On the stability of asymptotic property C
for products and some group extensions

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We show that Dranishnikov's asymptotic property C is preserved by direct products and the free product of discrete metric spaces. In particular, if G and H are groups with asymptotic property C, then both $G \times H$ and $G * H$ have asymptotic property C. We also prove that a group G has asymptotic property C if $1 \to K \to G \to H \to 1$ is exact, $\operatorname{asdim} K < \infty$ and H has asymptotic property C. The groups are assumed to have left-invariant proper metrics and need not be finitely generated. These results settle questions of Dydak and Virk (2016), of Bell and Moran (2015) and an open problem in topology.

1 Introduction

Asymptotic property C is a coarse invariant of metric spaces introduced by A N Dranishnikov in 2000 [11]. Recently it received a lot of attention; see Dranishnikov and Zarichnyi [13], Dydak [14], Dydak and Virk [15], Bell and Moran [7], Beckhardt [1], Beckhardt and Goldfarb [2] and Yamauchi [30].

Asymptotic property C is a coarse analog of topological property C; see Haver [21]. Topological property C fails to be preserved by direct products in very striking ways: There is an example of a separable metrizable space X with property C whose product with the irrationals fails to have property C (under the continuum hypothesis); see Pol [24]. There is also a separable complete metric space X with property C whose square $X \times X$ fails to have property C; see Pol and Pol [25]. The question whether asymptotic property C is preserved by direct products is widely known and was published by Pearl [23]. We answer it in the affirmative in the following theorem.

Theorem 1.1 *Let X and Y be metric spaces. If X and Y have asymptotic property C, then $X \times Y$ has asymptotic property C.*

Theorem 1.1 is an analog of a theorem of D M Rohm [27, Theorem 3], which states that topological property C is preserved by a direct product if one of the factors is compact. Asymptotic property C has a compactness-like property built into the definition, which requires that the sequence of constructed families is finite.

The technique used to prove Theorem 1.1 can also be used to prove a useful fibering lemma:

Fibering lemma   *If $f: X \to Y$ is a uniformly expansive map of metric spaces, Y has asymptotic property C, and coarse fibers of f have uniform asymptotic property C, then X has asymptotic property C.*

As an immediate consequence of this lemma, we can provide an affirmative answer to a question of J Dydak and Ž Virk [15] regarding preservation of asymptotic property C via maps with finite asymptotic dimension.

The fibering lemma is effective in settling various questions concerning asymptotic property C of countable groups or finitely generated groups. Consider the short exact sequences

$$1 \to G \to G \times H \to H \to 1$$

and

$$1 \to [\![G, H]\!] \to G * H \to G \times H \to 1$$

with groups G and H endowed with left-invariant proper metrics. We show that the surjective maps $G \times H \to H$ and $G * H \to G \times H$ satisfy the assumptions of the fibering lemma if G and H have asymptotic property C. In the latter sequence $[\![G, H]\!]$ is the commutator subgroup of $G * H$, which is quasi-isometric to a tree — see Serre [28] — hence it is of finite asymptotic dimension. Therefore the last exact sequence is a special case of the group extension

$$1 \to K \to G \to H \to 1,$$

which (as we show) satisfies assumptions of the fibering lemma if K has finite asymptotic dimension and H has asymptotic property C. These observations are summarized in Theorem 1.2.

Theorem 1.2   *Let G and H be groups with proper left-invariant metrics.*

(1)   *If G and H have asymptotic property C, then $G \times H$ has asymptotic property C.*

(2) If G and H have asymptotic property C, then $G * H$ has asymptotic property C.

(3) If $1 \to K \to G \to H \to 1$ is an exact sequence, H has asymptotic property C and K has finite asymptotic dimension, then G has asymptotic property C.

Theorem 1.2(2) settles the question of whether asymptotic property C is preserved by free products, which was posed by Bell and Moran [7]. We do not consider the more general question of free products with an amalgamated subgroup.

The result about free products of groups can be extended to free products of metric spaces.

Theorem 1.3 *Let X and Y be metric spaces that are discrete in the metric sense* (see Definition 6.7). *If X and Y have asymptotic property C, then $X * Y$ has asymptotic property C.*

The proof differs from the proof given in the case of groups; it relies on the following decomposition lemma and is of independent interest.

Decomposition lemma *Let X be a metric space. Assume that there exists k such that for each (infinite) sequence $R_1, R_2, \ldots$ of real numbers there exists a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ of families of subsets of X such that*

(1) $\bigcup_i \mathcal{U}_i$ *covers X,*

(2) $\mathcal{U}_i$ *is R_i–disjoint,*

(3) $\mathcal{U}_i$ *has asymptotic dimension bounded by k uniformly.*

Then X has asymptotic property C.

2 Preliminaries

Let U and V be subsets of a metric space X. Let $R > 0$ be a real number. The subsets U and V are said to be *R–disjoint* if $\operatorname{dist}(u, u') > R$ whenever $u \in U$ and $u' \in U'$.

A family of subsets $\mathcal{U}$ of a metric space X will be said to be *R–disjoint* if U and U' are R–disjoint whenever U and U' are distinct elements of $\mathcal{U}$. We define the *mesh* of a collection $\mathcal{U}$ of subsets of X by $\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}(U) : U \in \mathcal{U}\}$. A family of subsets $\mathcal{U}$ of a metric space X will be said to be *uniformly bounded* if $\mathrm{mesh}(\mathcal{U}) < \infty$. We will occasionally describe a family as being *B–bounded* when $\mathrm{mesh}(\mathcal{U}) \leq B$ for some $B > 0$.

Asymptotic dimension was defined by Gromov [18] as an asymptotic analog of covering dimension.

Definition 2.1 A metric space X is said to have *asymptotic dimension at most n*, asdim $X \leq n$, if for every $R > 0$ there are $n + 1$ uniformly bounded families of R–disjoint sets $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_n$ such that $\bigcup_i \mathcal{U}_i$ covers X.

Occasionally we will need more control over the disjointness and mesh of covers of a collection of sets. We say that a family $\{X_\alpha\}_\alpha$ of subsets of a metric space X has asymptotic dimension no more than n *uniformly* if for every R there is a B such that each X_α can be covered by at most $n + 1$ R–disjoint families of B–bounded sets.

Dranishnikov defined asymptotic property C [11] as an asymptotic analog of Haver's topological property C [21].

Definition 2.2 A metric space X has *asymptotic property C* if for each (infinite) sequence $R_1, R_2, \ldots$ of real numbers there exists a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ of families of subsets of X such that

(1) $\bigcup_i \mathcal{U}_i$ covers X,

(2) $\mathcal{U}_i$ is R_i–disjoint,

(3) $\mathcal{U}_i$ is uniformly bounded.

There is no loss of generality in assuming that the numbers R_i comprise a nondecreasing sequence. It is clear that spaces with finite asymptotic dimension have asymptotic property C; it is also true that metric spaces with asymptotic property C have Yu's property A [11, Theorem 7.11].

It is easy to see that the union of a finite collection of sets with asymptotic property C has asymptotic property C. In fact, this is true for certain types of infinite unions, too (see [7, Theorem 4.2]).

Proposition 2.3 *Let X and Y have asymptotic property C. Then $X \cup Y$ has asymptotic property C.*

Finally, we mention that the concepts of asymptotic dimension and asymptotic property C (among others) belong to the realm of so-called coarse geometry. It is possible to define these concepts in the context of the coarse category without reference to a metric (see [8; 17; 26]); however, we do not pursue this course further in this work.

