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DAHA and plane curve singularities

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We suggest a relatively simple and totally geometric conjectural description of uncolored DAHA superpolynomials of arbitrary algebraic knots (conjecturally coinciding with the reduced stable Khovanov–Rozansky polynomials) via the flagged Jacobian factors (new objects) of the corresponding unibranch plane curve singularities. This generalizes the Cherednik–Danilenko conjecture on the Betti numbers of Jacobian factors, the Gorsky combinatorial conjectural interpretation of superpolynomials of torus knots and that by Gorsky and Mazin for their constant term. The paper mainly focuses on nontorus algebraic knots. A connection with the conjecture due to Oblomkov, Rasmussen and Shende is possible, but our approach is different. A motivic version of our conjecture is related to p–adic orbital A–type integrals for anisotropic centralizers.

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0 Introduction

We propose a relatively simple and totally computable conjectural *geometric* description of uncolored DAHA superpolynomials of arbitrary algebraic knots in terms of *flagged Jacobian factors* (new objects) of the corresponding unibranch plane curve singularities, presumably coinciding with the corresponding *stable Khovanov–Rozansky polynomials*. This description significantly generalizes (a) Cherednik and Danilenko's conjecture on the Betti numbers of Jacobian factors (any unibranch singularities), (b) Gorsky's conjectural interpretation of superpolynomials of torus knots from [16] etc, and (c) that from Gorsky and Mazin [19; 20] for their a–constant term. Our conjecture is different from the *ORS conjecture* from Gorsky, Oblomkov, Rasmussen and Shende [22] and Oblomkov, Rasmussen and Shende [36] (though some connection is not impossible).

Motivation Algebrogeometric theory of topological invariants of algebraic links has a long history, starting with the well-known algebraic interpretation of the *Alexander polynomials*. This paper provides an algebrogeometric description of stable

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Khovanov–Rozansky polynomials via the DAHA-superpolynomials. See eg Khovanov [26], Khovanov and Rozansky [27; 28], Rasmussen [39] and Webster and Williamson [42]. The geometry of flagged Jacobian factors conjecturally provides the DAHA-superpolynomials from Cherednik [6; 7] and Cherednik and Danilenko [8; 9]. For instance, this explains the positivity of the latter in the uncolored case, conjectured in [8].

For uncolored *torus knots*, this positivity results from the combinatorial construction from Gorsky [16; 17], which conjecturally provides both stable *KhR-polynomials* and those via DAHA (and is closely related to rational DAHA). Our conjecture makes the positivity entirely geometric (generalizing [19; 20]) for torus and arbitrary algebraic knots. We expect important implications in the theory of plane curve singularities, the theory of p–adic *orbital integrals* and *affine Springer fibers*; see Conjecture 2.5(iii) and Section 5.

Algebraic knots Torus-type (quasihomogeneous) plane singularities are very special. Not much is actually known on the Jacobian factors of nontorus plane singularities; the paper by Piontkowski [38] still remains the main source of examples. It was an important development when the DAHA approach from [6; 7] and Gorsky and Neguț [21] was extended from torus knots to arbitrary algebraic knots in [8] and then to any algebraic link in [9]. The *Newton pairs* and the theory of *Puiseux expansion*, the key in the *topological* classification of plane curve singularities, naturally emerge in the DAHA approach.

One of the key advantages of the usage of DAHA is that adding colors is relatively direct (via the Macdonald polynomials), which is well understood for any *iterated torus link* (including all algebraic links). This is well ahead of any other approaches (topology included) for such links. We expect that our present paper can be enhanced by adding colors via (presumably) the curves suggested in Maulik [31]. The case of rectangle Young diagrams is exceptional due to the conjectured positivity of the corresponding *reduced* DAHA superpolynomials for algebraic knots [6; 8]. The switch from the rank-one torsion free modules in the definition of compactified Jacobians to arbitrary ranks is expected here (among other modifications), which is in progress.

The passage to arbitrary algebraic knots and links from torus knots is important because of multiple reasons. The generality is an obvious advantage, but not the only one. All algebraic links (not only torus knots) are necessary to employ the technique of the *resolved conifold* and similar tools used in [31] to prove the (colored generalization of) the *OS conjecture* concerning the HOMFLYPT polynomials; see Oblomkov and Shende [37]. Also, all algebraic *links* are needed for the theory of Hitchin and affine Springer fibers since *spectral curves* are generally not unibranch. Topologically, the class of iterated links is closed with respect to *cabling*, a major operation in knot theory.

ORS conjecture Let us briefly comment on Conjecture 2 from [36]; see Section 5 for some further discussion. It relates the geometry of *nested Hilbert schemes* of arbitrary (germs of) plane curve singularities to the Khovanov–Rozansky *unreduced* stable polynomials of the corresponding links. The main component of their conjecture versus the OS conjecture is the *weight filtration*. The polynomials there, $\mathscr{P}_{\mathrm{alg}}$, conjecturally coincide with uncolored ones in the q,t–DAHA theory (upon switching to the standard parameters); see [8]. They are connected with the *perverse filtration* on the cohomology of the compactified Jacobians from Maulik and Yun [32] and Migliorini and Shende [34] (see Proposition 4 in [36]).

Our approach is based on *admissible flags* of submodules $\{\mathcal{M}_i\}$ in the normalization ring; here $\dim_{\mathbb{C}}(\mathcal{M}_{i+1}/\mathcal{M}_i)=1$, but they are not full flags, and the admissibility is a very restrictive condition. The absence of (nested) Hilbert schemes is due to the *reduced setting* of our paper (continuing [8]); there are other important deviations from [36]. For instance, we do not need the weight filtration, and our approach is quite computable. There may be a connection with Section 9.1 from [22] (a reduced version of the construction of [36]), but this is unclear. Actually, the weight filtration appears naturally in (5-2), which follows from a modular variant (2-7) of our conjecture, but it is associated with a parameter different from that in [36].

Main results The key is Conjecture 2.5; anything else is about confirmations, examples and connections. It extends Conjecture 2.4(iii) from [8] for Betti numbers of Jacobians factors for unibranch plane curve singularities (the case $q=1$, $a=0$). It was essentially checked in Mellit [33] for torus knots. The Betti numbers for torus knots are due to Lusztig and Smelt [30] (see also [19] and [38]). We focus in this paper on nontorus knots.

The series of the plane curve singularities for *Puiseux exponents* $(4,2u,v)$ for odd u,v and $v>2u>4$ is the simplest of nontorus type; the corresponding links are $\mathrm{Cab}(2u+v,2)T(u,2)$. Here we generalize the formulas from [38] for the dimensions of cells in the corresponding CW-presentation. The most convincing demonstrations of our main conjecture are the examples where such cells are not all affine. Such examples are well beyond [38] and are actually a new vintage in the theory of compactified Jacobians as well as our flagged generalization.

Some perspectives An extension of the geometric approach to superpolynomials from this paper to *all* root systems is of obvious interest, especially due to connections with p–adic orbital integrals. Cherednik and Elliot [10] hint that such a uniform theory may exist, in spite of the fact that there can be no rank stabilization for the systems EFG. Such a theory can be expected to provide *refined* generalizations of *orbital integrals* from the geometric fundamental lemma; local spectral curves are taken here as plane curve singularities. See the end of the paper.

The case $q = 1$, $a = 0$ is directly related to p–adic orbital integrals of nil-elliptic type A. An immediate corollary of Conjecture 2.5 is that such orbital integrals are *topological* invariants of the corresponding plane curve singularities. This readily follows from [30] for torus knots, but seems beyond any existing approaches for nontorus singularities, especially in the presence of nonaffine cells (see the online supplement). This invariance, the refined orbital integrals, the connections with HOMFLYPT homology and an extension of our paper to arbitrary algebraic links, any colors and all root systems are natural challenges.

1 DAHA superpolynomials

We will provide here the main facts of DAHA theory needed for the definition of the DAHA–Jones polynomials and DAHA superpolynomials. See [6; 7; 5] for details. The construction is totally uniform for any root systems and weights.

1.1 Definition of DAHA

Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system of type $A_n, \dots, G_2$ with respect to a euclidean form $(\cdot,\cdot)$ on $\mathbb{R}^n$, W the Weyl group generated by the reflections s_α, and R_+ the set of positive roots corresponding to fixed simple roots $\alpha_1, \dots, \alpha_n$. The form is normalized by the condition $(\alpha,\alpha) = 2$ for *short* roots. The weight lattice is $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$, where $\{\omega_i\}$ are fundamental weights. The root lattice is $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_+ = \{\mathbb{Z} \ni m \ge 0\}$, we obtain P_+, Q_+; see eg [3] or [5].

Setting $\nu_\alpha := \frac{1}{2}(\alpha,\alpha)$, the vectors $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R$ and $j \in \mathbb{Z}$ form the *twisted affine root system* $\widetilde{R} \supset R$ ($z \in \mathbb{R}^n$ is identified with $[z, 0]$).

We add $\alpha_0 := [-\vartheta, 1]$ to the simple roots for the *maximal short root* $\vartheta \in R_+$. The corresponding set $\widetilde{R}_+$ of positive roots is $R_+ \cup \{[\alpha, \nu_\alpha j] \mid \alpha \in R,\ j > 0\}$.

The set of the indices of the images of α_0 by all automorphisms of the affine Dynkin diagram will be denoted by O (with $O = \{0\}$ for E_8, F_4, G_2). Let $O' := \{r \in O \mid r \neq 0\}$; then $O' = [1, \dots, n]$ for A_n. The elements ω_r for $r \in O'$ are *minuscule weights*, defined by the inequalities $(\omega_r, \alpha^\vee) \le 1$ for all $\alpha \in R_+$. We set $\omega_0 = 0$ for the sake of uniformity.

Affine Weyl groups Given $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}$ and $b \in P$, let

$$s_{\widetilde{\alpha}}(\widetilde{z}) = \widetilde{z} - (z, \alpha^\vee)\widetilde{\alpha}, \quad b'(\widetilde{z}) = [z, \zeta - (z, b)] \tag{1-1}$$

for $\widetilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$. The *affine Weyl group* $\widetilde{W} = \langle s_{\widetilde{\alpha}} \mid \widetilde{\alpha} \in \widetilde{R}_+ \rangle$ is the semidirect product $W \ltimes Q$ of its subgroups $W = \langle s_\alpha \mid \alpha \in R_+ \rangle$ and Q, where α is identified with

$$s_\alpha s_{[\alpha, \nu_\alpha]} = s_{[-\alpha, \nu_\alpha]} s_\alpha \quad \text{for } \alpha \in R \text{ considered in } \langle s_{\widetilde{\alpha}} \rangle.$$

Using the presentation of $\widetilde{W}$ as $W \ltimes Q$, the *extended Weyl group* $\widehat{W}$ can be defined as $W \ltimes P$, where the corresponding action is

$$(wb)([z, \zeta]) = [w(z), \zeta - (z, b)] \quad \text{for } w \in W,\ b \in P. \tag{1-2}$$

It is canonically isomorphic to $\widetilde{W} \ltimes \Pi$ for $\Pi := P/Q$. The latter group consists of $\pi_0 = \mathrm{id}$ and the images π_r of minuscule ω_r in P/Q.

The group Π will be naturally identified with the subgroup of $\widehat{W}$ of the elements of the length zero; the *length* is defined as follows:

$$l(\widehat{w}) = |\lambda(\widehat{w})| \quad \text{for } \lambda(\widehat{w}) := \widetilde{R}_+ \cap \widehat{w}^{-1}(-\widetilde{R}_+).$$

One has $\omega_r = \pi_r u_r$ for $r \in O'$, where u_r is the (unique) element $u \in W$ of minimal length such that $u(\omega_r) \in -P_+$.

Setting $\widehat{w} = \pi_r \widetilde{w} \in \widehat{W}$ for $\pi_r \in \Pi$ and $\widetilde{w} \in \widetilde{W}$, the length $l(\widehat{w})$ coincides with the length of any reduced decomposition of $\widetilde{w}$ in terms of the simple reflections $s_i, 0 \le i \le n$ (a standard and important fact).

Let m be the least natural number such that $(P, P) = (1/\mathsf{m})\mathbb{Z}$; then $\mathsf{m} = n+1$ for A_n. The double affine Hecke algebra, DAHA, depends on the parameters q, t_ν $(\nu \in \{\nu_\alpha\})$; to be exact, it is defined over the ring of polynomials in terms of $q^{\pm 1/\mathsf{m}}$ and $\{t_\nu^{\pm 1/2}\}$.

For $\widetilde{\alpha} = [\alpha, \nu_\alpha j] \in \widetilde{R}$, we set $t_{\widetilde{\alpha}} = t_\alpha = t_{\nu_\alpha}$ and $q_{\widetilde{\alpha}} = q_\alpha = q^{\nu_\alpha}$, and introduce $k_{\widetilde{\alpha}} = k_\alpha = k_{\nu_\alpha}$ from the relation $t_\nu = q^{\nu k_\nu}$. For $i = 1, \dots, n$, let

$$\rho_k := \frac{1}{2}\sum_{\alpha>0} k_\alpha \alpha = k_{\mathrm{sht}}\rho_{\mathrm{sht}} + k_{\mathrm{lng}}\rho_{\mathrm{lng}}, \quad \rho_\nu = \frac{1}{2}\sum_{\nu_\alpha=\nu} \alpha = \sum_{\nu_{\alpha_i}=\nu} \omega_i,$$

where sht and lng are used for short and long roots. We note that the specialization $k_{\mathrm{sht}} = 1 = k_{\mathrm{lng}}$ corresponds to quantum groups and provides the *WRT invariants* in the construction below; see [6].

For pairwise commutative $X_1, \dots, X_n$,

$$X_{\widetilde{b}} := \prod_{i=1}^{n} X_i^{l_i} q^j \quad \text{if } \widetilde{b} = [b, j],\ \widehat{w}(X_{\widetilde{b}}) = X_{\widehat{w}(\widetilde{b})}, \tag{1-3}$$

where

$$b = \sum_{i=1}^{n} l_i \omega_i \in P, \quad j \in (1/\mathsf{m})\mathbb{Z}, \quad \widehat{w} \in \widehat{W}.$$

For instance, $X_0 := X_{\alpha_0} = X_{[-\vartheta,1]} = qX_\vartheta^{-1}$.

Recall that $\omega_r = \pi_r u_r$ for $r \in O'$ (see above). Note that π_r^{-1} is $\pi_{\iota(i)}$, where ι is the standard involution of the *nonaffine* Dynkin diagram, induced by $\alpha_i \mapsto -w_0(\alpha_i)$; it is the reflection of $[1, \dots, n]$ in type A_n. Finally, we set $m_{ij} = 2, 3, 4, 6$ when the number of links between α_i and α_j in the affine Dynkin diagram is $0, 1, 2, 3$.

Definition 1.1 The double affine Hecke algebra $\mathcal{HH}$ is generated by the elements $\{T_i \mid 0 \le i \le n\}$, pairwise commutative $\{X_b \mid b \in P\}$ satisfying (1-3) and the group Π, where the following relations are imposed:

(o) $(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0 \quad$ for $0 \le i \le n$;

(i) $T_i T_j T_i \cdots = T_j T_i T_j \cdots, \quad$ with m_{ij} factors on each side;

(ii) $\pi_r T_i \pi_r^{-1} = T_j \quad$ if $\pi_r(\alpha_i) = \alpha_j$;

(iii) $T_i X_b = X_b X_{\alpha_i}^{-1} T_i^{-1} \quad$ if $(b, \alpha_i^\vee) = 1$ for $0 \le i \le n$;

(iv) $T_i X_b = X_b T_i \quad$ if $(b, \alpha_i^\vee) = 0$ for $0 \le i \le n$;

(v) $\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{\iota(r)}, b)} \quad$ for $r \in O'$.

Given $\widetilde{w} \in \widetilde{W}, r \in O$, the product

$$T_{\pi_r \widetilde{w}} := \pi_r T_{i_l} \cdots T_{i_1}, \quad \text{where } \widetilde{w} = s_{i_l} \cdots s_{i_1} \text{ and } l = l(\widetilde{w}), \tag{1-4}$$

does not depend on the choice of the reduced decomposition. Moreover,

$$T_{\widehat{v}} T_{\widehat{w}} = T_{\widehat{v}\widehat{w}} \quad \text{whenever} \quad l(\widehat{v}\widehat{w}) = l(\widehat{v}) + l(\widehat{w}), \qquad \text{for } \widehat{v}, \widehat{w} \in \widehat{W}. \tag{1-5}$$

In particular, we arrive at the pairwise commutative elements

$$Y_b := \prod_{i=1}^{n} Y_i^{l_i} \quad \text{if} \quad b = \sum_{i=1}^{n} l_i \omega_i \in P, \quad Y_i := T_{\omega_i}, \quad b \in P. \tag{1-6}$$

1.2 Main features

The following maps can be (uniquely) extended to automorphisms of $\mathcal{HH}$, where $q^{1/(2\mathrm{m})}$ must be added to the ring of constants [5, (3.2.10)–(3.2.15)]:

$$\tau_+ \colon \begin{cases} X_b \mapsto X_b, \\ Y_r \mapsto X_r Y_r q^{-(\omega_r, \omega_r)/2}, \\ T_i \mapsto T_i \ (i > 0), \\ T_0 \mapsto q^{-1} X_\vartheta T_0^{-1}, \\ \pi_r \mapsto q^{-(\omega_r,\omega_r)/2} X_r \pi_r \ (r \in O'), \end{cases} \qquad \tau_- \colon \begin{cases} Y_b \mapsto Y_b, \\ X_r \mapsto Y_r X_r q^{(\omega_r,\omega_r)/2}, \\ T_i \mapsto T_i \ (i \geq 0), \\ X_\vartheta \mapsto q T_0 X_\vartheta^{-1} T_{s_\vartheta}^{-1}, \end{cases} \tag{1-7}$$

$$\sigma := \tau_+ \tau_-^{-1} \tau_+ = \tau_-^{-1} \tau_+ \tau_-^{-1}. \tag{1-8}$$

These automorphisms fix t_ν, q and their fractional powers.

The span of $\tau_\pm$ is the *projective* $\mathrm{PSL}_2(\mathbb{Z})$ (due to Steinberg), which is isomorphic to the braid group B_3. Let us list the matrices corresponding to the automorphisms above upon the natural projection onto $\mathrm{SL}_2(\mathbb{Z})$, which is upon the specialization $t_\nu^{1/(2\mathrm{m})}, q^{1/(2\mathrm{m})} \mapsto 1$. The matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ will represent the map

$$X_b \mapsto X_b^\alpha Y_b^\gamma, \quad Y_b \mapsto X_b^\beta Y_b^\delta$$

for $b \in P$. One has $\tau_+ \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\tau_- \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and $\sigma \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We note that there are some simplifications with the definition of DAHA and $\tau_\pm$ for A_n and in Theorem 1.2(i), but they are not significant (the theory is very much uniform for any root systems). However, A_n is obviously needed in part (ii) in this theorem.

Following [5], we use the PBW Theorem to express any $H \in \mathcal{HH}$ in the form $\sum_{a,w,b} c_{a,w,b} X_a T_w Y_b$ for $w \in W$ and $a, b \in P$ (this presentation is unique). Then we substitute

$$\{\cdot\}_{\mathrm{ev}}\colon X_a \mapsto X_a(q^{-\rho_k}) = q^{-(\rho_k,a)}, \quad Y_b \mapsto q^{(\rho_k,b)}, \quad T_i \mapsto t_i^{1/2}. \tag{1-9}$$

The functional $\mathcal{HH} \ni H \mapsto \{H\}_{\mathrm{ev}}$, which is called *coinvariant*, acts via the projection $H \mapsto H\Downarrow := H(1)$ of $\mathcal{HH}$ onto the *polynomial representation* $\mathcal{V}$, which is the $\mathcal{HH}$–module induced from the one-dimensional character $T_i(1) = t_i^{-1/2} = Y_i(1)$ for $1 \le i \le n$ and $T_0(1) = t_0^{-1/2}$. Here $t_0 = t_{\mathrm{sht}}$; see [5; 6; 7].

The polynomial representation $\mathcal{V}$ is linearly generated by X_b $(b \in P)$, and the action of T_i $(0 \le i \le n)$ there is given by the *Demazure–Lusztig operators*:

$$T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \le i \le n. \tag{1-10}$$

The elements X_b become the multiplication operators and π_r $(r \in O')$ act via the general formula $\widehat{w}(X_b) = X_{\widehat{w}(b)}$ for $\widehat{w} \in \widehat{W}$.

Macdonald polynomials The Macdonald polynomials $P_b(X)$, $b \in P_+$ are uniquely defined as follows. For $c \in P$, let c_+ be a unique element such that $c_+ \in W(c) \cap P_+$. Given $b \in P_+$ and assuming that $c \in P$ is such that $b \ne c_+ \in b - Q_+$,

$$P_b - \sum_{a \in W(b)} X_a \in \bigoplus_c \mathbb{Q}(q, t_\nu) X_c \quad \text{and} \quad \mathrm{CT}\big(P_b X_{c^\iota} \mu(X; q, t)\big) = 0, \tag{1-11}$$

where

$$\mu(X; q, t) := \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}{(1 - X_\alpha t_\alpha q_\alpha^j)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{j+1})}.$$

Here CT is the constant term; μ is considered a Laurent series of X_b with the coefficients expanded in terms of positive powers of q. The coefficients of P_b belong to the field $\mathbb{Q}(q, t_\nu)$. The following evaluation formula (the Macdonald evaluation conjecture) is important to us:

$$(P_b(q^{-\rho_k})) = q^{-(\rho_k,b)} \prod_{\alpha > 0} \prod_{j=0}^{(\alpha^\vee,b)-1} \left(\frac{1 - q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1 - q_\alpha^j X_\alpha(q^{\rho_k})} \right). \tag{1-12}$$

1.3 Algebraic knots

Torus knots $T(\mathsf{r}, \mathsf{s})$ are defined for any integers $\mathsf{r}, \mathsf{s} > 0$ such that $\gcd(\mathsf{r}, \mathsf{s}) = 1$. One has the symmetry $T(\mathsf{r}, \mathsf{s}) = T(\mathsf{s}, \mathsf{r})$, where we use "$=$" for the ambient isotopy equivalence. Also $T(\mathsf{r}, 1) = \bigcirc$ for the *unknot*, denoted by $\bigcirc$.

