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On high-dimensional representations of knot groups

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Given a hyperbolic knot K and any $n \geq 2$ the abelian representations and the holonomy representation each give rise to an $(n-1)$–dimensional component in the $\mathrm{SL}(n,\mathbb{C})$–character variety. A component of the $\mathrm{SL}(n,\mathbb{C})$–character variety of dimension $\geq n$ is called high-dimensional.

It was proved by D Cooper and D Long that there exist hyperbolic knots with high-dimensional components in the $\mathrm{SL}(2,\mathbb{C})$–character variety. We show that given any nontrivial knot K and sufficiently large n the $\mathrm{SL}(n,\mathbb{C})$–character variety of K admits high-dimensional components.

57M25, 57M27; 57M50

1 Introduction

Given a knot $K \subset S^3$ we denote by $E_K = S^3 \setminus \nu K$ the knot exterior and we write $\pi_K = \pi_1(E_K)$. Furthermore, given a group G and $n \in \mathbb{N}$ we denote by $X(G,\mathrm{SL}(n,\mathbb{C}))$ the $\mathrm{SL}(n,\mathbb{C})$–character variety. We recall the precise definition in Section 2. It is straightforward to see that the abelian representations of a knot group π_K give rise to an $(n-1)$–dimensional subvariety of $X(\pi_K,\mathrm{SL}(n,\mathbb{C}))$ consisting solely of characters of abelian representations (see [17, Section 2]).

If K is hyperbolic, then we denote by $\widetilde{\mathrm{Hol}}\colon \pi_K \to \mathrm{SL}(2,\mathbb{C})$ a lift of the holonomy representation. For $n \geq 2$ we denote by $\zeta_n\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ the, up to conjugation, unique rational irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. P Menal-Ferrer and J Porti [20; 19] showed that for any n, the representation $\rho_n := \zeta_n \circ \widetilde{\mathrm{Hol}}$ is a smooth point of the $\mathrm{SL}(n,\mathbb{C})$–representation variety $R(\pi_k,\mathrm{SL}(n,\mathbb{C}))$. Moreover, Menal-Ferrer and Porti proved that the corresponding character χ_{ρ_n} is a smooth point on the character variety $X(\pi_K,\mathrm{SL}(n,\mathbb{C}))$; in particular it is contained in a unique component of dimension $n-1$ [20, Theorem 0.4]. Also, the deformations of reducible representations studied in [16] and [2] give rise to $(n-1)$–dimensional components in the character variety $X(\pi_K,\mathrm{SL}(n,\mathbb{C}))$.

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The above discussion shows that the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ of any given knot K contains an $(n-1)$–dimensional subvariety consisting of abelian representations, and if K is hyperbolic, $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ also contains an $(n-1)$–dimensional subvariety that contains characters of irreducible representations. Furthermore E Falbel and A Guilloux [13] showed that every component of $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ satisfying a mild technical hypothesis is of dimension at least $n-1$.

This motivates the following definition. Given a knot K we say that a component of $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ is *high-dimensional* if its dimension is greater than $n-1$. We summarize some known facts about the existence and nonexistence of high-dimensional components of character varieties of knot groups:

- For $n = 3$ and K a nonalternating torus knot, the variety $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has 3–dimensional components, whereas for alternating torus knots, $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has only 2–dimensional components. In particular for alternating torus knots, the variety $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ does not contain any high-dimensional components. For more details see [21, Theorem 1.1].
- For $n = 3$ and $K = 4_1$ the variety $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has five 2–dimensional components. Three of the five components contain characters of irreducible representations. There are no high-dimensional components. (See [17, Theorem 1.2].)
- It was proved by D Cooper and D Long [9, Section 8] that for a given n there exists an alternating hyperbolic knot K_n in S^3 such that the $\mathrm{SL}(2,\mathbb{C})$–character variety admits a component of dimension at least n.

The main result of this note is the proof that the $\mathrm{SL}(n,\mathbb{C})$–character variety of every nontrivial knot admits high-dimensional components for n sufficiently large. More precisely, building on Cooper, Long, and A Reid [10, Theorem 1.3] we prove the following theorem.

Theorem 1.1 *Let $K \subset S^3$ be a nontrivial knot. Then for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ contains a high-dimensional component.*

Given a group G we now denote by $X^{\mathrm{irr}}(G, \mathrm{SL}(n,\mathbb{C}))$ the character variety corresponding to irreducible representations. We refer the reader to Section 3 for the precise definition. The following is then a more refined version of Theorem 1.1.

Theorem 1.2 *Let $K \subset S^3$ be a nontrivial knot. Then given any $N \in \mathbb{N}$ there exists a $p \geq N$ such that $X^{\mathrm{irr}}(\pi_K, \mathrm{SL}(p,\mathbb{C}))$ contains a high-dimensional component.*

Remark It is straightforward to generalize the proofs of the above theorems to any knot K in a rational homology sphere Σ such that $\Sigma \setminus \nu K \neq S^1 \times D^2$ and such that $\Sigma \setminus \nu K$ is irreducible. We restrict ourselves in the exposition to knots in S^3, to simplify the notation and to make it easier to compare our results to earlier results.

In the case of the figure-eight knot we obtain in Section 4.2 a refined quantitative result:

Proposition 1.3 *Let $K \subset S^3$ be the figure-eight knot. Then for all $n \in \mathbb{N}$ the representation variety $X(\pi_K, \mathrm{SL}(10n, \mathbb{C}))$ has a component C of dimension at least $4n^2 - 1$. Moreover, C contains characters of irreducible representations.*

Remark For a free group F_r we have $\dim X(F_r, \mathrm{SL}(n, \mathbb{C}))/(n^2 - 1) = (r - 1)$; hence

$$\limsup_{n\to\infty} \frac{\dim X(G, \mathrm{SL}(n, \mathbb{C}))}{n^2 - 1} \leq (r - 1)$$

if G is generated by r elements. It follows from Proposition 1.3 that for the figure-eight knot $K = 4_1$ the following inequality holds:

$$\tfrac{1}{25} \leq \limsup_{n\to\infty} \frac{\dim X(\pi_K, \mathrm{SL}(n, \mathbb{C}))}{n^2 - 1} \leq 1.$$

At the end of Section 4.2 we discuss the question of the extent to which the result of Proposition 1.3 can be generalized to other 2–bridge knots.

2 Representation and character varieties

Before proving Theorem 1.1, we recall some definitions and facts. The general reference for representation and character varieties is the book [18] of A Lubotzky and A Magid.

Given two representations $\rho_1\colon G \to \mathrm{GL}(n_1, \mathbb{C})$ and $\rho_2\colon G \to \mathrm{GL}(n_2, \mathbb{C})$ we define the *direct sum* $\rho_1 \oplus \rho_2\colon G \to \mathrm{GL}(n_1 + n_2, \mathbb{C})$ by

$$(\rho_1 \oplus \rho_2)(\gamma) = \left(\begin{array}{c|c} \rho_1(\gamma) & 0 \\ \hline 0 & \rho_2(\gamma) \end{array}\right).$$

We also define the *tensor product* $\rho_1 \otimes \rho_2\colon G \to \mathrm{GL}(n_1 \cdot n_2, \mathbb{C}) = \mathrm{Aut}(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$ by

$$((\rho_1 \otimes \rho_2)(\gamma))(v_1 \otimes v_2) = \rho_1(\gamma)(v_1) \otimes \rho_2(\gamma)(v_2).$$

Definition We call a representation $\rho\colon G \to \mathrm{GL}(n, \mathbb{C})$ *reducible* if there exists a nontrivial, proper subspace $V \subset \mathbb{C}^n$ such that V is $\rho(G)$–stable. The representation ρ is called *irreducible* or *simple* if it is not reducible. A *semisimple* representation is a direct sum of simple representations.