Definition 2.4 Let $f\colon X \to Y$ be a map of metric spaces. We say that f is

(i) *effectively proper* if there is some proper (see Definition 2.5), nondecreasing $\rho_1\colon [0, \infty) \to [0, \infty)$ such that $\rho_1(\text{dist}_X(x, x')) \le \text{dist}_Y(f(x), f(x'))$;

(ii) *uniformly expansive* if there is a nondecreasing $\rho_2\colon [0, \infty) \to [0, \infty)$ such that $\text{dist}_Y(f(x), f(x')) \le \rho_2(\text{dist}_X(x, x'))$;

(iii) *coarsely uniform embedding* if it is both effectively proper and uniformly expansive;

(iv) *coarsely surjective* if there is some R such that for each $y \in Y$, there is some $x \in X$ for which $\text{dist}_Y(y, f(x)) < R$;

(v) a *coarse equivalence* if it is a coarsely uniform embedding that is coarsely surjective.

It is straightforward to verify that asymptotic dimension and asymptotic property C are invariants of coarse equivalence.

Definition 2.5 A metric space (X, dist) is said to be *proper* if closed balls are compact. We also say that the metric itself is proper in this case. A map $f\colon X \to Y$ between topological spaces is called *proper* if inverse images of compact sets are compact.

A metric d on a group Γ will be called *left-invariant* if $\text{dist}(gh, gh') = \text{dist}(h, h')$ for all g, h and h' in Γ.

Let Γ be a finitely generated group. Fix a finite generating set S that is symmetric in the sense that $s \in S$ implies $s^{-1} \in S$. The pair (Γ, S) carries a natural proper left-invariant metric called the word metric. This metric is given by first defining a norm on Γ as follows: $\|g\|_S = 0$ if and only if g is the identity element; otherwise $\|g\|_S$ is the length of the shortest word in the alphabet S that presents the element g. The (left-invariant) *word metric* is given by $\text{dist}_S(g, h) = \|g^{-1}h\|_S$. It is easy to see that any two word metrics corresponding to different (finite) generating sets give rise to

coarsely equivalent spaces. In the case of a general countable group Γ, which may not be finitely generated, Dranishnikov and Smith showed that up to coarse equivalence there is a unique left-invariant proper metric on Γ [12, Proposition 1.1]. Moreover, each such metric arises from a weight function as follows.

Definition 2.6 Let Γ be a countable group with (possibly infinite) generating set S. Give S the discrete topology. A *weight function* on S is a map $w: S \to [0, \infty)$ that is positive, proper and has the property that $w(s) = w(s^{-1})$.

Theorem 2.7 [12, Proposition 1.3] *Every weight function w on a countable group defines a proper norm* (which induces a proper left-invariant metric). *The norm is given by*

$$\|\gamma\|_w = \inf\left\{ \sum_{i=1}^{n} w(s_i) \,\middle|\, \gamma = s_1 \cdots s_n, \, s_i \in S \right\}$$

and the corresponding metric is $\mathrm{dist}_w(g, h) = \|g^{-1}h\|_w$.

Definition 2.8 An *action* of a group Γ on a topological space X is a homomorphism $\varphi: \Gamma \to \mathrm{Homeo}(X)$. We denote $\varphi(\gamma)(x)$ by $\gamma.x$. The *orbit* of $x_0 \in X$ is the equivalence class of all $\{x \in X : x = g.x_0 \text{ for some } g \in \Gamma\}$. The action is *transitive* if there is only one orbit. For $x \in X$, the *stabilizer* of x is denoted by Γ_x and is the subgroup of Γ defined by $\Gamma_x = \{\gamma \in \Gamma : \gamma.x = x\}$. Let Γ be a group acting on a metric space X. The action of Γ on X is said to be an *action by isometries* (or an *isometric action*) if $\varphi(\Gamma) \subset \mathrm{Isom}(X)$. For $R > 0$ define the *R–stabilizer* of $x_0 \in X$ by $W_R(x_0) = \{\gamma \in \Gamma : \mathrm{dist}_X(\gamma.x_0, x_0)\} \leq R$.

3   Direct products preserve asymptotic property C

Several authors have studied the so-called permanence properties — that is, the extent to which properties are preserved by unions, products, etc — of coarse invariants such as finite asymptotic dimension [4], finite decomposition complexity [19; 15], or property A [3]. The permanence properties of asymptotic property C have proved to be slightly more elusive than others, with incremental special cases being proved for unions [7] and certain types of group extensions and products [1; 2]. Indeed, although it had been conjectured for some time, only recently was a group with infinite asymptotic dimension and asymptotic property C shown to exist [30].

Let (X, dist_X) and (Y, dist_Y) be metric spaces. We can define a metric on the product $X \times Y$ by $\mathrm{dist}_{X \times Y}(x \times y, x' \times y') = \sqrt{(\mathrm{dist}_X(x, x'))^2 + (\mathrm{dist}_Y(y, y'))^2}$.

$$R_{21} \quad R_{27} \quad R_{34} \quad R_{42} \quad R_{51} \quad R_{61} \qquad\qquad R_{1,6} \quad R_{2,6} \quad R_{3,6} \quad R_{4,6} \quad R_{5,6} \quad R_{6,6}$$
$$R_{15} \quad R_{20} \quad R_{26} \quad R_{33} \quad R_{41} \quad R_{50} \qquad\qquad R_{1,5} \quad R_{2,5} \quad R_{3,5} \quad R_{4,5} \quad R_{5,5} \quad R_{6,5}$$
$$R_{10} \quad R_{14} \quad R_{19} \quad R_{25} \quad R_{32} \quad R_{40} \qquad\qquad R_{1,4} \quad R_{2,4} \quad R_{3,4} \quad R_{4,4} \quad R_{5,4} \quad R_{6,4}$$
$$R_{6} \quad R_{9} \quad R_{13} \quad R_{18} \quad R_{24} \quad R_{31} \qquad\qquad R_{1,3} \quad R_{2,3} \quad R_{3,3} \quad R_{4,3} \quad R_{5,3} \quad R_{6,3}$$
$$R_{3} \quad R_{5} \quad R_{8} \quad R_{12} \quad R_{17} \quad R_{23} \qquad\qquad R_{1,2} \quad R_{2,2} \quad R_{3,2} \quad R_{4,2} \quad R_{5,2} \quad R_{6,2}$$
$$R_{1} \quad R_{2} \quad R_{4} \quad R_{7} \quad R_{11} \quad R_{16} \qquad\qquad R_{1,1} \quad R_{2,1} \quad R_{3,1} \quad R_{4,1} \quad R_{5,1} \quad R_{6,1}$$

Figure 1: We arrange the given sequence as indicated and assign new labels corresponding to the row and column in which the number now appears.

The main goal of this section is to answer the following question from the Lviv Topological Seminar [23, Question 1.3, page 656]: Let X and Y be two metric spaces with asymptotic property C. Does $X \times Y$ have asymptotic property C?

Theorem 3.1 *Let X and Y be metric spaces with asymptotic property C. Then $X \times Y$ has asymptotic property C.*

The idea for the proof was inspired by the proof that a product of a compact space with (topological) property C and a space with property C has property C [27, Theorem 3].

Proof Let $R_1 \leq R_2 \leq \cdots$ be a given nondecreasing sequence of positive numbers. Rearrange the sequence into subsequences $R_{i,1} \leq R_{i,2} \leq \cdots$ using the rearrangement shown in Figure 1.

Fix $i = 1$, so that we are considering the column $R_{1,j}$ with $j = 1, 2, \ldots$. Let $\mathcal{U}_{1,1}, \mathcal{U}_{1,2}, \ldots, \mathcal{U}_{1,n_1}$ be uniformly bounded families of subsets of X such that $\mathcal{U}_{1,j}$ is $R_{1,j}$–disjoint and the union $\bigcup_j \mathcal{U}_{1,j}$ covers X. Similarly, for each $i = 2, 3, \ldots$, we construct finite collections of $R_{i,j}$–disjoint families $\mathcal{U}_{i,j}$ for $j = 1, 2, \ldots, n_i$ of uniformly bounded subsets of X whose union covers X. This construction yields a sequence of positive numbers R_{i,n_i} with the property that $R_{i,n_i} \geq R_{i,j}$ for all $j \leq n_i$. Using this sequence we obtain covers $\mathcal{V}_i$ for $i = 1, 2, \ldots, m$ of Y; see Figure 2.

