcal{M}^\gamma(x; x_0)$, which correspond to the parameter $t = e^{is} \in S^1$ approaching $1 \in S^1$ from the two sides. These domains are shown in Figure 5: as l goes from -1 to 0 , the two interior marked points move along the punctured paths. Observe that these points are crossing the node between the two disks at some intermediate value of l ; this does not cause any difficulty with the definitions because these marked points are only used to represent varying perturbation data consistent with the types of bubbling

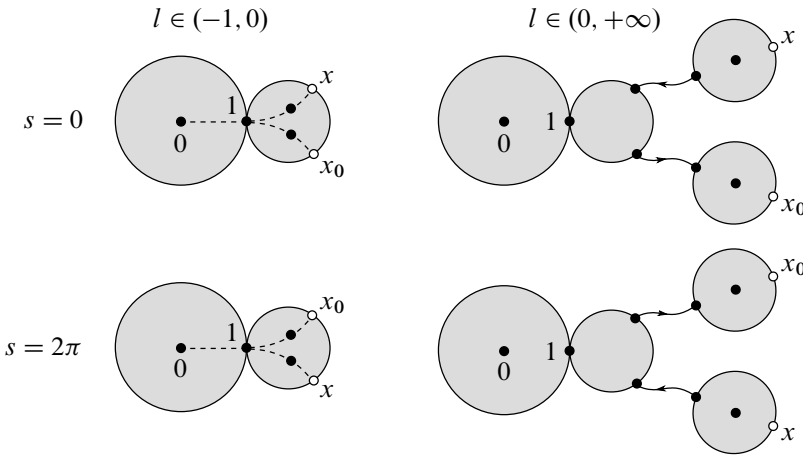


Figure 5: The domains for $s \in \{0, 2\pi\}$, $l \in [-1, \infty]$

we prescribe in the figures. We will soon mention what these varying data are in terms of stretching certain strips inside a fixed disk. When $l > 0$, the length of the paths equals l . When $l = \infty$, one introduces broken lines $[0, +\infty) \sqcup (-\infty, 0]$ as above.

Having specified the domains, we briefly explain how to equip the disks with suitable perturbed pseudoholomorphic equations and line segments with suitable gradient equations to get a moduli space of solutions. When $l = -1$, we choose the equations defining $\mathcal{M}^\gamma(x; x_0)$ as discussed above in [Step 1](#); in particular, this is consistent with bubbling at the unpunctured point as $s \rightarrow 0$ or $s \rightarrow 2\pi$. When $-1 < l < 0$, we choose the equations with the same properties as for $\mathcal{M}^\gamma(x; x_0)$, which are additionally consistent with bubbling at unpunctured points as $l \rightarrow 0$. When $l \geq 0$, we choose the equation on the disk with an input puncture to be the γ_t -pullback of the one appearing in the definition of the PSS map Ψ , and the equation on the disk with an output puncture to be exactly the equation from the PSS map Φ . Finally we fix two generic Morse–Smale functions $f, g: L \rightarrow \mathbb{R}$. On the line segments and rays, the equation is the gradient equation for g or the γ_t -pullback of the gradient equation for f , as shown in [Figure 4](#).

In general, we can arrange the equations on all disks to have a Hamiltonian perturbation. For the disks with punctures, we must do so anyway; however, for the disks without punctures (with unpunctured marked points only), we can choose the equation to be J -holomorphic without a Hamiltonian term, provided that we check that our moduli space can be made regular with this restricted choice. In order to carry out the computations below, we choose the zero Hamiltonian perturbation on all disks without boundary punctures, namely,

- the central disk in Figure 4, bottom left, top right and bottom right,
- the left disks in all domains of Figure 5, and the central disks in Figure 5, top right and bottom right.

We will now specify one more property of the equations that we choose. When $s = 0$ and $l < 0$, we require the perturbation data on the twice-punctured disk in Figure 5, top left, to be obtained by π -rotation from the perturbation data on the similar disk for $s = 2\pi$ (and the same parameter l) if we identify the two punctures with points 1 and -1 of the unit disk. When $s = 0$ and $l \geq 0$, we make a similar symmetric choice.

Finally, we specify the asymptotic conditions at the punctures. If x is a generator of $CF^1(L, L)$, we specify that the input puncture in Figure 4 must be asymptotic to the γ_t -pullback of x (as usual, if x is a linear combination of generators, we take the disjoint union of the relevant moduli spaces). The output puncture must be asymptotic to the unique generator $x_0 \in CF^0(L, L)$. When $l = \infty$, the first pair of rays in Figure 4, bottom right, must be asymptotic to a point p such that $\gamma_t^{-1}(p) \in C^1(L)$ (that is, p is an index-1 critical point of $f \circ \gamma_t$) and the second pair of rays must be asymptotic to $q \in C^0(L)$; we assume q is the unique minimum of g . The interior marked points on the disks are unconstrained.

Above, we have specified a 2-dimensional space of domains and the equations over them. This gives us a moduli space of solutions (“pearly trees”) which is 1-dimensional by our choice of indices. We remind the reader that a formal definition of this moduli space falls into the setup of moduli spaces of pearly trajectories given by Sheridan [44]. For our purposes, its description given above will suffice.

The boundary of our 1-dimensional moduli space consists of

- solutions whose domains have parameter $l = -1$ or $l = \infty$,
- solutions whose domains have parameter $s = 0$ or $s = 2\pi$.

We claim that solutions of the second type cancel pairwise. Indeed, recall that the disks without boundary punctures in Figure 5 are constant, and the perturbation data on the punctured disks for $s = 0, 2\pi$ are chosen in a way to provide the same solutions after a π -rotation on each disk. Let us describe more explicitly what happens when $l < 0$ as the case when $l \geq 0$ is clear enough from Figure 5, top right and bottom right. We can represent the domains shown in Figure 4, top left, with free $l < 0$ and free small $s > 0$, where $t = e^{is}$, by a disk with fixed boundary punctures, stretched with length parameters $-1/l$ and $1/s$ along the three strips shown in Figure 6, left. The stretching procedure

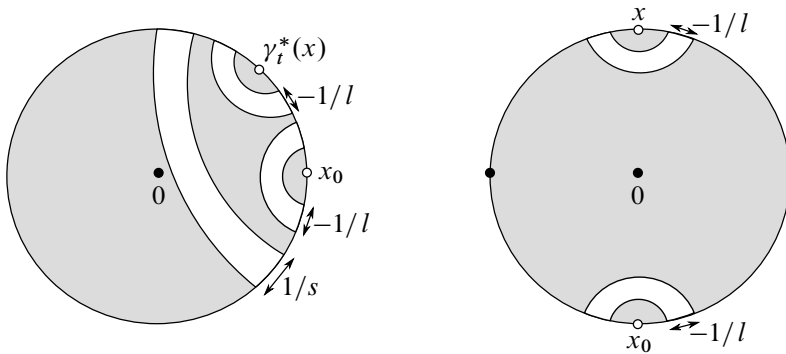


Figure 6: The domains (left) for $l < 0$, $s > 0$, seen as a fixed disk with three stretched strips, and (right) for $l < 0$, $s = 0$, when the principal disk is constant.

was described earlier, and our choice of perturbation data says that the stretched strips, and the subdomain to the left of the $1/s$ -strip, carry an unperturbed pseudoholomorphic equation. So for $s = 0$, we get the disks shown in Figure 6, right, with the unpunctured boundary marked point attached to a constant disk, which means this boundary marked point is unconstrained. (As usual, the domain is considered up to complex automorphisms, so the unconstrained point does not prevent us from having rigid solutions.) This way, Figures 5, top left, and 6, right, are drawings of the same configuration for any $l < 0$. If we rotate the disk in Figure 6, right, by π , we get precisely the disk with perturbation data we would have gotten for $s = 2\pi$, except that the boundary marked point (with the attached constant disk) is on the different side of the boundary.

We have shown that the solutions for $s = 0$ and $s = 2\pi$ are in a natural bijection; we claim that this bijection reverses the signs associated to those solutions as parts of the moduli space. This is ultimately related to the fact that the constant disk is being glued to those solutions at the opposite boundary components. One can adopt a proof (which we will not provide in detail here) from the following classical example where an analogous sign issue has been treated. Suppose L_0 and L_1 are monotone Lagrangian submanifolds with obstruction numbers $m_0(L_i) \in \mathbb{Z}$, $i = 0, 1$. Then the Floer differential d on $\text{CF}^*(L_0, L_1)$ satisfies $d^2 = m_0(L_0) - m_0(L_1)$. This relation arises from Maslov index 2 disks bubbling off the two sides of a 1-dimensional moduli space of Floer strips [33]; see also [45, (2.3.9)] and [46, Figure 2]. Here, indeed, the gluing of the side bubbles contributes with the opposite signs and results in the term $m_0(L_0) - m_0(L_1)$. The reference for the signs in this relation is [20]; see specifically Remark 3.2.21(1), Formula (3.3.4) and Chapter 8 from that book. Although in our case

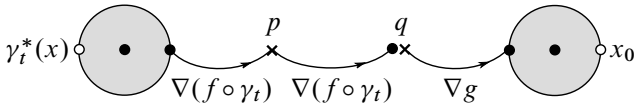


Figure 7: The domains when $l = +\infty$ and the principal disk together with a flowline are constant. Here p is an index-1 critical point of $f \circ \gamma_t$, and q is the minimum of f .

we would be gluing a constant disk rather than Maslov index 2 disk, and the nonconstant curve in Figure 6, right, satisfies a different equation than the standard Floer one (eg our equation is not \mathbb{R} -invariant), the required orientation analysis is essentially the same.

The outcome of the cancellation discussed above is that the count of configurations in Figure 4, top left, ie $\#\mathcal{M}^\gamma(x; x_0)$, equals the count of configurations in Figure 4, bottom right, and it remains to compute the latter.

Step 3 (a Morse-theoretic computation) Let us look at Figure 4, bottom right. Recall that $q \in L$ is the minimum of g , so the semi-infinite flowline of ∇g flowing into q must be constant. Second, we have arranged the principal (central) disk to be constant, as well. So the configurations in Figure 4, bottom right, reduce to those shown in Figure 7.

The free parameter $t = e^{is} \in S^1 \setminus \{1\}$ is “unseen” by the domain after the principal disk became ghost (ie constant), but the equations still depend on it. First, consider the left disk and the left flowline in Figure 7, forgetting the rest of the configuration. Those disk and flowline satisfy the γ_t -pullback of the equation defining the PSS map Ψ , so for each t , the linear combination of points p appearing as limits of such configurations equals $\gamma_t(\Psi(x))$, where $\Psi(x) \in C^1(L)$ is the PSS image which is a linear combination of index-1 critical points of f , so that $\gamma_t(\Psi(x))$ is a combination of critical points of $f \circ \gamma_t$.

Let us now add back the middle flowline, still forgetting the right flowline and the right disk, and count the resulting configurations. The middle flowline is a semi-infinite flowline of $\nabla(f \circ \gamma_t)$ ending at the point q ; note that q is *not* a critical point of f . Suppose for the moment that we allow the right end of the middle flowline to be free (not constrained to q) and denote the moduli space of such configurations by P . Then we can consider the evaluation map at the right end of the flowline, $ev: P \rightarrow L$. The image of ev , as a chain, is a linear combination of unstable manifolds, with respect to the function $f \circ \gamma_t$, associated with the linear combination of the critical points p which we have previously computed. Recall that this linear combination of points p equals $\gamma_t(\Psi(x))$. Consequently, if we denote by $C_{\Psi(x)} \subset L$ the disjoint union of

(oriented, codimension-1) unstable manifolds of the Morse cochain $\Psi(x) \in C^1(L)$ with respect to f , then

$$P = (S^1 \setminus \{1\}) \times C_{\Psi(x)}, \quad \text{ev}(t, z) = \gamma_t(z).$$

Those configurations which evaluate at $q \in L$ are the intersection points $C_{\Psi(x)} \cap l$, where $l = \{\gamma_t(q)\}_{t \in S^1}$ is the orbit of q . By perturbing γ_t and f , the intersections can be easily made transverse, and we get

$$\#(P \times_{\text{ev}} \{q\}) = [C_{\Psi(x)}] \cdot [l] = \langle \Psi(x), l \rangle.$$

Recall this is the count of the part of configurations in [Figure 7](#) which end up at q . Finally, the count of the rightmost flowlines (emerging from q) plus the right disks in [Figure 7](#) equals $1_L \in \text{CF}^0(L, L)$. Indeed, the g -unstable manifold of the minimum q is the whole manifold L (minus a codimension-2 subset), so the count is the same as the count of the rightmost disks only, and the latter produces 1_L by definition.

