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Algebraic & Geometric Topology 18 (2018) 409–439

Nonexistence of boundary maps for some hierarchically hyperbolic spaces

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We provide negative answers to questions posed by Durham, Hagen and Sisto on the existence of boundary maps for some hierarchically hyperbolic spaces, namely maps from right-angled Artin groups to mapping class groups. We also prove results on existence of boundary maps for free subgroups of mapping class groups.

57M07; 20F36, 20F65

1 Introduction

Let Γ be a finite graph with vertex set $V(\Gamma) = \{s_1, \dots, s_k\}$. The right-angled Artin group determined by Γ, denoted by $A(\Gamma)$, is the group with the presentation

$$A(\Gamma) = \langle s_1, \dots, s_k : [s_i, s_j] = 1 \iff s_i s_j \text{ is an edge in } \Gamma \rangle.$$

Let $S = S_{g,n}$ be a connected, oriented surface of genus g with n punctures, and let $\mathrm{Mod}(S)$ denote the mapping class group of S. Clay, Leininger and Mangahas [6] and Koberda [8] construct "nice" embeddings of right-angled Artin groups to mapping class groups. Behrstock, Hagen and Sisto [3; 2] introduced a geometric structure called a hierarchically hyperbolic space (HHS). Important examples of spaces that are HHSs include mapping class groups of surfaces and right-angled Artin groups. Durham, Hagen and Sisto [7] constructed a boundary for hierarchically hyperbolic spaces (see Section 2). In that paper, the authors ask the following question, motivated by a desire to develop a notion of geometrically finite subgroups of mapping class groups:

Question 1.1 *Let $A(\Gamma)$ be a right-angled Artin group embedded in* $\mathrm{Mod}(S)$ *in the sense of either Clay, Leininger and Mangahas [6] or Koberda [8]. Does the embedding $A(\Gamma) \to \mathrm{Mod}(S)$ extend continuously to an injective map $\partial A(\Gamma) \to \partial \mathrm{Mod}(S)$?*

We prove that in general the answer to Question 1.1 is no by providing, for each type of embedding, an explicit example where the embedding does not extend continuously.

Published: 19 January 2018 DOI: 10.2140/agt.2018.18.409

Theorem 1.2 *There exists a surface S, a right-angled Artin group Γ, a Clay–Leininger–Mangahas embedding $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$, and a Koberda embedding $\phi'\colon A(\Gamma) \to \mathrm{Mod}(S)$ such that, regardless of the HHS structure on $A(\Gamma)$, neither ϕ nor ϕ' extends continuously to a map $\partial A(\Gamma) \to \partial \mathrm{Mod}(S)$.*

Clay–Leininger–Mangahas (CLM) embeddings are quasi-isometric embeddings (see Theorem 2.5). Thus Theorem 1.2 shows that the sufficient conditions for extendability of maps between hierarchically hyperbolic spaces are different than those for existence of Cannon–Thurston maps for Gromov hyperbolic spaces. Indeed, quasi-isometric embeddings between Gromov hyperbolic spaces always extend continuously to maps between Gromov boundaries.

We also prove the following result, which gives a complete characterization of the Koberda embeddings of free groups sending all generators to powers of Dehn twists that have continuous extensions:

Theorem 1.3 *Let $\{\alpha_1, \dots, \alpha_k\}$ be a collection of pairwise intersecting curves in S and Γ the graph with $V(\Gamma) = \{s_1, \dots, s_k\}$ and no edges. For sufficiently large N, the homomorphism*

$$\phi\colon A(\Gamma) \to \mathrm{Mod}(S) \quad \text{defined by } \phi(s_i) = T_{\alpha_i}^N \text{ for all } i$$

is injective by the work of Koberda [8]. Moreover, ϕ extends continuously to a map $\partial A(\Gamma) \to \partial \mathrm{Mod}(S)$ if and only if $\{\alpha_1, \dots, \alpha_k\}$ pairwise fill S, where $A(\Gamma)$ is equipped with any HHS structure.

In fact, we prove something stronger than Theorem 1.3. We prove a nonexistence result (Theorem 5.3) for a class of Koberda embeddings of right-angled Artin groups that are not necessarily free groups. We also prove an existence result (Theorem 6.1) for a class of embeddings of free groups that includes the Koberda embeddings described in Theorem 1.3 as well as a class of CLM embeddings. The following question remains open:

Question 1.4 *Let $A(\Gamma) \to \mathrm{Mod}(S)$ be a CLM embedding of a free group that sends some pair of generators of $A(\Gamma)$ to mapping classes whose full supports together do not fill S. Is it always the case that ϕ does not extend? In other words, does the forward direction of Theorem 1.3 hold for CLM embeddings?* (Theorem 6.1 proves the backwards direction).

Idea behind the nonexistence proofs (Theorems 1.2, 1.3 and 5.3) All of the embeddings $\phi\colon A(\Gamma)\to \mathrm{Mod}(S)$ we present that do not extend share the following key feature: for some pair of noncommuting generators a and b of $A(\Gamma)$, the subsurface Y filled by the full supports of $\phi(a)$ and $\phi(b)$ is a *proper* subsurface of S. For the embeddings we consider, this allows us to produce two sequences in $A(\Gamma)$ that converge to the same point in $\partial A(\Gamma)$, but whose images do not converge to the same point in $\partial\mathrm{Mod}(S)$. We choose the n^{th} term of the first sequence so that the annular projection of its image to some boundary component γ of Y is distance $O(n)$ from a basepoint, while the projection to γ of the image of the second sequence has bounded diameter. We then show that $O(n)$ is fast enough to conclude that every accumulation point in $\partial\mathrm{Mod}(S)$ of the image of the first sequence has a term associated to γ. On the other hand, accumulation points of the image of the second sequence have no such term. Thus the images of the sequences do not converge to the same point in $\partial\mathrm{Mod}(S)$.

The following open question arises naturally from our nonexistence proofs:

Question 1.5 *Let* $\mathrm{Teich}(S)$ *denote the Teichmüller space of a surface* S, *equipped with the Weil–Petersson metric. There is a hierarchically hyperbolic space structure on* $\mathrm{Teich}(S)$, *where the set of domains is all nonannular subsurfaces of* S (see Brock [5]). *Given that we show annular subsurface projections can obstruct extendability, we wonder if an orbit map from* $A(\Gamma)$ *to* $\mathrm{Teich}(S)$ *corresponding to a CLM embedding* $A(\Gamma)\to\mathrm{Mod}(S)$ *extends continuously to a boundary map.* (Note that this is clearly not the case for the Koberda embeddings described in Theorem 2.6, since applying powers of a Dehn twist to a point in $\mathrm{Teich}(S)$ can move it only a bounded amount.)

In Section 2 we will recall relevant definitions and theorems and introduce notation. Section 3 will establish a handful of lemmas that will be used for proving Theorem 1.2. Section 4 is devoted to proving Theorem 1.2 for a CLM embedding, and in Section 5 we prove Theorem 1.2 for a Koberda embedding. Using similar techniques, we then prove that a more general class of Koberda embeddings of right-angled Artin groups do not extend continuously (Theorem 5.3), which will imply one direction of Theorem 1.3. In Section 6 we will prove Theorem 6.1, which will imply the other direction of Theorem 1.3.

Remark We call the embeddings that send generators of our right-angled Artin group to mapping classes that are pseudo-Anosov on subsurfaces CLM embeddings and those that send generators to powers of Dehn twists Koberda embeddings, even though

Koberda [8] proved that both these types of embeddings are injective. We do this primarily to distinguish the two types of embeddings, but also to emphasize that CLM embeddings have nice geometric properties (see Theorem 2.5).

Acknowledgments The author was supported by the Department of Defense (DoD) through the National Defense Science and Engineering Graduate Fellowship (NDSEG) Program and by a Research Assistantship through NSF Grant number DMS-1510034. The author would like to thank her PhD advisor Chris Leininger for his numerous insights, which inspired a great deal of this paper, and also for his patience and consistent support. The author would also like to thank Mark Hagen and Matt Durham for helpful conversations and the referees for their suggestions and careful reading of this paper.

2 Background

In this section, we recall some needed definitions and theorems.

Notation Let $f, g\colon X \to \mathbb{R}$ be functions. Given constants $A \geq 1$ and $B \geq 0$, we write $f \succ_{A,B} g$ to mean $f(x) \geq \frac{1}{A} g(x) - B$ for all $x \in X$, and will just write $f \succ g$ when the constants are understood.

2.1 Curves and subsurfaces

Throughout this paper, we let $S = S_{g,n}$ denote a connected, oriented surface of genus g with n punctures. Define the *complexity of* S to be $\xi(S) = 3g - 3 + n$. We will always assume $\xi(S) \geq 1$. Additionally, we fix a complete hyperbolic metric on S. That is, we assume that S is of the form $S = \mathbb{H}^2/\Lambda$, where $\Lambda \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ and Λ acts properly discontinuously and freely on $\mathbb{H}^2$.

For $i = 1, 2$, let $\widetilde{\gamma}_i$ be a bi-infinite path in $\mathbb{H}^2$ with ends limiting to distinct points x_i and y_i on $\partial\mathbb{H}^2$. We say that $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ *link* if the geodesic connecting x_1 to y_1 intersects the geodesic connecting x_2 to y_2 in the interior of $\mathbb{H}^2$.

By a *curve* in S, we will always mean the geodesic representative in the homotopy class of an essential, simple, closed curve in S. By a *multicurve* in S, we will always mean a collection of pairwise disjoint curves in S. We write $i(\alpha, \beta)$ to denote the geometric intersection number of curves α and β. We say that a pair of curves α and β *fills* S if for every curve γ in S we have $i(\gamma, \alpha) > 0$ or $i(\gamma, \beta) > 0$.

A *nonannular subsurface Y of S* is a component of S after removing a (possibly empty) collection of pairwise disjoint curves on S. Additionally, we require that Y satisfies $\xi(Y) \geq 1$; in particular, we do not consider a pair of pants to be a subsurface. We define ∂Y to be the collection of curves in S that are disjoint from Y and also are contained in the closure of Y, treating Y as a subset of S. When $Y \neq S$, the path metric completion of Y is a surface with boundary, and the image of this boundary under the map induced by the inclusion $Y \subseteq S$ is ∂Y.

An annular subsurface of S is defined as follows. Let α be a curve in S. Choose a component $\widetilde{\alpha}$ of the preimage of α in $\mathbb{H}^2$, and let $h \in \Lambda$ be a primitive isometry with axis $\widetilde{\alpha}$. Define

$$Y = (\overline{\mathbb{H}^2} - \{x, y\}) / \langle h \rangle,$$

where x and y are the fixed points of h on $\partial\mathbb{H}^2$. Observe that Y is a compact annulus and $\operatorname{int}(Y) \to S$ is a covering. We say that Y is the *annular subsurface of S with core curve α*. We define ∂Y to be α.