Let $\mathcal{W}_{i,j} = \{U \times V : U \in \mathcal{U}_{i,j}, V \in \mathcal{V}_i\}$.

Given $(x, y) \in X \times Y$ there is some i_0 such that $\mathcal{V}_{i_0}$ contains an element V containing y, since the union of the $\mathcal{V}_i$ covers Y. Then there is some j_0 such that $\mathcal{U}_{i_0,j_0}$ contains a set U containing x, since each $\bigcup_j \mathcal{U}_{i,j}$ is a cover of X for each i. Thus, $U \times V$ is some $W \in \mathcal{W}_{i_0,j_0}$ that contains (x, y). Thus, the collection $\bigcup_{i,j} \mathcal{W}_{i,j}$ covers $X \times Y$.
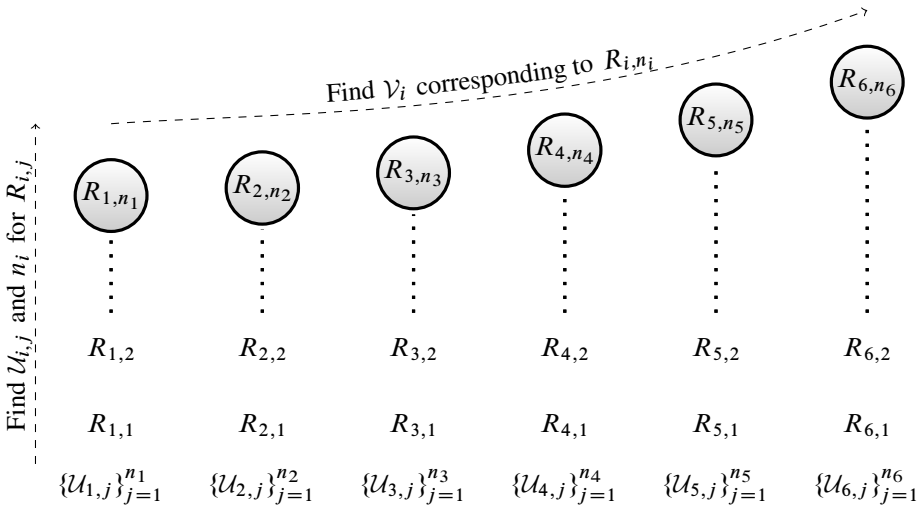
Figure 2: We find covers $\{\mathcal{U}_{i,j}\}_{j=1}^{n_i}$ of X for each column $R_{i,1}, R_{i,2}, \dots,$ and then construct a cover $\{\mathcal{V}_i\}_{i=1}^m$ of Y corresponding to R_{i,n_i}.

The family $\mathcal{W}_{i,j}$ is uniformly bounded; indeed, if $W \in \mathcal{W}_{i,j}$ then $\mathrm{diam}(W)^2 \le \mathrm{mesh}(\mathcal{U}_{i,j})^2 + \mathrm{mesh}(\mathcal{V}_i)^2$.

Next, we check that each family $\mathcal{W}_{i,j}$ is $R_{i,j}$–disjoint; we suppose that distinct elements W and W' are in some fixed $\mathcal{W}_{i,j}$. Write $W = U \times V$ and $W' = U' \times V'$. Take arbitrary elements $(u, v) \in W$ and $(u', v') \in W'$. If $U = U'$ then $\mathrm{dist}((u, v), (u', v')) \ge \mathrm{dist}_Y(v, v') > R_{i,n_i}$. If $V = V'$, then $\mathrm{dist}((u, v), (u', v')) \ge \mathrm{dist}_X(u, u') > R_{i,j}$. Otherwise, $\mathrm{dist}((u, v), (u', v')) \ge \min\{\mathrm{dist}_X(u, u'), \mathrm{dist}_Y(v, v')\} > \min\{R_{i,n_i}, R_{i,j}\} = R_{i,j}$, since $R_{i,j} \le R_{i,n_i}$ for each i and each $j \le n_i$. Thus, $\mathcal{W}_{i,j}$ is $R_{i,j}$–disjoint.

Finally, we rearrange the $\mathcal{W}_{i,j}$ back into a single sequence as in Figure 1 to obtain $\mathcal{W}_{k_1}, \mathcal{W}_{k_2}, \dots, \mathcal{W}_{k_N}$ (where some indices may have been omitted). Notice that this produces a finite sequence $\{k_1, k_2, \dots, k_N\}$ with the property that $k_i \ge i$ for each i. Thus, $R_{k_i} \ge R_i$ for each i. Thus, the collection $\mathcal{W}_{k_i}$ is R_{k_i}–disjoint and therefore it is R_i–disjoint. Therefore, we have a finite collection $\mathcal{W}_{k_1}, \mathcal{W}_{k_2}, \dots, \mathcal{W}_{k_N}$ of uniformly bounded R_i–disjoint families of subsets of $X \times Y$ with the property that $\bigcup \mathcal{W}_{k_i}$ covers $X \times Y$. $\qquad\square$

4   The fibering lemma

In this section we prove a Hurewicz-type theorem for asymptotic property C that is motivated by the technique of the proof of Theorem 3.1. The classical Hurewicz

theorem on mappings that lower dimension goes back to 1927 and can be stated as follows:

Hurewicz theorem [16, Theorem 1.12.4] *If* $f\colon X \to Y$ *is a closed mapping of separable metric spaces and there is an integer* $k \geq 0$ *such that* $\dim f^{-1}(y) \leq k$ *for every* $y \in Y$, *then* $\dim X \leq \dim Y + k$.

This theorem was first translated to asymptotic dimension for finitely generated groups in [5]. It was proved for countable groups in [12] and translated to Assouad–Nagata dimension in [9]. A version of the theorem for metric spaces appears in [15]. In a more general context, this theorem belongs to the class of fibering theorems (see for example [19]) in which a property is preserved by a mapping when the range and fibers have this property.

4.1   Fibers with uniform asymptotic property C

Definition 4.1   Let $f\colon X \to Y$ be a map of metric spaces. Let M be a real number. We say that $A \subset X$ is a *coarse fiber at scale* M if $\operatorname{diam} f(A) < M$.

Definition 4.2   Let $f\colon X \to Y$ be a map of metric spaces. We say that *coarse fibers of* f *have uniform asymptotic property C* if for each infinite sequence $R_1, R_2, R_3, \ldots$ of real numbers there exists an integer k such that, for each real number M, there exists a real number B such that, for every $A \subset X$ with $\operatorname{diam} f(A) < M$, there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$ of families of subsets of A that satisfies the following conditions:

(1) $\bigcup_{i=1}^{k} \mathcal{U}_i$ covers A.

(2) $\mathcal{U}_i$ is R_i–disjoint for each i.

(3) $\operatorname{mesh} \mathcal{U}_i \leq B$ for each i.

Comment 4.3   Note the order of quantifiers in Definition 4.2: the number k of families depends only on the sequence $R_1, R_2, \ldots$; the bound B on meshes of the families depends on the sequence $R_1, R_2, \ldots$, the number k and the scale M of fibers.

If $f\colon X \times Y \to Y$ is the projection map from the product of metric spaces $X \times Y$, then it is straightforward to show that coarse fibers of f have uniform asymptotic property C if X has asymptotic property C.

Fibering lemma   *If* $f\colon X \to Y$ *is a uniformly expansive map of metric spaces,* Y *has asymptotic property C and coarse fibers of* f *have uniform asymptotic property C, then* X *has asymptotic property C.*

It is known that topological property C is preserved by preimages if the map is closed, fibers have topological property C and the codomain is paracompact; see [20].