Putting everything together and noting that $\epsilon = \epsilon(l)$ by [Remark 2.2](#) (since the total boundary in [Figure 7](#) is contractible for a small Hamiltonian perturbation), we get the statement of [Proposition 2.8](#). One last thing is to argue that the moduli spaces we have been using are regular.

According to [\[45\]](#), the regularity of moduli spaces of pearly trajectories consisting of pseudoholomorphic disks and flowlines is equivalent to the regularity of the separate disks and flowlines not constrained to satisfy the incidence conditions, plus the transversality of the evaluation maps which account for the incidence conditions.

The nonconstant disks in the proof carry the pseudoholomorphic equation with a Hamiltonian perturbation which makes them regular. The constant disks are known to be regular on their own; and it is easy to see that for generic f and g , the flowlines are transverse to the evaluation maps for all appearing configurations. \square

2.4 Checking nontriviality in Hochschild cohomology

In this subsection, we prove [Theorem 1.7\(b\)–\(c\)](#) (recall that part (a) was proved earlier; see [Corollary 2.4](#)). We have computed in [Proposition 2.8](#) the map $\mathcal{CO}^1(S(\gamma))|_{\text{CF}^1(L, L)}$, and it remains to see when the result survives to something nontrivial on the level of Hochschild cohomology, and thus distinguishes $\mathcal{CO}^*(S(\gamma)) \in \text{HH}^*(L, L)$ from the unit in $\text{HH}^*(L, L)$.

First, let us quickly recall the definition of Hochschild cohomology. Let A be an A_∞ algebra, and assume it is $\mathbb{Z}/2$ graded if $\text{char } \mathbb{K} \neq 2$. When $\text{char } \mathbb{K} = 2$, we consider A as an ungraded algebra. The space of Hochschild cochains is, by definition,

$$\text{CC}^*(A, A) = \prod_{k \geq 0} \text{Hom}(A^{\otimes k}, A).$$

If A is $\mathbb{Z}/2$ -graded, then $\text{CC}^*(A, A)$ is $\mathbb{Z}/2$ -graded:

$$\text{CC}^r(A, A) = \prod_{k \geq 0} \text{Hom}(A^{\otimes k}, A[r - k]).$$

If $h = \{h^k\}_{k \geq 0} \in \text{CC}^*(A)$ with $h^k: A^{\otimes k} \rightarrow A$, then the Hochschild differential of h is the sequence of maps

$$\begin{aligned} (\partial h)^k(a_k, \dots, a_1) = & \sum_{i+j \leq k} (-1)^{(r+1)(|a_1| + \dots + |a_i| + i)} \mu^{k+1-i}(a_k, \dots, a_{i+j+1}, h^j(a_{i+j}, \dots, a_{i+1}), a_i, \dots, a_1) \\ & + \sum_{i+j \leq k} (-1)^{r+1+|a_1| + \dots + |a_i| + i} \cdot h^{k+1-i}(a_k, \dots, a_{i+j+1}, \mu^j(a_{i+j}, \dots, a_{i+1}), a_i, \dots, a_1). \end{aligned}$$

Here r is the $\mathbb{Z}/2$ -degree of h . (When $k = 0$, the agreement is that $\text{Hom}(A^{\otimes 0}, A) = A$, so h^0 is an element of A .) If $\text{char } \mathbb{K} = 2$, we do not need the gradings as the signs do not matter.

Let us return to the A_∞ algebra $\text{CF}^*(L, L)$. We continue to use the Morse \mathbb{Z} -grading on the vector space $\text{CF}^*(L, L)$ keeping in mind this grading is not respected by the A_∞ structure. If L is oriented, the reduced $\mathbb{Z}/2$ -grading is preserved by the A_∞ structure, so $\text{CF}^*(L, L)$ is a $\mathbb{Z}/2$ -graded A_∞ algebra. If L is not oriented, we must suppose $\text{char } \mathbb{K} = 2$.

Proof of Theorem 1.7(b) We continue to work with \mathcal{CO}^* on the chain level. Because the homological closed-map is unital [45, Lemma 2.3], the Hochschild cohomology unit is realised by the cochain $1_{\text{HH}} := \mathcal{CO}^*(1) \in \text{CC}^*(L, L)$, where 1 is the unit in $\text{QH}^*(X)$. The A_∞ category $\text{CF}^*(L, L)$ need not be strictly unital, so the maps $(1_{\text{HH}})^k = \mathcal{CO}^k(1)$ need not vanish for $k > 0$. However, because the identity Hamiltonian loop preserves L and has homologically trivial orbits on it, Proposition 2.8 applies to $1 = S(\text{Id}) \in \text{QH}^*(X)$ and says that $(1_{\text{HH}})^1(x) = 0$ for any Floer cocycle $x \in \text{CF}^1(L, L)$.

Suppose $\mathcal{CO}^*(S(\gamma)) + \alpha \cdot 1_{\text{HH}}$ is the coboundary of an element $h \in \text{CC}^*(L, L)$ for some $\alpha \in \mathbb{K}$. By comparing $(\partial h)^0$ with $\mathcal{CO}^0(S(\gamma)) + \alpha \cdot (1_{\text{HH}})^0$ (see Corollary 2.4),

we get

$$\mu^1(h^0) = (-1)^{\epsilon(l)} \rho(l) \cdot 1_L + \alpha \cdot 1_L.$$

Here μ^1 is the Floer differential and 1_L is a chain-level cohomology unit; see (2-3). The assumption $\mathrm{HF}^*(L, L) \neq 0$ implies that the Floer cohomology unit 1_L cannot be killed by the Floer differential. Therefore, we cannot solve the above equation unless $\alpha = -(-1)^{\epsilon(l)} \rho(l)$ and $\mu^1(h^0) = 0$. Next, by comparing $(\partial h)^1$ with $\mathcal{CO}^1(S(\gamma)) + \alpha \cdot (1_{\mathrm{HH}})^1$ (see Proposition 2.8) for any Floer cocycle $x \in \mathrm{CF}^1(L, L)$, we get

$$\begin{aligned} (-1)^{|h^0|+1} \mu^2(x, h^0) + (-1)^{(|h^0|+1)(|x|+1)} \mu^2(h^0, x) + \mu^1(h^1(x)) + (-1)^{|h^1|+1} h^1(\mu^1(x)) \\ = (-1)^{\epsilon(l)} \rho(l) \cdot \langle \Psi(x), l \rangle \cdot 1_L. \end{aligned}$$

Here we are using the version of Ψ as in (2-5). Because x is a Floer cocycle, the last summand of the left-hand side vanishes. If $\mathrm{char} \mathbb{K} = 2$, let $a := h^0 \in \mathrm{CF}^*(L, L)$. If $\mathrm{char} \mathbb{K} \neq 2$, let $a \in \mathrm{CF}^{\mathrm{odd}}(L, L)$ be the odd degree part of h^0 . By computing the signs in the above equality, we get for any Floer cocycle $x \in \mathrm{CF}^1(L, L)$,

$$\mu^2(x, a) + \mu^2(a, x) + \mu^1(h^1(x)) = (-1)^{\epsilon(l)} \rho(l) \cdot \langle \Psi(x), l \rangle \cdot 1_L.$$

Recall that $\mu^1(h^0) = 0$, so $\mu^1(a) = 0$ as well, and we get the following equality for Floer cohomology classes $[x], [a] \in \mathrm{HF}^*(L, L)$ and $\Psi(x) \in H^1(L)$:

$$\mu^2([x], [a]) + \mu^2([a], [x]) = (-1)^{\epsilon(l)} \rho(l) \cdot \langle \Psi(x), l \rangle \cdot 1_L \in \mathrm{HF}^*(L, L).$$

Now put $x = \Phi(y)$, where $y \in C^1(L)$ is a Morse cochain and Φ is the chain-level map from (2-4). The above equality means that for all $[y] \in H^1(L)$,

$$\mu^2(\Phi([y]), [a]) + \mu^2([a], \Phi([y])) = (-1)^{\epsilon(l)} \rho(l) \cdot \langle [y], l \rangle \cdot 1_L \in \mathrm{HF}^*(L, L).$$

This time, we have used the homology-level version of Φ from (2-5). The above equality is exactly prohibited by the hypothesis of Theorem 1.7(b), so Theorem 1.7(b) is proved. \square

Proof of Theorem 1.7(c) Note that, on the homology level,

$$\mathcal{CO}^0(S(\gamma) * Q) = \mathcal{CO}^0(S(\gamma)) \cdot \mathcal{CO}^0(Q) = 1_L \cdot \mathcal{CO}^0(Q) = \mathcal{CO}^0(Q)$$

(here the dot denotes the μ^2 product), so the only possible linear relation between $\mathcal{CO}^*(S(\gamma) * Q)$ and $\mathcal{CO}^*(Q)$ is the following, where 1 is the unit in $\mathrm{QH}^*(X)$:

$$\mathcal{CO}^*((S(\gamma) - 1) * Q) = 0.$$

We have $\mathcal{CO}^*((S(\gamma) - 1) * Q) = \mathcal{CO}^*(S(\gamma) - 1) \star \mathcal{CO}^*(Q)$, where the symbol \star denotes the Yoneda product in Hochschild cohomology.

Let us now return to working with \mathcal{CO}^* on the chain level. Recall that if $\phi = \{\phi^k\}_{k \geq 0}$, $\psi = \{\psi^k\}_{k \geq 0} \in \mathcal{CC}^*(L, L)$ are Hochschild cochains, the $k = 1$ part of their Yoneda product by definition equals

$$(\phi \star \psi)^1(x) = \pm \mu^2(\phi^1(x), \psi^0) \pm \mu^2(\phi^0, \psi^1(x)).$$

There is an explicit formula for the signs which we do not need. Let us apply this formula to $\mathcal{CO}^*(S(\gamma) - 1)$ and $\mathcal{CO}^*(Q)$. We know that $(\mathcal{CO}^*(S(\gamma) - 1))^0 = 0$ by [Corollary 2.4](#), and $(\mathcal{CO}^*(S(\gamma) - 1))^1(x) = \mathcal{CO}^1(S(\gamma))(x)$ is given by [Proposition 2.8](#) for any Floer cocycle $x \in \text{CF}^1(L, L)$. Consequently, we get

$$\mathcal{CO}^1((S(\gamma) - 1) * Q)(x) = (-1)^{\epsilon(l)} \rho(l) \cdot \langle \Psi(x), l \rangle \cdot \mathcal{CO}^0(Q).$$

From this point, the rest of the proof follows the one of [Theorem 1.7\(b\)](#). □

3 The closed-open map for real toric Lagrangians

In this section, after a short proof of [Theorem 1.13](#), we look for further examples of real toric Lagrangians where [Theorem 1.7](#) can be effectively applied. We also discover that [Proposition 2.8](#), after additional work, allows us to show that the Fukaya A_∞ algebra of some of the considered Lagrangians is not formal. In particular, we prove the results about real toric Lagrangians stated in [Section 1](#) (except for [Proposition 1.1](#) and [Corollary 1.2](#), which have been proved therein). We work with a coefficient field \mathbb{K} of characteristic 2 throughout this section.