For any subsurface Y of S, we will write $Y \subseteq S$, even though when Y is an annulus, Y is not a subset of S.

Given $f \in \operatorname{Mod}(S)$ and a curve or simple bi-infinite geodesic γ in S, we define $f(\gamma)$ to be the curve or simple bi-infinite geodesic obtained as follows. Consider a component $\widetilde{\gamma}$ of the preimage of γ in $\mathbb{H}^2$. Choose a representative ψ in the isotopy class of f and lift it to a map $\widetilde{\psi}\colon \mathbb{H}^2 \to \mathbb{H}^2$. We define $f(\gamma)$ to be the image in S of the geodesic in $\mathbb{H}^2$ that connects the endpoints of $\widetilde{\psi}(\widetilde{\gamma})$ on $\partial\mathbb{H}^2$. Given $Y \subseteq S$, if Y is nonannular, we let $f(Y)$ denote the nonannular subsurface in its isotopy class. If Y is an annulus with core curve α, we let $f(Y)$ denote the annular subsurface of S with core curve $f(\alpha)$.

2.2 Curve complex

Let Y be a subsurface of S. If Y satisfies $\xi(Y) \geq 1$, the *curve complex of Y*, denoted by $\mathcal{C}(Y)$, is the simplicial complex whose vertices are curves contained in Y, and if $\xi(Y) > 1$, a set of vertices forms a simplex if and only if they are pairwise disjoint. If $\xi(Y) = 1$, then we define the simplices of $\mathcal{C}(Y)$ differently. In the case that Y is a once-punctured torus, a set of vertices forms a simplex if and only if they pairwise intersect exactly once. If Y is a four-times-punctured sphere, a set of vertices forms a simplex if and only if they pairwise intersect exactly twice.

Now let Y be a compact annulus. Consider all embedded arcs in Y that connect one boundary component to the other. We define two arcs to be equivalent if one can be homotoped to the other, fixing the endpoints of the arcs throughout the homotopy. In this case, the *curve complex of* Y is the simplicial complex whose vertices are equivalence classes of arcs, and a set of vertices forms a simplex if and only if for each pair of vertices there exist representative arcs of each whose restrictions to $\mathrm{int}(Y)$ are disjoint. The following simple formula will be useful to us: given inequivalent arcs α and β in $\mathcal{C}(Y)$,

$$d_{\mathcal{C}(Y)}(\alpha, \beta) = |\alpha \cdot \beta| + 1, \tag{1}$$

where $\alpha \cdot \beta$ denotes the algebraic intersection number of α and β.

2.3 Markings and subsurface projection

A *marking* μ *on* S is a maximal collection of pairwise disjoint curves in S, denoted by $\mathrm{base}(\mu)$, together with another collection of associated curves called *transversals*: for each $\beta \in \mathrm{base}(\mu)$ its associated transversal γ_β is a curve that intersects β minimally (ie once or twice) and is disjoint from all other curves in $\mathrm{base}(\mu)$.

Let Y be a subsurface of S and β a multicurve in S. We will now define *the projection of* β *to* Y, which we will denote by $\pi_Y(\beta)$. Suppose Y is not an annulus and β is a single curve. If β is disjoint from Y, define $\pi_Y(\beta) = \varnothing$. If β is contained in Y, define $\pi_Y(\beta) = \beta$. Otherwise, $\beta \cap Y$ is a collection of essential arcs in Y with endpoints on ∂Y. For each such arc γ, take the geodesic representatives of the boundary components of a small regular neighborhood of $\gamma \cup \partial Y$ that are contained in Y. Define $\pi_Y(\beta)$ to be the collection of all such curves over all arcs γ in $\beta \cap Y$. If β is a multicurve, define $\pi_Y(\beta)$ to be the union of the projections to Y of each curve in β.

Now let Y be an annular subsurface with core curve α and $\mathrm{int}(Y) \to S$ the associated covering. Let β be a multicurve or a bi-infinite, simple geodesic in S. Consider the components of the full preimage of β in $\mathrm{int}(Y)$ that are arcs. We will view each such component as having endpoints on the boundary of Y. In this case, we define $\pi_Y(\beta)$ to be the (equivalence classes of) arcs in this collection that have an endpoint on each boundary component of Y. When convenient, we will write $\pi_\alpha(\beta)$ instead of $\pi_Y(\beta)$.

We now describe how to project a marking μ to $Y \subseteq S$. If Y is nonannular or Y is an annulus whose core curve is not contained in $\mathrm{base}(\mu)$, we define $\pi_Y(\mu) = \pi_Y(\mathrm{base}(\mu))$. Otherwise, Y is an annulus with core curve $\alpha \in \mathrm{base}(\mu)$, and we define $\pi_Y(\mu)$ to be $\pi_Y(\gamma_\alpha)$, where γ_α is the transversal associated to α.

Given any subsurface $Y \subseteq S$, we define

$$d_Y(\mu,\mu') = \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu) \cup \pi_Y(\mu')),$$

where μ and μ' are markings, collections of curves, or (when Y is an annulus) biinfinite simple geodesics in S. A useful fact about subsurface projection is the following: for all $f \in \operatorname{Mod}(S)$,

$$d_Y(f(\mu), f(\mu')) = d_{f^{-1}(Y)}(\mu,\mu').$$

In this paper, we utilize the following theorem, which involves subsurface projections:

Theorem 2.1 [11, Lemma 2.3] *For all subsurfaces W of S, given any marking or multicurve μ such that $\pi_W(\mu) \neq \varnothing$, we have that $\operatorname{diam}_{\mathcal{C}(W)}(\pi_W(\mu)) \le 2$. If W is an annulus, then $\operatorname{diam}_{\mathcal{C}(W)}(\pi_W(\mu)) \le 1$.*

Masur and Minsky [11] define the *marking graph of* S, denoted by $\widetilde{\mathcal{M}}(S)$, to be the graph whose vertices are markings and vertices are adjacent if one can be obtained from the other by an elementary move; see [11] for a complete definition. Giving $\widetilde{\mathcal{M}}(S)$ the path metric $d_{\widetilde{\mathcal{M}}(S)}$ and $\operatorname{Mod}(S)$ a word metric $d_{\operatorname{Mod}(S)}$, there is an action of $\operatorname{Mod}(S)$ on $\widetilde{\mathcal{M}}(S)$ by isometries for which every orbit map is a quasi-isometry. The following theorem gives a relationship between distances in $\widetilde{\mathcal{M}}(S)$ and subsurface projections:

Theorem 2.2 [11, Lemma 3.5] *For any subsurface W of S and any markings μ and μ' on S, we have that $d_W(\mu,\mu') \le 4 d_{\widetilde{\mathcal{M}}(S)}(\mu,\mu')$.*

We say that distinct subsurfaces X and Y are *disjoint* if $\pi_X(\partial Y) = \varnothing$ and $\pi_Y(\partial X) = \varnothing$. We say that X is a *proper subsurface of* Y, denoted by $X \subsetneq Y$, if $\pi_Y(\partial X) \neq \varnothing$ and $\pi_X(\partial Y) = \varnothing$. We say that X and Y are *overlapping*, denoted by $X \pitchfork Y$, if $\pi_Y(\partial X) \neq \varnothing$ and $\pi_X(\partial Y) \neq \varnothing$. In the case where X and Y are not annuli, these relationships, respectively, are disjointness, proper containment and intersection without containment as subsets of S. We say X and Y *fill* S if for every curve γ in S we have $\pi_X(\gamma) \neq \varnothing$ or $\pi_Y(\gamma) \neq \varnothing$.

The following theorems will be used to prove our results. The first theorem was proved in [1] and later a simpler proof with constructive constants appeared in [9].

Theorem 2.3 (Behrstock inequality [1, Theorem 4.3; 9, Lemma 2.13]) *Let X and Y be overlapping subsurfaces of S and μ a marking on S. Then*

$$d_X(\mu,\partial Y) \ge 10 \implies d_Y(\mu,\partial X) \le 4.$$

Theorem 2.4 (Bounded Geodesic Image Theorem [11, Theorem 3.1]) *There exists a constant K_0 depending only on S such that the following is true: Let Y and Z be subsurfaces of S with Y a proper subsurface of Z. Let $v_1, \dots, v_n$ be any geodesic segment in $\mathcal{C}(Z)$ satisfying $\pi_Y(v_i) \neq \varnothing$ for all $1 \leq i \leq n$. Then*

$$\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y(v_1) \cup \cdots \cup \pi_Y(v_n)) \leq K_0.$$

2.4 Partial order on subsurfaces

Let μ and μ' be markings on S and $K \geq 20$. Let $\Omega(K, \mu, \mu')$ denote the collection of subsurfaces Y of S such that $d_Y(\mu, \mu') \geq K$. Behrstock, Kleiner, Minsky and Mosher [4] define the following partial order on $\Omega(K, \mu, \mu')$: given $X, Y \in \Omega(K, \mu, \mu')$ such that $X \pitchfork Y$, define $X \prec Y$ if and only if one of the following equivalent conditions is satisfied:

$$d_X(\mu, \partial Y) \geq 10, \quad d_X(\partial Y, \mu') \leq 4, \quad d_Y(\mu, \partial X) \leq 4 \quad \text{or} \quad d_Y(\partial X, \mu') \geq 10.$$

That these conditions are equivalent is a consequence of Theorem 2.3; see Corollary 3.7 in [6].

2.5 Embedding RAAGs in Mod(S)

If $f \in \mathrm{Mod}(S)$ is such that there exists a representative in the isotopy class of f that pointwise fixes the complement of a nonannular subsurface Y, we say that f is *supported on* Y. Given such an f, we define the *translation length of f on $\mathcal{C}(Y)$* to be

$$\tau_Y(f) = \lim_{n \to \infty} \frac{d_Y(\mu, f^n(\mu))}{n},$$

where μ is any marking on S. If $f \in \mathrm{Mod}(S)$ is a power of a Dehn twist about a curve α, we say that f is *supported on* the annular subsurface Y with core curve α, and define $\tau_Y(f)$ to be the absolute value of the power. In either case, we say that Y *fully supports* f if $\tau_Y(f) > 0$. By the work of Masur and Minsky [10], when Y is nonannular, Y fully supports f if and only if f is pseudo-Anosov on Y.