Proof   Let $\rho\colon [0, \infty) \to [0, \infty)$ be strictly increasing and such that

$$\mathrm{dist}_Y(f(x), f(x')) \leq \rho(\mathrm{dist}_X(x, x')).$$

Let $R_1, R_2, R_3, \ldots$ be an infinite sequence of real numbers. Without any loss of generality we may assume that $R_1, R_2, R_3, \ldots$ is increasing.

Rearrange the sequence $R_1 \leq R_2 \leq R_3 \leq \cdots$ into subsequences $R_{i,1} \leq R_{i,2} \leq R_{i,3} \leq \cdots$ using the rearrangement shown in Figure 1. For each i, let n_i be the number of families obtained by applying Definition 4.2 to the map f and the sequence $R_{i,1} \leq R_{i,2} \leq R_{i,3} \leq \cdots$. Consider the sequence

$$\rho(R_{1,n_1}), \ \rho(R_{2,n_2}), \ \rho(R_{3,n_3}), \ \ldots$$

Using the assumption that Y has asymptotic property C, we obtain a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of families of subsets of Y such that:

(1) $\bigcup_{i=1}^m \mathcal{V}_i$ covers Y.

(2) $\mathcal{V}_i$ is $\rho(R_{i,n_i})$–disjoint for each i.

(3) $\mathcal{V}_i$ is uniformly bounded for each i.

Let $M_i = \mathrm{mesh}\, \mathcal{V}_i$. Apply Definition 4.2 to the map f, the sequence $R_{i,1} \leq R_{i,2} \leq \cdots \leq R_{i,n_i}$ and the scale M_i to obtain a bound B_i.

Let $V \in \mathcal{V}_i$. The set $f^{-1}(V)$ is a coarse fiber of f at scale M_i. By Definition 4.2 applied to the sequence $R_{i,1} \leq R_{i,2} \leq \cdots \leq R_{i,n_i}$ and the coarse fiber $f^{-1}(V)$ at scale M_i, there exists a sequence $\mathcal{U}_1^V, \mathcal{U}_1^V, \ldots, \mathcal{U}_{n_i}^V$ of families of subsets of $f^{-1}(V)$ such that:

(1) $\bigcup_{j=1}^{n_i} \mathcal{U}_j^V$ covers $f^{-1}(V)$.

(2) $\mathcal{U}_j^V$ is $R_{i,j}$–disjoint for each j.

(3) $\mathrm{mesh}\, \mathcal{U}_j^V \leq B_i$ for each j.

Observe that if $V, W \in \mathcal{V}_i$ with $v \in V$, $w \in W$ and $V \neq W$, then

$$\mathrm{dist}_X(f^{-1}(v), f^{-1}(w)) > R_{i,n_i}.$$

Since $R_{i,n_i} \geq R_{i,j}$ for $j \leq n_i$, the family

$$\mathcal{U}_{i,j} = \bigcup_{V \in \mathcal{V}_i} \mathcal{U}_j^V$$

is $R_{i,j}$–disjoint and mesh $\mathcal{U}_{i,j} < B_i$. Hence we obtain a collection of uniformly bounded families $\{\mathcal{U}_{i,j} : i \leq m, \ j \leq n_i\}$ that are $R_{i,j}$–disjoint and whose union covers X. Rearranging these families back into the sequence $R_1, R_2, \ldots$ (possibly leaving some places empty) shows that X has asymptotic property C. □

By the comments following Comment 4.3 we see that the product theorem (Theorem 3.1) can be deduced from the fibering lemma.

4.2 Fibers with uniformly bounded asymptotic dimension

We turn our attention to the preservation of coarse properties discussed in [15], from which we recall the following definition.

Definition 4.4 Given a function $f\colon X \to Y$ of metric spaces, its *asymptotic dimension* asdim(f) is the supremum of asdim(A) for $A \subset X$ such that asdim($f(A)) = 0$.

Lemma 4.5 Let $f\colon X \to Y$ be a uniformly expansive map of metric spaces. The following are equivalent:

(1) *For every $M > 0$ and every $R > 0$, there is a $B > 0$ such that if $U \subset X$ and diam$(f(U)) < M$, then U can be covered by $n+1$ R–disjoint families of B–bounded subsets of X.*

(2) asdim $f \leq n$.

Proof (1) $\Longrightarrow$ (2) Since f is uniformly expansive, there is an increasing control function $\rho\colon [0, \infty) \to [0, \infty)$ such that dist$_Y(f(x), f(x')) \leq \rho(\text{dist}_X(x, x'))$. Let $R > 0$ be given and take an arbitrary $A \subset X$ with the property that asdim $f(A) = 0$. We may cover $f(A)$ with uniformly bounded sets U_i that are $\rho^{-1}(R)$–disjoint. Let M be an upper bound for diam U_i and take $B > 0$ as above. Then each $f^{-1}(U_i)$ can be covered by the union of $n+1$ families $\mathcal{U}_{i,j}$ of R–disjoint, uniformly B–bounded subsets of $f^{-1}(U_i)$. By the choice of the U_i, we see that if $U_i \neq U_j$ then dist$(f^{-1}(u), f^{-1}(v)) > R$ whenever $u \in U_i$ and $v \in U_j$. Thus, by taking the collection $\mathcal{V}_j = \mathcal{U}_{i,j}$ for $j = 0, 1, \ldots, n$ we obtain a collection of $n+1$ R–disjoint families of

uniformly bounded sets whose union covers A. Thus, asdim $A \leq n$ and, since A was arbitrary, we conclude that asdim $f \leq n$.

(2) $\Longrightarrow$ (1) Suppose that asdim $f \leq n$. Fix $R > 0$ and $M > 0$. For each $U \subset X$ with diam$(f(U)) < M$, let $B(U)$ denote the smallest integer B for which U can be expressed as a union of B–bounded, R–disjoint families $\mathcal{U}_0, \ldots, \mathcal{U}_n$.

If sup$\{B(U) : \text{diam}(f(U)) < M\} < \infty$ then there is nothing to prove. Thus, we assume that we can find a sequence of subsets $U_1, U_2, \ldots$ of X with diam$(f(U_i)) < M$ for which $B(U_1), B(U_2), \ldots$ is unbounded. We may take the sequence of such $B(U_i)$ to be monotonically increasing. If the set $\{f(U_1), f(U_2), \ldots\}$ is bounded, then asdim$\left(\bigcup_i f(U_i)\right) = 0$ and therefore the sequence of the $B(U_i)$ is bounded from above. Otherwise, the sequence $f(U_i)$ is unbounded. Thus, we take $i_1 = 1$. Because the $f(U_i)$ are bounded, yet the sequence $\{f(U_i)\}$ is unbounded, for each $k > 1$ we can find an increasing sequence of indices i_k such that dist$\left(\bigcup_{j=1}^{k-1} f(U_{i_j}), f(U_k)\right) > 2^k$. (Here the distance between subsets is defined by dist$(A, B) = \inf\{\text{dist}(a, b) : a \in A, b \in B\}$.) Thus, $\bigcup_{k=1}^{\infty} f(U_{i_j})$ has asymptotic dimension 0, which contradicts the fact that the $B(U_i)$ increase monotonically to infinity since asdim $f \leq n$. $\qquad \square$

Dydak and Virk have the following theorem regarding such maps and property C:

Theorem 4.6 [15, Theorem 6.3] *Suppose that $f : X \to Y$ is coarsely surjective and asdim$(f) = 0$. Then, if Y has asymptotic property C, X has asymptotic property C.*

This naturally leads them to the following question, which we are able to settle in the affirmative [15, Question 6.4]: Suppose $f : X \to Y$ is uniformly expansive, coarsely surjective and of finite asymptotic dimension asdim(f). If Y has asymptotic property C, does X have asymptotic property C?