Let $G = \langle g_1, \dots, g_r \rangle$ be a finitely generated group. A $\mathrm{SL}(n,\mathbb{C})$–representation is a homomorphism $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$. The $\mathrm{SL}(n,\mathbb{C})$–representation variety is

$$R(G,\mathrm{SL}(n,\mathbb{C})) = \mathrm{Hom}(G,\mathrm{SL}(n,\mathbb{C})) \subset \mathrm{SL}(n,\mathbb{C})^r \subset M_n(\mathbb{C})^r \cong \mathbb{C}^{n^2 r}.$$

The representation variety $R(G,\mathrm{SL}(n,\mathbb{C}))$ is an affine algebraic set. It is contained in $\mathrm{SL}(n,\mathbb{C})^r$ via the inclusion $\rho \mapsto (\rho(g_1), \dots, \rho(g_r))$, and it is the set of solutions of a system of polynomial equations in the matrix coefficients.

The group $\mathrm{SL}(n,\mathbb{C})$ acts by conjugation on $R(G,\mathrm{SL}(n,\mathbb{C}))$. More precisely, for $A \in \mathrm{SL}(n,\mathbb{C})$ and $\rho \in R(G,\mathrm{SL}(n,\mathbb{C}))$ we define $(A.\rho)(g) = A\rho(g)A^{-1}$ for all $g \in G$. In what follows we will write $\rho \sim \rho'$ if there exists an $A \in \mathrm{SL}(n,\mathbb{C})$ such that $\rho' = A.\rho$, and we will call ρ and ρ' *equivalent*. For $\rho \in R(G,\mathrm{SL}(n,\mathbb{C}))$ we define its *character* $\chi_\rho\colon G \to \mathbb{C}$ by $\chi_\rho(\gamma) = \mathrm{tr}(\rho(\gamma))$. We have $\rho \sim \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$. Moreover, if ρ and ρ' are semisimple, then $\rho \sim \rho'$ if and only if $\chi_\rho = \chi_{\rho'}$. (See Theorems 1.27 and 1.28 in the book [18] of Lubotzky and Magid.)

The *algebraic quotient* or *GIT quotient* for the action of $\mathrm{SL}(n,\mathbb{C})$ on $R(G,\mathrm{SL}(n,\mathbb{C}))$ is called the *character variety*. This quotient will be denoted by $X(G,\mathrm{SL}(n,\mathbb{C})) = R(G,\mathrm{SL}(n,\mathbb{C}))/\!\!/\,\mathrm{SL}(n,\mathbb{C})$. The character variety is not necessarily an irreducible affine algebraic set. Work of C Procesi [23] implies that there exists a finite number of group elements $\{\gamma_i \mid 1 \le i \le M\} \subset G$ such that the image of $t\colon R(G,\mathrm{SL}(n,\mathbb{C})) \to \mathbb{C}^M$ given by

$$t(\rho) = (\chi_\rho(\gamma_1), \dots, \chi_\rho(\gamma_M))$$

can be identified with the affine algebraic set $X(G,\mathrm{SL}(n,\mathbb{C})) \cong t(R(G,\mathrm{SL}(n,\mathbb{C})))$; see also [18, page 27]. This justifies the name *character variety*. For an introduction to algebraic invariant theory see I Dolgachev's book [12]. For a brief introduction to $\mathrm{SL}(n,\mathbb{C})$–representation and character varieties of groups see [15].

Example 2.1 For a free group F_r of rank r we have $R(F_r,\mathrm{SL}(n,\mathbb{C})) \cong \mathrm{SL}(n,\mathbb{C})^r$ is an irreducible algebraic variety of dimension $r(n^2-1)$, and the dimension of the character variety $X(F_k,\mathrm{SL}(n,\mathbb{C}))$ is $(r-1)(n^2-1)$.

The first homology group of the knot exterior is isomorphic to $\mathbb{Z}$. A canonical surjection $\varphi\colon \pi_K \to \mathbb{Z}$ is given by $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$ where lk denotes the linking number in S^3 (see [7, 3.B]). Hence, every *abelian* representation of a knot group π_K factors through $\varphi\colon \pi_K \to \mathbb{Z}$. Here, we call ρ *abelian* if its image is abelian. Therefore, we obtain for each nonzero complex number $\eta \in \mathbb{C}^*$ an abelian representation

$\eta^\varphi\colon \pi_K \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$ given by $\gamma \mapsto \eta^{\varphi(\gamma)}$. Notice that a 1–dimensional representation is always irreducible.

Let W be a finite dimensional $\mathbb{C}$–vector space. For every representation $\rho\colon G \to \mathrm{GL}(W)$ the vector space W turns into a left $\mathbb{C}[G]$–module via ρ. This $\mathbb{C}[G]$–module will be denoted by W_ρ or simply W if no confusion can arise. Notice that every finite dimensional $\mathbb{C}$–vector space W which is a left $\mathbb{C}[G]$–module gives a representation $\rho\colon G \to \mathrm{GL}(W)$, and by fixing a basis of W we obtain a matrix representation.

The following lemma follows from Proposition 1.7 in [18] and the discussion therein.

Lemma 2.2 *Any group epimorphism $\alpha\colon G \twoheadrightarrow F$ between finitely generated groups induces a closed embedding $R(F,\mathrm{SL}(n,\mathbb{C})) \hookrightarrow R(G,\mathrm{SL}(n,\mathbb{C}))$ of algebraic varieties, and an injection*

$$X(F,\mathrm{SL}(n,\mathbb{C})) \hookrightarrow X(G,\mathrm{SL}(n,\mathbb{C})).$$

Let $H \le G$ be a subgroup of finite index. Then the restriction of a representation $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ to H will be denoted by $\mathrm{res}_H^G\,\rho$ or simply by $\rho|_H$ if no confusion can arise. This restriction is compatible with the action by conjugation and it induces a regular map $\nu\colon X(G,\mathrm{SL}(n,\mathbb{C})) \to X(H,\mathrm{SL}(n,\mathbb{C}))$. In what follows we will make use of the following result of A S Rapinchuk which follows directly from [24, Lemma 1].

Lemma 2.3 *If $H \le G$ is a subgroup of finite index k, then*

$$\nu\colon X(G,\mathrm{SL}(n,\mathbb{C})) \to X(H,\mathrm{SL}(n,\mathbb{C}))$$

has finite fibers.

