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Higher cohomology operations and R–completion

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Let R be either $\mathbb{F}_p$ or a field of characteristic 0. For each R–good topological space $\boldsymbol{Y}$, we define a collection of higher cohomology operations which, together with the cohomology algebra $H^*(\boldsymbol{Y};R)$, suffice to determine $\boldsymbol{Y}$ up to R–completion. We also provide a similar collection of higher cohomology operations which determine when two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ between R–good spaces (inducing the same algebraic homomorphism $H^*(\boldsymbol{Y};R) \to H^*(\boldsymbol{Z};R)$) are R–equivalent.

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Introduction

We describe complete sets of invariants for the R–homotopy types of R–good topological spaces and maps between them, consisting of systems of higher R–cohomology operations (where R is either $\mathbb{F}_p$ or a field of characteristic 0).

Higher homotopy or cohomology operations (see Toda [37], Spanier [35; 36], Adams [1] and Maunder [30]) should be thought of as inductively defined systems of obstructions to rectifying homotopy-commutative diagrams. In particular, an n^{th} *order cohomology operation* is attached to a diagram $F\colon I \to \mathrm{ho}\,\mathsf{Top}$ indexed by a *lattice* in the sense of Blanc and Markl [15, Section 2] — that is, a directed category I of length $n+1$ such that, for all but the initial object i_0 in I, $F(i)$ is a product of R–module Eilenberg–Mac Lane spaces (an R–GEM).

0.1 Example The simplest example is a Toda bracket [3], for a diagram of the form

$$\begin{tikzcd} \boldsymbol{Y} \arrow[r, "f"] \arrow[rr, bend left, "*"] & \boldsymbol{W}_0 \arrow[r, "g"] \arrow[rr, bend right, "*"'] & \boldsymbol{W}_1 \arrow[r, "h"] & \boldsymbol{W}_2, \end{tikzcd} \tag{0.2}$$

with $\boldsymbol{W}_0$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ R–GEMs, and each adjacent composition nullhomotopic (see Adams [1] and Harper [26]).

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This defines a secondary cohomology operation in the sense of Adams [1], since f represents a set of cohomology classes in $H^*(\boldsymbol{Y};R)$, on which the set of primary R–cohomology operations represented by g vanish. The fact that $h \circ g \sim *$ indicates a relation among primary operations.

Our general strategy for rectification of any $F\colon I \to \operatorname{ho}\mathsf{Top}$ is to inductively rectify, and then make fibrant, longer and longer final segments of the given diagram. In our example, we first make $\boldsymbol{W}_2$ fibrant and change h into a fibration (so the subdiagram $\boldsymbol{W}_1 \xrightarrow{h} \boldsymbol{W}_2$ is fibrant). We then change g up to homotopy so that $h \circ g = *$, using Blanc, Johnson and Turner [11, Lemma 5.11]. Factoring g as $\boldsymbol{W}_0 \xrightarrow{k'} \operatorname{Fib}(h) \overset{i}{\hookrightarrow} \boldsymbol{W}_1$, and then changing k' into a fibration k makes $\boldsymbol{W}_0 \xrightarrow{g} \boldsymbol{W}_1 \xrightarrow{h} \boldsymbol{W}_2$ fibrant. To simplify notation we denote $i \circ k$ simply by $g\colon \boldsymbol{W}_0 \to \boldsymbol{W}_1$.

We think of the following solid diagram of vertical and horizontal fibration sequences as the *template* for our Toda bracket (depending only on $\boldsymbol{W}_0 \xrightarrow{g} \boldsymbol{W}_1 \xrightarrow{h} \boldsymbol{W}_2$):

(0.3)
$$
\begin{tikzcd}
\boldsymbol{Y} \arrow[drr, dashed, bend left, "\varphi"] \arrow[dr, dotted, "\exists?\psi"] \arrow[ddr, dashed, bend right, "f"'] & & & \\
& \operatorname{Fib}(k) \arrow[r, hook, "\ell"'] \arrow[d, hook] & \operatorname{Fib}(\widetilde{g}) \arrow[r, two heads, "q"'] \arrow[d, hook] & \Omega\boldsymbol{W}_2 \arrow[d, hook] \\
& \boldsymbol{W}_0 \arrow[r, hook, "j", "\simeq"'] \arrow[d, two heads, "k"] \arrow[dr, "g"] & \boldsymbol{W}_0' \arrow[r, two heads, "G"] \arrow[d, two heads, "\widetilde{g}"] & P\boldsymbol{W}_2 \arrow[d, two heads, "p"] \\
& \operatorname{Fib}(h) \arrow[r, hook, "i"] & \boldsymbol{W}_1 \arrow[r, two heads, "h"] & \boldsymbol{W}_2
\end{tikzcd}
$$

The nullhomotopy G exists since $h \circ \widetilde{g} \circ j = g \circ h = *$ and j is a weak equivalence.

Now, any map $f\colon \boldsymbol{Y} \to \boldsymbol{W}_0$ with $g \circ f \sim *$ factors up to homotopy through a map $\varphi\colon \boldsymbol{Y} \to \operatorname{Fib}(\widetilde{g})$. To rectify (0.2), φ and f should induce a map $\psi\colon \boldsymbol{Y} \to \operatorname{Fib}(k) = \operatorname{Fib}(q)$. The obstruction to doing so — namely the homotopy class of the composite $q \circ \varphi\colon \boldsymbol{Y} \to \Omega\boldsymbol{W}_2$ — is called the *value* of the Toda bracket $\langle f, g, h\rangle$ (for the given choices of k and φ).

0.4 The basic construction Our object in this paper is to associate to each R–good space $\boldsymbol{Y}$ a sequence $\langle\langle \boldsymbol{Y}\rangle\rangle = (\langle\langle \boldsymbol{Y}\rangle\rangle_n)_{n=2}^{\infty}$ of higher cohomology operations which serve as a complete set of invariants for the R–homotopy type of $\boldsymbol{Y}$, constructed roughly as follows:

(a) We start with $H^*(Y;R)$ as a Θ_R*–algebra* — ie a graded R–algebra with an action of the primary R–cohomology operations (such as Steenrod squares). We then choose a *CW resolution* of this Θ_R–algebra, given by an inductively defined simplicial Θ_R–algebra $V_\bullet$, with the n^{th} stage obtained from the $(n-1)$–truncation by attaching a free Θ_R–algebra $\overline{V}_n$ along a map $\overline{\partial}^n_0\colon \overline{V}_n \to V_{n-1}$ (see Section 1.11 below). Our goal is to realize $V_\bullet \to H^*(Y;R)$ by a coaugmented cosimplicial space $Y \to W^\bullet = \lim_n W^\bullet_{[n]}$, for which $Y \to \operatorname{Tot} W^\bullet$ is the R–completion map of Bousfield and Kan [20].

(b) To carry out this program, we use a double induction: if $W^\bullet_{[n-1]}$ realizes $V_\bullet$ through simplicial dimension $n-1$, we can choose a map $\overline{d}^0_{n-1}\colon W^{n-1}_{[n-1]} \to \overline{W}^n$ realizing $\overline{\partial}^n_0\colon \overline{V}_n \to V_{n-1}$ (with $\overline{W}^n$ an R–GEM). Then $\overline{d}^0_{n-1} \circ d^0$ is nullhomotopic by a nullhomotopy $F^{n-2}\colon W^{n-2}_{[n-1]} \to P\overline{W}^n$, with $F^{n-2} \circ d^0$ factoring through $a^{n-3}\colon W^{n-3}_{[n-1]} \to \Omega\overline{W}^n$, as in Example 0.1 above. One can in fact choose F^{n-2} so that a^{n-3} is nullhomotopic.

(c) Step (b) can be repeated, and the k^{th} obstruction a^{n-k} again turns out to be nullhomotopic. We end up with an n^{th} order cohomology operation $\langle\langle Y\rangle\rangle_n$ with value $a^{-1}\colon Y \to \Omega^{n-1}\overline{W}^n$. The fact that this too vanishes allows us to extend $W^\bullet_{[n-1]}$ to a cosimplicial space $W^\bullet_{[n]}$ (which will be n–coskeletal, up to homotopy). We call the system $\mathcal{W} = (W^\bullet_{[n]})_{n\in\mathbb{N}}$ a *sequential realization* of $V_\bullet$ for the space Y, and think of it as a template for a sequence $\langle\langle Y\rangle\rangle = (\langle\langle Y\rangle\rangle_n)_{n=2}^\infty$ of higher cohomology operations.

(d) Finally, given a space Z with $H^*(Y;R) \cong H^*(Z;R)$, the augmentation of Θ_R–algebras $\varepsilon\colon V_0 \to H^*(Z;R)$ can be realized by a map $\varepsilon_{[0]}\colon Z \to W^0_{[0]}$. If we can extend this to a coaugmentation $\varepsilon\colon Z \to W^\bullet := \lim W^\bullet_{[n]}$, we would obtain an R–equivalence between Z and Y. The obstruction to extending the $(n-1)^{\text{st}}$ approximation $\varepsilon_{[n-1]}\colon Z \to W^\bullet_{[n]}$ for ε to $\varepsilon_{[n]}$ is a map $a^{-1}\colon Z \to \Omega^{n-1}\overline{W}^n$ as above — the value associated to Z for the n^{th} order cohomology operation $\langle\langle Y\rangle\rangle_n$.

0.5 Main results In order to apply the machinery described above, we need the following important technical result:

Theorem A *Any CW resolution $V_\bullet$ of the Θ_R–algebra $H^*(Y;R)$ can be realized by a coaugmented cosimplicial space $Y \to W^\bullet$ with each W^n an R–GEM, obtained as the limit of a sequential realization as above.*

See Theorem 2.33 below. This allows us to produce various templates for the system of higher cohomology operations $\langle\langle Y\rangle\rangle$, based on the algebraic resolution of our choice.

The sequence of higher cohomology operations presented here is dual to the higher homotopy operations of Blanc [6] and Blanc, Johnson and Turner [13], which correspond to the André–Quillen cohomology obstructions for distinguishing between different realizations $\pi_* \boldsymbol{Y}$ of a Π–algebra (see Blanc, Dywer and Goerss [10]). Such André–Quillen classes appear also in the dual context of distinguishing between different realizations of a given abstract Θ_R–algebra Γ (see Blanc [8] and Biedermann, Raptis and Stelzer [5]). However, the higher cohomology operations of Blanc [8] correspond to André–Quillen *cocycles* (for a specific algebraic resolution $V_\bullet$ of $H^*(\boldsymbol{Y}; R)$). We therefore would like to collect together the various higher order cohomology operations corresponding to a given André–Quillen cohomology class. We do this by means of suitable (split) weak equivalences, called *comparison maps*, between sequential realizations for various algebraic resolutions, and show:

Theorem B *Any two sequential realizations of two CW resolutions $V_\bullet$ for the same space $\boldsymbol{Y}$ are connected by a zigzag of comparison maps.*

See Theorem 3.20 below.

Our two main results may then be summarized as follows:

Theorem C *For R either $\mathbb{F}_p$ or a field of characteristic 0, let $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be R–good spaces and $\vartheta\colon H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$ an isomorphism of Θ_R–algebras. Then ϑ is realizable by a zigzag of R–equivalences between $\boldsymbol{Y}$ and $\boldsymbol{Z}$ if and only if the system of higher operations associated to this initial data vanishes.*

See Theorem 4.18 below.

By extending the ideas sketched in Section 0.4, one can use any sequential realization for $\boldsymbol{Y}$ to define a system of higher cohomology operations associated to any two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ which induce the same map in cohomology; although the construction is more complicated, these operations still take values in the groups $[\boldsymbol{Z}, \Omega^{n-1}\overline{\boldsymbol{W}}^n]$, and we have:

Theorem D *For R either $\mathbb{F}_p$ or a field of characteristic 0, let $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ be two maps between R–good spaces which induce the same morphism of Θ_R–algebras $H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$. Then f_0 is R–equivalent to f_1 if and only if the associated system of higher operations vanishes.*

See Theorem 5.29 below.

0.6 Organization Section 1 provides some background material on (co)simplicial constructions and on sketches and their algebras. In Section 2 we define and study sequential realizations of algebraic resolutions. In Section 3 we show how any two such sequential realizations may be connected by a zigzag of comparison maps. In Section 4 we construct the higher cohomology operations used to distinguish between spaces, including a detailed rational example in Section 4.24, while in Section 5 we define the analogous invariants for maps.

The appendix reviews the notions of enriched sketches and their mapping algebras, which are used to generalize and prove Theorem A.

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1 Background

In this section we present some background material on (co)simplicial theory and algebraic theories that will be used throughout the paper.

1.1 Definition Let $\mathbf{\Delta}$ denote the category of finite ordered sets and order-preserving maps (see May [31, Section I.2]), and $\mathbf{\Delta}_+$ the subcategory with the same objects, but only monic maps. A *cosimplicial object* $G^\bullet$ in a category $\mathcal{C}$ is a functor $\mathbf{\Delta} \to \mathcal{C}$, and a *restricted* cosimplicial object is a functor $\mathbf{\Delta}_+ \to \mathcal{C}$. More concretely, we write G^n for the value of $G^\bullet$ at the ordered set $[\boldsymbol{n}] = \{0 < 1 < \cdots < n\}$. The maps in the diagram $G^\bullet$ are generated by the *coface* maps $d^i = d^i_n\colon G^n \to G^{n+1}$ for $0 \le i \le n+1$, as well as *codegeneracy* maps $s^j = s^j_n\colon G^n \to G^{n-1}$ for $0 \le j < n$ in the nonrestricted case, satisfying the usual cosimplicial identities. Dually, a *simplicial object* $G_\bullet$ in $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$. The category of cosimplicial objects over $\mathcal{C}$ will be denoted by $\mathcal{C}^{\mathbf{\Delta}}$, and the category of simplicial objects over $\mathcal{C}$ will be denoted by $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$. However, the category of simplicial sets is denoted simply by $\mathcal{S} = s\mathsf{Set}$, and that of pointed simplicial sets by $\mathcal{S}_* = s\mathsf{Set}_*$. By a *space* we always mean a pointed simplicial set.

There are natural embeddings $c(-)^\bullet\colon \mathcal{C} \to \mathcal{C}^{\mathbf{\Delta}}$ and $c(-)_\bullet\colon \mathcal{C} \to \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$, defined by letting $c(A)^\bullet$ denote the constant cosimplicial object which is A in every cosimplicial dimension, and similarly for $c(A)_\bullet$.

1.2 Latching and matching objects For a cosimplicial object $G^\bullet \in \mathcal{C}^{\mathbf{\Delta}}$ in a complete category $\mathcal{C}$, the n^{th} *matching object* for $G^\bullet$ is defined to be

(1.3) $$M^n G^\bullet := \lim_{\phi\colon [\boldsymbol{n}]\to[\boldsymbol{k}]} G^k,$$

where ϕ ranges over the nonidentity surjective maps $[\boldsymbol{n}]\to[\boldsymbol{k}]$ in $\mathbf{\Delta}$. There is a natural map $\zeta^n\colon G^n \to M^n G^\bullet$ induced by the structure maps of the limit, and any iterated codegeneracy map $s^I = \phi_*\colon G^n \to G^k$ factors as

(1.4) $$s^I = \mathrm{proj}_\phi \circ \zeta^n,$$

where $\mathrm{proj}_\phi\colon M^n G^\bullet \to G^k$ is the structure map for the copy of G^k indexed by ϕ (see Bousfield and Kan [20, Section X.4.5]).

Note that the inclusion $\mathbf{\Delta}_+ \hookrightarrow \mathbf{\Delta}$ induces a forgetful functor $\mathcal{U}\colon \mathcal{C}^{\mathbf{\Delta}} \to \mathcal{C}^{\mathbf{\Delta}_+}$, and its right adjoint $\mathcal{F}\colon \mathcal{C}^{\mathbf{\Delta}_+} \to \mathcal{C}^{\mathbf{\Delta}}$ is given by $(\mathcal{F}G^\bullet)^n = G^n \times M^n G^\bullet$, with codegeneracies given by (1.4) and coface maps by the cosimplicial identities.

The n^{th} *latching object* for $G^\bullet \in \mathcal{C}^{\mathbf{\Delta}}$ is

$$L^n G^\bullet := \operatorname*{colim}_{\theta\colon [\boldsymbol{k}]\to[\boldsymbol{n}]} G^k,$$

where the maps θ are now nonidentity injective maps, with $\sigma^n\colon L^n G^\bullet \to G^n$ defined by the structure maps.

These two constructions have analogues for a simplicial object $G_\bullet$ over a (co)complete category $\mathcal{C}$: the *latching object*

$$L_n G_\bullet := \operatorname*{colim}_{\theta\colon [\boldsymbol{k}]\to[\boldsymbol{n}]} G_k,$$

and the *matching object*

$$M_n G_\bullet := \lim_{\phi\colon [\boldsymbol{n}]\to[\boldsymbol{k}]} G_k,$$

equipped with the obvious canonical maps.

1.5 Definition Let $\mathcal{C}$ be a pointed category. If it is complete, the n^{th} *Moore chain* object of $G_\bullet \in \mathcal{C}^{\mathbf{\Delta}}$ is

(1.6) $$C_n G_\bullet := \bigcap_{i=1}^{n} \mathrm{Ker}\{d_i\colon G_n \to G_{n-1}\},$$

with differential $\partial_n^{G_\bullet} = \partial_n := (d_0)|_{C_n G_\bullet}\colon C_n G_\bullet \to C_{n-1} G_\bullet$. The n^{th} *Moore cycle* object is $Z_n G_\bullet := \mathrm{Ker}(\partial_n^{G_\bullet})$.

Dually, if $\mathcal{C}$ is cocomplete, the n^{th} *Moore cochain* object of $G^\bullet \in \mathcal{C}^{\mathbf{\Delta}^{\text{op}}}$ is

$$C^n G^\bullet := \operatorname{Coker}\Big(\coprod_{i=1}^{n-1} G^n \xrightarrow{\perp_i d^i} G^n \Big) \tag{1.7}$$

with differential

$$\delta^{n-1} \colon C^{n-1} G^\bullet \to C^n G^\bullet$$

induced by d^0_{n-1}, and structure map $v^n \colon G^n \to C^n G^\bullet$.

We denote the cofiber of δ^{n-1} by $Z^n G^\bullet$, with structure map $w^n \colon C^n G^\bullet \to Z^n G^\bullet$.

1.8 Cochain complexes in $\mathcal{C}$ In general, a *cochain complex* in $\mathcal{C}$ is a commuting diagram A^* of the form

$$\begin{tikzcd}[column sep=small] A^0 \arrow[rr, "\delta^0"] \arrow[dr] & & A^1 \arrow[rr, "\delta^1"] \arrow[dr] & & A^2 \cdots & \cdots A^{n-1} \arrow[rr, "\delta^{n-1}"] \arrow[dr] & & A^n \arrow[rr, "\delta^n"] \arrow[dr] & & A^{n+1} \cdots \\ 0 \arrow[ur] & 0 \arrow[ur] & & 0 \cdots & & \cdots 0 \arrow[ur] & 0 \arrow[ur] & & 0 \cdots & \end{tikzcd} \tag{1.9}$$

(so $\delta^n \circ \delta^{n-1} = 0$ for all n).

We let $\mathsf{Ch}^{\mathcal{C}}$ denote the category of nonnegatively graded cochain complexes over $\mathcal{C}$, and by $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$ the category of *n–truncated* cochain complexes A^* in $\mathcal{C}$ (for which $A^i = 0$ unless $0 \leq i \leq n$).

The category $\mathsf{Ch}_{\mathcal{C}}$ of nonnegatively graded chain complexes over $\mathcal{C}$ is defined analogously.

The Moore cochain functor $C^* \colon \mathcal{C}^{\mathbf{\Delta}_+} \to \mathsf{Ch}^{\mathcal{C}}$ has a right adjoint (and left inverse) $\mathcal{E} \colon \mathsf{Ch}^{\mathcal{C}} \to \mathcal{C}^{\mathbf{\Delta}_+}$ with $(\mathcal{E} A_*)^n = A^n$, $d^0_n = \delta^n$, and $d^i_n = 0$ for $i \geq 1$. This holds also for $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$ if we truncate $\mathcal{C}^{\mathbf{\Delta}_+}$, too.

When $\mathcal{C}$ is a model category, we have several possible model category structures on $\mathcal{C}^{\mathbf{\Delta}}$, $\mathcal{C}^{\mathbf{\Delta}_+}$, $\mathsf{Ch}^{\mathcal{C}}$, and $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$ (see eg Hirschhorn [28, Section 15.3], Bousfield and Kan [19] and Chachólski and Scherer [21, Section 12]).

In particular, $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$ has two different Reedy model category structures, depending on how we choose the degrees in (1.9); in both cases, the weak equivalences are defined levelwise. In the *right Reedy* model structure, fibrations are also defined levelwise, and an n–cochain complexes A^* is cofibrant if for each $k \leq n$ the natural map $Z^k A^* := \operatorname{Cof}(\delta^{k-1}) \to A^{k+1}$ is a cofibration (with $A^{-1} := *$). In the *left Reedy* model structure, cofibrations are defined levelwise, and an n–cochain complex A^*

is fibrant if for each $k \leq n$ the natural map $A^k \to \mathrm{Ker}(\delta^{k+1})$ is a fibration (with $A^{-1} := *$). Evidently, right Reedy cofibrancy implies left Reedy cofibrancy, and left Reedy fibrancy implies right Reedy fibrancy.

Note that the Moore cochains functor $C^*\colon \mathcal{S}_*^{\mathbf{\Delta}} \to \mathsf{Ch}^{\mathcal{S}_*}$ preserves cofibrancy and weak equivalences among cofibrant objects in the (right) Reedy model structures (see Bousfield and Kan [20, Proposition X.6.3]).

By analogy with the usual fiber/cone construction for (co)chain complexes we have:

1.10 Fibers and cones (i) For any map $f\colon A^\bullet \to B^\bullet$ in $\mathcal{C}^{\mathbf{\Delta}_+}$ we define the restricted cosimplicial object $C^\bullet = \mathrm{Fib}(f)$ by setting $C^n := A^n \times B^{n-1}$, with coface maps

$$\begin{tikzcd} C^{n+1} \arrow[r, phantom, "="] & A^{n+1} \arrow[r, phantom, "\times"] & B^n \\ C^n \arrow[u, "d^0"] \arrow[r, phantom, "="] & A^n \arrow[u, "d^0"] \arrow[ur, "f^n"'] \arrow[r, phantom, "\times"] & B^{n-1} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} C^{n+1} \arrow[r, phantom, "="] & A^{n+1} \arrow[r, phantom, "\times"] & B^n \\ C^n \arrow[u, "d^i"] \arrow[r, phantom, "="] & A^n \arrow[u, "d^i"] \arrow[r, phantom, "\times"] & B^{n-1} \arrow[u, "d^{i-1}"] \end{tikzcd} \quad \text{for } i \geq 1$$

and a natural projection $\ell\colon C^\bullet \to A^\bullet$. For $d^j \circ d^0\colon C^{n-1} \to C^{n+1}$ we verify that

$$\begin{aligned} \mathrm{pr}_{B^n} \circ d^j \circ d^0 &= d^{j-1}_{B^{n-1}} \circ f^{n-1} \circ \mathrm{pr}_{A^{n-1}} \\ &= f^n \circ d^{j-1}_{A^{n-1}} \circ \mathrm{pr}_{A^{n-1}} = \mathrm{pr}_{B_n} \circ d^0 \circ d^{j-1} \end{aligned}$$

for all $j > 0$, while clearly $\mathrm{pr}_{B^n}(d^j \circ d^i) = \mathrm{pr}_{B^n}(d^i \circ d^{j-1})$ for $1 \leq i < j$.

(ii) Similarly, for a map $f\colon A_\bullet \to B_\bullet$ in $\mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}_+}$ we define the restricted simplicial object $C_\bullet = \mathrm{Cone}(f)$ by setting $C_n := A_{n-1} \amalg B_n$, with

$$d_i^{C_n} := \begin{cases} f_{n-1} \perp d_0^{B_n} & \text{if } i = 0, \\ d_{i-1}^{A_{n-1}} \perp d_i^{B_n} & \text{if } i \geq 1, \end{cases}$$

and a natural inclusion $m\colon B_\bullet \hookrightarrow C_\bullet$.

1.11 Simplicial CW objects A simplicial object $G_\bullet \in \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ over a pointed category $\mathcal{C}$ is called a *CW object* if it is equipped with a *CW basis* $(\overline{G}_n)_{n \in \mathbb{N}}$ in $\mathcal{C}$ such that $G_n = \overline{G}_n \amalg L_n G_\bullet$, and $d_i|_{\overline{G}_n} = 0$ for $1 \leq i \leq n$. In this case, $\overline{\partial}_0^{G_n} := d_0|_{\overline{G}_n}\colon \overline{G}_n \to G_{n-1}$ is called the *attaching map* for $\overline{G}_n$. By the simplicial identities $\overline{\partial}_0^{G_n}$ factors as

$$\overline{\partial}_0^{G_n}\colon \overline{G}_n \to Z_{n-1} G_\bullet \subset G_{n-1}. \tag{1.12}$$

Note that we have an explicit formula

$$L_n G_\bullet := \coprod_{0 \leq k < n} \; \coprod_{0 \leq i_1 < \cdots < i_{n-k} \leq n-1} \overline{G}_k, \tag{1.13}$$

where the iterated degeneracy map $s_{i_{n-k}} \dots s_{i_2} s_{i_1}$, restricted to the basis object $\bar{G}_k$, is the inclusion into the copy of $\bar{G}_k$ indexed by $(i_1, \dots, i_{n-k})$.

1.14 Remark Given $\bar{G} \in \mathcal{C}$, define $\bar{G} \otimes_* S^n$ in $\mathsf{Ch}_{\mathcal{C}}$ be the chain complex with $\bar{G}$ in dimension n and $*$ elsewhere. A CW object $G_\bullet$ over $\mathcal{C}$ with CW basis $(\bar{G}_n)_{n=0}^{\infty}$ is the colimit of an inductively constructed sequence of skeleta $\mathrm{sk}_0\, G_\bullet \hookrightarrow \mathrm{sk}_1\, G_\bullet \hookrightarrow \cdots$, in the usual sense (see Goerss and Jardine [25, Section VII.1]), starting with $\mathrm{sk}_0\, G_\bullet := c(\bar{G}_0)_\bullet$. To do so, note that the attaching map $\bar{\partial}_0^{G_n} \colon \bar{G}_n \to Z_{n-1} G_\bullet$ defines a chain map

$$\phi \colon \bar{G}_n \otimes_* S^{n-1} \to C_* \,\mathrm{sk}_{n-1}\, G_\bullet,$$

which has an adjoint $\tilde{\phi} \colon \mathcal{E}'(\bar{G}_n \otimes_* S^{n-1}) \to \mathcal{U}' \,\mathrm{sk}_{n-1}\, G_\bullet$, where $\mathcal{E}' \colon \mathsf{Ch}_{\mathcal{C}} \to \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is left adjoint to the Moore chain functor $C_* \colon \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}} \to \mathsf{Ch}_{\mathcal{C}}$ and $\mathcal{U}' \colon \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ is the forgetful functor (compare Section 1.8).

Note that $\mathcal{U}'$ has a left adjoint $\mathcal{F}' \colon \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}} \to \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ given by $(\mathcal{F}' G_\bullet)_n = G_n \amalg L_n G_\bullet$, (compare Section 1.2). If $\vartheta \colon \mathcal{F}'\mathcal{U}' \to \mathrm{Id}$ is the counit for the adjunction, we have $\mathrm{sk}_n\, G_\bullet$ as the pushout in

$$\begin{tikzcd}
\mathcal{F}'\mathcal{U}' \,\mathrm{sk}_{n-1}\, G_\bullet \arrow[rr, "\vartheta"] \arrow[d, "\mathcal{F}' m"'] & & \mathrm{sk}_{n-1}\, G_\bullet \arrow[d] \\
\mathcal{F}' \,\mathrm{Cone}(\tilde{\phi}) \arrow[rr] \arrow[urr, phantom, "\boxed{\mathrm{PO}}"] & & \mathrm{sk}_n\, G_\bullet
\end{tikzcd} \tag{1.15}$$

for m as in Section 1.10(ii).

1.16 Cosimplicial CW objects A cosimplicial CW object $G^\bullet \in \mathcal{C}^{\mathbf{\Delta}}$ with CW basis $(\bar{G}^n)_{n \in \mathbb{N}}$ may be defined analogously as the limit of a tower of coskeleta

$$\cdots \to \mathrm{csk}^2\, G^\bullet \to \mathrm{csk}^1\, G^\bullet \to \mathrm{csk}^0\, G^\bullet$$

(see [loc. cit.]), starting with $\mathrm{csk}^0\, G^\bullet := c(\bar{G}^0)^\bullet$, by thinking of its attaching maps as a cochain map $\varphi \colon C^* \,\mathrm{csk}^{n-1}\, G^\bullet \to \bar{G}^n \otimes^* S^{n-1}$ (where $\bar{G}^n \otimes^* S^{n-1}$ is the cochain complex with $\bar{G}^n$ in dimension $n-1$ and zero elsewhere).

The map φ has an adjoint $\hat{\varphi} \colon \mathcal{U}\,\mathrm{csk}^{n-1}\, G^\bullet \to \mathcal{E}(\bar{G}^n \otimes^* S^{n-1})$, and we have $\mathrm{csk}^n\, G^\bullet$ as the pullback in

$$\begin{tikzcd}
\mathrm{csk}^n\, G^\bullet \arrow[rr] \arrow[d] \arrow[drr, phantom, "\boxed{\mathrm{PB}}"] & & \mathcal{F}\,\mathrm{Fib}(\hat{\varphi}) \arrow[d, "\mathcal{F}\ell"] \\
\mathrm{csk}^{n-1}\, G^\bullet \arrow[rr, "\theta"] & & \mathcal{F}\mathcal{U}\,\mathrm{csk}^{n-1}\, G^\bullet
\end{tikzcd} \tag{1.17}$$

with θ the unit for $\mathcal{F}\mathcal{U}$ and ℓ as in Section 1.10(i), using the notation of Section 1.2.

1.18 List of functors For the reader's convenience we list the main functors we have defined for simplicial and cosimplicial objects in a category $\mathcal{C}$:

(a) The Moore cochain complex functor $C^*\colon \mathcal{C}^{\mathbf{\Delta}_+} \to \mathsf{Ch}^{\mathcal{C}}$ and its right adjoint (and left inverse) $\mathcal{E}\colon \mathsf{Ch}^{\mathcal{C}} \to \mathcal{C}^{\mathbf{\Delta}_+}$.

(b) The Moore chain complex functor $C_*\colon \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}} \to \mathsf{Ch}_{\mathcal{C}}$ and its left adjoint (and right inverse) $\mathcal{E}'\colon \mathsf{Ch}_{\mathcal{C}} \to \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$.

(c) The forgetful functor $\mathcal{U}\colon \mathcal{C}^{\mathbf{\Delta}} \to \mathcal{C}^{\mathbf{\Delta}_+}$ and its right adjoint $\mathcal{F}\colon \mathcal{C}^{\mathbf{\Delta}_+} \to \mathcal{C}^{\mathbf{\Delta}}$ (adding codegeneracies).

(d) The forgetful functor $\mathcal{U}'\colon \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}} \to \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}}$ and its left adjoint $\mathcal{F}'\colon \mathcal{C}^{\mathbf{\Delta}_+^{\mathrm{op}}} \to \mathcal{C}^{\mathbf{\Delta}^{\mathrm{op}}}$ (adding degeneracies).

When there is no danger of confusion, we denote $C^*(\mathcal{U}(\boldsymbol{W}^\bullet))$ simply by $C^*\boldsymbol{W}^\bullet$ and $C_*(\mathcal{U}'(G_\bullet))$ by $C_*G_\bullet$.

1.19 Definition A *simplicial* category $\mathcal{C}$ (in the sense of Quillen) is one in which, for each (finite) $K \in \mathcal{S}$ and $\boldsymbol{X} \in \mathcal{C}$, we have objects $\boldsymbol{X} \otimes K$ and $\boldsymbol{X}^K$ in $\mathcal{C}$ equipped with appropriate adjunction-like isomorphisms. In particular, such categories are simplicially enriched. A *simplicial model category* is a simplicial category with a model category structure satisfying axiom SM7 of Quillen [32, Sections II.1–II.2]). The basic examples are $\mathcal{S}$ and $\mathcal{S}_*$.

1.20 Assumption From now on $\mathcal{C}$ will be a pointed simplicial model category in which all objects are cofibrant — so in particular it is left proper.

The main example we shall be concerned with is $\mathcal{C} = \mathcal{S}_*$, so we shall sometimes refer to the objects of $\mathcal{C}$ — denoted by boldface letters $\boldsymbol{X}$, $\boldsymbol{Y}$, and so on — as "spaces".