Algebraic knots $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ are associated with two sequences of (strictly) positive integers:

$$\vec{\mathsf{r}}=\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\quad \vec{\mathsf{s}}=\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\}\quad \text{such that } \gcd(\mathsf{r}_i,\mathsf{s}_i)=1; \tag{1-13}$$

we will call ℓ the *length* of $\vec{\mathsf{r}},\vec{\mathsf{s}}$. The pairs $\{\mathsf{r}_i,\mathsf{s}_i\}$ are *characteristic* or *Newton pairs*.

We will need one more sequence:

$$\mathsf{a}_1=\mathsf{s}_1,\quad \mathsf{a}_i=\mathsf{a}_{i-1}\mathsf{r}_{i-1}\mathsf{r}_i+\mathsf{s}_i\ (i=2,\dots,m). \tag{1-14}$$

See eg [12] and [38]. Then

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}):=\operatorname{Cab}(\vec{\mathsf{a}},\vec{\mathsf{r}})(\bigcirc)=(\operatorname{Cab}(\mathsf{a}_\ell,\mathsf{r}_\ell)\cdots\operatorname{Cab}(\mathsf{a}_2,\mathsf{r}_2))(T(\mathsf{a}_1,\mathsf{r}_1)) \tag{1-15}$$

in terms of the *cabling* defined below. Note that the first iteration (application of Cab) is for $\{\mathsf{a}_1,\mathsf{r}_1\}$ (not for the last pair!).

Cabling The *cabling* $\operatorname{Cab}(\mathsf{a},\mathsf{b})(K)$ of any oriented knot K in (oriented) S^3 is defined as follows; see eg [35; 12]. We consider a small 2–dimensional torus around K and put there the torus knot $T(\mathsf{a},\mathsf{b})$ in the direction of K, which is $\operatorname{Cab}(\mathsf{a},\mathsf{b})(K)$ (up to ambient isotopy).

This procedure depends on the order of a,b and the orientation of K. We choose the latter in the standard way (compatible with almost all sources, including the Mathematica package “KnotTheory”); the parameter a gives the number of turns around K. This construction also depends on the framing of the cable knots; we take the natural one, associated with the parallel copy of the torus where a given cable knot sits (its parallel copy has zero linking number with this knot).

By construction, $\operatorname{Cab}(\mathsf{a},0)(K)=\bigcirc$ and $\operatorname{Cab}(\mathsf{a},1)(K)=K$ for any knot K and $\mathsf{a}\neq 0$. See [8] for further discussion of relations. The pairs $\{\mathsf{r}_i,\mathsf{a}_i\}$ are sometimes called *topological*; the isotopy equivalence of algebraic knots generally can be seen only at the level of r,a–parameters (not at the level of the Newton or Puiseux pairs).

Newton–Puiseux theory Given a sequence $\mathsf{r}_i,\mathsf{s}_i>0$ of Newton (characteristic) pairs, the knot $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ is the link of the *germ of the singularity*

$$y=x^{\mathsf{s}_1/\mathsf{r}_1}(c_1+x^{\mathsf{s}_2/(\mathsf{r}_1\mathsf{r}_2)}(c_2+\cdots+x^{\mathsf{s}_\ell/(\mathsf{r}_1\mathsf{r}_2\cdots\mathsf{r}_\ell)}))\quad \text{at } 0, \tag{1-16}$$

which is the intersection of the corresponding plane curve with a small 3–dimensional sphere in $\mathbb{C}^2$ around 0. We will always assume in this paper that this germ is *unibranch*.

The inequality $\mathsf{s}_1 < \mathsf{r}_1$ is commonly imposed here (otherwise x and y can be switched). Formula (1-16) is the celebrated *Newton–Puiseux expansion*; see eg [12]. All algebraic knots can be obtained in such a way.

Jacobian factors One can associate with a unibranch $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ the *Jacobian factor* $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$. Up to a homeomorphism, it can be introduced as the canonical compactification of the *generalized Jacobian* of an integral *rational* planar curve with $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ as its *only* singularity. It has a purely local definition, which we will use below. Its dimension is the δ*–invariant* of the singularity $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$, also called the *arithmetic genus*.

Calculating the *Euler number* $e(J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}))$, the topological Euler characteristic of $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$, and the corresponding *Betti numbers* in terms of $\vec{\mathsf{r}},\vec{\mathsf{s}}$ is a challenging problem. For torus knots $T(\mathsf{r},\mathsf{s})$, one has $e(J(\mathcal{C}_{\mathsf{r},\mathsf{s}})) = (1/(\mathsf{r}+\mathsf{s}))\binom{\mathsf{r}+\mathsf{s}}{\mathsf{r}}$ due to [2]. This formula is related to the perfect modules of rational DAHA and the combinatorics of generalized Catalan numbers; see eg [19].

The Euler numbers of $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$ were calculated in [38] (the Main Theorem) for the following triples of *Puiseux characteristic exponents*:

$$(4,2u,v),\ (6,8,v),\ (6,10,v) \quad \text{for odd } u,v>0, \tag{1-17}$$

where $4<2u<v$, $8<v$, and $10<v$, respectively. Here $\delta=\dim J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$ is $\frac{1}{2}(\mathsf{r}-1)(\mathsf{s}-1)$ for $T(\mathsf{r},\mathsf{s})$ and $2u+\frac{1}{2}(v-1)-1$ for the series $(4,2u,v)$. Generally, δ equals the cardinality $|\mathbb{N}\setminus\Gamma|$, where Γ is the *valuation semigroup* associated with $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$; see [38] and [45]. The Euler numbers of the Jacobian factors can be also calculated via the HOMFLYPT polynomials of the corresponding links (see below) due to [37; 31].

Concerning the Betti numbers for the torus knots and the series $(4,2u,v)$, the odd (co)homology of $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$ vanishes. The formulas for the corresponding even Betti numbers $h^{(2k)}=\mathrm{rk}(H^{2k}(J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})))$ were calculated explicitly for many values of k in [38], where $0\le k\le\delta$. Not much was and is known/expected beyond these two series.

1.4 DAHA–Jones theory

The following results and conjectures are mainly from [8]; see also [7, Theorem 1.2] and [6; 21].

The construction is given directly in terms of the parameters $\{\vec{\mathsf{r}},\vec{\mathsf{s}}\}$, though it actually depends only on the corresponding topological parameters $\{\vec{\mathsf{a}},\vec{\mathsf{r}}\}$. Recall that torus knots $T(\mathsf{r},\mathsf{s})$ are naturally represented by $\gamma_{\mathsf{r},\mathsf{s}}\in \mathrm{PSL}_2(\mathbb{Z})$ with the first column $(\mathsf{r},\mathsf{s})^{\mathrm{tr}}$

(tr denotes transposition), where $\mathsf{r},\mathsf{s}>0$ and we assume that $\gcd(\mathsf{r},\mathsf{s})=1$. Let $\widehat{\gamma}_{\mathsf{r},\mathsf{s}}$ be any pullback of $\gamma_{\mathsf{r},\mathsf{s}}$ to the projective $\mathrm{PSL}_2(\mathbb{Z})$.

For a polynomial F in terms of fractional powers of q and t_ν, the *tilde-normalization* $\widetilde{F}$ will be the result of the division of F by the lowest q,t_ν–monomial, assuming that it is well defined. We put $q^\bullet t^\bullet$ for a monomial factor (possibly fractional) in terms of q,t_ν. See [8] for the following theorem. We will also apply this definition to the superpolynomials, where the lowest q,t–monomial is picked from the constant a–term.

Theorem 1.2 *Let R be a reduced irreducible root system. Recall $H\mapsto H\Downarrow := H(1)$, where the action of $H\in\mathcal{HH}$ in $\mathcal{V}$ is used.*

(i) *Given two strictly positive sequences $\vec{\mathsf{r}},\vec{\mathsf{s}}$ of length ℓ as in (1-13), we lift $(\mathsf{r}_i,\mathsf{s}_i)^{\mathrm{tr}}$ to γ_i and then to $\widehat{\gamma}_i$ (acting in $\mathcal{HH}$) as above. For a weight $b\in P_+$, the* DAHA***–Jones polynomial*** *is*

(1-18)
$$\mathrm{JD}^{R}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)=\mathrm{JD}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t) := \Big\{\widehat{\gamma}_1\Big(\cdots\Big(\widehat{\gamma}_{\ell-1}\Big(\big(\widehat{\gamma}_\ell(P_b)/P_b(q^{-\rho_k})\big)\Downarrow\Big)\Downarrow\Big)\cdots\Big)\Big\}_{\mathrm{ev}}.$$

It does not depend on the particular choice of the lifts γ_i and $\widehat{\gamma}_i\in\mathrm{GL}_2^{\wedge}(\mathbb{Z})$. The ***tilde-normalization*** *$\widetilde{\mathrm{JD}}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ is well defined and is a polynomial in terms of q,t_ν with the constant term 1.*

(ii) *Let us switch to the root system A_n for $\mathfrak{sl}_{n+1}$, setting $t=t_{\mathrm{sht}}=q^k$ and considering $P_+\ni b=\sum_{i=1}^n b_i\omega_i$ as (dominant) weights for any A_m (for $\mathfrak{sl}_{m+1}$) with $m\ge n-1$, where we assume that $\omega_n=0$ upon the restriction to A_{n-1}.*

Then given $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ as above, there exists a DAHA***-superpolynomial*** *$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a)$ from $\mathbb{Z}[q,t^{\pm1},a]$ satisfying the relations*

(1-19)
$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a=-t^{m+1})=\widetilde{\mathrm{JD}}^{A_m}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)\quad \text{for any } m\ge n-1;$$

its a–constant term is automatically tilde-normalized. □

Topological connection Let us briefly discuss the conjectural relation of DAHA-superpolynomials to *stable Khovanov–Rozansky polynomials* denoted by $\mathrm{KhR}_{\mathrm{stab}}$; see [27; 28; 26; 39]. We consider only the reduced setting (actually not quite developed topologically).

The passage to the Khovanov–Rozansky polynomials for $\mathfrak{sl}_N$ for sufficiently large N is the substitution $a\mapsto t^N\sqrt{q/t}$. Note the relation to the *Heegaard–Floer homology* for $N=0$. Equivalently, this passage is $a_{\mathrm{st}}\mapsto q_{\mathrm{st}}^N$ in the *standard topological parameters* (also used in the *ORS conjecture*), which are related to the DAHA parameters as follows:

$$t = q_{\mathrm{st}}^2, \quad q = (q_{\mathrm{st}} t_{\mathrm{st}})^2, \quad a = a_{\mathrm{st}}^2 t_{\mathrm{st}}, \quad q_{\mathrm{st}}^2 = t, \quad t_{\mathrm{st}} = \sqrt{q/t}, \quad a_{\mathrm{st}}^2 = a\sqrt{t/q}. \tag{1-20}$$

For the DAHA-superpolynomials from (1-19),

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q, t, a)_{\mathrm{st}} = \widetilde{\mathrm{KhR}}_{\mathrm{stab}}(q_{\mathrm{st}}, t_{\mathrm{st}}, a_{\mathrm{st}}) \quad \text{where } \square = \omega_1, \tag{1-21}$$

and $\widetilde{\mathrm{KhR}}_{\mathrm{stab}}$ is *reduced* $\mathrm{KhR}_{\mathrm{stab}}$ divided by the smallest power of a_{st} and then by $q_{\mathrm{st}}^{\bullet} t_{\mathrm{st}}^{\bullet}$ such that $\widetilde{\mathrm{KhR}}_{\mathrm{stab}}(a_{\mathrm{st}} = 0) \in \mathbb{Z}_+[q_{\mathrm{st}}, t_{\mathrm{st}}]$ with the constant term 1. Here $\{\cdot\}_{\mathrm{st}}$ means the switch from the DAHA parameters to the *standard topological parameters*.

Also, the polynomials $\mathrm{KhR}_{\mathrm{stab}}$ are expected to coincide with the (reduced) physics *superpolynomials* based on the *BPS states* [11; 1; 14; 18] and those obtained in terms of *rational* DAHA [22; 21] for torus knots. The latter approach is developed so far only for torus knots and in the uncolored case; there is some progress for symmetric powers of the fundamental representation; see [18]. We will not touch the connections with rational DAHA in this paper. Concerning physics origins, let us mention that using the Macdonald polynomials at roots of unity q (for $t = q^k, k \in \mathbb{Z}_+$) instead of Schur functions in the usual construction of *knot operators* was suggested in [1].

Betti numbers $h^{(i)} = \mathrm{rk}\, H^i(J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}))$ We are very much motivated by the DAHA approach to these numbers. Technically, we generalize the interpretation of Gorsky's superpolynomials for torus knots at $a = 0$ from [19; 20] and the following conjecture from [8]:

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q = 1, t, a = 0) = \sum_{i=0}^{2\delta} h^{(i)} t^{i/2} \quad \text{for } \delta = \dim J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}). \tag{1-22}$$

It implies that $h^{\mathrm{odd}} = 0$ (the van Straten–Warmt conjecture). Relation (1-22) will be generalized below to the whole superpolynomial $\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q, t, a)$, which is the main result of our paper.

2 Geometric superpolynomials

2.1 Modules of semigroups

Let $\mathcal{R}$ be the complete local ring of the *unibranch* germ of the plane curve singularity, embedded into the normalization ring $\mathcal{O} = \mathbb{C}[\![z]\!]$. The *conductor* of $\mathcal{R}$ is the smallest c such that $z^{\mathsf{c}}\mathcal{O} \subset \mathcal{R}$; actually it is the ideal $(z^{\mathsf{c}}) \subset \mathcal{O}$, but we will call c the conductor in this paper. We set $\mathcal{K} := \mathbb{C}((z))$.

The corresponding *semigroup* $\Gamma_{\mathcal{R}}$ is formed by the orders of the smallest powers, ie *valuations* $\nu(x)$ (minimal z–degrees) $x \in \mathcal{R}$. The δ–invariant (the arithmetic genus) is then $\delta_{\mathcal{R}} = |\mathbb{Z}_+ \setminus \Gamma_{\mathcal{R}}|$. We will call $\mathbb{Z}_+ \setminus \Gamma_{\mathcal{R}}$ the *set of gaps* and denote it by $G_{\mathcal{R}}$; thus $\delta = |G_{\mathcal{R}}|$. Also, $\mathsf{c} = 2\delta$.

Compactified Jacobians for projective curves are generally defined as the varieties of coherent torsion free sheaves of rank one and fixed degree up to isomorphisms. The *Jacobian factor* $J_{\mathcal{R}}$, we are going to define, is a local version of the compactified Jacobian. It is (as a set) formed by all finitely generated $\mathcal{R}$–submodules $\mathcal{M} \subset \mathcal{K} = \mathbb{C}((z))$, of (any) prescribed degree, also called $\mathcal{R}$*–lattices*.

We define $\mathcal{O}$*–degree* $\deg_{\mathcal{O}}\mathcal{M}$ with respect to $\mathcal{O}$; it is $\dim_{\mathbb{C}}(\mathcal{O}/\mathcal{M})$ if $\mathcal{M} \subset \mathcal{O}$. For arbitrary submodules $\mathcal{M} \subset \mathcal{K}$,

$$\deg_{\mathcal{O}}(\mathcal{M}) = \dim_{\mathbb{C}}(\mathcal{O}/(\mathcal{O} \cap \mathcal{M})) - \dim_{\mathbb{C}}(\mathcal{M}/(\mathcal{O} \cap \mathcal{M})). \tag{2-1}$$

This definition is a natural counterpart of the degree of a divisor at a given point (here at $z = 0$) in the smooth situation. Actually we will mainly need below $-\deg_{\mathcal{R}}(\mathcal{M}) = \delta - \deg_{\mathcal{O}}(\mathcal{M})$.

The valuations $\nu(x)$ of the elements $x \in \mathcal{M} \subset \mathcal{K}$ form a $\Gamma_{\mathcal{R}}$–module $\Delta_{\mathcal{M}}$; the modules for semigroups Γ (with 0) are subsets $\Delta \subset \mathbb{Z}_+$ such that $\Gamma + \Delta \subset \Delta$. Unless stated otherwise, we assume that $\mathcal{M} \subset \mathcal{O}$ and that it contains the element $1 + \sum_{i=1}^{\mathsf{c}-1} \lambda_0^i z^i$ (of valuation 0), such an embedding can be achieved by the division by z^m for $m = \min(\Delta_{\mathcal{M}})$. Here the upper limit $\mathsf{c}-1$ is sufficient in the sum due to the definition of the conductor. The notation $\Delta_\circ$ is used for such a normalization in [38], we call it the *standard normalization*. See there for these and related definitions and facts.

For a *standard* $\mathcal{M}$, we will use the notation $D_{\mathcal{M}}$ or $D[\mathcal{M}]$ for $G_{\mathcal{R}} \cap \Delta_{\mathcal{M}}$ and call it the *set of added gaps* or simply the D*–set*. The square brackets will be used for the list of its elements. For instance, $D_{\mathcal{R}} = \varnothing = []$ corresponds to the (trivial) Γ–module $\Delta_{\mathcal{R}} = \Gamma$, and $D_{\mathcal{O}} = G_{\mathcal{R}} = [1, \dots, \mathsf{c}-1]$ for $\mathcal{M} = \mathcal{O}$ (recall that $\mathsf{c}-1$ is the last gap in G).

Due to the normalization we impose, one has

$$\deg_{\mathcal{O}}(\mathcal{M}) = \delta - |D_{\mathcal{M}}|, \quad -\deg_{\mathcal{R}}(\mathcal{M}) = |D_{\mathcal{M}}| \qquad \text{for standard } \mathcal{M}. \tag{2-2}$$

Not all Γ–modules Δ can be realized as $\Delta_{\mathcal{M}}$ for nontorus singularities. Recall that torus knots $T(\mathsf{r},\mathsf{s})$ are associated with the rings $\mathcal{R} = \mathbb{C}[\![z^{\mathsf{r}}, z^{\mathsf{s}}]\!]$. The simplest example of a nontorus singularity is $\mathcal{R} = \mathbb{C}[\![z^4, z^6 + z^7]\!]$ with $\Gamma = \langle 4, 6, 13\rangle$ and $\mathsf{c} = 16$. Then the sets of added gaps $D = [2, 15]$ and $D = [2, 11, 15]$ do not come from any modules $\mathcal{M}$; following [38], we call them *nonadmissible*. All other D are admissible in this case (ie can be obtained as $D_{\mathcal{M}}$).

2.2 $J_{\mathcal{R}}$ as a projective variety

Let $J_{\mathcal{R}}[D] \subset J_{\mathcal{R}}$ be the set of modules $\mathcal{M}$ with $D_{\mathcal{M}} = D$, $J_{\mathcal{R}}[d] = \bigcup_{|D|=d} J_r[D]$ and $\bar{J}_{\mathcal{R}}[d] = \bigcup_{d' \geq d} J_r[d']$ for $d \geq 0$. That is, the latter is the set of all standard submodules of $\mathcal{O}$–degree $\delta - d$ or smaller. The set $J_{\mathcal{R}}[0] = J_{\mathcal{R}}[\varnothing]$ is *the big cell*; it is formed by all invertible modules $\mathcal{M}$ (with one generator and of $\mathcal{O}$–degree δ). Also, $J_{\mathcal{R}}[\delta] = \bar{J}_{\mathcal{R}}[\delta] = \{\mathcal{M} = \mathcal{O}\}$ and $J_{\mathcal{R}}[> \delta] = \varnothing$. Finally, we note that the isomorphisms of a particular submodule $\mathcal{M}$ are those induced by the action of the group of units $\mathcal{R}^*$.

We can give $\bar{J}_{\mathcal{R}}[d]$ the structure of a projective subvariety in $J_{\mathcal{R}} = \bar{J}_{\mathcal{R}}[0]$ following [23]; actually they are projective schemes over $\mathbb{Z}$. Generally, the $\mathcal{O}$–degree of $z^a\mathcal{M}$ for an $\mathcal{R}$–module $\mathcal{M} \subset \mathcal{O}$ and $a \in \mathbb{Z}_+$ is $\deg_{\mathcal{O}}\mathcal{M} + a$. Indeed, $\deg_{\mathcal{O}}(z^a\mathcal{M}) = |\mathbb{Z}_+ \setminus \nu(z^a\mathcal{M})| = a + |\mathbb{Z}_+ \setminus \nu(\mathcal{M})|$.

Given $0 \leq d \leq \delta$, let $\mathcal{M}^\circ$ be a standard submodule of $\mathcal{O}$–degree $\delta - d^\circ$ for $d^\circ \geq d$ and $a = d^\circ - d$. Then the submodule $\mathcal{M} = z^a\mathcal{M}^\circ$ is of $\mathcal{O}$–degree $\delta - d^\circ + d^\circ - d = \delta - d$, and one has

$$z^{2\delta}\mathcal{O} \subset \mathcal{M} \subset \mathcal{O} \quad \text{and} \quad \dim_{\mathbb{C}}(\mathcal{O}/\mathcal{M}) = \delta - d.$$

Vice versa, $\mathcal{M}^\circ = z^{-a}\mathcal{M}$ are standard for any such $\mathcal{R}$–submodules $\mathcal{M}$, where a is the smallest evaluation $\nu(z)$ among all $z \in \mathcal{M}$. Such $\mathcal{M}$ for $d = 0$ are called δ*–normalized modules* in [38]; our standard submodules are called there 0*–normalized*.

Now consider the Grassmanian $\mathrm{Gr}(\mathcal{O}/z^{2\delta}\mathcal{O}, \delta + d)$. A subspace of dimension $\delta + d$ is an $\mathcal{R}$–module if and only if it is invariant under the action of $\mathcal{R}/z^{2\delta}\mathcal{O}$ by multiplication. Thus $\bar{J}_{\mathcal{R}}[d]$ is the set of fixed points in $\mathrm{Gr}(\mathcal{O}/z^{2\delta}\mathcal{O}, \delta + d)$ under the group action of $(\mathcal{R}/z^{2\delta}\mathcal{O})^\star$. To obtain the structure we desire, extend the action of $(\mathcal{R}/z^{2\delta}\mathcal{O})^\star$ to $\bigwedge^{\delta+d}\mathcal{O}/z^{2\delta}\mathcal{O}$ and consider the image under the Plücker embedding. The condition of being a fixed point under the action $(\mathcal{R}/z^{2\delta}\mathcal{O})^\star$ defines a linear subspace of the projective space $\mathbb{P}(\bigwedge^{\delta+d}\mathcal{O}/z^{2\delta}\mathcal{O})$, so $J_{\mathcal{R}}$ is the intersection of the image of the Plücker embedding of $\mathrm{Gr}(\mathcal{O}/z^{2\delta}\mathcal{O}, \delta + d)$ and this linear subspace. Finally, we go back to our standard modules using the identification maps above.