2.1 The induced representation

Let G be a group and let $H \le G$ be a subgroup of finite index k. Given a representation $\alpha\colon H \to \mathrm{GL}(m,\mathbb{C})$ we refer to the representation of G that is given by left multiplication by G on

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}^m$$

as the induced representation. We denote by $e_1,\dots,e_m$ the standard basis of $\mathbb{C}^m$ and we pick representatives $g_1,\dots,g_k$ of G/H. It is straightforward to see that $g_i \otimes e_j$ with $i \in \{1,\dots,k\}$ and $j \in \{1,\dots,m\}$ form a basis for $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}^m$ as a complex vector space. Using the ordered basis

$$g_1 \otimes e_1,\ \dots,\ g_1 \otimes e_m,\ \dots,\ g_k \otimes e_1,\ \dots,\ g_k \otimes e_m,$$

the induced representation can be viewed as a representation $\mathrm{ind}_H^G\,\alpha\colon G \to \mathrm{GL}(mk,\mathbb{C})$.

If $\alpha\colon H \to \mathrm{SL}(m, \mathbb{C})$ is a representation into the special linear group, then for $g \in G$ a priori the determinant of $\mathrm{ind}_H^G\, \alpha(g)$ is in $\{\pm 1\}$. But it is straightforward to see that if m is even, then $\mathrm{ind}_H^G\, \alpha$ defines in fact a representation $G \to \mathrm{SL}(mk, \mathbb{C})$.

Lemma 2.4 *Let m be even and $H \leq G$ a subgroup of finite index k. Then the map*

$$\iota\colon R(H, \mathrm{SL}(m, \mathbb{C})) \to R(G, \mathrm{SL}(mk, \mathbb{C}))$$

given by $\iota(\alpha) = \mathrm{ind}_H^G\, \alpha$ is an injective algebraic map. It depends on the choice of a system of representatives, and it is compatible with the action of $\mathrm{SL}(m, \mathbb{C})$ and $\mathrm{SL}(mk, \mathbb{C})$ respectively.

Moreover, the corresponding regular map (which does not depend on the choice of a system of representatives)

$$\overline{\iota}\colon X(H, \mathrm{SL}(m, \mathbb{C})) \to X(G, \mathrm{SL}(mk, \mathbb{C}))$$

has finite fibers.

Proof A very detailed proof of the first statement can be found in [11, Section 10.A] (see also [18, pages 9–10] and [24]). The second part is [24, Lemma 3]. □

Finally we recall the following theorem of Cooper, Long, and Reid [10, Theorem 1.3] (see also [8, Corollary 6] or [1, page 94]).

Theorem 2.5 *Let N be an irreducible 3–manifold with incompressible nonempty toroidal boundary.* (In our application N will be the exterior of a nontrivial knot in S^3.) *Then $\pi_1(N)$ admits a finite-index subgroup H that admits an epimorphism $\alpha\colon H \to F_2$ onto a free group on two generators.*

We are now in the position to prove our main result:

Proof of Theorem 1.1 Let K be a nontrivial knot. We write $G = \pi_1(E_K)$. By Theorem 2.5 the group G admits a finite-index subgroup H that admits an epimorphism $\alpha\colon H \to F_2$ onto a free group on two generators. It is clear (see Example 2.1) that $R(F_2, \mathrm{SL}(m, \mathbb{C})) \cong \mathrm{SL}(m, \mathbb{C})^2$, and

$$\dim X(F_2, \mathrm{SL}(m, \mathbb{C})) = m^2 - 1.$$

It follows from Lemma 2.2 that the variety $X(H, \mathrm{SL}(m, \mathbb{C}))$ has a component of dimension at least $m^2 - 1$. We denote by k the index of H in G, and we will suppose that m is even. Then it follows from Lemma 2.4 that $X(G, \mathrm{SL}(mk, \mathbb{C}))$ contains an irreducible component of dimension at least $m^2 - 1$.

Now for all $m > k$ we have $m^2 - 1 > mk - 1$. Therefore, for a given $N \in \mathbb{N}$ we choose an $m \in \mathbb{N}$ that is even and greater than k such that $n := mk \geq N$. The character variety $X(\pi_K, \mathrm{SL}(n, \mathbb{C}))$ contains an irreducible component whose dimension is bigger than $m^2 - 1 > mk - 1 = n - 1$. $\square$

3 Proof of Theorem 1.2

We let $R^{\mathrm{irr}}(G, \mathrm{SL}(n, \mathbb{C})) \subset R(G, \mathrm{SL}(n, \mathbb{C}))$ denote the Zariski-open subset of irreducible representations. The set $R^{\mathrm{irr}}(G, \mathrm{SL}(n, \mathbb{C}))$ is invariant by the $\mathrm{SL}(n, \mathbb{C})$–action, and we will denote by $X^{\mathrm{irr}}(G, \mathrm{SL}(n, \mathbb{C})) \subset X(G, \mathrm{SL}(n, \mathbb{C}))$ its image in the character variety. Notice that $X^{\mathrm{irr}}(G, \mathrm{SL}(n, \mathbb{C}))$ is an orbit space for the action of $\mathrm{SL}(n, \mathbb{C})$ on $R^{\mathrm{irr}}(G, \mathrm{SL}(n, \mathbb{C}))$ (see [22, Chapter 3, Section 3]).

Before we can give the proof of Theorem 1.2 we need to introduce several further definitions. These notations are classic (see [28; 4] for more details).

Let H and K be two subgroups of finite index of G, and let $\alpha\colon H \to \mathrm{GL}(W)$ be a linear representation. Then for all $g \in G$ we obtained the *twisted* representation $\alpha^g\colon gHg^{-1} \to \mathrm{GL}(W)$ given by

$$\alpha^g(x) = \alpha(g^{-1}xg) \quad \text{for } x \in gHg^{-1}. \tag{1}$$

Notice that the twisted representation α^g is irreducible or semisimple if and only if α is irreducible or semisimple respectively.

Now, we choose a set of representatives S of the (K, H) double cosets of G. For $s \in S$, we let $H_s = sHs^{-1} \cap K \leq K$. We obtain a homomorphism $\mathrm{res}_{H_s}^{sHs^{-1}} \alpha^s\colon H_s \to \mathrm{GL}(W)$ by restriction of α^s to H_s. The representation $\mathrm{res}_K^G \mathrm{ind}_H^G \alpha$ is equivalent to the direct sum of twisted representations:

$$\mathrm{res}_K^G \mathrm{ind}_H^G \alpha \cong \bigoplus_{s \in S} \mathrm{ind}_{H_s}^K \mathrm{res}_{H_s}^{sHs^{-1}} \alpha^s. \tag{2}$$

Equation (2) takes a simple form if $H = N = K$ is a normal subgroup of finite index of G. We obtain

$$\mathrm{res}_N^G \mathrm{ind}_N^G \alpha \cong \bigoplus_{s \in S} \alpha^s, \tag{3}$$

where S is a set of representatives of the N cosets of G.