1.21 $\mathcal{G}$–resolution model structure Let $\mathcal{G}$ be a class of homotopy group objects in a model category $\mathcal{C}$ as above, closed under loops. A map $i\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathrm{ho}\,\mathcal{C}$ is called *$\mathcal{G}$–monic* if $i^*\colon [\boldsymbol{B}, \boldsymbol{G}] \to [\boldsymbol{A}, \boldsymbol{G}]$ is onto for each $\boldsymbol{G} \in \mathcal{G}$. An object $\boldsymbol{Y}$ in $\mathcal{C}$ is called *$\mathcal{G}$–injective* if $i^*\colon [\boldsymbol{B}, \boldsymbol{Y}] \to [\boldsymbol{A}, \boldsymbol{Y}]$ is onto for each $\mathcal{G}$–monic map $i\colon \boldsymbol{A} \to \boldsymbol{B}$ in $\mathrm{ho}\,\mathcal{C}$. A fibration in $\mathcal{C}$ is called *$\mathcal{G}$–injective* if it has the right lifting property for the $\mathcal{G}$–monic cofibrations in $\mathcal{C}$.

The homotopy category $\mathrm{ho}\,\mathcal{C}$ is said to have *enough $\mathcal{G}$–injectives* if each object is the source of a $\mathcal{G}$–monic map to a $\mathcal{G}$–injective target. In this case, $\mathcal{G}$ is called a class of *injective models* in $\mathrm{ho}\,\mathcal{C}$.

Recall that a homomorphism in the category $s\mathsf{Gp}$ of simplicial groups is a weak equivalence or fibration when its underlying map in $\mathcal{S}_*$ is such.

A map $f\colon \boldsymbol{W}^\bullet \to \boldsymbol{Y}^\bullet$ in $\mathcal{C}^{\boldsymbol{\Delta}}$ is called a *$\mathcal{G}$–equivalence* if $f^*\colon [\boldsymbol{Y}^\bullet, \boldsymbol{G}] \to [\boldsymbol{W}^\bullet, \boldsymbol{G}]$ is a weak equivalence in $s\mathsf{Gp}$ for each $\boldsymbol{G} \in \mathcal{G}$. Bousfield [19, Theorem 3.3] showed that if $\mathcal{G}$ is a class of injective models in $\mathrm{ho}(\mathcal{C})$, then $\mathcal{C}^{\boldsymbol{\Delta}}$ has a left proper pointed simplicial model category structure with such maps as weak equivalences.

1.22 $\mathcal{G}$–completion Given a class $\mathcal{G}$ of injective models in $\mathcal{C}$ as above, a *$\mathcal{G}$–resolution* of an object $\boldsymbol{Y} \in \mathcal{C}$ is a $\mathcal{G}$–fibrant $\boldsymbol{W}^\bullet$ equipped with a $\mathcal{G}$–trivial cofibration $c(\boldsymbol{Y})^\bullet \hookrightarrow \boldsymbol{W}^\bullet$ (see Definition 1.1). In this case $\widehat{L}_{\mathcal{G}}\boldsymbol{Y} := \mathrm{Tot}\,\boldsymbol{W}^\bullet$ is called the *$\mathcal{G}$–completion* of Y, where $\mathrm{Tot}\,\boldsymbol{W}^\bullet \in \mathcal{C}$ is constructed as in [19, Section 2.8]. Moreover, a map $f\colon \boldsymbol{Y} \to \boldsymbol{Z}$ in $\mathcal{C}$ is a $\mathcal{G}$–equivalence if and only if $\widehat{L}_{\mathcal{G}} f\colon \widehat{L}_{\mathcal{G}}\boldsymbol{Y} \to \widehat{L}_{\mathcal{G}}\boldsymbol{Z}$ is a weak equivalence in $\mathcal{C}$. An object $\boldsymbol{Y} \in \mathcal{C}$ is called *$\mathcal{G}$–complete* if $\boldsymbol{Y} \to \widehat{L}_{\mathcal{G}}\boldsymbol{Y}$ is a weak equivalence, and it is *$\mathcal{G}$–good* if $\boldsymbol{Y} \to \widehat{L}_{\mathcal{G}}\boldsymbol{Y}$ is a $\mathcal{G}$–equivalence — so $\widehat{L}_{\mathcal{G}}\boldsymbol{Y}$ is $\mathcal{G}$–complete (see Bousfield [19, Section 8]). We say that two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ between $\mathcal{G}$–good objects in $\mathcal{C}$ are *$\mathcal{G}$–equivalent* if $\widehat{L}_{\mathcal{G}} f_0 \sim \widehat{L}_{\mathcal{G}} f_1\colon \widehat{L}_{\mathcal{G}}\boldsymbol{Z} \to \widehat{L}_{\mathcal{G}}\boldsymbol{Y}$ are homotopic (for a suitable fibrant and cofibrant model of the $\mathcal{G}$–completion).

A cosimplicial object $\boldsymbol{W}^\bullet \in \mathcal{C}^{\boldsymbol{\Delta}}$ is called *weakly $\mathcal{G}$–fibrant* if it is Reedy fibrant (see Hirschhorn [28, Section 15.3]), and every $\boldsymbol{W}^n$ is in $\mathcal{G}$ for $n \geq 0$. A *weak $\mathcal{G}$–resolution* of an object $\boldsymbol{Y} \in \mathcal{C}$ is a weakly $\mathcal{G}$–fibrant $\boldsymbol{W}^\bullet$ which is $\mathcal{G}$–equivalent to $c(\boldsymbol{Y})^\bullet$. In this case there is a natural weak equivalence $\widehat{L}_{\mathcal{G}}\boldsymbol{Y} \to \mathrm{Tot}\,\boldsymbol{W}^\bullet$, by [19, Theorem 6.5].

1.23 Example The example we have in mind is the class $\mathcal{G} = \mathcal{G}_R$ of all R–GEMs in $\mathcal{C} = \mathcal{S}_*$, for some ring R. In this case the $\mathcal{G}$–completion is the Bousfield–Kan R–completion [20].

1.24 Sketches and their algebras An (FP–)*sketch* in the sense of Ehresmann [22] (cf Lawvere [29] is a small category Θ with a specified set $\mathcal{P}$ of (finite) products. This generalizes Lawvere's notion of a *theory*, which requires that $\mathrm{Obj}(\Theta) = \mathbb{N}$. A *$\Theta$–algebra* (or Θ–model) is a functor $\Gamma\colon \Theta \to \mathsf{Set}_*$ preserving the products in $\mathcal{P}$, with natural transformations as model morphisms (see Borceux [18, Section 4.1]). We write $\Gamma\{\boldsymbol{B}\}$ for the value of Γ at $\boldsymbol{B} \in \Theta$. The category of Θ–algebras is denoted by Θ–Alg. Examples of categories of such models include varieties of universal algebras, such as groups, rings, and modules over rings.

1.25 Remark In homotopy theory such theories often arise by choosing a collection $\mathcal{A}$ of objects in a model category $\mathcal{C}$ (as in Assumption 1.20), and letting $\Theta = \Theta_{\mathcal{A}}$ denote the full subcategory of the homotopy category $\mathrm{ho}\,\mathcal{C}$ consisting of objects of $\mathcal{C}$ which are representable as *finite-type* products of objects in $\mathcal{A}$ — that is, products of the form

(1.26)
$$\prod_{A\in\mathcal{A}}\prod_{i\in I_A} A,$$

where each indexing set I_A is finite. When each $A\in\mathcal{A}$ is a homotopy group object in $\mathcal{C}$, Θ–algebras have a natural underlying group structure; in such cases we call the sketch Θ *algebraic*.

For every $Y\in\mathcal{C}$ we then have a *realizable* Θ–algebra $\Gamma=H^*_\Theta Y$, with $\Gamma\{B\}=[Y,B]_{\mathcal{C}}$ for every $B\in\Theta$. A Θ–algebra is called *free* if is isomorphic to $H^*_\Theta B$ for some $B\in\Theta$.

Note, however, that even though the forgetful functor $U\colon \Theta\text{–}\mathsf{Alg}\to\mathsf{Set}^{\mathcal{A}}$ to $\mathcal{A}$–graded sets has a left adjoint $F\colon \mathsf{Set}^{\mathcal{A}}\to\Theta\text{–}\mathsf{Alg}$, not all Θ–algebras in its image are free by our definition: because F preserves colimits, any Θ–algebra in the image of F is a coproduct of *monogenic* free Θ–algebras (those realizable by on object of $\mathcal{A}$), and conversely. For our purposes, however (see Theorem 2.33 below), it is necessary that any free Θ–algebra be realizable in $\mathcal{C}$.

1.27 Example For any commutative ring R, let $\mathcal{A}:=\{K(R,n)\}_{n=1}^\infty$ in $\mathcal{C}=\mathcal{S}_*$; we then have an FP–sketch Θ_R in $\mathrm{ho}\,\mathcal{S}_*$ whose objects are "finite-type" R–GEMs of the form $\prod_{n=1}^\infty K(V_n,n)$, where $V_n=R^{k_n}$ is a free R–module of dimension $k_n<\infty$. Since each $B\in\Theta_R$ is an R–module object, all Θ_R–algebras take values in R–modules.

Note that the realizable Θ_R–algebra $\Gamma=H^*_{\Theta_R}Y$ has $\Gamma\{K(R,n)\}=H^n(Y;R)$, so we denote it simply by $H^*(Y;R)$; it is the R–cohomology algebra of Y, equipped with the action of the primary R–cohomology operations. When $R=\mathbb{F}_p$, Θ_R is the homotopy category of such $\mathbb{F}_p$–GEMs, and a Θ_R–algebra is an unstable algebra over the mod p Steenrod algebra, as in Schwartz [34, Section 1.4]. When $R=\mathbb{Q}$, a Θ_R–algebra is just a graded commutative $\mathbb{Q}$–algebra.

More generally, for any limit cardinal λ, let $R\text{–}\mathsf{Mod}^\lambda$ denote the set of isomorphism types of free R–modules of dimension $<\lambda$, and let

$$\mathcal{A}^\lambda:=\{K(V,n)\mid V\in R\text{–}\mathsf{Mod}^\lambda,\ 1\le n\le\infty\}.$$

The FP–sketch Θ^λ_R then consists of the R–GEMs which are finite-type products of the form $\prod_{n=1}^\infty K(V_n,n)$ with V_n a free R–module of dimension $<\lambda$.

Note that since finite products in R–Mod are also coproducts, $\prod_{I_n} V_i$ is itself in R–Mod^{λ}, since I_n is finite and λ is infinite. However, we must be able to distinguish the generating set $\mathcal{A}$ from the resulting sketch Θ_{R}, in order to define what we mean by "finite-type" products or coproducts.

We note the following two standard facts about an algebraic sketch Θ:

1.28 Lemma *If Γ is an Θ–algebra and $\boldsymbol{B} \in \Theta$, there is a natural isomorphism* $\mathrm{Hom}_{\Theta\text{-}\mathsf{Alg}}(H^*_{\Theta}\boldsymbol{B}, \Gamma) \cong \Gamma\{\boldsymbol{B}\}$.

1.29 Proposition [16, Section 6] *The category $s\Theta$–Alg of simplicial Θ–algebras has a model category structure, in which the weak equivalences and fibrations are defined on the underlying (graded) simplicial groups.*

1.30 Definition For any algebraic sketch Θ, a *CW resolution* of a Θ–algebra Γ is a cofibrant replacement $\varepsilon\colon V_{\bullet} \xrightarrow{\simeq} c(\Gamma)_{\bullet}$ equipped with a CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$ as in Section 1.11, with each $\overline{V}_n$ a free Θ–algebra.

1.31 Remark In fact, any CW object $V_{\bullet}$ for which each $\overline{V}_n$ is a free Θ–algebra and each attaching map $\overline{\partial}_0^{V_n}$ for $n \geq 0$ surjects onto $Z_{n-1}V_{\bullet}$ is a CW resolution. Here we set $Z_{-1}V_{\bullet} := \Gamma$ and $\overline{\partial}_0^{V_0} := \varepsilon$, so that

$$\varepsilon \circ \overline{\partial}_0^{V_1} = 0. \tag{1.32}$$

Thus one can easily construct (nonfunctorial) CW resolutions of any Θ–algebra Γ.

Moreover, by dualizing [8, Proposition 3.12] (see also [7, Proposition 12]) we can choose a CW basis for any free simplicial resolution in Θ_R–Alg when R is a field. However, we shall not make use of this fact.

2 Realizing simplicial Θ–algebra resolutions

Let Θ be an algebraic sketch obtained as in Remark 1.25 from a set $\mathcal{A}$ of homotopy group objects in a model category $\mathcal{C}$. In this section we show how a CW resolution $V_{\bullet}$ of a realizable Θ–algebra Γ can itself be realized over $\mathcal{C}$.

Any algebraic resolution $V_{\bullet} \to \Gamma$ is clearly realizable by a cosimplicial object $\widetilde{\boldsymbol{W}}^{\bullet} \in c(\mathrm{ho}\,\mathcal{C})$ (for which the cosimplicial identities hold up to homotopy), but it is not clear a priori that this can be rectified to a strict cosimplicial object $\boldsymbol{W}^{\bullet}$ in $\mathcal{C}$. This is in fact possible in the two cases described in Theorem A.11 in the appendix. However, for the purposes of this paper, we need not only the existence of this realization $\boldsymbol{W}^\bullet$, but also the particular form it takes, described as follows:

2.1 Definition Let Θ be a sketch in $\operatorname{ho}\mathcal{C}$ as in Remark 1.25, and let $V_\bullet$ be a simplicial CW resolution of the Θ–algebra $H^*_\Theta \boldsymbol{Y}$ for some $\boldsymbol{Y} \in \mathcal{C}$, with CW basis $(\overline{V}_n)_{n=0}^\infty$. A *sequential realization*

$$\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]} \rangle_{n\in\mathbb{N}}$$

of $V_\bullet$ for $\boldsymbol{Y}$ consists of a tower of Reedy fibrant and cofibrant $\boldsymbol{Y}$–coaugmented cosimplicial objects

(2.2)
$$\cdots \to \boldsymbol{W}^\bullet_{[n]} \xrightarrow{\pi[n]} \boldsymbol{W}^\bullet_{[n-1]} \xrightarrow{\pi[n-1]} \boldsymbol{W}^\bullet_{[n-2]} \to \cdots \to \boldsymbol{W}^\bullet_{[0]}$$

in $\mathcal{C}^{\boldsymbol{\Delta}}$, with each $\pi_{[n]}$ a Reedy fibration. We call $\boldsymbol{W}^\bullet_{[n]}$ the n^{th} *stage* of $\mathcal{W}$, and set $\boldsymbol{W}^\bullet := \operatorname{holim}_n \boldsymbol{W}^\bullet_{[n]}$ to be the limit of the tower (2.2). This tower must satisfy the following requirements for each $n \geq 0$:

(a) The coaugmentation $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Y} \to \boldsymbol{W}^\bullet_{[n]}$ realizes $V_\bullet \to H^*_\Theta \boldsymbol{Y}$ through simplicial dimension n — that is, we have a natural isomorphism

(2.3)
$$H^*_\Theta \boldsymbol{W}^k_{[n]}\{\boldsymbol{B}\} \xrightarrow{\cong} V_k\{\boldsymbol{B}\} \quad \text{for all } \boldsymbol{B} \in \Theta \text{ and } -1 \leq k \leq n,$$

using the notation of Example 1.27.

(b) The coaugmentation $\boldsymbol{\varepsilon}_{[n-1]}\colon \boldsymbol{Y} \to \boldsymbol{W}^\bullet_{[n-1]}$ lifts along the $\pi_{[n]}\colon \boldsymbol{W}^\bullet_{[n]} \to \boldsymbol{W}^\bullet_{[n-1]}$ to $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Y} \to \boldsymbol{W}^\bullet_{[n]}$.

(c) Each $\boldsymbol{W}^\bullet_{[n]}$ is obtained from $\boldsymbol{W}^\bullet_{[n-1]}$ as follows: assume given a fibrant realization $\overline{\boldsymbol{W}}^n \in \mathcal{C}$ of $\overline{V}_n$ and let $\boldsymbol{D}^* \in \mathsf{Ch}^{\mathcal{C}}$ be a left Reedy fibrant replacement of $\overline{\boldsymbol{W}}^n \otimes^* S^{n-1}$. Assume that we can realize the n^{th} attaching map $\overline{\partial}{}_0^{V_n}\colon \overline{V}_n \to V_{n-1}$ of $V_\bullet$ by a cochain map $F\colon C^*\boldsymbol{W}^\bullet_{[n-1]} \to \boldsymbol{D}^*$ in the category of coaugmented cochain complexes (that is, defined in dimensions $-1 \leq k \leq n-1$). Note that $(\overline{\boldsymbol{W}}^n \otimes^* S^{n-1})^{-1} = 0$.

Let $\widetilde{F}\colon \mathcal{U}\boldsymbol{W}^\bullet_{[n-1]} \to \mathcal{E}\boldsymbol{D}^*$ the corresponding map of restricted cosimplicial objects as in Section 1.16, and set $\widetilde{\boldsymbol{W}}^\bullet_{[n]} := \operatorname{Fib}(\widetilde{F})$ as in Section 1.10, take a pullback

(2.4)
$$\begin{tikzcd}
\widehat{\boldsymbol{W}}^\bullet_{[n]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{\text{PB}}"] & \mathcal{F}\widetilde{\boldsymbol{W}}^\bullet_{[n]} \arrow[d, "\mathcal{F}\ell"] \\
\boldsymbol{W}^\bullet_{[n-1]} \arrow[r, "\theta"] & \mathcal{F}\mathcal{U}\boldsymbol{W}^\bullet_{[n-1]}
\end{tikzcd}$$

as in (1.17), with $W^{\bullet}_{[n]}$ a Reedy fibrant and cofibrant replacement (see Hirschhorn [28, Section 15.3]) for $\widehat{W}^n$.

We start the process with $W^{\bullet}_{[0]} := c(\overline{W}^0)^{\bullet}$.

(d) For reasons which will appear later, we sometimes require the chain map $F\colon C^*W^{\bullet}_{[n-1]} \to \boldsymbol{D}^*$ in step (c) to be a levelwise (ie left Reedy) cofibration. If this is true for each $n \geq 1$, we say that the sequential realization $\mathcal{W}$ is *cofibrant*.

2.5 Understanding the passage from $W^{\bullet}_{[n-1]}$ to $W^{\bullet}_{[n]}$ We now give an explicit description of the construction of $W^{\bullet}_{[n]}$ from $W^{\bullet}_{[n-1]}$, also introducing some auxiliary notation:

(i) What does a left Reedy fibrant replacement for $\overline{W}^n \otimes^* S^{n-1}$ look like? In order for the coaugmented cochain complex $\boldsymbol{D}^*$ in $\mathsf{Ch}^{\mathcal{C}}$ to be left Reedy fibrant, the structure map $\boldsymbol{D}^{k-1} \to \mathrm{Ker}(\delta^k_{\boldsymbol{D}})$ (into the "k–cocycles"—not the same as $Z^k\boldsymbol{D}^*$ of Definition 1.5) must be a fibration for each k. In addition, since $(\overline{W}^n \otimes^* S^{n-1})^k = 0$ for $k \neq n-1$, $\boldsymbol{D}^k$ must be contractible.

It is natural for such a $\boldsymbol{D}^*$ to be described by a downward induction, starting with $\boldsymbol{D}^n = 0$ and $\boldsymbol{D}^{n-1} = \overline{W}^n$. Thus $\mathrm{Ker}(\delta^{n-1}_{\boldsymbol{D}}) = \overline{W}^n$, too, so $\boldsymbol{D}^{n-2}$ must be a contractible object equipped with a fibration to $\overline{W}^n$—ie it is a path object for $\overline{W}^n$, in the sense of Quillen [32, Section I.2]. We could choose it to be the standard path object $P\overline{W}^n$ of Equation (A.3), but this may not be a good choice if we also want $F^k\colon C^kW^{\bullet}_{[n-1]} \to \boldsymbol{D}^k$ to be a cofibration for each k, as in Definition 2.1(d). Thus we let $\delta^{n-2}_{\boldsymbol{D}}\colon \boldsymbol{D}^{n-2} \to \boldsymbol{D}^{n-1}$ be some fibration $\overline{p}^0\colon \overline{PW^n} \to \overline{W}^n$ with $\overline{PW^n}$ contractible.

At the $(n-3)^{\text{rd}}$ stage we see that $\mathrm{Ker}(\delta^{n-2}_{\boldsymbol{D}})$ is the fiber $\overline{\Omega^1 W^n}$ of $\overline{p}^0$, and again $\delta^{n-3}_{\boldsymbol{D}}\colon \boldsymbol{D}^{n-3} \to \boldsymbol{D}^{n-2}$ must be a fibration $\overline{p}^1\colon \overline{P\Omega^1 W^n} \to \overline{\Omega^1 W^n}$ (with $\overline{P\Omega^1 W^n}$ contractible), followed by the inclusion $\overline{\iota}^1\colon \overline{\Omega^1 W^n} \hookrightarrow \overline{P\Omega^0 W^n}$.

Proceeding by induction, for each $-1 \leq j \leq n-1$ we obtain a fibration sequence

$$\overline{\Omega^{j+1} W^n} \xhookrightarrow{\overline{\iota}^{j+1}} \overline{P\Omega^j W^n} \xrightarrow{\overline{p}^j} \overline{\Omega^j W^n} \tag{2.6}$$

in $\mathcal{C}$, called the j^{th} *modified path-loop fibration*, with $\overline{P\Omega^j W^n}$ contractible. By convention we set $\overline{W}^n := \overline{\Omega^0 W^n} = \overline{P\Omega^{-1} W^n}$, with the identity map as

$$\overline{\iota}^0\colon \overline{\Omega^0 W^n} \xrightarrow{=} \overline{P\Omega^{-1} W^n}. \tag{2.7}$$

Thus $\boldsymbol{D}^k = \overline{P\Omega^{n-k-2} W^n}$ for each $-1 \leq k \leq n-1$, and the differential $\delta^k_{\boldsymbol{D}}\colon \boldsymbol{D}^k \to \boldsymbol{D}^{k+1}$ is the composite

$$\overline{P\Omega^{n-k-2} W^n} \xrightarrow{\overline{p}^{n-k-2}} \overline{\Omega^{n-k-2} W^n} \xrightarrow{\overline{\iota}^{n-k-2}} \overline{P\Omega^{n-k-3} W^n}. \tag{2.8}$$

For later use we may (and shall) assume (see Step VIII in proof of Theorem A.11) that for each $0 \leq j \leq n-1$, (2.6) fits into a commutative diagram

(2.9)
$$\begin{tikzcd}
\overline{\Omega^{j+1}\boldsymbol{W}^n} \arrow[r, hook, "\bar{\iota}^{j+1}"] \arrow[d, two heads, "\simeq"', "\sigma^{j+1}"] & \overline{P\Omega^j\boldsymbol{W}^n} \arrow[r, two heads, "\bar{p}^j"] \arrow[d, two heads, "\simeq"', "P\tau^j"] & \overline{\Omega^j\boldsymbol{W}^n} \arrow[d, two heads, "="] \\
\Omega\overline{\Omega^j\boldsymbol{W}^n} \arrow[r, hook, "\iota"] & P\overline{\Omega^j\boldsymbol{W}^n} \arrow[r, two heads, "p"] & \overline{\Omega^j\boldsymbol{W}^n}
\end{tikzcd}$$

in which the bottom sequence is the usual path-loop fibration for $\overline{\Omega^j\boldsymbol{W}^n}$, and the vertical maps are all trivial fibrations.

(ii) The map $F\colon C^*W^{\bullet}_{[n-1]} \to D^*$ and its fiber As in Definition 2.1(c), the n^{th} attaching map for $V_\bullet$ is to be realized by a cochain map $F\colon C^*\boldsymbol{W}^{\bullet}_{[n-1]} \to \boldsymbol{D}^*$ in the category of coaugmented cochain complexes given by maps $F^k\colon C^k\boldsymbol{W}^{\bullet}_{[n-1]} \to \boldsymbol{D}^k = \overline{P\Omega^{n-k-2}\boldsymbol{W}^n}$ for each $-1 \leq k \leq n-1$, so by Section 1.10 the restricted cosimplicial object $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]} = \operatorname{Fib}(\widetilde{F})$ is given by

$$\widetilde{\boldsymbol{W}}^k_{[n]} := \boldsymbol{W}^k_{[n-1]} \times \overline{P\Omega^{n-k-1}\boldsymbol{W}^n} \tag{2.10}$$

in dimension $0 \leq k \leq n$, while by (2.7) we have

$$\widetilde{\boldsymbol{W}}^n_{[n]} := \boldsymbol{W}^n_{[n-1]} \times \overline{\boldsymbol{W}}^n. \tag{2.11}$$

We denote the two structure maps for the product (2.10) by

$$\psi^k_{[n]}\colon \widetilde{\boldsymbol{W}}^k_{[n]} \to \boldsymbol{W}^k_{[n-1]}, \quad q^k_{[n]}\colon \widetilde{\boldsymbol{W}}^k_{[n]} \to \overline{P\Omega^{n-k-1}\boldsymbol{W}^n}, \tag{2.12}$$

respectively. By Section 1.10 we see that the coface maps $\tilde{d}^i_k\colon \widetilde{\boldsymbol{W}}^k_{[n]} \to \widetilde{\boldsymbol{W}}^{k+1}_{[n]}$ are determined by

$$F^k \circ v^k \circ \psi^k_{[n]} = q^{k+1}_{[n]} \circ \tilde{d}^0_k, \tag{2.13}$$

where $v^k\colon \boldsymbol{W}^k_{[n-1]} \to C^k\boldsymbol{W}^{\bullet}_{[n-1]}$ is the structure map for (1.7). For $i = 1$ we have

$$\delta^{k-1}_{\boldsymbol{D}} \circ q^k_{[n]} = \bar{\iota}^{n-k-1} \circ \bar{p}^{n-k-1} \circ q^k_{[n]} = q^{k+1}_{[n]} \circ \tilde{d}^1_k, \tag{2.14}$$

while for $i \geq 2$ we have

$$q^{k+1}_{[n]} \circ \tilde{d}^i_k = 0. \tag{2.15}$$

When $k = n-1$ we have $\bar{p}^0 \circ q^{n-1}_{[n]} = q^n_{[n]} \circ \tilde{d}^1_{n-1}$, in accordance with (2.7).

We note also that if the given $\boldsymbol{W}^{\bullet}_{[n-1]}$ is equipped with a coaugmentation $\boldsymbol{\varepsilon}_{[n-1]}\colon \boldsymbol{Y} \to \boldsymbol{W}^{\bullet}_{[n-1]}$, we also have a coaugmented version $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Y} \to \widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$ for the restricted cosimplicial object $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$, which is defined in the same way by setting $\widetilde{\boldsymbol{W}}^{-1}_{[n-1]} := \boldsymbol{Y}$.

(iii) Making $\widetilde{\boldsymbol{W}}^{\bullet}_{[\boldsymbol{n}]}$ into a full cosimplicial object $\boldsymbol{W}^{\bullet}_{[\boldsymbol{n}]}$ Given the restricted cosimplicial object $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$ obtained in step (ii), we first define a full cosimplicial object $\widehat{\boldsymbol{W}}^{\bullet}_{[n]}$ by setting

(2.16)
$$\overline{\boldsymbol{G}}^{k} := \begin{cases} \overline{P\Omega^{n-k-1}\boldsymbol{W}^{n}} & \text{if } k \le n, \\ * & \text{if } k > n \end{cases}$$

(using (2.7)). We then let

(2.17)
$$\begin{aligned} \widehat{\boldsymbol{W}}^{r}_{[n]} &:= \widetilde{\boldsymbol{W}}^{r}_{[n]} \times \prod_{0<k<r} \ \prod_{0\le i_1<\cdots<i_k\le r-1} \overline{\boldsymbol{G}}^{r-k} \\ &= \boldsymbol{W}^{r}_{[n-1]} \times \prod_{0\le k\le r} \ \prod_{0\le i_1<\cdots<i_k\le r} \overline{\boldsymbol{G}}^{r-k} \end{aligned}$$

be the construction denoted by $\operatorname{csk}^{n} G^{\bullet}$ in (1.17). See Section 1.2 and (2.10) (and compare (1.13) in the dual case).

The codegeneracy map $s^{t}\colon \widehat{\boldsymbol{W}}^{r+1}_{[n]} \to \widehat{\boldsymbol{W}}^{r}_{[n]}$ is defined into the factor $\overline{\boldsymbol{G}}^{r-k}$ of $\widehat{\boldsymbol{W}}^{r}_{[n]}$ indexed by the k–tuple $I = (i_1, \dots, i_k)$ by projecting $\widehat{\boldsymbol{W}}^{r+1}_{[n]}$ onto the factor $\overline{\boldsymbol{G}}^{r-k}$ indexed by the unique $(k+1)$–tuple $J = (j_1, \dots, j_{k+1})$ satisfying the cosimplicial identity $s^{I} \circ s^{t} = s^{J}$. The coface maps of $\widehat{\boldsymbol{W}}^{\bullet}_{[n]}$ are determined by those of $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$ and the cosimplicial identities, and we have a natural map of restricted cosimplicial objects $g\colon \mathcal{U}\widehat{\boldsymbol{W}}^{\bullet}_{[n]} \to \widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$, which is a dimensionwise trivial fibration.

2.18 Remark $\widehat{\boldsymbol{W}}^{\bullet}_{[n]}$ is obviously n–coskeletal. Moreover, it is Reedy fibrant, since the natural map
$$\widehat{\zeta}^{r}\colon \widehat{\boldsymbol{W}}^{r}_{[n]} \to M^{r}\widehat{\boldsymbol{W}}^{\bullet}_{[n]}$$
(cf Section 1.2) is just the product of $\zeta^{r}\colon \boldsymbol{W}^{r}_{[n-1]} \to M^{r}\boldsymbol{W}^{\bullet}_{[n-1]}$ (which is a fibration, since $\boldsymbol{W}^{\bullet}_{[n-1]}$ is Reedy fibrant) with the projection onto the appropriate factors in (2.17) (which is a fibration since all objects $\overline{\boldsymbol{G}}^{k}$ are fibrant). Moreover, the composites of
$$\widehat{\boldsymbol{W}}^{k}_{[n]} \xrightarrow{g^{k}} \widetilde{\boldsymbol{W}}^{k}_{[n]} \xrightarrow{\psi^{k}_{[n]}} \boldsymbol{W}^{k}_{[n-1]}$$
for each k fit together to define a map of cosimplicial objects ${}'\pi_{[n]}\colon \widehat{\boldsymbol{W}}^{\bullet}_{[n]} \twoheadrightarrow \boldsymbol{W}^{\bullet}_{[n-1]}$, which is a Reedy fibration (for the same reason).

Finally, we let $h_{[n]}\colon \boldsymbol{W}^{\bullet}_{[n]} \to \widehat{\boldsymbol{W}}^{\bullet}_{[n]}$ be a (functorial) Reedy cofibrant replacement (see Hirschhorn [28, Section 15], and compare Bousfield and Kan [20, Section X.4.2]), so

$h_{[n]}$ is a trivial Reedy fibration, and we set $\pi_{[n]}\colon \boldsymbol{W}^\bullet_{[n]} \to \boldsymbol{W}^\bullet_{[n-1]}$ to be the composite ${}'\pi_{[n]} \circ h_{[n]}$ — again a Reedy fibration. Thus the full cosimplicial object $\boldsymbol{W}^\bullet_{[n]}$ is Reedy fibrant and cofibrant, and dimensionwise weakly equivalent to $\widehat{\boldsymbol{W}}^\bullet_{[n]}$. Since the latter is n–coskeletal, we see that $\boldsymbol{W}^\bullet_{[n]}$ is n–coskeletal "up to homotopy".

2.19 Lemma *Let $(\boldsymbol{C}^*, \delta_{\boldsymbol{C}})$ be a cochain complex and $(\boldsymbol{D}^*, \delta_{\boldsymbol{D}})$ a left Reedy fibrant $(n-1)$–truncated cochain complex as in Section 2.5(i) in a pointed model category $\mathcal{C}$, and let $Z^j\boldsymbol{C}^* := \operatorname{Coker}(\delta^{j-1}_{\boldsymbol{C}})$ with structure map $w^j\colon \boldsymbol{C}^j \to Z^j\boldsymbol{C}^*$ as in Definition 1.5. Any cochain map $F^j\colon \boldsymbol{C}^j \to \boldsymbol{D}^j$ defined for $k \le j < n$ induces a unique map $a^{k-1}\colon Z^{k-1}\boldsymbol{C}^* \to \overline{\Omega^{n-k-1}\boldsymbol{W}^n}$ with*

$$\bar\iota^{n-k-1} \circ a^{k-1} \circ w^{k-1} = F^k \circ \delta^{k-1}_{\boldsymbol{C}}, \tag{2.20}$$

in the notation of (2.6) *and* (2.8).