Corollary 4.7 *Suppose that $f : X \to Y$ is a uniformly expansive, coarsely surjective map of finite asymptotic dimension asdim(f) and that Y has asymptotic property C. Then X has asymptotic property C.*

Proof Suppose that asdim$(f) = n$. Then, given a sequence $R_1 \leq R_2 \leq \cdots$, we take $k = n + 1$ and apply Lemma 4.5 to R_{n+1} to find that for any $M > 0$ there is a B such that whenever diam $f(A) < M$, we can cover A by $n + 1$ R_{n+1}–disjoint families $\mathcal{U}_1, \ldots, \mathcal{U}_{n+1}$ that are each B–bounded. Since the sequence R_i is nondecreasing, we see that this shows that the coarse fibers of f have uniform asymptotic property C. Thus, by the fibering lemma, we conclude that X has asymptotic property C. $\qquad \square$

5 Fibers of group homomorphisms

In the assumptions of the fibering lemma we impose the asymptotic property C condition *uniformly* on *coarse* fibers. The following example shows that this is indeed necessary. In the present section we show that, for group homomorphisms, in some cases the uniformity of the fiber condition is implicit.

Example 5.1 Let $\{0, 1\}^n$ be the set of vertices of an n–dimensional cube endowed with the ℓ_1–metric. The disjoint union $\bigsqcup_{n=1}^{\infty}\{0, 1\}^n$ can be metrized in such a way that it is a locally finite metric space that fails to have property A [22] and hence fails to have asymptotic property C as well [11, Theorem 7.11]. Consider the map $\bigsqcup_{n=1}^{\infty}\{0, 1\}^n \to \mathbb{Z}$ that collapses $\{0, 1\}^n$ into a single integer n^2. It is a uniformly expanding map onto a zero-dimensional space. Its coarse fibers are all bounded (hence they are zero-dimensional and have asymptotic property C).

5.1 Fibers vs coarse fibers

Throughout this section, G and H denote countable groups endowed with proper left-invariant metrics.

Lemma 5.2 *Let G be a countable group with a proper left-invariant metric acting by isometries on a metric space X. Let $x_0 \in X$ and define $\varphi\colon G \to X$ by $\varphi(g) = g.x_0$. Fix n and suppose that, for every $R > 0$, the R–stabilizer (see Definition 2.8) satisfies $\operatorname{asdim} W_R(x_0) \leq n$. Then coarse fibers of φ have uniform asymptotic property C.*

We remark that the R–stabilizer is a coarse fiber in the sense of Definition 4.1.

Proof We may assume that the action is transitive. Suppose that a sequence $R_1 \leq R_2 \leq \cdots$ is given and take $k = n + 1$. Now, if $M > 0$ is given, we may use the fact that $\operatorname{asdim} W_M(x_0) \leq k - 1$ to find families of R_k–disjoint sets $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$ such that $\bigcup_i \mathcal{U}_i$ covers $W_M(x_0)$ and $\operatorname{mesh}(\mathcal{U}_i) \leq B$ for some B.

Let $A \subset G$ have the property that $\operatorname{diam}(\varphi(A)) \leq M$. Then take any $g_A \in G$ and observe that $g_A(x_0) \in \varphi(A)$. Thus, $\varphi(A) \subset B_M(g_A.x_0)$ and so $g_A^{-1}A \subset W_M(x_0)$. Since the metric on G is left-invariant, the families $\mathcal{V}_i$ given by $\{g_A^{-1}U : U \in \mathcal{U}_i\}$ have the property that they are R_k–disjoint and B–bounded. Moreover, their union covers A, as required. □

We actually proved a stronger statement: for each R the coarse fibers at scale R have asymptotic dimension at most n uniformly.

Lemma 5.3 Let $\varphi \colon G \to X$ describe the transitive action by isometries of a countable discrete group G on a proper discrete metric space X. If the stabilizer G_x has finite asymptotic dimension for some $x \in X$, then coarse fibers of the map $\varphi \colon G \to X$ describing the action have uniform asymptotic property C.

Proof Let $R > 0$ be given and consider $B_R(x) \subset X$. Since this set is finite, it follows that $W_R(x)$ is a finite union of sets that are isometric to G_x. Hence, this union has the same asymptotic dimension as G_x and the result follows from Lemma 5.2. □

Lemma 5.4 If the kernel of a group homomorphism $f \colon G \to H$ has finite asymptotic dimension, then coarse fibers of f have uniform asymptotic property C.

We do not know whether the weaker assumption that the kernel of f has asymptotic property C is sufficient; see Question 7.2.

Proof Define an action of G on H by $g.h \mapsto f(g)h$. Notice that this action is isometric; ie $\operatorname{dist}(g.h, g.h') = \operatorname{dist}(h, h')$ for all $g \in G$. Also, the stabilizer of this action at e is $\ker f$. Since H is assumed to be a countable group with a proper metric, as a metric space it has bounded geometry, and the result follows from Lemma 5.3. □

Lemma 5.5 If the kernel of a projection $p \colon G \times H \to G$ has asymptotic property C, then coarse fibers of p have uniform asymptotic property C.

Proof We observe that the kernel of p is $\{e\} \times H$, which is isometric to H. Let $R_1 \leq R_2 \leq \cdots$ be a given sequence. Take families $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$ of subsets of H such that $\mathcal{U}_i$ is R_i–disjoint, mesh $\mathcal{U}_i$ is finite for each i, and $\bigcup_i \mathcal{U}_i$ covers H. Given $M > 0$, we take $B = M + \max_i\{\operatorname{mesh} \mathcal{U}_i\}$. If $A \subset G \times H$ has the property that $\operatorname{diam}(p(A)) \leq M$, then there is some $g \in G$ such that $p(A) \subset B^G(g, M)$, where the superscript indicates that the M–ball is taken in the group G. Consider the collection $\mathcal{V}_i$ of subsets of $G \times H$ given by $\{B^G(g, M) \times U : U \in \mathcal{U}_i\}$. Observe that this collection covers A, mesh $\mathcal{V}_i \leq \operatorname{mesh} \mathcal{U}_i + M \leq B$, and each $\mathcal{V}_i$ is R_i–disjoint. □

We would like to remark that we did not require the bounded geometry condition on G or H in the previous proof. Indeed, the same technique can be used to show that if point fibers of a projection $p \colon X \times Y \to X$ have asymptotic property C, then coarse fibers of p have uniform asymptotic property C.

In [1] it is shown that a finitely generated group acting on a space with asymptotic property C will have asymptotic property C provided the R–stabilizers have finite

asymptotic dimension. We can use our fibering theorem to recover this result, which we state for arbitrary countable groups with proper left-invariant metrics.

Theorem 5.6 *Let G be a countable group with a proper left-invariant metric acting by isometries on a metric space X. Let $x_0 \in X$ and define $\phi\colon G \to X$ by $\phi(g) = g.x_0$. If, for every $R > 0$, $\operatorname{asdim} W_R(x_0) \le n$ and, X has asymptotic property C, then G has asymptotic property C.*

Proof This will follow from the fibering lemma and Lemma 5.2 provided we can show that ϕ is uniformly expansive.

To this end, by [12] we know that the metric on G is a weighted word metric. Therefore, we assume that $S = \{s_i\}$ is a generating set for G with corresponding weights $w(s_i)$. By their definition and the fact that the metric is proper, the weights have the following property. For each fixed $N \in \mathbb{N}$, there are only finitely many (finite) sequences of weights whose sums are bounded from above by N; that is, the collection of all sequences $(w(s_{i_1}), w(s_{i_2}), \dots)$ for which $\sum w(s_{i_j}) \le N$ is a finite collection. Thus, it makes sense to define a function $\rho\colon \mathbb{N} \to \mathbb{N}$ by $\rho(N) = \max\{\sum \operatorname{dist}(s_{i_j}.x_0, x_0)\}$, where the maximum is taken over all sequences of s_i for which $\sum w(s_{i_j}) \le N$.