In what follows we will make use of the following lemmas:

Lemma 3.1 *Let G be a group, $H \leq G$ a subgroup of finite index, and $\rho\colon G \to \mathrm{GL}(V)$ a representation. If $\mathrm{res}^G_H \rho\colon H \to \mathrm{GL}(V)$ is semisimple, then $\rho\colon G \to \mathrm{GL}(V)$ is semisimple.*

Proof This is [30, Theorem 1.5]. □

Lemma 3.2 *Let G be a group, $H \leq G$ a subgroup of finite index, and $\alpha\colon H \to \mathrm{GL}(W)$ a representation. If α is irreducible, then $\mathrm{ind}^G_H \alpha$ is semisimple.*

Proof We can choose a normal subgroup $N \trianglelefteq G$ of finite index such that $N \leq H$. More precisely, we can take

$$N = \bigcap_g gHg^{-1}$$

to be the *normal core* of H in G. We choose a set of representatives S of the (N, H) double cosets of G. In this case we obtain that $H_s = sHs^{-1} \cap N = s(H \cap N)s^{-1} = N$, and the double coset NsH is equal to sH since $N \subset H$ is normal. Therefore, (2) gives

$$\mathrm{res}^G_N \mathrm{ind}^G_H \alpha \cong \bigoplus_{s \in S} \mathrm{res}^{sHs^{-1}}_N \alpha.$$

Now, $\mathrm{res}^{sHs^{-1}}_N \alpha\colon N \to \mathrm{GL}(W)$ is a *twist* of $\alpha|_N$ ie for all $g \in N$ we have

$$\mathrm{res}^{sHs^{-1}}_N \alpha(g) = \alpha(s^{-1}gs) = (\alpha|_N)^s(g).$$

By Clifford's theorem [30, Theorem 1.7], we obtain that $\alpha|_N$ is semisimple. We have that $\alpha|_N = \alpha_1 \oplus \cdots \oplus \alpha_k$ is a direct sum of simple representations. Therefore,

$$(\mathrm{ind}^G_H \alpha)|_N \cong \bigoplus_{s \in S} \alpha_1^s \oplus \cdots \oplus \alpha_k^s$$

is the direct sum of irreducible representations. This proves that $(\mathrm{ind}^G_H \alpha)|_N$ is semisimple, and it follows from Lemma 3.1 that $\mathrm{ind}^G_H \alpha$ is semisimple. □

Corollary 3.3 *Let G be a group, and let $N \lhd G$ be a normal subgroup of finite index. If $\alpha\colon N \to \mathrm{SL}(V)$ is irreducible, then $\mathrm{ind}^G_N(\alpha)$ is semisimple.*

Moreover, if $\mathrm{ind}^G_N(\alpha) \cong \rho_1 \oplus \cdots \oplus \rho_l$ is a decomposition of $\mathrm{ind}^G_N(\alpha)$ into irreducible representations $\rho_j\colon G \to \mathrm{SL}(V_j)$, then $\dim V$ divides $\dim V_j$ and hence

$$\dim V \leq \dim V_j \leq \dim(V) \cdot [G : N].$$

Proof The first part follows directly from (3) and Lemma 3.1 since α^s is irreducible for all $s \in G$. Notice that S is now a set of representatives of the cosets G/N. Moreover, we obtain

$$\bigoplus_{s\in S}\alpha^s \cong \operatorname{res}_N^G \operatorname{ind}_N^G \alpha \cong (\rho_1|_N)\oplus\cdots\oplus(\rho_l|_N).$$

If $\rho_j|_N$ is irreducible, then it must be isomorphic to one of the twisted representations α^s. Otherwise $\rho_j|_N$ is isomorphic to a direct sum of twisted representations α^s, $s\in S$, and hence $\dim V_j$ is a multiple of $\dim V$. □

Lemma 3.4 *Let G be a group and let $H\le G$ be a finite-index subgroup.*

If there exists a surjective homomorphism $\varphi\colon H\twoheadrightarrow F_2$ onto a free group of rank two, then there exists a normal subgroup $N\trianglelefteq G$ of finite index such that $N\le H$, and $\varphi(N)\le F_2$ is a free group of finite rank $r\ge 2$.

Proof Let N be the normal core of H ie $N=\bigcap_{g\in G} gHg^{-1}\trianglelefteq G$. The normal subgroup is a finite-index subgroup of G, and $N\le H$. Now, $H\ge\varphi^{-1}(\varphi(N))\ge N$, and therefore $\varphi^{-1}(\varphi(N))$ is also of finite index in H (and hence in G). Hence, $\varphi(N)\trianglelefteq F_2$ is of finite index, and $\varphi(N)$ is a free group of rank $r=[F_2:\varphi(N)]+1\ge 2$. □

We conclude this section with the following theorem.

Theorem 3.5 *Let $K\subset S^3$ be a nontrivial knot. Then there exists $k\in\mathbb{N}$ such that for all even $m\in\mathbb{N}$ there exists $p\in\mathbb{N}$ such that $m\le p\le mk$, and*

$$\dim X^{\mathrm{irr}}(\pi_K,\mathrm{SL}(p,\mathbb{C}))\ge\frac{m^2-k}{k}.$$

In particular, for m even with $m>k^2$ there exists $p\in\mathbb{N}$ such that $m\le p<m\sqrt{m}$, and

$$\dim X^{\mathrm{irr}}(\pi_K,\mathrm{SL}(p,\mathbb{C}))\ge\frac{m^2-k}{k}>km-1\ge p-1.$$

The last sequence of inequalities in Theorem 3.5 implies that $X^{\mathrm{irr}}(\pi_K,\mathrm{SL}(p,\mathbb{C}))$ contains a high-dimensional component. In particular, Theorem 3.5 implies Theorem 1.2 from the introduction.

Proof By Lemma 3.4 there exists a finite-index normal subgroup $N\trianglelefteq\pi_K$ of the knot group π_K, and an epimorphism $\psi\colon N\twoheadrightarrow F_2$. We put $k=[\pi_K:N]$.

For each even $m\in\mathbb{N}$, we obtain a regular map

$$\psi^*\colon X^{\mathrm{irr}}(F_2,\mathrm{SL}(m,\mathbb{C}))\to X(N,\mathrm{SL}(m,\mathbb{C}))$$

and we let $C\subset X(N,\mathrm{SL}(m,\mathbb{C}))$ denote the image of ψ^*. By Chevalley's theorem [3, Corollary 10.2], the set $C\subset X(N,\mathrm{SL}(m,\mathbb{C}))$ is constructible. Again, by Chevalley's theorem the image $D:=\overline{\iota}(C)$, where $\overline{\iota}\colon X(N,\mathrm{SL}(m,\mathbb{C}))\to X(\pi_K,\mathrm{SL}(km,\mathbb{C}))$, is also a constructible set. Notice that $\dim C = \dim D = m^2 - 1$ since ψ^* is an embedding by Lemma 2.2 and since $\bar{\iota}$ has finite fibers by Lemma 2.3.

If D contains a character of an irreducible representation, then

$$D \cap X^{\mathrm{irr}}(\pi_K, \mathrm{SL}(km, \mathbb{C}))$$

contains a Zariski-open subset of $\bar{D}$ which is of dimension $m^2 - 1 \geq (m^2 - k)/k$. Hence the conclusion of the theorem is satisfied for $p = km$.