Clay, Leininger and Mangahas [6] proved the following result, which allows us to find quasi-isometrically embedded right-angled Artin subgroups inside $\mathrm{Mod}(S)$:

Theorem 2.5 [6, Theorem 2.2] *Let Γ be a finite graph with $V(\Gamma) = \{s_1, \dots, s_k\}$, and let $\{X_1, \dots, X_k\}$ be a collection of nonannular subsurfaces of S. Suppose $s_i s_j$ is an edge in Γ if and only if X_i and X_j are disjoint, and $s_i s_j$ is not an edge in Γ*

if and only if $X_i \pitchfork X_j$ or $i = j$. Then there exists a constant $C > 0$ such that the following holds: Let $\{f_1, \dots, f_k\}$ be a set of mapping classes of S such that f_i is pseudo-Anosov on X_i and satisfies $\tau_{X_i}(f_i) \geq C$ for all i. Then the homomorphism

$$\phi\colon A(\Gamma) \to \mathrm{Mod}(S) \quad \textit{defined by } \phi(s_i) = f_i \textit{ for all } i$$

is a quasi-isometric embedding, implying that ϕ is injective, since $A(\Gamma)$ is torsion-free.

Koberda [8] also has a result which produces right-angled Artin subgroups of $\mathrm{Mod}(S)$. Below we give a special case of Koberda's result that we will use.

Theorem 2.6 [8, Theorem 1.1] *Let $\{\alpha_1, \dots, \alpha_k\}$ be a collection of distinct curves in S. Let Γ be the graph with $V(\Gamma) = \{s_1, \dots, s_k\}$ and with $s_i s_j$ an edge in Γ if and only if $i(\alpha_i, \alpha_j) = 0$. Then, for sufficiently large N, the homomorphism*

$$\phi\colon A(\Gamma) \to \mathrm{Mod}(S) \quad \textit{defined by } \phi(s_i) = T_{\alpha_i}^N \textit{ for all } i$$

is injective, where T_{α_i} denotes a Dehn twist about α_i.

2.6 Gromov boundary of hyperbolic spaces

A geodesic metric space X is *Gromov hyperbolic* (or just hyperbolic) if there exists a $\delta \geq 0$ such that, given any geodesic triangle in X, each side is contained in the δ–neighborhood of the union of the other two sides. Given a Gromov hyperbolic space (X, d_X) and points $x, y, z \in X$, the *Gromov product* of x and y with respect to z is defined as

$$(x, y)_z = \tfrac{1}{2}(d_X(x, z) + d_X(y, z) - d_X(x, y)).$$

We say that a sequence (x_n) in X *converges at infinity* if $\liminf_{i,j\to\infty}(x_i, x_j)_z = \infty$ for some (any) $z \in X$. We define two such sequences (x_n) and (y_n) to be equivalent if $\liminf_{i,j\to\infty}(x_i, y_j)_z = \infty$ for some (any) $z \in X$. The *Gromov boundary of X* is the collection of all such sequences up to this equivalence, and is denoted by $\partial_G X$ or just ∂X when it is clear from context that we are using the Gromov boundary.

One Gromov hyperbolic space that this paper is concerned with is the curve complex of S, which was proved to be Gromov hyperbolic by Masur and Minsky [11]. We can now state a corollary of Theorem 2.4 that will be useful later.

Corollary 2.7 *Let X and Y be subsurfaces of S with X a proper subsurface of Y. Suppose $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of markings on S such that $\pi_Y(\mu_n) \to \lambda$ for some $\lambda \in \partial\mathcal{C}(Y)$. Then $\mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \pi_X(\mu_2) \cup \cdots) < \infty$.*

Proof For each n, choose $\alpha_n \in \pi_Y(\mu_n)$. Because $\pi_Y(\mu_n) \to \lambda \in \partial\mathcal{C}(Y)$, we can choose L large so that for all $n \geq L$ we have

$$(\alpha_n, \alpha_L)_{\alpha_1} \geq 2 + d_Y(\partial X, \alpha_1), \tag{2}$$

where the Gromov product is computed in $\mathcal{C}(Y)$. Consider $n \geq L$. Let γ_n be a geodesic in $\mathcal{C}(Y)$ with endpoints α_n and α_L. If there exists a vertex v on γ_n with $\pi_X(v) = \varnothing$, then v and ∂X form a multicurve in Y, which implies that

$$\begin{aligned}
&(\alpha_n, \alpha_L)_{\alpha_1} \\
&\quad = \tfrac{1}{2}(d_Y(\alpha_n, \alpha_1) + d_Y(\alpha_L, \alpha_1) - d_Y(\alpha_n, \alpha_L)) \\
&\quad \leq \tfrac{1}{2}\big(d_Y(\alpha_n, v) + d_Y(v, \alpha_1) + d_Y(\alpha_L, v) + d_Y(v, \alpha_1) - (d_Y(\alpha_n, v) + d_Y(v, \alpha_L))\big) \\
&\quad = d_Y(v, \alpha_1) \leq d_Y(v, \partial X) + d_Y(\partial X, \alpha_1) \leq 1 + d_Y(\partial X, \alpha_1).
\end{aligned}$$

But this contradicts (2), so we conclude that $\pi_X(v) \neq \varnothing$ for all v on γ_n. We can now apply Theorems 2.1 and 2.4 to see that, for all $n \geq L$,

$$d_X(\mu_n, \mu_L) \leq \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_n)) + d_X(\alpha_n, \alpha_L) + \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_L)) \leq 2 + K_0 + 2,$$

where K_0 is as in Theorem 2.4. Therefore,

$$\begin{aligned}
\mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \pi_X(\mu_2) \cup \cdots) &\leq \mathrm{diam}_{\mathcal{C}(X)}(\pi_X(\mu_1) \cup \cdots \cup \pi_X(\mu_L)) + 2(K_0 + 4) \\
&< \infty.
\end{aligned}$$

□

2.7 Hierarchically hyperbolic spaces

In [3], Behrstock, Hagen and Sisto define the notion of a hierarchically hyperbolic space. Roughly, a hierarchically hyperbolic space is a quasi-geodesic metric space $\mathcal{X}$, equipped with additional structure which we will call a hierarchically hyperbolic space (HHS) structure. An HHS structure consists of an index set $\mathcal{G}$ and for each $W \in \mathcal{G}$ a Gromov hyperbolic space $\widehat{C}W$ and a projection map $\pi_W\colon \mathcal{X} \to 2^{\widehat{C}W}$. The elements of $\mathcal{G}$ and the projection maps must satisfy a long list of properties. See [3; 2].

The first example of a hierarchically hyperbolic space is $\mathrm{Mod}(S)$, where here $\mathcal{G}$ is the collection of all subsurfaces of S, $\widehat{C}W$ is the curve graph of W for $W \in \mathcal{G}$, and projection π_W is given by composing an orbit map for the action of $\mathrm{Mod}(S)$ on $\widetilde{\mathcal{M}}(S)$ with the subsurface projection map defined in Section 2.3. The works of Masur and Minsky [10; 11], Behrstock [1] and Behrstock, Kleiner, Minsky and Mosher [4] imply that $\mathrm{Mod}(S)$ is a hierarchically hyperbolic space. See [2, Section 11] for details. In fact, the notion of hierarchical hyperbolicity was motivated by a desire to generalize some of the machinery surrounding mapping class groups.

In [3], it is shown that a large class of CAT(0) cube complexes can be equipped with a hierarchically hyperbolic structure, including the universal covers of Salvetti complexes associated to right-angled Artin groups. The CAT(0) cube complex we are primarily concerned with is the Cayley graph X of $A(\Gamma)$ when Γ has no edges (that is, $A(\Gamma)$ is a free group). We equip $A(\Gamma)$ with a hierarchically hyperbolic structure by equipping X with such a structure and then associating $A(\Gamma)$ with X.

2.8 Boundary of hierarchically hyperbolic spaces

In [7], the authors construct a boundary for hierarchically hyperbolic spaces. Here we will describe convergence in this boundary for $\mathrm{Mod}(S)$ and for free groups. With the exception of Theorem 5.3, these will be the only examples we will need.

As a set, the HHS boundary of $\mathrm{Mod}(S)$ is defined as follows:

$$\partial\mathrm{Mod}(S) = \Big\{ \sum_{Y\subseteq S} c_Y\lambda_Y \;\Big|\; c_Y \geq 0 \text{ and } \lambda_Y \in \partial\mathcal{C}(Y) \text{ for all } Y,\ \sum_{Y\subseteq S} c_Y = 1, \\ \text{and if } c_{Y'}, c_Y > 0, \text{ then } Y \text{ and } Y' \text{ are disjoint or equal}\Big\}.$$

In [7], the authors define a topology on $\mathrm{Mod}(S)\cup\partial\mathrm{Mod}(S)$. In this topology, [7, Definition 2.10] tells us that a sequence of mapping classes $(g_n)_{n\in\mathbb{N}}$ in $\mathrm{Mod}(S)$ converges to a point $\sum_{i=1}^k c_i\lambda_i$ in $\partial\mathrm{Mod}(S)$, where $c_i > 0$ for all i, $\sum_{i=1}^k c_i = 1$ and $\lambda_i \in \partial\mathcal{C}(Y_i)$ for pairwise disjoint subsurfaces $Y_1,\dots,Y_k$, if and only if the following statements hold: for a fixed marking μ on S,

(1) $\lim_{n\to\infty} \pi_{Y_i}(g_n\mu) = \lambda_i$ for each $i = 1,\dots,k$,

(2) $\lim_{n\to\infty} d_{Y_i}(\mu, g_n\mu)/d_{Y_j}(\mu, g_n\mu) = c_i/c_j$ for each $i, j = 1,\dots,k$, and

(3) $\lim_{n\to\infty} d_W(\mu, g_n\mu)/d_{Y_i}(\mu, g_n\mu) = 0$ for every (any) $i = 1,\dots,k$ and every subsurface $W\subseteq S$ that is disjoint from Y_j for all $j = 1,\dots,k$.

Let Γ be a graph with no edges, and let $A(\Gamma)$ be the corresponding free group, equipped with an HHS structure. The HHS boundary of $A(\Gamma)$ will be denoted by $\partial A(\Gamma)$. We do not define $\partial A(\Gamma)$ here because Theorem 4.3 in [7] implies that the identity map $A(\Gamma)\to A(\Gamma)$ extends to a homeomorphism $A(\Gamma)\cup\partial_G A(\Gamma)\to A(\Gamma)\cup\partial A(\Gamma)$. Thus, two sequences in $A(\Gamma)$ converge to the same point in $\partial_G A(\Gamma)$ if and only if they converge to the same point in $\partial A(\Gamma)$. (See Section 2 of [7] for the definition of $\partial A(\Gamma)$.)

Another useful fact on convergence is that $\mathrm{Mod}(S)\cup\partial\mathrm{Mod}(S)$ and $A(\Gamma)\cup\partial A(\Gamma)$ are sequentially compact (see Theorem 3.4 of [7]).

To understand Question 1.1 and the statements of our theorems, one last definition is needed.