Proof By assumption, we have the following solid commuting diagram:

$$\begin{tikzcd}
\boldsymbol{C}^{k+1} \arrow[rr, "F^{k+1}"] & & \overline{P\Omega^{n-k-3}\boldsymbol{W}^n} & = & \boldsymbol{D}^{k+1} \\
& & \overline{\Omega^{n-k-2}\boldsymbol{W}^n} \arrow[u, hook, "\bar\iota^{n-k-2}"] & & \\
\boldsymbol{C}^k \arrow[uu, "\delta^k_{\boldsymbol{C}}"] \arrow[rr, "F^k"] & & \overline{P\Omega^{n-k-2}\boldsymbol{W}^n} \arrow[u, two heads, "\bar p^{n-k-2}"] & = & \boldsymbol{D}^k \arrow[uu, "\delta^k_{\boldsymbol{D}}"'] \\
& Z^{k-1}\boldsymbol{C}^* \arrow[ul, "\bar d^0"'] \arrow[r, dotted, "a^{k-1}"] & \overline{\Omega^{n-k-1}\boldsymbol{W}^n} \arrow[u, hook, "\bar\iota^{n-k-1}"] \arrow[uu, bend right, "0"'] & & \\
\boldsymbol{C}^{k-1} \arrow[uu, "\delta^{k-1}_{\boldsymbol{C}}"] \arrow[ur, "w^{k-1}"] \arrow[urr, dotted, "a"'] \arrow[rr, dotted, "F^{k-1}"'] & & \overline{P\Omega^{n-k-1}\boldsymbol{W}^n} \arrow[u, two heads, "\bar p^{n-k-1}"] & = & \boldsymbol{D}^{k-1} \arrow[uu, "\delta^{k-1}_{\boldsymbol{D}}"']
\end{tikzcd} \tag{2.21}$$

Then

$$\bar\iota^{n-k-2} \circ \bar p^{n-k-2} \circ F^k \circ \delta^{k-1}_{\boldsymbol{C}} = \delta^k_{\boldsymbol{D}} \circ F^k \circ \delta^{k-1}_{\boldsymbol{C}} = F^{k+1} \circ \delta^k_{\boldsymbol{C}} \circ \delta^{k-1}_{\boldsymbol{C}} = 0.$$

Since $\bar\iota^{n-k-2}$ is a monomorphism, in fact $\bar p^{n-k-2} \circ F^k \circ \delta^{k-1}_{\boldsymbol{C}} = 0$, Therefore, since (2.6) is a fibration sequence, $F^k \circ \delta^{k-1}_{\boldsymbol{C}}$ factors through $a\colon \boldsymbol{C}^{k-1} \to \overline{\Omega^{n-k-1}\boldsymbol{W}^n}$ as indicated in (2.21). Moreover, since $\bar\iota^{n-k-1} \circ a \circ \delta^{k-2}_{\boldsymbol{C}} = F^k \circ \delta^{k-1}_{\boldsymbol{C}} \circ \delta^{k-2}_{\boldsymbol{C}} = 0$ and $\bar\iota^{n-k-1}$ is a monomorphism, too, the map a factors through $a^{k-1}\colon Z^{k-1}\boldsymbol{C}^* \to \overline{\Omega^{n-k-1}\boldsymbol{W}^n}$ as in (2.21), satisfying (2.20). □

We next note the following technical fact about Moore chain objects:

2.22 Lemma *Let $\boldsymbol{W}^\bullet \in \mathcal{C}^{\boldsymbol{\Delta}}$ be a Reedy cofibrant cosimplicial object over a model category $\mathcal{C}$ as in Assumption 1.20, and $\boldsymbol{B}$ a homotopy group object in $\mathcal{C}$. Then for any Moore chain $\alpha \in C_n[\boldsymbol{W}^\bullet, \boldsymbol{B}]$ for the simplicial group $[\boldsymbol{W}^\bullet, \boldsymbol{B}]$:*

- (a) *α can be realized by a map $a\colon \boldsymbol{W}^n \to \boldsymbol{B}$ with $a \circ d^i_{n-1} = 0$ for all $1 \le i \le n$, and thus induces a map $\overline{a}\colon C^n\boldsymbol{W}^\bullet \to \boldsymbol{B}$ with $\overline{a} \circ v^n = a$.*
- (b) *If α is a Moore* cycle, *we can choose a nullhomotopy $H\colon \boldsymbol{W}^{n-1} \to P\boldsymbol{B} \subseteq \boldsymbol{B}^{[0,1]}$ for $a \circ d^0_{n-1}$ such that $H \circ d^j_{n-2} = 0$ for $1 \le j \le n-1$, and thus induces a map $\overline{H}\colon C^n\boldsymbol{W}^\bullet \to P\boldsymbol{B}$ with $\overline{H} \circ v^n = H$.*

Proof Since $\boldsymbol{W}^\bullet$ is Reedy cofibrant, the simplicial space $\boldsymbol{U}_\bullet = \mathrm{map}_*(\boldsymbol{W}^\bullet, \boldsymbol{B}) \in s\mathcal{S}_*$ is Reedy fibrant, so we have an isomorphism

(2.23) $$\iota_\star\colon \pi_i C_n \boldsymbol{U}_\bullet \to C_n \pi_i \boldsymbol{U}_\bullet \quad \text{for all } i \ge 0$$

(see [20, Proposition X.6.3]). Thus we can represent $\alpha \in C_n\pi_0\boldsymbol{U}_\bullet$ by a map $a \in C_n\boldsymbol{U}_\bullet$, which implies (i).

If α is a cycle, then $\partial_n(\alpha) = [a \circ d^0_{n-1}]$ vanishes in $\pi_0 C_{n-1}\boldsymbol{U}_\bullet$, so we have a nullhomotopy H for $a \circ d^0_{n-1}$ in

$$PC_{n-1}\,\mathrm{map}_*(\boldsymbol{W}^\bullet, \boldsymbol{B}) = C_{n-1}\,\mathrm{map}_*(\boldsymbol{W}^\bullet, P\boldsymbol{B}) \subseteq \mathrm{map}_*(\boldsymbol{W}^{n-1}, P\boldsymbol{B}),$$

which implies (ii). □

From the description in Section 2.5 we can actually deduce:

2.24 Proposition *Any $\boldsymbol{W}^\bullet_{[n]}$ obtained from $\boldsymbol{W}^\bullet_{[n-1]}$ as in Definition 2.1(c) will satisfy Definition 2.1(a)–(b), and the limit $\boldsymbol{W}^\bullet$ of* (2.2) *is a Reedy fibrant cosimplicial object over $\mathcal{C}$ which realizes the given algebraic resolution $V_\bullet \to H^*_\Theta \boldsymbol{Y}$.*

Proof In the setting of Definition 2.1, with $\boldsymbol{C}^* = C^*\boldsymbol{W}^\bullet_{[n-1]}$ for Reedy cofibrant $\boldsymbol{W}^\bullet_{[n-1]}$, we can use Lemma 2.22(a) to represent the attaching map $\overline{\partial}{}^n_0\colon \overline{V}_n \to Z_{n-1}V_\bullet$ for the CW resolution $V_\bullet$ by a map $F^{n-1}\colon C^{n-1}\boldsymbol{W}^\bullet_{[n-1]} \to \overline{\boldsymbol{W}}{}^n$ (see Step IV in the proof of Theorem A.11 below). From (2.16) and (2.17), we then see that $\boldsymbol{W}^\bullet_{[n]}$ realizes $V_\bullet$ through simplicial dimension n.

Moreover, $\boldsymbol{W}^\bullet$ is as stated because the maps $\pi_{[n]}$ restrict to trivial fibrations

$$\pi^k_{[n]}\colon \boldsymbol{W}^k_{[n]} \to \boldsymbol{W}^k_{[n-1]}$$

for each $0 \le k < n$, so $\boldsymbol{W}^k = \mathrm{holim}_n\, \boldsymbol{W}^k_{[n]}$ in $\mathcal{C}$. □

2.25 Higher cohomology operations When F^{k-1} also exists that makes (2.21) commute, we have

$$a^{k-1} \circ w^{k-1} = \bar{p}^{n-k-1} \circ F^{k-1}, \tag{2.26}$$

so F^{k-1} is a nullhomotopy for $a^{k-1} \circ w^{k-1}$ in the sense of Quillen [32, Section I.2]. Note that Lemma 2.19 also makes sense for $k = n-1$, where $F^n = 0$.

If we choose F^{n-1} as in the proof of Proposition 2.24, the map $F^{n-1} \circ \delta^{n-2}$, which induces $a^{n-2} \circ w^{n-2}$, is nullhomotopic, with nullhomotopy F^{n-2}. We can then think of $a^{n-3} \circ w^{n-3}\colon C^{n-2} \to \overline{\Omega^1 \boldsymbol{W}^n}$ as the value of the secondary cohomology operation corresponding to the diagram

$$\begin{tikzcd} \boldsymbol{C}^{n-3} \arrow[r, "\delta^{n-3}"] \arrow[rr, bend left, "0"] & \boldsymbol{C}^{n-2} \arrow[r, "\delta^{n-2}"] \arrow[rr, bend right, "0"'] & \boldsymbol{C}^{n-1} \arrow[r, "F^{n-1}"] & \overline{\boldsymbol{W}}^n \end{tikzcd} \tag{2.27}$$

as in (0.2). Only $\overline{\boldsymbol{W}}^n$ is an R–GEM, but this suffices to let us think of each value $a^{n-3} \circ w^{n-3}$ of this Toda bracket in $[\boldsymbol{C}^{n-3}, \Omega\overline{\boldsymbol{W}}^n]$ as a collection of cohomology classes for $\boldsymbol{C}^{n-3}$. This nevertheless qualifies as a higher cohomology operation as described in the introduction, if we use a truncation of $\boldsymbol{\Delta}_+$ as our indexing category I.

By what we say above, if $a^{n-3} \circ w^{n-3} \sim *$ — that is, the secondary operation corresponding to (2.27) vanishes — then the choice of a nullhomotopy F^{n-3} yields a value $a^{n-4} \circ w^{n-4}\colon \boldsymbol{C}^{n-4} \to \overline{\Omega^2 \boldsymbol{W}^n}$ for the corresponding third order operation, and so on. This observation is the key to what we are doing in this paper.

2.28 Definition Let Θ be an algebraic sketch, as in Remark 1.25, associated to $\mathcal{A} \subseteq \operatorname{Obj} \mathcal{C}$, so that by definition any $\boldsymbol{B} \in \Theta$ is of the form $\boldsymbol{B} := \prod_{\boldsymbol{A} \in \mathcal{A}} \prod_{i \in I_{\boldsymbol{A}}} \boldsymbol{A}$ with each $I_{\boldsymbol{A}}$ a finite indexing set (see (1.26)).

We then say that Θ is *allowable* if the natural map

$$\coprod_{\boldsymbol{A} \in \mathcal{A}} \coprod_{i \in I_{\boldsymbol{A}}} H^*_\Theta \boldsymbol{A} \to H^*_\Theta \boldsymbol{B} \tag{2.29}$$

is an isomorphism for any such $\boldsymbol{B} \in \Theta$.

2.30 Remark If we write $I := \coprod_{\boldsymbol{A} \in \mathcal{A}} I_{\boldsymbol{A}}$ and denote the copy of $\boldsymbol{A}$ indexed by $i \in I_{\boldsymbol{A}}$ by $\boldsymbol{B}_i$, we have $\boldsymbol{B} = \prod_{i \in I} \boldsymbol{B}_i$. For any Θ–algebra Γ we then have, by

Lemma 1.28 and (2.29),

(2.31)
$$\prod_{i\in I}\Gamma\{\boldsymbol{B}_i\}=\prod_{i\in I}\operatorname{Hom}_{\Theta\text{-}\mathsf{Alg}}(H^*_\Theta\boldsymbol{B}_i,\Gamma)$$
$$=\operatorname{Hom}_{\Theta\text{-}\mathsf{Alg}}\Big(\coprod_{i\in I}H^*_\Theta\boldsymbol{B}_i,\Gamma\Big)=\operatorname{Hom}_{\Theta\text{-}\mathsf{Alg}}(H^*_\Theta\boldsymbol{B},\Gamma).$$

2.32 Lemma *If R is $\mathbb{F}_p$ or a field of characteristic 0 and λ is any limit cardinal, the FP–sketch Θ^λ_R of Example 1.27 is allowable.*

Proof Every $\boldsymbol{B}\in\Theta^\lambda_R$ has the form $\boldsymbol{B}=\prod_{n=1}^\infty\prod_{I_n}\boldsymbol{K}(V_i,n)$ for $V_i\in R\text{–}\mathsf{Mod}^\lambda$ and finite indexing sets I_n. Then $H^*(\boldsymbol{B};R)=\coprod_{n=1}^\infty\coprod_{I_n}H^*(\boldsymbol{K}(V_i,n);R)$ (see Blanc and Sen [17, Lemma 4.17]). □

We are now in a position to state our first important technical result:

2.33 Theorem *Let Θ be an allowable algebraic sketch in $\mathcal{C}$ and let $V_\bullet$ be any CW resolution of the realizable Θ–algebra $H^*_\Theta\boldsymbol{Y}$. Then there is a cofibrant sequential realization $\mathcal{W}=\langle\boldsymbol{W}^\bullet_{[n]},\widetilde{\boldsymbol{W}}^\bullet_{[n]}\rangle_{n\in\mathbb{N}}$ of $V_\bullet$ for $\boldsymbol{Y}$, with each $\boldsymbol{W}^k_{[n]}$ in Θ.*

We defer the proof to the appendix, where we actually prove a more general result (which is needed elsewhere).

2.34 Remark If we want to use the allowability of Θ^λ_R in Lemma 2.32 for Theorem 2.33, the choice of the cardinal λ may depend on the size of graded R–vector space $H^*(\boldsymbol{Y};R)$ (see Blanc and Sen [17, Section 3]). However, in the most commonly encountered case, $H^*(\boldsymbol{Y};R)$ will be of finite type, and we may choose the CW resolution $V_\bullet$ to be finite type in each simplicial dimension, too. In this case we can make do with the original $\Theta_R=\Theta^\omega_R$ of Example 1.27.

3 Comparing cosimplicial resolutions

Cosimplicial resolutions of the type constructed in Section 2 play a central role in our theory of higher cohomology operations, but they depend on many particular choices. In this section we shall show how any two such cosimplicial objects are related by a zigzag of maps of a particularly simple form.

Although many of the results hold more generally, from now on we restrict attention to the algebraic sketch Θ_R for R either $\mathbb{F}_p$ or a field of characteristic 0 (see Example 1.27), with $\mathcal{C} = \mathcal{S}_*$. This allows us to assume for convenience that all the objects in each stage of our sequential realizations are simplicial R–modules (though the maps between them need not be strict simplicial homomorphisms).

We shall also assume from here on that all spaces are connected. The modifications needed for the nonconnected case should be clear.

We first note the following general facts about model categories:

3.1 Lemma *Let X and Y be two weakly equivalent fibrant and cofibrant objects in a simplicial model category $\mathcal{C}$.*

(a) *There is a diagram of weak equivalences*

(3.2)
$$\begin{tikzcd}
& Z \arrow[dl, bend right, "s"'] \arrow[dr, bend left, "t"] & \\
X \arrow[ur, "h"] \arrow[rr, "f"] & & Y \arrow[ul, "j"'] \arrow[ll, bend left, "g"]
\end{tikzcd}$$

where $h := f' \circ i_0$, $s \circ j = g$, $t \circ h = f$, $s \circ h = \mathrm{Id}_X$, $t \circ j = \mathrm{Id}_Y$ and the notation f', i_0 is as in the proof.

(b) *There are maps $X \amalg Y \xrightarrow{F} Z \xrightarrow{G} X \times Y$, with F a cofibration which is a trivial cofibration on each summand, G a fibration which is a trivial fibration onto each factor, and the induced maps $X \to X$ and $Y \to Y$ are identities.*

Proof (a) By Quillen [32, Section I.1]) we have homotopy equivalences $f\colon X \to Y$ and $g\colon Y \to X$ with a homotopy $H\colon g \circ f \sim \mathrm{Id}_X$ fitting into a commutative diagram

(3.3)
$$\begin{tikzcd}
X \arrow[r, hook, "i_1"] \arrow[d, "f"'] \arrow[drrr, bend left=40, "="] \arrow[dr, phantom, "\boxed{\mathrm{PO}}"] & X \otimes \Delta[1] \arrow[d, "f'"'] \arrow[drr, "p"] \arrow[ddr, "H"] & & X \arrow[ll, hook', "i_0"'] \arrow[d, "="] & \\
Y \arrow[r, hook, "j"] \arrow[dr, "g"'] \arrow[drrrr, bend right=40, "="'] & Z \arrow[d, dotted, "s"] \arrow[drrr, dotted, "t"] & & X \arrow[dr, "f"] & \\
& & X & & Y
\end{tikzcd}$$

with all maps weak equivalences, where Z is the pushout, the maps i_0 and i_1 are induced by the inclusions $\Delta[0] \hookrightarrow \Delta[1]$ and p is induced by $\Delta[1] \twoheadrightarrow \Delta[0]$.

(b) Choose a weak equivalence $f\colon X \to Y$, and factor it as $X \xrightarrow{k} \widehat{Z} \xrightarrow{\ell} Y$, with k a trivial cofibration and ℓ a trivial fibration. By the LLP and fibrancy of X we have a retraction $r\colon \widehat{Z} \to X$ for k, and by RLP and cofibrancy of Y we have a section $u\colon Y \to \widehat{Z}$ for ℓ, both weak equivalences. Set

$$\phi := (\mathrm{Id}_X \top (r \circ u)) \perp (f \top \mathrm{Id}_Y)\colon X \amalg Y \to X \times Y.$$

Factor $k \perp u\colon X \amalg Y \to \widehat{Z}$ as $X \amalg Y \xrightarrow{k' \perp u'} Z' \xrightarrow{p} \widehat{Z}$ (a cofibration followed by a trivial fibration), and $r \top \ell\colon \widehat{Z} \to X \times Y$ as $\widehat{Z} \xrightarrow{i} Z'' \xrightarrow{r' \top \ell'} X \times Y$ (a trivial cofibration followed by a fibration). Finally, factor $i \circ p\colon Z' \xrightarrow{\simeq} Z''$ as $Z' \xrightarrow{e} Z \xrightarrow{q} Z'$ (a trivial cofibration followed by a trivial fibration):

(3.4)
$$\begin{tikzcd}
X \arrow[r, hook, "\mathrm{inc}"] \arrow[dd, "\mathrm{Id}"'] \arrow[rrd, "k"', "\simeq"] & X \amalg Y \arrow[r, hook, "k' \perp u'"] \arrow[dd, "\phi"] \arrow[rd, "k \perp u"] \arrow[rrrd, bend left, "F"] & & Z' \arrow[ld, two heads, "p", "\simeq"'] \arrow[dd, "i \circ p", "\simeq"'] \arrow[rd, hook, "e", "\simeq"'] & \\
& & \widehat{Z} \arrow[llld, "r", "\simeq"'] \arrow[ld, "r \top \ell"] \arrow[rd, hook, "i", "\simeq"'] & & Z \arrow[ld, two heads, "q", "\simeq"'] \arrow[llld, bend left, "G"] \\
X & X \times Y \arrow[l, two heads, "\mathrm{proj}"] & & Z'' \arrow[ll, two heads, "r' \top \ell'", "\simeq"'] &
\end{tikzcd}$$

Then $F := e \circ (k' \perp u')$ is a cofibration, $G := (r' \top \ell') \circ q$ is a fibration, and the claim follows by tracking the weak equivalences in (3.4) (and similarly for Y). $\square$

3.5 Definition Given two CW resolutions $\varepsilon\colon V_\bullet \to \Gamma$ and ${}'\varepsilon\colon {}'V_\bullet \to \Gamma$ of a Θ_R–algebra Γ, with CW bases $(\overline{V}_n)_{n\in\mathbb{N}}$ and $({}'\overline{V}_n)_{n\in\mathbb{N}}$, an *algebraic comparison map* $\Psi\colon V_\bullet \to {}'V_\bullet$ is a system

$$\Psi = \langle \varphi, \rho, (\overline{\varphi}_n, \overline{\rho}_n)_{n\in\mathbb{N}} \rangle, \tag{3.6}$$

where $\varphi\colon V_\bullet \to {}'V_\bullet$ is a split monic weak equivalence of simplicial Θ_R–algebras with retraction $\rho\colon {}'V_\bullet \to V_\bullet$ (with ${}'\varepsilon \circ \overline{\varphi}_0 = \varepsilon$), induced by inclusions of coproduct summands $\overline{\varphi}_n\colon \overline{V}_n \hookrightarrow {}'\overline{V}_n$ with retractions $\overline{\rho}_n$ for each $n \geq 0$.

3.7 Lemma *Any two CW resolutions $\varepsilon^{(0)}\colon V_\bullet^{(0)} \to \Gamma$ and $\varepsilon^{(1)}\colon V_\bullet^{(1)} \to \Gamma$ of the same Θ_R–algebra Γ have a common "algebraic h–cobordism" CW resolution $\varepsilon\colon V_\bullet \to \Gamma$, with algebraic comparison maps $\Psi^{(i)}\colon V_\bullet^{(i)} \to V_\bullet$ for $i = 0, 1$.*

Proof Let $(\overline{V}_n^{(i)})_{n\in\mathbb{N}}$ be CW bases for $V_\bullet^{(i)}$ for $i=0,1$. Since $X=V_\bullet^{(0)}$ and $Y=V_\bullet^{(1)}$ are fibrant and cofibrant in $s\Theta_R$–Alg (cf Proposition 1.29), they have homotopy equivalences $f\colon X\to Y$ and $g\colon Y\to X$ as in Lemma 3.1(a). We make explicit the construction of the lemma by producing a CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$ for $Z=V_\bullet$ in (3.3), together with inclusions of coproduct summands $\overline{\varphi}_{(i)}^n\colon \overline{V}_n^{(i)}\hookrightarrow \overline{V}_n$ for each $n\geq 0$ as in Definition 3.5.

If we write $e_0,e_1\in\Delta[1]_0$ and $\sigma\in\Delta[1]_1$ for the nondegenerate simplices, we have $V_n={}'\overline{V}_n\amalg L_nV_\bullet$, with

$$\textbf{(3.8)}\quad {}'\overline{V}_n:=\overline{V}_n^{(0)}\otimes(\boldsymbol{s}^{n-1}e_0)\amalg\coprod_{k=0}^{n-1}[\overline{V}_n^{(0)}\otimes(\boldsymbol{s}_{\hat{k}}^{n-1}\sigma)]\amalg[\overline{V}_n^{(1)}\otimes(\boldsymbol{s}^{n-1}e_1)]$$
$$\amalg\coprod_{k=0}^{n-1}[s_k\overline{V}_{n-1}^{(0)}\otimes(\boldsymbol{s}_{\hat{k}}^{n-1}\sigma)].$$

Here $\boldsymbol{s}^n$ is the iterated degeneracy map $s_n\dots s_0$ and $\boldsymbol{s}_{\hat{k}}^n:=s_n\dots s_{k+1}\hat{s}_ks_{k-1}\dots s_0$.

The face maps are calculated as usual on each factor of $a\otimes b$, except that

$$\textbf{(3.9)}\qquad d_n(s_{n-1}u\otimes(\boldsymbol{s}^{n-2}\sigma))=f_{n-1}u\otimes(\boldsymbol{s}^{n-2}e_1)\in[\overline{V}_{n-1}^{(1)}\otimes(\boldsymbol{s}^{n-2}e_1)]$$

for $k=n-1$ in the second line of (3.8), by (3.3), where $f\colon V_\bullet^{(0)}\to V_\bullet^{(1)}$ is the chosen homotopy equivalence.

Note that ${}'\overline{V}_n$ is not a CW basis object for $V_\bullet$, since the summands in the second line of (3.8) always have at least two nonvanishing face maps.

However, for any $v\in{}'\overline{V}_n$ we can define $v^{[0]}:=v$ and $v^{[k+1]}:=v^{[k]}-s_{n-k-1}d_{n-k}v^{[k]}$ by induction on k, and find that $d_iv^{[k]}=0$ for $n-k<i\leq n$, so $v^{[n]}$ is a Moore chain. Note that for $v\in U$ in summands U on the first line of (3.8) we have simply $v=v^{[n]}$.

Explicitly, we replace each generator $v=s_ku\otimes(\boldsymbol{s}_{\hat{k}}^{n-1}\sigma)$ of a summand in the second line of (3.8) for ${}'\overline{V}_n$ by

$$\textbf{(3.10)}\quad v^{[n]}=\begin{cases}\sum_{i=0}^{k}(-1)^i[s_{k-i}u]\otimes[(\boldsymbol{s}_{\hat{k}}^{n-1}\sigma)-(\boldsymbol{s}_{\widehat{k+1}}^{n-1}\sigma)] & \text{if } k<n-1,\\ \sum_{i=1}^{n}(-1)^i[s_{n-i}f_{n-1}u\otimes(\boldsymbol{s}^{n-2}e_1)-s_{n-i}u\otimes(\boldsymbol{s}^{n-2}\sigma)] & \text{if } k=n-1.\end{cases}$$

Thus $v^{[n]}$ always has the form $v+\sum_{i=1}^{k}u_i$, with the elements u_i all degenerate.

The Θ_R–algebra retraction $\rho'\colon V_n\to{}'\overline{V}_n$ onto the summand ${}'\overline{V}_n$ therefore takes $v^{[n]}$ to v. This implies that if $\{v_i\}_{i\in I}$ are generators of ${}'\overline{V}_n$, the new elements $\{v_i^{[n]}\}_{i\in I}$

still generate a free sub-Θ_R–algebra of V_n, because any relation of the form

$$\psi(v_1^{[n]},\dots,v_k^{[n]})=0$$

in V_n, where ψ is some primary R–cohomology operation, implies that also

$$0=\rho'(\psi(v_1^{[n]},\dots,v_k^{[n]}))=\psi(\rho'(v_1^{[n]}),\dots,\rho'(v_k^{[n]}))=\psi(v_1,\dots,v_k),$$

which can hold only if $\psi\equiv 0$ since the elements v_i are generators of a free Θ_R–algebra.

Therefore, if we write $\overline{V}_n$ for the sub-Θ_R–algebra of V_n generated by all elements $v^{[n]}$ as v varies over a set of generators for (each summand of) $'\overline{V}_n$, we still have a coproduct of free Θ_R–algebras $V_n=\overline{V}_n\amalg L_nV_\bullet$, where $\overline{V}_n$ now serves as an n^{th} CW basis object for $V_\bullet$.

Moreover, we still have inclusions of coproduct summands $\overline{\varphi}^n_{(i)}\colon \overline{V}_n^{(i)}\hookrightarrow \overline{V}_n$ inducing split trivial cofibrations of simplicial Θ_R–algebras $\varphi^{(i)}\colon V_\bullet^{(i)}\to V_\bullet$ for $i=0,1$. We can use these to further write $\overline{V}_n:=\overline{U}_n\amalg\overline{V}_n^{(0)}\amalg\overline{V}_n^{(1)}$. □

3.11 Definition Given an algebraic comparison map $\Psi=\langle\varphi,\rho,(\overline{\varphi}_n,\overline{\rho}_n)_{n\in\mathbb{N}}\rangle$ between two CW resolutions $\varepsilon\colon V_\bullet\to\Gamma$ and $'\varepsilon\colon {}'V_\bullet\to\Gamma$ of a realizable Θ_R–algebra Γ (see Definition 3.5), and sequential realizations $\mathcal{W}$ and $'\mathcal{W}$ of $V_\bullet$ and $'V_\bullet$, respectively, a *comparison map* $\Phi\colon\mathcal{W}\to{}'\mathcal{W}$ *over* Ψ is a system

(3.12)
$$\Phi=\langle e_{[n]},r_{[n]},(\overline{Pe}^k_n)_{k=0}^{n-1},(\bar{e}^k_n)_{k=0}^{n-1},(\overline{Pr}^k_n)_{k=0}^{n-1},(\bar{r}^k_n)_{k=0}^{n-1}\rangle_{n\in\mathbb{N}}$$

consisting of:

(i) Split fibrations of the modified path-loop fibrations of (2.6) fitting into a diagram

(3.13)
$$\begin{tikzcd}
\overline{\Omega^{k+1}{}'\boldsymbol{W}^n} \arrow[r, hook, "{}'\bar{\iota}^{k+1}"] \arrow[d, shift left, "\bar{e}^{k+1}_n"] & \overline{P\Omega^k{}'\boldsymbol{W}^n} \arrow[r, two heads, "{}'\bar{p}^k"] \arrow[d, shift left, "\overline{Pe}^k_n"] & \overline{\Omega^k{}'\boldsymbol{W}^n} \arrow[d, shift left, "\bar{e}^k_n"] \\
\overline{\Omega^{k+1}\boldsymbol{W}^n} \arrow[r, hook, "\bar{\iota}^{k+1}"] \arrow[u, bend left, "\bar{r}^{k+1}_n"] & \overline{P\Omega^k\boldsymbol{W}^n} \arrow[r, two heads, "\bar{p}^k"] \arrow[u, bend left, "\overline{Pr}^k_n"] & \overline{\Omega^k\boldsymbol{W}^n} \arrow[u, bend left, "\bar{r}^k_n"]
\end{tikzcd}$$

for each $0\le k<n$, in which both upward and downward squares commute, as well as

(3.14)
$$\overline{Pe}^k_n\circ\overline{Pr}^k_n=\mathrm{Id}\quad\text{and}\quad\bar{e}^k_n\circ\bar{r}^k_n=\mathrm{Id}\quad\text{for all } 0\le k<n.$$

We require that for all $0\le k<n$, the maps $\bar{e}^k_n$ realize $\Omega^k\overline{\varphi}_n$ and the maps $\bar{r}^k_n$ realize $\Omega^k\overline{\rho}_n$.

(ii) A cosimplicial map $e_{[n]}\colon {'\boldsymbol{W}}^{\bullet}_{[n]} \to \boldsymbol{W}^{\bullet}_{[n]}$ realizing $\varphi\colon V_{\bullet} \to {'V_{\bullet}}$ through simplicial dimension n, with section $r_{[n]}\colon \boldsymbol{W}^{\bullet}_{[n]} \to {'\boldsymbol{W}}^{\bullet}_{[n]}$ realizing ρ, such that for each $0 \le k < n$, both squares in the following diagram commute:

(3.15)
$$\begin{tikzcd}
C^k({'\boldsymbol{W}}^{\bullet}_{[n-1]}) \arrow[r, two heads, "C^k(e^k_{[n-1]})"] \arrow[d, "'F^k"'] & C^k(\boldsymbol{W}^{\bullet}_{[n-1]}) \arrow[d, "F^k"] & C^k({'\boldsymbol{W}}^{\bullet}_{[n-1]}) \arrow[d, "'F^k"'] & C^k(\boldsymbol{W}^{\bullet}_{[n-1]}) \arrow[l, hook', "C^k(r^k_{[n-1]})"'] \arrow[d, "F^k"] \\
\overline{P\Omega^{n-k-2}{'\boldsymbol{W}}^n} \arrow[r, two heads, "\overline{Pe}^{n-k-2}_n"'] & \overline{P\Omega^{n-k-2}\boldsymbol{W}^n} & \overline{P\Omega^{n-k-2}{'\boldsymbol{W}}^n} & \overline{P\Omega^{n-k-2}\boldsymbol{W}^n} \arrow[l, hook', "\overline{Pr}^{n-k-2}_n"]
\end{tikzcd}$$

When each map $\bar{e}^k_n\colon \overline{\Omega^k {'\boldsymbol{W}}^n} \twoheadrightarrow \overline{\Omega^k \boldsymbol{W}^n}$ and $e^k_{[n]}\colon {'\boldsymbol{W}}^k_{[n]} \twoheadrightarrow \boldsymbol{W}^k_{[n]}$ is a trivial fibration (and thus each map $\bar{r}^k_n\colon \overline{\Omega^k \boldsymbol{W}^n} \hookrightarrow \overline{\Omega^k {'\boldsymbol{W}}^n}$ and $r^k_{[n]}\colon {'\boldsymbol{W}}^k_{[n]} \hookrightarrow \boldsymbol{W}^k_{[n]}$ is a trivial cofibration in $\mathcal{S}_*$), we say that Φ is a *trivial* comparison map.

If we only have

(3.16)
$$\Phi = \big\langle e_{[n]}, r_{[n]}, (\overline{Pe}^k_n)^{n-1}_{k=0}, (\bar{e}^k_n)^{n-1}_{k=0}, (\overline{Pr}^k_n)^{n-1}_{k=0}, (\bar{r}^k_n)^{n-1}_{k=0} \big\rangle^N_{n=0}$$

as above, we say that $\Phi\colon \mathcal{W} \to {'\mathcal{W}}$ is an *N–stage comparison map* over Ψ.

3.17 Remark If we let $\bar{j}^k_{[n]}\colon \overline{\Omega^k \boldsymbol{X}^n} \hookrightarrow \overline{\Omega^k {'\boldsymbol{W}}^n}$ denote the inclusion of the fiber of $\bar{e}^k_n\colon \overline{\Omega^k {'\boldsymbol{W}}^n} \twoheadrightarrow \overline{\Omega^k \boldsymbol{W}^n}$, we see that the splitting $\bar{r}^k_n\colon \overline{\Omega^k \boldsymbol{W}^n} \hookrightarrow \overline{\Omega^k {'\boldsymbol{W}}^n}$ induces a retraction $\bar{s}^k_{[n]}\colon \overline{\Omega^k {'\boldsymbol{W}}^n} \twoheadrightarrow \overline{\Omega^k \boldsymbol{X}^n}$ for $\bar{j}^k_{[n]}$, defined $x \mapsto x - \bar{r}^k_n \bar{e}^k_n(x)$, and thus a map

(3.18)
$$\overline{\Omega^k {'\boldsymbol{W}}^n} \xrightarrow[\simeq]{\bar{s}^k_{[n]} \top \bar{e}^k_n} \overline{\Omega^k \boldsymbol{X}^n} \times \overline{\Omega^k \boldsymbol{W}^n},$$

which is a weak equivalence for each $0 \le k < n$ (using the abelian group structure on all spaces). As we shall see, in many cases we can assume (3.18) is actually an equality.