There is an alternative *intrinsic* interpretation of the projective structure of $J_{\mathcal{R}}$. Let us consider $\mathcal{R}$–submodules $\widetilde{\mathcal{M}} \subset \mathcal{K} \otimes_{\mathbb{C}} \mathbb{C}[\![u]\!]$, finitely generated over $\mathbb{C}[\![u]\!]$, ie a family of submodules $\mathcal{M}$ (formally) analytically depending on a parameter u. Let $\widetilde{\mathcal{M}}$ contain a principle ideal $\mathbb{C}[\![u,z]\!]z^p$ for some p (arbitrarily large).

We define the *enhanced* (*flat*) *limit* $\widetilde{\mathcal{M}}_0$ of $\widetilde{\mathcal{M}}$ as the linear subspace of $\mathcal{K}$ of *all* linear combinations of the vectors in $\widetilde{\mathcal{M}}$ divided by the smallest possible power of u and then evaluated at $u=0$. This is one of the key definitions in the theory of sheaves and bundles over curves.

Such a limit obviously contains the straightforward specialization $\mathcal{M}_0 := \widetilde{\mathcal{M}}(u=0)$. The space $\widetilde{\mathcal{M}}_0 \subset \mathcal{O}$ has a natural structure of a $\mathcal{R}$–submodule by construction. Therefore it becomes a standard module (an element of $J_{\mathcal{R}}$) upon the division by a proper power z^n $(n \geq 0)$.

As in the definition of the standard normalization, we assume here that $\widetilde{\mathcal{M}}$ contains an element with zero z–valuation; its z–constant term can be any nonzero element (possibly noninvertible) from $\mathbb{C}[\![u]\!]$. By the way, one can check that the enhanced limit will remain the same if the principle ideal $\mathbb{C}[\![u,z]\!]z^{\mathsf{c}}$ is added to $\widetilde{\mathcal{M}}$; ie p above can be assumed to be no greater than the conductor c. This results from a standard theory of flat limits. Without general theory here, this fact can be justified by enlarging $\widetilde{\mathcal{M}}$ with all linear combinations of the vectors in $\widetilde{\mathcal{M}}$ divided by the corresponding minimal power of u and verifying that this procedure will eventually add $\mathbb{C}[\![u,z]\!]z^{\mathsf{c}}$ to $\mathcal{M}$. The limit $\widetilde{\mathcal{M}}_0$ will remain unchanged under such an "enhancement" of $\widetilde{\mathcal{M}}$.

Finally, the *boundary* of any family of modules considered in $J_{\mathcal{R}}$ is the collection of enhanced limits for all one-parametric subfamilies $\widetilde{\mathcal{M}}$. Using one u is sufficient in this approach (another general fact).

Proposition 2.1 *Let $\mathcal{M}'$ be the enhanced limit $\widetilde{\mathcal{M}}_0$ of a u–family $\widetilde{\mathcal{M}}$ of modules invertible over $\mathbb{C}((u))$, where we assume that $0 \in \Delta_{\widetilde{\mathcal{M}}}$ as above. Setting $\widetilde{\mathcal{M}}_0 \subset (z^{d'}) = \mathcal{O}z^{d'}$ for the maximal such d', one has that $d' \geq d$ for the number of added gaps for the standard module $\mathcal{M}$ corresponding to $\mathcal{M}'$ and $d' = d$ for sufficiently general such $\widetilde{\mathcal{M}}$.*

Proof For generic $\widetilde{\mathcal{M}}$, the limit $\widetilde{\mathcal{M}}_0$ is of degree 0 relative to $\mathcal{R} \in \mathcal{O}$; we use that $J_{\mathcal{R}}$ is a closed subvariety in $\mathrm{Gr}(\mathcal{O}/z^{2\delta}\mathcal{O}, \delta)$. Therefore $z^{-d}\mathcal{M}'$ is standard for $d = |D_{\mathcal{M}}|$ and $d' \geq d$ for any (nongeneric) $\widetilde{\mathcal{M}}$. □

Two examples of enhanced limits To illustrate Proposition 2.1, let us consider $\mathcal{R} = \langle z^4, z^6 + z^9, z^7\rangle$ with $\Gamma = \{0, 4, 6, 7, 8, 10, 11, \dots\}$, $G = [1, 2, 3, 5, 9]$, $\delta = |G| = 5$, and the conductor $\mathsf{c} = 10$. We take the following family of modules invertible over $\mathbb{C}((u))$:

$$\widetilde{\mathcal{M}} = \mathcal{R}[\![u, z]\!](u^4 + u(z + z^3) + u^3 z^2) + \mathbb{C}[\![u, z]\!]z^{10},$$

which is an invertible $\mathcal{R}$–module over $\mathbb{C}((u))$ (but not over $\mathbb{C}[\![u]\!]$ due to u^4). Then its *enhanced* limit $\widetilde{\mathcal{M}}_0$ is a module with the corresponding set of added gaps $D = [2, 3, 5, 9]$, so $d = |D| = 4$. However, it is contained in (z^5), which is smaller than (z^4) guaranteed by the proposition.

Taking here $\widetilde{\mathcal{M}} = \mathcal{R}[\![u, z]\!](u(1 + z^5) + z^9) + \mathbb{C}[\![u, z]\!]z^{10}$, the limit $\widetilde{\mathcal{M}}_0$ has the same $D = [2, 3, 5, 9]$. Now it strictly belongs to (z^4), so this is the case of a general position for such D.

The cardinalities of the D–sets will be associated below with the parameter q. If $J_{\mathcal{R}}[D]$ are all affine spaces, their dimensions give the powers of t. The third parameter a of our construction will be due to the flag-length of the *flagged Jacobian factors* defined as follows.

2.3 Flagged Jacobian factors

Definition 2.2 For a D–set D and the sequence $\vec{g} = \{g_i \mid 1 \le i \le m\}$ in $G \setminus D$ such that $g_i < g_{i+1}$, the D–*flag* is

$$\mathcal{D} = \{D_0 = D, D_1 = D \cup g_1, D_2 = D \cup \{g_1, g_2\}, \dots, D_m = D \cup \{g_1, \dots, g_m\}\},$$

provided that all $\Delta_i = \Gamma \cup D_i$ are standard Γ–modules. Then the corresponding *flagged Jacobian factor* $J^m_{\mathcal{R}}$ is defined as the union of the following varieties of flags of standard submodules $\mathcal{M} \subset \mathcal{O}$:

$$J^m_{\mathcal{R}}[\mathcal{D}] := \{\mathscr{M} = \{\mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots \subset \mathcal{M}_m\}, \text{ where } D_i = D_{\mathcal{M}_i}\} \tag{2-3}$$

for all $0 \le i \le m$; we will sometimes omit m here. If at least one such flag $\mathscr{M}$ exists, we call the corresponding $\mathcal{D}$ and $\{\Delta_i\}$ *admissible.*

Considering D–flags $\mathcal{D}$ of length m, we set $J^m_{\mathcal{R}}[d > \delta] = \varnothing$,

$$J^m_{\mathcal{R}}[d] := \bigcup_{\mathcal{D}, |D_0| = d} J^m_{\mathcal{R}}[\mathcal{D}] \subset J^m_{\mathcal{R}}, \quad \bar{J}^m_{\mathcal{R}}[d] = \bigcup_{d' \ge d} J^m_{\mathcal{R}}[d'], \tag{2-4}$$

where $d = 0, 1, \dots, \delta$. For $m = 1$, the D–flag $\mathcal{D}$ is obtained from D by adding one gap g and we use the notation $J^1_{\mathcal{R}}[D, g]$. When $m = 0$, we use $J_{\mathcal{R}}[D]$, which is from [38].

Let us begin with some general properties of flags $\mathscr{M}$. We note that our usage of Nakayama's lemma in Proposition 2.3 below is actually similar to that in [36], Section 2.1.

For an arbitrary module $\mathcal{M}$, we set $\mathcal{M}^{(i)} = \mathcal{M} \cap z^i\mathcal{O}$, which is obviously an $\mathcal{R}$–module, and $\overline{\mathcal{M}} = \mathcal{M}/\mathfrak{m}\mathcal{M}$ for the maximal ideal $\mathfrak{m} \subset \mathcal{R}$. Accordingly, $\overline{\mathcal{M}}^{(i)}$ is the image of $\mathcal{M}^{(i)}$ in $\overline{\mathcal{M}}$. Obviously, $\dim \mathcal{M}^{(g)}/\mathcal{M}^{(g+1)} = 1$ if and only if $g \in \Delta_{\mathcal{M}}$, and $\dim \overline{\mathcal{M}}^{(g)}/\overline{\mathcal{M}}^{(g+1)} \le 1$.

Proposition 2.3 (i) *For a D–flag*

$$\mathcal{D} = \big\{D_0 \subset D_1 = D_0 \cup \{g_1\} \subset \cdots \subset D_m = D_0 \cup \{g_1, \dots, g_m\}\big\}$$

from Definition 2.2, all $D_0 \cup \{g_i\}$ for $1 \le i \le m$ are D–sets. Let $\mathscr{M} = \{\mathcal{M}_i\}$ be a flag of (standard) modules corresponding to a D–flag (admissible). Picking an arbitrary $m_i \in \mathcal{M}_i$ such that $\upsilon(m_i) = g_i$, the space $\mathcal{M}_0 \oplus \mathbb{C}m_i$ is an $\mathcal{R}$–module.

Then for $1 \le i \le m$, we have $\mathcal{M}_i = \mathcal{M}_0 \oplus \mathbb{C}m_1 \oplus \cdots \oplus \mathbb{C}m_i$, $\mathfrak{m}\mathcal{M}_m \subset \mathcal{M}_0$ and $\dim \overline{\mathcal{M}}^{(g_i)}/\overline{\mathcal{M}}^{(g_i+1)} = 1$. The elements m_i modulo $\mathcal{M}_0$ are uniquely determined up to proportionality, ie depend only on the flag $\mathscr{M}$.

(ii) *Vice versa, for the D–flag $\mathcal{D}$ as above, let us assume the existence of a module $\mathcal{M}_{\mathrm{top}}$ such that $D[\mathcal{M}_{\mathrm{top}}] = D_m$ and $\dim \overline{\mathcal{M}}^{(g_i)}_{\mathrm{top}}/\overline{\mathcal{M}}^{(g_i+1)}_{\mathrm{top}} = 1$ for all i. We do not impose the admissibility of the whole $\mathcal{D}$. Then $\mathcal{M}_m = \mathcal{M}_{\mathrm{top}}$ can be extended to a flag $\mathscr{M}$ corresponding to $\mathcal{D}$ (so it is admissible) and all such flags can be described as follows.*

(a) *For any subspace $\overline{\mathcal{M}}_0 \subset \overline{\mathcal{M}}_{\mathrm{top}}$ such that for $1 \le i \le m$,*

$$\dim \overline{\mathcal{M}}_{\mathrm{top}}/\overline{\mathcal{M}}_0 = m, \quad \dim(\overline{\mathcal{M}}_0 + \overline{\mathcal{M}}^{(g_i)}_{\mathrm{top}})/(\overline{\mathcal{M}}_0 + \overline{\mathcal{M}}^{(g_i+1)}_{\mathrm{top}}) = 1, \quad \overline{\mathcal{M}}_0 \not\subset \overline{\mathcal{M}}^{(1)}_{\mathrm{top}},$$

let $\mathfrak{m}\mathcal{M}_{\mathrm{top}} \subset \mathcal{M}_0 \subset \mathcal{M}_{\mathrm{top}}$ be a unique lift of $\overline{\mathcal{M}}_0$ (the Nakayama lemma).

(b) *Then for any pullbacks, denoted by $\bar{m}_i$, of the generators of the latter quotients to $(\overline{\mathcal{M}}_0 + \overline{\mathcal{M}}^{(g_i)}_{\mathrm{top}})/\overline{\mathcal{M}}_0$, we take $\mathscr{M} = \{\mathcal{M}_i\}$, where $\mathcal{M}_i$ is a similar lift of the space $\overline{\mathcal{M}}_i = \overline{\mathcal{M}}_0 \oplus \mathbb{C}\bar{m}_1 \oplus \cdots \oplus \mathbb{C}\bar{m}_i$ to an $\mathcal{R}$–submodule of $\mathcal{M}_{\mathrm{top}}$ containing $\mathfrak{m}\mathcal{M}_{\mathrm{top}}$. These imply that $D[\mathcal{M}_0] = D_0$, $\mathcal{M}_{i=m} = \mathcal{M}_{\mathrm{top}}$.*

(iii) *Continuing* (ii), *all flags* $\mathscr{M}$ *in* (i) *are uniquely described by the following data:* (α) *the space* $\overline{\mathcal{M}}_0 \subset \overline{\mathcal{M}}_{\text{top}}$ *as above,* (β) *the set of elements* $\bar{m}_i \in (\overline{\mathcal{M}}_0 + \overline{\mathcal{M}}_{\text{top}}^{(g_i)})/\overline{\mathcal{M}}_0$ *considered up to proportionality. That is, for a fixed* $\mathcal{M}_{\text{top}}$ *and* $\overline{\mathcal{M}}_0$ *as in* (ii), *such flags of modules are naturally parametrized by unipotent complex matrices of size* $m \times m$, *which is due to the action of the Borel subgroup preserving the full flag* $\{(\overline{\mathcal{M}}_0 + \overline{\mathcal{M}}_{\text{top}}^{(g_i)})/\overline{\mathcal{M}}_0\}$ *in* $\overline{\mathcal{M}}_{\text{top}}/\overline{\mathcal{M}}_0$.

Proof Part (iii) formally follows from (i,ii), including the equivalence of the admissibility of a D–flag $\mathcal{D}$ and the existence of the pair $\{\mathcal{M}_0, \mathcal{M}_m\}$ for D_0, D_m such that $\mathfrak{m}\mathcal{M}_m \subset \mathcal{M}_0 \subset \mathcal{M}_m$. Let us justify (i,ii). Below $\{g_i\}$ means a single element g_i considered as a set.

First of all, $D_0 \cup \{g_i\}$ correspond to certain Γ–modules for any $1 \le i \le m$. Indeed, adding g_i to D_{i-1} cannot add anything new to D_i (since the corresponding Δ_i is assumed a Γ–module). The same holds for $D_0 \cup \{g_i\}$, since extra elements in the minimal Γ–module containing the latter can be only greater than g_i, ie can be only in $D_i \setminus D_{i-1}$

This reasoning is equally applicable to the flags $\mathscr{M}$ for admissible $\mathcal{D}$. We claim that $D_0 \cup \{g_i\}$ is admissible for each g_i. Indeed, adding $\mathcal{R}m_i \subset \mathcal{M}_i$ to $\mathcal{M}_0$ for an element m_i with the valuation $\upsilon(m_i) = g_i$ cannot create any new valuations versus Δ_0 but g_i, since this does not create them when going from $\mathcal{M}_{i-1}$ to $\mathcal{M}_i$. Thus $\mathcal{M}_0 + \mathcal{R}m_i = \mathcal{M}_0 + \mathbb{C}m_i$ is an $\mathcal{R}$–module corresponding to $D_0 \cup \{g_i\}$.

Next, ψm_i must belong to $\mathcal{M}_0$ for any $\psi \in \mathfrak{m}$. We use that for any $N > 0$ there exists an element $m' \in \mathcal{M}_0$ such that $\upsilon(\psi m_i - m') > N$; thus this difference can be assumed in $\mathcal{M}_0$.

Similarly, $\mathcal{M}_i$ is the linear span of $\mathcal{M}_{i-1}$ and m_i for the elements m_i introduced above, since for every $m \in \mathcal{M}_i$, there exists $m' \in \mathcal{M}_{i-1} + \mathcal{R}m_i$ such that $\upsilon(m - m')$ is greater than any given number. Thus $m - m'$ can be assumed in $\mathcal{M}_{i-1}$. We obtain that adding the elements $\{m_i \mid 1 \le i \le j\}$ to $\mathcal{M}_0$ generates $\mathcal{M}_j$. Combining this claim for $j = m$ with $\mathfrak{m}m_i \in \mathcal{M}_0$ checked above, we conclude that $\mathfrak{m}\mathcal{M}_m \subset \mathcal{M}_0$.

Part (ii) uses the same arguments and the Nakayama lemma. We lift $\bar{m}_i$ to arbitrary elements in $\mathcal{M}_{\text{top}}^{(g_i)}$. The corresponding $\mathcal{R}$–modules does not depend on such choices. Note that the condition $\overline{\mathcal{M}}_m = \overline{\mathcal{M}}_{\text{top}}$ provides that all $\mathcal{M}_i$ are standard ($\Delta_i \ni 0$). We need only to check that these modules have the required D–sets, which results from the following lemma by induction with respect to the length m of $\mathscr{M}$.

Lemma 2.4 *For two modules $\mathcal{M}' \subset \mathcal{M}$, where $\mathcal{M}'$ is not necessarily under the standard normalization, let $g \in D[\mathcal{M}]$, $\mathfrak{m}\mathcal{M} \subset \mathcal{M}'$, $\dim(\overline{\mathcal{M}}' + \overline{\mathcal{M}}^{(g)})/(\overline{\mathcal{M}}' + \overline{\mathcal{M}}^{(g+1)}) = 1$ and $\overline{\mathcal{M}} = \overline{\mathcal{M}}' \oplus \mathbb{C}\bar{m}$ for a pullback $\bar{m}$ of the generator of the latter quotient. Then we claim that $\mathcal{M}'$ is a standard module and $D[\mathcal{M}'] = D[\mathcal{M}] \setminus \{g\}$.*

Proof The module $\mathcal{M}$ is linearly generated by $\mathcal{M}'$ and any lift of $\bar{m}$ to $\mathcal{M}$ (the Nakayama lemma). Since $\dim(\overline{\mathcal{M}}' + \overline{\mathcal{M}}^{(g)})/(\overline{\mathcal{M}}' + \overline{\mathcal{M}}^{(g+1)})$ is 1, the gap g cannot be among the valuations of $\mathcal{M}'$ and this is the only missing gap there versus $\mathcal{M}$, which gives the required. □

Part (iii) follows from (ii). The inequalities $\dim \overline{\mathcal{M}}_{\mathrm{top}}^{(g_i)} / \overline{\mathcal{M}}_{\mathrm{top}}^{(g_i+1)} > 0$ are obviously necessary and sufficient to ensure the existence of the required $\overline{\mathcal{M}}_0$. The rest is a combination of (i) and (ii). □

Thus given a D–flag $\mathcal{D}$ as in (i), and $\mathcal{M}_{\mathrm{top}}$ such that

$$D[\mathcal{M}_{\mathrm{top}}] = D_m = D_0 \cup \{g_1, \dots, g_m\},$$

the inequalities $\dim \overline{\mathcal{M}}_{\mathrm{top}}^{(g_i)} / \overline{\mathcal{M}}_{\mathrm{top}}^{(g_i+1)} = 1$ are equivalent to the admissibility of $\mathcal{D}$. We note that in all examples we calculated, the admissibility of a D–flag is equivalent to that of every D_i. We cannot justify this in general.

This proposition is of independent interest and can be used to count the dimensions of $J_{\mathcal{R}}[\mathcal{D}]$ and to check (in some cases) that the latter are biregular to affine spaces.

We note that there is a natural (biregular) action of the algebraic group $J_{\mathcal{R}}[\varnothing] = \{\mathcal{L}\}$ in $J_{\mathcal{R}}$. Without going into details, Theorem 1 from [38] can be extended to flagged Jacobian factors as follows. If the GIT quotient $J_{\mathcal{R}}^m[\mathcal{D}]/J_{\mathcal{R}}[\varnothing]$ is biregular to an affine space and with the stabilizers of points in $J_{\mathcal{R}}^m[\mathcal{D}]$ of the same dimension, then the latter space is biregular to some $\mathbb{A}^N$. This is not always the case (see the online supplement), but it is not impossible that $J_{\mathcal{R}}^m[d]$ can be always paved by affine spaces.

2.4 The main conjecture

The flagged Jacobian factor $J_{\mathcal{R}}^m$ has a natural structure of a quasiprojective variety. Accordingly, $J_{\mathcal{R}}^m[\mathcal{D}]$ and $\bar{J}_{\mathcal{R}}^m[d]$ are its subvarieties; the latter is closed. Proposition 2.3 gives that $J_{\mathcal{R}}^m$ is a certain *subvariety* in a proper *parahoric Springer fibers* in the terminology from Section 2.2.9 from [44]. They are formed by *partial* periodic flags of $\mathcal{R}$–invariant lattices. We do not use them in the present work. By H_i, we will mean singular (relative) homology with the $\mathbb{C}$–coefficients.