Definition 2.8 Let $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$ be an injective homomorphism and let $A(\Gamma)$ and $\mathrm{Mod}(S)$ be equipped with any fixed HHS structures. We say that ϕ *extends continuously to a map* $\partial A(\Gamma) \to \partial\mathrm{Mod}(S)$ if there exists a function $\overline{\phi}\colon A(\Gamma) \cup \partial A(\Gamma) \to \mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ such that

(1) $\overline{\phi}|_{A(\Gamma)} = \phi$,

(2) $\overline{\phi}(\partial A(\Gamma)) \subseteq \partial\mathrm{Mod}(S)$, and

(3) $\overline{\phi}$ is continuous at each point in $\partial A(\Gamma)$.

Remark 2.9 To establish that $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$ extends continuously, it is enough to show that, for all $x \in \partial A(\Gamma)$, given any two sequences (x_n) and (y_n) in $A(\Gamma)$ that converge to x, we have that $(\phi(x_n))$ and $(\phi(y_n))$ converge to the same point in $\partial\mathrm{Mod}(S)$. This follows from a diagonal sequence argument (see the end of the proof of Theorem 5.6 in [7] for details).

3 Lemmas on subsurface projections

The following lemmas are the heart of our proof of Theorem 1.2:

Lemma 3.1 *Suppose X and Y are disjoint subsurfaces of S, and if Y is an annulus, then the core of Y is not contained in ∂X. If μ and μ' are markings and $f \in \mathrm{Mod}(S)$ a mapping class supported on X, then $|d_Y(\mu, f(\mu')) - d_Y(\mu, \mu')| \leq 4$.*

Proof If Y is not an annulus, then $\pi_Y(f(\mu')) = \pi_Y(\mu')$, so the claim clearly holds. Assume then that Y is an annular subsurface of S with core α, and let $\mathrm{int}(Y) \to S$ be the associated covering. If X is not an annulus, define Z to be the component of $S - X$ that contains α. If X is an annulus with core β, let Z be the component of S containing α after removing a small regular neighborhood of β. Let $\widetilde{\alpha}$ be the component of the preimage of α in $\mathrm{int}(Y)$ that is a closed curve. Let $\widetilde{Z}$ be the component of the preimage of Z in $\mathrm{int}(Y)$ that contains $\widetilde{\alpha}$.

Abusing notation, we let f denote a representative in the isotopy class of f that fixes Z pointwise. Consider the lift of f to $\mathrm{int}(Y)$ that fixes a point on $\widetilde{\alpha}$, and thus fixes

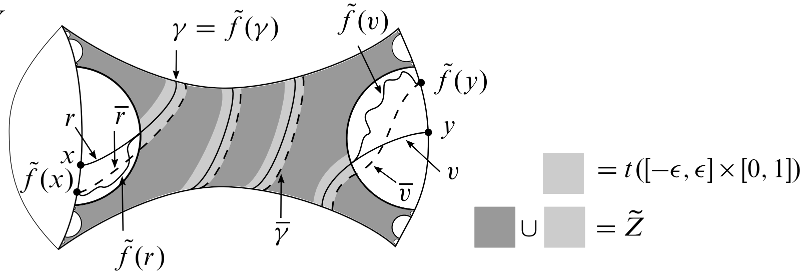

Figure 1: The arc β' is the concatenation of r, γ and v. The concatenation of $\overline{r}, \overline{\gamma}$ and $\overline{v}$ is equivalent to $\tilde{f}(\beta')$, and that representative of $\tilde{f}(\beta')$ intersects β' at most once (drawn is the exactly once case). See the proof of Lemma 3.1.

$\widetilde{Z}$ pointwise. Let $\tilde{f}\colon Y \to Y$ denote the continuous extension of that lift. Consider $\beta' \in \pi_Y(\mu')$. We will show

$$d_{\mathcal{C}(Y)}(\beta', \tilde{f}(\beta')) \leq 2. \tag{3}$$

This will complete the proof because (3), the triangle inequality and Theorem 2.1 imply that

$$\begin{aligned} |d_Y(\mu, f(\mu')) - d_Y(\mu, \mu')| \\ &\leq d_Y(\mu', f(\mu')) \\ &\leq \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(\mu')) + d_{\mathcal{C}(Y)}(\beta', \tilde{f}(\beta')) + \operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(f(\mu'))) \\ &\leq 1 + 2 + 1 = 4. \end{aligned}$$

Inequality (3) holds if β' is contained in $\widetilde{Z}$ because, in that case, $\tilde{f}(\beta') = \beta'$. Thus, we assume β' is not contained in $\widetilde{Z}$. We break β' up into three parts. Let γ be the largest subpath of β' contained in $\widetilde{Z}$. Let x and y denote the endpoints of β' on ∂Y. Removing γ from β' yields rays r and v that limit to x and y, respectively.

We now construct an arc equivalent to $\tilde{f}(\beta')$ that intersects β' at most once. Figure 1 illustrates the construction. Let $t\colon [-\epsilon, \epsilon] \times [0, 1] \to Y$ be a small tubular neighborhood of γ such that $t|_{\{0\}\times[0,1]} = \gamma$ and $t([-\epsilon, \epsilon] \times \{0, 1\}) \subseteq \partial\widetilde{Z}$. Let R and V denote the components of $\operatorname{int}(Y) - \widetilde{Z}$ containing r and v, respectively. Because $\tilde{f}$ fixes $\widetilde{Z}$ pointwise, $\tilde{f}$ restricts to homeomorphisms of both R and V, implying that $\tilde{f}(r)$ and $\tilde{f}(v)$ are contained in R and V, respectively. Consequently, in R there exists a ray $\overline{r}$ based at $t(-\epsilon, 0)$ or $t(\epsilon, 0)$ that limits to $\tilde{f}(x)$ and is disjoint from r. If $\overline{r}$ is based at $t(-\epsilon, 0)$, define $\overline{\gamma} = t|_{\{-\epsilon\}\times[0,1]}$. Otherwise, define $\overline{\gamma} = t|_{\{\epsilon\}\times[0,1]}$. Choose $\overline{v}$ to be an arc in V from $\overline{\gamma}(1)$ to $\tilde{f}(y)$ that intersects v at most once. Observe that the arc obtained by concatenating $\overline{r}$, $\overline{\gamma}$ and $\overline{v}$ is equivalent to $\tilde{f}(\beta')$ and intersects β' at most once.

Therefore, by (1) we have $d_{\mathcal{C}(Y)}(\tilde{f}(\beta'), \beta') = 1 + |\tilde{f}(\beta') \cdot \beta'| \leq 2$, as desired. □

Lemma 3.2 *Given a homomorphism $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$ and a marking μ on S, there exists a constant $M \geq 1$ such that the following holds: Let $y_1 \dots y_n \in A(\Gamma)$, where each $y_i \in V(\Gamma)$. Then $d_W(\mu, \phi(y_1 \dots y_n)\mu) \leq Mn$ for all subsurfaces $W \subseteq S$.*

Proof Define $M = 4\max\{d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(x)\mu) : x \in V(\Gamma)\}$. By the triangle inequality and Theorem 2.2,

$$\begin{aligned} d_W(\mu, \phi(y_1 \dots y_n)\mu) &\leq \sum_{i=1}^{n} d_W(\phi(y_1 \dots y_{i-1})\mu, \phi(y_1 \dots y_i)\mu) \\ &\leq \sum_{i=1}^{n} 4 d_{\widetilde{\mathcal{M}}(S)}(\phi(y_1 \dots y_{i-1})\mu, \phi(y_1 \dots y_i)\mu) \\ &= \sum_{i=1}^{n} 4 d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(y_i)\mu) \leq Mn. \end{aligned}$$ □

Lemma 3.3 *Let $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$ be a homomorphism. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements in $A(\Gamma)$ and μ a marking on S. Suppose, for some subsurface $W \subseteq S$, there exist constants $A \geq 1$ and $B \geq 0$ that do not depend on n such that $d_W(\mu, \phi(g_n)\mu) \succ_{A,B} \|g_n\|$, where $\|g_n\|$ denotes the word length of g_n with respect to the standard generating set $V(\Gamma)$ for $A(\Gamma)$. Further suppose that $\lim_{n\to\infty} \|g_n\| = \infty$ and that $(\pi_W(\phi(g_n)\mu))_{n\in\mathbb{N}}$ converges to some point λ_W in $\partial\mathcal{C}(W)$. Then all accumulation points of $(\phi(g_n))_{n\in\mathbb{N}}$ in $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ are in $\partial\mathrm{Mod}(S)$ and are of the form $\sum_{Y\subseteq S} c_Y \lambda_Y$, where $c_W > 0$.*

Proof After passing to a subsequence, we may assume that $(\phi(g_n))_{n\in\mathbb{N}}$ converges. By assumption, $\lim_{n\to\infty} d_W(\mu, \phi(g_n)\mu) = \infty$. Combine this with Theorem 2.2 to see that $\lim_{n\to\infty} d_{\widetilde{\mathcal{M}}(S)}(\mu, \phi(g_n)\mu) = \infty$. Because $\widetilde{\mathcal{M}}(S)$ is quasi-isometric to $\mathrm{Mod}(S)$ via orbit maps, it follows that $\lim_{n\to\infty} d_{\mathrm{Mod}(S)}(1, \phi(g_n)) = \infty$. Thus, it must be that $\lim_{n\to\infty} \phi(g_n) \in \partial\mathrm{Mod}(S)$.

Suppose $\lim_{n\to\infty} \phi(g_n) = \sum_{Y\subseteq S} c_Y \lambda_Y$ for constants $c_Y \geq 0$ and $\lambda_Y \in \partial\mathcal{C}(Y)$. We will now argue that $c_W > 0$. Let $Z \subseteq S$ be such that $c_Z > 0$. If $W = Z$, we are done. So we assume $W \neq Z$. By definition of the topology on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, we have that $\lim_{n\to\infty}\pi_Z(\phi(g_n)\mu)=\lambda_Z$. If $W\subsetneq Z$, then Corollary 2.7 implies that $\operatorname{diam}_{\mathcal{C}(W)}(\pi_W(\phi(g_1)\mu)\cup\pi_W(\phi(g_2)\mu)\cup\cdots)<\infty$. But this cannot be since $\pi_W(\phi(g_n)\mu)\to\lambda_W\in\partial\mathcal{C}(W)$. Similarly, we cannot have $Z\subsetneq W$ for then Corollary 2.7 implies that $\operatorname{diam}_{\mathcal{C}(Z)}(\pi_Z(\phi(g_1)\mu)\cup\pi_Z(\phi(g_2)\mu)\cup\cdots)<\infty$, contradicting that $\pi_Z(\phi(g_n)\mu)\to\lambda_Z\in\partial\mathcal{C}(Z)$. Now suppose that $Z\pitchfork W$. Then by Theorem 2.3, after passing to a subsequence, we have that

$$d_W(\partial Z,\phi(g_n)\mu)\le 10\quad\text{for all } n\qquad\text{or}\qquad d_Z(\partial W,\phi(g_n)\mu)\le 10\quad\text{for all } n.$$

If $d_W(\partial Z,\phi(g_n)\mu)\le 10$ for all n, then, for all n,

$$d_W(\mu,\phi(g_n)\mu)\le d_W(\mu,\partial Z)+d_W(\partial Z,\phi(g_n)\mu)\le d_W(\mu,\partial Z)+10,$$

contradicting that $\pi_W(\phi(g_n)\mu)\to\lambda_W\in\partial\mathcal{C}(W)$. Similarly, if $d_Z(\partial W,\phi(g_n)\mu)\le 10$ for all n, then $d_Z(\mu,\phi(g_n)\mu)$ is bounded independent of n, contradicting that $\pi_Z(\phi(g_n)\mu)\to\lambda_Z\in\partial\mathcal{C}(Z)$. So it is not the case that $Z\pitchfork W$. Therefore it must be that W and Z are disjoint for all $Z\subseteq S$ with $c_Z>0$.