3.19 Definition A *zigzag of comparison maps* between two sequential realizations $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ of a realizable Θ_R–algebra Γ is a (possibly infinite) sequence of cospans of comparison maps starting from $\mathcal{W}^{(0)}$ and ending at $\mathcal{W}^{(1)}$, which is *locally finite* in the sense that for each $n \ge 0$, only finitely many of the comparison maps in the zigzag between the n^{th} stages $(\boldsymbol{W}^{\bullet}_{[n]})^{(0)}$ and $(\boldsymbol{W}^{\bullet}_{[n]})^{(1)}$ are not the identity map.

We say that two abstract sequential realizations $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ (of arbitrary spaces $\boldsymbol{Y}^{(0)}$ and $\boldsymbol{Y}^{(1)}$) are *weakly equivalent* if they are related by a zigzag of comparison maps.

Similarly, if only the n^{th} stages $(\mathbf{W}^\bullet_{[n]})^{(0)}$ and $(\mathbf{W}^\bullet_{[n]})^{(1)}$ of two such sequential realizations $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ are related by a zigzag of comparison maps, we say that $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ are *n–equivalent*, or that $(\mathbf{W}^\bullet_{[n]})^{(0)}$ and $(\mathbf{W}^\bullet_{[n]})^{(1)}$ are *weakly equivalent*.

3.20 Theorem *Any two cofibrant sequential realizations $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ of $\mathbf{Y}$ are weakly equivalent.*

Proof We prove the theorem in two main steps:

(i) Different algebraic resolutions We first show that, given an algebraic comparison map $\Psi\colon V_\bullet \to {}'V_\bullet$ for $\mathbf{Y}$ and a cofibrant sequential realization $\mathcal{W}$ of $V_\bullet$, there is a cofibrant sequential realization ${}'\mathcal{W}$ of ${}'V_\bullet$ with a comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ over Ψ, constructed (with the maps $e_{[n]}\colon {}'\mathbf{W}^\bullet_{[n]} \to \mathbf{W}^\bullet_{[n]}$ and sections $r_{[n]}\colon \mathbf{W}^\bullet_{[n]} \to {}'\mathbf{W}^\bullet_{[n]}$) by induction on $n \geq 0$:

At the n^{th} stage, we may assume by Lemma 3.1 that $e_{[n-1]}$ is a fibration and $r_{[n-1]}$ is a cofibration in the resolution model category $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$, so in particular $e^j_{[n-1]}$ is a fibration and $r^j_{[n-1]}$ a cofibration for $0 \leq j \leq n-1$ (see Bousfield [19, Section 3.2]).

Since $\overline{V}_n$ is a coproduct summand in ${}'\overline{V}_n = \overline{V}_n \amalg \overline{U}_n$, the map $\overline{\varphi}_n\colon \overline{V}_n \hookrightarrow {}'\overline{V}_n$ is simply the inclusion, while $\overline{\rho}_n\colon {}'\overline{V}_n \hookrightarrow \overline{V}_n$ has the form $\mathrm{Id} \perp \zeta$. If we realize $\overline{V}_n$ by $\overline{\mathbf{W}}^n$ and $\overline{U}_n$ by $\overline{\mathbf{X}}^n$ then ${}'\overline{V}_n$ is realized by ${}'\widehat{\mathbf{W}}^n := \overline{\mathbf{X}}^n \times \overline{\mathbf{W}}^n$. By Definition 2.1, the n^{th} stage of $\mathcal{W}$ is determined by the choice of left Reedy fibrant replacement $\mathbf{D}^*$ of $\overline{\mathbf{W}}^n \otimes^* S^{n-1}$, equipped with a left Reedy cofibration $F\colon C^*\mathbf{W}^\bullet_{[n-1]} \to \mathbf{D}^*$ realizing the given attaching map $\overline{\partial}^n_0\colon \overline{V}_n \to C_{n-1}V_\bullet$.

If $\mathbf{K}^*$ is similarly a left Reedy fibrant replacement for $\overline{\mathbf{X}}^n \otimes^* S^{n-1}$, the attaching map ${}'\overline{\partial}^n_0\colon {}'\overline{V}_n \to C_{n-1}{}'V_\bullet$ has the form $\overline{\partial}^n_0 \perp \tau$, and we may realize $\tau\colon \overline{U}_n \to C_{n-1}{}'V_\bullet$ by $T\colon C^*{}'\mathbf{W}^\bullet_{[n-1]} \to \mathbf{K}^*$ (not a cofibration) and $\zeta\colon \overline{U}_n \to \overline{V}_n$ by $Z\colon \mathbf{D}^* \to \mathbf{K}^*$.

Consider the following diagram in the left Reedy model category of n–truncated cochain complexes over $\mathcal{C}$, in which $\mathbf{P}^*$ is the pushout of the upper left square and the map p with section r is induced by $C^*r_{[n-1]}$:

(3.21)
$$\begin{tikzcd}
C^*\mathbf{W}^\bullet_{[n-1]} \arrow[r, hook, "C^*r_{[n-1]}"'] \arrow[d, hook, "F"] \arrow[dr, phantom, "\boxed{\mathrm{PO}}"] & C^*{}'\mathbf{W}^\bullet_{[n-1]} \arrow[l, bend right, "C^*e_{[n-1]}"'] \arrow[d, hook, "j"] \arrow[dr, bend left, "T"] & \\
\mathbf{D}^* \arrow[r, hook, "r"] \arrow[rr, bend right=40, "Z"'] & \mathbf{P}^* \arrow[l, bend left, "p"] \arrow[r, dotted, "S"'] & \mathbf{K}^*
\end{tikzcd}$$

Since, by Definition 3.5, $C_{n-1}\rho \circ {}'\overline{\partial}{}^n_0 = \overline{\partial}{}^n_0 \circ \overline{\rho}_n$, also $C_{n-1}\rho \circ \tau = \overline{\partial}{}^n_0 \circ \overline{\rho}_n|_{\overline{U}_n} = \overline{\partial}{}^n_0 \circ \zeta$, so the outer square in (3.21) commutes up to homotopy. Since F is a cofibration, we may change Z up to homotopy to make it commute on the nose by Blanc, Johnson and Turner [11, Lemma 5.11]. The maps Z and T then induce S as indicated. This allows us to extend (3.21) to a commuting diagram

(3.22)
$$\begin{tikzcd}
C^*\boldsymbol{W}^\bullet_{[n-1]} \arrow[r, hook, "C^*r_{[n-1]}"'] \arrow[d, hook, "F"] & C^*{}'\boldsymbol{W}^\bullet_{[n-1]} \arrow[l, bend right, "C^*e_{[n-1]}"'] \arrow[d, hook, "j"] \arrow[dr, bend left, "T\top(p\circ j)"] & \\
\boldsymbol{D}^* \arrow[r, hook, "r"] & \boldsymbol{P}^* \arrow[r, "S\top p"] \arrow[l, bend left, "p"] & \boldsymbol{K}^*\times\boldsymbol{D}^* \arrow[ll, bend left=40, "\mathrm{proj}"]
\end{tikzcd}$$

We now factor $S\top p$ as a cofibration $G'\colon \boldsymbol{P}^* \hookrightarrow \boldsymbol{E}^*$ followed by a trivial fibration $t\colon \boldsymbol{E}^* \twoheadrightarrow \boldsymbol{K}^*\times\boldsymbol{D}^*$ (in the left Reedy model structure on truncated cochain complexes). If we set $G\colon C^*{}'\boldsymbol{W}^\bullet_{[n-1]} \to \boldsymbol{E}^*$ equal to $G'\circ j$, $\bar{r}\colon \boldsymbol{E}^* \to \boldsymbol{D}^*$ equal to $\mathrm{proj}\circ t$, and $\bar{e}\colon \boldsymbol{D}^* \to \boldsymbol{E}^*$ equal to the cofibration $G'\circ e$, we see that $\boldsymbol{E}^*$ is a left Reedy fibrant replacement for ${}'\widehat{\boldsymbol{W}}{}^n \otimes^* S^{n-1}$ (since $\boldsymbol{K}^*\times\boldsymbol{D}^*$ is a product of fibrant objects), G is a left Reedy cofibration, and they fit into a diagram

(3.23)
$$\begin{tikzcd}
C^*\boldsymbol{W}^\bullet_{[n-1]} \arrow[r, hook, "C^*r_{[n-1]}"'] \arrow[d, hook, "F"] & C^*{}'\boldsymbol{W}^\bullet_{[n-1]} \arrow[l, bend right, "C^*e_{[n-1]}"'] \arrow[d, hook, "G"] \\
\boldsymbol{D}^* \arrow[r, hook, "\bar{e}"] & \boldsymbol{E}^* \arrow[l, bend left, "\bar{r}"]
\end{tikzcd}$$

in which both the left and right squares commute, and $\bar{r}\circ\bar{e} = \mathrm{Id}$.

Applying the functorial procedure of Definition 2.1(c) to the two vertical arrows in (3.23) again, we obtain n–stage comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ extending the given $(n-1)$–stage comparison map.

(ii) Reducing to the case of one algebraic resolution Assume $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ are associated respectively to the two CW resolutions $V_\bullet^{(0)}$ and $V_\bullet^{(1)}$ of the Θ_R–algebra $\Gamma = H^*(\boldsymbol{Y};R)$, with CW bases $(\overline{V}{}^{(i)}_n)_{n\in\mathbb{N}}$ for $i=0,1$.

By Lemma 3.7, there is a third CW resolution ${}'\varepsilon\colon {}'V_\bullet \to \Gamma$, with CW basis $({}'\overline{V}_n)_{n\in\mathbb{N}}$, equipped with algebraic comparison maps $\Psi^{(i)}\colon V_\bullet^{(i)} \to {}'V_\bullet$ for $i=0,1$. By step (i), there are then two sequential realizations $'\mathcal{W}^{(i)}$ of $'V_{\bullet} \to \Gamma$, for $i = 0, 1$, each equipped with a comparison map $\Psi^{(i)}\colon \mathcal{W}^{(i)} \to {}'\mathcal{W}^{(i)}$ over $\Psi^{(i)}$. Thus we are reduced to dealing with the case where the two (cofibrant) sequential realizations $\mathcal{W}^{(0)}$ and $\mathcal{W}^{(1)}$ (ie the $'\mathcal{W}^{(i)}$ just constructed) are of the same CW resolution $V_{\bullet} \to \Gamma$, with CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$. We construct a zigzag of comparison maps between them, by induction on $n \geq 0$:

We assume by induction the existence of a cospan of $(n-1)$–stage trivial comparison maps $\Phi^{(i)}\colon \mathcal{W}^{(i)} \to \mathcal{W}$ for $i = 0, 1$ over $\mathrm{Id}_{V_{\bullet}}$. For $n = 0$ this is simply

$$\boldsymbol{W}^{\bullet(0)}_{[-1]} = c(\boldsymbol{Y})^{\bullet} = \boldsymbol{W}^{\bullet(1)}_{[-1]}.$$

By Definition 2.1(c), the n^{th} stage for $\mathcal{W}^{(i)}$ is determined by the choice of left Reedy fibrant replacements $\boldsymbol{D}^*_{(i)}$ of $\overline{\boldsymbol{W}}^n \otimes^* S^{n-1}$ (where $\overline{\boldsymbol{W}}^n$ is some realization of the n^{th} algebraic CW basis object $\overline{V}_n$), together with left Reedy cofibrations $F^{(i)}\colon C^*\boldsymbol{W}^{\bullet(i)}_{[n-1]} \to \boldsymbol{D}^*_{(i)}$ for $i = 0, 1$ realizing the given attaching map $\overline{\partial}^n_0\colon \overline{V}_n \to C_{n-1}V_{\bullet}$.

For $i = 0, 1$, consider the following diagram in the left Reedy model category $\mathsf{Ch}^{\mathcal{C}}_{\leq n}$, in which $\boldsymbol{P}^*$ is again the pushout of the upper left square

(3.24)
$$\begin{tikzcd}
C^*\boldsymbol{W}^{\bullet(i)}_{[n-1]} \arrow[r, hook, "\simeq", "C^*r_{[n-1]}"'] \arrow[d, hook, "F_{(i)}"] & C^*{}'\boldsymbol{W}^{\bullet}_{[n-1]} \arrow[l, bend right, "C^*e_{[n-1]}"'] \arrow[d, hook, "j_{(i)}"] \arrow[dr, dotted, "G_{(i)}"] \arrow[dl, phantom, "\boxed{\mathrm{PO}}"] & & \\
\boldsymbol{D}^*_{(i)} \arrow[r, hook, "r_{(i)}", "\simeq"'] & \boldsymbol{P}^*_{(i)} \arrow[r, hook, "S_{(i)}", "\simeq"'] \arrow[l, bend left, "p_{(i)}"] & {}'\boldsymbol{E}^*_{(i)} \arrow[r, two heads] \arrow[ll, bend left=40, "\pi_{(i)}", "\simeq"'] & *
\end{tikzcd}$$

The retraction $p_{(i)}$ for the trivial cofibration $r_{(i)}$ is induced by the retraction $C^*e^{(i)}_{[n-1]}$ for the trivial cofibration $C^*r^{(i)}_{[n-1]}$, so $p_{(i)}$ is a weak equivalence. Factor $p_{(i)}$ as a trivial cofibration $k_{(i)}\colon \boldsymbol{P}^*_{(i)} \to {}'\boldsymbol{E}^*_{(i)}$ followed by a fibration $\pi_{(i)}\colon {}'\boldsymbol{E}^*_{(i)} \to \boldsymbol{D}^*_{(i)}$ (also a weak equivalence), so $'\boldsymbol{E}^*_{(i)}$ is in particular a fibrant replacement for $\boldsymbol{P}^*_{(i)}$ since $'\boldsymbol{E}^*_{(i)}$ is fibrant.

Now set $G_{(i)} := S_{(i)} \circ j_{(i)}\colon C^*{}'\boldsymbol{W}^{\bullet}_{[n-1]} \hookrightarrow {}'\boldsymbol{E}^*_{(i)}$ (a cofibration). Because the maps $\pi_i \circ G_{(i)} = F_{(i)} \circ C^*e_{[n-1]}$ realize the same algebraic attaching map $\phi\colon \overline{V}_n \otimes_* S^{n-1} \to C^*V_{\bullet}$ for $i = 0, 1$, they are weakly equivalent in the arrow category of $\mathsf{Ch}^{\mathcal{C}}_{\leq n-1}$. Thus $G^{(0)}$ and $G^{(1)}$ are weakly equivalent fibrant and cofibrant objects in the under category $C^*{}'\boldsymbol{W}^{\bullet}_{[n-1]} \backslash \mathsf{Ch}^{\mathcal{C}}_{\leq n-1}$ with its standard model category structure (see Hirschhorn [28, Theorem 7.6.5(a)]). We can therefore apply Lemma 3.1(b) to obtain an intermediate object G fitting into the following diagram, in which all four triangles commute, and $s_{(i)} \circ f_{(i)} = \mathrm{Id}$ for $i = 0, 1$:

(3.25)
$$\begin{tikzcd}
& C^{*\prime}\boldsymbol{W}^{\bullet}_{[n-1]} \arrow[dl, hook', "G_{(0)}"'] \arrow[d, hook, "G"] \arrow[dr, hook, "G_{(1)}"] & \\
{}'\boldsymbol{E}^{*}_{(0)} \arrow[r, hook, "f_{(0)}"] & {}'\boldsymbol{E}^{*} \arrow[l, bend left, "\simeq" description, "s_{(0)}"] \arrow[r, bend right, "\simeq" description, "s_{(1)}"'] & {}'\boldsymbol{E}^{*}_{(1)} \arrow[l, hook', "f_{(1)}"']
\end{tikzcd}$$

Again applying the functors of Definition 2.1(c) to all three downward arrows of (3.25) yields a new n–stage sequential realization $'\mathcal{W}$ (corresponding to $G\colon C^{*}\boldsymbol{W}^{\bullet}_{[n-1]} \hookrightarrow {}'\boldsymbol{E}^{*}$) with two new n–stage trivial comparison maps $'\Phi_{(i)}\colon {}'\mathcal{W}^{(i)} \to {}'\mathcal{W}$ for $i = 0, 1$.

The two composites

(3.26)
$$\mathcal{W}^{(0)} \xrightarrow{\Phi^{(0)}} {}'\mathcal{W}^{(0)} \xrightarrow{'\Phi_{(0)}} {}'\mathcal{W} \xleftarrow{'\Phi_{(1)}} {}'\mathcal{W}^{(1)} \xleftarrow{\Phi^{(1)}} \mathcal{W}^{(1)}$$

then yield the required cospan of n–stage comparison maps. □

4 Higher cohomology operations

The notion of secondary and higher cohomology operations has a long history in homotopy theory, going back to the 1950s, but there is no completely satisfactory general theory of such operations. Here we follow the point of view taken in Blanc and Markl [15] and Blanc, Johnson and Turner [12], where they are subsumed under the notion of general pointed higher homotopy operations.

We want to think of a cofibrant sequential realization $\mathcal{W}$ for a space $\boldsymbol{Y}$ as providing a template for an infinite sequence of operations of order n for $n = 2, 3, \ldots$, potentially acting on any space $\boldsymbol{Z}$ with $H^*(\boldsymbol{Z}; R)$ (abstractly) isomorphic to $H^*(\boldsymbol{Y}; R)$. The operation of order n is defined only when specific choices have been made inductively for all lower order operations in such a way that they all vanish.

The sequence of such choices is called a "strand" of the higher cohomology operation associated to the given sequential realization $\mathcal{W}$, and the corresponding "system of higher cohomology operations" will be an equivalence class of strands under comparison maps.

4.1 Definition Let $\Gamma = H^*(\boldsymbol{Y}; R)$ be the Θ_R–algebra associated to a space $\boldsymbol{Y}$, and $\varepsilon\colon V_\bullet \to \Gamma$ a CW resolution with CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$. Assume we are also given *initial data* consisting of a sequential realization $\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]}\rangle_{n\in\mathbb{N}}$ of $V_\bullet$, and an isomorphism of Θ_R–algebras $\vartheta\colon \Gamma \to H^*(\boldsymbol{Z}; R)$ for some space $\boldsymbol{Z}$.

An *n–strand* for $0 \le n \le \infty$ $\mathcal{S}_{[n]} = (\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \dots, \boldsymbol{\varepsilon}_{[n]})$ for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ consists of a compatible collection of coaugmentations $\boldsymbol{\varepsilon}_{[k]}\colon \boldsymbol{Z} \to \boldsymbol{W}^\bullet_{[k]}$ for $k = 0, \dots, n$ realizing $\vartheta \circ \varepsilon\colon V_\bullet \to H^*(\boldsymbol{Z}; R)$ through simplicial dimension n. Compatibility means that $\boldsymbol{\varepsilon}_{[k-1]} = \pi_{[k]} \circ \boldsymbol{\varepsilon}_{[k]}$ (cf Definition 2.1(c)). In particular, an ∞*–strand* for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ is an infinite sequence $\mathcal{S}_{[\infty]} := (\boldsymbol{\varepsilon}_{[0]}, \dots, \boldsymbol{\varepsilon}_{[n]}, \dots)$ of such compatible coaugmentations.

Given an $(n-1)$–strand $\mathcal{S}_{[n-1]}$ for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$, consider the composite ξ of

$$\boldsymbol{Z} \xrightarrow{\boldsymbol{\varepsilon}_{[n-1]}} \boldsymbol{W}^0_{[n-1]} \xrightarrow{F^0_{[n-1]}} \overline{P\Omega^{n-2}\boldsymbol{W}^n} \xrightarrow{\bar{p}^{n-2}} \overline{\Omega^{n-2}\boldsymbol{W}^n} \xrightarrow{\bar{\iota}^{n-2}} \overline{P\Omega^{n-3}\boldsymbol{W}^n},$$

which by (2.14) represents the component

$$q^2_{[n-1]} \circ d^1 \circ d^0 \circ \boldsymbol{\varepsilon}_{[n-1]}$$

of the iterated coface map from $\boldsymbol{Z}$ into $\overline{P\Omega^{n-3}\boldsymbol{W}^n}$. As in Lemma 2.19, since

$$d^1 \circ d^0 \circ \boldsymbol{\varepsilon}_{[n-1]} = d^2 \circ d^1 \circ \boldsymbol{\varepsilon}_{[n-1]}$$

and $q^2_{[n-1]} \circ d^2 = 0$ by (2.15), we see that ξ is the zero map. Since $\bar{\iota}^{n-2}$ is monic, this means that the composite

$$\bar{p}^{n-2} \circ F^0_{[n-1]} \circ \boldsymbol{\varepsilon}_{[n-1]}$$

is already zero, so $F^0_{[n-1]} \circ \boldsymbol{\varepsilon}_{[n-1]}$ factors through the fiber $\overline{\Omega^{n-1}\boldsymbol{W}^n}$ of $\bar{p}^{n-2}$. We denote the resulting map by

$$a^{-1}_{[n-1]}\colon \boldsymbol{Z} \to \overline{\Omega^{n-1}\boldsymbol{W}^n},$$

with

$$\bar{\iota}^{n-1} \circ a^{-1}_{[n-1]} = F^0_{[n-1]} \circ \boldsymbol{\varepsilon}_{[n-1]}, \tag{4.2}$$

as in (2.20).

Note that ϑ induces an isomorphism $[\boldsymbol{Z}, \overline{\Omega^{n-1}\boldsymbol{W}^n}] \cong \Gamma\{\overline{\Omega^{n-1}\boldsymbol{W}^n}\}$ so the homotopy class $[a^{-1}_{[n-1]}]$ may be identified with an element

$$\mathrm{Val}(\mathcal{S}_{[n-1]}) \in \Gamma\{\overline{\Omega^{n-1}\boldsymbol{W}^n}\} \cong \Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^n\}, \tag{4.3}$$

called the *value* of the $(n-1)$–strand $\mathcal{S}_{[n-1]}$.

4.4 Remark We do not need the full sequential realization $\mathcal{W}$ to define $\mathrm{Val}(\mathcal{S}_{[n]})$, but only the restricted cosimplicial set $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$ of its $(n{-}1)^{\text{st}}$ stage $\boldsymbol{W}^{\bullet}_{[n-1]}$. Thus a 0–*strand* is completely determined by a choice of a realization $\boldsymbol{\varepsilon}\colon \boldsymbol{Z} \to \boldsymbol{W}^{0}_{[0]}$ of $\vartheta \circ \varepsilon\colon V_0 \to H^*(\boldsymbol{Z}; R)$. Such an $\boldsymbol{\varepsilon}$ always exists, and is unique up to homotopy.

Note also that Definition 4.1 can be stated purely in the language of $\boldsymbol{\Theta}_R$–mapping algebras — see Baues and Blanc [2] and the appendix below.

4.5 Lemma *Given an* $(n{-}1)$*–strand* $\mathcal{S}_{[n-1]} = (\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \dots, \boldsymbol{\varepsilon}_{[n-1]})$ *for* $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$, *the coaugmentation* $\boldsymbol{\varepsilon}_{[n-1]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[n-1]}$ *extends to a coaugmentation* $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[n]}$ *if and only if* $\mathrm{Val}(\mathcal{S}_{[n-1]}) = 0$ *in* $\Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^{n}\}$.

Proof If the value is zero, we can choose a nullhomotopy $F^{-1}_{[n]}\colon \boldsymbol{Z} \to \overline{P\Omega^{n-1}\boldsymbol{W}^{n}}$ for $a^{-1}_{[n-1]}$, (in the sense of Quillen [32, Section I.2]), with $\bar{p}^{n-1} \circ F^{-1}_{[n]} = a^{-1}_{[n-1]}$. This extends $\boldsymbol{\varepsilon}_{[n-1]}$ to a coaugmentation $\widetilde{\boldsymbol{\varepsilon}}_{[n]}\colon \boldsymbol{Z} \to \widetilde{\boldsymbol{W}}^{0}_{[n]}$ as in the proof of Theorem A.11, which also shows how to extend $\widetilde{\boldsymbol{\varepsilon}}_{[n]}$ to a coaugmentation for $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$, and thus (after making it Reedy cofibrant) for $\boldsymbol{W}^{\bullet}_{[n]}$.

Conversely, since $\boldsymbol{W}^{0}_{[n]} = \widetilde{\boldsymbol{W}}^{0}_{[n]} = \boldsymbol{W}^{0}_{[n-1]} \times \overline{P\Omega^{n-1}\boldsymbol{W}^{n}}$ by (2.10) and step (iii) of Section 2.5, given an extension $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{0}_{[n]}$ of $\boldsymbol{\varepsilon}_{[n-1]}$, we can compose it with the projection $q^{0}_{[n]}\colon \widetilde{\boldsymbol{W}}^{0}_{[n]} \to \overline{P\Omega^{n-1}\boldsymbol{W}^{n}}$ to obtain a nullhomotopy $F^{-1}_{[n]}$ for $a^{-1}_{[n-1]}$. $\square$

4.6 Correspondence of strands Note that if $\Psi\colon V_\bullet \to {}'V_\bullet$ is an algebraic comparison map between two CW resolutions for a Θ_R–algebra Γ, as in (3.6), the mutually inverse weak equivalences $\varphi\colon V_\bullet \to {}'V_\bullet$ and $\rho\colon {}'V_\bullet \to V_\bullet$ induce mutually inverse isomorphisms of Θ_R–algebras $\varphi_\#\colon \pi_0 V_\bullet \cong \Gamma \cong \pi_0 {}'V_\bullet :\rho_\#$.

On the other hand, if $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ is an n–stage comparison map between two sequential realizations over Ψ, as in (3.16), then $(\bar{r}^{n-1})_*\colon \Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^{n}\} \hookrightarrow \Gamma\{\Omega^{n-1}{}'\overline{\boldsymbol{W}}^{n}\}$ is just a split inclusion, with retraction $(\bar{e}^{n-1})_*\colon \Gamma\{\Omega^{n-1}{}'\overline{\boldsymbol{W}}^{n}\} \twoheadrightarrow \Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^{n}\}$.

Let $\vartheta\colon \Gamma \cong H^*(\boldsymbol{Z}; R)$ be an isomorphism of Θ_R–algebras, and let $\mathcal{S}_{[n]}$ and ${}'\mathcal{S}_{[n]}$ be two n–strands for the initial data $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ and $({}'\mathcal{W}, \boldsymbol{Z}, \vartheta)$, respectively. If $\Phi = \big\langle e_{[n]}, r_{[n]}, (\overline{Pe}^{k}_{n})_{k=0}^{n-1}, (\bar{e}^{k}_{n})_{k=0}^{n-1}, (\overline{Pr}^{k}_{n})_{k=0}^{n-1}, (\bar{r}^{k}_{n})_{k=0}^{n-1}\big\rangle_{n=0}^{N}$ is an n–stage comparison map as above, we write ${}'\mathcal{S}_{[n]} = r^\#(\mathcal{S}_{[n]})$ if ${}'\boldsymbol{\varepsilon}_{[k]} = r^{0}_{[k]} \circ \boldsymbol{\varepsilon}_{[k]}$ for each $0 \le k \le n$, and $\mathcal{S}_{[n]} = e^\#({}'\mathcal{S}_{[n]})$ if $\boldsymbol{\varepsilon}_{[k]} = e^{0}_{[k]} \circ {}'\boldsymbol{\varepsilon}_{[k]}$ for each $0 \le k \le n$.

From (3.13), (3.15), and (4.2) we see that

$$\textbf{(4.7)} \qquad \mathrm{Val}(r^\#(\mathcal{S}_{[n]})) = \bar{r}^{n}_{*}(\mathrm{Val}(\mathcal{S}_{[n]})) \quad \text{and} \quad \mathrm{Val}(e^\#({}'\mathcal{S}_{[n]})) = \bar{e}^{n}_{*}(\mathrm{Val}({}'\mathcal{S}_{[n]})).$$

Therefore, given an n–stage comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ as above, an n–strand $\mathcal{S}_{[n]}$ for $\mathcal{W}$ and an n–strand ${}'\mathcal{S}_{[n]}$ for ${}'\mathcal{W}$, we see that:

4.8 (a) $\mathrm{Val}(\mathcal{S}_{[n]}) = 0$ if and only if $\mathrm{Val}(r^{\#}(\mathcal{S}_{[n]})) = 0$.

(b) If $\mathrm{Val}({}'\mathcal{S}_{[n]}) = 0$ then $\mathrm{Val}(e^{\#}({}'\mathcal{S}_{[n]})) = 0$, but not necessarily conversely.

This explains the need for the following:

4.9 Definition Given spaces $\boldsymbol{Y}$ and $\boldsymbol{Z}$ with $\vartheta\colon H^*(\boldsymbol{Y};R) \xrightarrow{\cong} H^*(\boldsymbol{Z};R)$, we define two equivalence relations $\sim$ and $\approx$ on n–strands for $\boldsymbol{Z}$ (with respect to various sequential realizations):

The *weak equivalence* relation of strands $\sim$ is generated by $e^{\#}({}'\mathcal{S}_{[n]}) \sim {}'\mathcal{S}_{[n]}$ for any n–strand ${}'\mathcal{S}_{[n]}$ for $({}'\mathcal{W}, \boldsymbol{Z}, \vartheta)$ and any comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$. We denote the corresponding equivalence class by $[\mathcal{S}_{[n]}]$.

The *strong equivalence* relation of strands $\approx$ is generated by the relation $e^{\#}({}'\mathcal{S}_{[n]}) \approx {}'\mathcal{S}_{[n]}$ for any n–strand ${}'\mathcal{S}_{[n]}$ for $({}'\mathcal{W}, \boldsymbol{Z}, \vartheta)$ and any comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ satisfying

$$(\bar{s}^{n}_{[n+1]})_{\#}(\mathrm{Val}({}'\mathcal{S}_{[n]})) = 0, \tag{4.10}$$

in the notation of Remark 3.17. We denote strong equivalence classes by $[\![\mathcal{S}_{[n]}]\!]$.

4.11 Remark Clearly $\mathcal{S}_{[n]} \approx {}'\mathcal{S}_{[n]}$ implies that $\mathcal{S}_{[n]} \sim {}'\mathcal{S}_{[n]}$, and both notions coincide if the comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$ is trivial — that is, if in the underlying algebraic comparison map Ψ, $V_\bullet$ and ${}'V_\bullet$ have isomorphic CW bases.

Also $\mathcal{S}_{[n]} \approx r^{\#}(\mathcal{S}_{[n]})$ (and thus $\mathcal{S}_{[n]} \sim r^{\#}(\mathcal{S}_{[n]})$) for *any* comparison map $\Phi\colon \mathcal{W} \to {}'\mathcal{W}$, since $e^{\#}r^{\#}(\mathcal{S}_{[n]}) = \mathcal{S}_{[n]}$ and $(\bar{s}^{n}_{[n+1]})_{\#}\big(\mathrm{Val}(r^{\#}(\mathcal{S}_{[n]}))\big) = (\bar{s}^{n}_{[n+1]})_{\#}(\bar{r}^{n})_{*}(\mathrm{Val}(\mathcal{S}_{[n]})) = 0$ by (4.7) and (3.18).

4.12 Lemma *When* $\mathcal{S}_{[n]} \approx {}'\mathcal{S}_{[n]}$, $\mathrm{Val}(\mathcal{S}_{[n]}) = 0$ *if and only if* $\mathrm{Val}({}'\mathcal{S}_{[n]}) = 0$.

Proof This follows from (4.7), since by (4.10) we see that $\mathrm{Val}({}'\mathcal{S}_{[n]}) \in \Gamma\{\Omega^{n\,\prime}\overline{\boldsymbol{W}}^{n+1}\}$ is uniquely determined by its image $\bar{e}^{n}_{*}(\mathrm{Val}({}'\mathcal{S}_{[n]}))$ under the projection in $\Gamma\{\Omega^{n}\overline{\boldsymbol{W}}^{n+1}\}$. □

4.13 Definition Given a space $\boldsymbol{Y}$ with $\Gamma := H^*(\boldsymbol{Y};R)$, we want to think of each sequential realization $\mathcal{W}$ for $\boldsymbol{Y}$ as a template for a countable sequence $\langle\!\langle \boldsymbol{Y} \rangle\!\rangle = (\langle\!\langle \boldsymbol{Y} \rangle\!\rangle_n)_{n=2}^{\infty}$ of higher operations, where for each $n \geq 2$, we define the *universal*

n^{th} *order cohomology operation* $\langle\langle \boldsymbol{Y} \rangle\rangle_n$ associated to the space $\boldsymbol{Y}$ to be the function that assigns to every $(n-1)$–strand $\mathcal{S}_{[n-1]}$ for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ the class

$$\langle\langle \boldsymbol{Y} \rangle\rangle_n(\mathcal{S}_{[n-1]}) := \mathrm{Val}(\mathcal{S}_{[n-1]}) \in \Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^n\}.$$

We say that the operation $\langle\langle \boldsymbol{Y} \rangle\rangle_n$ *vanishes for* $\boldsymbol{Z}$ if there is a cofibrant sequential realization $\mathcal{W}$ and an $(n-1)$–strand $\mathcal{S}_{[n-1]}$ for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ such that $\langle\langle \boldsymbol{Y} \rangle\rangle_n(\mathcal{S}_{[n-1]}) = 0$. By Lemma 4.12, this notion of the vanishing depends only the strong equivalence classes $[\![\mathcal{S}_{[n-1]}]\!]$ of the strand.