Conjecture 2.5 *Let $\mathcal{R}$ be the ring of a unibranch plane curve singularity $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ from Section 1.3, and $\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q,t,a)$ the* DAHA *uncolored superpolynomial from (1-19). Recall that $\delta = |G_{\mathcal{R}}|$ is the arithmetic genus of $\mathcal{R}$; we assume that $\mathsf{r}_1 > \mathsf{s}_1$.*

(i) *We conjecture that the relative homology $H_{2i+1}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1])$ vanishes for all $i, d \geq 0$, and*

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q,t,a) = \sum_{d,i,m} \mathrm{rk}\big(H_{2i}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1])\big) q^{d+m} t^{\delta-i} a^m, \tag{2-5}$$

where $0 \leq d, i \leq \delta$, and the range of m is from 0 to $\mathsf{s}_1\mathsf{r}_2\cdots\mathsf{r}_\ell - 1$, which is the number of (*admissible*) *D such that $\dim(J_{\mathcal{R}}[D]) = \delta - 1$. The right-hand side of (2-5) will be called $\mathcal{H}^{\mathrm{hom}}(q,t,a)$. Also, we conjecture that $J^m_{\mathcal{R}}[d]$ are paved by affine spaces.*

(ii) *If all varieties $J^m_{\mathcal{R}}[\mathcal{D}]$ are affine spaces $\mathbb{A}^i$, then their total number for any fixed i, d gives the corresponding rank $\mathrm{rk}(H_{2i})$ from (2-5), and this relation reads*

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q,t,a) = \sum_{\mathcal{D}} q^{d+m} t^{\delta - \dim(J^m_{\mathcal{R}}[\mathcal{D}])} a^m, \tag{2-6}$$

where the summation is over all admissible D–flags $\mathcal{D}$, $d = |D_0|$. Here the cells $J^m_{\mathcal{R}}[\mathcal{D}]$ contribute to $H_{2i}(\bar{J}^m_{\mathcal{R}}[d]) \subset H_{2i}(J^m_{\mathcal{R}})$ for $i = \dim(J^m_{\mathcal{R}}[\mathcal{D}])$, and then to $H_{2i}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1])$. Formula (2-6) can be readily extended to any affine cell decompositions of $J^m_{\mathcal{R}}[d]$ (*not only those via $J^m_{\mathcal{R}}[\mathcal{D}]$*)*.*

(iii) *Let $1/t = p^\ell$ for prime p and $\ell \in \mathbb{N}$, let $\mathbb{F} = \mathbb{F}_{1/t}$ be the field with p^ℓ elements, and let $|X(\mathbb{F})|$ be the number of $\mathbb{F}$–points of a scheme X defined over $\mathbb{F}$. One can assume that $\mathcal{R}$ is defined over $\mathbb{Z}$ and consider $J^m_{\mathcal{R}}[d]$ as schemes over $\mathbb{F}$. We conjecture that apart from finitely many p,*

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q,t,a) = t^\delta \sum_{d,m} |J^m_{\mathcal{R}}[d](\mathbb{F})| q^{d+m} a^m := \mathcal{H}^{\mathrm{mod}}(q,t,a). \tag{2-7}$$

If $J^m_{\mathcal{R}}[d]$ are (a) *paved by affine spaces over $\mathbb{F}$, and* (b) (*non*)*admissible $\mathcal{D}$ remain such over this field, then (2-7) is equivalent to (2-5).* $\square$

Relative homology Vanishing $H_{2i+1}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1])$ generalizes the *van Straten–Warmt conjecture*, which claims that odd Betti numbers of $J_{\mathcal{R}}$ vanish. This assumption and the exact sequences for relative (singular) homology imply that the natural maps $H_{2i+1}(\bar{J}^m_{\mathcal{R}}[d+1]) \to H_{2i+1}(\bar{J}^m_{\mathcal{R}}[d])$ are surjective, which readily gives that $H_{2i+1}(\bar{J}^m_{\mathcal{R}}[d]) = \{0\}$ for any i, d. Using this, the following sequence is exact:

$$0 \to H_{2i}(\bar{J}^m_{\mathcal{R}}[d+1]) \to H_{2i}(\bar{J}^m_{\mathcal{R}}[d]) \to H_{2i}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1]) \to 0. \tag{2-8}$$

As an application, (2-5) upon the substitution $q=1$ becomes

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q=1,t,a)=\sum_{i,m}\mathrm{rk}\, H_{2i}(J^m_{\mathcal{R}})t^{\delta-i}a^m,$$

where the special case $a=0$ is [8, Conjecture 2.4(iii)]. Note that $H_{2i}(\bar{J}^m_{\mathcal{R}}[d], \bar{J}^m_{\mathcal{R}}[d+1])$ are potentially connected with compactly supported cohomology $H^{2k}_c(J^m_{\mathcal{R}}[d])$ for $k=j$ or $k+j=\dim J^m_{\mathcal{R}}[d]$, though Poincaré duality fails for singular varieties. Compare with (5-2).

Motivic approach Relation (2-7) readily follows from part (ii). We think that it can generally hold. In examples, it suffices to take any $p^\ell=1/t$ such that Γ remains the same over $\mathbb{F}_{1/t}$, but we do not know how far this goes. Say that (2-7) holds for $\mathcal{R}=\mathbb{F}_2[\![z^4+z^5,z^6]\!]$ under $a=0$, but not for $\mathbb{F}_3$, which changes Γ. However, $\mathcal{R}=\mathbb{F}_3[\![z^4,z^6+z^7]\!]$ is fine, and we note that $\nu(\mathbb{F}_2[\![z^4+z^5,z^6]\!])=\nu(\mathbb{F}_3[\![z^4,z^6+z^7]\!])$. These rings over $\mathbb{C}$ correspond to coinciding DAHA-superpolynomials. See (5-2) for a reformulation of (2-7) in terms of the weight filtration.

When $a=0$ and $q=1$, part (iii) is closely related to Theorems 0.1 and 0.2 from [15] in the case of A_n for affine Springer fibers. The latter are not always paved by affine spaces for other types [25]. The A_n–case is exceptional; the positivity of the orbital integrals and vanishing odd rational homology are widely expected to hold for anisotropic centralizers (ie in the nil-elliptic case). This matches well the conjectured positivity of uncolored DAHA superpolynomials.

Knowing the spectral curve is sufficient here. However, it is far from obvious beyond the torus case (quasihomogeneous plane curve singularities) that only the *topological* type of the singularity matters here. This follows from of our conjecture. The analytic equivalence of plane curve singularities is generally very different from the topological perspective. See [29; 4] and Section 5.3 for the identification of affine Springer fibers of anisotropic type with the compactified Jacobian of rational curves such that $\mathcal{R}$ is the local ring at its *unique* singular point.

In full generality, with the nonzero parameters q and a in (2-7), we count points in $J^m_{\mathcal{R}}[d](\mathbb{F})$ with q–weights $q^{|D_m|}$, where $|D|=\delta-\deg_{\mathcal{O}}\mathcal{M}$ for the corresponding *standard* $\mathcal{M}$. This seems a new turn in the theory of affine Springer fibers and related p–adic *orbital integrals*. Possible (expected) adding colors and the multibranch generalization of our construction make this even more interesting. Generally, we think that using the powerful modern theory of *topological* invariants of plane curve singularities can be expected to impact the theory of orbital integrals (at least in type A) and related part of the fundamental lemma; see Sections 5.3 and 5.4 for further discussion.

The case $\mathcal{R} = \langle z^4, z^6 + z^7\rangle$ We will stick to *admissible pairs* $\{\mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1\}$, where $D_1 = D_0 \cup \{g_1\}$ (corresponding to the coefficient of a^1 in the main conjecture). Recall that $D_i = \Delta_i \cap G$ and $G = \mathbb{Z}_+ \setminus \Gamma$. Using the definition of $J_{\mathcal{R}}[D_0 \subset D_1] = J^1_{\mathcal{R}}[D_0, g_1]$ directly, we obtain the following lemma.

Lemma 2.6 *Provided the admissibility of the Γ–modules D_0 and $D_1 = D_0 \cup \{g_1\}$ for $g_1 \notin \Gamma \cup D_0$, one has*

$$\dim J_{\mathcal{R}}[D_0 \subset D_1] \le \dim J_{\mathcal{R}}[D_1] + \big|\{g \in \Delta_0 \mid g < g_1\}\big|, \tag{2-9}$$

$$\dim J_{\mathcal{R}}[D_0 \subset D_1] = \dim J_{\mathcal{R}}[D_1] + 1 \quad \text{if } g_1 < g \text{ for all } g \in D_0,$$

$$\dim J_{\mathcal{R}}[D_0 \subset D_1] = \dim J_{\mathcal{R}}[D_0] \qquad \text{if } \{g \in G \mid g > g_1\} \subset D_0. \tag{2-10}$$

In the case of $\mathcal{R} = \langle z^4, z^6 + z^7\rangle$, the pairs $\{D_0, g_1\}$ that are not included in the latter two formulas are all governed by the first one where "$\le$" is replaced by "$=$" there. Thus these formulas provide all dimensions of the varieties of admissible pairs $D_0 \subset D_1$. Using this for such $\mathcal{R}$, one obtains the formula

$$\begin{aligned}\sum_{D_0 \subset D_1} q^{|D_0|+1} t^{\delta - \dim(J^m_{\mathcal{R}}[D_0 \subset D_1])} &= q + q^2(1+t) + q^3(1+2t+t^2) + q^4(3t+2t^2+t^3) \\ &\quad + q^5(t+4t^2+2t^3+t^4) + q^6(t^2+4t^3+2t^4+t^5) \\ &\quad + q^7(t^3+3t^4+2t^5+t^6) + q^8(t^5+t^6+t^7),\end{aligned}$$

which matches the coefficient of a^1 of the uncolored DAHA superpolynomial for $\mathcal{R} = \langle z^4, z^6 + z^7\rangle$ from [8].

Proof To justify the first inequality, we begin with any parametric family of modules $\mathcal{M}_0$ corresponding to D_0, assuming that they can be extended to $\mathcal{M}_{g_1}$ by adding certain m_{g_1} with $\upsilon(m_{g_1}) = g_1$ and that they are different (as submodules of $\mathcal{O}$) for different values of parameters. The dimension of the resulting family of modules $\mathcal{M}_1$ is no greater than $\dim J_{\mathcal{R}}[D_1]$. Given $\mathcal{M}_1 \ni m_{g_1}$, its different submodules $\mathcal{M}_0$ can be only obtained by adding the terms from $\mathbb{C}m_{g_1}$ to the generators $m_g \in \mathcal{M}_0$ with $\upsilon(m_g) = g$, where $g \in D_0$ such that $g < g_1$. This gives the required inequality.

If $g_1 < D_0$, then only $m_0 = 1 + z(\cdot)$ can be altered by $\mathbb{C}m_{g_1}$ and there is a one-dimensional family of (pairwise distinct) such submodules $\mathcal{M}_0$ inside fixed $\mathcal{M}_1$, which gives the second formula in (2-9). Here we use that the relations for the coefficients of the generators $m \in \mathcal{M}$ necessary for the equality $D_{\mathcal{M}} = D$ allow such a deformation.

Generally, if there several such m_g (not just $g = 0$) with g before g_1, then it is not true that the increase of dimension for $D[\mathcal{M}_0]$ will coincide with the number of such g. It really occurs with five such admissible pairs $\{D_0, D_1\}$ for $\mathcal{R} = \langle z^4, z^6 + z^{13}\rangle$:

(2-11)

D_0	D_1	$\dim_0$	$\dim_1$
$[2, 9, 15]$	$[2, 9, 11, 15]$	7	6
$[2, 7, 11, 15]$	$[2, 7, 9, 11, 15]$	6	5
$[2, 9, 11, 15]$	$[2, 7, 9, 11, 15]$	6	5
$[2, 3, 7, 9, 11, 15]$	$[2, 3, 5, 7, 9, 11, 15]$	6	5
$[1, 2, 5, 7, 9, 11, 15]$	$[1, 2, 3, 5, 7, 9, 11, 15]$	3	0

where $\dim_i = \dim(J_{\mathcal{R}}[D_i])$. However, in all these cases, $\dim(J_{\mathcal{R}}[D_0 \subset D_1])$ coincides with $\dim_0$, which is (2-9) with strict equality there.

Finally, formula (2-10) holds because adding *any* element $m_{g_1} \in \mathcal{O}$ with such a valuation g_1 to $\mathcal{M}_0$ results in the required $\mathcal{M}_1$ in this case. *All* cases from (2-11) are of the latter type. Thus any admissible pair for $\mathcal{R} = \langle z^4, z^6 + z^7\rangle$ satisfies either (2-9) with the equality there or (2-10). Then one can use the formulas for $\dim J_{\mathcal{R}}^{m=0}[D]$ from [38]. □

3 The family $\mathbb{C}[\![z^4, z^{2u} + z^v]\!]$

3.1 Dimensions of $J_{\mathcal{R}}^m[\mathcal{D}]$

It is shown in [38] that the varieties $J_{\mathcal{R}}[D]$ are isomorphic to affine spaces for the family $\mathcal{R} = \mathbb{C}[\![z^4, z^{2u} + z^v]\!]$, where $(uv, 2) = 1$, $v > 2u$ and their dimensions are computed. We will generalize these claims to admissible D–flags and the corresponding varieties $J_{\mathcal{R}}^m[\mathcal{D}]$.

To write our formula for the dimensions we will need a few definitions. Let

$$\mu_{D,g} := \dim(J_{\mathcal{R}}^1[D, g]) - \dim(J_{\mathcal{R}}[D])$$

be the *dimension change*. Also, we write

$$\gamma_\Delta(\ell) := |[\ell, \infty) \setminus \Delta|$$

for the *gap counting function*. Recall that $D = \Delta \setminus \Gamma$ is called a D–set corresponding to a *standard* Γ–module Δ (ie $0 \in \Delta$).

Theorem 3.1 *Let $\mathcal{R} = \mathbb{C}[\![z^4, z^{2u} + z^v]\!]$, where $(uv, 2) = 1$, $v > 2u$, and $\mathcal{D}$ be an admissible D–flag (with D_i corresponding to admissible Δ_i). Then $J^m_{\mathcal{R}}[\mathcal{D}]$ (with the notation as above) is isomorphic to an affine space $\mathbb{A}^N$ with*

$$N = \dim(J_{\mathcal{R}}[D_0]) + \sum_{i=0}^{m-1} \mu_{D_i, g_{i+1}},$$

where

$$\mu_{D,g} = \begin{cases} \gamma_{\Delta\cup\{g\}}(g) - \gamma_{\Delta\cup\{g\}}(g+4) & \text{for } g \equiv 1 \text{ or } 3 \bmod 4, \\ \gamma_{\Delta\cup\{g\}}(g) - \gamma_{\Delta\cup\{g\}}(g+4) & \\ \qquad - (\gamma_{\Delta\cup\{g\}}(g+n) - \gamma_{\Delta\cup\{g\}}(g+4+n)) & \text{for } g \equiv 2 \bmod 4, \end{cases}$$

and $n = n(D)$ is the smallest odd number in $(\Delta \cup v) \cap [2u, \infty)$. Here $D = D_i$, $g = g_{i+1}$ or D is any admissible D–set such that $D \cup \{g\}$ is admissible.

3.2 Basic definitions

Before we proceed with the proof, we will briefly summarize and adjust the approach taken in [38] to prove that the $J_{\mathcal{R}}[D]$ are affine; our argument relies heavily on his method. Let Δ be a Γ–module and begin by choosing a_0, a_1, a_2 and a_3 such that $a_i = \min\{k \in \Delta \mid k \equiv i \bmod 4\}$. Consider the following elements in $\mathcal{O}$:

$$(3\text{-}1)\qquad \begin{aligned} m_0 &= 1 + \sum_{k\in\mathbb{N}\setminus\Delta} \lambda^0_k z^k, & m_1 &= z^{a_1} + \sum_{k\in[a_1,\infty)\setminus\Delta} \lambda^1_{k-a_1} z^k, \\ m_2 &= z^{a_2} + \sum_{k\in[a_2,\infty)\setminus\Delta} \lambda^2_{k-a_2} z^k, & m_3 &= z^{a_3} + \sum_{k\in[a_3,\infty)\setminus\Delta} \lambda^3_{k-a_3} z^k, \end{aligned}$$

where the λ–coefficients are treated as variables. The valuation Γ–module of the module $\mathcal{M}$ generated by $\{m_i\}$ will then contain Δ, since any element of Δ has the form $a_i + 4n$ for some $n \in \mathbb{N}$ and because $z^4 \in \mathcal{R}$. Thus $\{a_i\}$ form a *basis* for $\upsilon(\mathcal{M})$ in a natural sense.

An important component of Piontkowski's method is the observation that $\upsilon(\mathcal{M}) = \Delta$ if and only if the relations among the elements m_i do not produce elements with valuation not in Δ. Thus the *syzygies* of the set $\{m_i\}$ as well as the syzygies of the set of their leading terms are of importance. Lemma 11 of [38] uses this basic idea to give an equivalent condition for $\upsilon(\mathcal{M}) = \Delta$; it will be provided. We need the notion of *initial vector*, which is also from [38], to state the aforementioned lemma:

Definition 3.2 For $\vec{r} = (r_0, \dots, r_3) \in \mathcal{R}^4$, let $\sigma = \min\{\upsilon(r_j) + a_j\}$. The *initial vector* $\mathrm{in}(\vec{r})$ is as follows: $\mathrm{in}(\vec{r}) = (\zeta_j)$ with ζ_j equal to the monomial of lowest degree in r_j if $\upsilon(r_j) + a_j = \sigma$ and 0 otherwise.

Lemma 3.3 *Let $\mathcal{M}$ be an $\mathcal{R}$–module generated by $\{m_0, m_1, m_2, m_3\}$, and V an $\mathcal{R}$–submodule in $\bigoplus_{j=0}^{3}\mathcal{R}$ such that the initial vectors $\{\mathrm{in}(\vec{r}) \mid \vec{r} \in V\}$ of V linearly generate the* syzygies *of the set (z^{a_j}) of $\mathbb{C}[\Delta]$. Here $\mathbb{C}[\Delta]$ is the vector space generated by the elements $\{z^k\}$ for $k \in \Delta$. We use $\sigma = \min\{\upsilon(r_j m_j)\} = \min\{\upsilon(r_j) + a_j\}$ from Definition 3.2.*

Then $\upsilon(\mathcal{M}) = \Delta$ if and only if, for each $\vec{r} = (r_j) \in V$, the initial terms in $\sum_{j=0}^{3} r_j m_j$ cancel, ie $\upsilon\big(\sum_{j=0}^{3} r_j m_j\big) > \sigma$, and for every j, there exists $p_j \in \mathcal{R}$ such that $\upsilon(p_j m_j) > \sigma$ and $\sum_{j=0}^{3} r_j m_j = \sum_{j=0}^{3} p_j m_j$. If such p_j exist, then the element $\sum_{j=0}^{3} p_j m_j \in \mathcal{M}$ is called a ***higher-order expression*** *for $\sum_{j=0}^{3} r_j m_j \in \mathcal{M}$, which is generally not uniquely determined by the latter element.* □

We will need the following *reduction procedure* from [38] for series $y \in \mathcal{O}$. Let $y_0 = y$. Then define inductively $y_{i+1} = y_i$ if $i \notin \Delta$, and we also set $s_i = 0$ in this case. If $i \in \Delta$, then find the monomial $c_i z^i$ with power i in y_i and the element $s_i \in \mathcal{R}$ such that $s_i m_{j_i} = c_i z^i + \cdots$ for one of the generators m_{j_i} (it can be nonunique). Then we let $y_{i+1} = y_i - s_i m_{j_i}$. The sequence of elements $y_i \in \mathcal{O}$ converges to an element y_∞, which has the form $\sum_{k \in \mathbb{N}\setminus\Delta} d_k z^k$ for some coefficients d_k.

The key facts from [38] about the reduction procedure are that (a) $y_\infty = 0$ if and only if $y \in \mathcal{M}$, and (b) if $y_\infty = 0$, then $\sum s_i m_{j_i}$ is a higher-order expression for y.

Definition 3.4 For any element $y \in \mathcal{O}$, the element $y^\dagger$ will be the result of the reduction procedure applied to y. One has $(y^\dagger)^\dagger = y^\dagger$.

The reduction procedure depends on the choices above. We can standardize the procedure by always taking the elements s_i involved to be of the form $(z^4)^k$ for some nonnegative integers k, which then makes the reduction procedure unique. To see that picking such s_i is possible, observe that when eliminating $c_i z^i$ for $i \in \Delta$, we can choose a unique generator m_j such that $a_j \equiv i \mod 4$. We will call such a procedure *standard*, and we will always assume such a standardization in what will follow.

One has $x^\dagger = x + f$ for some $f \in \mathcal{M}$. If for two elements $x, y \in \mathcal{O}$, we know that $x^\dagger = x + f_1$ and $y^\dagger = y + f_2$, then the standard reduced form $(x+y)^\dagger$ of $x+y$ is $x + y + f_1 + f_2$. Generally, if $(x+y)^\dagger = x + y + f$, the uniqueness guarantees $f = f_1 + f_2$, and we obtain that $x^\dagger + y^\dagger = (x+y)^\dagger$ for the standardization we will always impose. Thus $\dagger$ is a $\mathbb{C}$–linear projection.

Piontkowski proves that $J_{\mathcal{R}}[D]$ are affine by using the previous lemma and the reduction procedure. In Lemma 3.3, it is sufficient to consider the degrees of syzygies of (z^{a_j}) less than $\max\{a_j \mid j=0,\dots,3\}-3$, and there are only finitely many of such syzygies. Thus it is sufficient to check that only finitely many linear combinations of the $\{m_i\}$ prescribed by the syzygies have higher-order expressions to ensure that $\Delta = \upsilon(\mathcal{M})$. To obtain the higher-order expressions of these elements the reduction procedure is used and the resulting elements of $\mathcal{O}$ must vanish since these elements are in $\mathcal{M}$. The coefficients of the higher-order expressions are polynomials in terms of the coefficients of $\{m_i\}$, which have the form

$$\lambda_k^j - \lambda_k^i + \text{polynomial in } \lambda_\ell^*, \quad \ell < k.$$

These expressions must vanish due to the properties of the reduction procedure. Since they vanish and are linear in the parameters λ_k^j, λ_k^i, we can express λ_k^j in terms of λ_ℓ^* for $\ell < k$ and λ_k^i.

Applying the same process to λ_ℓ^* such that $\ell < k$, we can eventually show that $J_{\mathcal{R}}[D]$ is a graph of a regular function on an affine space $\mathbb{A}^n$. Since the graph of a regular function is always isomorphic to the domain of the function, we have that $J_{\mathcal{R}}[D] \cong \mathbb{A}^n$. We will use a similar technique to prove Theorem 3.1.

3.3 The cells $J_{\mathcal{R}}^m[\mathcal{D}]$ are affine

Let $\mathcal{D}$ be an admissible D–flag with $\vec{g} = \{g_i \mid 1 \le i \le m\}$. Define $\mathcal{E}$ to be the admissible D–flag of length $m-1$ such that $E_0 = D_0$ and $E_j = D_0 \cup \{g_i \mid 1 \le i \le j\}$ where $1 \le j \le m-1$; ie $\mathcal{E}$ is a truncation of $\mathcal{D}$.