If D does not contain irreducible representations, then

$$D \subset X^{\mathrm{red}}(\pi_K, \mathrm{SL}(km, \mathbb{C})).$$

In this case we can choose for a given $\chi \in D$ a semisimple representation ρ such that $\chi_\rho = \chi$. Now, we follow the argument in the proof of Corollary 3.3 and we obtain $\rho \sim \rho_1 \oplus \cdots \oplus \rho_l$, where $\rho_j \colon \pi_K \to \mathrm{GL}(p_j, \mathbb{C})$ and $m \leq p_j < km$. For the l–tuple $(p_1, \dots, p_l)$ we consider the regular map

$$\Phi_{(p_1,\dots,p_l)} \colon R(\pi_K, \mathrm{SL}(p_1, \mathbb{C})) \times \cdots \times R(\pi_K, \mathrm{SL}(p_l, \mathbb{C})) \times (\mathbb{C}^*)^{l-1} \to R(\pi_K, \mathrm{SL}(km, \mathbb{C}))$$

given by

$$\Phi_{(p_1,\dots,p_l)}(\rho_1, \dots, \rho_l, \lambda_1, \dots, \lambda_{l-1}) = \bigoplus_{i=1}^{l-1} (\rho_i \otimes \lambda_i^{p_l \varphi}) \oplus (\rho_l \otimes (\lambda_1^{-p_1} \cdots \lambda_{l-1}^{-p_{l-1}})^{\varphi}).$$

The map $\Phi_{(p_1,\dots,p_l)}$ induces a map $\bar{\Phi} := \bar{\Phi}_{(p_1,\dots,p_l)}$ between the character varieties

$$\bar{\Phi} \colon X(\pi_K, \mathrm{SL}(p_1, \mathbb{C})) \times \cdots \times X(\pi_K, \mathrm{SL}(p_l, \mathbb{C})) \times (\mathbb{C}^*)^{l-1} \to X^{\mathrm{red}}(\pi_K, \mathrm{SL}(km, \mathbb{C})) \tag{4}$$

(see [18, page 20]).

By Lemma 3.7, the map $\bar{\Phi}_{(p_1,\dots,p_l)}$ also has finite fibers; denote its image by $D_{(p_1,\dots,p_l)}$. Again, by Chevalley's theorem, the image $D_{(p_1,\dots,p_l)} \subset X^{\mathrm{red}}(\pi_K, \mathrm{SL}(km, \mathbb{C}))$ is a constructible set, and

$$\dim D_{(p_1,\dots,p_l)} = \sum_{j=1}^{l} \dim X(\pi_K, \mathrm{SL}(p_j, \mathbb{C})) + l - 1. \tag{5}$$

By Corollary 3.3, D is covered by finitely many sets of the form $D_{(p_1,\dots,p_l)}$ (see also [18, Proposition 1.29]). Since $\dim D = m^2 - 1$, there must be at least one set $D_{(p_1,\dots,p_l)}$ of dimension at least $m^2 - 1$. If we apply (5) to this choice we obtain that

$$\sum_{j=1}^{l} \dim X(\pi_K, \mathrm{SL}(p_j, \mathbb{C})) \geq m^2 - l.$$

In particular there is a j such that the corresponding summand is greater than or equal to $(m^2-l)/l$. Note that from $m \le p_j \le mk$ for $j=1,\dots,l$ and $p_1+\cdots+p_l=mk$ it follows that $l \le k$ which in turn implies that $(m^2-l)/l \ge (m^2-k)/k$. Summarizing we see that

$$\dim X^{\mathrm{irr}}(\pi_K, \mathrm{SL}(p_j,\mathbb{C})) \ge \frac{m^2-k}{k}.$$

This concludes the proof of the first statement of the theorem.

The second statement follows from the first statement using some elementary algebraic inequalities. □

In order to prove Lemma 3.7 we will make use of Schur's lemma (see [14, 1.7]).

Lemma 3.6 (Schur's lemma) *If V and W are irreducible representations of G and $f\colon V \to W$ is a $\mathbb{C}[G]$–module homomorphism, then:*

(1) *Either f is an isomorphism or $f=0$.*
(2) *If $V=W$, then $f=\lambda\cdot\mathrm{id}$ for some $\lambda\in\mathbb{C}$.*

Lemma 3.7 *The map $\overline{\Phi} := \overline{\Phi}_{(p_1,\dots,p_l)}$ from* (4),

$$\overline{\Phi}\colon X(\pi_K, \mathrm{SL}(p_1,\mathbb{C})) \times\cdots\times X(\pi_K, \mathrm{SL}(p_l,\mathbb{C})) \times (\mathbb{C}^*)^{l-1} \to X(\pi_K, \mathrm{SL}(km,\mathbb{C})),$$

has finite fibers.

Proof We show that $\overline{\Phi}_{(p_1,\dots,p_l)}$ is a composition of maps having finite fibers.

Let G be a finitely generated group, and let $n_1+\cdots+n_k=n$ be a partition of n. Then the map

$$\Psi\colon R(G,\mathrm{GL}(n_1,\mathbb{C}))\times\cdots\times R(G,\mathrm{GL}(n_k,\mathbb{C})) \to R(G,\mathrm{GL}(n,\mathbb{C}))$$

given by $\Psi(\varrho_1,\dots\varrho_k)=\varrho_1\oplus\cdots\oplus\varrho_k$ induces a regular map

$$\overline{\Psi}\colon X(G,\mathrm{GL}(n_1,\mathbb{C}))\times\cdots\times X(G,\mathrm{GL}(n_k,\mathbb{C})) \to X(G,\mathrm{GL}(n,\mathbb{C}))$$

(see [18, page 20]). Let $\chi\in X(G,\mathrm{GL}(n,\mathbb{C}))$ be in the image of $\overline{\Psi}$. We will use that a character $\chi=\chi_\varrho$ is the character of a semisimple representation ϱ, which is unique up to conjugation [18, Theorem 1.28]. The representation $\varrho\sim\varrho_1\oplus\cdots\oplus\varrho_l$ decomposes into irreducible representations. Schur's lemma implies that the representations ϱ_j are unique up to equivalence [14, Proposition 1.8]. If $\overline{\Psi}(\chi_1,\dots,\chi_k)=\chi$ then each χ_i is also the character of a semisimple representation α_i, and $\alpha_1\oplus\cdots\oplus\alpha_k\sim\varrho$. Now, the irreducible representations which can occur in the decomposition of α_i are equivalent to the representations ϱ_j. This means that the semisimple representations α_i such that $\Psi(\alpha_1,\dots,\alpha_k)$ is equivalent to ϱ, can have among their irreducible components only representations equivalent to $\varrho_1,\dots,\varrho_l$. Hence $\overline{\Psi}^{-1}(\chi)$ is finite.

Next, for $G=\pi_K$ and the surjection $\varphi\colon \pi_K \twoheadrightarrow \mathbb{Z}$ we consider the map

$$\Lambda_p\colon R(\pi_K,\mathrm{SL}(p,\mathbb{C}))\times\mathbb{C}^*\to R(\pi_K,\mathrm{GL}(p,\mathbb{C})),\quad \Lambda_p(\rho,\lambda)=\rho\otimes\lambda^{\varphi}.$$

The cyclic group $C_p=\langle\omega\rangle$ of p^{th} roots of unity acts on $R(\pi_K,\mathrm{SL}(p,\mathbb{C}))\times\mathbb{C}^*$ by $\omega(\rho,\lambda)=(\rho\otimes\omega^{\varphi},\varpi\lambda)$. This action commutes with the action of $\mathrm{SL}(p,\mathbb{C})$ by conjugation. We obtain an action of C_p on $X(\pi_K,\mathrm{SL}(p,\mathbb{C}))\times\mathbb{C}^*$. The map

$$\overline{\Lambda}_p\colon X(\pi_K,\mathrm{SL}(p,\mathbb{C}))\times\mathbb{C}^*\to X(\pi_K,\mathrm{GL}(p,\mathbb{C}))$$

factors through $(X(\pi_K,\mathrm{SL}(p,\mathbb{C}))\times\mathbb{C}^*)/C_p\cong X(\pi_K,\mathrm{GL}(p,\mathbb{C}))$ (see Lemma 2.2 in [17]). Hence $\overline{\Lambda}_p$ has finite fibers.