3.1 A proof of [Theorem 1.13](#)

Let X be a compact, smooth toric Fano variety, and $D \subset X$ be a toric divisor corresponding to one of the facets of the polytope defining X . There is a Hamiltonian circle action γ on X associated with D , which comes from the toric action by choosing a Hamiltonian which achieves maximum on D . A theorem of McDuff and Tolman [\[32\]](#) says the following.

Theorem 3.1 $S(\gamma) = D^*$, where $D^* \in \text{QH}^*(X)$ is the Poincaré dual of D . □

The loop γ never preserves the real Lagrangian $L \subset X$, but if we parametrise $\gamma = \{\gamma_t\}_{t \in [0,1]}$, then $\gamma_{1/2}(L) = L$; see [\[23\]](#). Consequently, $\alpha = \{\gamma_t(L)\}_{t \in [0,1/2]}$ is a

loop of Lagrangian submanifolds, and moreover, we have $\alpha^2 = \{\gamma_t(L)\}_{t \in [0,1]}$ in the space of Lagrangian loops. There is an associated Lagrangian Seidel element $S_L(\alpha) \in \text{HF}^*(L, L)$, which counts pseudoholomorphic disks with rotating boundary condition α , and a single boundary puncture which evaluates to an element of $\text{HF}^*(L, L)$. A theorem of Hyvriier [25, Theorem 1.13], based on the disk doubling trick, computes $S_L(\alpha)$.

Theorem 3.2 We have $S_L(\alpha) = [L \cap D]^*$, where $L \cap D$ is the clean intersection that has codimension 1 in L , and $[L \cap D]^* \in H^1(L) \subset \text{HF}^*(L, L)$ is its dual class. \square

The inclusion $H^1(L) \subset \text{HF}^*(L, L)$ is the PSS map Φ from Section 2, which is injective because $\text{HF}^*(L, L) \cong H^*(L)$ by Theorem 1.12.

Proof of Theorem 1.13 It suffices to prove that $\mathcal{CO}^0(D^*) = \mathcal{D}(\mathcal{F}(D^*))$, where $D \subset X$ is a toric divisor as above and $D^* \in \text{QH}^*(X)$ is its dual class, because such D^* generate $\text{QH}^*(X)$ as an algebra [32]. Let γ be the Hamiltonian loop corresponding to D as above, and α be the Lagrangian loop as above, such that $\alpha^2 = \{\gamma_t(L)\}_{t \in [0,1]}$. It follows from Theorem 2.1 that

$$\mathcal{CO}^0(S(\gamma)) = S_L(\alpha^2),$$

and the latter can be rewritten as $\mathcal{F}(S_L(\alpha))$, where \mathcal{F} is the Frobenius map on $\text{HF}^*(L, L)$. By Theorem 3.1, $S(\gamma) = D^*$, and by Theorem 3.2, $S_L(\alpha) = [L \cap D]^*$. Finally, if we look at Haug’s construction [23] of the Duistermaat isomorphism \mathcal{D} , we will see that $[L \cap D]^* = \mathcal{D}(D^*)$. Putting everything together, we get

$$\mathcal{CO}^0(D^*) = \mathcal{F}(\mathcal{D}(D^*)).$$

Because \mathcal{D} is a ring map, it commutes with the Frobenius maps on $\text{HF}^*(L, L)$ and $\text{QH}^*(X)$, and the theorem follows. \square

3.2 Split-generation for toric varieties with Picard rank 2

It is known that the unique toric variety with Picard number 1 is the projective space. By a theorem of Kleinschmidt [28] (see also [14]), every n -dimensional toric Fano variety whose Picard group has rank 2 (ie whose fan has $n + 2$ generators) is isomorphic to the projectivisation of a sum of line bundles over $\mathbb{C}P^{n-k}$

$$(3-1) \quad X(a_1, \dots, a_k) := \mathbb{P}_{\mathbb{C}P^{n-k}}(\mathcal{O} \oplus \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_k))$$

$$\text{with } a_i \geq 0, \sum_{i=1}^k a_i \leq n - k - 1.$$

(The imposed conditions on the a_i are equivalent to X being toric Fano.) The $n + 2$ vectors in \mathbb{Z}^n generating the fan of $X(a_1, \dots, a_k)$ are the columns of the matrix

$$(3-2) \quad \begin{pmatrix} & -1 & a_1 \\ & \vdots & \vdots \\ I_{n \times n} & -1 & a_k \\ & 0 & -1 \\ & \vdots & \vdots \\ & 0 & -1 \end{pmatrix}$$

The minimal Chern number of $X(a_1, \dots, a_k)$ equals $\gcd(k + 1, n - k + 1 - \sum a_i)$; see [36]. Some of these varieties provide further examples where, using Theorems 1.13 and 1.7, we can prove the injectivity of \mathcal{CO}^* and deduce split-generation.

Theorem 3.3 *Let $X := X(a_1, \dots, a_k)$ be as above, $L \subset X$ the real Lagrangian, and \mathbb{K} a field of characteristic 2. Suppose all a_i are odd and $\gcd(k + 1, n - k + 1 - \sum a_i) \geq 2$.*

- (a) *If $n - k + 1$ is odd, then $\mathcal{CO}^0: \text{QH}^*(X) \rightarrow \text{HF}^*(L, L)$ is injective.*
- (b) *If $n - k + 1$ is even, k is even and the numbers a_i come in equal pairs, then $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(L, L)$ is injective while \mathcal{CO}^0 is not.*

In both cases, L split-generates $\mathcal{Fuk}(X)_0$.

Proof Let $x, y \in H^2(X)$ be the generators corresponding to the last two columns of the matrix (3-2). They generate $\text{QH}^*(X)$ as an algebra and satisfy the following relations when $\text{char } \mathbb{K} = 2$:

$$x(x + y)^k = 1, \quad y^{n-k+1}(x + y)^{-\sum a_i} = 1.$$

(For brevity, we no longer use the symbol $*$ to denote the quantum product.) If $n - k + 1$ is odd, one can show that the Frobenius endomorphism \mathcal{F} on $\text{QH}^*(X)$ is an isomorphism, so \mathcal{CO}^0 is injective by Theorem 1.13. It follows that \mathcal{CO}^* is also injective, and split-generation follows from Theorem 1.10. Part (a) is proved.

In the rest of the proof we work with case (b), so we let $n - k + 1 = 2r$, $k = 2q$, and $\sum a_i = 2p$. The rewritten relations in $\text{QH}^*(X)$ are

$$(3-3) \quad x(x + y)^{2q} = 1, \quad y^{2r}(x + y)^{-2p} = 1.$$

Lemma 3.4 *For the ring $\text{QH}^*(X)$ as in (3-3), the kernel of \mathcal{F} is the ideal generated by $y^r(x + y)^{-p} + 1$.*

Proof Equations (3-3) are equivalent to

$$x^{-p} = y^{2rq}, \quad y^{4rq+2r} + y^{4rp+2p} + 1 = 0,$$

where the second equation is rewritten from the second equation in (3-3) using the substitution $x^{-p} = y^{2rq}$. This means if we let

$$R(y) = y^{2rq+r} + y^{2rq+p} + 1,$$

then $R(y) = y^r(x + y)^{-p} + 1$. Let $g = \gcd(2rq, p)$ and let $\alpha, \beta \in \mathbb{Z}$ be such that

$$-2rq \cdot \alpha + p \cdot \beta = g.$$

Consider the map $\phi: \mathbb{K}[u] \rightarrow \text{QH}^*(X)$ given by $u \mapsto x^\alpha y^\beta$; this map is onto because

$$(3-4) \quad \phi(u^{p/g}) = y, \quad \phi(u^{2qr/g}) = x^{-1}$$

using the given relations (note that the powers $p/g, 2qr/g$ are integral). Further, $\ker \phi$ is obviously the ideal generated by $V(u)^2$, where $V(u) := R(u^{p/g})$, and we conclude that ϕ provides an isomorphism

$$(3-5) \quad \phi: \mathbb{K}[u]/V(u)^2 \xrightarrow{\cong} \text{QH}^*(X), \quad V(u) = u^{(p/g)(2rq+r)} + u^{(p/g)(2rq+p)} + 1.$$

It is clear that $V(u)$ generates the kernel of the Frobenius map on $\mathbb{K}[u]/V(u)^2$. Because $V(u)$ corresponds to $y^r(x + y)^{-p} + 1$ under ϕ , Lemma 3.4 follows. \square

We continue the proof of Theorem 3.3(b). It turns out that, similarly to the case of $\mathbb{R}P^n \subset \mathbb{C}P^n$ studied in the introduction, the generator of $\ker \mathcal{F}$ from Lemma 3.4 equals $S(\gamma) + 1$ for a real Hamiltonian loop γ on X which preserves L setwise and has homologically nontrivial orbits on it. To construct γ , we will need the additional assumption that the a_i come in equal pairs, so we assume the sequence $(a_i)_{i=1}^{2q}$ is $(a_1, a_1, \dots, a_q, a_q)$.

Recall that X , being a toric manifold, is a quotient of $\mathbb{C}^{2r+2q+1}$ minus some linear subspaces determined by the fan, by an action of $(\mathbb{C}^*)^2$. Using the common notation, this action is given by $z \mapsto t_1^{v_1} t_2^{v_2} z$, where $z \in \mathbb{C}^{2r+2q+1}$ and v_1, v_2 are the vectors in $\mathbb{Z}^{2r+2q+1}$ given by the following two rows:

$$(3-6) \quad \begin{array}{cccc|ccc|cc} & \text{2q entries} & & & & \text{2r-1 entries} & & \text{2 entries} & & \\ & a_1 & a_1 & \cdots & a_q & a_q & -1 & \cdots & -1 & 0 & -1 \\ & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & 0 & -1 & 0 \end{array}$$

Let $(z_1, \dots, z_{2r+2q+1})$ be the coordinates on $\mathbb{C}^{2r+2q+1}$. The action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^{2r+2q+1}$ commutes with the action of

$$G = \mathrm{SU}(2)^q \times \mathrm{SU}(2r),$$

where the $\mathrm{SU}(2)$ factors act respectively on $(z_1, z_2), \dots, (z_{2q-1}, z_{2q})$, and $\mathrm{SU}(2r)$ acts on $(z_{2q+1}, \dots, z_{2q+2r-1}, z_{2q+2r+1})$; note we have omitted z_{2q+2r} . (If we view X as a projective bundle over $\mathbb{C}P^{2r-1}$ as in (3-1), the coordinates on which $\mathrm{SU}(2r)$ acts are the homogeneous coordinates on the base.) Denote the real form of G by $G^{\mathbb{R}} = \mathrm{SO}(2)^q \times \mathrm{SO}(2r)$. Because all a_i are odd, the action of $(-1, +1) \in (\mathbb{C}^*)^2$ coincides with the action of $-I \in G$. Consequently, the action of G descends to a Hamiltonian action of $G/\pm I$ on X . Its real form $G^{\mathbb{R}}/\pm I$ preserves the real Lagrangian $L \subset X$, and we let γ be the S^1 -subgroup of $G^{\mathbb{R}}/\pm I$ defined as follows. This subgroup lifts to the path from I to $-I$ in $G^{\mathbb{R}}$ which is the image of the rotation $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in \mathrm{SO}(2)$, $t \in [0, \pi]$, under the diagonal inclusions

$$\mathrm{SO}(2) \subset \mathrm{SO}(2)^q \times \mathrm{SO}(2)^r \subset \mathrm{SO}(2)^q \times \mathrm{SO}(2r) = G^{\mathbb{R}}.$$

Recall that we are assuming $\mathrm{char} \mathbb{K} = 2$.