Fix $Z\subseteq S$ with $c_Z>0$. Lemma 3.2 together with the fact that $d_W(\mu,\phi(g_n)\mu)\succ_{A,B}\|g_n\|$ implies that

$$\frac{d_W(\mu,\phi(g_n)\mu)}{d_Z(\mu,\phi(g_n)\mu)}\ge\frac{\frac{1}{A}\|g_n\|-B}{M\|g_n\|},\tag{4}$$

where $M\ge 1$ is as in Lemma 3.2. Since $\|g_n\|\to\infty$, (4) implies

$$\lim_{n\to\infty}\frac{d_W(\mu,\phi(g_n)\mu)}{d_Z(\mu,\phi(g_n)\mu)}\ge\lim_{n\to\infty}\frac{\frac{1}{A}\|g_n\|-B}{M\|g_n\|}>0.$$

Therefore, by definition of the topology of $\operatorname{Mod}(S)\cup\partial\operatorname{Mod}(S)$, we have $c_W>0$, as desired. □

4 Clay–Leininger–Mangahas RAAGs

In this section, we prove the first part of Theorem 1.2. We begin with a description of a CLM embedding $\phi\colon A(\Gamma)\to\operatorname{Mod}(S)$.

Embedding construction Let Γ be the graph with vertex set $V(\Gamma)=\{a,b\}$ and no edges. Let $S=\mathbb{H}^2/\Lambda$, X_a and X_b be the surfaces indicated in Figure 2. For short, let X_{ab} denote $X_a\cup X_b$. Let $\widetilde{S-X_{ab}}$ be a component of the preimage of $S-X_{ab}$ in $\mathbb{H}^2$, and let $\widetilde{\partial X_{ab}}$ be a geodesic in $\mathbb{H}^2$ that is in the boundary of $\widetilde{S-X_{ab}}$.

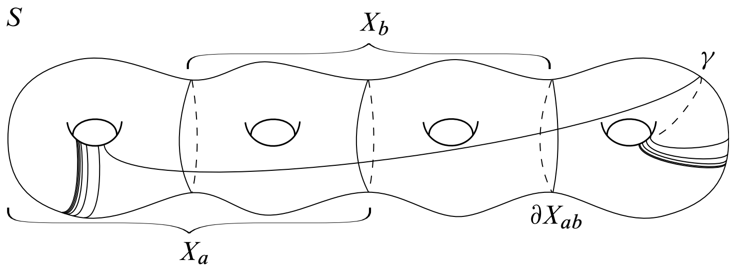

Figure 2: Overlapping subsurfaces X_a and X_b of surface S, curve ∂X_{ab} and bi-infinite simple geodesic γ

Let $\tilde{\gamma}$ be a geodesic in $\mathbb{H}^2$ that links with $\widetilde{\partial X_{ab}}$ and maps to a simple bi-infinite geodesic γ in S. Further suppose that $\tilde{\gamma} \cap (\widetilde{S - X_{ab}})$ is an infinite ray and let p be its endpoint on $\partial\mathbb{H}^2$. For example, take γ to be the simple bi-infinite geodesic in S with one end spiraling around a curve essential in $S - X_{ab}$ and the other end spiraling around a curve in X_a, as in Figure 2, and take $\tilde{\gamma}$ to be an appropriate lift of γ. Choose $f_b \in \mathrm{Mod}(S)$ so that f_b is pseudo-Anosov on X_b. To simplify arguments, we abuse notation and let f_b denote a representative in the isotopy class of f_b that fixes all points outside X_b. This ensures that $\tilde{f}_b$ fixes $\widetilde{S - X_{ab}}$ pointwise, where $\tilde{f}_b\colon \mathbb{H}^2 \to \mathbb{H}^2$ is the lift of f_b fixing some point on $\widetilde{\partial X_{ab}}$. Thus, the extension of $\tilde{f}_b$ to $\partial\mathbb{H}^2$ fixes pointwise p and the endpoints x and y of $\widetilde{\partial X_{ab}}$. Additionally, we choose f_b to have the following properties:

(1) $\tilde{f}_b(\tilde{\gamma})$ links with $h(\tilde{\gamma})$, where $h \in \Lambda$ is a primitive isometry with axis $\widetilde{\partial X_{ab}}$; and

(2) $\tau_{X_b}(f_b) \geq C$, where C is as in Theorem 2.5.

We note that a pseudo-Anosov on X_b satisfying (1) can be obtained from any mapping class that is pseudo-Anosov on X_b by postcomposing with some number of Dehn twists (or inverse Dehn twists) about ∂X_{ab}. Finally, a pseudo-Anosov on X_b satisfying (1) and (2) can be obtained from one satisfying (1) by passing to a sufficiently high power.

Let $f_a \in \mathrm{Mod}(S)$ be any mapping class that is pseudo-Anosov on X_a and satisfies $\tau_{X_a}(f_a) \geq C$. Theorem 2.5 says that the homomorphism

$$\phi\colon A(\Gamma) \to \mathrm{Mod}(S) \quad \text{defined by } \phi(a) = f_a \text{ and } \phi(b) = f_b$$

is a quasi-isometric embedding.

Equip $A(\Gamma)$ with any HHS structure. In the remainder of this section, we will prove the following theorem, which proves the first part of Theorem 1.2:

Theorem 4.1 *The sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^n b^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, but $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n b^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial \mathrm{Mod}(S)$.*

We will divide the proof of Theorem 4.1 into two propositions.

Proposition 4.2 *The sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^n b^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$.*

Proof Let X be the Cayley graph of $A(\Gamma)$. By the discussion in Section 2.8, to show that $(a^n)_{n\in\mathbb{N}}$ and $(a^n b^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, it is enough to show that they converge to the same point in $\partial_G X$. Now the Gromov product satisfies

$$(a^i, a^j b^j)_1 = \min(i, j) \to \infty \quad \text{as } i, j \to \infty.$$

Thus, $\lim_{n\to\infty} a^n = \lim_{n\to\infty} a^n b^n$ in $\partial_G X$, as desired. □

Throughout the rest of this section, μ will denote a fixed marking on S. To continue, we require the following lemma:

Lemma 4.3 *There exist constants $A \geq 1$ and $B \geq 0$ such that for all $n \geq 1$ we have $d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu) \succ_{A,B} n$. Consequently, after passing to a subsequence, $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial \mathcal{C}(\partial X_{ab})$.*

Proof We begin by establishing the following claim:

Claim 1 *Let $n \geq 1$. Then $\tilde{f}_b^n(\tilde{\gamma})$ has endpoint p and links with $h^i(\tilde{\gamma})$ for all $1 \leq i \leq n$.*

Proof of Claim 1 By our choice of $\tilde{f}_b$ and $\tilde{\gamma}$, we know the claim holds for $n = 1$. Let $n \geq 2$. Inductively, suppose that $\tilde{f}_b^{n-1}(\tilde{\gamma})$ has endpoint p and links with $h^i(\tilde{\gamma})$ for all $1 \leq i \leq n-1$. Let I be the interval in $\partial\mathbb{H}^2$ that connects the endpoints of $\widetilde{\partial X_{ab}}$ and does not contain p, oriented from the repelling fixed point of h to the attracting fixed point. We will use interval notation when speaking about connected subsets of I. Now $\tilde{f}_b$ extends continuously to a homeomorphism of $\partial\mathbb{H}^2$, which we will also denote by $\tilde{f}_b$, and because $\tilde{f}_b$ fixes the endpoints of $\widetilde{\partial X_{ab}}$, this extension restricts to a homeomorphism of I. Let z be the endpoint of $\tilde{\gamma}$ in I, and let $x \in \partial I$ be the attracting fixed point of h. Because $\tilde{f}_b^{n-1}(\tilde{\gamma})$ links with $h^i(\tilde{\gamma})$ for all $1 \leq i \leq n-1$ and has endpoint p, we have

$$(\tilde{f}_b^{n-1}(z), x] \subseteq (h^i(z), x] \quad \text{for all } 0 \leq i \leq n-1. \tag{5}$$

Since $\tilde{f}_b(\tilde{\gamma})$ has endpoint p and links with $h(\tilde{\gamma})$, it must be that $\tilde{f}_b(z) \in (hz, x]$. It follows from this, the fact that $\tilde{f}_b$ and h fix x, that $\tilde{f}_b$ and h commute by uniqueness of map lifting, and (5) that, for all $0 \le i \le n-1$,

$$(6) \qquad \tilde{f}_b^n(z) = \tilde{f}_b^{n-1}(\tilde{f}_b(z)) \in \tilde{f}_b^{n-1}(h(z), x] = h(\tilde{f}_b^{n-1}(z), x] \subseteq h(h^i(z), x] = (h^{i+1}(z), x].$$

Because $\tilde{f}_b$ fixes p, we have $\tilde{f}_b^n(p) = p$. This combined with (6) implies that $\tilde{f}_b^n(\tilde{\gamma})$ links with $h^{i+1}(\tilde{\gamma})$ for all $0 \le i \le n-1$, proving Claim 1. □

By Claim 1, after replacing $\tilde{f}_b^n(\tilde{\gamma})$ with the geodesic connecting its endpoints, the images of $\tilde{f}_b^n(\tilde{\gamma})$ and $\tilde{\gamma}$ in $(\overline{\mathbb{H}}^2 - \{x, y\})/\langle h \rangle$ intersect each other at least n times, and all these intersections have the same sign. Now apply (1) to see that

$$d_{\partial X_{ab}}(\gamma, \phi(b^n)\gamma) \ge n + 1.$$

It follows that

$$(7) \quad d_{\partial X_{ab}}(\mu, \phi(b^n)\mu) \ge d_{\partial X_{ab}}(\gamma, \phi(b^n)\gamma) - d_{\partial X_{ab}}(\mu, \gamma) - d_{\partial X_{ab}}(\phi(b^n)\mu, \phi(b^n)\gamma) \ge n+1-2d_{\partial X_{ab}}(\mu, \gamma).$$

Lemma 3.1 says that $|d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu) - d_{\partial X_{ab}}(\mu, \phi(b^n)\mu)| \le 4$. This together with (7) implies that

$$d_{\partial X_{ab}}(\mu, \phi(a^n b^n)\mu)) \succ n.$$

From this and the fact that $\mathcal{C}(\partial X_{ab})$ is quasi-isometric to $\mathbb{R}$, it is immediate that $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ has a subsequence converging to a point in $\partial\mathcal{C}(\partial X_{ab})$. □

Proposition 4.4 *The sequences $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n b^n))_{n\in\mathbb{N}}$ do not converge to the same point in* $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$.