Then, by applying the triangle inequality and the fact that the metric is left-invariant, we find $\operatorname{dist}_X(g'.x_0, g.x_0) = \operatorname{dist}_X(g^{-1}g'.x_0, x_0) \le \sum_{j=1}^{k} \operatorname{dist}_X(s_{i_j}.x_0, x_0)$, where $g^{-1}g' = s_{i_1}s_{i_2}\cdots s_{i_k}$. But now

$$\sum_{j=1}^{k} \operatorname{dist}_X(s_{i_j}.x_0, x_0) \le \rho(\|g^{-1}g'\|) = \rho(\operatorname{dist}_G(g', g))$$

and so we conclude that $\operatorname{dist}_X(\phi(g'), \phi(g)) \le \rho(\operatorname{dist}_G(g', g))$.   □

Theorem 5.7 *Let $\phi\colon G \to H$ be a surjective homomorphism of finitely generated groups; suppose that H has property C. If $\operatorname{asdim} \ker \phi < \infty$ then G has asymptotic property C.*

Proof Let S be a generating set for H with corresponding weights W. For each $s \in S$, let $\tilde{s} \in \phi^{-1}(s)$ be some element of G. Extend the set $\{\tilde{s} : s \in S\}$ to a generating set T for G with the property that the weight of each $\tilde{s}$ is the same as the weight of the corresponding s. This yields a left-invariant proper metric on G which is coarsely equivalent to any other choice of left-invariant proper metric on G, so it suffices to show that G has asymptotic property C in this metric. In this metric, ϕ is coarsely expansive and so the result follows from the fibering lemma and Lemma 5.4.   □

Corollary 5.8 *If G and H have asymptotic property C, then $G * H$ has asymptotic property C.*

Proof We consider the exact sequence

$$1 \to [\![G, H]\!] \to G * H \to G \times H \to 1.$$

The commutator group $[\![G, H]\!]$ is free [28] and so as a subspace of $G * H$ it is quasi-isometric to a tree (albeit one in which edges can have arbitrarily long length). Thus, its asymptotic dimension is 1. By Theorem 3.1, $G \times H$ has asymptotic property C. We apply Theorem 5.7 to complete the proof. □

6 Free products preserve asymptotic property C

In the present section we use the decomposition lemma to prove that the free product of metric spaces preserves asymptotic property C; however, the decomposition lemma has generalizations that make it interesting in its own right.

6.1 The decomposition lemma

Lemma 6.1 *Let X be a metric space. Assume that there exists k such that for each (infinite) sequence $R_1, R_2, \ldots$ of real numbers there exists a finite sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ of families of subsets of X such that*

(1) $\bigcup_i \mathcal{U}_i$ *covers X,*

(2) $\mathcal{U}_i$ *is R_i–disjoint,*

(3) $\mathcal{U}_i$ *has asymptotic dimension bounded by $k - 1$ uniformly.*

Then X has asymptotic property C.

Proof Let $R_1 \leq R_2 \leq \cdots$ be a given sequence of real numbers. For $i \in \{1, 2, \ldots, n\}$, take $\mathcal{U}_i$ to be an R_{ik}–disjoint collection of subsets of X that has asymptotic dimension less than k uniformly such that $\bigcup_i \mathcal{U}_i$ covers X.

Fix some i and suppose that $U \in \mathcal{U}_i$. Using the uniformity of the asymptotic dimension of the $\mathcal{U}_i$, take some B and take families $\mathcal{V}_j^U$ for $j = 1, \ldots, k$ that comprise an R_{ik}–disjoint collection of subsets of U whose diameter is bounded by B such that the union covers U. Let $\mathcal{V}_{(i-1)k+j}$ denote the collection $\{V \in \mathcal{V}_j^U : U \in \mathcal{U}_i\}$. Since the family $\mathcal{U}_i$ is itself R_{ik}–disjoint, it is clear that the collection $\mathcal{V}_{(i-1)k+j}$ for $j = 1, \ldots, k$

will be an R_{ik}–disjoint, B–bounded cover of $\bigcup_{U \in \mathcal{U}_i} U$. Since we assume the given sequence to be nondecreasing, we have found a uniformly bounded cover of X by nk families $\mathcal{V}_1, \ldots, \mathcal{V}_{nk}$ of subsets of X with the property that $\mathcal{V}_t$ is R_t–bounded for each $t = 1, \ldots, nk$. Thus, X has property C, as required. □

Corollary 6.2 [29, Theorem III.8] *If X has asymptotic property C and Y has finite asymptotic dimension, then $X \times Y$ has asymptotic property C.*

6.2 Free product of metric spaces

Definition 6.3 Let (X, x_0) be a metric space with a fixed basepoint x_0. The *free product* $*X$ is the metric space whose elements are words in the alphabet $X \setminus \{x_0\}$ along with the trivial word ϵ.

We denote $X \setminus \{x_0\}$ by X^*. We identify elements of X^* with the set of single-letter (nontrivial) words in $*X$.

There is a natural notion of *concatenation* of words, which we usually denote by juxtaposition. When we want to emphasize the concatenation of two elements u and v in $*X$, we write $u \cdot v$. If $u \in *X$ and $A \subset *X$, we write $u \cdot A$ for the set of all elements of $*X$ that are obtained by concatenating an element of A on the left of the element $u \in *X$. The notation X^m is used to denote those elements of $*X$ that are obtained by concatenating exactly m elements of X^*. We define $X^0 = \{\epsilon\}$ and use the notation $X^{\leq m}$ for $\bigcup_{i=0}^m X^i$. Finally, if A and B are subsets of $*X$, then we define their *concatenation* $A \cdot B$ to be all elements of the form $a \cdot b$ with $a \in A$ and $b \in B$.

We define the *norm* of the trivial word to be 0 and the *norm* of a letter $x \in X^*$ to be $\|x\| = \mathrm{dist}(x_0, x)$. The *norm* of a composite word is the sum of norms of its letters: $\|x_1 x_2 \cdots x_k\| = \sum_{i=1}^k \|x_i\|$. The *distance* between two elements of $*X$ is determined by eliminating their common prefix: if $w = u \cdot x \cdot v$ and $w' = u \cdot x' \cdot v'$ (with $u, v, v' \in *X$ and $x, x' \in X^*$ with $x \neq x'$), then $\mathrm{dist}(w, w') = \mathrm{dist}(x, x') + \|v\| + \|v'\|$. The distance from w to the trivial word ϵ is equal to the norm $\|w\|$ of w.

Comment 6.4 For simplicity of proofs we defined the *unary* free product of a *single* metric space X. For applications, we use the fact that if X and Y are metric spaces, then $X * Y$ embeds isometrically into $*(X \vee Y)$, where $X \vee Y$ denotes the wedge of X and Y and $X * Y$ denotes the free product of X and Y, defined as in [4].

Definition 6.5 A subset $A \subset *X$ is said to be *flat* if there exists some $x \in *X$ such that $A \subset x \cdot (X^*)$.

Definition 6.6 Let $R > 0$. The R–*cone* of a set $A \subset *X$ is defined to be

$$\operatorname{con}_R A = A \cdot (*\bar{B}(x_0, R)) \subset *X,$$

where $\bar{B}(x_0, R)$ denotes the closed ball with center x_0 and radius R.

Definition 6.7 We will say that the metric space X is *discrete in the metric sense* if $\inf\{\operatorname{dist}(x, y) : x, y \in X, x \neq y\} > 0$.