Lemma 3.5 *The homology class of γ -orbits on L is nonzero in $H_1(L; \mathbb{K})$.*

Proof Indeed, L is a real projective bundle over $\mathbb{R}P^{2r-1}$, and the orbits project to the nontrivial cycle on the base, provided $\mathrm{char} \mathbb{K} = 2$. □

Lemma 3.6 *We have $S(\gamma) + 1 = y^r(x + y)^{-p} + 1$ (which is the generator of $\ker \mathcal{F}$ from Lemma 3.4).*

Proof Inside the complex group $G/\pm I$, the loop γ is homotopic to the loop γ' lifting to the path from I to $-I$ in G which is the image of the path $\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \in \mathrm{SU}(2)$, $t \in [0, \pi]$, under the diagonal inclusions

$$\mathrm{SU}(2) \subset \mathrm{SU}(2)^q \times \mathrm{SU}(2)^r \subset \mathrm{SU}(2)^q \times \mathrm{SU}(2r) = G.$$

By using the action of $\mathbb{C}^* \subset (\mathbb{C}^*)^2$ corresponding to the first vector in (3-6), we see that γ' descends to the same Hamiltonian loop in X as the loop γ'' in G which acts on $\mathbb{C}^{2r+2q+1}$, for $t \in [0, 2\pi]$, as follows:

$$(z_1, \dots, z_{2r+2q+1}) \mapsto (e^{it(a_1+1)/2} z_1, e^{it(a_1-1)/2} z_2, \dots, e^{it(a_q+1)/2} z_{2q-1}, e^{it(a_q-1)/2} z_{2q}, z_{2q+1}, e^{-it} z_{2q+2}, \dots, e^{-it} z_{2r+2q-2}, z_{2r+2q-1}, z_{2r+2q}, e^{-it} z_{2r+2q+1}).$$

Note that here t runs through $[0, 2\pi]$, hence the $\frac{1}{2}$ -factors. Because all a_i are odd, γ'' is now a closed loop in G , not only in $G/\pm I$. So by [32], its Seidel element $S(\gamma'') \in \text{QH}^*(X)$ can be computed as the quantum product of powers of the divisors corresponding to the coordinates on $\mathbb{C}^{2r+2q+1}$, where the powers are the multiplicities of rotations. Given $\text{char } \mathbb{K} = 2$, and recalling that $S(\gamma'') = S(\gamma') = S(\gamma)$, we get

$$\begin{aligned} S(\gamma) &= (x+y)^{(a_1+1)/2}(x+y)^{(a_1-1)/2} \dots (x+y)^{(a_q+1)/2}(x+y)^{(a_q-1)/2} y^{-1} \dots y^{-1} \\ &= (x+y)^p y^{-r}. \end{aligned}$$

This element squares to 1 by (3-3) (in agreement with the fact γ has order 2 in $\pi_1(G/\pm I) \cong \mathbb{Z}/2$), so it also equals $y^r(x+y)^{-p}$, which proves Lemma 3.6. \square

We conclude the proof of Theorem 3.3(b). By Lemmas 3.4 and 3.6, $\ker \mathcal{F}$ is the ideal generated by $S(\gamma) + 1$. Suppose $P \in \text{QH}^*(X)$ such that $\mathcal{CO}^*(P) = 0 \in \text{HH}^*(L, L)$. Then $\mathcal{CO}^0(P) = 0$, so $P \in \ker \mathcal{F}$ by Theorem 1.13. Consequently, $P = (S(\gamma) + 1) * Q$, and if $P \neq 0$, then $Q \notin \ker \mathcal{F}$ (because otherwise we would get $P \in (\ker \mathcal{F})^2 = \{0\}$). Apply Theorem 1.7(b) to the product $(S(\gamma) + 1) * Q$; the left-hand side of (***) vanishes because μ^2 is commutative on $\text{HF}^*(L, L)$ [23], and the right-hand side is nontrivial for some y by Lemma 3.5 and because $\mathcal{CO}^0(Q) \neq 0$. It follows that $\mathcal{CO}^*(P) \neq 0$. We have shown that \mathcal{CO}^* is injective, and split-generation follows from Theorem 1.10. Note that $w(L) = 0$ holds for all real Lagrangians, as Maslov index 2 disks come in pairs because of the action of the antiholomorphic involution; see [23]. \square

The following corollary in particular implies Proposition 1.4 from the introduction.

Corollary 3.7 *Let $X = \text{Bl}_{\mathbb{C}P^{2q-1}} \mathbb{C}P^{2r+2q-1}$, and let $L \subset X$ be the real Lagrangian (diffeomorphic to $\text{Bl}_{\mathbb{R}P^{2q-1}} \mathbb{R}P^{2r+2q-1}$). Assume $\text{gcd}(2q + 1, 2r - 2q) \geq 2$ and that either r or q are odd. Then $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HF}^*(L, L)$ is injective, although \mathcal{CO}^0 is not. Consequently, L split-generates $\mathcal{Fuk}(X)_0$.*

Proof of Proposition 1.4 Take X as in (3-1) with $a_1 = \dots = a_k = 1$. Then $X = \text{Bl}_{\mathbb{C}P^{k-1}} \mathbb{C}P^n$; see eg [17, Proposition 11.14]. The additional hypotheses of the current corollary make sure X satisfies all conditions of Theorem 3.3(b), which together with the split-generation criterion (Theorem 1.10(b)) implies the corollary. \square

In order to deduce nondisplaceability results between the real Lagrangian L and other Lagrangians with arbitrary obstruction numbers, we need the following lemma.

Lemma 3.8 Suppose $\text{char } \mathbb{K} = 2$, L is an object of $\mathcal{Fuk}(X)_w$ and $\mathcal{CO}^*: \text{QH}^*(X) \rightarrow \text{HH}^*(L, L)$ is injective. Assuming [Hypothesis 3.9](#) below, $L \times L$ split-generates $\mathcal{Fuk}(X \times X)_0$.

Note that by [Lemma 1.9](#), the condition of [Lemma 3.8](#) can only hold if $\text{QH}^*(X) = \text{QH}^*(X)_w$ or L is nonorientable.

Proof First, observe that $w(L \times L) = 2w(L) = 0$. By [\[45\]](#), the injectivity of \mathcal{CO}^* is equivalent to the fact that the open-closed map $\mathcal{OC}^*: \text{HH}_*(L, L) \rightarrow \text{QH}^*(X)$ hits the unit $1 \in \text{QH}^*(X)$.

Hypothesis 3.9 There is a commutative diagram

$$\begin{array}{ccc} \text{HH}_*(L, L) \otimes \text{HH}_*(L, L) & \longrightarrow & \text{HH}_*^{\text{split}}(L \times L, L \times L) \\ \downarrow \mathcal{OC}^* \otimes \mathcal{OC}^* & & \downarrow \mathcal{OC}_{\text{prod}}^* \\ \text{HF}^*(X) \otimes \text{HF}^*(X) & \xrightarrow{=} & \text{HF}^*(X \times X) \end{array}$$

where $\mathcal{OC}_{\text{prod}}^*$ is the open-closed map on the product, and the HF^* are Hamiltonian Floer cohomologies, isomorphic to the quantum cohomologies of the corresponding spaces.

Here $\text{HH}_*^{\text{split}}(L \times L, L \times L)$ indicates that the A_∞ structure on $L \times L$ is computed using a split Hamiltonian perturbation and a product almost complex structure; such a choice can be made regular. We expect the hypothesis to hold following [Ganatra \[22, Remark 11.1\]](#), who stated it on chain level, in the setup of the wrapped Fukaya category of an exact manifold. A slight complication is that the A_∞ algebra of $L \times L$ appearing in the top right corner of the diagram had to be equipped with so-called one-sided homotopy units; their presence is denoted by a tilde in [\[22, Remark 11.1\]](#). This does not affect the diagram on the homology level [\[22, Proposition 10.10\]](#), but we have not checked how this subtlety carries over to the monotone setup; therefore we leave [Hypothesis 3.9](#) as a conjecture.

Given [Hypothesis 3.9](#), if \mathcal{OC}^* hits the unit, then $\mathcal{OC}^* \otimes \mathcal{OC}^*$ and $\mathcal{OC}_{\text{prod}}^*$ also do. The latter fact implies that \mathcal{CO}^* is injective on the product, and split-generation follows from [Theorem 1.10\(b\)](#). □

[Corollary 1.5](#) from the introduction is a particular case of the following.

Corollary 3.10 *Let \mathbb{K} be a field of characteristic 2 and $L \subset X$ be as in [Theorem 3.3\(a\)](#) or (b), or as in [Corollary 3.7](#). Suppose $L' \subset X$ is another monotone Lagrangian, perhaps equipped with a local system $\pi_1(L) \rightarrow \mathbb{K}^\times$, with minimal Maslov number at least 2 and such that $\text{HF}^*(L', L') \neq 0$. If $w(L') \neq 0$, assume [Hypothesis 3.9](#). Then $L \cap L' \neq \emptyset$.*

Proof If $w(L') = 0$, this follows from the fact that L split-generates $\mathcal{Fuk}(X)_0$ and from [Lemma 1.11](#). If $w(L') \neq 0$, we have that $w(L' \times L') = 2w(L') = 0$, so $L' \times L'$ is an object of $\mathcal{Fuk}(X \times X)_0$ which is split-generated by $L \times L$ by [Lemma 3.8](#). Then $(L \times L) \cap (L' \times L') \neq 0$ by [Lemma 1.11](#), and so $L \cap L' \neq \emptyset$. \square

3.3 An application to nonformality

Recall that if $\mathcal{A} \rightarrow \mathcal{A}'$ is a quasi-isomorphism of A_∞ categories, it induces an isomorphism $\text{HH}^*(\mathcal{A}) \rightarrow \text{HH}^*(\mathcal{A}')$; see eg Seidel [[42](#), (1.14)]. We will need an explicit chain-level formula for this isomorphism, which can be obtained by combining Seidel’s argument with Ganatra’s functoriality formulas [[22](#), Section 2.9], and this requires a short account. We are assuming the reader is familiar with the basic language of A_∞ categories from eg [[40](#); [45](#); [22](#)], so that we can skip some basic definitions and present the other ones rather informally. For simplicity, we are working with $\text{char } \mathbb{K} = 2$ so we won’t have to worry about signs, and we restrict to A_∞ algebras rather than categories.

Recall that if \mathcal{A} is an A_∞ algebra, its Hochschild cohomology $\text{HH}^*(\mathcal{A})$ can be seen as Hochschild cohomology $\text{HH}^*(\mathcal{A}, \mathcal{A})$ of \mathcal{A} as an \mathcal{A} – \mathcal{A} bimodule. If $F: \mathcal{A} \rightarrow \mathcal{A}'$ is a quasi-isomorphism between A_∞ algebras, it induces quasi-isomorphisms

$$(3-7) \quad \text{CC}^*(\mathcal{A}, \mathcal{A}) \xrightarrow{F_*} \text{CC}^*(\mathcal{A}, F^*\mathcal{A}') \xleftarrow{F^*} \text{CC}^*(\mathcal{A}', \mathcal{A}'),$$

which proves that $\text{HH}^*(\mathcal{A}, \mathcal{A}) \cong \text{HH}^*(\mathcal{A}', \mathcal{A}')$. Chain-level formulas for the two intermediate quasi-isomorphisms, which we will now recall, were written down eg by Ganatra [[22](#), Section 2.9] (in the context of Hochschild homology, but these are easily adjusted to cohomology).