Let $\mathscr{M}$ be an element of $J_{\mathcal{R}}^{m-1}[\mathcal{E}]$, and define

$$h_{g_m} = z^{g_m} + \sum_{k \in [g_m,\infty)\setminus\Delta_m} \lambda_{k-g_m}^{h_{g_m}} z^k.$$

By adjoining the module $\mathcal{M}_m = \mathcal{M}_{m-1} \oplus \mathcal{R} h_{g_m}$ to the end of $\mathscr{M}$, we can extend $\mathscr{M}$ to an element of $J_{\mathcal{R}}^m[\mathcal{D}]$ (of course there are restrictions on the λ coefficients which are addressed below). The other way around, every element of $J_{\mathcal{R}}^m[\mathcal{D}]$ can be obtained by this procedure from flags of length $m-1$.

To ensure that $\upsilon(\mathcal{M}_m) = \Delta_m$ and that $\mathcal{M}_{m-1} \subset \mathcal{M}_m$, we need to use Lemma 3.3 above and Lemma 3.5 below. The hypotheses for these lemmas only concern $\mathcal{M}_{m-1}$ and g_m. For this reason, we can reduce the argument to the case of $m=1$.

From now on we set $D = D_0$, so it corresponds to an admissible Γ–module $\Delta = \Delta_0$. Accordingly, set $g = g_1$ and $D_1 = D \cup \{g\}$, where D_1 will be assumed admissible. Also, we let $\Delta' = \Delta \cup \{g\}$.

Recall that we let the module generated by the $\{m_i\}$ as in (3-1) be denoted by $\mathcal{M}$. Then we consider adding

$$h := z^g + \sum_{k \in [g,\infty) \setminus \Delta} \lambda^h_{k-g} z^k$$

to the set of generators $\{m_i\}$ and set $\mathcal{M}' = \langle \mathcal{M}, h \rangle$. Note that h will replace m_ℓ where $g \equiv \ell \bmod 4$.

The modules $\mathcal{M}$ and $\mathcal{M}'$ must satisfy $\upsilon(\mathcal{M}) = \Delta$ and $\upsilon(\mathcal{M}') = \Delta'$. By Lemma 3.3, this holds for $\mathcal{M}$ if and only if the following elements have higher-order expressions:

$$\begin{aligned} T^1 &:= (z^{2u} + z^v) m_0 - z^{4\alpha_2} m_2, \\ T^2 &:= z^{4(u-\alpha_2)} m_0 - (z^{2u} + z^v) m_2, \\ T^3 &:= \left(z^{2u+v} + \tfrac{1}{2} z^{2v}\right) m_0 - z^{4\alpha_1} m_j, \end{aligned} \tag{3-2}$$

where α_i is the unique integer such that $a_i = \beta_i(2u+v) + \gamma_i(2u) - 4\alpha_i$ for $\beta_i, \gamma_i \in \{0,1\}$ and $j \equiv 2u + v \bmod 4$. To obtain the higher-order expressions, the reduction procedure is applied to T^i. As a result of the reduction procedure polynomial relations among the λ variables are obtained (as discussed before the beginning of the proof). For $\mathcal{M}'$, we use a similar approach. When we consider the syzygies (3-2) in the context of $\mathcal{M}'$, we will denote the resulting T by $(T^i)'$. See pages 14–17 of [38] concerning the existence of the higher-order expressions.

Given a module $\mathcal{M}$ with $\upsilon(\mathcal{M}) = \Delta$, not every module $\mathcal{N}$ with $\upsilon(\mathcal{N}) = \Delta'$ is its extension. In order to understand when $\mathcal{M} \subset \mathcal{N}$, let

$$F := m_i - (z^4) h \in \mathcal{N},$$

where $i \equiv g \bmod 4$. Note that $\upsilon(F) > a_i$, which gives a new type of syzygy (not from [38]); there is one such syzygy for each pair $\mathcal{M} \subset \mathcal{N}$.

Let us take i such that $g \equiv i \bmod 4$. Then replacing a_i with g results in a basis for Δ'; let us check this. If there were ℓ such that $\ell \equiv a_i \bmod 4$, $g < \ell < a_i$ and $\ell \notin \Delta$, then the presence of g in Δ' implies that $\ell \in \Delta'$ since Δ' is a Γ–module. This contradicts $\Delta' = \Delta \cup \{g\}$. Thus $g + 4 = a_i$, as required.

The following lemma provides necessary and sufficient conditions for $\mathcal{M} \subset \mathcal{N}$ when $\upsilon(\mathcal{N}) = \Delta'$ and $\upsilon(\mathcal{M}) = \Delta$. It is important to distinguish performing the reduction procedure with respect to $\mathcal{M}$ or $\mathcal{N}$ and we will use $\dagger_1$ to denote reduction with respect to the generators (3-1) of $\mathcal{M}$ and $\dagger_2$ for reduction with respect to (3-1) except with the changes necessary for $\mathcal{N}$.

Lemma 3.5 *Suppose $\mathcal{M}$ and $\mathcal{N}$ are standard $\mathcal{R}$–modules in $\mathcal{O}$ such that $\upsilon(\mathcal{M}) = \Delta$ and $\upsilon(\mathcal{N}) = \Delta \cup \{g\}$. Furthermore, suppose that $\mathcal{N}$ contains the generators m_j of $\mathcal{M}$ from (3-1) satisfying $g \not\equiv j \mod 4$. Then $\mathcal{M} \subset \mathcal{N}$ if and only if $F^{\dagger_2} = 0$.*

Proof Since $g \in \upsilon(\mathcal{N})$, we see that $h \in \mathcal{N}$ for some choice of values for the variables λ_k^h and that h is a normalized generator of $\mathcal{N}$. For some i, we have that $g \equiv a_i \mod 4$ (ie g replaces a_i in the basis for $\upsilon(\mathcal{M})$). Observe that $M \subset \mathcal{N}$ if and only if $m_i \in \mathcal{N}$, which will happen if and only if $(m_i)^{\dagger_2} = 0$. The first step in the reduction of m_i is F, and so $(m_i)^{\dagger_2} = 0$ if and only if $F^{\dagger_2} = 0$. □

Let us now return to considering $\mathcal{M} \subset \mathcal{M}'$ with $\upsilon(\mathcal{M}) = \Delta$ and $\upsilon(\mathcal{M}') = \Delta'$. We only need to consider the equations resulting from $F^{\dagger_2}$ since the equations resulting from $\upsilon(\mathcal{M}) = \Delta$ and $\upsilon(\mathcal{M}') = \Delta'$ are already solved for in [38]. We may write $F^{\dagger_2} = \sum_{k=1}^{\infty} \widetilde{c}_k z^{a_i+k}$. Recall that the only powers of z present in $F^{\dagger_2}$ are those greater than a_i that are not in Δ. By Lemma 3.5, we have $F^{\dagger_2} = 0$, which implies $\widetilde{c}_k = 0$. Similar to the analysis of $(T^j)^{\dagger_1}$ from the discussion before the proof, the $\widetilde{c}_k$ are in the form

$$\widetilde{c}_k = \lambda_k^i - \lambda_k^h + (\text{a polynomial in terms of } \lambda_p^\bullet \text{ for } p < k).$$

For a given k, we can then express λ_k^h in terms of $\lambda_p^\bullet$ for $p < k$. This gives that $J_{\mathcal{R}}^1[D, g]$ is an affine space because it is a graph of a regular function on an affine space. Since the $\gamma_{\Delta'}(g+4)$ equations $\widetilde{c}_k = 0$ are solvable, we see that $\mu_{D,g} \le \gamma_{\Delta'}(g) - \gamma_{\Delta'}(g+4)$. The exact value of $\mu_{D,g}$ depends on the congruence class of g modulo 4. We will now obtain the formulas for the dimensions.

3.4 Calculating dimensions

We are going now to justify the dimension formulas in Theorem 3.1. Recall that our approach extends the formulas and techniques used in [38] to the case of flags of modules.

First assume g is odd. Since $2u+v$ is odd, it is either congruent to 1 or $3 \bmod 4$. If $g \not\equiv 2u+v \bmod 4$, then $T^i = (T^i)'$ for all i, and hence we do not need to impose any further relations on the coefficients. Thus $\mu_{D,g} = \gamma_{\Delta'}(g) - \gamma_{\Delta'}(g+4)$ in this case.

When $g \equiv 2u+v \bmod 4$, we have $(T^3)' = z^{2u+v} g_0 - z^{4\alpha_h} h$ and $T^i = (T^i)'$ for $i \neq 3$. At the bottoms of pages 15 and 16 of [38], it is shown that a higher-order expression exists for T^3 when the smallest odd number $n \in \Delta \cap [2u, \infty)$ is less than or equal to v. When $n > v$ it is also shown that we can use the higher-order expressions of T^1 and T^2 to obtain a higher-order expression for T^3 (without imposing any new relations among the λ–parameters). Hence $\mu_{D,g} = \gamma_{\Delta'}(g) - \gamma_{\Delta'}(g+4)$. This finishes the proof when $g \equiv 1$ or $3 \bmod 4$.

In the last case, $g \equiv 2 \bmod 4$ implies that

$$\begin{aligned} (T^1)' &= (z^{2u} + z^v) m_0 - z^{4\alpha_h} h, \\ (T^2)' &= z^{4(u-\alpha_h)} m_0 - (z^{2u} + z^v) h, \end{aligned} \tag{3-3}$$

and $T^3 = (T^3)'$. Following [38], $\gamma_{\Delta'}(2u)$ equations for the λ–parameters result from the coefficients of $((T^1)')^{\dagger_2}$, and $\gamma_{\Delta'}(g+n)$ distinct equations result from the coefficients of $((T^2)')^{\dagger_2}$. We claim that the $\gamma_{\Delta'}(2u)$ equations from $((T^1)')^{\dagger_2}$ are equal to those from $(T^1)^{\dagger_1}$, and all of the $\gamma_{\Delta'}(a_2+n)$ equations from $(T^2)^{\dagger_1}$ can be obtained from the coefficients of $((T^2)')^{\dagger_2}$. To prove this we introduce the following definition and lemma.

Let P be any power series in z whose coefficients are polynomials in the λ variables. We let $\mathfrak{I}(P)$ be the ideal generated by the coefficients of P in the polynomial ring over the λ variables. We have the following basic result concerning $\mathfrak{I}$ and $\dagger_2$.

Lemma 3.6 *$\mathfrak{I}((rP)^{\dagger_2}) \subset \mathfrak{I}(P^{\dagger_2})$, where r is a polynomial in $\mathcal{R}$.*

Proof Since $\dagger_2$ is linear, it is sufficient to prove the lemma when r is a monomial. The reduction procedure for rP is exactly the same as for P until the first k such that $k \notin \Delta \ni k + v(r)$. Beyond this range, a multiple of the coefficient of z^k in $P^{\dagger_2}$ may be added to the remaining coefficients of $(rP)^{\dagger_2}$. Because of such k, all coefficients of $(rP)^{\dagger_2}$ will be coefficients of $P^{\dagger_2}$ plus some multiples of the previous coefficients of $P^{\dagger_2}$. Thus we have $\mathfrak{I}[(rP)^{\dagger_2}] \subset \mathfrak{I}[P^{\dagger_2}]$. □

On page 14 of [38], we see that the valuations of T^1, T^2, $(T^1)'$ and $(T^2)'$ are greater than $2u$ and therefore greater than g. Let us use this.

Lemma 3.7 *If $\mathcal{M} \subset \mathcal{M}'$, then $(T^1)^{\dagger_1} = (T^1)^{\dagger_2}$ and $(T^2)^{\dagger_1} = (T^2)^{\dagger_2}$.*

Proof Observe that $(T^1)^{\dagger_1} - (T^1)^{\dagger_2} = r_0 m_0 + r_1 m_1 + r_2 F + r_3 m_3$, where the $r_i \in R$. When we apply $\dagger_2$ to the left-hand side, we get

$$((T^1)^{\dagger_1} - (T^1)^{\dagger_2})^{\dagger_2} = (T^1)^{\dagger_1} - (T^1)^{\dagger_2}$$

since $(T^1)^{\dagger_1}$ has valuations greater than g, which implies the left-hand side is an eigenvector of $\dagger_2$. Applying $\dagger_2$ to the right-hand side, we have

$$(r_0 m_0 + r_1 m_1 + r_2 F + r_3 m_3)^{\dagger_2} = (r_2 F)^{\dagger_2}$$

since m_0, m_1, m_3 are in $\mathcal{M}'$. Hence we have

$$(T^1)^{\dagger_1} - (T^1)^{\dagger_2} = (r_2 F)^{\dagger_2}.$$

Note that r_2 may be a power series, but only a truncation of it determines $(r_2 F)^{\dagger_2}$ because terms with valuation greater than the conductor will all eventually be eliminated. Therefore, by Lemma 3.6, we have $\mathfrak{I}((r_2 F)^{\dagger_2}) \subset \mathfrak{I}(F^{\dagger_2})$. Now $\mathcal{M} \subset \mathcal{M}'$, which means $F^{\dagger_2} = 0$ by Lemma 3.5, and so we have $(r_2 F)^{\dagger_2} = 0$. The proof for T^2 is identical because T^2 has valuation greater than $2u$. □

Now we prove that $(T^1)^{\dagger_1} = ((T^1)')^{\dagger_2}$ when $\mathcal{M} \subset \mathcal{M}'$. Notice that $T^1 - (T^1)' = -z^{4\alpha_2} F$, where α_2 and α_h are defined in (3-2) and (3-3), respectively, and we have used that $\alpha_h = \alpha_2 + 1$. Thus

$$(T^1)^{\dagger_2} - ((T^1)')^{\dagger_2} = (-z^{4\alpha_2} F)^{\dagger_2}.$$

By Lemmas 3.5 and 3.6 we have $(-z^{4\alpha_2} F)^{\dagger_2} = 0$ so by the previous Lemma 3.7 we have $(T^1)^{\dagger_1} = ((T^1)')^{\dagger_2}$.

To prove the equations from $(T^2)^{\dagger_1}$ are redundant we first observe that $T^2 - z^4 (T^2)' = -(z^{2u} + z^v) F$. This implies that

$$(T^2)^{\dagger_2} - (z^4 (T^2)')^{\dagger_2} = (-(z^{2u} + z^v) F)^{\dagger_2}.$$

Again, by Lemmas 3.5 and 3.6, we see that $(-(z^{2u} + z^v) F)^{\dagger_2} = 0$. By Lemma 3.7, we have $(T^2)^{\dagger_2} = (T^2)^{\dagger_1}$, which means $(z^4 (T^2)')^{\dagger_2} = (T^2)^{\dagger_1}$. Finally, by Lemma 3.6, $\mathfrak{I}((z^4 (T^2)')^{\dagger_2}) \subset \mathfrak{I}(((T^2)')^{\dagger_2})$, which readily implies that the equations from $(T^2)^{\dagger_1}$ are redundant.

Thus we have shown that the $\gamma_{\Delta'}(2u)+\gamma_{\Delta'}(a_2+n)$ equations from $(T^1)^{\dagger_1}$ and $(T^2)^{\dagger_1}$ are actually redundant. So, as required,

$$\begin{aligned}\mu_{D,g} &= \gamma_{\Delta'}(g)-\gamma_{\Delta'}(a_2)-\gamma_{\Delta'}(2u)-\gamma_{\Delta'}(g+n)+\gamma_{\Delta'}(2u)+\gamma_{\Delta'}(a_2+n)\\ &= \gamma_{\Delta'}(g)-\gamma_{\Delta'}(a_2)-(\gamma_{\Delta'}(g+n)-\gamma_{\Delta'}(a_2+n))\\ &= \gamma_{\Delta\cup\{g\}}(g)-\gamma_{\Delta\cup\{g\}}(g+4)-(\gamma_{\Delta\cup\{g\}}(g+n)-\gamma_{\Delta\cup\{g\}}(g+4+n)).\end{aligned}$$

4 Numerical support

We provide here dimensions of cells for some basic examples and the corresponding nonadmissible D–flags $\mathcal{D}$. It is important to know whether these dimensions, nonadmissible $\mathcal{D}$ and admissible ones with nonaffine $J_r^m[\mathcal{D}]$ are *topological* properties of the singularity. We found no counterexamples, but formally this can be wrong even if our main conjecture holds. Zariski proved in Chapter IV, Section 3 of [45] that $\Gamma=\langle 4,6,13+2v\rangle$ for $v\geq 0$ uniquely determines the corresponding *analytic* singularity (ie that each equisingularity class is one point), but generally such questions can be (very) involved. Also, these examples seem absolutely necessary for (restarting) the theory of Jacobian factors beyond torus knots, which is of obvious importance for topology and geometry of plane curve singularities and for orbital integrals.

4.1 Two simplest cables

We begin with formula (2-6) for the knots $\mathrm{Cab}(13,2)T(3,2)$ and $\mathrm{Cab}(15,2)T(3,2)$. Here one can use the general Theorem 3.1 or the special Lemma 2.6. The latter can be extended to any m for these two cases (with minor adjustments).

Recall that $\mathcal{R}=\langle z^4,z^6+z^7\rangle$, $\Gamma=\langle 4,6,13\rangle$ and $\mathcal{R}=\langle z^4,z^6+z^9\rangle$, $\Gamma=\langle 4,6,15\rangle$ in these cases. The first DAHA superpolynomial is as follows:

$$\vec{\mathsf{r}}=\{3,2\},\quad \vec{\mathsf{s}}=\{2,1\},\quad \mathcal{T}=\mathrm{Cab}(13,2)T(3,2);$$

$$\begin{aligned}&\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a)=\\ &1+qt+q^8t^8+q^2(t+t^2)+a^3(q^6+q^7t+q^8t^2)+q^3(t+t^2+t^3)\\ &\quad+q^4(2t^2+t^3+t^4)+q^5(2t^3+t^4+t^5)+q^6(2t^4+t^5+t^6)+q^7(t^5+t^6+t^7)\\ &\quad+a^2\big(q^3+q^4(1+t)+q^5(1+2t+t^2)+q^6(2t+2t^2+t^3)+q^7(2t^2+2t^3+t^4)\\ &\qquad\qquad+q^8(t^3+t^4+t^5)\big)\\ &\quad+a\big(q+q^2(1+t)+q^3(1+2t+t^2)+q^4(3t+2t^2+t^3)+q^5(t+4t^2+2t^3+t^4)\\ &\qquad\qquad+q^6(t^2+4t^3+2t^4+t^5)+q^7(t^3+3t^4+2t^5+t^6)+q^8(t^5+t^6+t^7)\big).\end{aligned}$$

D–sets	dim
$\varnothing$	8
15	7
11, 15	6
7, 11, 15	6
9, 15	7
9, 11, 15	5
7, 9, 11, 15	4
3, 7, 9, 11, 15	4
5, 9, 11, 15	5
5, 7, 9, 11, 15	3
3, 5, 7, 9, 11, 15	2
1, 5, 7, 9, 11, 15	4

D–sets	dim
1, 3, 5, 7, 9, 11, 15	2
2, 7, 11, 15	6
2, 9, 15	7
2, 9, 11, 15	6
2, 7, 9, 11, 15	5
2, 3, 7, 9, 11, 15	4
2, 5, 9, 11, 15	5
2, 5, 7, 9, 11, 15	3
2, 3, 5, 7, 9, 11, 15	1
1, 2, 5, 7, 9, 11, 15	3
1, 2, 3, 5, 7, 9, 11, 15	0

Table 1: Dimensions for $\Gamma = \langle 4, 6, 13\rangle$, $m = 0$

Here and henceforth, we use [8]. Let us list the necessary information to verify (2-6). For the greatest possible $m = 3$, there are only three admissible D–sets D_0 that can occur in such a (long) flag. Namely, these flags and the dimensions $\dim J^3_{\mathcal{R}}[\mathcal{D}]$ are

$$\begin{aligned} &D_0 = [9, 11, 15], && \vec{g} = (2, 5, 7), && \dim = 8 && \rightsquigarrow q^6 t^0 a^3, \\ &D_0 = [7, 9, 11, 15], && \vec{g} = (2, 3, 5), && \dim = 7 && \rightsquigarrow q^7 t^1 a^3, \\ &D_0 = [5, 7, 9, 11, 15], && \vec{g} = (1, 2, 3), && \dim = 6 && \rightsquigarrow q^8 t^2 a^3; \end{aligned} \tag{4-1}$$

we show their contributions to the corresponding superpolynomial.

Tables 1, 2, 3 show D_0, the corresponding $\vec{g}$ and the dimensions of $J^m_{\mathcal{R}}[\mathcal{D}]$ for all admissible flags as $m = 0, 1, 2$.

There are no admissible extensions of degree 4, so we have

$$\begin{aligned} &\sum_D q^{|D|+m} t^{\delta - \dim(J^m_{\mathcal{R}}[D, D'])} a^m = \\ &1 + qt + q^2(t + t^2) + q^3(t + t^2 + t^3) + q^4(2t^2 + t^3 + t^4) \\ &\qquad + q^5(2t^3 + t^4 + t^5) + q^6(2t^4 + t^5 + t^6) + q^7(t^5 + t^6 + t^7) + q^8 t^8 \\ &\quad + a^3\big(q^6 + q^7 t + q^8 t^2\big) \\ &\quad + a^2\big(q^3 + q^4(1 + t) + q^5(1 + 2t + t^2) + q^6(2t + 2t^2 + t^3) \\ &\qquad + q^7(2t^2 + 2t^3 + t^4) + q^8(t^3 + t^4 + t^5)\big) \\ &\quad + a\big(q + q^2(1 + t) + q^3(1 + 2t + t^2) + q^4(3t + 2t^2 + t^3) + q^5(t + 4t^2 + 2t^3 + t^4) \\ &\qquad + q^6(t^2 + 4t^3 + 2t^4 + t^5) + q^7(t^3 + 3t^4 + 2t^5 + t^6) + q^8(t^5 + t^6 + t^7)\big), \end{aligned}$$

which coincides with $\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q, t, a)$ from Section 3.1 of [8].