Lemma 6.8 *Let X be a metric space that is discrete in the metric sense. If $\{A_\alpha\}$ is a family of flat uniformly bounded subsets of $*X$, then, for each M, the family $\{\operatorname{con}_M A_\alpha\}$ has asymptotic dimension bounded by 1 uniformly.*

Proof Let $E = \inf\{\operatorname{dist}(x, y) : x, y \in X, x \neq y\}$. By the assumption, $E > 0$. Let $D = \sup_\alpha\{\operatorname{diam}(A_\alpha)\}$. By the assumption, $D < \infty$.

For each α we define a simplicial tree T_α. The set of vertices of T_α consists of elements of $\operatorname{con}_M A_\alpha$ and of an additional element r_α, which we call the root vertex. The root vertex r_α is connected by edges to all vertices from the base A_α of the cone $\operatorname{con}_M A_\alpha$. Two vertices p and t of the cone $\operatorname{con}_M A_\alpha$ are connected by an edge if and only if there exists $x \in \bar{B}(x_0, M)$ with $x \neq x_0$ such that $p = t \cdot x$ or $t = p \cdot x$. The metric $\operatorname{dist}_{T_\alpha}$ on T_α is the geodesic metric with edge lengths equal to 1.

Consider the map $f_\alpha : \operatorname{con}_M A_\alpha \to T_\alpha$ that maps elements of $\operatorname{con}_M A$ to their corresponding vertices of T_α. We will show that f_α satisfies the inequalities

$$\frac{\operatorname{dist}(x, y)}{M} - \frac{D}{M} \leq \operatorname{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)) \leq \frac{\operatorname{dist}(x, y)}{E} + 3$$

for each $x, y \in \operatorname{con}_M A_\alpha$.

As a first case we consider $x, y \in \operatorname{con}_M A_\alpha$ such that $x = ax_1x_2\cdots x_k$ and $y = by_1y_2\cdots y_l$, where $a, b \in A_\alpha$ with $a \neq b$ and $x_i, y_i \in \bar{B}(x_0, R)^*$ with $k, l \geq 0$. We have

$$\operatorname{dist}(x, y) = \left(\sum_{i=1}^k \|x_i\|\right) + \operatorname{dist}(a, b) + \left(\sum_{i=1}^l \|y_i\|\right)$$

and

$$\operatorname{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)) = 2 + k + l.$$

We have

$$E \leq \operatorname{dist}(a, b) \leq D, \quad E \leq \|x_i\| \leq M, \quad E \leq \|y_i\| \leq M.$$

Therefore,

$$(k + l + 1)E \leq \text{dist}(x, y) \leq (k + l)M + D.$$

As a second case consider $x, y \in \text{con}_M A_\alpha$ such that $x = c \cdot x_1 x_2 \cdots x_k$, $y = c \cdot y_1 y_2 \cdot y_l$, $c \in \text{con}_M A_\alpha$, $x_i, y_i \in \bar{B}(x_0, R)^*$ and $x_1 \neq y_1$. We have

$$\text{dist}(x, y) = \left(\sum_{i=2}^{k} \|x_i\| \right) + \text{dist}(x_1, y_1) + \left(\sum_{i=2}^{l} \|y_i\| \right)$$

and

$$\text{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)) = k + l.$$

We have

$$E \leq \text{dist}(x_1, y_1) \leq D, \quad E \leq \|x_i\| \leq M, \quad E \leq \|y_i\| \leq M.$$

Therefore,

$$(k + l - 1)E \leq \text{dist}(x, y) \leq (k + l)M.$$

Combining the inequalities from both cases we have

$$\text{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)) \leq k + l + 2 \leq \frac{\text{dist}(x, y)}{E} + 3$$

and

$$\frac{\text{dist}(x, y)}{M} - \frac{D}{M} \leq k + l \leq \text{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)).$$

Hence, the inequalities that we sought:

$$\frac{\text{dist}(x, y)}{M} - \frac{D}{M} \leq \text{dist}_{T_\alpha}(f_\alpha(x), f_\alpha(y)) \leq \frac{\text{dist}(x, y)}{E} + 3.$$

Therefore, each f_α is a quasi-isometry with constants that do not depend on α.

In the example following Theorem 19 of [6], it is shown that for a given $r > 0$ one can take a cover of any simplicial tree by $3r$–bounded, r–disjoint families of subsets. It follows that any family of simplicial trees has asymptotic dimension bounded by 1 uniformly. Since the family $\{\text{con}_M A_\alpha\}$ is uniformly quasi-isometric to a family of simplicial trees, it has asymptotic dimension bounded by 1 uniformly. □

Definition 6.9 We say that a set A is R–*connected* if for every $x, y \in A$ there exists a finite sequence $x_0 = x, x_1, x_2, \ldots, x_k = y$ such that $\text{dist}(x_i, x_{i+1}) \leq R$ for each $i = 1, 2, \ldots, k - 1$. An R–*connected component* of A (or simply an R–*component*) is a maximal R–connected subset of A. We recall that in Section 2 we described the sets A and B as being R–*disjoint* if, for each $x \in A$ and $y \in B$, we have $\text{dist}(x, y) > R$.

Definition 6.10 The *order* of an element x of $*X$ is the nonnegative integer m such that $x \in X^m$. We write $\mathrm{ord}(x) = m$ in this case.

Lemma 6.11 *Let* X *be a discrete metric space and let* $R > 0$. *Let* $A \subset *X$ *be such that* $A \subset *X \cdot (A \setminus \bar{B}(x_0, R))$. *Assume that* R*–connected components of* A *are uniformly bounded. Then, for each* M, R*–connected components of* $\mathrm{con}_M A$ *have asymptotic dimension bounded by* 1 *uniformly.*

Proof As a first step we will show that

C is an R–connected component of $\mathrm{con}_M A$

$\quad \Longleftrightarrow \quad$ $C \cap X^{\leq n}$ is an R–connected component of $\mathrm{con}_M A \cap X^{\leq n}$ for each n.

The implication $\Leftarrow$ is trivial.

We will say that two words w_1, w_2 in $*X$ are *adjacent* if

(1) $w_1 = w_2 \cdot x$ for some $x \in X$, or

(2) $w_2 = w_1 \cdot x$ for some $x \in X$, or

(3) $w_1 = w \cdot x_1$ and $w_2 = w \cdot x_2$ for some $w \in *X$ and $x_1, x_2 \in X$.

In other words, w_1 and w_2 are adjacent if they differ by their last letter or if one is equal to the other with a single letter appended at the end.

Let $x = p \cdot x_1 x_2 \cdot x_k$ and $y = p \cdot y_1 y_2 \cdot y_l$ with $x, y \in \mathrm{con}_M A$ and $\mathrm{dist}(x, y) \leq R$, where p is the common prefix of x and y. We have

$$\mathrm{dist}(x, y) = \left(\sum_{i=2}^{k} \|x_i\| \right) + \mathrm{dist}(x_1, y_1) + \left(\sum_{i=2}^{l} \|y_i\| \right) \leq R.$$

Therefore $x_i, y_i \in \bar{B}(x_0, R)$ for $i \geq 2$, so $x_i, y_i \notin A$ for $i \geq 2$. Hence, $p \cdot x_1 x_2 \cdots x_i \in \mathrm{con}_M A$ and $p \cdot y_1 y_2 \cdots y_i \in \mathrm{con}_M A$ for each i. Therefore, $x = p \cdot x_1 x_2 \cdots x_k$, $p \cdot x_1 x_2 \cdots x_{k-1}, \ldots, p \cdot x_1, p \cdot y_1, p \cdot y_1 y_2, \ldots, p \cdot y_1 y_2 \cdots y_l = y$ is a sequence of R–close adjacent elements of $\mathrm{con}_M A$.

Therefore, if C is an R–connected component of $\mathrm{con}_M A$, then for each $x, y \in C$ there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ of adjacent R–close points of C.