Note that the map $\varepsilon\colon(\mathbb{C}^*)^{l-1}\to(\mathbb{C}^*)^l$ given by

$$\varepsilon(\lambda_1,\dots,\lambda_{l-1})=(\lambda_1^{p_l},\dots,\lambda_{l-1}^{p_l},(\lambda_1^{p_1}\cdots\lambda_{l-1}^{p_{l-1}})^{-1})$$

has finite fibers. Finally,

$$\overline{\Phi}_{(p_1,\dots,p_l)}=\overline{\Psi}\circ(\overline{\Lambda}_{p_1}\times\cdots\times\overline{\Lambda}_{p_l})\circ(\mathrm{id}\times\cdots\times\mathrm{id}\times\varepsilon)$$

has finite fibers. □

4 The character variety of the figure-eight knot

The aim of this section is to prove Proposition 1.3. Before studying the character variety of the figure-eight knot we will extend Mackey's irreducibility criterion to infinite groups. Suppose that $H\le G$ is a subgroup, and that W is a representation of H. Mackey's irreducibility criterion gives necessary and sufficient conditions under which $\mathrm{ind}_H^G W$ is irreducible as a $\mathbb{C}[G]$–module.

Let $H\le G$ be a subgroup of finite index, and let $\alpha\colon H\to\mathrm{GL}(W)$ be a representation. For $s\in G$ we obtain the twisted representation $\alpha^s\colon sHs^{-1}\to\mathrm{GL}(W)$ (see (1)). Define

$$H_s:=sHs^{-1}\cap H\quad\text{and}\quad W_s:=\mathrm{res}_{H_s}^{sHs^{-1}}W_{\alpha^s}.$$

In what follows we call two semisimple representations V and V' of G *disjoint* if $\mathrm{Hom}_{\mathbb{C}[G]}(V,V')=0$. The aim of the next subsection is to prove the following:

Proposition 4.1 (Mackey's irreducibility criterion) *Let $H \le G$ be a subgroup of finite index. We suppose that $\alpha\colon H \to \mathrm{GL}(W)$ is an irreducible representation. Then $\mathrm{ind}_H^G \alpha$ is irreducible if and only if for all $s \in G - H$ the $\mathbb{C}[H_s]$–modules W_s and $\mathrm{res}_{H_s}^H W$ are disjoint.*

Remark Proposition 4.1 is well known for finite groups (see [28]). We will see that it also holds for infinite groups under the assumption that the subgroup $H \le G$ has finite index. The main difference is that for infinite groups the induced and *coinduced* modules are in general not isomorphic. We will make use of Lemma 4.2 in place of the classical Frobenius reciprocity theorem.

4.1 Mackey's criterion

We follow Serre's notation and exposition in [28]. First let us recall the adjoint isomorphism. Let Q and R be rings and let ${}_QS$, ${}_RT_Q$, and ${}_RU$ be modules (here the location of the indices indicates whether these are left or right module structures). Then $\mathrm{Hom}_R(T,U)$ is a left Q–module via $qf(v) = f(vq)$ for all $v \in V$, and

$$\mathrm{Hom}_Q(S, \mathrm{Hom}_R(T,U)) \cong \mathrm{Hom}_R(T \otimes_Q S, U) \tag{6}$$

(see [27, Theorem 2.76]).

Lemma 4.2 *Let G be a group and $H \le G$ a subgroup of finite index. For each left $\mathbb{C}[H]$–module W and a left $\mathbb{C}[G]$–module V we obtain*

$$\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathrm{ind}_H^G W) \cong \mathrm{Hom}_{\mathbb{C}[H]}(\mathrm{res}_H^G V, W), \tag{7}$$

and

$$\mathrm{Hom}_{\mathbb{C}[H]}(W, \mathrm{res}_H^G V) \cong \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{ind}_H^G W, V). \tag{8}$$

Proof For proving (7) we apply (6) with $Q = \mathbb{C}[G]$, $R = \mathbb{C}[H]$, $S = V$, $T = \mathbb{C}[G]$, and $U = W$:

$$\mathrm{Hom}_{\mathbb{C}[G]}(V, \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)) \cong \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G] \otimes_{\mathbb{C}[G]} V, W).$$

Since $H \le G$ is of finite index we obtain that the coinduced module $\mathrm{coind}_H^G(W) := \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)$ and the induced module $\mathrm{ind}_H^G(W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ are isomorphic as left $\mathbb{C}[G]$–modules (see [4, Lemma III.5.9]). Moreover, $\mathrm{res}_H^G(U)$ and $\mathbb{C}[G] \otimes_{\mathbb{C}[G]} U$ are isomorphic as left $\mathbb{C}[H]$–modules, and (7) follows.

To prove (8) we apply (6) with $Q=\mathbb{C}[H]$, $R=\mathbb{C}[G]$, $S=W$, $T=\mathbb{C}[G]$, $U=V$:

$$\mathrm{Hom}_{\mathbb{C}[H]}(W,\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G],V))\cong\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W,V).$$

The left $\mathbb{C}[H]$–module $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G],V)$ is isomorphic to $\mathrm{res}_H^G(V)$, and hence (8) follows. □

Before proving Mackey's criterion we state the following:

Lemma 4.3 *Let V be a semisimple G–module. Then V is irreducible if and only if* $\mathrm{Hom}_{\mathbb{C}[G]}(V,V)\cong\mathbb{C}$.

Proof Let $V\cong V_1\oplus\cdots\oplus V_k$ be a decomposition of V into irreducible modules. Then

$$\mathrm{Hom}_{\mathbb{C}[G]}(V,V)=\bigoplus_{i,j}\mathrm{Hom}_{\mathbb{C}[G]}(V_i,V_j).$$

Now, the lemma follows from Schur's lemma since $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[G]}(V,V)=1$ if V is irreducible and $\dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[G]}(V,V)>1$ if V is not irreducible. □

Proof of Proposition 4.1 It follows from Lemma 3.2 that $\mathrm{ind}_H^G\alpha$ is semisimple. Therefore, by Lemma 4.3 it follows that $\mathrm{ind}_H^G\alpha$ is irreducible if and only if

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{ind}_H^G W,\mathrm{ind}_H^G W)\cong\mathbb{C}.$$

We choose a system S of the (H,H) double cosets of G. Then we obtain that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{ind}_H^G W,\mathrm{ind}_H^G W)&\cong\mathrm{Hom}_{\mathbb{C}[H]}(W,\mathrm{res}_H^G\,\mathrm{ind}_H^G W) && \text{(by (8))}\\ &\cong\bigoplus_{s\in S}\mathrm{Hom}_{\mathbb{C}[H]}(W,\mathrm{ind}_{H_s}^H W_s) && \text{(by (2) for } K=H)\\ &\cong\bigoplus_{s\in S}\mathrm{Hom}_{\mathbb{C}[H_s]}(\mathrm{res}_{H_s}^H W,W_s) && \text{(by (7)).}\end{aligned}$$