Proof After passing to a subsequence, we may assume $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n b^n))_{n\in\mathbb{N}}$ converge to points p and q, respectively, in $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ and, by Lemma 4.3, that $(\pi_{\partial X_{ab}}(\phi(a^n b^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(\partial X_{ab})$. Lemmas 3.3 and 4.3 imply that q is in $\partial\mathrm{Mod}(S)$. Say $q = \sum_{Y\subseteq S} c_Y^q \lambda_Y^q$, where $c_Y^q \ge 0$ and $\lambda_Y^q \in \partial\mathcal{C}(Y)$ for all $Y \subseteq S$. Then Lemmas 3.3 and 4.3 also imply that $c_{\partial X_{ab}}^q > 0$.

Now, if p were in $\mathrm{Mod}(S)$, then we would be done since clearly then $p \ne q$. So we will assume that $p \in \partial\mathrm{Mod}(S)$, and let $p = \sum_{Y\subseteq S} c_Y^p \lambda_Y^p$. Now observe that, by Lemma 3.1 and Theorem 2.1,

$$d_{\partial X_{ab}}(\mu, \phi(a^n)\mu) \le d_{\partial X_{ab}}(\mu, \mu) + 4 \le 5.$$

Thus, $(\pi_{\partial X_{ab}}(\phi(a^n)\mu))_{n\in\mathbb{N}}$ does not limit to a point on $\partial\mathcal{C}(\partial X_{ab})$. So, by definition of the topology of $\mathrm{Mod}(S)\cup\partial\mathrm{Mod}(S)$, it must be that $c^p_{\partial X_{ab}}=0$. Since $c^q_{\partial X_{ab}}>0$, we see that $p\neq q$, which completes the proof. □

5 Koberda RAAGs

In this section we complete the proof of Theorem 1.2. Following this, we will discuss how to use similar techniques to prove a large class of Koberda embeddings do not extend.

Let α and β be the pair of intersecting curves on $S=\mathbb{H}^2/\Lambda$ depicted in Figure 3. Let Γ be the graph with $V(\Gamma)=\{a,b\}$ and no edges. For sufficiently large N, Theorem 2.6 says that the homomorphism

$$\phi\colon A(\Gamma)\to\mathrm{Mod}(S)\quad\text{defined by } \phi(a)={T_\alpha}^N \text{ and } \phi(b)={T_\beta}^N$$

is injective, where T_α and T_β denote Dehn twists about α and β, respectively. Throughout this section, we let μ be a fixed marking on S. Equip $A(\Gamma)$ with an HHS structure.

In this section we prove the following theorem, which will complete the proof of Theorem 1.2:

Theorem 5.1 *There exists $g\in A(\Gamma)$ such that the sequences $(a^n)_{n\in\mathbb{N}}$ and $(a^ng^n)_{n\in\mathbb{N}}$ converge to the same point in $\partial A(\Gamma)$, but $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^ng^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial\mathrm{Mod}(S)$.*

As a step towards proving Theorem 5.1, we prove the following lemma, in which we construct $g\in A(\Gamma)$:

Lemma 5.2 *There exist constants $A\geq 1$ and $B\geq 0$ and a word $g\in A(\Gamma)$ such that for all $n\geq 1$ we have $d_\eta(\mu,\phi(a^ng^n)\mu)\succ_{A,B} n$, where η is the curve shown in Figure 3. Consequently, after passing to a subsequence, $(\pi_\eta(\phi(a^ng^n)\mu))_{n\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(\eta)$.*

Proof We will prove that there exist constants c_1, c_2 and c_3 such that $g=b^{c_1}a^{c_2}b^{c_3}$ has the desired properties.

Let A be the annulus in Figure 3. Let $\widetilde{A}$ be a component of the preimage of A in $\mathbb{H}^2$. Let $\widetilde{\beta}$ be a component of the preimage of β such that a segment of $\widetilde{\beta}$ is in the boundary of $\widetilde{A}$, and let $\widetilde{\eta}$ denote the component of the preimage of η in the boundary of $\widetilde{A}$. Let

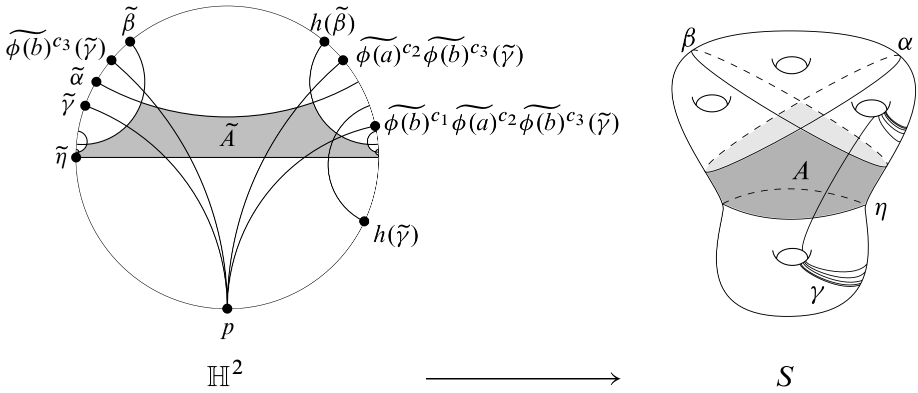

Figure 3: Curves α, β and η bounding an annulus A, and simple bi-infinite geodesic γ on surface S, and the universal cover $\mathbb{H}^2$ of S as in Lemma 5.2

$h \in \Lambda$ be a primitive isometry with axis $\tilde{\eta}$. Let $\tilde{\alpha}$ be the component of the preimage of α that links with $\tilde{\beta}$ and $h(\tilde{\beta})$ and contains a segment that is in the boundary of $\tilde{A}$.

Let Y_α be the component of $S-\alpha$ that contains η. To simplify arguments, we let $\phi(a)$ denote a representative in its isotopy class that fixes Y_α pointwise. Let $\widetilde{\phi(a)}\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the lift of $\phi(a)$ that fixes some point on $\tilde{\alpha}$. Similarly define Y_β to be the component of $S-\beta$ containing η, choose a representative in the isotopy class of $\phi(b)$ that fixes Y_β pointwise, and let $\widetilde{\phi(b)}$ be the lift of $\phi(b)$ that fixes some point on $\tilde{\beta}$. It then follows that

$$\widetilde{\phi(a)} = \mathbb{1} \quad \text{on } \tilde{Y}_\alpha \qquad \text{and} \qquad \widetilde{\phi(b)} = \mathbb{1} \quad \text{on } \tilde{Y}_\beta,$$

where for $i \in \{\alpha, \beta\}$ we let $\tilde{Y}_i$ denote the component of the preimage of Y_i in $\mathbb{H}^2$ whose boundary contains $\tilde{\imath}$. Observe that $\widetilde{\phi(i)}$ fixes the endpoints of $\tilde{\eta}$ for $i \in \{a, b\}$.

Choose a geodesic $\tilde{\gamma}$ in $\mathbb{H}^2$ that links with both $\tilde{\beta}$ and $\tilde{\eta}$ and maps to a simple bi-infinite geodesic in S. Further, suppose that $\tilde{\gamma} \cap \tilde{Y}_\alpha \cap \tilde{Y}_\beta$ is an infinite ray, and let p denote its endpoint on $\partial\mathbb{H}^2$. For example, take γ to be the simple bi-infinite geodesic in S with one end spiraling around a curve essential in $Y_\alpha \cap Y_\beta$ and the other end spiraling around a curve essential in $S - Y_\beta$ as in Figure 3, and take $\tilde{\gamma}$ to be an appropriate component of the preimage of γ. Observe that $\widetilde{\phi(a)}$ and $\widetilde{\phi(b)}$ must fix p.

Now choose $c_3 \in \mathbb{Z}$ so that $\widetilde{\phi(b)}^{c_3}(\tilde{\gamma})$ links with $\tilde{\alpha}$. Then choose $c_2 \in \mathbb{Z}$ so that $\widetilde{\phi(a)}^{c_2}\widetilde{\phi(b)}^{c_3}(\tilde{\gamma})$ links with $h(\tilde{\beta})$. Finally, choose $c_1 \in \mathbb{Z}$ so that

$$\widetilde{\phi(b)}^{c_1}\widetilde{\phi(a)}^{c_2}\widetilde{\phi(b)}^{c_3}(\tilde{\gamma})$$

links with $h(\tilde{\gamma})$. See Figure 3.

To simplify notation, define

$$g = b^{c_1} a^{c_2} b^{c_3} \in A(\Gamma) \quad \text{and} \quad \widetilde{\phi(g)} = \widetilde{\phi(b)}^{c_1} \widetilde{\phi(a)}^{c_2} \widetilde{\phi(b)}^{c_3}.$$

As in Lemma 4.3, we have that $\widetilde{\phi(g)}^n(\widetilde{\gamma})$ has endpoint p and links with $h^i(\widetilde{\gamma})$ for all $1 \le i \le n$, implying that $d_\eta(\gamma, \phi(g^n)\gamma) \ge n+1$. It follows that

$$\begin{aligned} (8) \qquad d_\eta(\mu, \phi(g^n)\mu) &\ge d_\eta(\gamma, \phi(g^n)\gamma) - d_\eta(\mu, \gamma) - d_\eta(\phi(g^n)\mu, \phi(g^n)\gamma) \\ &\ge n + 1 - 2d_\eta(\mu, \gamma). \end{aligned}$$

Now Lemma 3.1 says that $|d_\eta(\mu, \phi(a^n g^n)\mu) - d_\eta(\mu, \phi(g^n)\mu)| \le 4$. This together with (8) implies that $d_\eta(\mu, \phi(a^n g^n)\mu) \succ n$. From this and the fact that $\mathcal{C}(\eta)$ is quasi-isometric to $\mathbb{R}$, it is immediate that $(\pi_\eta(\phi(a^n g^n)\mu))_{n\in\mathbb{N}}$ has a subsequence converging to a point in $\partial\mathcal{C}(\eta)$. □

Proof of Theorem 5.1 Let $g \in A(\Gamma)$ be as in Lemma 5.2. By the discussion in Section 2.8, to show that $(a^n)_{n\in\mathbb{N}}$ and $(a^n g^n)_{n\in\mathbb{N}}$ converge to the same point $\partial A(\Gamma)$ it is enough to show that they converge to the same point in $\partial_G X$, where X is the Cayley graph of $A(\Gamma)$. Now the Gromov product satisfies

$$(a^i, a^j g^j)_1 = (a^i, a^j(b^{c_1} a^{c_2} b^{c_3})^j)_1 = \min(i, j) \to \infty \quad \text{as } i, j \to \infty.$$

Therefore, $\lim_{n\to\infty} a^n = \lim_{n\to\infty} a^n g^n$ in $\partial_G X$, as desired.