D–sets	g	dim	D–sets	g	dim
$\varnothing$	15	8	5, 7, 9, 11, 15	2	4
15	9	8	5, 7, 9, 11, 15	3	3
15	11	7	3, 5, 7, 9, 11, 15	1	3
11, 15	7	7	3, 5, 7, 9, 11, 15	2	2
11, 15	9	6	1, 5, 7, 9, 11, 15	2	5
7, 11, 15	2	7	1, 5, 7, 9, 11, 15	3	4
7, 11, 15	9	6	1, 3, 5, 7, 9, 11, 15	2	2
9, 15	2	8	2, 7, 11, 15	9	6
9, 15	11	7	2, 9, 15	11	7
9, 11, 15	2	7	2, 9, 11, 15	5	7
9, 11, 15	5	6	2, 9, 11, 15	7	6
9, 11, 15	7	5	2, 7, 9, 11, 15	3	6
7, 9, 11, 15	2	6	2, 7, 9, 11, 15	5	5
7, 9, 11, 15	3	5	2, 3, 7, 9, 11, 15	5	4
7, 9, 11, 15	5	4	2, 5, 9, 11, 15	7	5
3, 7, 9, 11, 15	2	5	2, 5, 7, 9, 11, 15	1	4
3, 7, 9, 11, 15	5	4	2, 5, 7, 9, 11, 15	3	3
5, 9, 11, 15	2	6	2, 3, 5, 7, 9, 11, 15	1	1
5, 9, 11, 15	7	5	1, 2, 5, 7, 9, 11, 15	3	3
5, 7, 9, 11, 15	1	5			

Table 2: Dimensions for $\Gamma = \langle 4, 6, 13\rangle$, $m = 1$

For $\mathcal{R}=\mathbb{C}[\![t^4, t^6+t^9]\!]$ corresponding to the $\Gamma=\langle 4,6,15\rangle$ and cable $\mathrm{Cab}(15,2)T(3,2)$, the situation is very similar. We checked that

$$\sum_{m=0}^{\infty}\sum_{\{D_0=D,\dots,D_m\}} q^{|D|+m}t^{\delta-\dim(J^m_{\mathcal{R}}[D])}a^m = \mathcal{H}_{\{3,2\},\{2,3\}}(\square; q,t,a) =$$

$$\begin{aligned}
1 &+ qt + q^9t^9 + q^2(t+t^2) + q^3(t+t^2+t^3) + a^3\big(q^6 + q^7t + q^8t^2 + q^9t^3\big)\\
&+ q^4(2t^2+t^3+t^4) + q^5(2t^3+t^4+t^5) + q^6(2t^4+t^5+t^6)\\
&\qquad\qquad + q^7(2t^5+t^6+t^7) + q^8(t^6+t^7+t^8)\\
&+ a^2\big(q^3 + q^4(1+t) + q^5(1+2t+t^2) + q^6(2t+2t^2+t^3)\\
&\qquad\qquad + q^7(2t^2+2t^3+t^4) + q^8(2t^3+2t^4+t^5) + q^9(t^4+t^5+t^6)\big)\\
&+ a\big(q + q^2(1+t) + q^3(1+2t+t^2)\\
&\qquad + q^4(3t+2t^2+t^3) + q^5(t+4t^2+2t^3+t^4) + q^6(t^2+4t^3+2t^4+t^5)\\
&\qquad\qquad + q^7(t^3+4t^4+2t^5+t^6) + q^8(t^4+3t^5+2t^6+t^7) + q^9(t^6+t^7+t^8)\big).
\end{aligned}$$

D–sets	$\vec{g}$	dim
15	9, 11	8
11, 15	7, 9	7
7, 11, 15	2, 9	7
9, 15	2, 11	8
9, 11, 15	2, 5	8
9, 11, 15	2, 7	7
9, 11, 15	5, 7	6
7, 9, 11, 15	2, 3	7
7, 9, 11, 15	2, 5	6
7, 9, 11, 15	3, 5	5

D–sets	$\vec{g}$	dim
3, 7, 9, 11, 15	2, 5	5
5, 9, 11, 15	2, 7	6
5, 7, 9, 11, 15	1, 2	6
5, 7, 9, 11, 15	1, 3	5
5, 7, 9, 11, 15	2, 3	4
3, 5, 7, 9, 11, 15	1, 2	3
1, 5, 7, 9, 11, 15	2, 3	5
2, 9, 11, 15	5, 7	7
2, 7, 9, 11, 15	3, 5	6
2, 5, 7, 9, 11, 15	1, 3	4

Table 3: Dimensions for $\Gamma = \langle 4, 6, 13\rangle$, $m = 2$

See the same section of [8]. Note that formulas (3.1) and (3.2) there for the DAHA–Betti polynomials are obtained from the superpolynomials as $a = 0, q = 1$.

4.2 The case of $\Gamma = \langle 4, 14, 31\rangle$

We checked the main conjecture in many cases for the series $(4, 2u, v)$ of *Puiseux exponents*. This example is of importance because quite a few features of Theorem 3.1 (and our proof) cannot be seen for the subfamily $(4, 6, v)$. For $\Gamma = \langle 4, 14, 31\rangle$, we list all admissible D–flags $\mathcal{D}$ and the dimensions $\dim J^m_{\mathcal{R}}[\mathcal{D}]$ for $m = 3$ (the greatest possible value). The calculation of dimensions at the maximal m as a matter of fact includes a lot of information about the dimensions for previous lengths m, so this is a good test of our conjecture. However, we restrict ourselves with a^3 here due to practical reasons. The total number of admissible $\mathcal{D}$ is 1071, but there are "only" 85 such D–flags of the top length for $m = 3$. The dimensions are in Tables 4 and 5 (its continuation) in the same format as above. Compare them with the coefficients of a^3 of the corresponding superpolynomial from [8], which we provide:

$$\mathcal{R} = \mathbb{C}[\![z^4, z^{14} + z^{17}]\!], \quad \mathcal{T} = \mathrm{Cab}(31, 2)T(7, 2);$$

$\mathcal{H}_{\{7,2\},\{2,3\}}(\square; q, t, a) =$

$1+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^5t^2+q^6t^2+q^3t^3+q^4t^3+2q^5t^3+2q^6t^3+2q^7t^3+ q^8t^3+q^9t^3+q^4t^4+q^5t^4+2q^6t^4+2q^7t^4+3q^8t^4+2q^9t^4+2q^{10}t^4+q^5t^5+q^6t^5+2q^7t^5+ 2q^8t^5+3q^9t^5+3q^{10}t^5+3q^{11}t^5+q^6t^6+q^7t^6+2q^8t^6+2q^9t^6+3q^{10}t^6+3q^{11}t^6+4q^{12}t^6+ q^7t^7+q^8t^7+2q^9t^7+2q^{10}t^7+3q^{11}t^7+3q^{12}t^7+4q^{13}t^7+q^8t^8+q^9t^8+2q^{10}t^8+2q^{11}t^8+ 3q^{12}t^8+3q^{13}t^8+4q^{14}t^8+q^9t^9+q^{10}t^9+2q^{11}t^9+2q^{12}t^9+3q^{13}t^9+3q^{14}t^9+4q^{15}t^9+$

$$q^{10}t^{10}+q^{11}t^{10}+2q^{12}t^{10}+2q^{13}t^{10}+3q^{14}t^{10}+3q^{15}t^{10}+3q^{16}t^{10}+q^{11}t^{11}+q^{12}t^{11}+2q^{13}t^{11}+2q^{14}t^{11}+3q^{15}t^{11}+3q^{16}t^{11}+2q^{17}t^{11}+q^{12}t^{12}+q^{13}t^{12}+2q^{14}t^{12}+2q^{15}t^{12}+3q^{16}t^{12}+2q^{17}t^{12}+q^{18}t^{12}+q^{13}t^{13}+q^{14}t^{13}+2q^{15}t^{13}+2q^{16}t^{13}+3q^{17}t^{13}+q^{18}t^{13}+q^{14}t^{14}+q^{15}t^{14}+2q^{16}t^{14}+2q^{17}t^{14}+2q^{18}t^{14}+q^{15}t^{15}+q^{16}t^{15}+2q^{17}t^{15}+2q^{18}t^{15}+q^{19}t^{15}+q^{16}t^{16}+q^{17}t^{16}+2q^{18}t^{16}+q^{19}t^{16}+q^{17}t^{17}+q^{18}t^{17}+2q^{19}t^{17}+q^{18}t^{18}+q^{19}t^{18}+q^{20}t^{18}+q^{19}t^{19}+q^{20}t^{19}+q^{20}t^{20}+q^{21}t^{21}$$

$$+a^3\big(q^6+q^7t+q^8t+q^9t+q^8t^2+q^9t^2+2q^{10}t^2+q^{11}t^2+q^{12}t^2+q^9t^3+q^{10}t^3+2q^{11}t^3+2q^{12}t^3+2q^{13}t^3+q^{10}t^4+q^{11}t^4+2q^{12}t^4+2q^{13}t^4+3q^{14}t^4+q^{11}t^5+q^{12}t^5+2q^{13}t^5+2q^{14}t^5+3q^{15}t^5+q^{12}t^6+q^{13}t^6+2q^{14}t^6+2q^{15}t^6+3q^{16}t^6+q^{13}t^7+q^{14}t^7+2q^{15}t^7+2q^{16}t^7+3q^{17}t^7+q^{14}t^8+q^{15}t^8+2q^{16}t^8+2q^{17}t^8+2q^{18}t^8+q^{15}t^9+q^{16}t^9+2q^{17}t^9+2q^{18}t^9+q^{19}t^9+q^{16}t^{10}+q^{17}t^{10}+2q^{18}t^{10}+q^{19}t^{10}+q^{17}t^{11}+q^{18}t^{11}+2q^{19}t^{11}+q^{18}t^{12}+q^{19}t^{12}+q^{20}t^{12}+q^{19}t^{13}+q^{20}t^{13}+q^{20}t^{14}+q^{21}t^{15}\big)$$

$$+a^2\big(q^3+q^4+q^5+q^4t+2q^5t+3q^6t+2q^7t+q^8t+q^5t^2+2q^6t^2+4q^7t^2+4q^8t^2+4q^9t^2+2q^{10}t^2+q^{11}t^2+q^6t^3+2q^7t^3+4q^8t^3+5q^9t^3+6q^{10}t^3+5q^{11}t^3+3q^{12}t^3+q^7t^4+2q^8t^4+4q^9t^4+5q^{10}t^4+7q^{11}t^4+7q^{12}t^4+5q^{13}t^4+q^8t^5+2q^9t^5+4q^{10}t^5+5q^{11}t^5+7q^{12}t^5+8q^{13}t^5+6q^{14}t^5+q^9t^6+2q^{10}t^6+4q^{11}t^6+5q^{12}t^6+7q^{13}t^6+8q^{14}t^6+6q^{15}t^6+q^{10}t^7+2q^{11}t^7+4q^{12}t^7+5q^{13}t^7+7q^{14}t^7+8q^{15}t^7+6q^{16}t^7+q^{11}t^8+2q^{12}t^8+4q^{13}t^8+5q^{14}t^8+7q^{15}t^8+8q^{16}t^8+5q^{17}t^8+q^{12}t^9+2q^{13}t^9+4q^{14}t^9+5q^{15}t^9+7q^{16}t^9+7q^{17}t^9+3q^{18}t^9+q^{13}t^{10}+2q^{14}t^{10}+4q^{15}t^{10}+5q^{16}t^{10}+7q^{17}t^{10}+5q^{18}t^{10}+q^{19}t^{10}+q^{14}t^{11}+2q^{15}t^{11}+4q^{16}t^{11}+5q^{17}t^{11}+6q^{18}t^{11}+2q^{19}t^{11}+q^{15}t^{12}+2q^{16}t^{12}+4q^{17}t^{12}+5q^{18}t^{12}+4q^{19}t^{12}+q^{16}t^{13}+2q^{17}t^{13}+4q^{18}t^{13}+4q^{19}t^{13}+q^{20}t^{13}+q^{17}t^{14}+2q^{18}t^{14}+4q^{19}t^{14}+2q^{20}t^{14}+q^{18}t^{15}+2q^{19}t^{15}+3q^{20}t^{15}+q^{19}t^{16}+2q^{20}t^{16}+q^{21}t^{16}+q^{20}t^{17}+q^{21}t^{17}+q^{21}t^{18}\big)$$

$$+a\big(q+q^2+q^3+q^2t+2q^3t+3q^4t+2q^5t+q^6t+q^3t^2+2q^4t^2+4q^5t^2+4q^6t^2+4q^7t^2+2q^8t^2+q^9t^2+q^4t^3+2q^5t^3+4q^6t^3+5q^7t^3+6q^8t^3+5q^9t^3+4q^{10}t^3+q^{11}t^3+q^5t^4+2q^6t^4+4q^7t^4+5q^8t^4+7q^9t^4+7q^{10}t^4+7q^{11}t^4+2q^{12}t^4+q^6t^5+2q^7t^5+4q^8t^5+5q^9t^5+7q^{10}t^5+8q^{11}t^5+9q^{12}t^5+3q^{13}t^5+q^7t^6+2q^8t^6+4q^9t^6+5q^{10}t^6+7q^{11}t^6+8q^{12}t^6+10q^{13}t^6+3q^{14}t^6+q^8t^7+2q^9t^7+4q^{10}t^7+5q^{11}t^7+7q^{12}t^7+8q^{13}t^7+10q^{14}t^7+3q^{15}t^7+q^9t^8+2q^{10}t^8+4q^{11}t^8+5q^{12}t^8+7q^{13}t^8+8q^{14}t^8+10q^{15}t^8+3q^{16}t^8+q^{10}t^9+2q^{11}t^9+4q^{12}t^9+5q^{13}t^9+7q^{14}t^9+8q^{15}t^9+9q^{16}t^9+2q^{17}t^9+q^{11}t^{10}+2q^{12}t^{10}+4q^{13}t^{10}+5q^{14}t^{10}+7q^{15}t^{10}+8q^{16}t^{10}+7q^{17}t^{10}+q^{18}t^{10}+q^{12}t^{11}+2q^{13}t^{11}+4q^{14}t^{11}+5q^{15}t^{11}+7q^{16}t^{11}+7q^{17}t^{11}+4q^{18}t^{11}+q^{13}t^{12}+2q^{14}t^{12}+4q^{15}t^{12}+5q^{16}t^{12}+7q^{17}t^{12}+5q^{18}t^{12}+q^{19}t^{12}+q^{14}t^{13}+2q^{15}t^{13}+4q^{16}t^{13}+5q^{17}t^{13}+6q^{18}t^{13}+2q^{19}t^{13}+q^{15}t^{14}+2q^{16}t^{14}+4q^{17}t^{14}+5q^{18}t^{14}+4q^{19}t^{14}+q^{16}t^{15}+2q^{17}t^{15}+4q^{18}t^{15}+4q^{19}t^{15}+q^{20}t^{15}+q^{17}t^{16}+2q^{18}t^{16}+4q^{19}t^{16}+2q^{20}t^{16}+q^{18}t^{17}+2q^{19}t^{17}+3q^{20}t^{17}+q^{19}t^{18}+2q^{20}t^{18}+q^{21}t^{18}+q^{20}t^{19}+q^{21}t^{19}+q^{21}t^{20}\big).$$

4.3 The series $(6,8,v)$

The series with *Puiseux exponents* $(4,2u,v)$ above corresponds to the somewhat special links $\mathrm{Cab}(2u+v,2)T(u,2)$; torus knots and cables for $(2p+1,2)$ are known to have some special symmetries.

D–sets	$\vec{g}$	dim
27, 37, 41	10, 23, 33	21
27, 33, 37, 41	10, 23, 29	20
27, 29, 33, 37, 41	10, 23, 25	19
25, 27, 29, 33, 37, 41	10, 21, 23	18
21, 25, 27, 29, 33, 37, 41	10, 17, 23	18
17, 21, 25, 27, 29, 33, 37, 41	10, 13, 23	18
23, 27, 33, 37, 41	10, 19, 29	20
23, 27, 29, 33, 37, 41	10, 19, 25	19
23, 25, 27, 29, 33, 37, 41	10, 19, 21	17
21, 23, 25, 27, 29, 33, 37, 41	10, 17, 19	16
17, 21, 23, 25, 27, 29, 33, 37, 41	10, 13, 19	16
13, 17, 21, 23, 25, 27, 29, 33, 37, 41	9, 10, 19	16
19, 23, 27, 29, 33, 37, 41	10, 15, 25	19
19, 23, 25, 27, 29, 33, 37, 41	10, 15, 21	17
19, 21, 23, 25, 27, 29, 33, 37, 41	10, 15, 17	15
17, 19, 21, 23, 25, 27, 29, 33, 37, 41	10, 13, 15	14
13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	9, 10, 15	14
9, 13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 10, 15	15
15, 19, 23, 25, 27, 29, 33, 37, 41	10, 11, 21	17
15, 19, 21, 23, 25, 27, 29, 33, 37, 41	10, 11, 17	15
15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	10, 11, 13	13
13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	9, 10, 11	12
9, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 10, 11	13
5, 9, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 10, 11	14
11, 15, 19, 21, 23, 25, 27, 29, 33, 37, 41	7, 10, 17	15
11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	7, 10, 13	13
11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	7, 9, 10	11
9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 7, 10	11
5, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 7, 10	12
7, 11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 10, 13	14
7, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 9, 10	12
7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 5, 10	11
5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 3, 10	11
10, 17, 21, 25, 27, 29, 33, 37, 41	6, 13, 23	18
10, 23, 27, 33, 37, 41	6, 19, 29	20
10, 23, 27, 29, 33, 37, 41	6, 19, 25	19
10, 23, 25, 27, 29, 33, 37, 41	6, 19, 21	18
10, 21, 23, 25, 27, 29, 33, 37, 41	6, 17, 19	17
10, 17, 21, 23, 25, 27, 29, 33, 37, 41	6, 13, 19	17
10, 13, 17, 21, 23, 25, 27, 29, 33, 37, 41	6, 9, 19	16
10, 19, 23, 27, 29, 33, 37, 41	6, 15, 25	19
10, 19, 23, 25, 27, 29, 33, 37, 41	6, 15, 21	18
10, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 15, 17	16

Table 4: Dimensions for $\Gamma = \langle 4, 14, 31 \rangle$, $m = 3$ (I)

D–sets	$\vec{g}$	dim
10, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 13, 15	15
10, 13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 9, 15	14
9, 10, 13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 6, 15	14
10, 15, 19, 23, 25, 27, 29, 33, 37, 41	6, 11, 21	17
10, 15, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 11, 17	16
10, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 11, 13	14
10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 9, 11	12
9, 10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 6, 11	12
5, 9, 10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 6, 11	13
10, 11, 15, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 7, 17	15
10, 11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 7, 13	13
10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	6, 7, 9	10
9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	5, 6, 7	9
5, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 6, 7	10
7, 10, 11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 6, 13	13
7, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 6, 9	10
7, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	3, 5, 6	8
5, 7, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 3, 6	8
6, 10, 17, 21, 25, 27, 29, 33, 37, 41	2, 13, 23	18
6, 10, 17, 21, 23, 25, 27, 29, 33, 37, 41	2, 13, 19	17
6, 10, 13, 17, 21, 23, 25, 27, 29, 33, 37, 41	2, 9, 19	16
6, 10, 19, 23, 27, 29, 33, 37, 41	2, 15, 25	19
6, 10, 19, 23, 25, 27, 29, 33, 37, 41	2, 15, 21	18
6, 10, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 15, 17	17
6, 10, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 13, 15	16
6, 10, 13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 9, 15	15
6, 9, 10, 13, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 5, 15	14
6, 10, 15, 19, 23, 25, 27, 29, 33, 37, 41	2, 11, 21	17
6, 10, 15, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 11, 17	16
6, 10, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 11, 13	15
6, 10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 9, 11	13
6, 9, 10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 5, 11	12
5, 6, 9, 10, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 2, 11	12
6, 10, 11, 15, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 7, 17	15
6, 10, 11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 7, 13	14
6, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 7, 9	11
6, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 5, 7	9
5, 6, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 2, 7	9
6, 7, 10, 11, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 3, 13	13
6, 7, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 3, 9	10
6, 7, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	2, 3, 5	7
5, 6, 7, 9, 10, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 33, 37, 41	1, 2, 3	6

Table 5: Dimensions for $\Gamma = \langle 4, 14, 31 \rangle$, $m = 3$ (II)

We will now consider the series $(6, 8, v)$ for odd $v \geq 9$ from (1-17) with the link $\mathrm{Cab}(25 + (v-9), 2)T(4, 3)$. The corresponding ring, semigroup, δ–invariant, and Euler number of $J_{\mathcal{R}}$ are

$$\mathcal{R} = \mathbb{C}[\![z^6, z^8 + z^v]\!], \qquad \Gamma = \langle 6, 8, 25 + (v-9)\rangle,$$
$$\delta = 18 + \tfrac{1}{2}(v-9), \quad e(J_{\mathcal{R}}) = 227 + \tfrac{25}{2}(v-9).$$

Piontkowski [38] provides the Euler number, the total number of Γ–modules, which is $273 + \frac{25}{2}(v-9)$, and the number of *nonadmissible* ones. The latter is 46 for any v. The difference $273 + \frac{25}{2}(v-9) - 46$ is exactly the Euler number, since all admissible cells in the decomposition from [38] are diffeomorphic to $\mathbb{A}^N$ in the case under consideration and the homology can be readily calculated.

For our conjecture, we need to know the set of all Γ–modules Δ, those that are nonadmissible, and the dimensions $\dim(J_{\mathcal{R}}[\mathcal{D}])$. It is not too difficult to find all nonadmissible modules following [38], but they are not provided in his paper. There are "generic" nonadmissible modules and three exceptional ones, which we are going to describe.

We examine the elements

$$T_{ij}^{pq} = \phi_i m_p - \phi_j m_q \in \mathcal{M},$$

where $D = D_{\mathcal{M}}$ is the set of gaps of $\mathcal{M}$, $i < j \in \Gamma$, $p > q \in \Delta_{\mathcal{M}} = v(\mathcal{M})$, $v(\phi_i) = i$, and $v(m_p) = p$, where we use the valuation $v\colon \mathcal{O} \to \mathbb{Z}_+$. Recall that modules $\mathcal{M}$ are submodules in $\mathcal{O} = \mathbb{C}[\![z]\!]$ with an element of valuation 0.

We will assume that the leading z–monomial in ϕ_i, m_k has the coefficient 1. For instance, $m_0 = 1 + \sum_{p>0} \lambda_0^p z^p$. The choice of these elements is of course nonunique (higher terms can be added to them).

Proposition 4.1 *All nonadmissible $\mathcal{M}$ for $\mathcal{R} = \mathbb{C}[\![t^6, t^8 + t^v]\!]$ can be described as follows. In the differences from Section 4.3, let*

(a) $q = 0,\ p = 2, 4, 10$; (b) $q = 2,\ p = 4$; (c) $p > q \in \{0, 2, 4\}$.