If $\mathcal{B}, \mathcal{B}'$ are two \mathcal{A} – \mathcal{A} bimodules, a morphism $G: \mathcal{B} \rightarrow \mathcal{B}'$ is a sequence of maps $G^k: \mathcal{A}^{\otimes i} \otimes \mathcal{B} \otimes \mathcal{A}^{\otimes j} \rightarrow \mathcal{B}'$, $i + j + 1 = k$, satisfying a sequence of relations which we informally write down as

$$\sum_{\star} G^*(\text{Id}^{\otimes \star} \otimes \mu_{\mathcal{A} \text{ or } \mathcal{B}}^{\star} \otimes \text{Id}^{\otimes \star}) = \sum_{\star} \mu_{\mathcal{B}'}^{\star}(\text{Id}^{\otimes \star} \otimes G^{\star} \otimes \text{Id}^{\otimes \star}).$$

Here \star are positive integers which are mutually independent but are such that the total number of inputs on both sides of the equation is the same, the sum is over all such

possibilities, and the structure map on the left is $\mu_{\mathcal{A}}^*$ or $\mu_{\mathcal{B}}^*$ depending on whether one of its arguments is in \mathcal{B} . In its full form, the above relation should be written as follows:

$$\begin{aligned} & \sum G^{i_1+i_4-i_3+1}(a_1, \dots, a_{i_1}, \mu_{\mathcal{B}}^{i_3-i_1+1}(a_{i_1+1}, \dots, a_{i_2}, b, a_{i_2+1}, \dots, a_{i_3}), a_{i_3+1}, \dots, a_{i_4}) \\ & + \sum G^{i_1+i_4-i_2+2}(a_1, \dots, a_{i_1}, \mu_{\mathcal{A}}^{i_2-i_1}(a_{i_1+1}, \dots, a_{i_2}), a_{i_2+1}, \dots, a_{i_3}, b, a_{i_3+1}, \dots, a_{i_4}) \\ & + \sum G^{i_2+i_4-i_3+2}(a_1, \dots, a_{i_1}, b, a_{i_1+1}, \dots, a_{i_2}, \mu_{\mathcal{A}}^{i_3-i_2}(a_{i_2+1}, \dots, a_{i_3}), a_{i_3+1}, \dots, a_{i_4}) \\ & = \sum \mu_{\mathcal{B}'}^{i_1+i_4-i_3+1}(a_1, \dots, a_{i_1}, G^{i_3-i_1+1}(a_{i_1+1}, \dots, a_{i_2}, b, a_{i_2+1}, \dots, a_{i_3}), a_{i_3+1}, \dots, a_{i_4}) \end{aligned}$$

for $a_i \in \mathcal{A}$ and $b \in \mathcal{B}$, where the sums are all over $i_1 + i_2 + i_3 + i_4 = l$. We will keep the informal style of notation, in which the inputs are omitted and the valencies are replaced by \star . The induced map $G_*: \text{CC}^*(\mathcal{A}, \mathcal{B}) \rightarrow \text{CC}^*(\mathcal{A}, \mathcal{B}')$ is defined by

$$(3-8) \quad (G_*(h))^{\star} = \sum_{\star} G^{\star}(\text{Id}^{\otimes \star} \otimes h^{\star} \otimes \text{Id}^{\otimes \star}),$$

where $h^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{B}$ and $(G_*(h))^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{B}'$. If G is a quasi-isomorphism, so is G_* .

If $\mathcal{A}, \mathcal{A}'$ are two A_{∞} algebras, a morphism $F: \mathcal{A} \rightarrow \mathcal{A}'$ is a sequence of maps $F^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{A}'$ such that

$$\sum_{\star} \mu_{\mathcal{A}'}^{\star}(F^{\star} \otimes \dots \otimes F^{\star}) = \sum_{\star} F^{\star}(\text{Id}^{\otimes \star} \otimes \mu_{\mathcal{A}}^{\star} \otimes \text{Id}^{\otimes \star}).$$

Next, if \mathcal{B} is an $\mathcal{A}' - \mathcal{A}'$ bimodule, its two-sided pullback $F^*\mathcal{B}$ is an $\mathcal{A} - \mathcal{A}$ bimodule based on the same vector space \mathcal{B} , whose structure maps [22, Section 2.8] are

$$(3-9) \quad \mu_{F^*\mathcal{B}}^{\star} = \sum_{\star} \mu_{\mathcal{B}}^{\star}(F^{\star} \otimes \dots \otimes F^{\star} \otimes \text{Id}_{\mathcal{B}} \otimes F^{\star} \otimes \dots \otimes F^{\star}).$$

There is also a morphism $F^*: \text{CC}^*(\mathcal{A}', \mathcal{B}) \rightarrow \text{CC}^*(\mathcal{A}, F^*\mathcal{B})$ defined by

$$(3-10) \quad (F^*(h))^{\star} = \sum_{\star} h^{\star}(F^{\star} \otimes \dots \otimes F^{\star}),$$

where $h^{\star}: (\mathcal{A}')^{\otimes \star} \rightarrow \mathcal{B}$ and $(F^*(h))^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{B}$. The total number of inputs here can be zero, and $F^*(h)^0 = h^0$. If F is a quasi-isomorphism, so is F^* .

If, again, $F: \mathcal{A} \rightarrow \mathcal{A}'$ is a morphism of A_{∞} algebras, let $F^*\mathcal{A}'$ be the $\mathcal{A} - \mathcal{A}$ bimodule which is the pullback of \mathcal{A}' seen as an $\mathcal{A}' - \mathcal{A}'$ bimodule.

Lemma 3.11 *The same sequence of maps $F^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{A}'$ provides a morphism of $\mathcal{A} - \mathcal{A}$ bimodules $\mathcal{A} \rightarrow F^*\mathcal{A}'$, also denoted by f .*

Proof We must check $\sum_{\star} F^{\star}(\text{Id}^{\otimes \star} \otimes \mu_{\mathcal{A}}^{\star} \otimes \text{Id}^{\otimes \star}) = \sum_{\star} \mu_{F^{\star}\mathcal{A}}^{\star}(\text{Id}^{\otimes \star} \otimes F^{\star} \otimes \text{Id}^{\otimes \star})$. If we apply formula (3-9) to rewrite the right-hand sum, the unique Id -factor in (3-9), which in our case is $\text{Id}_{\mathcal{A}'}$, gets applied to the F^{\star} -factor. So our right-hand sum equals $\sum_{\star} \mu_{\mathcal{A}'}^{\star}(F^{\star} \otimes \cdots \otimes F^{\star} \otimes \cdots \otimes F^{\star})$, which is exactly the condition that F is a morphism of A_{∞} algebras $\mathcal{A} \rightarrow \mathcal{A}'$. \square

This lemma explains the precise meaning of (3-7): if $F: \mathcal{A} \rightarrow \mathcal{A}'$ is a morphism of A_{∞} algebras, then the first map F_{\star} from (3-7) is the pushforward of F considered as a morphism of modules $\mathcal{A} \rightarrow F^{\star}\mathcal{A}'$ given by formula (3-8). The second map in (3-7) is the pullback as in (3-10).

Next, recall that a general property of quasi-isomorphisms between A_{∞} algebras (bimodules, etc.) is that they have quasi-inverses [40, Chapter 1]. If $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}'$ is a quasi-isomorphism, then the \mathcal{A} - \mathcal{A} bimodule morphism F from Lemma 3.11 is also a quasi-isomorphism; hence there is an \mathcal{A} - \mathcal{A} bimodule quasi-isomorphism $G: F^{\star}\mathcal{A}' \rightarrow \mathcal{A}$ which is a quasi-inverse of F , so we have quasi-isomorphisms

$$(3-11) \quad \text{CC}^*(\mathcal{A}, \mathcal{A}) \xleftarrow{G_{\star}} \text{CC}^*(\mathcal{A}, F^{\star}\mathcal{A}') \xleftarrow{F^{\star}} \text{CC}^*(\mathcal{A}', \mathcal{A}').$$

Their composition acts on Hochschild cochains by

$$(G_{\star}F^{\star}(h))^{\star} = \sum_{\star} G^{\star}(\text{Id}^{\otimes \star} \otimes h^{\star}(F^{\star} \otimes \cdots \otimes F^{\star}) \otimes \text{Id}^{\otimes \star}),$$

where $h^{\star}: (\mathcal{A}')^{\otimes \star} \rightarrow \mathcal{A}'$ and $(G_{\star}F^{\star}(h))^{\star}: \mathcal{A}^{\otimes \star} \rightarrow \mathcal{A}$. In particular, $(G_{\star}F^{\star}(h))^0 = G^1(h^0)$, and if $h^0 = 0 \in \mathcal{A}'$, then

$$(3-12) \quad (G_{\star}F^{\star}(h))^1(u) = G^1(h^1(F^1(u))), \quad u \in \mathcal{A}'.$$

Note that $G^1: \mathcal{A}' \rightarrow \mathcal{A}$ and $F^1: \mathcal{A} \rightarrow \mathcal{A}'$ are chain maps with respect to $\mu_{\mathcal{A}}^1$ and $\mu_{\mathcal{A}'}^1$, and they are cohomology inverses of each other.

Assume $L \subset X$ is a Lagrangian preserved by a Hamiltonian loop γ which together satisfy the conditions of Theorem 1.7 (those conditions which are common to all parts of the theorem). Assume the A_{∞} algebra $\text{CF}^*(L, L)$ is formal, ie there is an A_{∞} quasi-isomorphism $F: \text{HF}^*(L, L) \rightarrow \text{CF}^*(L, L)$. Let

$$(3-13) \quad h := (G_{\star}F^{\star})(\text{CO}^*(S(\gamma)) - 1) \in \text{CC}^*(\text{HF}^*(L, L), \text{HF}^*(L, L)),$$

where G is a quasi-inverse of F , and G_{\star} and F^{\star} are as in (3-11). So h is a Hochschild cochain for the associative algebra $\text{HF}^*(L, L)$. Then by Corollary 2.4, $h^0 = 0$, Proposition 2.8 and (3-12), we have

$$h^1(x) = \rho(l) \cdot \langle \Psi(F^1(x)), l \rangle \cdot G^1(1_L).$$

(We have dropped the extra sign, working in characteristic two.) Let us additionally assume that L is *wide* [9, Definition 1.2.1], ie there is a vector space isomorphism between $H^*(L)$ and $\text{HF}^*(L, L)$, and that L admits a perfect Morse function. These conditions enable us to identify $\text{CF}^*(L, L) \cong \text{HF}^*(L, L)$ as vector spaces. Because G^1 is cohomologically unital, $G^1(1_L) = 1_L \in \text{HF}^*(L, L)$, so

$$(3-14) \quad h^1(x) = \rho(l) \cdot \langle \Psi(F^1(x)), l \rangle \cdot 1_L \in \text{HF}^*(L, L).$$

Under our identifications, Ψ becomes an isomorphism between the vector spaces below, and F^1 can be considered as an algebra isomorphism from $\text{HF}^*(L, L)$ to itself:

$$(3-15) \quad \text{HF}^*(L, L) \xrightarrow[\text{algebra iso.}]{F^1} \text{HF}^*(L, L) \xrightarrow[\text{linear iso.}]{\Psi} H^*(L).$$

We now turn the discussion to Hochschild cohomology of monic algebras. Let $f(u) \in \mathbb{K}[u]$ be a polynomial and $A := \mathbb{K}[u]/(f)$ the quotient algebra; it is called a monic algebra. This is an algebra in the ordinary associative sense; we can consider it as an A_∞ algebra by equipping it with trivial higher-structure maps. The Hochschild cohomology algebra $\text{HH}^*(A)$ was computed by Holm [24]. Recall that Hochschild cohomology of ungraded associative algebras is \mathbb{Z} -graded (unlike Hochschild cohomology of non- \mathbb{Z} -graded A_∞ algebras): cochains $A^{\otimes k} \rightarrow A$ are said to have degree k , and the differential has degree 1. By [24, Proposition 2.2],

$$\text{HH}^k(A) = \begin{cases} A & \text{if } k = 0, \\ \text{Ann}_A(f') & \text{if } k > 0 \text{ is odd,} \\ A/(f') & \text{if } k > 0 \text{ is even.} \end{cases}$$

Although we already know the answer, let us compute $\text{HH}^1(A)$ explicitly, as this will be helpful later.