To finish this proof, we mimic the proof of Proposition 4.4. Replacing b with g, and ∂X_{ab} with η, and Lemma 4.3 with Lemma 5.2, we find that $(\phi(a^n))_{n\in\mathbb{N}}$ and $(\phi(a^n g^n))_{n\in\mathbb{N}}$ do not converge to the same point in $\partial\mathrm{Mod}(S)$. □

Our techniques used to prove Theorem 5.1 can be used to prove a more general statement on nonexistence of boundary maps for right-angled Artin groups that are not necessarily free groups. To prove this more general statement, one needs to understand HHS structures for all right-angled Artin groups. In the following theorem, by a *standard HHS structure on* $A(\Gamma)$, we mean one induced by a factor system generated by a rich family of subgraphs of Γ. We refer the reader to [3], specifically Proposition 8.3 and Remark 13.2, for details and to [7] for a general description of the corresponding HHS boundary. In the proof of the following theorem, we freely use definitions and notations used in [3; 7].

Theorem 5.3 *Let $\{\alpha_1, \dots, \alpha_k\}$ be any collection pairwise distinct of curves in S. Let Γ be the graph with $V(\Gamma) = \{s_1, \dots, s_k\}$ and $s_i s_j$ an edge in Γ if and only if*

$i(\alpha_i, \alpha_j) = 0$. Give $A(\Gamma)$ a standard HHS structure, or if $A(\Gamma)$ is a free group, any HHS structure. If there exist distinct intersecting curves α_i and α_j that do not fill S, then any corresponding Koberda embedding $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$ does not extend continuously to a map $\partial A(\Gamma) \to \mathrm{Mod}(S)$.

Proof Consider the subgraph Λ of Γ with $V(\Lambda) = \{s_i, s_j\}$. Contained in the Salvetti complex S_Γ associated to Γ there is a subcomplex that is the Salvetti complex associated to $A(\Lambda)$. We let $\widetilde{S}_\Lambda$ denote the lift of this subcomplex to the universal cover $\widetilde{S}_\Gamma$ of S_Γ that contains 1. Let $\mathcal{R}$ be a rich family of induced subgraphs of Γ, and let $\mathcal{F}$ be the corresponding factor system in $\widetilde{S}_\Gamma$. Lemma 8.4 of [3] tells us that

$$\mathcal{F}' = \{F \cap \widetilde{S}_\Lambda : F \in \mathcal{F}\}$$

is a factor system in $\widetilde{S}_\Lambda$. Associating $A(\Gamma)$ and $A(\Lambda)$ with $\widetilde{S}_\Gamma$ and $\widetilde{S}_\Lambda$, respectively, we equip each with the HHS structures corresponding to their respective factor systems. We first argue that the inclusion map $A(\Lambda) \to A(\Gamma)$ extends continuously to a map $\partial A(\Lambda) \to \partial A(\Gamma)$. If ϕ extends continuously to a map $\partial A(\Gamma) \to \partial\mathrm{Mod}(S)$, it will follow that $A(\Lambda) \to \mathrm{Mod}(S)$ extends continuously to a map $\partial A(\Lambda) \to \partial\mathrm{Mod}(S)$; we will show that this is impossible.

First, consider $A(\Lambda) \to A(\Gamma)$. Given $U \in \mathcal{F}'$ such that U is not a 0–cube, define $\pi(U)$ to be the parallelism class of the $\subseteq$–minimal $F \in \mathcal{F}$ such that $U = F \cap \widetilde{S}_\Lambda$. Observe that U and V are nested (respectively orthogonal) if and only if $\pi(U)$ and $\pi(V)$ are nested (respectively orthogonal). This together with Lemma 10.11 of [7] implies that $(A(\Lambda) \to A(\Gamma), \pi)$ is a hieromorphism. Theorem 5.6 of [7] gives a condition guaranteeing that a hieromorphism extends continuously. In our case, if the following claims are true, we can apply Theorem 5.6 to conclude that $A(\Lambda) \to A(\Gamma)$ extends continuously.

Claim 1 *π is injective.*

Proof of Claim 1 Suppose $U, V \in \mathcal{F}'$ and $\pi(U) = \pi(V)$. Then $\pi(U) \sqsubseteq \pi(V)$ and $\pi(V) \sqsubseteq \pi(U)$. Thus, $U \subseteq V$ and $V \subseteq U$, implying $U = V$, as desired. □

Claim 2 *If $[F] \in \overline{\mathcal{F}}$ is not a class of 0–cubes and there exists no $U \in \mathcal{F}'$ satisfying $\pi(U) = [F]$, then $\mathrm{diam}_{\widehat{C}F}(\pi_F(\widetilde{S}_\Lambda))$ is bounded above uniformly for some (any) $F \in [F]$.*

Proof of Claim 2 Let $[F] \in \overline{\mathcal{F}}$ be as in Claim 2. First, suppose there exists $F \in [F]$ such that $F \cap \widetilde{S}_\Lambda \neq \varnothing$. By Lemma 8.5 in [3], we have $\mathfrak{g}_F(\widetilde{S}_\Lambda) \subseteq F \cap \widetilde{S}_\Lambda$. If $F \cap \widetilde{S}_\Lambda$

is a 0–cube, then $\operatorname{diam}_{\widehat{C}F}(\pi_F(\widetilde{S}_\Lambda)) \le 1$, so the claim holds. Otherwise, there must exist $\bar{F} \in \mathcal{F}$ such that $\bar{F} \subsetneq F$ and $\bar{F} \cap \widetilde{S}_\Lambda = F \cap \widetilde{S}_\Lambda$. It follows that $C\bar{F}$ is coned off in $\widehat{C}F$ and that $\mathfrak{g}_F(\widetilde{S}_\Lambda) \subseteq \bar{F}$. This implies that $\operatorname{diam}_{\widehat{C}F}(\pi_F(\widetilde{S}_\Lambda)) \le 4$.

Now assume $F \cap \widetilde{S}_\Lambda = \varnothing$ for all $F \in [F]$. An argument like that in the proof of Proposition 8.3 of [3] shows that we can find $g \in A(\Gamma)$, $\Gamma' \in \mathcal{R}$, and $x \in A(\Lambda)$ so that $g\widetilde{S}_{\Gamma'} \in [F]$ and

$$\mathfrak{g}_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda) \subseteq g(\widetilde{S}_{\Gamma' \cap \Lambda \cap \operatorname{Lk}\bar{g}}) \subseteq g(\widetilde{S}_{\Gamma' \cap \operatorname{Lk}\bar{g}}), \tag{9}$$

where $\bar{g} = g^{-1}x$ and $\operatorname{Lk}\bar{g}$ denotes the link of $\bar{g}$. Now if $\Gamma' \cap \Lambda \cap \operatorname{Lk}\bar{g} = \varnothing$, then $\mathfrak{g}_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda) = \{g\}$, implying that $\operatorname{diam}_{\widehat{C}(g\widetilde{S}_{\Gamma'})}(\pi_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda)) \le 1$. Assume then that $\Gamma' \cap \Lambda \cap \operatorname{Lk}\bar{g} \ne \varnothing$. Then by definition of $\mathcal{R}$ and $\mathcal{F}$, we have that $\Gamma' \cap \operatorname{Lk}\bar{g} \in \mathcal{R}$ and $g(\widetilde{S}_{\Gamma' \cap \operatorname{Lk}\bar{g}}) \in \mathcal{F} - \{0\text{–cubes}\}$. If $g(\widetilde{S}_{\Gamma' \cap \operatorname{Lk}\bar{g}})$ is not a proper subcomplex of $g\widetilde{S}_{\Gamma'}$, then $\Gamma' \subseteq \operatorname{Lk}\bar{g}$, implying that $x\widetilde{S}_{\Gamma'}$ is parallel to $g\widetilde{S}_{\Gamma'}$ (see Lemma 2.4 in [3]). But this cannot be because $(x\widetilde{S}_{\Gamma'}) \cap \widetilde{S}_\Lambda = x(\widetilde{S}_{\Gamma' \cap \Lambda}) \ne \varnothing$ and no factor parallel to $g\widetilde{S}_{\Gamma'}$ intersects $\widetilde{S}_\Lambda$ nontrivially. Therefore, $g(\widetilde{S}_{\Gamma' \cap \operatorname{Lk}\bar{g}})$ must be a proper subcomplex of $g\widetilde{S}_{\Gamma'}$. Thus, $Cg(\widetilde{S}_{\Gamma' \cap \operatorname{Lk}\bar{g}})$ is coned off in $\widehat{C}(g\widetilde{S}_{\Gamma'})$. This together with (9) implies that $\operatorname{diam}_{\widehat{C}(g\widetilde{S}_{\Gamma'})}(\pi_{g\widetilde{S}_{\Gamma'}}(\widetilde{S}_\Lambda)) \le 4$, completing the proof of Claim 2. □

We now argue that $A(\Lambda) \to \operatorname{Mod}(S)$ does not extend continuously to a map $\partial A(\Lambda) \to \partial\operatorname{Mod}(S)$. Let η denote a geodesic representative of an essential boundary component of a small regular neighborhood of $\alpha_i \cup \alpha_j$. Using the proof techniques of Lemma 5.2, we can construct $g \in A(\Lambda)$ so that $d_\eta(\mu, \phi(s_i^n g^n)\mu)$ grows linearly in n. For later convenience, we construct g so that when written in reduced form, the first letter of g is $s_j^{\pm 1}$. As in Proposition 4.4, we see that the sequences $(\phi(s_i^n))$ and $(\phi(s_i^n g^n))$ do not converge to the same point in $\operatorname{Mod}(S) \cup \partial\operatorname{Mod}(S)$. Now observe that (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial_G A(\Lambda)$. Therefore, by the discussion in Section 2.8, (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial A(\Lambda)$. We have now established that $A(\Lambda) \to \operatorname{Mod}(S)$ does not extend continuously to a map $\partial A(\Lambda) \to \partial\operatorname{Mod}(S)$. Therefore, $A(\Gamma) \to \operatorname{Mod}(S)$ does not extend continuously when $A(\Gamma)$ is equipped with a standard HHS structure.