Observe that if $x_1, x_2, \ldots, x_k$ is a sequence of adjacent points with $\mathrm{ord}(x_1) = \mathrm{ord}(x_k)$ and $\mathrm{ord}(x_i) > \mathrm{ord}(x_1)$ for all i with $2 \leq i \leq k-1$, then $x_1 = x_k$. This implies that if $x, y \in \mathrm{con}_M A$ are in the same R–connected component and $x, y \in X^{\leq n}$, then x and y are in the same R–connected component of $C \cap X^{\leq n}$. The implication $\Rightarrow$ is proven.

Let D be a bound on the diameters of the R–connected components of A. Let C be an R–connected component of $\mathrm{con}_M A$. Let $C_k = C \cap X^{\leq k}$. We have proven that each C_k is R–connected. Let k_0 be the smallest integer such that $C_{k_0} \neq \varnothing$. The diameter of C_{k_0} is bounded by $2M + 2R + 2D$. We will show by induction on k that

$$C_k \subset \mathrm{con}_{M+R+D} C_{k_0}.$$

Consider $x \in (C_{k+1} \setminus C_k)$. Since $C \subset \mathrm{con}_M A$, we have $x \in A$ or $x \in \mathrm{con}_M C_k$.

Let $x \in A$. Then x has to be in an R–connected component of A that is R–close to $\mathrm{con}_M C_k$. Therefore $x \in \mathrm{con}_{M+R+D} C_k$.

Let $x \in \mathrm{con}_M C_k$. Then $\mathrm{con}_M C_k \subset \mathrm{con}_{M+R+D} C_k$. Hence, $x \in \mathrm{con}_{M+R+D} C_k \subset \mathrm{con}_{M+R+D} \mathrm{con}_{M+R+D} C_{k_0} = \mathrm{con}_{M+R+D} C_{k_0}$.

We proved by induction that if C is an R–connected component of $\mathrm{con}_M A$, then $C \subset \mathrm{con}_{M+R+D} C_{k_0}$. Observe that C_{k_0} is flat and has uniformly bounded diameter (by $2M + 2R + 2D$). Therefore, by Lemma 6.8, the collection of R–connected components of $\mathrm{con}_M A$ has asymptotic dimension bounded by 1 uniformly. $\square$

Theorem 6.12 *Let X be a discrete metric space with a fixed basepoint x_0. If X has asymptotic property C, then $*X$ has asymptotic property C.*

Proof Let $R_1 \leq R_2 \leq \cdots$ be a nondecreasing sequence of real numbers. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ be a finite sequence of R_i–disjoint uniformly bounded families of subsets of X such that $\bigcup \mathcal{U}_i$ covers X. For $i = 1, 2, \ldots, n$, let

$$\mathcal{V}_i = \{x \cdot (U \setminus \bar{B}(x_0, R_{n+1})) : U \in \mathcal{U}_i,\ x \in *X\}$$

and

$$\mathcal{V}_{n+1} = \{\{x_0\}\}.$$

We will prove that

(1) $\bigcup_{i=1}^{n+1} \left(\mathrm{con}_{R_{n+1}} \bigcup \mathcal{V}_i\right) = *X$.

(2) $\mathcal{V}_i$ is R_i–disjoint, uniformly bounded and its elements are flat for each i.

To prove (1), consider an element $x = x_1 x_2 \cdots x_k \in *X$. If $\|x_i\| \leq R_{n+1}$ for each i, then $x \in \mathrm{con}_{R_{n+1}} \{x_0\}$. Otherwise, let m be the largest integer such that $\|x_m\| > R_{n+1}$. Since $\bigcup \mathcal{U}_i$ covers X, there exists an l and a $U \in \mathcal{U}_l$ such that $x_m \in U$. We have $x \in x_1 x_2 \cdots x_{m-1} \cdot (U \setminus \bar{B}(x_0, R_{n+1})) \cdot *\bar{B}(x_0, R_{n+1})$; hence, $x \in \mathrm{con}_{R_{n+1}} \bigcup \mathcal{V}_l$.

To prove (2), assume that i is given, with $1 \leq i \leq n$. Observe that elements of $\mathcal{V}_i$ are flat and uniformly bounded directly from the definition. So, it remains only to show that $\mathcal{V}_i$ is R_i–disjoint. Suppose that V_1 and V_2 are distinct elements of $\mathcal{V}_i$, say $V_1 = x_1 \cdot U_1$ and $V_2 = x_2 \cdot U_2$. We consider two cases: (a) If $x_1 = x_2$, then the result follows from the fact that the family $\mathcal{U}_i$ is R_i–disjoint. (b) If $V_1 \cup V_2$ is not flat, then we take arbitrary elements $v_1 = x_1 u_1 \in V_1 = x_1 \cdot U_1$ and $v_2 = x_2 u_2 \in V_2 = x_2 \cdot U_2$. Then $\mathrm{dist}(v_1, v_2) \geq \min\{\|u_1\|, \|u_2\|\} > R_{n+1} \geq R_i$, by the definition of $\mathcal{V}_i$ and the definition of the metric in $*X$.

By Lemma 6.11, the R_i–connected components of $\mathrm{con}_{R_{n+1}+1} \bigcup \mathcal{V}_i$ have asymptotic dimension bounded by 1 uniformly and are R_i–disjoint. By the decomposition lemma, $*X$ has asymptotic property C. $\square$

Corollary 6.13 *Let X and Y be metric spaces with fixed basepoints. If X and Y have asymptotic property C, then $X * Y$ has asymptotic property C.*

Proof By the union theorem (Proposition 2.3), $X \vee Y$ has asymptotic property C. By Theorem 6.12, $(X \vee Y) * (X \vee Y)$ has asymptotic property C. Since $X * Y$ embeds isometrically into $(X \vee Y) * (X \vee Y)$ and asymptotic property C is inherited by subspaces, we are done. $\square$

7 Open questions

Although we were able to resolve several questions related to asymptotic property C, we would like to mention the following problems, which remain open.

We were able to show that the free product of two metric spaces (or groups) with asymptotic property C has asymptotic property C, but this invites the obvious question for amalgamated products [7, Questions 4.3]: Is asymptotic property C preserved by amalgamated free products? Generalizing this a bit more, one could consider the following question:

Let Γ be a finite graph. To each vertex v of Γ we assign a group G_v. We let $\mathfrak{G}\Gamma$ denote the graph product of these groups; ie the group generated by the G_v along with the additional relation that two elements of the groups G_v and G_w commute precisely when an edge of Γ connects v and w. If all the G_v have asymptotic property C, does $\mathfrak{G}\Gamma$ have asymptotic property C?

These questions and the question of whether a semidirect product preserves asymptotic property C could be addressed via a true fibering theorem. We state this in two forms, both as a statement for metric spaces and a statement in terms of exact sequences of groups.

Question 7.1 Let $f\colon X \to Y$ be a uniformly expansive map between metric spaces. Assume that Y has asymptotic property C and that $f^{-1}(A)$ has asymptotic property C for every bounded subset $A \subset Y$. Does X have to have asymptotic property C?

Question 7.2 Let $f\colon G \to H$ be a group homomorphism such that H and $\ker f$ have asymptotic property C. Does G have to have asymptotic property C?

One could also ask several questions related to the construction of Yamauchi [30]. In particular, one could ask [30, Question 3.2]: Does every countable direct union of groups with finite asymptotic dimension have asymptotic property C? More generally, is asymptotic property C preserved by countable direct unions?

Dydak and Virk showed that the notions of countable asymptotic dimension and straight finite decomposition complexity are equivalent [15]. It is easy to see that asymptotic property C implies either of these two notions, but the exact nature of the relationship between these notions is unknown for groups. We believe that the asymptotic property C can be generalized to a property we propose to call ω–APC. This generalization enables us to see that our fibering lemma and decomposition lemma are both special cases of a more general framework of which straight finite decomposition complexity and asymptotic property C are the first nontrivial special cases.

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