Lemma 3.12 *For a Hochschild cocycle $h: A \rightarrow A$ in $\text{CC}^1(A)$, we must have*

$$(3-16) \quad h(u^m) = am u^{m-1}$$

for some fixed $a \in A$. Note that $a = h(u)$.

Proof Let us compute the Hochschild differential $\partial h: A \otimes A \rightarrow A$ on the two elements u and u^{m-1} for some $m \in \mathbb{N}$. Because ∂h vanishes by assumption, we get

$$0 = (\partial h)(u, u^{m-1}) = uh(u^{m-1}) + u^{m-1}h(u) + h(u^m).$$

The desired formula follows by induction on m . □

So any cocycle $h \in \text{CC}^1(A)$ is completely determined by a single element $a = h(u) \in A$, and h must meet an additional condition that $h(f(u)) = h(0) = 0$, which is equivalent to $a \in \text{Ann}_A(f')$. As the differential $\text{CC}^0(A) \rightarrow \text{CC}^1(A)$ vanishes, we get an isomorphism $\text{HH}^1(A) \rightarrow \text{Ann}_A(f')$, $h \mapsto h(u)$.

We will further assume that $\text{char } \mathbb{K} = 2$ and $f' = 0$. The latter condition means that f is a sum of even powers of u . Denote by

$$\psi: \text{HH}^1(A) \rightarrow A$$

the isomorphism $\phi(h) = h(u)$ from above. Note that if $s(u) \in A$ is an arbitrary element given by a polynomial with derivative $s'(u)$, then by (3-16), we get

$$(3-17) \quad \psi(h) = s'(u) \cdot h(s(u)).$$

For $k > 1$, we also have isomorphisms $\psi: \text{HH}^k(A) \rightarrow A$, all of which we denote by the same letter by abusing notation; we will not need an explicit formula for these isomorphisms when $k > 1$.

Moreover, [24, Lemma 5.1] computes the Yoneda product on $\text{HH}^*(A)$. In particular, given $h_1, h_2 \in \text{HH}^1(A)$, their Yoneda product $h_1 \star h_2$ is determined by

$$(3-18) \quad \psi(h_1 \star h_2) = \psi(h_1) \cdot \psi(h_2) \sum_{j \text{ odd}} f_{2j} u^{2j-2} \in A,$$

where $f = \sum_j f_j u^j$, $f_j \in \mathbb{K}$.

The two strands of discussion can be combined in the following theorem.

Theorem 3.13 *Let \mathbb{K} be a field of characteristic 2 and $L \subset X$ a Lagrangian preserved by a Hamiltonian loop γ which together satisfy the conditions of Theorem 1.7 (those conditions which are common to all parts of the theorem). Assume there is an algebra isomorphism $\text{HF}^*(L, L) \cong \mathbb{K}[u]/(f)$, where $f(u) = \sum_{j \geq 0} f_j u^j$ is a polynomial, and also that L is wide and admits a perfect Morse function, so that we can identify the vector spaces $\text{HF}^*(L, L) \cong \text{CF}^*(L, L)$, and $\Psi: \text{HF}^*(L, L) \rightarrow H^*(L)$ becomes an isomorphism of vector spaces. Further, assume that*

- $f' = 0$ and $\sum_{j \text{ odd}} f_{2j} u^{2j-2}$ is invertible in $\mathbb{K}[u]/(f)$,
- $\langle \Psi(r(u)), l \rangle = 1$ for an element $r(u) \in \mathbb{K}[u]/(f) \cong \text{HF}^*(L, L)$ which generates $\text{HF}^*(L, L)$ as an algebra,
- $S(\gamma)^2 = 1 \in \text{QH}^*(X)$.

Then the Fukaya A_∞ algebra of L is not formal over \mathbb{K} .

Proof Supposing $\text{CF}^*(L, L)$ is formal, let h be as in (3-13) and F^1 as in (3-15). Then there exists $s(u) \in \text{HF}^*(L, L)$ (we view this element as a polynomial in $\mathbb{K}[u]/(f)$) such that $F^1(s(u)) = r(u)$. Then by (3-14), $h^1(s(u)) = \rho(l) \cdot 1 \in \text{HF}^*(L, L)$, so by (3-17),

$$\psi(h^1) = \rho(l) \cdot s'(u) \in \text{HF}^*(L, L).$$

Further, note that $h \star h = 0$ because $(S(\gamma) + 1)^2 = 0$, so (3-18) yields

$$\rho(l)^2 \cdot (s'(u))^2 \sum_{j \text{ odd}} f_{2j} u^{2j-2} = 0 \in \text{HF}^*(L, L).$$

By hypothesis, this implies $(s'(u))^2 = 0$, so $s'(u) \in \ker \mathcal{F}$, where $\mathcal{F}: \mathbb{K}[u]/(f) \rightarrow \mathbb{K}[u]/(f)$ is the Frobenius endomorphism. In general, over $\text{char } \mathbb{K} = 2$, it is always true that $s'(u)$ is a sum of even powers of u , so $s'(u)$ is a square of another polynomial: $s'(u) = (t(u))^2$. Then $t(u)^2 \in \ker \mathcal{F}$, which implies $t(u) \in \ker \mathcal{F}$ because $\ker \mathcal{F}$, being an ideal in $\mathbb{K}[u]/(f)$, is necessarily prime. Consequently, $s'(u) = 0$. So $s(u)$ is a sum of even powers of u , hence the subalgebra generated by $s(u)$ lies in the subalgebra of $\mathbb{K}[u]/(f)$ generated by u^2 , which is smaller than the whole $\mathbb{K}[u]/(f)$: for example, it does not contain the element u . (Recall that f is also a sum of even powers of u .) On the other hand, we know that F^1 is an algebra isomorphism, $F^1(s(u)) = r(u)$ and $r(u)$ generates the whole $\text{HF}^*(L, L)$ by hypothesis. So $s(u)$ should also generate $\text{HF}^*(L, L)$, which is a contradiction. \square

Proof of Proposition 1.3 Take the real loop γ preserving $\mathbb{R}P^{4n+1}$ defined in the proof of Proposition 1.1, and let $L = \mathbb{R}P^{4n+1}$ and $X = \mathbb{C}P^{4n+1}$. Recall that if $x \in H^2(X)$ is the generator, then $\text{QH}^*(X) \cong \mathbb{K}[x]/(x^{4n+2} + 1)$ and $S(\gamma) = x^{2n+1}$, so $S(\gamma)^2 = 1$. Also recall that $l \in H_1(L) \cong \mathbb{K}$ is nonzero. By Theorem 1.12, $\text{HF}^*(L, L) \cong \mathbb{K}[u]/(u^{4n+2} + 1)$, where $u \in \text{CF}^1(L, L) \cong \mathbb{K}$, and we have $\langle \Psi(u), l \rangle = 1$. Now apply Theorem 3.13 taking $r(u) = u$ to conclude the proof. \square

Proposition 3.14 Let X be $\text{Bl}_{\mathbb{C}P^{2q-1}} \mathbb{C}P^{2r+2q-1}$ and $L \subset X$ the real Lagrangian (diffeomorphic to $\text{Bl}_{\mathbb{R}P^{2q-1}} \mathbb{R}P^{2r+2q-1}$). Assume $\text{gcd}(2q+1, 2r-2q) \geq 2$ and either r or q is odd. Then the A_∞ algebra of L is not formal over a characteristic-2 field.

Proof We recall that all real Lagrangians are wide by Theorem 1.12 and admit a perfect Morse function by [23]. The fact that $\text{gcd}(2q+1, 2r-2q) \geq 2$ means we are in the situation of Theorem 3.3(b) with $k = 2q = 2p$ and $a_1 = \dots = a_k = 1$. We have already seen by (3-5) that $\text{HF}^*(L, L) \cong \mathbb{K}[u]/(f)$ with $f' = 0$, and it is easy to check

that $\sum_{j \text{ odd}} f_{2j} u^{2j-2}$ is invertible provided that either r or q is odd (otherwise this element would vanish). Moreover, via (3-4) and Haug's isomorphism (Theorem 1.12), $u^{p/g}$ corresponds to the generator of $\text{CF}^1(L, L) \cong \mathbb{K}^2$ such that $\langle \Psi(u), l \rangle = 1$. Now apply Theorem 3.13 taking $r(u) = u^{p/g}$. \square

3.4 Nonformality of the equator on the sphere

Proposition 1.3 says in particular that the A_∞ algebra of an equatorial circle on S^2 is not formal over $\text{char } \mathbb{K} = 2$. This is an especially simple case which can be verified by hand, and it is worth discussing it in more detail. Let $L_1 \subset S^2$ be a fixed equator and L_2, L_3, \dots a sequence of its small Hamiltonian perturbations; assume $|L_i \cap L_j| = 2$ for each i, j . Then $\text{CF}^0(L_i, L_j) \cong \mathbb{K}$ is generated by an element which we denote by 1 (this is the cohomological unit), and $\text{CF}^1(L_i, L_j) \cong \mathbb{K}$ is generated by an element which we denote by u (we use the same letter for all i, j). Of the two intersection points $L_i \cap L_j$, the point u is the one at which $T_u L_j$ is obtained from $T_u L_i$ by a small positive rotation with respect to the ω -induced orientation on S^2 . Consider the A_∞ structure maps between the consecutive Lagrangians:

$$(3-19) \quad \mu^k: \text{CF}^*(L_k, L_{k+1}) \otimes \cdots \otimes \text{CF}^*(L_1, L_2) \rightarrow \text{CF}^*(L_1, L_{k+1})$$

given by counting immersed polygons as in [40]. These give a model for the A_∞ algebra of L , because all the L_i differ by small perturbations, and we can canonically identify the spaces $\text{CF}^*(L_i, L_{i+1})$ with each other. The A_∞ maps will depend on the particular arrangement of the L_i , although up to quasi-isomorphism they give the same A_∞ algebra.

Remark 3.15 The fact the A_∞ algebra of L defined using the count of polygons is quasi-isomorphic to the one defined using Hamiltonian perturbations seems not to have been written down in detail, but it is widely accepted. An approach is sketched in [41, Remark 7.2] and also performed in [44] in a slightly different setup.

Let us compute some of the A_∞ structure maps using a specific choice of the L_i . Fix a Hamiltonian H whose flow is the rotation of $S^2 \subset \mathbb{R}^3$ around an axis which is not orthogonal to the plane intersecting S^2 along the equator L_1 . Let L_2, L_3, \dots be obtained from L_1 by applying that rotation by small but consecutively increasing angles, ie L_i are time- t_i pushoffs of L_1 under the flow of H for $0 = t_1 < t_2 < t_3 < \dots$. The first four resulting circles L_i are represented in Figure 8, left. The pairwise intersections of the L_i are contained in two opposite patches of the sphere; those

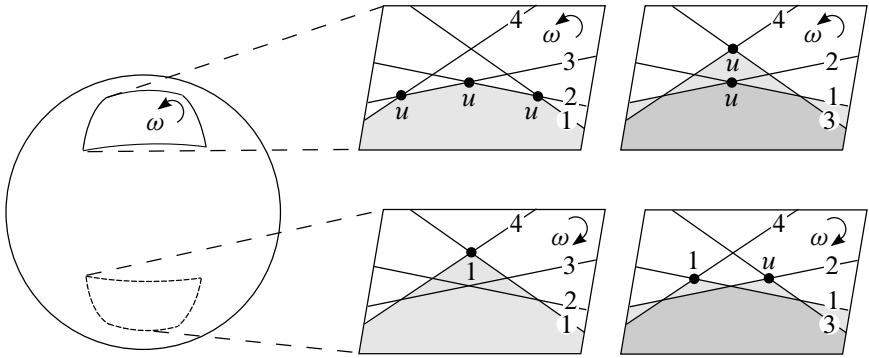


Figure 8: Two different configurations (left) and (right) consisting of four small Hamiltonian pushoffs L_1, \dots, L_4 (marked by numbers) of an equatorial circle on S^2 . The image of the disk contributing to $\mu^3(u, u, u) = 1$ is shaded.

patches are shown in the top and bottom of Figure 8, left, together with the L_i on them, which are depicted by straight lines. Both patches are drawn as if we look at them from the same point “above” the sphere, so that the positive rotation (with respect to the orientation on S^2) is counterclockwise on the upper patch and clockwise on the lower patch. For this particular choice of perturbations, and for each $i < j$, all degree-one points $u \in CF^*(L_i, L_j)$ are located on the upper patch, and all points $1 \in CF^*(L_i, L_j)$ are on the lower patch.