Then, let us impose the following relations for i, j in Section 4.3:

$$i + p + 1 \notin \Delta_{\mathcal{M}} \quad \text{and} \quad i + p = j + q \qquad \text{for } i, j \in \Gamma. \tag{4-2}$$

Here the latter results in the inequalities $v(T_{ij}^{pq}) \geq 1 + p + 1$, which can be only strict due to the former since $T_{ij}^{pq} \in \mathcal{M}$.

The nonadmissibility of $\mathcal{M}$ of type (a) *or* (b) *occurs if and only if there exist no $m_p, m_q \in \mathcal{M}$ in Section 4.3 satisfying (4-2) for* ***all possible*** *choices of i, j there. In* *the case of* (c), *the absence of* $m_0, m_2, m_4 \in \mathcal{M}$ *must be checked for each of the three choices of* p, q *there altogether and every possible* i, j *satisfying* Section 4.3. □

Let us list the D–sets (the sets of gaps) for all 46 nonadmissible modules $\mathcal{M}$. In Table 6, $(v-9)$ must be added to all gaps in the second half of the first column (clearly visible). The second column contains the smallest $p > 0$ in D that ensures the nonadmissibility in the case of (a), the letters b or c stand for the remaining three exceptional cases. The third column contains the first $g = g_{\min} = i + p + 1 \not\in D$ such that its absence in D contradicts the absence of the previous gaps $g' i' + p + 1 < g_{\min}$ in D, which is for a given pair from (a, b, c). Such a gap is provided only for the pair $(0, 2)$ in the case of (c).

The nonadmissibility can occur only if there are at least two possible pairs (i, j) satisfying (4-2). The corresponding conditions are simple equalities for the differences $\lambda^1_p - \lambda^1_q$, where $m_p = z^p + \lambda^1_p z^{p+1} + \cdots$. If the difference $\lambda^1_p - \lambda^1_q$ takes different values for different pairs (i, j), then the module cannot be admissible. In the case of c, such a contradiction can be reached only if *three* such sequences of equalities are considered together, ie for $\lambda^1_0 - \lambda^1_2$, $\lambda^1_0 - \lambda^1_4$, and $\lambda^1_2 - \lambda^1_4$. Generally, higher-order z–expansions can be necessary here, ie $\lambda^{i>1}_p$ may occur; however this is not the case with $(6, 8, v)$.

Using this table and the list of all $273+\frac{25}{2}(v-9)$ modules Δ, we checked Conjecture 2.5 for $t = 1$ (ie ignoring the dimensions) for quite a few v. Here and in *all* calculations we performed it appeared sufficient to replace the admissibility of $\mathcal{D}$ by the admissibility of all D_i (separately), a potentially weaker condition. Generalizing [38], we verified here that all subvarieties $J^m_{\mathcal{R}}[\mathcal{D}]$ are diffeomorphic to proper $\mathbb{A}^N$, so we are in the situation of (2-6).

Let us provide some details. Recall that the D–flag of length m originated at D is by definition the sequence of D–sets for Γ–modules:

$$\mathcal{D} = \big\{D_0 = D,\ D_1 = D \cup \{g_1\},\ \ldots,\ D_m = D \cup \{g_1, g_2, \ldots, g_m\}\big\}, \tag{4-3}$$

where the inequalities $g_1 < g_2 < \cdots < g_m \in G \setminus D$ are imposed. Let us illustrate numerically the importance of this very ordering in our definition of D–flags.

As it results from Proposition 2.3, each set $D \cup \{g_i\}$ corresponds to a certain Γ–module for any $1 \le i \le m$. However, apart from torus knots, these conditions (imposed together) are generally significantly weaker than the conditions we need. The Γ–modules D_i from (4-3) are not always admissible (coming from some $\mathcal{M}$) if all $D \cup \{g_i\}$ are.

D–sets	under $\{\cdot\}+(v-9)$	p	g	D–sets	under $\{\cdot\}+(v-9)$	p	g
10,	35	10	19	2, 10,	21, 23, 27, 29, 35	2	19
10,	27, 35	10	19	4, 10,	19, 21, 27, 29, 35	4	17
10,	29, 35	10	19	4, 10,	19, 23, 27, 29, 35	4	17
2, 10,	27, 35	2	19	4, 10,	21, 23, 27, 29, 35	4	17
4, 10,	29, 35	4	17	10,	15, 21, 23, 27, 29, 35	10	19
10,	23, 29, 35	10	19	2, 4, 10,	17, 23, 27, 29, 35	2	19
10,	27, 29, 35	10	19	2, 4, 10,	19, 21, 27, 29, 35	4	17
2, 10,	27, 29, 35	2	19	2, 4, 10,	19, 23, 27, 29, 35	4	17
4, 10,	23, 29, 35	4	17	2, 4, 10,	21, 23, 27, 29, 35	2	19
4, 10,	27, 29, 35	4	17	2, 10,	15, 21, 23, 27, 29, 35	2	19
10,	21, 27, 29, 35	10	19	2, 10,	17, 21, 23, 27, 29, 35	2	19
10,	23, 27, 29, 35	10	19	4, 10,	15, 21, 23, 27, 29, 35	4	17
2, 4, 10,	27, 29, 35	2	19	4, 10,	19, 21, 23, 27, 29, 35	4	17
2, 10,	21, 27, 29, 35	2	19	2, 4, 10,	15, 21, 23, 27, 29, 35	2	19
2, 10,	23, 27, 29, 35	2	19	2, 4, 10,	17, 19, 23, 27, 29, 35	$\boldsymbol{b}$	21
4, 10,	19, 27, 29, 35	4	17	2, 4, 10,	17, 21, 23, 27, 29, 35	2	19
4, 10,	21, 27, 29, 35	4	17	2, 4, 10,	19, 21, 23, 27, 29, 35	4	17
4, 10,	23, 27, 29, 35	4	17	2, 10,	15, 17, 21, 23, 27, 29, 35	2	19
10,	21, 23, 27, 29, 35	10	19	4, 10,	15, 19, 21, 23, 27, 29, 35	4	17
2, 4, 10,	19, 27, 29, 35	4	17	2, 4, 10,	15, 17, 21, 23, 27, 29, 35	2	19
2, 4, 10,	21, 27, 29, 35	2	19	2, 4, 10,	15, 19, 21, 23, 27, 29, 35	4	17
2, 4, 10,	23, 27, 29, 35	2	19	2, 4, 10,	17, 19, 21, 23, 27, 29, 35	$\boldsymbol{c}$	11
2, 10,	17, 23, 27, 29, 35	2	19	2, 4, 10,	15, 17, 19, 21, 23, 27, 29, 35	$\boldsymbol{c}$	11

Table 6: Nonadmissible modules for $(6, 8, v)$

We set $\epsilon(D) = 1, 0$ correspondingly for admissible and nonadmissible D and $\epsilon(\mathcal{D}) = \prod_{i=1}^{m} \epsilon(D_i)$, where $D_i = D_{\mathcal{M}_i}$. Then $\epsilon(\mathcal{D}) = 1$ is equivalent to the admissibility of $\mathcal{D}$ (in the example under consideration). We expect this implication to hold in general, but cannot prove this at the moment. Thus $\epsilon(\mathcal{D}) = 1$ implies that $\prod_{i=1}^{m} \epsilon(D \cup g_i) = 1$, but the latter condition is not generally insufficient for the former.

Namely, if the admissibility of D–flags were defined as $\prod_{i=1}^{m} \epsilon(D \cup \{g_i\}) = 1$ instead of $\epsilon(\mathcal{D}) = 1$, ie separately for each and every $D \cup \{g_i\}$, then there would be 14 extra (wrong) terms (with multiplicities) in (4-4) below. This clearly demonstrates that our flags are generally more subtle than using "marks" for torus knots in [16] and other related works.

The smallest example is as follows. Using Table 6 with $v = 9$, the set of all g_i for $D = [10, 17, 19, 23, 27, 29, 35]$ such that $D \cup \{g_i\}$ is admissible is $\{2, 4, 11, 21\}$.

However, $D \cup \{2,4\} = [2,4,10,17,19,23,27,29,35]$ is a nonadmissible D–set (it is marked by b in the table).

Finally, we arrive at the following identity:

$$\sum_{m=0}^{\infty} \sum_{\mathcal{D}=\{D_0,\dots,D_m\}} \epsilon(\mathcal{D}) q^{|D_0|+m} a^m = \tag{4-4}$$

$$\begin{aligned}
&1+q+2q^2+3q^3+5q^4+7q^5+10q^6+12q^7+16q^8+19q^9+22q^{10}\\
&\qquad +24q^{11}+25q^{12}+24q^{13}+22q^{14}+17q^{15}+11q^{16}+5q^{17}+q^{18}\\
&+a\big(q+2q^2+4q^3+7q^4+12q^5+18q^6+26q^7+35q^8+46q^9+56q^{10}\\
&\qquad +66q^{11}+72q^{12}+74q^{13}+70q^{14}+59q^{15}+41q^{16}+21q^{17}+5q^{18}\big)\\
&+a^2\big(q^3+2q^4+5q^5+9q^6+16q^7+24q^8+36q^9+48q^{10}+62q^{11}\\
&\qquad +74q^{12}+82q^{13}+83q^{14}+76q^{15}+58q^{16}+34q^{17}+10q^{18}\big)\\
&+a^3\big(q^6+2q^7+5q^8+9q^9+15q^{10}+22q^{11}+31q^{12}+38q^{13}\\
&\qquad +44q^{14}+44q^{15}+38q^{16}+26q^{17}+10q^{18}\big)\\
&+a^4\big(q^{10}+2q^{11}+4q^{12}+6q^{13}+9q^{14}+11q^{15}+11q^{16}+9q^{17}+5q^{18}\big)\\
&+a^5\big(q^{15}+q^{16}+q^{17}+q^{18}\big).
\end{aligned}$$

The latter sum exactly coincides with $\mathcal{H}_{\vec{r},\vec{s}}(\square; q, t=1, a)$ from formula (4) in Section 3.2 of [8], where

$$\vec{r} = \{4,2\}, \quad \vec{s} = \{3,1\}, \quad \mathcal{T} = \mathrm{Cab}(25,2)T(4,3).$$

Also see formula (3.4) there for the corresponding DAHA–Betti polynomial, which is the a–constant term of (4-4) upon the substitution $q \mapsto 1/t$ and multiplication by t^{18}:

$$\begin{aligned}
&1+5t+11t^2+17t^3+22t^4+24t^5+25t^6+24t^7+22t^8+19t^9\\
&\qquad +16t^{10}+12t^{11}+10t^{12}+7t^{13}+5t^{14}+3t^{15}+2t^{16}+t^{17}+t^{18}.
\end{aligned}$$

Here we use the superduality.

We also checked that $J^m_{\mathcal{R}}[\mathcal{D}]$ are always affine spaces, found formulas for the dimensions of $J^{m=0,1}_{\mathcal{R}}[D]$ and correspondingly verified (2-6) for the coefficients of superpolynomials of a^0, a^1 in the considered case.

4.4 Beyond “Piontkowski”

Examples are given in [38] where his approach does not work because the corresponding cells $J_r[D]$ are nonaffine spaces. Thus the count of such cells and knowing their dimensions is insufficient for obtaining the Betti numbers and the Euler number of J_r. Some “negative” examples are provided in the table after Theorem 13 in [38]. We found that not always the cells in his table are really nonaffine, but the phenomenon he discovered is of course important. It is unclear whether nonaffine cells and nonadmissible D and $\mathcal{D}$ are topological. This is of obvious interest due to our main conjecture and because of the exciting link to orbital integral in type A.

We mostly consider a relatively simple singularity with Puiseux exponents $(6,9,10)$ and the cable $\mathrm{Cab}(19,3)T(3,2)$. Its ring is $\mathcal{R}=\mathbb{C}[\![t^6,t^9+t^{10}]\!]$, the valuation semigroup is $\Gamma=\langle 6,9,19\rangle$ and $\delta=|\mathbb{N}\setminus\Gamma|=21$ in this case. See [8, Section 3.2] for details and the formula for the DAHA superpolynomial; the DAHA–Betti polynomial ($a=0, q=1$) is formula (3.3) there, which does provide correct Betti numbers for the corresponding $J_{\mathcal{R}}$.

This example is transitional in a sense; all cells are still affine spaces, but the justification of this fact (straightforward, using computers) becomes more involved. We considered some deformations of parameters of $\mathcal{R}$ and think that Table 7 depends only on the corresponding Γ (ie this table is of a topological nature) but this is not clear.

The simplest example of nonaffine $J_{\mathcal{R}}[D]$ in this case given in [38] is

$$D=[3,7,10,13,16,17,20,22,23,26,29,32,35,41],$$

but we found that the corresponding $J_{\mathcal{R}}[D]$ is biregular to $\mathbb{A}^{14}$. The only problem with this and two other similar sets D (our program obtained) is that the natural λ–variables Piontkowski uses (as do we) are inconvenient to parametrize $J_{\mathcal{R}}[D]$; a certain (linear) change of variables is necessary. The other two D–sets with a similar behavior (when a straight elimination of the λ–variables is insufficient) are

$$[3,10,13,16,20,22,23,26,29,32,35,41],$$
$$[3,10,13,16,17,20,22,23,26,29,32,35,41].$$

All three (and any other cells) are affine spaces in this case. The next example of a nonaffine cell from [38] is for $\mathbb{C}[\![t^6,t^9+t^{13}]\!]$; we confirm it. See Appendix A in the online supplement.

We note that our program routinely calculates $|J_{\mathcal{R}}[D](\mathbb{F}_3)|$ for “suspicious” D to double check the direct verification of the affineness of the cells (mostly automated).

22, 41	2, 3, 5, 8, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 41	1, 3, 4, 7, 10, 13, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
22, 35, 41	3, 5, 8, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 32, 41	2, 3, 8, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 35, 41	2, 3, 5, 8, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
16, 22, 35, 41	1, 4, 7, 10, 13, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 35, 41	1, 3, 4, 7, 10, 13, 16, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 29, 35, 41	3, 8, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 32, 35, 41	3, 5, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
16, 22, 32, 35, 41	3, 5, 8, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 29, 35, 41	3, 4, 7, 10, 13, 16, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 32, 35, 41	2, 3, 8, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 26, 32, 35, 41	1, 4, 7, 10, 13, 16, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 29, 32, 35, 41	4, 7, 10, 13, 16, 20, 22, 23, 26, 29, 32, 35, 41
13, 16, 22, 32, 35, 41	3, 11, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 13, 16, 22, 32, 35, 41	3, 8, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 26, 32, 35, 41	3, 8, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 29, 32, 35, 41	3, 5, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 22, 23, 29, 32, 35, 41	3, 4, 7, 10, 13, 16, 22, 23, 26, 29, 32, 35, 41
3, 22, 26, 29, 32, 35, 41	4, 7, 10, 13, 16, 22, 23, 26, 29, 32, 35, 41
13, 16, 22, 29, 32, 35, 41	3, 14, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 13, 16, 22, 29, 32, 35, 41	3, 11, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 23, 29, 32, 35, 41	3, 11, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 26, 29, 32, 35, 41	3, 8, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 20, 22, 26, 29, 32, 35, 41	3, 7, 10, 13, 16, 22, 23, 26, 29, 32, 35, 41
3, 22, 23, 26, 29, 32, 35, 41	7, 10, 13, 16, 22, 23, 26, 29, 32, 35, 41
10, 13, 16, 22, 29, 32, 35, 41	3, 16, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 10, 13, 16, 22, 29, 32, 35, 41	3, 14, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 20, 22, 26, 29, 32, 35, 41	3, 14, 16, 20, 22, 23, 26, 29, 32, 35, 41
3, 16, 22, 23, 26, 29, 32, 35, 41	3, 11, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 17, 22, 23, 26, 29, 32, 35, 41	3, 7, 10, 13, 16, 22, 26, 29, 32, 35, 41
3, 20, 22, 23, 26, 29, 32, 35, 41	7, 10, 13, 16, 22, 26, 29, 32, 35, 41
10, 13, 16, 22, 26, 29, 32, 35, 41	3, 17, 20, 22, 23, 26, 29, 32, 35, 41
3, 10, 13, 16, 22, 26, 29, 32, 35, 41	3, 16, 20, 22, 23, 26, 29, 32, 35, 41
3, 14, 20, 22, 23, 26, 29, 32, 35, 41	3, 16, 17, 22, 23, 26, 29, 32, 35, 41

Table 7: Nonadmissible D for $\Gamma = \langle 6, 9, 19\rangle$

These cardinalities must be $3^{\dim}$ for affine cells. The prime 3 is a "place of good reduction" for this $\mathcal{R}$. The reduction is bad modulo $p = 2$ and one needs to switch to the *topologically* equivalent ring $\mathbb{C}[\![t^6+t^7,t^9]\!]$ before replacing $\mathbb{C}\mapsto\mathbb{F}_2$. We omit general analysis of places of bad reduction in this paper.

We will not discuss much the flags here, but let us mention that all cells are affine for $J^{m=1}_{\mathcal{R}}[\mathcal{D}]$ (their dimensions are all calculated) in the case of $\mathcal{R}=\mathbb{C}[\![t^6,t^9+t^{10}]\!]$. The list of (all) 70 nonadmissible D (ie for $m=0$) will be provided. The total number of modules Δ is $447=377+70$ in this case, and the Euler number is $e(J_{\mathcal{R}})=377$. Accordingly, we checked (numerically) the coincidence from (2-6) in the following two cases: (a) for all a when $t=1$ (for the admissibility of $\mathcal{D}$ understood as the admissibility of all D_i in this flag), and (b) for the coefficients of $\mathcal{H}(\Box;q,t,a)$ from [8] of $a^{0,1}$.

The calculation in (a) greatly demonstrates the role of admissibility and the implications of the ordering $g_1<\cdots<g_m$, which are quite nontrivial combinatorially. The corresponding reduction of the superpolynomial for $\mathcal{R}=\mathbb{C}[\![t^6,t^9+t^{10}]\!]$ is

$$\begin{aligned}
&\mathcal{H}(\Box;q,t=1,a)=\\
&1+q+2q^2+3q^3+5q^4+7q^5+10q^6+13q^7+17q^8+21q^9+25q^{10}+29q^{11}+33q^{12}\\
&\qquad+36q^{13}+37q^{14}+37q^{15}+34q^{16}+28q^{17}+20q^{18}+12q^{19}+5q^{20}+q^{21}\\
&+a^5\big(q^{15}+q^{16}+2q^{17}+2q^{18}+3q^{19}+2q^{20}+q^{21}\big)\\
&+a^4\big(q^{10}+2q^{11}+4q^{12}+7q^{13}+11q^{14}+15q^{15}+19q^{16}\\
&\qquad+22q^{17}+23q^{18}+21q^{19}+13q^{20}+5q^{21}\big)\\
&+a\big(q+2q^2+4q^3+7q^4+12q^5+18q^6+27q^7+37q^8+50q^9+63q^{10}+78q^{11}+91q^{12}\\
&\qquad+105q^{13}+113q^{14}+118q^{15}+114q^{16}+100q^{17}+76q^{18}+48q^{19}+22q^{20}+5q^{21}\big)\\
&+a^3\big(q^6+2q^7+5q^8+9q^9+16q^{10}+24q^{11}+36q^{12}+47q^{13}+61q^{14}+71q^{15}\\
&\qquad+81q^{16}+82q^{17}+76q^{18}+57q^{19}+32q^{20}+10q^{21}\big)\\
&+a^2\big(q^3+2q^4+5q^5+9q^6+16q^7+25q^8+38q^9+53q^{10}+71q^{11}+90q^{12}+109q^{13}\\
&\qquad+126q^{14}+138q^{15}+143q^{16}+134q^{17}+111q^{18}+75q^{19}+38q^{20}+10q^{21}\big).
\end{aligned}$$

5 Some perspectives

The topics below are mostly open projects, but we believe that this section can be of interest; the relation to orbital integrals, affine Springer fibers and the motivic reformulation (5-2) of (2-7) are the key. We will first address the absence of colors and links in our main conjecture and related issues.

5.1 Adding colors

In contrast to [9] and previous paper, we restrict this one to algebraic uncolored knots and only type A is addressed in our conjecture. Let us briefly comment on this. Adding colors is expected via the curves in [31], so algebraic *links* are/seem necessary for this. Apart from rectangle Young diagrams, the coefficients of DAHA-superpolynomials are nonpositive, which is an obvious challenge for the geometric interpretation. This is the same for links (included uncolored ones). The Jacobian factors become *ind-schemes* for algebraic links and we need to follow [25] (divide by certain split tori) to make them *proper*.

One can try to bypass the nonpositivity issues switching to some *Hilbert schemes* instead of our flagged Jacobians, which corresponds to the unreduced topological setting. This can be similar to [36; 13]; see also Conjecture 5.3(ii) from [9].

Superpolynomials beyond A_n Our flagged Jacobian factors are related to the Hitchin and affine Springer fibers. The hope is that the DAHA-superpolynomials can be directly and geometrically determined by spectral curves for types B_n, C_n, D_n via the corresponding flagged Jacobian factors and/or Hilbert schemes. The rank stabilization was conjectured in [6]; the name "hyperpolynomials" is used instead of superpolynomials for non-A types. The spectral curves are generally nonunibranch, so the passage to links is necessary here. A certain confirmation is the following observation.

It is true in all known examples that $\mathcal{H}(b; q, t, u, a = 0)$ for the hyperpolynomials from Section 4.2 of [6] depend only on the corresponding knot and the color b. This holds for the hyperpolynomials of type C, D calculated there for $T(3,2), T(7,2), T(4,3)$ and $b = \omega_1, 2\omega_1, \omega_2$. The hyperpolynomials of type D are the specialization of those of type C upon $u = 1$ (see (4.7) there; u is the second t for B, C). The same holds for type B under $q^2 \mapsto q$, $u \mapsto t$.

Furthermore, the hyperpolynomials $\mathcal{H}^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}(q, t, a)$ for the *Deligne–Gross series* (extending the series $E_{6,7,8}$) introduced in Section 4.2 of [10] coincide at $a = 0$ with $\mathcal{H}_{\mathsf{r},\mathsf{s}}(\square; q, t, a = 0)$ of type A. The existence of $\mathcal{H}^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}$ was confirmed only partially in [10] (and only for $T(3,2), T(4,3)$). These $\mathcal{H}^{\mathfrak{ad}}_{\mathsf{r},\mathsf{s}}$ are mysterious from the viewpoint of [6], since there can be no rank stabilization here. We hope that our present paper can shed some light on this.