Now, if $s\in H$, then $H_s=H$ and $W_s\cong W$, and $\mathrm{Hom}_H(W,W)\cong\mathbb{C}$ since W is an irreducible $\mathbb{C}[H]$–module (see Lemma 4.3). Thus $\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{ind}_H^G W,\mathrm{ind}_H^G W)\cong\mathbb{C}$ if and only if $\mathrm{Hom}_{\mathbb{C}[H_s]}(\mathrm{res}_{H_s}^H W,W_s)=0$ for all $s\in G-H$. □

4.2 The character varieties of the figure-eight knot

Let us consider the figure-eight knot K_{4_1} and its group π_{4_1}:

(9) $\quad \pi_{4_1}\cong\langle s,t\mid st^{-1}s^{-1}ts=tst^{-1}s^{-1}t\rangle\cong\langle s,a\mid a^{-1}s^{-1}asa^{-1}sas^{-1}a^{-1}\rangle,$

where $a = ts^{-1}$. Following Reidemeister [26, Section 15], we obtain a representation $\delta\colon \pi_{4_1} \to S_5$ into the symmetric group S_5 given by

$$\delta(s) = \sigma := (1)(2,5)(3,4) \quad \text{and} \quad \delta(a) = \tau := (1,2,3,4,5). \tag{10}$$

The image of δ is a dihedral group. We adopt the convention that permutations act on the right on $\{1,\dots,\alpha\}$, and hence π_{4_1} acts on the right. We put $N = \mathrm{Ker}(\delta)$ and $H = \mathrm{Stab}(1) = \{g \in G \mid 1^{\delta(g)} = 1\}$. We have $N \le H$, $N \trianglelefteq \pi_{4_1}$, $[\pi_{4_1} : H] = 5$, and $[H:N] = 2$. We obtain

$$\pi_{4_1} = H \sqcup Ha \sqcup Ha^2 \sqcup Ha^3 \sqcup Ha^4,$$

and $g \in Ha^i$ if and only if $1^{\delta(g)} = i+1$ for $0 \le i \le 4$. Notice that $\delta(H) = \langle\sigma\rangle$ is the cyclic group generated by σ.

Now we can use the Reidemeister–Schreier method [31] for finding a presentation for H: $\{1, a, a^2, a^3, a^4\}$ is a Schreier representative system for the right cosets modulo H. Hence generators of H are

$$y_0 = s, \quad y_i = a^i s a^{i-5} \text{ for } i = 1,2,3,4, \quad \text{and} \quad y_5 = a^5. \tag{11}$$

We obtain defining relations r_i, where $i = 0,1,2,3,4$, for H by expressing each

$$r_i = a^i(a^{-1}s^{-1}asa^{-1}sas^{-1}a^{-1})a^{-i}$$

as a word in the y_j:

$$\begin{aligned}
&r_0 = y_5^{-1}y_1^{-1}y_2^2y_1^{-1}, \quad r_1 = y_0^{-1}y_1y_3y_2^{-1}, \qquad r_2 = y_4^{-1}y_5y_0y_5^{-1}y_4y_3^{-1},\\
&r_3 = y_3^{-1}y_4y_0y_4^{-1}, \qquad r_4 = y_2^{-1}y_3y_1y_5y_0^{-1}y_5^{-1}.
\end{aligned}$$

It follows that $H/\langle\!\langle y_0\rangle\!\rangle \cong \langle y_1, y_2, y_3, y_4, y_5 \mid y_3 = y_5 = 1, y_1 = y_2\rangle \cong F(y_1, y_4)$. Therefore, a surjection $\psi\colon H \twoheadrightarrow F_2 = F(x,y)$ is given by

$$\psi(y_0) = \psi(y_3) = \psi(y_5) = 1, \quad \psi(y_1) = \psi(y_2) = x, \quad \text{and} \quad \psi(y_4) = y. \tag{12}$$

We have $y_0 \in H - N$ since by (10) the permutation $\delta(y_0) = \delta(s) = \sigma$ fixes 1 but $\delta(y_0) = \sigma \neq \mathrm{id}$. Moreover, $y_0 \in \mathrm{Ker}(\psi)$.

We need also generators of N: we have $y_0 \notin N$, $\delta(y_5) = \mathrm{id}$, and $\delta(y_i) = \sigma$ for $0 \le i \le 4$. Hence Reidemeister–Schreier gives that N is generated by

$$y_iy_0^{-1}, \quad y_5, \quad y_0^2, \quad y_0y_i, \quad y_0y_5y_0^{-1} \qquad \text{where } i = 1,2,3,4. \tag{13}$$

Lemma 4.4 *Let $\beta\colon F_2 \to \mathrm{SL}(2m,\mathbb{C})$ be irreducible. Then $\alpha = \beta\circ\psi\colon H \to F_2 \to \mathrm{SL}(2m,\mathbb{C})$ and $\alpha|_N\colon H \to F_2 \to \mathrm{SL}(2m,\mathbb{C})$ are also irreducible.*

Proof If $\beta\colon F_2 \to \mathrm{SL}(2m,\mathbb{C})$ is irreducible, then $\alpha = \beta\circ\psi$ is also irreducible since $\psi\colon H \to F_2$ is surjective (the representations α and β have the same image). Similarly, (13) and (12) give that $\psi|_N\colon N \to F_2$ is also surjective (β and $\alpha|_N$ have the same image). □

We obtain a component of representations $X_0 \subset X(H, \mathrm{SL}(2m,\mathbb{C}))$ with $\dim X_0 \geq 4m^2-1$, and X_0 contains irreducible representations. In order to apply Proposition 4.1 we notice that $\{1, a, a^2\}$ is a representative system for the (H, H) double cosets

$$\pi_{4_1} = H \sqcup HaH \sqcup Ha^2H.$$

More precisely, we have that $HaH = Ha \sqcup Ha^4$ and $Ha^2H = Ha^2 \sqcup Ha^3$ since $1^{\delta(hah')} \in \{2,5\}$ and $1^{\delta(ha^2h')} \in \{3,4\}$ for all $h, h' \in H$ (recall that $\delta(H) = \langle\sigma\rangle$).

We have also

$$H_a = H \cap aHa^{-1} = N = H_{a^2} = H \cap a^2Ha^{-2},$$

since an element in the image of the dihedral representation $\delta\colon \pi_{4_1} \to S_5$ given by (10) which fixes two numbers is the identity, and $N = \mathrm{Ker}(\delta)$.

For the rest of this section we let $G := \pi_{4_1}$ denote the group of the figure-eight knot.