We claim that in this model, we get

$$\mu^3(u, u, u) = 1, \quad \mu^3(u, u, 1) = 0, \quad \mu^3(u, 1, u) = 0, \quad \mu^3(1, u, u) = 0.$$

For grading reasons, $\mu^k(u, \dots, u)$ is a multiple of 1 and is determined by counting Maslov index 2 disks. There is a unique such disk; for $k = 3$, it is shown in grey shade in Figure 8, left, on the two patches; away from the patches, this disc is just a strip between L_1 and L_4 . Also for grading reasons, the only other products which can possibly be nontrivial are $\mu^k(u, \dots, u, 1, u, \dots, u) \in \{0, u\}$, where exactly one input is 1. It possible to check that these vanish for our configuration of the circles L_i , at least when $k = 3$. Now note that

$$\mu^2(1 + u, 1 + u) = 0, \quad \mu^3(1 + u, 1 + u, 1 + u) = 1.$$

The latter equality exhibits a nontrivial Massey product seen as a well-defined element of

$$\mathbb{K}[u]/(1 + u) \cong \mathbb{K}.$$

An explanation of where the Massey products generally belong is found in Remark 1.2 of [40], which explains the quotient by $1 + u$ above. The presence of a nontrivial Massey product is invariant under quasi-isomorphisms. To see this, recall that the analogous fact for dg algebras is easy, and any A_∞ algebra is quasi-isomorphic to a dg algebra. Moreover, the Massey products for the A_∞ and dg models satisfy a simple relation [31, Theorem 3.1 and Corollary A.5]; in particular, if triple Massey products of an A_∞ algebra are nontrivial, they remain nontrivial for its dg model. This gives us an alternative proof of the fact that the A_∞ algebra of the equator on S^2 is not formal.

For any other arrangement of the L_i , we will necessarily have $\mu^3(1 + u, 1 + u, 1 + u) = 1$ modulo $1 + u$ because of invariance of Massey products, meaning that

$$\mu^3(1 + u, 1 + u, 1 + u) \in \{1, u\}.$$

For example, another possible configuration of L_1, \dots, L_4 is shown in Figure 8, right; it is simply obtained from the earlier configuration by changing the ordering of the L_i . In this new model, the maps μ^k from (3-19) are now

$$\mu^3(u, u, u) = 1, \quad \mu^3(u, u, 1) = 0, \quad \mu^3(u, 1, u) = u, \quad \mu^3(1, u, u) = u.$$

The unique disk contributing to $\mu^3(u, u, u)$ is shown in Figure 8, right, by grey shade. It is an immersed disk, and the domain over which it self-overlaps has darker shade. Note that in this model, the degree-one generators $u \in \text{CF}^1(L_1, L_2)$, $\text{CF}^1(L_3, L_4)$, $\text{CF}^1(L_1, L_4)$ correspond to the intersection points on the upper patch, and the degree-one generator $u \in \text{CF}^1(L_2, L_3)$ corresponds to the intersection point on the lower patch. We see that we again get $\mu^3(1 + u, 1 + u, 1 + u) = u$.

The existence of the Massey product above crucially required $\text{char } \mathbb{K} = 2$, because otherwise we would not get $\mu^2(1 + u, 1 + u) = 0$, which is necessary to speak of the triple Massey product of $1 + u$ with itself. If $\text{char } \mathbb{K} \neq 2$, then

$$\text{HF}^*(L_1, L_1) \cong \mathbb{K}[u]/(u^2 - 1) \cong \mathbb{K}[u]/(u - 1) \oplus \mathbb{K}[u]/(u + 1)$$

is a direct sum of fields, whose Hochschild cohomology as an ordinary algebra vanishes [27] in degree 2, in contrast to the case where $\text{char } \mathbb{K} = 2$. So any A_∞ algebra over $\mathbb{K}[u]/(u - 1) \oplus \mathbb{K}[u]/(u + 1)$ is formal by [26] or [43, Section 3]; in particular, the A_∞ algebra of the equator on S^2 is formal. For example, the product $\mu^3(1 + u, 1 + u, 1 + u)$ can be made to vanish after a formal diffeomorphism. Because of the nontrivial Massey product in characteristic 2, such a formal diffeomorphism, say over \mathbb{Q} , will necessarily involve division by 2, and cannot be realised by any geometric choice of the pushoffs L_i .

In comparison, the topological A_∞ algebra of the circle is formal over a field of any characteristic. Indeed, the topological A_∞ algebra is \mathbb{Z} -graded, so if we make this algebra to be based on the cohomology ring $H^*(S^1) \cong \mathbb{K}[x]/x^2$ where $|x| = 1$, the only possibly nontrivial products will be $\mu^k(x, \dots, x, 1, x, \dots, x)$ for grading reasons. On the other hand, every A_∞ algebra is quasi-isomorphic to a minimal, strictly unital one over a field of any characteristic [40, Lemma 2.1; 30, Theorem 3.1.1]. In a minimal strictly unital model, those products vanish by definition when $k \geq 3$.

4 The closed-open map for monotone toric fibres

4.1 The mechanism of Theorem 1.7 for toric fibres

Let X be an n -dimensional compact toric Fano variety, and $T \subset X$ the unique monotone toric fibre. Evans and Lekili [19] proved (after this paper had appeared as a preprint) that if $\text{char } \mathbb{K} = 0$, the Fukaya category $\mathcal{Fuk}(X)_w$ is split-generated by several copies of T , equipped with the local systems corresponding to the critical points of the Landau–Ginzburg superpotential with critical value $w \in \mathbb{K}$. We shall recall the formula for the superpotential of a toric manifold in the next subsection; the common general references are [13; 7; 21].

Prior to [19], the split-generation by toric fibres had been proved only in the case when the superpotential is Morse; see Ritter [37]. (For Ritter, proving split-generation requires considerable effort, even in the Morse case, if W has several critical points with the same critical value. However, the difficulty is mainly related to the fact that he allows some noncompact toric varieties, where the injectivity of \mathcal{CO}^* is no longer a criterion for split-generation and one must look at \mathcal{OC}^* instead. If we work with compact manifolds, checking that \mathcal{CO}^* is injective for an arbitrary Morse potential is easy; see Corollary 4.4.) An example of a toric Fano variety with non-Morse superpotential over \mathbb{C} has been obtained by Ostrover and Tyomkin [35], and one can check that the superpotential in their case has an A_3 singularity.

To complete the literature overview, we should mention the work in progress by Abouzaid, Fukaya, Oh, Ohta and Ono that will prove the split-generation result for toric manifolds that are not necessarily Fano.

Because the toric fibre T is invariant under all the Hamiltonian loops coming from the torus action, it is an obvious example where Theorem 1.7 can be put to the test. It turns out that it does allow one to prove split-generation away from the Morse case, though

not too far from it: the superpotential is required to have at worst A_2 singularities, and the extra condition that $\text{char } \mathbb{K} \neq 2, 3$ is required; see [Corollary 4.6](#).

Although our result is much weaker than the general one from [\[19\]](#), we find it interesting in our approach that the ability to solve equation [\(*\)](#) from [Theorem 1.7](#) depends on whether W is Morse or not. Equip T with a local system ρ which corresponds to a critical point of W ; then (T, ρ) is wide, and we can identify the vector spaces $\text{HF}^*(T, \rho) \cong H^*(T)$ via the PSS map Φ . For convenience, let us rewrite equation [\(*\)](#) from [Theorem 1.7](#):

$$(*)' \quad \mu^2(a, y) + \mu^2(y, a) = \rho(l) \cdot \langle y, l \rangle \cdot 1_T \quad \text{for each } y \in H^1(T).$$

We are using the standard spin structure on T , and hence the sign $(-1)^{\epsilon(l)}$ is positive; see [Remark 1.8](#). Recall that [Theorem 1.7\(b\)](#) can be applied if there exists no $a \in \text{HF}^*(T, \rho)$ making [\(*\)'](#) hold. The Floer cohomology algebra of (T, ρ) is a Clifford algebra determined by the Hessian of W at the point ρ , so the left-hand side of [\(*\)'](#) is equal to $\text{Hess}_\rho W(a, y) \cdot 1_T$, at least when $a \in H^1(T)$; we are using informal notation for the moment. Therefore, finding an element a solving [\(*\)'](#) reduces to finding an a such that

$$(4-1) \quad \text{Hess}_\rho W(a, -) = \text{const} \cdot \langle -, l \rangle.$$

The ability to find such an a depends on how degenerate $\text{Hess}_\rho W$ is. If ρ is a Morse point of W , such an a can always be found, so [Theorem 1.7\(b\)](#) does not apply. However, the Morse case can actually be covered by [Theorem 1.7\(a\)](#) as we explain below. On the other hand, when $\text{Hess}_\rho W$ has kernel, we will have some elements $l \in H_1(T)$ for which [\(4-1\)](#) has no solution a . If we consider the S^1 -action whose orbit is such an element l , [Theorem 1.7\(b\)](#) can be applied to the Seidel element of this S^1 -action to get some new information on \mathcal{CO}^* which is not seen by \mathcal{CO}^0 . This information turns out to be sufficient only when the superpotential has A_2 singularities, however, there is a possible way of improvement which we speculate upon in the end of this section.

4.2 The results

Recall [\[12; 13; 21; 20\]](#) that the Landau–Ginzburg superpotential of X is a Laurent polynomial $W: (\mathbb{K}^\times)^n \rightarrow \mathbb{K}$ given by

$$W(x_1, \dots, x_n) = \sum_e \prod_{j=1}^n x_j^{e^j},$$

where the first sum is over the outer normals $e \in \mathbb{Z}^n$ to the facets of the polyhedron defining X , and $e^j \in \mathbb{Z}$ are their coordinates. (Sometimes, the superpotential is written

down with a Novikov parameter, but we can ignore it because we will only be working with the monotone torus T .) We identify $(\mathbb{K}^\times)^n$ with the space of all local systems $H_1(T; \mathbb{Z}) \rightarrow \mathbb{K}^\times$. For $\rho \in (\mathbb{K}^\times)^n$, we write (T, ρ) for the torus equipped with this local system. Also, we will abbreviate

$$\mathrm{HF}^*(T, \rho) := \mathrm{HF}^*((T, \rho), (T, \rho)),$$

and the same for Hochschild cohomology. It is known (see [35, Proposition 3.3] for example) that

$$(4-2) \quad \mathrm{QH}^*(X) \cong \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / \mathrm{Jac}(W) = \mathcal{O}(Z),$$

where the Jacobian ideal $\mathrm{Jac}(W)$ is generated by $(\partial W / \partial x_1, \dots, \partial W / \partial x_n)$, and Z is the subscheme of $\mathrm{Spec} \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ defined by the ideal sheaf $\mathrm{Jac}(W)$. Then Z is a 0-dimensional scheme supported at the critical points of W ,

$$\{\rho_1, \dots, \rho_q\} = \mathrm{Crit} W, \quad \rho_i \in (\mathbb{K}^\times)^n.$$

The obstruction number of the torus is given by

$$w(T, \rho) = W(\rho) \in \mathbb{K}.$$

Under the isomorphism (4-2), the quantum product is the usual product on $\mathcal{O}(Z)$, and the first Chern class of X is given by the function W itself. The generalised eigenspace decomposition with respect to $- * c_1(X)$ is simply the decomposition into the local rings at the points ρ_i :

$$\mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / \mathrm{Jac}(W) \cong \bigoplus_{\rho_i \in \mathrm{Crit} W} \mathcal{O}_{\rho_i}(Z),$$

the eigenvalue of the ρ_i -summand being the critical value $W(\rho_i)$. From Lemma 1.9, we see that $\mathrm{HF}^*(T, \rho) = 0$ if $\rho \notin \mathrm{Crit} W$. On the other hand, it is known that (T, ρ_i) is wide for $\rho_i \in \mathrm{Crit} W$; ie $\mathrm{HF}^*(T, \rho_i)$ is isomorphic as a vector space to $H^*(T)$.