4.14 Definition Given spaces $\boldsymbol{Y}$ and $\boldsymbol{Z}$ with $\vartheta\colon H^*(\boldsymbol{Y}; R) \xrightarrow{\cong} H^*(\boldsymbol{Z}; R)$ and a sequential realization $\mathcal{W}$ of $\boldsymbol{Y}$, we say that $\langle\langle \boldsymbol{Y} \rangle\rangle$ *vanishes coherently for* $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ if there is an ∞–strand $\mathcal{S}_{[\infty]}$ for $\mathcal{W}$ as in Definition 4.1; that is, for each $n \geq 2$, we have a given $(n-1)$–strand $\mathcal{S}_{[n-1]}$ for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ such that $\mathrm{Val}(\mathcal{S}_{[n-1]}) = 0$, which extends to the next n–strand $\mathcal{S}_{[n]}$ using Lemma 4.5.

4.15 Example For any sequential realization $\mathcal{W}$ of a space $\boldsymbol{Y}$, the sequence $\langle\langle \boldsymbol{Y} \rangle\rangle$ vanishes coherently for $(\mathcal{W}, \boldsymbol{Y}, \mathrm{Id}_\Gamma)$, since then we have a given coaugmentation $\boldsymbol{\varepsilon}\colon \boldsymbol{Y} \to \boldsymbol{W}^\bullet$, which we can then project to each $\boldsymbol{W}^\bullet_{[n]}$ (see Proposition 2.24) to yield $\boldsymbol{\varepsilon}_{[n]}$ showing that $\mathrm{Val}(\mathcal{S}_{[n-1]}) = 0$ for the corresponding $(n-1)$–strand $\mathcal{S}_{[n-1]}$, by Lemma 4.5.

4.16 Remark A priori, each individual strand $\mathcal{S}_{[n-1]}$ has a different template for $\langle\langle \boldsymbol{Y} \rangle\rangle$ — namely, the restricted cosimplicial set $\widetilde{\boldsymbol{W}}^\bullet_{[n]}$ of the $(n-1)^{\mathrm{st}}$ stage $\boldsymbol{W}^\bullet_{[n-1]}$ of $\mathcal{W}$. However, the following result, which follows from Theorem 3.20, shows that we can in fact use any one cofibrant sequential realization to calculate $\langle\langle \boldsymbol{Y} \rangle\rangle_n$:

4.17 Key Lemma *Given* $\boldsymbol{Y}$ *and* $\vartheta\colon H^*(\boldsymbol{Y}; R) \xrightarrow{\cong} H^*(\boldsymbol{Z}; R)$ *as above,* $\langle\langle \boldsymbol{Y} \rangle\rangle_n$ *vanishes for* $\boldsymbol{Z}$ *if and only if for* every *n–stage cofibrant sequential realization* $\mathcal{W}$ *of* $\boldsymbol{Y}$*, there is an* $(n-1)$*–strand* $\mathcal{S}_{[n-1]}$ *for* $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ *such that* $\mathrm{Val}(\mathcal{S}_{[n-1]}) = 0$.

Proof By definition, $\langle\langle \boldsymbol{Y} \rangle\rangle_n$ vanishes for $\boldsymbol{Z}$ if there is *some* cofibrant n–stage sequential realization $'\mathcal{W}$ of $\boldsymbol{Y}$ and an $(n-1)$–strand $\mathcal{S}'_{[n-1]}$ for $('\mathcal{W}, \boldsymbol{Z}, \vartheta)$ such that $\mathrm{Val}(\mathcal{S}'_{[n-1]}) = 0$. By Theorem 3.20 we know that there is a finite zigzag of cospans of comparison maps connecting $'\mathcal{W}$ to $\mathcal{W}$, say

$$'\Phi_{(1)}\colon \mathcal{W}^{(0)} = {}'\mathcal{W} \to \mathcal{W}^{(1)}, \quad '\Phi_{(2)}\colon \mathcal{W}^{(2)} \to \mathcal{W}^{(1)}, \quad '\Phi_{(3)}\colon \mathcal{W}^{(2)} \to \mathcal{W}^{(3)},$$

and so on until ${}'\Phi_{(N)}\colon \mathcal{W}^{(N-1)} \to \mathcal{W}^{(N)} = \mathcal{W}$. If ${}'\Phi_{(1)} = \langle e_{[k]}, r_{[k]}, \dots \rangle_{k=0}^{n}$ as in (3.16), we set $\mathcal{S}^{(1)}_{[n-1]} := r^{\#}(\mathcal{S}'_{[n-1]})$ (an $(n-1)$–strand for $\mathcal{W}^{(1)}$), and see from (4.7) that $\mathrm{Val}(\mathcal{S}^{(1)}_{[n-1]}) = 0$. Similarly, if ${}'\Phi_{(2)} = \langle {}'e_{[k]}, {}'r_{[k]}, \dots \rangle_{k=0}^{n}$ we set $\mathcal{S}^{(2)}_{[n-1]} := {}'e^{\#}(\mathcal{S}^{(1)}_{[n-1]})$ (an $(n-1)$–strand for $\mathcal{W}^{(2)}$), and again see from (4.7) that $\mathrm{Val}(\mathcal{S}^{(2)}_{[n-1]}) = 0$. Continuing in this way we finally obtain an $(n-1)$–strand $\mathcal{S}_{[n-1]} = \mathcal{S}^{(N)}_{[n-1]}$ for $\mathcal{W}^{(N)} = \mathcal{W}$ with $\mathrm{Val}(\mathcal{S}_{[n-1]}) = 0$, as required.

The converse follows from the fact that $H^*(\boldsymbol{Y}; R)$ has at least one CW resolution by Remark 1.31, and thus there is at least one cofibrant sequential realization for $\boldsymbol{Y}$ by Theorem 2.33. $\square$

Clearly, the $(n-1)$–strands $\mathcal{S}'_{[n-1]}$ and $\mathcal{S}_{[n-1]}$ are weakly equivalent. However, they are not necessarily strongly equivalent, since there is no reason for (4.10) to hold for the even-numbered comparison maps above ${}'\Phi_{(2)}$, ${}'\Phi_{(4)}$, and so on.

4.18 Theorem *For R either $\mathbb{F}_p$ or a field of characteristic 0, let $\boldsymbol{Y}$ and $\boldsymbol{Z}$ be R–good spaces with isomorphic Θ_R–algebras. Then the following are equivalent:*

(a) *The system of higher cohomology operations $\langle\langle \boldsymbol{Y} \rangle\rangle$ vanishes coherently for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ for* some *cofibrant sequential realization $\mathcal{W}$ of $\boldsymbol{Y}$ and some ϑ.*

(b) *$\langle\langle \boldsymbol{Y} \rangle\rangle$ vanishes coherently for* every *cofibrant sequential realization of $\boldsymbol{Y}$.*

(c) *$\boldsymbol{Y}$ and $\boldsymbol{Z}$ are R–equivalent.*

Proof (a) $\Longleftrightarrow$ (b) This is by Key Lemma 4.17.

(a) $\Longrightarrow$ (c) Assume that $\langle\langle \boldsymbol{Y} \rangle\rangle$ vanishes coherently — that is, there is an ∞–strand $\mathcal{S}_{[\infty]}$ for some $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ (where $\mathcal{W}$ need not be cofibrant), and thus coaugmentations $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[n]}$ for all $n \geq 0$. These fit together to define a coaugmentation $\boldsymbol{\varepsilon}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}$ for $\boldsymbol{W}^{\bullet} := \mathrm{holim}\, \boldsymbol{W}^{\bullet}_{[n]}$, which induces an isomorphism

$$H^*(\mathrm{Tot}\, \boldsymbol{W}^{\bullet}; R) \to H^*(\boldsymbol{Z}; R).$$

Since $\boldsymbol{Y}$ is R–good, it is R–equivalent to the total space $\mathrm{Tot}\, \boldsymbol{W}^{\bullet} \simeq \widehat{L}_{\mathcal{G}} \boldsymbol{Y}$ (see Section 1.22), and thus the map $f\colon \boldsymbol{Z} \to \mathrm{Tot}\, \boldsymbol{W}^{\bullet}$ induced by the coaugmentation $\boldsymbol{\varepsilon}$ realizes ϑ, so $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are related by a cospan of R–equivalences.

(c) $\Longrightarrow$ (a) Conversely, if $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are R–equivalent, we have a zigzag of R–equivalences from $\boldsymbol{Y}$ to $\boldsymbol{Z}$ inducing an isomorphism of Θ_R–algebras $\vartheta\colon H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$, so it suffices to consider the following two special cases:

(i) Given a R–equivalence $f\colon \boldsymbol{Z} \to \boldsymbol{Y}$, and some (not necessarily cofibrant) sequential realization $\mathcal{W}$ for $\boldsymbol{Y}$, by precomposing the coaugmentations $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Y} \to \boldsymbol{W}^{\bullet}_{[n]}$ with f we obtain coaugmentations $\boldsymbol{\varepsilon}_{[n]} \circ f\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[n]}$, still realizing $V_{\bullet} \to \Gamma$, since $f^{\#}\colon \Gamma \to H^*(\boldsymbol{Z}; R)$ is an isomorphism. This yields an ∞–strand for $(\mathcal{W}, \boldsymbol{Z}, \vartheta)$ by Lemma 4.5.

(ii) On the other hand, given a R–equivalence $g\colon \boldsymbol{Y} \to \boldsymbol{Z}$ and any cofibrant sequential realization ${}'\mathcal{W}$ for $\boldsymbol{Z}$, by precomposing the coaugmentations $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to {}'\boldsymbol{W}^{\bullet}_{[n]}$ with g as in (a) we obtain coaugmentations $\boldsymbol{\varepsilon}_{[n]} \circ g\colon \boldsymbol{Y} \to {}'\boldsymbol{W}^{\bullet}_{[n]}$ realizing $V_{\bullet} \to \Gamma$, and thus making ${}'\mathcal{W}$ itself with the new coaugmentations into a cofibrant sequential realization ${}''\mathcal{W}$ for $\boldsymbol{Y}$. The coaugmentations $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to {}'\boldsymbol{W}^{\bullet}_{[n]}$ form an ∞–strand $\mathcal{S}_{[\infty]}$ for ${}''\mathcal{W}$, showing that $\langle\langle \boldsymbol{Y} \rangle\rangle$ vanishes coherently for $({}''\mathcal{W}, \boldsymbol{Z}, \vartheta)$.

This completes the proof. □

4.19 Corollary *If $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ are R–equivalent R–good spaces, any cofibrant sequential realization $\mathcal{W} = \langle \boldsymbol{W}^{\bullet}_{[n]}, \widetilde{\boldsymbol{W}}^{\bullet}_{[n]} \rangle_{n \in \mathbb{N}}$ for $\boldsymbol{Y}$ is also a (cofibrant) sequential realization for $\boldsymbol{Y}'$.*

Proof The R–equivalence implies that there is an isomorphism $\vartheta\colon H^*(\boldsymbol{Y}; R) \cong H^*(\boldsymbol{Y}'; R)$, so by the theorem there is an ∞–strand $\mathcal{S}_{[\infty]}$ for $(\mathcal{W}, \boldsymbol{Y}', \vartheta)$, and thus a coaugmentation $\boldsymbol{\varepsilon}'\colon \boldsymbol{Y}' \to \boldsymbol{W}^{\bullet}$. □

4.20 Low-dimensional cases As noted in Remark 4.4, given a simplicial set $\boldsymbol{Z}$ equipped with an isomorphism of Θ_R–algebras $\vartheta\colon \Gamma \to H^*(\boldsymbol{Z}; R)$ and a sequential realization $\mathcal{W}$ for Γ as above, we can always define a coaugmentation $\boldsymbol{\varepsilon}_{[0]}\colon \boldsymbol{Z} \to \boldsymbol{W}^0_{[0]}$ realizing $\phi = \vartheta \circ \varepsilon\colon V_0 \to H^*(\boldsymbol{Z}; R)$, which is unique up to homotopy, as in the proof of Lemma 4.5. Moreover, if $\overline{d}^0_0\colon \overline{\boldsymbol{W}}^0 \to \overline{\boldsymbol{W}}^1$ realizes the first attaching map $\overline{\partial}^1_0\colon \overline{V}_1 \to V_0 = \overline{V}_0$, then $a^{-1}_{[0]} := \overline{d}^0_0 \circ \boldsymbol{\varepsilon}_{[0]}$ is nullhomotopic, since it realizes $\varepsilon \circ \overline{\partial}^1_0$ (see (A.12)). Thus we can always choose a nullhomotopy $F^{-1}_{[0]}$ for $a^{-1}_{[0]}$, and use it do define $\boldsymbol{\varepsilon}_{[1]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[1]}$, as in the proof of the lemma. Note however that while the map $\boldsymbol{\varepsilon}_{[0]}$ is unique up to homotopy, the map $\boldsymbol{\varepsilon}_{[1]}$ depends on our choice of $F^{-1}_{[0]}$.

This explains why our definition of n^{th} order cohomology operations only makes sense for $n \geq 2$.

Thus the first case of interest is $n = 2$. Since $\boldsymbol{\Theta}_R$ consists of abelian group objects, we can replace our 2–truncated restricted cosimplicial diagram $\boldsymbol{W}^{\bullet}_{[1]} \to \overline{\boldsymbol{W}}^2$ by

(4.21)
$$\boldsymbol{Z} \xrightarrow{\boldsymbol{\varepsilon}_{[1]}} \boldsymbol{W}^0_{[1]} \xrightarrow{d^0 - d^1} \boldsymbol{W}^1_{[1]} \xrightarrow{\overline{d}^0_1} \overline{\boldsymbol{W}}^2,$$

where the fact that $d^0 \circ \boldsymbol{\varepsilon}_{[1]} = d^1 \circ \boldsymbol{\varepsilon}_{[1]}$ means that the first composite is nullhomotopic, while the fact that $\overline{d}_1^0 \circ d^0$ is nullhomotopic and $\overline{d}_1^0 \circ d^1 = 0$ means that the second composite is also nullhomotopic.

In particular, the first map in (4.21) represents (a collection of) R–cohomology classes α, while the remaining two represent R–cohomology operations ξ and ζ, with $\xi(\alpha) = 0$ and $\zeta \circ \xi = 0$. Thus our universal secondary operation is just a (collection of) secondary cohomology operations in the sense of Adams (see [1]), taking values in $[\boldsymbol{Z}, \Omega \overline{\boldsymbol{W}}^2] = H^{*-1}(\boldsymbol{Z}; R)$.

4.22 Remark Our higher cohomology operations are modeled on Adams' (stable) secondary cohomology operations (see also Harper [26]). However, the more delicate questions involving the prerequisites for an $(n+1)^{\mathrm{st}}$ order operation to be defined, and the dependence on various choices made, are hidden here in the two components of the n–strand $\mathcal{S}_{[n]}$, consisting of:

(a) The n^{th} stage $\boldsymbol{W}^{\bullet}_{[n]}$ in the sequential approximation $\mathcal{W}$ encodes a preliminary choice of nullhomotopies for that part of the diagram consisting only of spaces in $\boldsymbol{\Theta}_R$ (the representing spaces for cohomology).

(b) The data associated to the specific simplicial set $\boldsymbol{Z}$ consists of the coaugmentation $\boldsymbol{\varepsilon}_{[n]}\colon \boldsymbol{Z} \to \boldsymbol{W}^{\bullet}_{[n]}$, which itself is determined by:

   i. The coherent system of earlier choices made, encoded in the $(n-1)$–strand $\mathcal{S}_{[n-1]} = (\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}_{[1]}, \dots, \boldsymbol{\varepsilon}_{[n-1]})$ (essentially, the single map $\boldsymbol{\varepsilon}_{[n-1]}$).

   ii. The single choice of the nullhomotopy $F^{-1}_{[n]}$ for $a^{-1}_{[n]}$ (ie the value of the previously defined n^{th} order operation, which must necessarily vanish in order to proceed to the $(n+1)^{\mathrm{st}}$ step).

4.23 Models for rational homotopy theory When working with $R = \mathbb{Q}$ it is convenient to use some of the known models for rational homotopy theory (see Quillen [33]). In particular, finite-type, simply connected rational spaces $\boldsymbol{Y} \in \mathcal{S}_{\mathbb{Q}}$ can be modelled in the category CDGA of differential graded commutative $\mathbb{Q}$–algebras (CDGAs), using a suitable Sullivan model $(A^*, d) \in \mathsf{CDGA}$ for $\boldsymbol{Y}$ (see Félix and Halperin [24, Section 12]).

The equivalence of homotopy categories $\operatorname{ho}\mathcal{S}_{\mathbb{Q}} \to \operatorname{ho}\mathsf{CDGA}$ is contravariant and takes products to coproducts and path or loop spaces to cone or suspension objects. Thus if we try to apply Theorem 2.33 to the model $(A^*, d) \in \mathsf{CDGA}$ directly, rather than to $\boldsymbol{Y} \in \mathcal{S}_{\mathbb{Q}}$, we will end up with a *simplicial* CDGA $W_\bullet$, obtained as the homotopy colimit of sequential simplicial realization (see Blanc, Johnson and Turner [14] for full details of the simplicial version). We could in fact replace this simplicial CDGA by a bigraded CDGA (see Félix [23], and compare Blanc [9]).

In particular, we have an adjunction $\Lambda\colon \mathsf{Ch}^*_{\mathbb{Q}} \rightleftharpoons \mathsf{CDGA} :U$ between cochain complexes and CDGAs, where U is the forgetful functor and $\Lambda(V^*, d)$ the free graded commutative algebra on the graded vector space V^*, with $d^i\colon V^i \to V^{i+1}$ extended as a derivation. Note that each $\Lambda(V^*, d)$ is a Sullivan algebra, and thus a cofibrant CDGA (see Hess [27, Section 1]).

The functor Λ yields formal minimal models for each $\mathbb{Q}$–GEM, as well as their cylinders, cones and suspensions. For example, if V^* is a graded $\mathbb{Q}$–vector space which is degreewise finite-dimensional, then $\Lambda(V^*, 0)$ is a minimal model for $\prod_{i=0}^{\infty} \boldsymbol{K}(V^i, i)$. Similarly, if $A^* := \Lambda(V^*, d)$ is a Sullivan model for some space $\boldsymbol{Y}$, and

$$i\colon (V^*, d) \hookrightarrow C(V^*, d)$$

is the inclusion into the cone (see Weibel [38, Section 1.5]), then $i_*\colon \Lambda(V^*, d) \to \Lambda(C(V^*, d))$, the corresponding cone inclusion in CDGA, is a CDGA model for the path fibration $p\colon P\boldsymbol{Y} \to \boldsymbol{Y}$ (see Félix and Halperin [24, Section 14]).

4.24 A rational example Even though the above discussion was stated in terms of the sketch Θ_R in $\mathrm{ho}\,\mathcal{C}$ (for $\mathcal{C} = \mathcal{S}_*$), when $R = \mathbb{Q}$, as we just pointed out, we can also apply it (mutatis mutandis) to the corresponding CDGA models.

For example, let $\boldsymbol{Y}$ be the simply connected, $\mathbb{Q}$–local, finite-type space represented by the free CDGA (A^*, d) with:

i. $A^n = \mathbb{Q}\langle x, y, z\rangle$ with $dx = dy = dz = 0$.

ii. $A^{2n-1} = \mathbb{Q}\langle u, v\rangle$ with $du = xy$ and $dv = xz$.

iii. $A^{3n-2} = \mathbb{Q}\langle q, r, s, t\rangle$ with $dq = xu$, $dr = xv$, $ds = yu$ and $dt = zv$.

iv. A^i for $i > 3n$ chosen so that $H^i(A^*) = 0$ for $i \geq 3n$.

Here $n > 1$ is odd.

Thus (A^*, d) has rational cohomology $\Gamma = H^*(\boldsymbol{Y}; Q)$ (as a $\Theta_{\mathbb{Q}}$–algebra — that is, a graded $\mathbb{Q}$–algebra) with:

i. $\Gamma^n = \mathbb{Q}\langle [x], [y], [z]\rangle$.

ii. $\Gamma^{2n} = \mathbb{Q}\langle [y]\cdot[z]\rangle$.

iii. $\Gamma^{3n-1} = \mathbb{Q}\langle \omega\rangle$, where ω is represented in A^* by $zu + yv$.

iv. $\Gamma^i = 0$ for $i \neq n, 2n, 3n-1$.

Note that the (formal) rational space $\boldsymbol{Z} := (\boldsymbol{S}^n \vee (\boldsymbol{S}^n \times \boldsymbol{S}^n) \vee \boldsymbol{S}^{3n-1})_{\mathbb{Q}}$ also has $H^*(\boldsymbol{Z}; Q) \cong \Gamma$ (as $\Theta_{\mathbb{Q}}$–algebras). It is represented by the Sullivan model (B^*, d) with:

i. $B^n = \mathbb{Q}\langle x, y, z\rangle$ with $dx = dy = dz = 0$.

ii. $B^{2n-1} = \mathbb{Q}\langle u, v\rangle$ with $du = xy$ and $dv = xz$.

iii. $B^{3n-2} = \mathbb{Q}\langle p, q, r, s, t\rangle$ with $dp = zu + yv$, $dq = xu$, $dr = xv$, $ds = yu$, and $dt = zv$.

iv. $B^{3n-1} = \mathbb{Q}\langle w\rangle$ with $dw = 0$, where ω is represented in B^* by w.

v. Again, B^i for $i > 3n$ chosen so that $H^i(B^*) = 0$ for $i \geq 3n$.

Let us denote by $\mathcal{F}_{\mathbb{Q}}\langle x_{n_1}, \dots, x_{n_k}\rangle$ the free $\Theta_{\mathbb{Q}}$–algebra generated by elements x_{n_i} in degree n_i for $i = 1, \dots, k$ — so that $\mathcal{F}_{\mathbb{Q}}\langle x_{n_1}, \dots, x_{n_k}\rangle \cong H^*\big(\prod_{i=1}^k \boldsymbol{K}(\mathbb{Q}, n_i); \mathbb{Q}\big)$ (cf Section 4.23).

We may choose a (minimal) CW resolution of $\Theta_{\mathbb{Q}}$–algebras $V_\bullet \to \Gamma$ with CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$ as follows:

(a) $\overline{V}_0 = \mathcal{F}_{\mathbb{Q}}\langle x_n, y_n, z_n, w_{3n-1}\rangle$, with the obvious augmentation $\varepsilon\colon \overline{V}_0 \to \Gamma$.

(b) $\overline{V}_1 = \mathcal{F}_{\mathbb{Q}}\langle u_{2n}, v_{2n}\rangle \amalg \overline{U}_1$, where $\overline{U}_1$ is a free $\Theta_{\mathbb{Q}}$–algebra with generators in degrees $> 3n$. The attaching map $\overline{\partial}_0\colon \overline{V}_1 \to V_0 = \overline{V}_0$ is defined by $u_{2n} \mapsto x_n y_n$ and $v_{2n} \mapsto x_n z_n$.

(c) $\overline{V}_2 = \mathcal{F}_{\mathbb{Q}}\langle p_{3n}, q_{3n}, r_{3n}, s_{3n}, t_{3n}\rangle \amalg \overline{U}_2$, where $\overline{U}_2$ is again a free $\Theta_{\mathbb{Q}}$–algebra with generators in degrees $> 3n$. The attaching map $\overline{\partial}_0\colon \overline{V}_2 \to V_1 = \overline{V}_1 \amalg s_0\overline{V}_0$ is defined by

$$\begin{gathered} p_{3n} \mapsto (s_0 z_n) u_{2n} + (s_0 y_n) v_{2n}, \quad q_{3n} \mapsto (s_0 x_n) u_{2n}, \\ r_{3n} \mapsto (s_0 x_n) v_{2n}, \quad s_{3n} \mapsto (s_0 y_n) u_{2n}, \quad t_{3n} \mapsto (s_0 z_n) v_{2n}. \end{gathered} \tag{4.25}$$

(d) For $k \geq 3$ the basis $\Theta_{\mathbb{Q}}$–algebra $\overline{V}_k$ has generators in degrees $> 3n$.

Denote by $\Lambda[\boldsymbol{x}_n] = \Lambda(M^*, 0)$ the formal free CDGA model for $\boldsymbol{K}(\mathbb{Q}, n)$ (where M^* is the graded vector space concentrated in degree n with basis $\{\boldsymbol{x}_n\}$; see Section 4.23).

We can realize $V_\bullet \to \Gamma$ through (co)simplicial dimension 2 and degree $3n$ by an augmented simplicial CDGA $W_\bullet^{[2]} \to B^*$ with CW basis $(\overline{W}_n)_{n\in\mathbb{N}}$ constructed as follows:

Step A First, we construct the simplicial analogue of $W_\bullet^{[1]}$ through simplicial dimension 1:

(a) We let

$$\overline{W}_0 = \Lambda[\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \boldsymbol{w}_{3n-1}] := \Lambda[\boldsymbol{x}_n] \amalg \Lambda[\boldsymbol{y}_n] \amalg \Lambda[\boldsymbol{z}_n] \amalg \Lambda[\boldsymbol{w}_{3n-1}]$$

with augmentation $\boldsymbol{\varepsilon}\colon \overline{W}_0 \to A^*$ defined by

$$\boldsymbol{x}_n \mapsto x, \quad \boldsymbol{y}_n \mapsto y, \quad \boldsymbol{z}_n \mapsto z, \quad \boldsymbol{w}_{3n-1} \mapsto zu + yv.$$

(b) We let

$$\widetilde{W}_1^{[1]} = \overline{W}_1 := \Lambda[\boldsymbol{u}_{2n}, \boldsymbol{v}_{2n}],$$

where the untruncated version of $\overline{W}_1$ has an additional free CDGA coproduct summand D_2 with generators in degrees $> 3n$. The attaching map $\overline{\partial}_0\colon \overline{W}_1 \to W_0^{[2]}$ is defined by $\boldsymbol{u}_{2n} \mapsto \boldsymbol{x}_n\boldsymbol{y}_n$, $\boldsymbol{v}_{2n} \mapsto \boldsymbol{x}_n\boldsymbol{z}_n$, and all other generators sent to 0.

(c) Dually to Step III in the proof of Theorem 2.33, we must add a coproduct summand $C\overline{W}_1$ to obtain $\widetilde{W}_0^{[1]} := \overline{W}_0 \amalg C\overline{W}_1$, where the cone on the formal CDGA $\overline{W}_1 = \Lambda[\boldsymbol{u}_{2n}, \boldsymbol{v}_{2n}]$, which models the path space $P\overline{\boldsymbol{W}}^1$, is the CDGA

$$C\overline{W}_1 = \Lambda(\mathbb{Q}(\boldsymbol{u}'_{2n}, \boldsymbol{v}'_{2n}, \overline{\boldsymbol{u}}_{2n-1}, \overline{\boldsymbol{v}}_{2n-1}), d)$$

with differential

$$d(\overline{\boldsymbol{u}}_{2n-1}) = -\boldsymbol{u}'_{2n} \quad \text{and} \quad d(\overline{\boldsymbol{v}}_{2n-1}) = -\boldsymbol{v}'_{2n}$$

(see Section 4.23). We will denote this simply by $\Lambda(\boldsymbol{u}'_{2n}, \boldsymbol{v}'_{2n}, \overline{\boldsymbol{u}}_{2n-1}, \overline{\boldsymbol{v}}_{2n-1})$.

The augmentation $\boldsymbol{\varepsilon}\colon C\overline{W}_1 \to A^*$ is given by

$$\boldsymbol{u}'_{2n} \mapsto xy, \quad \boldsymbol{v}'_{2n} \mapsto xz, \quad \overline{\boldsymbol{u}}_{2n-1} \mapsto u, \quad \overline{\boldsymbol{v}}_{2n-1} \mapsto v.$$

(d) Finally, we must add the degeneracies to $\widetilde{W}_\bullet^{[1]}$ (as in step (iii) of Section 2.5), that is, we add two coproduct summands

$$s_0\overline{W}_0 = \Lambda[s_0\boldsymbol{x}_n, s_0\boldsymbol{y}_n, s_0\boldsymbol{z}_n, s_0\boldsymbol{w}_{3n-1}],$$
$$s_0C\overline{W}_1 = \Lambda(s_0\boldsymbol{u}'_{2n}, s_0\boldsymbol{v}'_{2n}, s_0\overline{\boldsymbol{u}}_{2n-1}, s_0\overline{\boldsymbol{v}}_{2n-1}),$$

to obtain the 1–truncation of the augmented simplicial CDGA $W_\bullet^{[1]}$ (dual to (A.12)) in degrees $\leq 3n$ given by

$$\begin{tikzcd}
W_1^{[1]} \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & = & s_0\overline{W}_0 \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] & \amalg & \overline{W}_1 = \Lambda[\boldsymbol{u},\boldsymbol{v}] \arrow[dll, "\bar{\partial}_0"'] \arrow[d, "d_1=\iota"] & \amalg & s_0 C\overline{W}_1 \arrow[dll, shift right, "\mathrm{Id}"'] \arrow[dll, shift left, "\mathrm{Id}"] \\
W_0^{[1]} \arrow[d, "\varepsilon"] & = & \overline{W}_0 = \Lambda[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}] & \amalg & C\overline{W}_1 = \Lambda(\boldsymbol{u}',\boldsymbol{v}',\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) & & \\
A^* & = & \Lambda[x,y,z,u,v,q,r,s,t] & & & &
\end{tikzcd}$$

Step B To extend $W_\bullet^{[1]}$ to $W_\bullet^{[2]}$, we proceed as follows:

(a) First, we set
$$\widetilde{W}_2^{[2]} = \overline{W}_2 := \Lambda[\boldsymbol{p}_{3n},\boldsymbol{q}_{3n},\boldsymbol{r}_{3n},\boldsymbol{s}_{3n},\boldsymbol{t}_{3n}]$$
(where again we omit generators in degrees $> 3n$).

As a first approximation, we would like to use (4.25) to determine $\bar{\partial}_0^{\overline{W}_2}\colon \overline{W}_2 \to W_1^{[2]}$. However, although this will guarantee that $d_0 \circ \bar{\partial}_0^{\overline{W}_2} = 0$, we would then not have $d_1 \circ \bar{\partial}_0^{\overline{W}_2} = 0$ (see Step VII in the proof of Theorem 2.33).

We therefore define $\bar{\partial}_0^{\overline{W}_2}$ by

$$\textbf{(4.26)}\qquad \begin{gathered}
\boldsymbol{p}_{3n} \mapsto (s_0\boldsymbol{z}_n)\boldsymbol{u}_{2n} + (s_0\boldsymbol{y}_n)\boldsymbol{v}_{2n} - s_0(\boldsymbol{z}_n\boldsymbol{u}'_{2n}) - s_0(\boldsymbol{y}_n\boldsymbol{v}'_{2n}),\\
\boldsymbol{q}_{3n} \mapsto (s_0\boldsymbol{x}_n)\boldsymbol{u}_{2n} - s_0(\boldsymbol{x}_n\boldsymbol{u}'_{2n}), \quad \boldsymbol{r}_{3n} \mapsto (s_0\boldsymbol{x}_n)\boldsymbol{v}_{2n} - s_0(\boldsymbol{x}_n\boldsymbol{v}'_{2n}),\\
\boldsymbol{s}_{3n} \mapsto (s_0\boldsymbol{y}_n)\boldsymbol{u}_{2n} - s_0(\boldsymbol{y}_n\boldsymbol{u}'_{2n}), \quad \boldsymbol{t}_{3n} \mapsto (s_0\boldsymbol{z}_n)\boldsymbol{v}_{2n} - s_0(\boldsymbol{z}_n\boldsymbol{v}'_{2n}),
\end{gathered}$$

with $d_1\colon \overline{W}_2 \to \widetilde{W}_1^{[2]}$ the inclusion into the new cone summand
$$C\overline{W}_2 = \Lambda(\boldsymbol{p}'_{3n},\boldsymbol{q}'_{3n},\boldsymbol{r}'_{3n},\boldsymbol{s}'_{3n},\boldsymbol{t}'_{3n},\overline{\boldsymbol{p}}_{3n-1},\overline{\boldsymbol{q}}_{3n-1},\overline{\boldsymbol{r}}_{3n-1},\overline{\boldsymbol{s}}_{3n-1},\overline{\boldsymbol{t}}_{3n-1})$$
in $\widetilde{W}_1^{[2]} = W_1^{[1]} \amalg C\overline{W}_2$ (with $d(\overline{\boldsymbol{p}}_{3n-1}) = \boldsymbol{p}'_{3n}$, and so on).