Lemma 4.5 *Let $\beta\colon F(x,y) \to \mathrm{SL}(2m,\mathbb{C})$ be given by $\beta(x) = A$ and $\beta(y) = B$. If β is irreducible, then $\rho = \mathrm{ind}_H^G(\beta\circ\psi)$ is also irreducible.*

Proof We let $\alpha = \beta\circ\psi$ denote the corresponding representation of H. We want to apply Proposition 4.1 to prove that $\rho = \mathrm{ind}_H^G\,\alpha$ is irreducible. For $s \in \{a, a^2\}$ we have

$$\mathrm{res}_{H_s}^H\,\alpha = \alpha|_N \quad \text{since } H_a = H_{a^2} = N.$$

Moreover, for $s \in \{a, a^2\}$ we obtain

$$W_s = \mathrm{res}_{H_s}^{sHs^{-1}} W_{\alpha^s} = (\alpha|_N)^s.$$

By Lemma 4.4 we obtain that α, $\alpha|_N$, $(\alpha|_N)^a$, and $(\alpha|_N)^{a^2}$ are irreducible. Recall also that by Schur's lemma (Lemma 3.6) two irreducible representations are disjoint if and only if they are not equivalent. Hence by Proposition 4.1 we obtain that $\rho = \mathrm{ind}_H^G\,\alpha$ is irreducible if and only if

$$\alpha|_N \not\sim (\alpha|_N)^a \quad \text{and} \quad \alpha|_N \not\sim (\alpha|_N)^{a^2}.$$

In order to see this, we consider the element $y_0^2 \in N$. We have $\psi(y_0) = 1$ and therefore $\alpha|_N(y_0^2) = I_{2m}$, where $I_{2m} \in \mathrm{SL}(2m,\mathbb{C})$ denotes the identity matrix. Recall that $\beta(x) = A$ and $\beta(y) = B$, where $A, B \in \mathrm{SL}(2m,\mathbb{C})$. We have

$$\begin{aligned} a^{-1}y_0^2a &= a^{-1}s^2a = a^{-5}\cdot a^4sa^{-1}\cdot asa^{-4}\cdot a^5 = y_5^{-1}y_4y_1y_5, \\ a^{-2}y_0^2a^2 &= a^{-2}s^2a^2 = a^{-5}\cdot a^3sa^{-2}\cdot a^2sa^{-3}\cdot a^5 = y_5^{-1}y_3y_2y_5. \end{aligned}$$

This gives $\psi(a^{-1}y_0^2a) = yx$ and $\psi(a^{-2}y_0^2a^2) = x$.

Therefore, $(\alpha|_N)^a(y_0^2) = BA$ and $(\alpha|_N)^{a^2}(y_0^2) = A$. Now, if β is irreducible, then $A \neq I_{2m}$, and $AB \neq I_{2m}$. Hence $\alpha|_N \not\sim (\alpha|_N)^a$ and $\alpha|_N \not\sim (\alpha|_N)^{a^2}$. □

Proof of Proposition 1.3 The subgroup $H \leq \pi_{4_1}$ is of index 5, and $\psi\colon H \to F_2$ is a surjective homomorphism onto a free group of rank 2. Now, by Lemma 4.4 and Lemma 4.5, and the same argument as in the proof of Theorem 1.2, we obtain that for all $m \in \mathbb{N}$ the character variety $X(\pi_{4_1}, \mathrm{SL}(10m,\mathbb{C}))$ has a component C of dimension at least $4m^2 - 1$. Finally, Lemma 4.5 implies that C contains characters of irreducible representations. □

More explicitly, if $\beta\colon F(x,y) \to \mathrm{SL}(2m,\mathbb{C})$ is a representation given by $\beta(x) = A$ and $\beta(y) = B$, then, following the construction given in Section 2.1, the induced representation $\rho = \mathrm{ind}_H^G(\beta \circ \psi)\colon \pi_{4_1} \to \mathrm{GL}(10m,\mathbb{C})$ is given by

$$\rho(s) = \begin{pmatrix} I_{2m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & I_{2m} & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & A & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(t) = \begin{pmatrix} 0 & A & 0 & 0 & 0 \\ I_{2m} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & I_{2m} & 0 \\ 0 & 0 & A & 0 & 0 \end{pmatrix}.$$

Here, s and t are the generators of π_{4_1} from (9), and $I_{2m} \in \mathrm{SL}(2m,\mathbb{C})$ is the identity matrix.

Remark We finish the section with some historical comments. Subgroups similar to H and N from Section 4.2 are more generally defined for two-bridge knots. Let $K = \mathfrak{b}(p,q) \subset S^3$ be a two-bridge knot. A presentation of the knot group $\pi_{p,q}$ is given by the following (see [7, E 12.1]):

$$\pi_{p,q} = \langle s, t \mid l_s s = t l_s \rangle \qquad \text{where } l_s = s^{\epsilon_1}t^{\epsilon_2}\cdots t^{\epsilon_{\alpha-1}} \text{ and } \epsilon_k = (-1)^{[k(\beta/\alpha)]}.$$

K Reidemeister [26, Section 15] defined the representation $\delta\colon \pi_{p,q} \to S_p$ into the

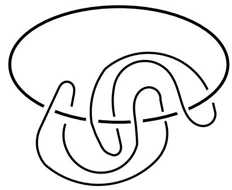

Figure 1: The link $\widehat{K}_{4_1} \subset \widehat{S}^3$

symmetric group S_p by

(14) $\quad \delta(s) = (1)(2, 2n+1)(3, 2n) \cdots (n+1, n+2) \quad \text{and} \quad \delta(a) = (1, 2, \dots, p),$

where $p = 2n+1$ and $a = ts^{-1}$. The image of δ is a dihedral group. We put $N = \operatorname{Ker}(\delta)$ and $H = \operatorname{Stab}(1) = \{g \in G \mid 1^{\delta(g)} = 1\}$. We have $N \le H$, $N \trianglelefteq \pi_{p,q}$, $[\pi_{p,q} : H] = p$, and $[H : N] = 2$.

The irregular covering of E_K corresponding to H has been studied since the beginning of knot theory. Reidemeister calculated a presentation of H. Moreover, he showed that the total space of the corresponding irregular branched covering $(\widehat{S}^3, \widehat{K}) \to (S^3, \mathfrak{b}(p,q))$ is simply connected. He proved also that the branching set $\widehat{K}$ consists of $(n+1)$ unknotted components (see [25; 26]). G Burde [5] proved that $\widehat{S}^3$ is in fact the 3–sphere and he determined the nature of the branching set explicitly in [6]. More recently, G Walsh studied the regular branched covering corresponding to N [29]. She proved that the corresponding branching set is a great circle link in S^3.

For the figure-eight knot $K_{4_1} = \mathfrak{b}(5,3)$, the link $\widehat{K}_{4_1} \subset S^3$ has a particularly simple form (see Figure 1 and [7, Example 14.22]). If we fill in the component $\widehat{k}_0$, then $\widehat{K}_{4_1}$ transforms into the trivial link of two components. Therefore the isomorphism $H/\langle\langle y_0 \rangle\rangle \cong F_2$, where $\langle\langle y_0 \rangle\rangle$ denotes the normal subgroup of H generated by the meridian y_0 of $\widehat{k}_0$, has a geometric origin.

We expect similar methods can be used to prove that for any hyperbolic 2–bridge knot $\mathfrak{b}(p,q)$, where $q \ge 2$, the character variety $X(\pi_{p,q}, \mathrm{SL}(2pm, \mathbb{C}))$ admits an at least $(4m^2-1)$–dimensional component. In [16, Section 4] it was proved that for the trefoil knot $\mathfrak{b}(3,1)$, the variety $X(\pi_{3,1}, \mathrm{SL}(6, \mathbb{C}))$ contains an at least 7–dimensional component. The same construction can be used to prove that for the torus knots $\mathfrak{b}(p,1)$, the variety $X(\pi_{p,1}, \mathrm{SL}(2pk, \mathbb{C}))$, where $k \in \mathbb{N}$, is at least $(2k^2p^2-4pk^2+1)$–dimensional.

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