Lemma 4.1 *Under the isomorphism (4-2), the map $\mathcal{CO}^0: \mathrm{QH}^*(X) \rightarrow \mathrm{HF}^*(T, \rho_i)$ is given by*

$$\mathcal{CO}^0(f) = f(\rho_i) \cdot 1_T.$$

Here $f(x_1, \dots, x_n) \in \mathrm{QH}^*(X)$, $f(\rho_i) \in \mathbb{K}$ is the value of the function at $\rho_i \in \mathrm{Crit} W$, and $1_T \in \mathrm{HF}^*(T, \rho_i)$ is the unit.

Proof Because \mathcal{CO}^0 is a map of algebras, it suffices to prove the lemma when $f = x_k$ is a linear function for $1 \leq k \leq n$. By [32], $f = S(\gamma)$ for a Hamiltonian loop γ coming from the Hamiltonian torus action such that the value of the local system ρ_i on an orbit of γ equals the k^{th} coordinate ρ_i^k , which is the same as the value $f(\rho_i)$. So $\mathcal{CO}^0(f) = f(\rho_i) \cdot 1_T$ by Theorem 1.7(a). \square

Corollary 4.2 For $\rho_i \neq \rho_j \in \text{Crit } W$, the map $\mathcal{CO}^*|_{\mathcal{O}_{\rho_i}(Z)} \rightarrow \text{HH}^*(T, \rho_j)$ vanishes.

Remark 4.3 If $W(\rho_i) \neq W(\rho_j)$, Corollary 4.2 follows from Lemma 1.9. When $W(\rho_i) = W(\rho_j)$, the statement is implicit in [37, Proof of Theorem 6.17] where it is shown that, dually,

$$\mathcal{OC}: \text{HH}_*(T, \rho_j) \rightarrow \text{QH}^*(X)$$

hits at most one summand of the form \mathcal{O}_{ρ_i} , and we know by Lemma 4.1 that this summand must actually be \mathcal{O}_{ρ_j} . The proof in [37] is very different and relies on the variation of the symplectic form.

Proof Let $f \in \mathbb{K}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be such that $f(\rho_i) \neq 0$ and $f(\rho_j) = 0$. Then as an element of $\mathcal{O}_{\rho_i}(Z)$, f is invertible. If the corollary does not hold, $\mathcal{CO}^*(f)$ is also invertible. On the other hand, $\mathcal{CO}^0(f) = 0 \in \text{HF}^*(T, \rho_j)$ by Lemma 4.1. The map $\text{HH}^*(T, \rho_j) \rightarrow \text{HF}^*(T, \rho_j)$, which takes a Hochschild cochain to its zeroth-order term, is a map of unital algebras by the formula for the Yoneda product and because the Hochschild cohomology unit is represented by a cochain whose zeroth-order term is the Floer cohomology unit (this follows, for example, from the unitality of \mathcal{CO}^*). We have determined that f lies in the kernel of $\text{HH}^*(T, \rho_j) \rightarrow \text{HF}^*(T, \rho_j)$, but that contradicts the fact that f is invertible. This implies the corollary. \square

For $w \in \mathbb{K}$, let

$$\text{Crit}_w W = \{\rho \in \text{Crit } W : W(\rho) = w\}$$

denote the set of all critical points of W with the same critical value w . We will sometimes denote the restrictions of \mathcal{CO}^0 and \mathcal{CO}^* to subalgebras of $\text{QH}^*(X)$ by the same symbol when it is otherwise clear that we are considering a restriction.

Corollary 4.4 If $\text{char } \mathbb{K} \neq 2$, the map

$$\mathcal{CO}^0: \text{QH}^*(X)_w \rightarrow \bigoplus_{\rho_i \in \text{Crit}_w W} \text{HF}^*(T, \rho_i)$$

is injective if and only if all points of $\text{Crit}_w W$ are Morse.

Proof By [Corollary 4.2](#), \mathcal{CO}^0 is injective if and only if its restrictions $\mathcal{CO}^0: \mathcal{O}_{\rho_i}(Z) \rightarrow \mathrm{HF}^*(T, \rho_i)$ are injective for each ρ_i . The map $\mathcal{O}_{\rho_i}(Z) \rightarrow \mathbb{K}$ which takes $f \in \mathcal{O}_{\rho_i}(Z)$ to its value $f(\rho_i)$ is injective if and only if $\mathcal{O}_{\rho_i}(Z)$ is a field, which is equivalent to the fact that ρ_i is a Morse point of W when $\mathrm{char} \mathbb{K} \neq 2$. Now apply [Lemma 4.1](#). \square

Proposition 4.5 *Suppose that $\mathrm{char} \mathbb{K} \neq 2, 3$ and that W has an A_2 singularity at a point ρ ; then $\mathcal{CO}^*: \mathcal{O}_\rho(Z) \rightarrow \mathrm{HH}^*(T, \rho)$ is injective.*

Proof After an integral linear change of coordinates, we may assume that the Hessian of W at ρ is the diagonal matrix $\mathrm{Hess}_\rho W = \mathrm{diag}(1, \dots, 1, 0)$. We claim that $\mathcal{O}_\rho(Z)$ is generated, as a vector space, by the two elements 1 and x_n , where the linear function x_n corresponds to the kernel of $\mathrm{Hess}_\rho W$. Indeed, after a further nonlinear change of coordinates with the identity linear part, we can bring W to the canonical form

$$W(\tilde{x}_1, \dots, \tilde{x}_n) = W(\rho) + \sum_{i=1}^{n-1} (\tilde{x}_i - \rho^i)^2 + (\tilde{x}_n - \rho^n)^3.$$

Here $\rho^i \in \mathbb{K}$ are the coordinates of ρ . Then

$$\mathrm{Jac}(W) = ((\tilde{x}_1 - \rho^1), \dots, (\tilde{x}_{n-1} - \rho^{n-1}), (\tilde{x}_n - \rho^n)^2),$$

so $\mathcal{O}_\rho(Z)$ is generated, as a vector space, by 1 and \tilde{x}_n . Because x_n , as a function of $\tilde{x}_1, \dots, \tilde{x}_n$, equals \tilde{x}_n plus terms of order at least 2, it is easy to see that the elements 1 and x_n also generate the vector space $\mathcal{O}_\rho(Z)$.

Let us identify $\mathrm{HF}^*(T, \rho)$ with $H^*(T)$ via the PSS map Φ . Recall that, in general, $\mathrm{HF}^*(T, \rho)$ is the algebra generated by $y_1, \dots, y_n \in H^1(T)$ with relations

$$y_p y_q + y_q y_p = \partial_{x_p x_q}^2 W(\rho).$$

In particular, in our case, we get $y_p y_n + y_n y_p = 0$ for any $1 \leq p \leq n$, so $y_n \in \mathrm{HF}^1(T, \rho)$ anticommutes with any element of $\mathrm{HF}^*(T, \rho)$ of odd degree. Consequently, the left-hand side of equation (*) from [Theorem 1.7](#) vanishes if we put $y = y_n$ and allow a to be of arbitrary odd degree.

Returning to our generator $x_n \in \mathcal{O}_\rho(Z)$, we have $x_n = S(\gamma)$ for a Hamiltonian S^1 -action (coming from the toric action) such that the element $y_n \in \mathrm{HF}^1(T, \rho)$ is dual to the orbit $l \in H_1(T)$ of γ , so $\langle y_n, l \rangle = 1$. Thus if we put $y = y_n$, the right-hand side of equation (*) from [Theorem 1.7](#) becomes $\rho^n \cdot 1_T \neq 0$. Hence (*) has no solution, and [Theorem 1.7\(b\)](#) says that $\mathcal{CO}^*(x_n)$ and $1_{\mathrm{HH}} = \mathcal{CO}^*(1)$ are linearly independent. \square

Combining the above discussion with the split-generation criterion, we get the following:

Corollary 4.6 *Suppose that $\text{char } \mathbb{K} \neq 2, 3$ and that each critical point $\rho_i \in \text{Crit}_w W$ is either Morse or an A_2 singularity. Then the copies of the monotone toric fibre with local systems $\{(T, \rho_i)\}_{\rho_i \in \text{Crit}_w W}$ split-generate $\mathcal{Fuk}(X)_w$. \square*

4.3 A way of extending Theorem 1.7

It is in fact not surprising that Theorem 1.7 turned out to be efficient only for A_2 singularities. The main result on which Theorem 1.7 is based upon is Proposition 2.8, which computes the linear part \mathcal{CO}^1 of the closed-open map, while the only non-Morse singularity whose local Jacobian is generated as a vector space by constant and linear functions is the A_2 singularity (for which the Jacobian is generated by 1 and x_n as above). One could extend the computation in Proposition 2.8 to all orders of \mathcal{CO}^* when applied to products of 1-cochains on L ; we conjecture that the following holds.

Conjecture 4.7 *The restriction*

$$\mathcal{CO}^k(S(\gamma))|_{\text{CF}^1(L, L)^{\otimes k}} : \text{CF}^1(L, L)^{\otimes k} \rightarrow \text{CF}^0(L, L)$$

equals

$$(4-3) \quad (-1)^{\epsilon(l)} \rho(l) \cdot (l^*)^{\otimes k} \cdot 1_L$$

on symmetrised tensor products of Floer 1-cocycles. Here $l^ : \text{CF}^1(L, L) \rightarrow \mathbb{K}$ is given by $l^*(x) = \langle \Psi(x), l \rangle$, and $l \in H_1(L)$ is the orbit of γ .*

Remark 4.8 As in Proposition 2.8, part of the statement is that the image of this restriction necessarily lands in $\text{CF}^0(L, L)$; this follows for degree reasons. Although the proof of the above formula should be analogous to Proposition 2.8, one new issue arises which we have not checked in detail. Consider the moduli spaces $\mathcal{M}^\gamma(x_1, \dots, x_k; x_0)$ from Section 2 and the pearly moduli spaces analogous to Figure 7 but with more inputs. The new issue is a different type of domain degenerations coming from the collision of input points: eg several punctured inputs for a curve in $\mathcal{M}^\gamma(x_1, \dots, x_k; x_0)$ may collide and create a bubble. To prove (4-3), one would need to argue that these collisions cancel out when the input string is symmetrised.

Formula (4-3) is a chain level computation, and whether it survives to something nontrivial in Hochschild cohomology will be governed by equations generalising

equation (*) from Theorem 1.7; those equations will be determined by the A_∞ structure maps on L up to order $k + 1$. When L is the monotone toric fibre, the A_∞ structure maps have been related to higher-order partial derivatives of W by Cho [12], and intuitively, the more degenerate the superpotential is, the more nontrivial information from (4-3) survives to Hochschild cohomology. Consequently, these observations are a possible starting point for proving split-generation results for toric Fano varieties with other degenerate superpotentials. However, further development of this discussion seems both complicated and not particularly demanded, given the general results of [19].

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Received: 8 April 2015 Revised: 19 May 2017