The map $d_0 = F_1\colon C\overline{W}_2 \to W_0^{[1]}$ is given by

$$\textbf{(4.27)}\qquad \begin{aligned}
\boldsymbol{p}'_{3n} &\mapsto -\boldsymbol{z}_n\boldsymbol{u}'_{2n} - \boldsymbol{y}_n\boldsymbol{v}'_{2n}, & \overline{\boldsymbol{p}}_{3n-1} &\mapsto \boldsymbol{z}_n\overline{\boldsymbol{u}}_{2n-1} + \boldsymbol{y}_n\overline{\boldsymbol{v}}_{2n-1} - \boldsymbol{w}_{3n-1},\\
\boldsymbol{q}'_{3n} &\mapsto -\boldsymbol{x}_n\boldsymbol{u}'_{2n}, & \overline{\boldsymbol{q}}_{3n-1} &\mapsto \boldsymbol{x}_n\overline{\boldsymbol{u}}_{2n-1},\\
\boldsymbol{r}'_{3n} &\mapsto -\boldsymbol{x}_n\boldsymbol{v}'_{2n}, & \overline{\boldsymbol{r}}_{3n-1} &\mapsto \boldsymbol{x}_n\overline{\boldsymbol{v}}_{2n-1},\\
\boldsymbol{s}'_{3n} &\mapsto -\boldsymbol{y}_n\boldsymbol{u}'_{2n}, & \overline{\boldsymbol{s}}_{3n-1} &\mapsto \boldsymbol{y}_n\overline{\boldsymbol{u}}_{2n-1},\\
\boldsymbol{t}'_{3n} &\mapsto -\boldsymbol{z}_n\boldsymbol{v}'_{2n}, & \overline{\boldsymbol{t}}_{3n-1} &\mapsto \boldsymbol{z}_n\overline{\boldsymbol{v}}_{2n-1},
\end{aligned}$$

as in (4.25).

(b) In dimension 0 we have $\widetilde{W}_0^{[2]} = W_0^{[1]} \amalg C\Sigma\overline{W}_2$, where

$C\Sigma\overline{W}_2 = \Lambda(\overline{\boldsymbol{p}}'_{3n-1}, \overline{\boldsymbol{q}}'_{3n-1}, \overline{\boldsymbol{r}}'_{3n-1}, \overline{\boldsymbol{s}}'_{3n-1}, \overline{\boldsymbol{t}}'_{3n-1}, \overline{\overline{\boldsymbol{p}}}_{3n-2}, \overline{\overline{\boldsymbol{q}}}_{3n-2}, \overline{\overline{\boldsymbol{r}}}_{3n-2}, \overline{\overline{\boldsymbol{s}}}_{3n-2}, \overline{\overline{\boldsymbol{t}}}_{3n-2})$

(with $d(\overline{\overline{\boldsymbol{p}}}_{3n-2}) = \overline{\boldsymbol{p}}'_{3n-1}$, and so on).

The face map

$$d_1\colon \widetilde{W}_1^{[2]} \to \widetilde{W}_0^{[2]}$$

is defined on the new summand $C\overline{W}_2$ to be the quotient $C\overline{W}_2 \twoheadrightarrow \Sigma\overline{W}_2$ followed by the inclusion $\Sigma\overline{W}_2 \hookrightarrow C\Sigma\overline{W}_2$, which is given by $\boldsymbol{p}_{3n} \mapsto 0$, $\overline{\boldsymbol{p}}_{3n-1} \mapsto -\overline{\boldsymbol{p}}'_{3n-1}$, and so on.

The augmentation $\boldsymbol{\varepsilon}\colon C\Sigma\overline{W}_2 \to A^*$ is given by

$$\textbf{(4.28)} \qquad \begin{array}{ll} \overline{\boldsymbol{p}}'_{3n-1} \mapsto 0, & \overline{\overline{\boldsymbol{p}}}_{3n-2} \mapsto 0, \\ \overline{\boldsymbol{q}}'_{3n-1} \mapsto x_n u_{2n-1}, & \overline{\overline{\boldsymbol{q}}}_{3n-2} \mapsto -q_{3n-2}, \\ \overline{\boldsymbol{r}}'_{3n-1} \mapsto x_n v_{2n-1}, & \overline{\overline{\boldsymbol{r}}}_{3n-2} \mapsto -r_{3n-2}, \\ \overline{\boldsymbol{s}}'_{3n-1} \mapsto y_n u_{2n-1}, & \overline{\overline{\boldsymbol{s}}}_{3n-2} \mapsto -s_{3n-2}, \\ \overline{\boldsymbol{t}}'_{3n-1} \mapsto z_n v_{2n-1}, & \overline{\overline{\boldsymbol{t}}}_{3n-2} \mapsto -t_{3n-2}. \end{array}$$

The 2–truncation of $\widetilde{W}_\bullet^{[2]}$ in degrees $\leq 3n$ is given by

$$\begin{tikzcd}
\overline{W}_2 = \Lambda[\boldsymbol{p},\boldsymbol{q},\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}] \arrow[d, "\bar{\partial}_0"] \arrow[drr, "\bar{\partial}_0"] \arrow[drrrr, bend left, "\bar{\partial}_0"] \arrow[drrrrrr, bend left, "d_1"] & & & & & & \\
s_0\overline{W}_0 \arrow[d, shift left, "="] \arrow[d, shift right, "="'] & \amalg & \overline{W}_1 = \Lambda[\boldsymbol{u},\boldsymbol{v}] \arrow[dll, "\bar{\partial}_0"'] \arrow[d, "d_1"] & \amalg & s_0C\overline{W}_1 \arrow[dll, shift left, "="'] \arrow[dll, shift right, "="] & \amalg & C\overline{W}_2 \arrow[dllll, "d_0"] \arrow[dll, "d_1"] \\
\overline{W}_0 = \Lambda[\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{w}] \arrow[d, "\boldsymbol{\varepsilon}"] & \amalg & C\overline{W}_1 = \Lambda(\boldsymbol{u}',\boldsymbol{v}',\overline{\boldsymbol{u}},\overline{\boldsymbol{v}}) \arrow[dll, "\boldsymbol{\varepsilon}"'] & \amalg & C\Sigma\overline{W}_2 & & \\
A^* = \Lambda[x,y,z,u,v,q,r,s,t] & & & & & &
\end{tikzcd}$$

Step C When we try to map $W_\bullet^{[2]}$ to the CDGA B^*, we must modify the augmentation $\boldsymbol{\varepsilon}\colon \widetilde{W}_0^{[2]} \to A^*$ as follows:

In order to realize $\varepsilon\colon V_0 \to \Gamma$ we must have $\boldsymbol{\varepsilon}(\boldsymbol{w}_{3n-1}) = w$ and otherwise $\boldsymbol{\varepsilon}$ is the same as in Step A. Therefore,

$$\boldsymbol{\varepsilon}(\overline{\boldsymbol{p}}'_{3n-1}) = \boldsymbol{\varepsilon}(d_1(\overline{\boldsymbol{p}}_{3n-1})) = \boldsymbol{\varepsilon}(\boldsymbol{z}_n\overline{\boldsymbol{u}}_{2n-1} + \boldsymbol{y}_n\overline{\boldsymbol{v}}_{2n-1} - \boldsymbol{w}_{3n-1}) = zu + yv - w,$$

by (4.27). Therefore, we must map $\overline{\overline{\boldsymbol{p}}}_{3n-2}$ to an element a in B^{3n-2} with $d(a) = zu + yv - w$. Since $d(p) = zu + yv$ but w represents a nonzero element in $\Gamma = H^{3n-1}(\boldsymbol{Z};\mathbb{Q})$, no such a exists. Thus the two rational spaces $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are not weakly equivalent by Corollary 4.19.

Intuitively, the element $\omega \in \Gamma^{3n-1}$ is represented by the Massey product $[zu + yv] = \langle [y], [x], [z] \rangle$ in $H^{3n-1}(\boldsymbol{Y};\mathbb{Q})$, while ω is not a Massey product in $H^{3n-1}(\boldsymbol{Z};\mathbb{Q})$ since $\boldsymbol{Z}$ is formal. In our language these facts are represented by the nullhomotopies $\overline{\boldsymbol{p}}_{3n-1} \mapsto \boldsymbol{z}_n\overline{\boldsymbol{u}}_{2n-1} + \boldsymbol{y}_n\overline{\boldsymbol{v}}_{2n-1} - \boldsymbol{w}_{3n-1}$ in $\boldsymbol{Y} \to \boldsymbol{W}^{\bullet}$, where in the analogous construction for $\boldsymbol{Z}$ we would have had $\overline{\boldsymbol{p}}_{3n-1} \mapsto \boldsymbol{z}_n\overline{\boldsymbol{u}}_{2n-1} + \boldsymbol{y}_n\overline{\boldsymbol{v}}_{2n-1}$.

5 Higher cohomology invariants for maps

The system of higher cohomology operations associated to a Θ_R–algebra $H^*(\boldsymbol{Y};R)$ described in the previous section may be thought of as a sequence of obstructions to realizing an algebraic isomorphism $\vartheta\colon H^*(\boldsymbol{Y};R) \xrightarrow{\cong} H^*(\boldsymbol{Z};R)$ by a map $f\colon \boldsymbol{Z} \to \boldsymbol{Y}$ (necessarily an R–equivalence) — as well as constituting a complete collection of higher invariants for the weak R–homotopy type of spaces.

In this section we address the analogous problem for arbitrary maps: given two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ which induce the same morphism of Θ_R–algebras $\psi\colon H^*(\boldsymbol{Y};R) \to H^*(\boldsymbol{Z};R)$, we define a sequence of higher cohomology operations which vanish coherently if and only if f_0 and f_1 are R–equivalent.

5.1 Obstructions for lifting homotopies We start with the *initial data* $(\mathcal{W}, f_0, f_1)$, consisting of a sequential realization $\mathcal{W}$ for $\boldsymbol{Y}$, and two maps f_0 and f_1 as above. Since f_0 and f_1 induce the same map of Θ_R–algebras, there is a homotopy $H^0_{[0]}\colon \boldsymbol{Z} \otimes I \to \overline{\boldsymbol{W}}^0 = \boldsymbol{W}^0_{[0]}$ between $\boldsymbol{\varepsilon}_{[0]} \circ f_0$ and $\boldsymbol{\varepsilon}_{[0]} \circ f_1$, so

$$\boldsymbol{\varepsilon}_{[0]} \circ (f_0 \perp f_1) = H^0_{[0]} \circ (i_0 \perp i_1). \tag{5.2}$$

We call $\mathcal{H}_{[0]} = (H^0_{[0]})$ a 0–*strand* for $(\mathcal{W}, f_0, f_1)$.

Recall that the standard cosimplicial space $\boldsymbol{\Delta}^{\bullet}$ is given by the diagram of n–simplices with the standard maps between them, where $\eta^i\colon \boldsymbol{\Delta}[k-1] \hookrightarrow \boldsymbol{\Delta}[k]$ is the inclusion of the i^{th} face, and $\sigma^j\colon \boldsymbol{\Delta}[k] \twoheadrightarrow \boldsymbol{\Delta}[k-1]$ is the j^{th} collapse map (see Bousfield and Kan [20, Section X.2.2]). Applying the simplicial structure operation $- \otimes \boldsymbol{\Delta}^{\bullet}$ to a fixed space $\boldsymbol{Z}$ yields a cosimplicial space $\boldsymbol{Z} \otimes \boldsymbol{\Delta}^{\bullet}$.

5.3 Definition An *n–strand* $\mathcal{H}_{[n]} = (H_{[0]}, \dots, H_{[m]}, \dots, H_{[n]})$ for $(\mathcal{W}, f_0, f_1)$ is a compatible sequence of maps of cosimplicial spaces $H_{[m]}\colon \boldsymbol{Z} \otimes I \otimes \boldsymbol{\Delta}^\bullet \to \boldsymbol{W}^\bullet_{[m]}$ for $m = 0, \dots, n$, each determined by the collection of k–homotopies $H^k_{[m]}\colon \boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[k]) \to \boldsymbol{W}^k_{[m]}$ for $0 \le k \le m$ such that all the downward and upward squares in the diagram

(5.4)
$$
\begin{tikzcd}
\boldsymbol{Z} \amalg \boldsymbol{Z} \arrow[r, "f_0 \perp f_1"] \arrow[d, "i_0 \perp i_1"] & \boldsymbol{Y} \arrow[d, "\boldsymbol{\varepsilon}_{[m]}"] \\
\boldsymbol{Z} \otimes I \arrow[r, "H^0_{[m]}"] \arrow[d, dotted, no head] & \boldsymbol{W}^0_{[m]} \arrow[d, dotted, no head] \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[k-1]) \arrow[r, "H^{k-1}_{[m]}"] \arrow[d, "\eta^i_*"'] & \boldsymbol{W}^{k-1}_{[m]} \arrow[d, "d^i"'] \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[k]) \arrow[r, "H^k_{[m]}"] \arrow[u, bend right, "\sigma^j_*"'] & \boldsymbol{W}^k_{[m]} \arrow[u, bend right, "s^j"']
\end{tikzcd}
$$

commute for all choices of $0 \le i \le k \le m$ and $0 \le j \le k-1$.

More precisely, the choices of $H^k_{[m]}$ for $0 \le k \le m$ uniquely determine a map of cosimplicial spaces $\widehat{H}_{[m]}\colon \boldsymbol{Z} \otimes I \otimes \boldsymbol{\Delta}^\bullet \to \widehat{\boldsymbol{W}}^\bullet_{[m]}$, since the target is m–coskeletal. We then use the left lifting property for the solid commuting square of cosimplicial spaces

(5.5)
$$
\begin{tikzcd}
* \arrow[r] \arrow[d, hook] & \boldsymbol{W}^\bullet_{[m]} \arrow[d, two heads, "h_{[m]}", "\simeq"'] \\
\boldsymbol{Z} \otimes I \otimes \boldsymbol{\Delta}^\bullet \arrow[r, "\widehat{H}_{[m]}"'] \arrow[ur, dotted, "H_{[m]}"] & \widehat{\boldsymbol{W}}^\bullet_{[m]}
\end{tikzcd}
$$

to obtain the required map $H_{[m]}$, unique up to weak equivalence, using the fact that $\boldsymbol{Z} \otimes I \otimes \boldsymbol{\Delta}^\bullet$ is cofibrant and the map $h_{[m]}\colon \boldsymbol{W}^\bullet_{[m]} \to \widehat{\boldsymbol{W}}^\bullet_{[m]}$ is a trivial fibration by Remark 2.18.

We say that an n–strand $\mathcal{H}_{[n]}$ *extends* a given $(n-1)$–strand $\mathcal{H}_{[n-1]}$ if

(5.6)
$$H_{[n-1]} = \pi_{[n]} \circ H_{[n]}\colon \boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}^\bullet) \to \boldsymbol{W}^\bullet_{[n-1]}$$

(cf (2.2)).

An *∞–strand* is a sequence $\mathcal{H}_{[\infty]} := (H_{[n]})_{n=1}^\infty$ satisfying (5.6) for each $n > 0$.

5.7 Remark In order to extend a given $(n-1)$–strand $\mathcal{H}_{[n-1]}$ for a sequential realization $\mathcal{W}$ of $\boldsymbol{Y}$ to an n–strand, we need to produce maps $\widehat{H}^k_{[n]}\colon \boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[k]) \to \overline{P\Omega^{n-k-1}\boldsymbol{W}^n}$ for $0 \le k \le n$ satisfying

$$\text{(5.8)} \qquad \begin{cases} \widehat{H}^k_{[n]} \circ \eta^0_* = F^{k-1} \circ v^{k-1} \circ H^{k-1}_{[n-1]}, \\ \widehat{H}^k_{[n]} \circ \eta^1_* = \delta^{k-2} \circ \widehat{H}^{k-1}_{[n]}, \\ \widehat{H}^k_{[n]} \circ \eta^i_* = 0 \quad \text{for } i \ge 2, \end{cases}$$

where $\delta^j = \delta^j_{\boldsymbol{D}}$ is the differential of $\boldsymbol{D}^*$ as in (2.8).

Thus, in the following diagram we are given the solid $(n-1)$–strand for $\boldsymbol{W}^{\bullet}_{[n-1]}$, which we wish to extend by the dashed maps to the (given) restricted cosimplicial space $\widetilde{\boldsymbol{W}}^{\bullet}_{[n]}$:

$$\text{(5.9)} \qquad \begin{tikzcd}
\boldsymbol{Z} \amalg \boldsymbol{Z} \arrow[rrr, "(f_0 \perp f_1)"] \arrow[d, "(i_0 \perp i_1)"] & & & \boldsymbol{Y} \arrow[d, "\boldsymbol{\varepsilon}"] \arrow[dll, "F^{-1}"'] & \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[0]) \arrow[r, dashed, "\widehat{H}^0_{[n]}"] \arrow[rrr, bend left, "H^0_{[n-1]}"] \arrow[d, shift right, "\eta^0_*"'] \arrow[d, shift left, "\eta^1_*"] & \overline{P\Omega^{n-1}\boldsymbol{W}^n} \arrow[d, "\delta^{-1}F^0 \circ v^0"] & \times & \boldsymbol{W}^0_{[n-1]} \arrow[d, shift right, "d^0"'] \arrow[d, shift left, "d^1"] \arrow[dll] & = \widetilde{\boldsymbol{W}}^0_{[n]} \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[1]) \arrow[r, dashed, "\widehat{H}^1_{[n]}"] \arrow[rrr, bend left, "H^1_{[n-1]}"] \arrow[d, shift right=2, "\eta^0_*"'] \arrow[d, "\eta^1_*"] \arrow[d, shift left=2, "\eta^2_*"] & \overline{P\Omega^{n-2}\boldsymbol{W}^n} \arrow[d, "\delta^0 F^1 \circ v^1"] & \times & \boldsymbol{W}^1_{[n-1]} \arrow[d, shift right=2, "d^0"'] \arrow[d, "d^1"] \arrow[d, shift left=2, "d^2"] \arrow[dll] & = \widetilde{\boldsymbol{W}}^1_{[n]} \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[2]) \arrow[r, dashed, "\widehat{H}^2_{[n]}"] \arrow[d, dotted] & \overline{P\Omega^{n-3}\boldsymbol{W}^n} \arrow[d, dotted] & \times & \boldsymbol{W}^2_{[n-1]} \arrow[d, dotted] & = \widetilde{\boldsymbol{W}}^2_{[n]} \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[n-1]) \arrow[r, dashed, "\widehat{H}^{n-1}_{[n]}"] \arrow[d, shift right=2, "\eta^0_*"'] \arrow[d, phantom, "\cdots"] \arrow[d, shift left=2, "\eta^n_*"] & \overline{P\Omega^0\boldsymbol{W}^n} \arrow[d, "p"', "\delta^{n-2}"] & \times & \boldsymbol{W}^{n-1}_{[n-1]} \arrow[dll, "\bar{d}^0_n"] & = \widetilde{\boldsymbol{W}}^{n-1}_{[n]} \\
\boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[n]) \arrow[r, dashed, "\widehat{H}^n_{[n]}"] & \overline{\boldsymbol{W}}^n & & &
\end{tikzcd}$$

5.10 Folding polytopes Consider the iterated trivial fibration of (2.9),

$$\text{(5.11)} \qquad \xi^j := P\Omega^{j-1}\sigma^1 \circ \cdots \circ P\Omega\sigma^{j-1} \circ P\sigma^j \circ P\tau^j\colon \overline{P\Omega^j\boldsymbol{W}^n} \twoheadrightarrow P\Omega^j\overline{\boldsymbol{W}}^n.$$

If we identify the k–simplex $\boldsymbol{\Delta}[k]$ with (a quotient of) the k–cube I^k, each map

$$\widehat{H}^k_{[n]}\colon \boldsymbol{Z} \otimes (I \times \boldsymbol{\Delta}[k]) \to \overline{P\Omega^{n-k-1}\boldsymbol{W}^n},$$

after postcomposing with $\xi^{n-k-1}\colon \overline{P\Omega^{n-k-1}\boldsymbol{W}^n} \to P\Omega^{n-k-1}\overline{\boldsymbol{W}}^n$, can be identified by adjunction with a pointed map $\widetilde{H}^k\colon (\boldsymbol{Z}\otimes I)\otimes I^n \to \overline{\boldsymbol{W}}^n$ taking certain facets of I^n to the basepoint.

Moreover, the compatibility conditions of (5.8) translate into requirements that the restrictions of the maps $\widetilde{H}^k$ to certain facets of I^n match up in an appropriate way. This information can be encoded by gluing together $n+1$ n–cubes (corresponding to cosimplicial dimensions $0,1,\dots,n$) along their facets to obtain a single n–dimensional cubical complex, as follows:

5.12 Definition The barycentric subdivision, as a triangulation of the standard n–simplex $\boldsymbol{\Delta}[n]$, exhibits it as a PL-cone on its boundary $\partial\boldsymbol{\Delta}[n]$. We may similarly define by induction a triangulation of the standard n–cube $I^n=[0,1]^n$ obtained as the cone on its boundary ∂I^n (more precisely, the join of the barycenter of I^n with the inductively defined triangulation of ∂I^n).

This allows us to define PL-homeomorphisms $\zeta^n\colon I^n \to \boldsymbol{\Delta}[n]$, starting with the obvious isomorphism for $n=1$, taking boundary to boundary, and extending to the interior by applying the join with the respective barycenters.

For each $n\geq 2$, we consider the corner C of ∂I^n consisting of all $(n-1)$–facets $(E_k)_{k=0}^{n-1}$ incident with the fixed vertex $v=(0,\dots,0)$, where

$$E_k=\{(t_1,\dots,t_n)\in I^n : t_{k+1}=0\}.$$

We use the previously defined ζ^{n-1} to identify $E_k\cong I^{n-1}$ with the k^{th} face $\boldsymbol{\Delta}_k[n]$ of $\boldsymbol{\Delta}[n]$. For the complementary corner C' (incident with the vertex $v'=(1,\dots,1)$ opposite v), we use the orthogonal projection from the last vertex of $\boldsymbol{\Delta}[n]$ onto the face $\boldsymbol{\Delta}_n[n]$ opposite it to obtain a subdivision of $\boldsymbol{\Delta}_n[n]$ into n $(n-1)$–dimensional simplices, which we identify with the n $(n-1)$–dimensional facets of $\mathbb{C}'$ using $(\zeta^{n-1})^{-1}$.

Now, for each $n\geq 1$ consider $n+1$ standard n–cubes $I^n_{(0)},\dots,I^n_{(n)}$, where we have a PL-isomorphism ζ^n_k given as follows:

$$I^n_{(k)}\cong I^k\times I^{n-k}\xrightarrow{\zeta^k\times\mathrm{Id}}\boldsymbol{\Delta}[k]\times I^{n-k}.$$

For $k<n$ we think of the first direction of I^{n-k} as the *path* direction, and the remaining $n-k-1$ directions as *loop* directions. This allows us to represent a pointed map $h\colon \widehat{\boldsymbol{Z}}\otimes\boldsymbol{\Delta}[k]\to P\Omega^{n-k-1}\overline{\boldsymbol{W}}$ by a map $\widehat{h}\colon \widehat{\boldsymbol{Z}}\otimes I^n_{(k)}\to\overline{\boldsymbol{W}}$ in a canonical way (sending certain facets of $I^n_{(k)}$ to the basepoint).

For any $1 \leq k \leq n$ we have two $(k-1)$–faces $\mathbf{\Delta}_0[k]$ and $\mathbf{\Delta}_1[k]$ of $\mathbf{\Delta}[k]$, and the $(n-1)$–dimensional prisms $\mathbf{\Delta}_0[k] \times I^{n-k}$ and $\mathbf{\Delta}_1[k] \times I^{n-k}$ are identified under the map $(\zeta^n_k)^{-1}$ with two $(n-1)$–dimensional facets of $I^n_{(k)}$, which we denote by B^k_0 and B^k_1, respectively. By our convention, if $0 \leq k < n$ we have another special $(n-1)$–dimensional facet of $I^n_{(k)}$, denoted by

$$C^k := \{(t_1, \dots, t_n) \in I^k \times I^{n-k} \mid t_{k+1} = 0\}$$

(the zero-face in the "path direction").

We now define the n^{th} *folding polytope* $\mathcal{P}^n$ for each $n \geq 2$ by taking the disjoint union of the $n+1$ n–cubes $I^n_{(0)}, \dots, I^n_{(n)}$, and identifying B^k_1 with C^{k-1} for each $1 \leq k \leq n$.

5.13 Lemma *For each $n \geq 2$, the folding polytope $\mathcal{P}^n$ is homeomorphic to an n–ball, with boundary $\partial\mathcal{P}^n$ homeomorphic to an $(n-1)$–sphere.*

5.14 Remark All the faces $(B^k_1)^n_{k=1}$ and $(C^k)^{n-1}_{k=0}$ are now interior to $\mathcal{P}^n$, while the remaining facets of the cubes $I^n_{(k)}$, including $(B^k_0)^n_{k=1}$, constitute the boundary $\partial\mathcal{P}^n$.

5.15 Example The four constituent 3–cubes of $\mathcal{P}^3$ are illustrated in Figure 1, with the dotted arrows indicating glued faces. Note that the two faces B^k_0 and B^k_1 are adjacent for $2 \leq k \leq n$, while B^1_0 and B^1_1 are opposite each other (since the same is true of $\mathbf{\Delta}_0[k]$ and $\mathbf{\Delta}_1[k]$ in $\mathbf{\Delta}[k]$). On the other hand, C^k is always adjacent to both B^k_0 and B^k_1.

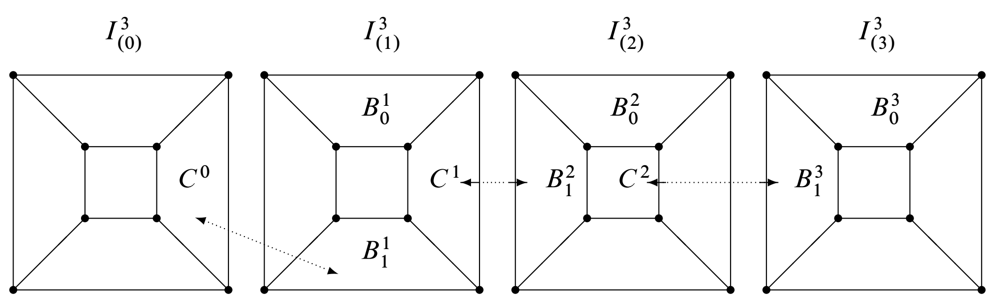

Figure 1: The four 3–cubes of $\mathcal{P}^3$

5.16 Lemma *Given two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ which induce the same algebraic homomorphism of Θ_R–algebras $H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$ and a sequential realization*

$\mathcal{W}$ for $\boldsymbol{Y}$, let $H_{[n-1]}\colon \boldsymbol{Z}\otimes I\otimes\boldsymbol{\Delta}^\bullet\to \boldsymbol{W}^\bullet_{[n-1]}$ be an $(n-1)$–strand as in Section 5.1. Then there is a one-to-one correspondence between collections of maps

$$\widehat{H}^k_{[n]}\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to \overline{P\Omega^{n-k-1}\boldsymbol{W}^n}\quad \text{for } 0\le k\le n$$

as in Remark 5.7 (satisfying (5.8)) *and maps $h\colon (\boldsymbol{Z}\otimes I)\otimes\mathcal{P}^n\to\overline{\boldsymbol{W}}^n$ such that*

(5.17) $$h|_{(\boldsymbol{Z}\otimes I)\otimes B^k_0}=\xi^{n-k-1}\circ F^{k-1}\circ v^{k-1}\circ H^{k-1}_{[n-1]}$$
$$\text{for } 1\le k\le n \text{ and } h|_{(\boldsymbol{Z}\otimes I)\otimes E}=*,$$

where $E:=\partial\mathcal{P}^n\setminus\bigcup_{k=1}^n B^k_0$.

Proof Given a map $\widehat{H}^k_{[n]}\colon \boldsymbol{Z}\otimes I\otimes\boldsymbol{\Delta}[k]\to\overline{P\Omega^{n-k-1}\boldsymbol{W}^n}$, we obtain a map

$$\widetilde{H}^k_{(n)}\colon (\boldsymbol{Z}\otimes I)\otimes I^n\to\overline{\boldsymbol{W}}^n,$$

where we identify I^k with $\boldsymbol{\Delta}[k]$ using ζ^k and take the $(k+1)^{\text{st}}$ coordinate for the path direction and the remaining $n-k-1$ coordinates for the loop directions, as in Definition 5.12.

The first condition in (5.8) says that on B^k_0 (corresponding to the 0–face of $\boldsymbol{\Delta}[k]$), $\widehat{H}^k_{[n]}$ equals $F^{k-1}\circ v^{k-1}\circ H^{k-1}_{[n-1]}$. The second condition there says that on B^k_1 (corresponding to the 1–face of $\boldsymbol{\Delta}[k]$), $\widehat{H}^k_{[n]}$ equals $\delta^{n-k}\circ\widehat{H}^{k-1}_{[n]}$ (where δ^{n-k} is defined in (2.8)), which stated in terms of cubes means that it coincides with $\widehat{H}^{k-1}_{[n-1]}$ restricted to C^{k-1}. Since the coface maps d^i into $P\Omega^{n-k-1}\overline{\boldsymbol{W}}^n$ vanish for $i\ge 2$, and $\widehat{H}^{k-1}_{[n]}$ also vanishes at the other end of the path direction, and at both ends of the loop directions, we obtain the description above.

Conversely, given such a map $\widetilde{H}$, we use its restrictions to the n–cubes $I^0_{(n)},\dots,I^n_{(n)}$ to define the maps $\widetilde{H}^k$, and thus maps ${}'\widehat{H}^k_{[n]}\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to P\Omega^{n-k-1}\overline{\boldsymbol{W}}^n$ satisfying

(5.18) $$\begin{cases} {}'\widehat{H}^k_{[n]}\circ\eta^0_*=\xi^{n-k-1}\circ F^{k-1}\circ v^{k-1}\circ H^{k-1}_{[n-1]},\\ {}'\widehat{H}^k_{[n]}\circ\eta^1_*=\xi^{n-k-1}\circ\delta^{n-k}\circ{}'\widehat{H}^{k-1}_{[n]},\\ {}'\widehat{H}^k_{[n]}\circ\eta^i_*=0\quad\text{for } i\ge 2,\end{cases}$$

for ξ^j as in (5.11). We now show by induction on $0\le k$ that these lift to maps $\widehat{H}^k_{[n]}\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to\overline{P\Omega^{n-k-1}\boldsymbol{W}^n}$ satisfying (5.8), and

(5.19) $${}'\widehat{H}^k_{[n]}=\xi^{n-k-1}\circ\widehat{H}^k_{[n]}.$$

Indeed, the inductively defined lift $\widehat{H}^{k-1}_{[n]}$ induces a map $L\colon \boldsymbol{Z}\otimes(I\times\partial\boldsymbol{\Delta}[k])\to\overline{P\Omega^{n-k-1}\boldsymbol{W}^n}$ fitting into the solid commutative diagram

(5.20)
$$
\begin{tikzcd}
\boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k-1]) \arrow[d, hook, "\eta^0_*"'] \arrow[rrr, bend left, "H^{k-1}_{[n-1]}"] & \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k-1]) \arrow[dl, hook', "\eta^1_*"'] \arrow[dr, "\widehat{H}^{k-1}_{[n]}"] & & \boldsymbol{W}^{k-1}_{[n-1]} \arrow[d, "v^{k-1}"] \\
 & \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k-1]) \arrow[dl, hook', "\eta^i_*\ (i\geq 2)"] \arrow[drr, "0"] & \overline{P\Omega^{n-k}\boldsymbol{W}^n} \arrow[dr, "\delta^{n-k}"] & C^{k-1}\boldsymbol{W}^{\bullet}_{[n-1]} \arrow[d, "F^{k-1}"] \\
\boldsymbol{Z}\otimes(I\times\partial\boldsymbol{\Delta}[k]) \arrow[d, hook, "\mathrm{inc}_*"'] \arrow[rrr, "L"] & & & \overline{P\Omega^{n-k-1}\boldsymbol{W}^n} \arrow[d, two heads, "\xi^{n-k-1}", "\simeq"'] \\
\boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k]) \arrow[urrr, dotted, "\widehat{H}^k_{[n]}"] \arrow[rrr, "{}'\widehat{H}^k_{[n]}"'] & & & P\Omega^{n-k-1}\overline{\boldsymbol{W}}^n
\end{tikzcd}
$$

where the upper squares fit together to define L by induction, using (5.8), and the bottom solid square then commutes by (5.18) and (5.19).

Since $\boldsymbol{Z}$ is cofibrant in $\mathcal{C}$ by Assumption 1.20, the map inc_* is a cofibration; see [32, Section II.2]. Moreover ξ^{n-k-1} is a trivial fibration, so we have the lifting $\widehat{H}^k_{[n]}$ by the LLP. The fact that (5.20) commutes implies that (5.8) holds for k, too. To start the induction for $k=0$, we just need the fact that ξ^{n-1} is a trivial fibration and $Z\otimes I$ is cofibrant with $L=0$, since (5.8) is then vacuous. $\square$

5.21 Definition Assume given initial data $(\mathcal{W}, f_0, f_1\colon \boldsymbol{Z}\to\boldsymbol{Y})$ with a corresponding $(n-1)$–strand $\mathcal{H}_{[n-1]}=(H_{[m]})_{m=0}^{n-1}$, as in Section 5.1. We associate to this a map $g\colon(\boldsymbol{Z}\otimes I)\otimes\partial\mathcal{P}^n\to\overline{\boldsymbol{W}}^n$ which sends $(\boldsymbol{Z}\otimes I)\otimes B^k_0$ to $\overline{\boldsymbol{W}}^n$ by $F^k\circ H^k_{[n]}$ for each $1\leq k\leq n$, and all other $(n-1)$–cubes of $\partial\mathcal{P}^n$ to the basepoint. Here we use the convention of (2.7), so $F^n=\bar{d}^0_n$.