5.2 Perfect DAHA modules

A challenge is an interpretation of DAHA superpolynomials for arbitrary algebraic knots, similar to that in [17] for torus ones via perfect modules of rational DAHA. See also Theorem 9.5 from [22]. There are some obstacles.

One can use here the classification results from [41; 40; 43] in terms of K–theory or homology (for the rational and trigonometric DAHA) of *Iwahoric Springer fibers*. See paper [41] for general theory, including a comprehensive analysis in type A.

Following Gorsky, let us consider the root system $A_{\mathsf{s}-1}$ for a given torus knot $T(\mathsf{r},\mathsf{s})$ (so we adjust its rank to the knot) and take $e = e_1 + \cdots + e_{\mathsf{s}-1} + z^{\mathsf{r}} f_\theta$ in the notations from Section 4.1 from [40]. Then $H_*(\mathcal{Y}_e)$ for the corresponding Iwahoric Springer fiber $\mathcal{Y}_e$ can be supplied with a natural structure of the *perfect* $\mathcal{HH}$–module for $t = q^{-\mathsf{r}/\mathsf{s}}$ of dimension $\mathsf{r}^{\mathsf{s}-1}$, which is a very explicit quotient of the polynomial representation of rational DAHA. This module becomes simpler with q, t (via K–theory), but the grading then will be missing.

The coefficients of a^m in the superpolynomial are associated with isotypic components of this $\mathcal{HH}$–module for the wedge m^{th} powers of the standard $\mathsf{s}-1$ dimensional representation of $W = \boldsymbol{S}_{\mathsf{s}}$. This construction is quite combinatorial; Gorsky relates a^m to m–sets of *marks*, which are corners in Dyck paths below the diagonal in the $\mathsf{s}{\times}\mathsf{r}$–rectangle. In the absence of marks (when $a = 0$), the space of W–coinvariants is the image of the projection of $H_*(\mathcal{Y}_e)$ onto $H_*(\mathcal{X}_e)$ for the affine Springer fiber $\mathcal{X}_e$ (see also Section 5.3 below).

The number of marks corresponds to m in our $J^m_{\mathcal{R}}$. The latter is a special subvariety of the *parahoric Springer fiber* of full m–subflags starting from the top, defined as the space of flags of $\mathcal{R}$–modules $\{\mathcal{M}_i \mid i = 0, \dots, m\}$, where $\mathfrak{m}\mathcal{M}_m \subset \mathcal{M}_i$ and $\dim_{\mathbb{C}}(\mathcal{M}_{i+1}/\mathcal{M}_i) = 1$; here $\mathfrak{m} \subset \mathcal{R}$ is the maximal ideal. They can be extended to full *periodic* flags via the forgetful map from $\mathcal{Y}_e$ to the parahoric one.

This construction involves the root system $A_{\mathsf{s}-1}$, and the topological $\mathsf{r} \leftrightarrow \mathsf{s}$ symmetry of the superpolynomial generally becomes far from obvious. Our geometric superpolynomials are defined directly in terms of the singularity and are manifestly $\mathsf{r}{\leftrightarrow}\mathsf{s}$–symmetric.

Furthermore, the (finite-dimensional) spaces $H_*(\mathcal{Y}_e)$ for arbitrary nil-elliptic (anisotropic) e are not generally related to any DAHA-modules unless in the torus case. These spaces are needed for general unibranch plane curve singularities, but their dimensions and other features are very different from those of the finite-dimensional modules. It is not impossible that they can appear as some “remarkable” subspaces in infinite-dimensional $\mathcal{HH}$–modules, but this is questionable.

5.3 Affine Springer fibers

Conjecture 2.5 leads to the following connection with the orbital integrals from the fundamental lemma (in the geometric setting). We will mainly follow Section 2.4 from [44]. The example from Section 2.4.2 there is for the spectral curve $y^{\mathsf{r}} = x^{\mathsf{s}}$ [30], which can be actually generalized to any (germs of) plane curve singularities $\mathcal{C}$ such that the Jacobian factor $J_{\mathcal{R}}$ at $(0,0)$ has the Piontkowski-type decomposition with affine cells only.

The following discussion will be mostly restricted to type A in the anisotropic case. Note that fractional ideals of $\mathcal{R}$ of *degree zero* are considered in [44] (and in the fundamental lemma) in the definition of the compactified Jacobians.

Let $1/t = p^{\ell}$ be the cardinality of a finite field $\mathbb{F}$. The choice $t = 1/|\mathbb{F}|$ is standard when connecting the q,t–theory of spherical functions with the p–adic theory (see eg [5]), though using q instead of $1/t$ is equally fine due to the superduality of the DAHA superpolynomials.

Because we stick to the unibranch case, $\gamma \in \mathfrak{g}(\mathbb{F}((x)))$ is assumed *nil-elliptic*: no split tori over a local field $\mathbb{F}((x))$ in the stabilizer G_{γ} ($\gamma = e$ was used above). The affine Springer fiber $\mathcal{X}_{\gamma}$ is then formed by the classes of g in the *affine Grassmannian* $G(\mathbb{F}((x)))/G(\mathbb{F}[\![x]\!])$ over the field $\mathbb{F}$ such that $\mathrm{Ad}_g^{-1}(\gamma) \in \mathfrak{g}(\mathbb{F}[\![x]\!])$. Here $\mathrm{Lie}(G) = \mathfrak{g}$; see eg [44]. Yun denotes the corresponding *parahoric* ones by $\mathcal{X}_{P,\gamma}$, which contain our flagged Jacobian factors (though we do not need any $\mathfrak{g}, G$). The Iwahoric Springer fibers are for $P = I$ in his notation.

The affine Springer fiber $\mathcal{X}_{\gamma}$ can be naturally identified with the compactified Jacobians for rational curves with the local ring $\mathcal{R}$ at its (unique) singularity; see eg [29; 4]. A general construction is from [4]. Let $\mathscr{G}$ be a *factorizable* Lie group schemes over a smooth projective curve E, defined by the conditions $H^1(E, \mathrm{Lie}(\mathscr{G})) = \{0\} = H^0(E, \mathrm{Lie}(\mathscr{G}))$ for the corresponding sheaf of Lie algebras $\mathrm{Lie}(\mathscr{G}) \to E$. For a scheme subtorus $\mathscr{T} \subset \mathscr{G}$ such that its generic fiber is a maximal torus, embeddings $f\colon \mathscr{T} \hookrightarrow \mathscr{G}$ become conjugations over sufficiently general open $U \subset E$: specifically, $f(\xi) = \phi^{-1}\xi\phi$ for $\xi \in H^0(U, \mathscr{T})$ and meromorphic sections $\phi = \phi_U$ of $\mathscr{G}$ over U. Čech cohomology is used.

Assuming that ϕ_i exist for open U_i such that $E = \bigcup_i U_i$, the map

$$\{f\} \ni f \mapsto \{\phi_i \phi_j^{-1} \in H^0(U_i \cap U_j, \mathscr{G})\} \to H^1(E, \mathscr{T})$$

is an isomorphism. Actually, we use this map in the opposite direction, from $H^1(E, \mathscr{T})$ to $\{f\}$. Here $H^1(E, \mathscr{T})$ is the generalized Jacobian of the cover $F \to E$ if $\mathscr{T} = \mathcal{O}_F^* \subset \mathscr{G}$. Note that E can be only $\mathbb{CP}^1$ or an elliptic curve; generalized Prim varieties appear for the latter.

Let F be rational with exactly one singular point that is the whole fiber F_o over some $o \in E = \mathbb{CP}^1$; here $\mathscr{G}_o$ must be G. Then $f(\mathscr{T})$ can be obtained as $g^{-1}\mathscr{T} g$ for *rational sections* g of $\mathscr{G}$ such that g are regular at $E \setminus o$ and $g^{-1}\mathscr{T} g$ are regular at F_o. Thus such g form an open subset in the affine Springer fiber $\mathscr{X}_\gamma$ at o for a sufficiently general section γ of Lie$(\mathscr{T})$ regular at o. For any fiber F_o, the map $g \mapsto g^{-1}\mathscr{T} g$ goes through the quotient $L_\gamma \setminus \mathscr{X}_\gamma$ for the group L_γ of rational sections of F regular at $F \setminus F_o$; see [44, Theorem 2.9] and [25].

Type A is not necessary here, but we need this in our paper (and anisotropic G_γ). Recall that our *flagged* Jacobian factors deviate from the usual ones. First, we consider only standard $\mathcal{R}$–modules. Second, admissible D–flags $\mathcal{D} = [D_0 \subset D_1 = D_0 \cup \{g_1\} \subset \cdots \subset D_m]$ are subtle; D_i must be D–sets and the ordering $g_1 < g_2 < \cdots < g_m$ is imposed. This ordering and the admissibility are quite nontrivial geometrically.

5.4 Motivic approach

Our construction results in the following generalization of orbital integrals:

$$t^\delta \sum_{\mathcal{D}} q^{m+|D_0|} a^m |J_{\mathcal{R}}^m[\mathcal{D}](\mathbb{F}_{1/t})|$$

in type A (the nil-elliptic case), where the summation is over all admissible D–flags $\mathcal{D}$. Their interpretation as "natural" orbital integrals is one of the main challenges triggered by this work. It seems doable due to an entirely geometric nature of our approach. Let us also mention here potential adding colors to our construction, another challenge for us and the specialists in orbital integrals.

For sufficiently general prime p, there is solid evidence that such sums coincide with the DAHA superpolynomials $\mathcal{H}_{\mathcal{C}}(\square; q, t, a)$ associated with the singularity $\mathcal{C}$, which is stated in (2-7). For $q = 1$ and $a = 0$, we arrive at *orbital integrals* O_γ. They are expected to be $|\mathbb{F}|$–integral and positive, which matches our main conjecture.

It is not impossible that all classes $[J^m_{\mathcal{R}}[d]]$ in $K_0(\mathrm{Sch}/\mathbb{C})$ are sums of classes $[\mathbb{A}^1]^N$ over $\mathbb{Z}_+$, the strongest possible (motivic) assumption. This would match known and conjectured properties of $\mathcal{H}_{\mathcal{C}}$. It is possible that $J^m_{\mathcal{R}}[d]$ are always paved by affine spaces (even when $J^m_{\mathcal{R}}[\mathcal{D}]$ are not all affine); see Appendix A in the online supplement.

If (2-7) holds, then DAHA superpolynomials provide all virtual Hodge numbers of $J^m_{\mathcal{R}}[d]$. This results from part (3) of Theorem 1 from the appendix by N M Katz in [24]. Following it, let $E(X; x, y) = \sum_{r,s} e_{r,s} x^r y^s$ for a separated scheme $X/\mathbb{C}$ of finite type and *virtual Hodge numbers* $e_{r,s} = \sum_i (-1)^i h^{r,s}(\mathrm{gr}^{r+s}_W(H^i_c(X^{\mathrm{an}}, \mathbb{C})))$.

Then (2-7) gives that $X = J^m_{\mathcal{R}}[d]$ is *strongly polynomial-count* and the following formula holds:

$$(5\text{-}1)\quad (xy)^{\delta}\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q, 1/(xy), a) = \sum_{d,m} E(J^m_{\mathcal{R}}[d]; x, y) q^{d+m} a^m$$
$$= \sum_{d,m,i,r} (-1)^i q^{d+m} a^m (xy)^r \,\mathrm{rk}(\mathrm{gr}^{2r}_W(H^i_c(J^m_{\mathcal{R}}[d], \mathbb{C}))),$$

which directly links the DAHA superpolynomials to the weight filtration in $H^i_c(X^{\mathrm{an}}, \mathbb{C})$. We use that $\mathcal{R}$ can be assumed over $\mathbb{Z}$ and then $J^m_{\mathcal{R}}[d]$ are defined over a localization of $\mathbb{Z}$ by finitely many primes.Thus $e_{r,s} = 0$ for $r \neq s$ (ie this is a Tate–Hodge structure) and the right-hand side in (2-7) can be replaced by the following $\mathcal{H}^{\mathrm{wt}}(q, t, a)$:

$$(5\text{-}2)\qquad \mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square; q, t, a) = \sum_{d,m,i,r} (-1)^i q^{d+m} t^{\delta-r} a^m \,\mathrm{rk}(\mathrm{gr}^{2r}_W(H^i_c(J^m_{\mathcal{R}}[d], \mathbb{C}))).$$

Conclusion In spite of some similarity between (5-2) and [22, Theorem 9.5], we do not see how they can be connected. First, $t_{\mathrm{st}}^2 = q/t$ is used there for the weight filtration instead of t in (5-2). Second, our construction does not require affine Springer fibers and picking the corresponding *isotypic* components in their homology. Lie groups do not appear in our approach; in a sense, this corresponds to the "endoscopy part" of the fundamental lemma. Third, related to the second, our admissible flags are new and different from those in [22]. Fourth, our approach is fully computational and we calculate well beyond torus knots and the series in [38]; the examples provided in [36] were only for some simple torus knots. In spite of these differences, the ORS conjecture can be still compatible with our one, but this is not clear.

Conceptually, *nested Hilbert schemes* from [36] and similar objects in related areas of geometry/topology and physics result in some *infinite* Poincaré series. Generally the "ultimate" problem is to transform them into polynomials or some *finite* expressions so that the resulting coefficients are positive integers. The latter positivity is generally much more subtle than the positivity (if any) of the coefficients of the initial series. In this paper, we provide the conjectural geometric interpretation for the DAHA superpolynomials via the *Jacobian factors* instead of Hilbert schemes, which is therefore ultimate in the sense above. There is recent progress with the geometric interpretation of the positivity conjecture for DAHA superpolynomials colored by symmetric or wedge powers, but so far not with arbitrary rectangles.

We note that not many formulas are known for the stable KhR–polynomials (the celebrated Khovanov polynomials for $\mathfrak{sl}_2$ are exceptional). They are mostly for $T(2\mathsf{m}+1,2)$ and no formulas are known for iterated nontorus knots. Thus checking our geometric superpolynomials versus the DAHA ones is actually the only way for iterated torus knots; though see [8] for some conditional verifications of our topological conjecture using the reduction to the Khovanov polynomials.

We do not pay any special attention to torus knots in our work. For such knots, we generalize the approach from [19; 20], where $a=0$. Our usage of *standard modules* is the key; this is not fully understood geometrically, but we already have some motivic interpretation of complete geometric superpolynomials (with all three parameters). We note that even for torus knots, our *flagged Jacobian factors* are new. The conditions on the corresponding Γ–modules Δ in their definition are not clear by now from the viewpoint of usual flagged constructions.

We plan to approach arbitrary *colored* algebraic *links* and possibly reach arbitrary root systems our further papers. It is a must if we want to realize the potential of DAHA in full and for connections with p–adic orbital integrals. Also, there is a realistic program of justifying our conjecture (related to [31]). It requires knowing the behavior of our geometric superpolynomials under monoidal-type transformations of the corresponding singularities. All of them must be considered here, not only those for torus knots.

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References

[1] **M Aganagic**, **S Shakirov**, *Refined Chern–Simons theory and knot homology*, from "String-Math 2011" (J Block, J Distler, R Donagi, E Sharpe, editors), Proc. Sympos. Pure Math. 85, Amer. Math. Soc., Providence, RI (2012) 3–31 MR

[2] **A Beauville**, *Counting rational curves on $K3$ surfaces*, Duke Math. J. 97 (1999) 99–108 MR

[3] **N Bourbaki**, *Éléments de mathématique, XXXIV: Groupes et algèbres de Lie, Chapitres 4–6*, Actualités Scientifiques et Industrielles 1337, Hermann, Paris (1968) MR

[4] **I V Cherednik**, *On the group-theoretical interpretation of Baker functions and τ–functions*, Uspekhi Mat. Nauk 38 (1983) 133–134 MR In Russian; translated in Russ. Math. Surv. 6 (1983) 113–114

[5] **I Cherednik**, *Double affine Hecke algebras*, London Math. Soc. Lec. Note Ser. 319, Cambridge Univ. Press (2005) MR

[6] **I Cherednik**, *Jones polynomials of torus knots via DAHA*, Int. Math. Res. Not. 2013 (2013) 5366–5425 MR

[7] **I Cherednik**, *DAHA-Jones polynomials of torus knots*, Selecta Math. 22 (2016) 1013–1053 MR

[8] **I Cherednik**, **I Danilenko**, *DAHA and iterated torus knots*, Algebr. Geom. Topol. 16 (2016) 843–898 MR

[9] **I Cherednik**, **I Danilenko**, *DAHA approach to iterated torus links*, from "Categorification in geometry, topology, and physics" (A Beliakova, A D Lauda, editors), Contemp. Math. 684, Amer. Math. Soc., Providence, RI (2017) 159–267 MR

[10] **V Cherednik**, **R Elliot**, *Refined composite invariants of torus knots via DAHA*, Ann. Fac. Sci. Toulouse Math. 25 (2016) 433–471 MR

[11] **N M Dunfield**, **S Gukov**, **J Rasmussen**, *The superpolynomial for knot homologies*, Experiment. Math. 15 (2006) 129–159 MR

[12] **D Eisenbud**, **W Neumann**, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies 110, Princeton Univ. Press (1985) MR

[13] **P Etingof**, **E Gorsky**, **I Losev**, *Representations of rational Cherednik algebras with minimal support and torus knots*, Adv. Math. 277 (2015) 124–180 MR

[14] **H Fuji**, **S Gukov**, **P Sułkowski**, *Super- A–polynomial for knots and BPS states*, Nuclear Phys. B 867 (2013) 506–546 MR

[15] **M Goresky**, **R Kottwitz**, **R MacPherson**, *Purity of equivalued affine Springer fibers*, Represent. Theory 10 (2006) 130–146 MR

[16] **E Gorsky**, *q,t–Catalan numbers and knot homology*, from "Zeta functions in algebra and geometry" (A Campillo, G Cardona, A Melle-Hernández, W Veys, editors), Contemp. Math. 566, Amer. Math. Soc., Providence, RI (2012) 213–232 MR

[17] **E Gorsky**, *Arc spaces and DAHA representations*, Selecta Math. 19 (2013) 125–140 MR

[18] **E Gorsky**, **S Gukov**, **M Stosic**, *Quadruply-graded colored homology of knots*, preprint (2013) arXiv

[19] **E Gorsky**, **M Mazin**, *Compactified Jacobians and q,t–Catalan numbers, I*, J. Combin. Theory Ser. A 120 (2013) 49–63 MR

[20] **E Gorsky**, **M Mazin**, *Compactified Jacobians and q,t–Catalan numbers, II*, J. Algebraic Combin. 39 (2014) 153–186 MR

[21] **E Gorsky**, **A Neguț**, *Refined knot invariants and Hilbert schemes*, J. Math. Pures Appl. 104 (2015) 403–435 MR

[22] **E Gorsky**, **A Oblomkov**, **J Rasmussen**, **V Shende**, *Torus knots and the rational DAHA*, Duke Math. J. 163 (2014) 2709–2794 MR

[23] **G-M Greuel**, **G Pfister**, *Moduli spaces for torsion free modules on curve singularities, I*, J. Algebraic Geom. 2 (1993) 81–135 MR

[24] **T Hausel**, **F Rodriguez-Villegas**, *Mixed Hodge polynomials of character varieties*, Invent. Math. 174 (2008) 555–624 MR With an appendix by N M Katz

[25] **D Kazhdan**, **G Lusztig**, *Fixed point varieties on affine flag manifolds*, Israel J. Math. 62 (1988) 129–168 MR

[26] **M Khovanov**, *Triply-graded link homology and Hochschild homology of Soergel bimodules*, Internat. J. Math. 18 (2007) 869–885 MR

[27] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology*, Fund. Math. 199 (2008) 1–91 MR

[28] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425 MR

[29] **G Laumon**, *Fibres de Springer et jacobiennes compactifiées*, from "Algebraic geometry and number theory" (V Ginzburg, editor), Progr. Math. 253, Birkhäuser, Boston (2006) 515–563 MR

[30] **G Lusztig**, **J M Smelt**, *Fixed point varieties on the space of lattices*, Bull. London Math. Soc. 23 (1991) 213–218 MR

[31] **D Maulik**, *Stable pairs and the HOMFLY polynomial*, Invent. Math. 204 (2016) 787–831 MR

[32] **D Maulik**, **Z Yun**, *Macdonald formula for curves with planar singularities*, J. Reine Angew. Math. 694 (2014) 27–48 MR

[33] **A Mellit**, *Toric braids and (m,n)–parking functions*, preprint (2016) arXiv

[34] **L Migliorini**, **V Shende**, *A support theorem for Hilbert schemes of planar curves*, J. Eur. Math. Soc. 15 (2013) 2353–2367 MR

[35] **H R Morton**, *The coloured Jones function and Alexander polynomial for torus knots*, Math. Proc. Cambridge Philos. Soc. 117 (1995) 129–135 MR

[36] **A Oblomkov**, **J Rasmussen**, **V Shende**, *The Hilbert scheme of a plane curve singularity and the HOMFLY homology of its link*, preprint (2012) arXiv

[37] **A Oblomkov**, **V Shende**, *The Hilbert scheme of a plane curve singularity and the HOMFLY polynomial of its link*, Duke Math. J. 161 (2012) 1277–1303 MR

[38] **J Piontkowski**, *Topology of the compactified Jacobians of singular curves*, Math. Z. 255 (2007) 195–226 MR

[39] **J Rasmussen**, *Some differentials on Khovanov–Rozansky homology*, Geom. Topol. 19 (2015) 3031–3104 MR

[40] **M Varagnolo**, **E Vasserot**, *Finite-dimensional representations of DAHA and affine Springer fibers: the spherical case*, Duke Math. J. 147 (2009) 439–540 MR

[41] **E Vasserot**, *Induced and simple modules of double affine Hecke algebras*, Duke Math. J. 126 (2005) 251–323 MR

[42] **B Webster**, **G Williamson**, *A geometric construction of colored HOMFLYPT homology*, Geom. Topol. 21 (2017) 2557–2600 MR

[43] **Z Yun**, *Global Springer theory*, Adv. Math. 228 (2011) 266–328 MR

[44] **Z Yun**, *Lectures on Springer theories and orbital integrals*, preprint (2016) arXiv

[45] **O Zariski**, *Le problème des modules pour les branches planes*, 2nd edition, Hermann, Paris (1986) MR

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