Since at most two additional $(n-1)$–facets of $I^n_{(k)}$ are identified with $(n-1)$–facets of $I^n_{(k\pm1)}$, we may think of $P\Omega^{n-k}\overline{\boldsymbol{W}}^n$ as contained in $\mathrm{map}_*(d^0_1 I^n_{(k)}, \overline{\boldsymbol{W}}^n)$, so the map induced by $H^{k-1}_{[n]}\circ F^k$ is well-defined. Moreover, these maps are compatible for adjacent values of k by (5.8).

By Lemma 5.13 we can think of g as a map $(\boldsymbol{Z}\otimes I)\otimes\boldsymbol{S}^{n-1}\to\overline{\boldsymbol{W}}^n$, and because all maps are pointed, this actually factors through the half-smash

$$(\boldsymbol{Z}\otimes I)\ltimes\boldsymbol{S}^{n-1} := ((\boldsymbol{Z}\otimes I)\times\boldsymbol{S}^{n-1})/(*\times\boldsymbol{S}^{n-1}),$$

which can be canonically identified with $\Sigma^{n-1}(\boldsymbol{Z}\otimes I)\vee(\boldsymbol{Z}\otimes I)$ (see [4]). Moreover, the map $(\boldsymbol{Z}\otimes I)\to\overline{\boldsymbol{W}}^n$ in question is nullhomotopic for $n\geq 1$, so we may restrict attention to the factor $g'\colon \Sigma^{n-1}(\boldsymbol{Z}\otimes I)\to\overline{\boldsymbol{W}}^n$, and define the *value* of the $(n-1)$–strand $\mathcal{H}_{[n-1]}$ to be the class

(5.22) $$\mathrm{Val}(\mathcal{H}_{[n-1]}) := [g'] \in [\Sigma^{n-1}(\boldsymbol{Z}\otimes I),\overline{\boldsymbol{W}}^n] \cong [\boldsymbol{Z},\Omega^{n-1}\overline{\boldsymbol{W}}^n],$$

so it consists of a set of cohomology classes for $\boldsymbol{Z}$.

5.23 Proposition *Under the assumptions of Lemma 5.16,* $\mathrm{Val}(\mathcal{H}_{[n-1]})=0$ *if and only if* $\mathcal{H}_{[n-1]}$ *extends to an* n*–strand* $\mathcal{H}_{[n]}$*.*

Proof The $(n-1)$–strand $\mathcal{H}_{[n-1]}$ extends to an n–strand $\mathcal{H}_{[n]}$ if and only if we have a collection of maps

$$\widehat{H}^k_{[n]}\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to\overline{P\Omega^{n-k-1}\boldsymbol{W}^n}$$

for $0\leq k\leq n$ as in Remark 5.7 satisfying (5.8), and by Lemma 5.16 this corresponds to a map $(\boldsymbol{Z}\otimes I)\otimes\mathcal{P}^n\to\overline{\boldsymbol{W}}^n$ whose restriction to $(\boldsymbol{Z}\otimes I)\otimes\partial\mathcal{P}^n$ is the map g determined by $\mathcal{H}_{[n-1]}$ as in Definition 5.21. The map g extends to $(\boldsymbol{Z}\otimes I)\otimes\mathcal{P}^n$ if and only if g' is nullhomotopic. □

5.24 Correspondence of strands for maps Given $f_0,f_1\colon \boldsymbol{Z}\to\boldsymbol{Y}$ with $f_0^*=f_1^*\colon H^*(\boldsymbol{Y};R)\to H^*(\boldsymbol{Z};R)$, an n–stage comparison map $\Phi\colon\mathcal{W}\to{}'\mathcal{W}$ between two sequential realizations for $\boldsymbol{Y}$ as in (3.16), and n–strands $\mathcal{H}_{[n]}$ and ${}'\mathcal{H}_{[n]}$ for $\mathcal{W}$ and ${}'\mathcal{W}$, respectively, we write

$${}'\mathcal{H}_{[n]} = r^{\#}(\mathcal{H}_{[n]})$$

if $(H^k_{[m]})' = r^k_{[m]}\circ H^k_{[m]}\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to{}'\boldsymbol{W}^k_{[m]}$ and

$$\mathcal{H}_{[n]} = e^{\#}({}'\mathcal{H}_{[n]})$$

if $H^k_{[m]} = e^k_{[m]}\circ(H^k_{[m]})'\colon \boldsymbol{Z}\otimes(I\times\boldsymbol{\Delta}[k])\to\boldsymbol{W}^k_{[m]}$ for each $0\leq k\leq m\leq n$ (compare Section 4.6).

By comparing (3.13) and (3.15) with (5.9) and Definition 5.21, we see that

(5.25) $$\mathrm{Val}(r^{\#}(\mathcal{H}_{[n]})) = \bar{r}^n_*(\mathrm{Val}(\mathcal{H}_{[n]})) \quad\text{and}\quad \mathrm{Val}(e^{\#}({}'\mathcal{H}_{[n]})) = \bar{e}^n_*(\mathrm{Val}({}'\mathcal{H}_{[n]})),$$

as in (4.7), so

5.26 (a) $\mathrm{Val}(\mathcal{H}_{[n]})=0$ if and only if $\mathrm{Val}(r^{\#}(\mathcal{H}_{[n]}))=0$,

(b) if $\mathrm{Val}({}'\mathcal{H}_{[n]})=0$ then $\mathrm{Val}(e^{\#}({}'\mathcal{H}_{[n]}))=0$, as in (4.8).

We define weak and strong equivalences relations on strands as in Definition 4.9.

5.27 Definition Given two maps $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ inducing the same homomorphism of Θ_R–algebras $\phi\colon H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$, the associated *universal n^{th} order cohomology operation* $\langle\langle f_{(0,1)}\rangle\rangle_n$ which assigns to an $(n-1)$–strand $\mathcal{H}_{[n-1]}$ for this data the value

$$\langle\langle f_{(0,1)}\rangle\rangle_n(\mathcal{H}_{[n-1]}) := \mathrm{Val}(\mathcal{H}_{[n-1]}) \in \Gamma'\{\Omega^{n-1}\overline{\boldsymbol{W}}^n\},$$

where $\Gamma' := H^*(\boldsymbol{Z}; R)$. We say that $\langle\langle f_{(0,1)}\rangle\rangle_n$ *vanishes* if there is a cofibrant $\mathcal{W}$ with an $(n-1)$–strand $\mathcal{H}_{[n-1]}$ for $(\mathcal{W}, f_0, f_1)$ such that $\langle\langle f_{(0,1)}\rangle\rangle_n(\mathcal{H}_{[n-1]}) = 0$. Note that this depends only on the strong equivalence class of $\mathcal{H}_{[n-1]}$.

As in Definition 4.14, we then say that the system $\langle\langle f_{(0,1)}\rangle\rangle = (\langle\langle f_{(0,1)}\rangle\rangle_n)_{n=2}^{\infty}$ of n^{th} order cohomology operations for (f_0, f_1) *vanishes coherently* for $(\mathcal{W}, f_0, f_1)$ if there is an ∞–strand $\mathcal{H}_{[\infty]}$ for this data — that is, for each $n \geq 1$, we have an n–strand $\mathcal{H}_{[n]}$ for $(\mathcal{W}, f_0, f_1)$ such that $\mathrm{Val}(\mathcal{H}_{[n]} = 0)$, which extends to the $(n+1)$–strand $\mathcal{H}_{[n+1]}$ using Proposition 5.23.

The proof of Key Lemma 4.17 shows also:

5.28 Lemma *Given $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ as above, $\langle\langle f_{(0,1)}\rangle\rangle_n$ vanishes if and only if, for* every *n–stage cofibrant sequential realization $\mathcal{W}$ of $\boldsymbol{Y}$, there is an $(n-1)$–strand $\mathcal{H}_{[n-1]}$ for $(\mathcal{W}, f_0, f_1)$ with $\mathrm{Val}(\mathcal{H}_{[n-1]}) = 0$.*

Moreover, if $\langle\langle f_{(0,1)}\rangle\rangle_n$ vanishes at the $(n-1)$–strand $\mathcal{H}_{[n-1]}$ for $(\mathcal{W}, f_0, f_1)$, then for any other n–stage cofibrant sequential realization ${}'\mathcal{W}$ of $\boldsymbol{Y}$ we can choose the $(n-1)$–strand ${}'\mathcal{H}_{[n-1]}$ for ${}'\mathcal{W}$ to be weakly equivalent to $\mathcal{H}_{[n-1]}$.

In analogy to Theorem 4.18 we therefore have:

5.29 Theorem *For R either $\mathbb{F}_p$ or a field of characteristic 0, let $f_0, f_1\colon \boldsymbol{Z} \to \boldsymbol{Y}$ be two maps between R–good spaces which induce the same map of Θ_R–algebras $H^*(\boldsymbol{Y}; R) \to H^*(\boldsymbol{Z}; R)$. Then the following are equivalent:*

(a) *The system of higher cohomology operations $\langle\langle f_{(0,1)}\rangle\rangle$ vanishes coherently for* some *cofibrant sequential realization $\mathcal{W}$ of $\boldsymbol{Y}$.*

(b) *$\langle\langle f_{(0,1)}\rangle\rangle$ vanishes coherently for* every *cofibrant sequential realization of $\boldsymbol{Y}$.*

(c) *The maps f_0 and f_1 are R–equivalent* (cf Section 1.22).

Proof (a) $\Longleftrightarrow$ (b) By Lemma 5.28.

(a) $\Longrightarrow$ (c) Note that the projection $p_X\colon X \otimes \Delta^\bullet \to c(X)^\bullet$ is a trivial Reedy fibration for any $X \in \mathcal{C}$. Since $Z \amalg Z \xrightarrow{i_0 \perp i_1} Z \otimes I \xrightarrow{\sigma} Z$ is a cylinder object in $\mathcal{C}$ (see Quillen [32, Section I.1]), the same is true after applying $(-) \otimes \Delta^\bullet$. An ∞–strand $\mathcal{H}_{[\infty]}$ for a sequential realization $\mathcal{W}$ (with associated $\varepsilon\colon Y \to W^\bullet$) defines a map $H\colon (Z \otimes I) \otimes \Delta^\bullet \to W^\bullet$, fitting into a commutative diagram of cosimplicial spaces

(5.30)
$$\begin{tikzcd}
Z \otimes \Delta^\bullet \arrow[r, two heads, "p_Z", "\simeq"'] \arrow[d, "i_j \otimes \mathrm{Id}"] & c(Z)^\bullet \arrow[drr, "c(f_j)^\bullet"] & & \\
(Z \amalg Z) \otimes \Delta^\bullet \arrow[rr, "(f_{(0)} \perp f_{(1)}) \otimes \mathrm{Id}"'] \arrow[d, "(i_0 \perp i_1) \otimes \mathrm{Id}"'] & & Y \otimes \Delta^\bullet \arrow[r, two heads, "p_Y", "\simeq"'] & c(Y)^\bullet \arrow[d, "\varepsilon"] \\
(Z \otimes I) \otimes \Delta^\bullet \arrow[rrr, "H"] & & & W^\bullet
\end{tikzcd}$$

for $j = 0, 1$. Applying Tot yields a cylinder object

$$\mathrm{Tot}((Z \otimes I) \otimes \Delta^\bullet)$$

for $\mathrm{Tot}((Z \amalg Z) \otimes \Delta^\bullet)$, and a homotopy $\mathrm{Tot}\, H$ between $\varepsilon_* \circ f_0 \circ \mathrm{Tot}(p_Z)$ and $\varepsilon_* \circ f_1 \circ \mathrm{Tot}(p_Z)$. Since $\mathrm{Tot}(p_Z)$ is a weak equivalence and $\varepsilon_*\colon Y \to \mathrm{Tot}\, W^\bullet$ is an R–equivalence, we see that f_0 and f_1 are R–equivalent.

(c) $\Longrightarrow$ (b) Let $\mathcal{W}$ be any cofibrant sequential realization for Y with associated $\varepsilon\colon Y \to W^\bullet$. By Definition 2.1, $W^\bullet$ is Reedy fibrant. Thus $\mathrm{Tot}\, W^\bullet$ is an R–complete Kan complex, with $\varepsilon_*\colon Y \to \mathrm{Tot}\, W^\bullet$ the R–completion map, and so the R–equivalent maps $\varepsilon_* \circ f_0$ and $\varepsilon_* \circ f_1$ are actually homotopic (see [20, Lemma I.5.5]). We may therefore choose a homotopy $F\colon Z \otimes I \to \mathrm{Tot}\, W^\bullet$ between them, whose adjoint is the map of cosimplicial spaces $\widetilde{F}\colon Z \otimes I \otimes \Delta^\bullet \to W^\bullet$ (see [20, Section I.3.3]). Composing $\widetilde{F}$ with the structure maps $W^\bullet \to W^\bullet_{[n]}$ for the limit of (2.2) yields a compatible sequence of cosimplicial maps $H_{[n]}\colon Z \otimes I \otimes \Delta^\bullet \to W^\bullet_{[n]}$.

This defines compatible n–strands for $\mathcal{W}$ and all $n \geq 1$, showing that the system $\langle\langle f_{(0,1)} \rangle\rangle$ of higher-order operations vanishes by Proposition 5.23. $\square$

5.31 Corollary *If $f_0, f_1\colon Z \to Y$ are maps between R–complete Kan complexes inducing the same map of Θ_R–algebras $\psi\colon H^*(Y; R) \to H^*(Z; R)$, the system of higher operations (f_0, f_1) is a complete set of invariants for the homotopy classes $[f_0]$ and $[f_1]$.*

5.32 A rational example for a map As in Section 4.24, we now consider an example of the obstruction to a map $f\colon \boldsymbol{Z} \to \boldsymbol{Y}$ being rationally trivial when $f^*\colon H^*(\boldsymbol{Y}; Q) \to H^*(\boldsymbol{Z}; Q)$ is the zero map:

Let $\boldsymbol{Z} := \boldsymbol{S}^{2n-1}_{\mathbb{Q}}$ and $\boldsymbol{Y} := (\boldsymbol{S}^n \vee \boldsymbol{S}^n)_{\mathbb{Q}}$ for $n > 1$ odd, with $f := [\iota_n, \iota'_n]\colon \boldsymbol{Z} \to \boldsymbol{Y}$ the Whitehead product map. The free CDGA model for $\boldsymbol{Y}$ is (A^*, d) given in degrees $\leq 2n$ by $A^* = \mathbb{Q}\langle x_n, y_n, u_{2n-1}\rangle$ with $d(u) = xy$, while $\boldsymbol{Z}$ has the formal CDGA model $B^* = \Lambda[z_{2n-1}]$. The CDGA model for f is $\varphi\colon A^* \to B^*$ mapping u to z.

Realizing the obvious minimal free algebraic resolution of $H^*(\boldsymbol{Y}; Q)$, we obtain the 1–truncated augmented simplicial CDGA $\widetilde{W}^{[1]}_\bullet \to A^* \to B^*$ in degrees $\leq 2n$ depicted in Figure 2.

$$
\begin{tikzcd}
\widetilde{W}^{[1]}_1 \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & = & & \overline{W}_1 = \Lambda[\boldsymbol{u}_{2n}] \arrow[dll, "d_0\colon \boldsymbol{u}\mapsto \boldsymbol{xy}"'] \arrow[dr, "d_1\colon \boldsymbol{u}\mapsto \boldsymbol{u}'"] & \\
W^{[1]}_0 \arrow[d, "\boldsymbol{\varepsilon}"] & = & \overline{W}_0 = \Lambda[\boldsymbol{x}_n, \boldsymbol{y}_n] \arrow[d, "\boldsymbol{\varepsilon}\colon \boldsymbol{x}\mapsto x,\ \boldsymbol{y}\mapsto y"] & \amalg & C\overline{W}_1 = \Lambda(\boldsymbol{u}'_{2n}, \bar{\boldsymbol{u}}_{2n-1}) \arrow[dll, "\boldsymbol{\varepsilon}\colon \bar{\boldsymbol{u}}\mapsto -u,\ \boldsymbol{u}'\mapsto xy"] \\
A^* \arrow[d, "\varphi"] & = & \Lambda[x, y, u] \arrow[d, "\varphi\colon \boldsymbol{u}\mapsto z"] & & \\
B^* & = & \Lambda[z] & &
\end{tikzcd}
$$

Figure 2: $\widetilde{W}^{[1]}_\bullet$ in degrees $\leq 2n$

As noted above, the original map $f\colon \boldsymbol{Z} \to \boldsymbol{Y}$ is nullhomotopic if and only if we can extend the composite $\widetilde{W}^{[1]}_\bullet \to B^*$ in the diagram in Figure 2 to the cone $C\widetilde{W}^{[1]}_\bullet \to B^*$, which by Theorem 5.29 is equivalent to the vanishing of the associated system of higher operations.

The model for the cofiber of $W^{[1]}_\bullet \hookrightarrow C\overline{W}_1$ (corresponding to the loop space $\Omega\overline{\boldsymbol{W}}{}^1$) is the formal CDGA $\Sigma\overline{W}_1 := \Lambda(\bar{\boldsymbol{u}}'_{2n-1})$, and its cone (corresponding to the path space $P\Omega\overline{\boldsymbol{W}}{}^1$) is $C\Sigma\overline{W}_1 := \Lambda(\bar{\boldsymbol{u}}''_{2n-1}, \bar{\bar{\boldsymbol{u}}}_{2n-2})$, with $d(\bar{\bar{\boldsymbol{u}}}_{2n-2}) = \bar{\boldsymbol{u}}''_{2n-1}$.

Thus in order to extend the given map $\varphi \circ \boldsymbol{\varepsilon}\colon W^{[1]}_0 \to B^*$ (sending $\bar{\boldsymbol{u}}_{2n-1}$ to $-z$) to its cone, we need a map $F\colon C\Sigma\overline{W}_1 \to B^*$ sending $\bar{\boldsymbol{u}}''_{2n-1}$ to $-z$. However, since necessarily $F(\bar{\bar{\boldsymbol{u}}}) = 0$, this is impossible — that is, the secondary cohomology operation does not vanish: its value is represented by the map $\psi\colon \Sigma\overline{W}_1 \to B^*$ defined by $\psi(\bar{\boldsymbol{u}}'_{2n-1}) = z$.

Appendix: Proof of Theorem 2.33

In this appendix we state and prove Theorem 2.33 in a more general form needed in Blanc and Sen [17]. For this purpose we recall the notion of a *mapping algebra*, which encodes the extra structure on the mapping spaces $\mathrm{map}_*(\boldsymbol{Y}, \boldsymbol{K}(R,n))$ needed to recover the R–completion of $\boldsymbol{Y}$ from them (see Blanc and Sen [17]).

A.1 Definition An *enriched sketch* $(\boldsymbol{\Theta}, \mathcal{P}, \mathcal{K})$ is a small subcategory $\boldsymbol{\Theta}$ of a simplicial category $\mathcal{C}$ (see Definition 1.19), with $\boldsymbol{\Theta}$ closed under a given set of limits $\mathcal{P}$ and under $(-)^K$ for K in a given subcategory $\mathcal{K}$ of $\mathcal{S}$. We assume that all mapping spaces $\mathrm{map}_{\boldsymbol{\Theta}}(\boldsymbol{B}, \boldsymbol{B}')$ are Kan complexes.

A $\boldsymbol{\Theta}$*–mapping algebra* is a pointed simplicial functor $\mathfrak{X}\colon \boldsymbol{\Theta} \to \mathcal{S}_*$ (written $\mathfrak{X}\colon \boldsymbol{B} \mapsto \mathfrak{X}\{\boldsymbol{B}\}$) which preserves the limits in $\mathcal{P}$ and with $\mathfrak{X}\{(\boldsymbol{B})^K\} = (\mathfrak{X}\{\boldsymbol{B}\})^K$ for any $\boldsymbol{B} \in \boldsymbol{\Theta}$ and $K \in \mathcal{K}$. The category of $\boldsymbol{\Theta}$–mapping algebras will be denoted by $\mathcal{M}ap_{\boldsymbol{\Theta}}$. See Baues and Blanc [2, Section 8] and Blanc and Sen [17, Section 1] for further details.

For any $\boldsymbol{Y} \in \mathcal{C}$ we have a *realizable* $\boldsymbol{\Theta}$–mapping algebra $\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{Y}$ defined for each $\boldsymbol{B} \in \boldsymbol{\Theta}$ by $\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{Y}\{\boldsymbol{B}\} := \mathrm{map}_{\mathcal{C}}(\boldsymbol{Y}, \boldsymbol{B})$.

A.2 Remark Assume that in our enriched sketch $(\boldsymbol{\Theta}, \mathcal{P}, \mathcal{K})$, the category $\mathcal{K}$ includes $\Delta[0] \hookrightarrow \Delta[1]$, and $\mathcal{P}$ includes all finite products and the pullback squares

(A.3)
$$
\begin{tikzcd}
P\boldsymbol{B} \arrow[r, hook] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & \boldsymbol{B}^{\Delta[1]} \arrow[d, two heads, "\mathrm{ev}_0"', "\simeq"] \\
* \arrow[r, hook] & \boldsymbol{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega\boldsymbol{B} \arrow[r, hook, "\iota_{\boldsymbol{B}}"] \arrow[d, two heads] \arrow[dr, phantom, "\boxed{\mathrm{PB}}"] & P\boldsymbol{B} \arrow[d, two heads, "\mathrm{ev}_1"'] \\
* \arrow[r, hook] & \boldsymbol{B}
\end{tikzcd}
$$

for any $\boldsymbol{B} \in \boldsymbol{\Theta}_R$. For each $\boldsymbol{\Theta}$–mapping algebra $\mathfrak{X}$ and objects $\boldsymbol{B}$ and $\{\boldsymbol{B}_i\}_{i=1}^n$ in $\boldsymbol{\Theta}$ we then have natural isomorphisms

(a) $i_\Pi\colon \mathfrak{X}\{\prod_{i=1}^n \boldsymbol{B}_i\} \xrightarrow{\cong} \prod_{i=1}^n \mathfrak{X}\{\boldsymbol{B}_i\}$;

(b) $i_P\colon \mathfrak{X}\{P\boldsymbol{B}\} \xrightarrow{\cong} P\mathfrak{X}\{\boldsymbol{B}\}$;

(c) $i_\Omega\colon \mathfrak{X}\{\Omega\boldsymbol{B}\} \xrightarrow{\cong} \Omega\mathfrak{X}\{\boldsymbol{B}\}$.

We assume that the objects $B \in \boldsymbol{\Theta}$ are fibrant in $\mathcal{C}$. In order to simplify the proof of Theorem A.11 below, we make the following ad hoc assumption: if $f\colon \boldsymbol{W}^\bullet \to \boldsymbol{Y}^\bullet$ is a map of cosimplicial object over $\mathcal{C}$ with each $\boldsymbol{W}^n$ and $\boldsymbol{Y}^n$ in $\boldsymbol{\Theta}$ and $\boldsymbol{W}^\bullet \to \boldsymbol{Z}^\bullet \to \boldsymbol{Y}^\bullet$ is the functorial factorization of f as a (trivial) Reedy cofibration followed by a (trivial) Reedy fibration, then each $\boldsymbol{Z}^n$ is in $\boldsymbol{\Theta}$, too.

This will hold, for instance, when $\mathcal{C} = \mathcal{S}_*$ and $\boldsymbol{\Theta}$ consists of all simplicial R–modules of cardinality $< \lambda$ (for some limit cardinal λ). This is the example for which case (1) of Theorem A.11 is needed in Blanc and Sen [17].

We have the following enriched version of Lemma 1.28:

A.4 Lemma [17, Lemma 1.9] *For any $\boldsymbol{\Theta}$–mapping algebra $\mathfrak{Y}$ and $\boldsymbol{B} \in \boldsymbol{\Theta}$, there is a natural isomorphism* $\mathrm{Hom}_{\mathcal{M}ap_{\boldsymbol{\Theta}}}(\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{B}, \mathfrak{Y}) \xrightarrow{\cong} \mathfrak{Y}\{\boldsymbol{B}\}_0$.

A.5 Definition To any enriched sketch $(\boldsymbol{\Theta}, \mathcal{P}, \mathcal{K})$ we associate a sketch $\pi_0\boldsymbol{\Theta}$ as in Remark 1.25, with the same objects and products as $\boldsymbol{\Theta}$, where $\mathrm{Hom}_{\Theta}(\boldsymbol{B}, \boldsymbol{B}') := \pi_0 \,\mathrm{map}_{\boldsymbol{\Theta}}(\boldsymbol{B}, \boldsymbol{B}')$. A map of $\boldsymbol{\Theta}$–mapping algebras $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a *weak equivalence* if it induces a weak equivalence $f_{\boldsymbol{B}}\colon \mathfrak{X}\{\mathcal{B}\} \to \mathfrak{Y}\{\mathcal{B}\}$ for any $\mathcal{B} \in \boldsymbol{\Theta}$. This means that f induces an isomorphism $f_\#\colon \pi_0\mathfrak{X} \to \pi_0\mathfrak{Y}$ of the corresponding $\pi_0\boldsymbol{\Theta}$–algebras.

A.6 Example For any commutative ring R, let $\boldsymbol{\Theta}_R \subseteq \mathcal{S}_*$ denote the enriched sketch whose objects are finite-type R–GEMs of the form $\prod_{i=1}^{\infty} \boldsymbol{K}(V_i, m_i)$ for $m_i \geq 0$, with V_i a finite-dimensional free R–module where $\mathcal{K}$ is as above and $\mathcal{P}$ includes all such finite-type products. Since each $\boldsymbol{B} \in \boldsymbol{\Theta}_R$ is an R–module object in $\mathcal{S}_*$, the same is true of $\mathfrak{X}\{\boldsymbol{B}\}$, so $\boldsymbol{\Theta}_R$–mapping algebras actually take value in simplicial R–modules. The enriched sketch $\boldsymbol{\Theta}_R^{\lambda}$ is defined analogously (see Example 1.27).

Note that $\pi_0\boldsymbol{\Theta}_R$ includes also 0–dimensional Eilenberg–Mac Lane spaces, while the algebraic sketch Θ_R consists of R–GEMs in dimensions ≥ 1 (to avoid having to deal with the nonreduced cohomology of nonconnected spaces). Thus $\Theta_R \subset \pi_0\boldsymbol{\Theta}_R$, which motivates the following:

A.7 Definition For $\boldsymbol{\Theta}$ an enriched sketch in $\mathcal{C}$ and $\mathfrak{X}$ a $\boldsymbol{\Theta}$–mapping algebra, let $\Theta \subseteq \pi_0\boldsymbol{\Theta}$ be a subalgebraic sketch (still closed under finite products, but not necessarily under loops), and let $V_\bullet \to \pi_0\mathfrak{X}$ be a CW resolution of the corresponding Θ–algebra. A *sequential realization* $\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]}\rangle_{n\in\mathbb{N}}$ of $V_\bullet$ for $\mathfrak{X}$ consists of a tower of Reedy fibrant and cofibrant cosimplicial objects as in (2.2), such that:

(a) We have an augmentation $\varepsilon_{[n]}\colon \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n]} \to \mathfrak{X}$ for the simplicial $\boldsymbol{\Theta}$–mapping algebra $\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^\bullet_{[n]}$, realizing $V_\bullet \to \pi_0\mathfrak{X}$ through simplicial dimension n — ie we have a natural isomorphism as in (2.3).

(b) The augmentation $\varepsilon_{[n-1]}\colon \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]} \to \mathfrak{X}$ extends along the $\boldsymbol{\Theta}$–mapping algebra map $\pi^*_{[n]}\colon \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]} \to \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n]}$ to $\varepsilon_{[n]}\colon \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n]} \to \mathfrak{X}$.

(c) Each $\boldsymbol{W}^\bullet_{[n]}$ is obtained from $\boldsymbol{W}^\bullet_{[n-1]}$ as in Definition 2.1(c).

A.8 Remark We do not require the sequential realization $\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]} \rangle_{n\in\mathbb{N}}$ for $\mathfrak{X}$ to be cofibrant, as in Definition 2.1(d). Therefore, in the explicit description of Section 2.5 we may take the fibration sequences (2.6) in step (i) to be the standard path-loop fibrations, so

$$\overline{\Omega^j \boldsymbol{W}^n} := \Omega^j \overline{\boldsymbol{W}}^n \tag{A.9}$$

for all $0 \le j \le n-1$. Thus (2.10) becomes simply

$$\widetilde{\boldsymbol{W}}^k_{[n]} = \boldsymbol{W}^k_{[n-1]} \times P\Omega^{n-k-1} \overline{\boldsymbol{W}}^n \tag{A.10}$$

for all $0 \le k \le n$ (with (2.11) still holding for $k = n$).

A.11 Theorem *Let $\boldsymbol{\Theta}$ be an enriched sketch $\boldsymbol{\Theta}$ in a model category $\mathcal{C}$ as in Remark A.2, and $\Theta \subseteq \pi_0\boldsymbol{\Theta}$ an algebraic sketch.*

1. *If $\mathfrak{X}$ is a $\boldsymbol{\Theta}$–mapping algebra, and $V_\bullet$ a CW resolution of $\Gamma := \pi_0\mathfrak{X}$ with CW basis $(\overline{V}_n)_{n\in\mathbb{N}}$ such that each $\overline{V}_n$ is realizable by an object $\overline{\boldsymbol{W}}^n \in \boldsymbol{\Theta}$, then there is a sequential realization $\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]} \rangle_{n\in\mathbb{N}}$ of $V_\bullet$ as in Definition A.7.*
2. *If Θ is allowable* (see Definition 2.28), *$\boldsymbol{Y} \in \mathcal{C}$, and $V_\bullet$ is any CW resolution of $\Gamma := H^*_\Theta \boldsymbol{Y} = \pi_0 \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{Y}$, then $V_\bullet$ has a cofibrant sequential realization $\mathcal{W} = \langle \boldsymbol{W}^\bullet_{[n]}, \widetilde{\boldsymbol{W}}^\bullet_{[n]} \rangle_{n\in\mathbb{N}}$ of $V_\bullet$ for $\boldsymbol{Y}$.*

Thus, in both cases $V_\bullet$ is realizable by $\boldsymbol{W}^\bullet = \operatorname{holim}_n \boldsymbol{W}^\bullet_{[n]}$. In case (2), if Θ is contained in some class of injective models $\mathcal{G}$ in $\mathcal{C}$, then $\boldsymbol{Y} \to \boldsymbol{W}^\bullet$ is a weak $\mathcal{G}$–resolution (see Section 1.22).

Proof In case (2), for each $n \ge 0$ we choose an object $\overline{\boldsymbol{W}}^n \in \Theta$ realizing $\overline{V}_n$. In both cases, we then construct a sequential realization $\mathcal{W}$ by a double induction, where in the outer induction $\boldsymbol{W}^\bullet_{[n]}$ is obtained from $\boldsymbol{W}^\bullet_{[n-1]}$ as in Definition 2.1(c), using an inner descending induction on $0 \le k \le n$:

I Step $n = 0$ of the outer induction We start the induction with $\boldsymbol{W}^\bullet_{[0]} := c(\overline{\boldsymbol{W}}^0)^\bullet$ (the constant cosimplicial object), which is weakly $\mathcal{G}$–fibrant. Because $\overline{V}_0$ is a free Θ–algebra, in case (1) the Θ–algebra augmentation $\varepsilon\colon \overline{V}_0 \to \Gamma$ corresponds by Lemma 1.28 to a unique element in $[\overline{\varepsilon}_{[0]}] \in \Gamma\{\overline{\boldsymbol{W}}^0\} = \pi_0\mathfrak{X}\{\overline{\boldsymbol{W}}^0\}$, for which we may choose a representative $\overline{\varepsilon}_{[0]} \in \mathfrak{X}\{\overline{\boldsymbol{W}}^0\}_0$, corresponding to a map of $\boldsymbol{\Theta}$–mapping algebras $\varepsilon_{[0]}\colon \mathfrak{M}_{\boldsymbol{\Theta}} \overline{\boldsymbol{W}}^0 \to \mathfrak{X}$ by Lemma A.4. In case (2), we may realize ε by a map $\boldsymbol{\varepsilon}\colon \boldsymbol{Y} \to \overline{\boldsymbol{W}}^0$ by Lemma 1.28 and (2.31), since Θ is allowable.

II Step $n = 1$ of the outer induction We choose a map $C^0\boldsymbol{W}^\bullet_{[0]} = \overline{\boldsymbol{W}}^0 \to \overline{\boldsymbol{W}}^1$ realizing the first attaching map $\overline{\partial}^1_0\colon \overline{V}_1 \to V_0 = \overline{V}_0$, with $\boldsymbol{W}^\bullet_{[1]}$ given in dimensions

≤ 1 by

(A.12)
$$\begin{tikzcd}
W^0_{[1]} \arrow[d, bend right, "d_0^0"'] \arrow[d, bend left, "d_0^1"] & = & \overline{W}^0 \arrow[d, bend right, "d_0^0=d_0^1=\mathrm{Id}"'] \arrow[d, bend left] \arrow[drr, "\bar{d}_0^0"] & & \times & & P\overline{W}^1 \arrow[d, bend right] \arrow[d, bend left, "d_0^0=d_0^1=\mathrm{Id}"] \arrow[dll, "d_0^1=p"'] \\
W^1_{[1]} \arrow[u, "s^0"] & = & \overline{W}^0 \arrow[u] & \times & \overline{W}^1 & \times & P\overline{W}^1 \arrow[u]
\end{tikzcd}$$

In case (1), to define the augmentation $\bar{\varepsilon}_{[1]}$ as a 0–simplex in $\mathfrak{X}\{W^0_{[1]}\}_0$ extending $\bar{\varepsilon}_{[0]} \in \mathfrak{X}\{W^0_{[0]}\} = \mathfrak{X}\{\overline{W}^0\}$, we use the fact that $\mathfrak{X}\{W^0_{[1]}\} = \mathfrak{X}\{\overline{W}^0\} \times P\mathfrak{X}\{\overline{W}^1\}$, by Remark A.2(a)-(b), so we need only to find a 0–simplex H in $P\mathfrak{X}\{\overline{W}^1\}$ — which, by (A.3), is a 1–simplex in $\mathfrak{X}\{\overline{W}^1\}$ with $d_1 H = 0$.

In order to qualify as an augmentation $\mathfrak{M}_{\mathbf{\Theta}} W^0_{[1]} \to \mathfrak{X}$ of simplicial $\mathbf{\Theta}$–mapping algebras, $\varepsilon_{[1]}$ must satisfy the simplicial identity

(A.13)
$$\varepsilon_{[1]} \circ d_0 = \varepsilon_{[1]} \circ d_1 \colon \mathfrak{M}_{\mathbf{\Theta}} W^1_{[1]} \to \mathfrak{X}$$

as maps of $\mathbf{\Theta}$–mapping algebras — or equivalently, these must correspond to the same 0–simplex in $\mathfrak{X}\{W^1_{[1]}\} = \mathfrak{X}\{\overline{W}^0\} \times \mathfrak{X}\{\overline{W}^1\} \times P\mathfrak{X}\{\overline{W}^1\}$. In the first and third factors this obviously holds, so we need only consider the two 0–simplices in $\mathfrak{X}\{\overline{W}^1\}$: in other words, since the path fibration p in (A.12) (induced by the inclusion $\Delta[0] \hookrightarrow \Delta[1]$) becomes d_0 in $\mathfrak{X}\{\overline{W}^1\}$, we must choose H so that $d_0 H$ is the 0–simplex $(\bar{d}_0^0)_{\#}\bar{\varepsilon}_{[0]}$. By (1.32), $\varepsilon \circ \bar{\partial}_0^0 = 0$ in Θ–Alg, which implies (by our choices of $\bar{d}_0^0$ and $\bar{\varepsilon}_{[0]}$ representing $\bar{\partial}_0^0$ and ε, respectively) that $(\bar{d}_0^0)_{\#}\bar{\varepsilon}_{[0]}$ is nullhomotopic, so the required H exists.

In case (2), we choose a nullhomotopy for $\bar{d}_0^0 \circ \boldsymbol{\varepsilon}$ to extend $\boldsymbol{\varepsilon}_{[0]}$ to the factor $P\overline{W}^1$, and thus define $\boldsymbol{\varepsilon}_{[1]} \colon \boldsymbol{Y} \to W^0_{[1]}$.

III Step n of the outer induction $n \geq 2$ To construct $W^\bullet_{[n]}$ given $W^\bullet_{[n-1]}$, by Proposition 2.24 it suffices to produce a cochain map $F \colon C^* W^\bullet_{[n-1]} \to \boldsymbol{D}^*$ as in Section 2.5 (where the left Reedy fibrant replacement $\boldsymbol{D}^*$ for $\overline{W}^n \otimes^* S^{n-1}$ is given by (A.9) and (2.8)). We do so by a descending induction on the cosimplicial dimension $0 \leq k \leq n-1$, starting with Step IV below for $k = n-1$, under the following induction hypotheses:

In stage k we assume the existence of $F^j \colon C^j W^\bullet_{[n-1]} \to P\Omega^{n-j-2}\overline{W}^n$ (constituting a cochain map for $k+1 \leq j \leq n-1$) and $a^j \colon Z^j W^\bullet_{[n-1]} \to \Omega^{n-j-2}\overline{W}^n$ for $k \leq j \leq n-1$, with a nullhomotopy
$$\widehat{F}^k \colon C^k W^\bullet_{[n-1]} \to P\Omega^{n-k-2}\overline{W}^n$$

such that

$$a^k \circ w^k = p \circ \widehat{F}^k \colon C^k W^\bullet_{[n-1]} \to \Omega^{n-k-2} \overline{W}^n, \tag{A.14}$$

as in (2.26). By Lemma 2.19, $\widehat{F}^k$ induces $\widehat{a}^{k-1} \colon Z^{k-1} W^\bullet_{[n-1]} \to \Omega^{n-k-1} \overline{W}^n$ with

$$\iota \circ \widehat{a}^{k-1} \circ w^{k-1} = \widehat{F}^k \circ \delta^{k-1}, \tag{A.15}$$

as in (2.20), where $\iota = \bar{\iota}^{n-k-1} \colon \Omega^{n-k-1} \overline{W}^n \hookrightarrow P\Omega^{n-k-2} \overline{W}^n$ is the inclusion. However, we do not assume in our induction hypothesis that $\widehat{a}^{k-1} \circ w^{k-1}$ is nullhomotopic.

IV Step $k = n-1$ of the inner induction To define $F^{n-1} \colon C^{n-1} W^\bullet_{[n-1]} \to \overline{W}^n$, note that the simplicial space $U_\bullet := \mathrm{map}_{\mathcal{C}}(W^\bullet_{[n-1]}, \overline{W}^n)$ is Reedy fibrant, since $W^\bullet_{[n-1]}$ is Reedy cofibrant. Moreover, since $H^*_\Theta W^k_{[n-1]} \cong V_k$ for all $0 \le k < n$ by Definition 2.1(a), the attaching map $\overline{\partial}^n_0 \colon \overline{V}_n \to V_{n-1}$ yields a homotopy class

$$\begin{aligned} \alpha \in [W^{n-1}_{[n-1]}, \overline{W}^n] &= \pi_0 \mathfrak{M}_{\boldsymbol{\Theta}} W^{n-1}_{[n-1]}\{\overline{W}^n\} = V_{n-1}\{\overline{W}^n\} \\ &= \mathrm{Hom}_{\Theta\text{-}\mathsf{Alg}}(H^*_\Theta \overline{W}^n, V_{n-1}) = \mathrm{Hom}_{\Theta\text{-}\mathsf{Alg}}(\overline{V}_n, V_{n-1}), \end{aligned} \tag{A.16}$$

where the next-to-last equality follows from Lemma 1.28, as extended in (2.31).

This α is a Moore chain in $\pi_0 U_\bullet$ by Section 1.11, so by Lemma 2.22(a), it can be represented by a map $F^{n-1} \colon C^{n-1} W^\bullet_{[n-1]} \to \overline{W}^n$, which induces $a^{n-2} \colon Z^{n-2} W^\bullet_{[n-1]} \to \overline{W}^n$ by Lemma 2.19 with $a^{n-2} \circ w^{n-2} = F^{n-1} \circ \delta^{n-2}$. Moreover, by (1.12) α is in fact a Moore $(n-1)$–cycle, so we have a nullhomotopy $\widehat{F}^{n-2} \colon C^{n-2} \to P\overline{W}^n$ for $a^{n-2} \circ w^{n-2}$, as in (A.14).

V Step k of the inner induction $1 < k \le n-2$ Let $A := \Omega^{n-k-2} \overline{W}^n$. By assumption (Step III), we have a nullhomotopy $\widehat{F}^k \colon C^k W^\bullet_{[n-1]} \to PA$ for $a^k \circ w^k$, and $\widehat{F}^k \circ \delta^{k-1}$ determines $\widehat{a}^{k-1} \colon Z^{k-1} W^\bullet_{[n-1]} \to \Omega A$ satisfying (A.15), which is thus a $(k-1)$–cycle for the Reedy fibrant bisimplicial set $\mathrm{map}_{\mathcal{C}}(W^\bullet_{[n-1]}, \Omega A)$.

Since $W^\bullet_{[n-1]}$ realizes $V_\bullet$ through simplicial dimension $n-1 > k$, by Definition 2.1(a), $\widehat{a}^{k-1} \circ w^{k-1} \circ v^{k-1} \colon W^\bullet_{[n-1]} \to \Omega A$ represents a $(k-1)$–cycle $[\widehat{a}^{k-1}]$ for $V_\bullet\{\Omega A\}$, as in (A.16). Because $V_\bullet \to \Gamma$ is a resolution, and thus acyclic, there is a Moore chain γ_k in $C_k V_\bullet\{\Omega A\}$ with $\partial^{V_k}_0(\gamma_k) = [\widehat{a}^{k-1}]$. Since $V_k\{\Omega A\} \cong \pi_0 \mathfrak{M}_{\boldsymbol{\Theta}} W^k_{[n-1]}\{\Omega A\} = [W^k_{[n-1]}, \Omega A]$, by Lemma 2.22(a) we can represent γ_k by a map $g^k \colon C^k W^\bullet_{[n-1]} \to \Omega A$, while by Lemma 2.22(b) we have a homotopy

$$G \colon g^k \circ \delta^{k-1} \sim \widehat{a}^{k-1} \circ w^k \colon C^{k-1} W^\bullet_{[n-1]} \to \Omega A. \tag{A.17}$$

Next, concatenation of homotopies gives an action of $\mathrm{Hom}(\boldsymbol{C}, \Omega \boldsymbol{A})$ on $\mathrm{Hom}(\boldsymbol{C}, P\boldsymbol{A})$ (see [35, Section 1]), which we use to define a new nullhomotopy

(A.18) $$F^k := \widehat{F}^k \star (\iota \circ g^k)^{-1} \colon C^k \boldsymbol{W}^{\bullet}_{[n-1]} \to P\boldsymbol{A}$$

for $a^k \circ w^k$ (where $\iota \colon \Omega \boldsymbol{A} \hookrightarrow P\boldsymbol{A}$ is the inclusion).

By Lemma 2.19, F^k induces a map $a^{k-1} \colon Z^{k-1} \boldsymbol{W}^{\bullet}_{[n-1]} \to \Omega \boldsymbol{A}$ satisfying

(A.19) $$\begin{aligned} \iota \circ \widetilde{a} \circ w^{k-1} &= \widetilde{F} \circ \delta^{k-1} = (\widehat{F}^k \star (\iota \circ g^k)^{-1}) \circ \delta^{k-1} \\ &= (\widehat{F}^k \circ \delta^{k-1}) \star (\iota \circ g^k \circ \delta^{k-1})^{-1} \\ &= \iota \circ [(\widehat{a}^{k-1} \circ w^{k-1}) \star (g^k \circ \delta^{k-1})^{-1})] \end{aligned}$$

by (2.20), (A.18), (A.15) and the fact that the H–space structure $\star$ and $(-)^{-1}$ commutes with precomposition of maps into $\Omega \boldsymbol{A}$.

Since ι is a monomorphism, we conclude that

(A.20) $$\begin{aligned} a^{k-1} \circ w^{k-1} &= (\widehat{a}^{k-1} \circ w^{k-1}) \star (g^k \circ \delta^{k-1})^{-1}) \\ &\sim (\widehat{a}^{k-1} \circ w^{k-1}) \star (\widehat{a}^{k-1} \circ w^{k-1})^{-1} \sim 0 \end{aligned}$$

by (A.17), so a^{k-1} satisfies (A.14) for $k-1$.

In case (2), the last two steps of the downward induction are no different from those for $k \geq 2$ if we set $\widetilde{\boldsymbol{W}}^{-1}_{[n]} = \boldsymbol{W}^{-1}_{[n-1]} := \boldsymbol{Y}$, with $\widetilde{d}^{\,0}_{-1} \colon \widetilde{\boldsymbol{W}}^{-1}_{[n]} \to \widetilde{\boldsymbol{W}}^{0}_{[n]}$ as the coaugmentation $\boldsymbol{\varepsilon}_{[n]}$. However, in case (1) we no longer have an object $\boldsymbol{W}^{k-1}_{[n-1]}$ in $\mathcal{C}$ for $k = 0$, so we must modify our construction somewhat, using the language of $\boldsymbol{\Theta}$–mapping algebras, as follows:

VI Step $k = 1$ of the descending induction By the descending induction hypotheses III for $k = 1$ we have some nullhomotopy $\widehat{F}^1 \colon a^1 \circ w^1 \sim 0$ and $\widehat{a}^0$ with $\iota \circ \widehat{a}^0 \circ w^0 = \widehat{F}^1 \circ d^0_0$ by (A.15). Let $\boldsymbol{B} := \Omega^{n-3} \overline{\boldsymbol{W}}^n$.

We can think of $a^1 \circ w^1 \circ v^1$ as a 0–simplex $\underline{a}^1 \in \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^1_{[n-1]}\{\boldsymbol{B}\}_0$, of $\widehat{a}^0 \circ w^0 \circ v^0$ as a 0–simplex $\underline{\widehat{a}}^0 \in \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^0_{[n-1]}\{\Omega \boldsymbol{B}\}_0$, and of $\widehat{F}^1 \circ v^1$ as a 1–simplex $\underline{\widehat{F}}^1 \in \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^1_{[n-1]}\{\boldsymbol{B}\}_1$, implicitly using the natural isomorphism $i_{\boldsymbol{P}} \colon \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^0_{[n-1]}\{P\boldsymbol{B}\} \cong P\mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}$ of Remark A.2(b), and the inclusion of $j \colon (PK)_i \subseteq K_{i+1}$ for any Kan complex K and $i \geq 0$ (see [31, Section 23.3]). Thus $\widehat{F}^1 \colon a^1 \circ w^1 \sim 0$ means $d_0 \underline{\widehat{F}}^1 = \underline{a}^1$ and $d_1 \underline{\widehat{F}}^1 = 0$ (simplicial face maps in the mapping space). The fact that the domain of $\widehat{F}^1$ is $C^1 \boldsymbol{W}^{\bullet}_{[n-1]}$ implies that

(A.21) $$(d^1)^*(\underline{\widehat{F}}^1) = 0 \quad \text{in } \mathfrak{M}_{\boldsymbol{\Theta}} \boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}_1,$$

and (A.15) becomes

(A.22) $$j\circ\iota\circ i_{\Omega}(\underline{\widehat{a}}^0)=(d^0)^*(\underline{\widehat{F}}^1)\quad\text{in } \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}_1,$$

again using $i_{\boldsymbol{P}}$, the isomorphism i_{Ω} of Remark A.2(c), the inclusion $\iota\colon \Omega K\hookrightarrow PK$ and $j\colon (PK)_0\subseteq K_1$ as above for $K=\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}$.

Moreover, we have an augmentation of simplicial $\boldsymbol{\Theta}$–mapping algebras

$$\varepsilon_{[n-1]}\colon \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^\bullet_{[n-1]}\to\mathfrak{X}$$

by Definition 2.1(b) in the outer induction hypothesis. Thus $\varepsilon_{[n-1]}$ satisfies the simplicial identity

(A.23) $$\varepsilon_{[n-1]}\circ(d^0)^*=\varepsilon_{[n-1]}\circ(d^1)^*.$$

In order to apply Lemma 2.19, since $\boldsymbol{Y}$ is not available in cosimplicial dimension -1, we need to verify that $\underline{\widehat{a}}^0$ is a "Moore 0–cycle" in the augmented simplicial $\boldsymbol{\Theta}$–mapping algebra $\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^\bullet_{[n-1]}\to\mathfrak{X}$ — that is,

(A.24) $$\varepsilon_{[n-1]}(\underline{\widehat{a}}^0)=0\quad\text{in } \mathfrak{X}\{\Omega\boldsymbol{B}\}_0,$$

so after postcomposition with the monic maps

$$\mathfrak{X}\{\Omega\boldsymbol{B}\}_0\xrightarrow{\iota_\Omega}\Omega\mathfrak{X}\{\boldsymbol{B}\}_0\xrightarrow{\iota}P\mathfrak{X}\{\boldsymbol{B}\}_0\xrightarrow{j}\mathfrak{X}\{\boldsymbol{B}\}_1$$

it suffices to show

(A.25) $$j\circ\iota\circ\iota_\Omega\circ\varepsilon_{[n-1]}(\underline{\widehat{a}}^0)=0,$$

Note that the following diagram of sets commutes:

(A.26) $$\begin{tikzcd}
 & \underline{\widehat{F}}^1\in\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^1_{[n-1]}\{\boldsymbol{B}\}_1 \arrow[d, "(d^0_1)^*"] \arrow[dr, "(d^1_1)^*"] & \\
\underline{\widehat{a}}^0\in\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\Omega\boldsymbol{B}\}_0 \arrow[r, "j\circ\iota\circ\iota_\Omega"] \arrow[d, "\varepsilon_{[n-1]}"] & \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}_1 \arrow[d, "\varepsilon_{[n-1]}"] & \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\boldsymbol{B}\}_1 \arrow[dl, "\varepsilon_{[n-1]}"] \\
\mathfrak{X}\{\Omega\boldsymbol{B}\}_0 \arrow[r, "j\circ\iota\circ\iota_\Omega"] & \mathfrak{X}\{\boldsymbol{B}\}_1 &
\end{tikzcd}$$

where the left-hand square commutes by naturality of $j\circ\iota\circ\iota_\Omega$, and the right-hand side commutes by (A.23). By (A.22), the simplices $\underline{\widehat{F}}^1$ and $\underline{\widehat{a}}^0$ thus map to the same value in $\mathfrak{X}\{B\}_1$, which is zero by (A.21), proving that (A.25), and thus (A.24), in fact hold.

By the outer induction hypothesis that Definition 2.1(b) holds, the augmented simplicial abelian group

(A.27) $$V_\bullet\{\Omega\boldsymbol{B}\} \xrightarrow{\varepsilon} \pi_0\mathfrak{X}\{\Omega\boldsymbol{B}\} = \Gamma\{\Omega^{n-2}\overline{\boldsymbol{W}}^n\}$$

is realized by $\boldsymbol{W}^\bullet_{[n-1]}$ through simplicial dimension $n-1$, so that

(A.28) $$V_k\{\Omega\boldsymbol{B}\} \cong \pi_0\mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^k_{[n-1]}\{\Omega\boldsymbol{B}\} = [\boldsymbol{W}^k_{[n-1]}, \Omega\boldsymbol{B}]$$

for $k=1$ and 0.

Since, by (A.24), $[\underline{\hat{a}}^0]$ is a 0–cycle in (A.27), which is acyclic, there is a Moore 1–chain $\gamma_1 \in V_1\{\Omega\boldsymbol{B}\}$ with $\partial_0^{V_1}(\gamma_1) = [\underline{\hat{a}}^0]$. As in (A.18) of Step V, this can be used to produce a map $F^1\colon C^1\boldsymbol{W}^\bullet_{[n-1]} \to P\boldsymbol{B}$ such that

(A.29) $$F^1 \circ d^1 = 0,$$

yielding $a^0\colon \boldsymbol{W}^0_{[n-1]} \to \Omega\boldsymbol{B}$ (by Step III) having a nullhomotopy $\widehat{F}^0\colon a^0 \sim 0$ (again, as in Step V). Moreover, (A.29) and (A.23) together imply that (after replacing $\widehat{F}^1$ by F^1 in (A.26)) one can deduce that

(A.30) $$\varepsilon_{[n-1]}(\underline{a}^0) = 0 \quad \text{in } \mathfrak{X}\{\Omega\boldsymbol{B}\},$$

as required.

VII Step $k=0$ of the descending induction For $k=0$, let $\boldsymbol{C} := \Omega^{n-2}\overline{\boldsymbol{W}}^n$. The nullhomotopy $\widehat{F}^0\colon a^0 \circ w^0 \sim 0$ is represented as in Step VI by a 1–simplex $\underline{\widehat{F}}^0 \in \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\boldsymbol{C}\}_1$ with $d_0\underline{\widehat{F}}^0 = \underline{a}^0$, and $d_1\underline{\widehat{F}}^0 = 0$. Thus for $\varepsilon_{[n-1]}(\underline{\widehat{F}}^0) \in \mathfrak{X}\{\boldsymbol{C}\}_1$ we have $d_0(\varepsilon_{[n-1]}(\underline{\widehat{F}}^0)) = \varepsilon_{[n-1]}(\underline{a}^0) = 0$, by (A.30), as well as $d_1(\varepsilon_{[n-1]}(\underline{\widehat{F}}^0)) = 0$.

Therefore, as in (A.22) we obtain $\underline{\hat{a}} \in \mathfrak{X}\{\Omega\boldsymbol{C}\}_0$ with $j \circ \iota \circ \iota_\Omega(\underline{\hat{a}}) = \varepsilon_{[n-1]}(\underline{\widehat{F}}^0)$.

Note that this argument fails if we do not have the mapping algebra $\mathfrak{X}$ (with $\boldsymbol{\Theta}$ as in Remark A.2), which allows us to obtain an element $[\underline{\hat{a}}] \in \Gamma\{\Omega\boldsymbol{C}\} = \Gamma\{\Omega^{n-1}\overline{\boldsymbol{W}}^n\}$ from the nullhomotopy $\widehat{F}^0\colon \boldsymbol{W}^0_{[n-1]} \to P\boldsymbol{C}$.

Again we use the acyclicity of (A.27) to deduce the existence of a Moore 0–chain $\gamma_0 \in V_0\{\Omega\boldsymbol{C}\}$ with

(A.31) $$\varepsilon(\gamma_0) = [\underline{\hat{a}}] \in \Gamma\{\Omega\boldsymbol{C}\}.$$

We no longer require Lemma 2.22 to deduce that we can represent γ_0 by a map $g^0\colon \boldsymbol{W}^0_{[n-1]} \to \Omega\boldsymbol{C}$, corresponding to a 0–simplex $\underline{g}^0 \in \mathfrak{M}_{\boldsymbol{\Theta}}\boldsymbol{W}^0_{[n-1]}\{\Omega\boldsymbol{C}\}_0$, and thus a 1–simplex

(A.32) $$\underline{H} := (j \circ \iota \circ \iota_{\Omega})(\underline{g}^0) \in \mathfrak{M}_{\Theta} W^0_{[n-1]}\{C\}_1 \quad \text{with } d_0\underline{H} = d_1\underline{H} = 0.$$

Since $\mathfrak{M}_{\Theta} W^0_{[n-1]}\{C\}$ is a homotopy group object in $\mathcal{S}_*$, with homotopy group structure $\star$ and inverse $(-)^{-1}$ induced from those of $C \in \Theta \subset \mathcal{C}$, we may define a new 1–simplex

(A.33) $$\underline{F}^0 := \widehat{\underline{F}}^0 \star \underline{H}^{-1} \in \mathfrak{M}_{\Theta} W^0_{[n-1]}\{C\}_1 \quad \text{with } d_0\underline{F}^0 = \underline{a}^0 \text{ and } d_1\underline{F}^0 = 0.$$

Since $\varepsilon_{[n-1]}(\underline{a}^0) = 0$, the 1–simplex $\varepsilon_{[n-1]}(\underline{F}^0) \in \mathfrak{X}\{C\}_1$ is in the image of the composite inclusion

$$\mathfrak{X}\{\Omega C\}_0 \xrightarrow{\iota_{\Omega}} \Omega\mathfrak{X}\{C\}_0 \xrightarrow{\iota} P\mathfrak{X}\{C\}_0 \xrightarrow{j} \mathfrak{X}\{C\}_1.$$

Thus, there is a 0–simplex $\underline{a} \in \mathfrak{X}\{\Omega C\}_0$ with

(A.34) $$(j \circ \iota \circ \iota_{\Omega})(\underline{a}) = \varepsilon_{[n-1]}(\underline{F}^0).$$

Moreover, since $(j \circ \iota \circ \iota_{\Omega})$ and $\varepsilon_{[n-1]}$ commute with the homotopy group structure,

$$(j\circ\iota\circ\iota_{\Omega})(\underline{a}) = \varepsilon_{[n-1]}(\widehat{\underline{F}}^0)\star(\varepsilon_{[n-1]}(\underline{H})^{-1}) = [(j\circ\iota\circ\iota_{\Omega})(\widehat{\underline{a}})]\star[(j\circ\iota\circ\iota_{\Omega})(\varepsilon_{[n-1]}\underline{g}^0)]^{-1},$$

by (A.32) and (A.34). Since $j \circ \iota \circ \iota_{\Omega}$ is monic, this implies that

(A.35) $$\underline{a} = \widehat{\underline{a}} \star (\varepsilon_{[n-1]}\underline{g}^0)^{-1}.$$

By (A.31) there is a 1–simplex $G \in \mathfrak{X}\{\Omega C\}$ with $d_0 G = \widehat{\underline{a}}$ and $d_1 G = \varepsilon_{[n-1]}(\underline{g}^0)$ in $\mathfrak{X}\{\Omega C\}_0$. Therefore, $G' := G \star (s_0\varepsilon_{[n-1]}(\underline{g}^0))^{-1} \in \mathfrak{X}\{\Omega C\}_1$ satisfies

(A.36) $$d_0 G' = \widehat{\underline{a}} \star (\varepsilon_{[n-1]}(\underline{g}^0))^{-1} \quad \text{and} \quad d_1 G' = \varepsilon_{[n-1]}(\underline{g}^0) \star (\varepsilon_{[n-1]}(\underline{g}^0))^{-1}.$$

Since ΩC is a homotopy group object, there is a 1–simplex $K \in \mathfrak{X}\{\Omega C\}_1$ with

(A.37) $$d_0 K = \varepsilon_{[n-1]}(\underline{g}^0) \star (\varepsilon_{[n-1]}(\underline{g}^0))^{-1} \quad \text{and} \quad d_1 K = 0.$$

Since $\mathfrak{X}\{\Omega C\}$ is a Kan complex, we have $\sigma \in \mathfrak{X}\{\Omega C\}_2$ with $d_0\sigma = G'$ and $d_1\sigma = K$, so if we set $\widehat{\underline{F}} := d_2\sigma$, we find

(A.38) $$d_0\widehat{\underline{F}} = \widehat{\underline{a}} \star (\varepsilon_{[n-1]}(\underline{g}^0))^{-1} \quad \text{and} \quad d_1\widehat{\underline{F}} = 0.$$

We deduce from (A.35) that $\widehat{\underline{F}}$ is a nullhomotopy for $\underline{a}$ in $\mathfrak{X}\{\Omega C\}$, as required in Step III, thus completing the construction of the restricted cosimplicial object $\widetilde{W}^{\bullet}_{[n]}$.

Since making the cosimplicial object $\widehat{W}^{\bullet}_{[k]}$ as in Section 2.5(iii) Reedy fibrant or cofibrant does not change $\widehat{W}^{0}_{[k]}$, we see by induction from (2.17) that

$$\textbf{(A.39)}\qquad \widetilde{W}^{0}_{[n]} = W^{0}_{[n-1]} \times P\Omega C = \prod_{k=0}^{n} P\Omega^{k-1}\overline{W}^{k}$$

(using the convention of (2.7)). Note that all the factors on the right-hand side of (A.39) are in $\mathbf{\Theta}$ by Remark A.2, so $\widetilde{W}^{0}_{[n]}$ is, too.

Thus, by Lemma A.4, the augmentation $\widetilde{\varepsilon}_{[n]}\colon \mathfrak{M}_{\mathbf{\Theta}}\widetilde{W}^{0}_{[n]} \to \mathfrak{X}$ is determined by the choice of a suitable 0–simplex

$$e = (e', e'') \in \mathfrak{X}\{\widetilde{W}^{0}_{[n]}\}_0 = \mathfrak{X}\{W^{0}_{[n-1]} \times P\Omega C\}_0 = \mathfrak{X}\{W^{0}_{[n-1]}\}_0 \times P\mathfrak{X}\{\Omega C\}_0$$

by (A.10), using Remark A.2(a), where the component $e' \in \mathfrak{X}\{W^{0}_{[n-1]}\}$ corresponds to the given $\varepsilon_{[n-1]}\colon \mathfrak{M}_{\mathbf{\Theta}}W^{0}_{[n-1]} \to \mathfrak{X}$.

On the other hand, as before, e'' corresponds to the nullhomotopy $\underline{\widehat{F}} \in \mathfrak{X}\{\Omega C\}_1$, which actually lands in $P\mathfrak{X}\{\Omega C\}_0$.

The cosimplicial identity

$$\textbf{(A.40)}\qquad \widetilde{\varepsilon}_{[n]} \circ (d^0_0)^* = \widetilde{\varepsilon}_{[n]} \circ (d^1_0)^*\colon \mathfrak{M}_{\mathbf{\Theta}}\widetilde{W}^{1}_{[n]} \to \mathfrak{X}$$

may be verified using Lemma A.4 by representing both sides of the equation by elements in

$$\mathfrak{X}\{\widetilde{W}^{1}_{[n]}\}_0 = \mathfrak{X}\{W^{1}_{[n-1]}\}_0 \times \mathfrak{X}\{PC\}_0,$$

where the components in $\mathfrak{X}\{W^{1}_{[n-1]}\}_0$ agree because $(\boldsymbol{\varepsilon}_{[n-1]})^*\colon \mathfrak{M}_{\mathbf{\Theta}}W^{\bullet}_{[n-1]} \to \mathfrak{X}$ is an augmented simplicial $\mathbf{\Theta}$–mapping algebra, by the outer induction hypothesis. Here we are using the last assumption in Remark A.2 to deduce that $W^{1}_{[n-1]} \in \mathbf{\Theta}$.

Thus it remains to identify the corresponding components in $\mathfrak{X}\{PC\}_0$. On the one hand, $q^1_{[n]} \circ d^0_0\colon \widetilde{W}^{0}_{[n-1]} \to PC$ is the map $F^0 \circ \psi^0_{[n]}$, and $\widetilde{\varepsilon}_{[n]} \circ (\psi^0_{[n]})^*\colon \mathfrak{M}_{\mathbf{\Theta}}W^{0}_{[n-1]} \to \mathfrak{X}$ is the given $\varepsilon_{[n-1]}$ by construction, and since $(F^0)^*\colon \mathfrak{M}_{\mathbf{\Theta}}PC \to \mathfrak{M}_{\mathbf{\Theta}}W^{0}_{[n-1]}$ is represented according to Lemma A.4 by $\underline{F}^0 \in \mathfrak{M}_{\mathbf{\Theta}}W^{0}_{[n-1]}\{PC\}$, we see that the composite $\widetilde{\varepsilon}_{[n]} \circ (q^1_{[n]} \circ d^0_0)^*$ is represented in $\mathfrak{X}\{PC\}_0$ by

$$\varepsilon_{[n-1]}(\underline{F}^0) = (j \circ \iota_{\Omega})(\underline{a}),$$

according to (A.34).

On the other hand, $q^1_{[n]} \circ d^1_0\colon \widetilde{\boldsymbol{W}}^0_{[n-1]} \to P\boldsymbol{C}$ equals $\iota \circ p \circ q^0_{[n]}$, so

$$\widetilde{\varepsilon}_{[n]} \circ (d^1_0)^* = \widetilde{\varepsilon}_{[n]} \circ (q^0_{[n]})^* \circ (\iota \circ p)^*,$$

where $\widetilde{\varepsilon}_{[n]} \circ (q^0_{[n]})^*\colon \mathfrak{M}_{\boldsymbol{\Theta}} P\Omega\boldsymbol{C} \to \mathfrak{X}$ is represented by $\underline{\widehat{F}} \in \mathfrak{X}\{\Omega\boldsymbol{C}\}_1$, so

$$\widetilde{\varepsilon}_{[n]} \circ (q^0_{[n]})^* \circ (\iota \circ p)^*$$

is represented by

$$(j \circ \iota_\Omega)(d_0\underline{\widehat{F}}) = (j \circ \iota_\Omega)(\underline{a}) \in \mathfrak{X}\{P\boldsymbol{C}\}_0$$

as above.

This completes the n^{th} step of the outer induction in case (1).

VIII Making $\mathcal{W}$ cofibrant in case (2) Note that when we represent the attaching map $\overline{\partial}^n_0\colon \overline{V}_n \to Z_{n-1}V_\bullet$ for $V_\bullet$ by a map $\phi\colon \overline{V}_n \otimes_* S^{n-1} \to C_*V_\bullet$ as in Remark 1.14, ϕ extends to a map of *augmented* $(n-1)$–truncated chain complexes in Θ–Alg, ending in $C_{-1}V_\bullet := \Gamma$. Moreover, in case (2) the cochain map $F\colon C^*\boldsymbol{W}^\bullet_{[n-1]} \to \boldsymbol{D}^*$ realizing ϕ which we have constructed in the outer induction above also extends to a map of coaugmented $(n-1)$–truncated cochain complexes, with $C^{-1}\boldsymbol{W}^\bullet_{[n-1]} := \boldsymbol{Y}$, δ^{-1} given by $\boldsymbol{\varepsilon}_{[n-1]}$, and F^{-1} given by Lemma 4.5.

In the left Reedy model category structure on the category $\mathsf{Ch}^{\mathcal{C}}_{-1\le *\le n-1}$ of $(n-1)$–truncated coaugmented cochain complexes in $\mathcal{C}$, factor F as a cofibration

$$G\colon C^*\boldsymbol{W}^\bullet_{[n-1]} \hookrightarrow \boldsymbol{E}^*$$

followed by a trivial fibration $T\colon \boldsymbol{E}^* \twoheadrightarrow \boldsymbol{D}^*$. Applying the functorial construction of Definition 2.1(c) to G yields $\boldsymbol{W}^\bullet_{[n]}$ as required for the cofibrant sequential realization $\mathcal{W}$. The vertical trivial fibrations of (2.9) are induced by T. □

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