Now suppose $A(\Gamma)$ is a free group equipped with any HHS structure. Then, by the discussion in Section 2.8, because (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial_G A(\Gamma)$, we have that (s_i^n) and $(s_i^n g^n)$ converge to the same point in $\partial A(\Gamma)$. Because $(\phi(s_i^n))$ and $(\phi(s_i^n g^n))$ do not converge to the same point in $\partial\operatorname{Mod}(S)$, it follows that $A(\Gamma) \to \operatorname{Mod}(S)$ does not extend continuously. □

6 Existence of boundary maps for some free groups

In this section, we show that a class of embeddings of free groups in $\mathrm{Mod}(S)$, which includes a class of Koberda embeddings and a class of CLM embeddings, extend continuously.

Throughout this section, let Γ be the graph with $V(\Gamma)=\{s_1,\dots,s_k\}$ and no edges, and let $A(\Gamma)$ denote the corresponding right-angled Artin group (a rank k free group). Equip $A(\Gamma)$ with an HHS structure. Let $\{X_1,\dots,X_k\}$ be a collection of distinct, pairwise overlapping and pairwise filling subsurfaces of S and $\{f_1,\dots,f_k\}$ a collection of mapping classes such that f_i is fully supported on X_i. Let μ be a fixed marking on S. The main theorem of this section is the following, which implies the remaining direction of Theorem 1.3 in the introduction:

Theorem 6.1 *Let $A(\Gamma)$ be the rank k free group equipped with any HHS structure. Let $\{X_1,\dots,X_k\}$ be a collection of distinct, pairwise overlapping and pairwise filling subsurfaces of S, and $\{f_1,\dots,f_k\}$ a collection of mapping classes such that f_i is fully supported on X_i. There exists a $C>0$ such that, if $\tau_{X_i}(f_i)\geq C$ for all i, then the homomorphism*

$$\phi\colon A(\Gamma)\to \mathrm{Mod}(S) \quad \text{defined by } \phi(s_i)=f_i \text{ for all } i$$

is a quasi-isometric embedding and extends continuously to a map $\partial A(\Gamma)\to\partial\mathrm{Mod}(S)$.

We emphasize the arguments we will use to establish that ϕ is a quasi-isometric embedding are essentially the same as those used by Clay, Leininger and Mangahas to prove Theorem 2.5. In particular, when the X_i are all nonannular, that ϕ is a quasi-isometric embedding is Theorem 2.5. To prove Theorem 6.1, we require the following proposition:

Proposition 6.2 *There exists $K>0$ such that the following holds: For each $1\leq i\leq k$, assume $\tau_{X_i}(f_i)\geq 2K$. Let $\phi\colon A(\Gamma)\to\mathrm{Mod}(S)$ be the homomorphism defined by $\phi(s_i)=f_i$ for all i. Consider $g_1\dots g_k\in A(\Gamma)$, where for each i we have $g_i=x_i^{e_i}$ for some $x_i\in\{s_1^{\pm1},\dots,s_k^{\pm1}\}$ and $e_i>0$, and $x_i\neq x_{i+1}$, and $x_1^{e_1}\dots x_k^{e_k}$ is a reduced word. Let Y_i be the subsurface of S that fully supports $\phi(x_i)$. Then:*

1. *For each $1\leq i\leq k$, we have $d_{\phi(g_1\dots g_{i-1})Y_i}(\mu,\phi(g_1\dots g_k)\mu)\geq Ke_i$.*
2. *For all $1\leq i<j\leq k$, we have $\phi(g_1\dots g_{i-1})Y_i\prec\phi(g_1\dots g_{j-1})Y_j$, where $\prec$ denotes the partial order on $\Omega(K,\mu,\phi(g_1\dots g_k)\mu)$.*
3. *The homomorphism $\phi\colon A(\Gamma)\to\mathrm{Mod}(S)$ is a quasi-isometric embedding.*

Proof Define $K = K_0 + 20 + 2\max\{d_{X_i}(\mu, \partial X_j) : 1 \le i, j \le k \text{ and } i \ne j\}$, where K_0 is maximum of the constants in Theorem 6.12 of [11] and Theorem 2.4. Statements (1) and (2) of this proposition are essentially Theorem 5.2 in [6]. The difference is that Theorem 5.2 does not allow for the homomorphism to send a generator to a power of a Dehn twist. The only obstruction to Theorem 5.2 holding for homomorphisms ϕ of this type is the following: Suppose X_i is the subsurface that fully supports $\phi(s_i)$, and let $\sigma \in A(\Gamma)$ be a nonempty word in letters commuting with s_i, not including s_i. If X_i is nonannular, then $d_{X_i}(\phi(\sigma)\mu', \mu'') = d_{X_i}(\mu', \mu'')$ for any markings μ' and μ''. This not necessarily true if X_i is an annulus. However, this issue does not arise for us because $A(\Gamma)$ a free group implies no such σ exists. Thus, the arguments used to prove Theorem 5.2 in [6] also prove our statements (1) and (2). The proof of our statement (3) is the same as the proof in [6] of Theorem 2.5, using our statement (1) instead of their Theorem 5.2. □

The proof of the next lemma is essentially contained in the proof of Theorem 6.1 in [6]. We include a proof here for completeness.

Lemma 6.3 *Let $\phi\colon A(\Gamma) \to \mathrm{Mod}(S)$, $g_1 \dots g_k \in A(\Gamma)$ and Y_i be as in Proposition 6.2. Let $\mathcal{G}$ be a geodesic in $\mathcal{C}(S)$ with one end in $\pi_S(\mu)$ and the other end in $\pi_S(\phi(g_1 \dots g_k)\mu)$. Then, for each $1 \le i \le k$, there exists a curve γ_i on $\mathcal{G}$ such that $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(\gamma_i) = \varnothing$. If $|i-j| \ge 3$ and γ_i and γ_j are two such curves, then $\gamma_i \ne \gamma_j$.*

Proof Fix $1 \le i \le k$. By way of contradiction, suppose for all curves v on $\mathcal{G}$, we have $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(v) \ne \varnothing$. Then Theorems 2.4 and 2.1 together imply that

$$d_{\phi(g_1 \dots g_{i-1})Y_i}(\mu, \phi(g_1 \dots g_k)\mu) \le 4 + K_0.$$

But Proposition 6.2 says $d_{\phi(g_1 \dots g_{i-1})Y_i}(\mu, \phi(g_1 \dots g_k)\mu) \ge K > K_0 + 4$, a contradiction. Thus, there must exist a curve γ_i on $\mathcal{G}$ such that $\pi_{\phi(g_1 \dots g_{i-1})Y_i}(\gamma_i) = \varnothing$, as desired. Note that this implies that γ_i and $\partial\phi(g_1 \dots g_{i-1})Y_i$ form a multicurve.

Now consider γ_i and γ_j, where $1 \le i < j \le k$ and $|i-j| \ge 3$. We will show that γ_i and γ_j are distinct curves. To the contrary, suppose $\gamma_i = \gamma_j$. Because of the filling assumption on $\{X_1, \dots, X_k\}$, the pair of subsurfaces Y_{i+1} and Y_{i+2} fill S. Thus, $\phi(g_1 \dots g_{i+1})Y_{i+1} = \phi(g_1 \dots g_i)Y_{i+1}$ and $\phi(g_1 \dots g_{i+1})Y_{i+2}$ are also a pair of subsurfaces that fill S. Thus, it must be that $\pi_{\phi(g_1 \dots g_{n-1})Y_n}(\gamma_i) \ne \varnothing$ for some $n \in \{i+1, i+2\}$. In any case, $i < n < j$.

In the remainder of this proof, to simplify notation, for each l we define $\overline{Y}_l = \phi(g_1 \dots g_{l-1})Y_l$. By Proposition 6.2, we have

$$\overline{Y}_i \prec \overline{Y}_n \prec \overline{Y}_j,$$

where $\prec$ is the partial order on $\Omega(K, \mu, \phi(g_1 \dots g_k)\mu)$. In particular, these three subsurfaces are pairwise overlapping. This together with the assumption that $\gamma_i = \gamma_j$ and Theorem 2.1 implies that

$$d_{\overline{Y}_n}(\partial \overline{Y}_i, \partial \overline{Y}_j) \le d_{\overline{Y}_n}(\partial \overline{Y}_i, \gamma_i) + d_{\overline{Y}_n}(\gamma_j, \partial \overline{Y}_j) \le 2 + 2 = 4.$$

It follows from this and the definition of $\prec$ that

$$\begin{aligned} d_{\overline{Y}_n}(\mu, \phi(g_1 \dots g_k)\mu) &\le d_{\overline{Y}_n}(\mu, \partial \overline{Y}_i) + d_{\overline{Y}_n}(\partial \overline{Y}_i, \partial \overline{Y}_j) + d_{\overline{Y}_n}(\partial \overline{Y}_j, \phi(g_1 \dots g_k)\mu) \\ &\le 4 + 4 + 4 = 12. \end{aligned}$$

But this cannot be, because $d_{\overline{Y}_n}(\mu, \phi(g_1 \dots g_k)\mu) \ge K \ge 20$ by Proposition 6.2. Therefore, γ_i and γ_j are distinct curves. □

We have now developed the tools we will need to prove Theorem 6.1.

Proof of Theorem 6.1 Define $C = 2K$, where K is as in Proposition 6.2 and, for each $1 \le i \le k$, assume that $\tau_{X_i}(f_i) \ge C$. By Proposition 6.2, ϕ is a quasi-isometric embedding.

Let X denote the Cayley graph of $A(\Gamma)$. Choose $x \in \partial_G X$. Let γ be the infinite geodesic ray in X based at 1 limiting to x in $\partial_G X$. We think of γ as an infinite word of the form $y_1 y_2 y_3 \dots$, where each $y_i \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ and the word $y_1 y_2 \dots y_i$ is a reduced word for all i. By construction, the sequence $(y_1 \dots y_n)$ converges to x in $X \cup \partial_G X$. Let (h_n) be another sequence in $A(\Gamma)$ that converges to x in $X \cup \partial_G X$. We will show that $(\phi(h_n))$ and $(\phi(y_1 \dots y_n))$ converge to the same point in $\partial \mathrm{Mod}(S)$. By the discussion in Section 2.8, this will prove the theorem. We will consider two cases: (1) there does not exist $N \ge 1$ such that $y_i = y_N$ for all $i \ge N$, and (2) such an N exists. In both cases, we will assume each h_n is written in the form $h_n = g_{n,1} \dots g_{n,N(n)}$, where for all i we have $g_{n,i} = x_{n,i}^{e_{n,i}}$ for some $e_{n,i} > 0$ and $x_{n,i} \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ satisfying $x_{n,i} \ne x_{n,i+1}$, and $x_{n,1}^{e,1} \dots x_{n,N(n)}^{e_{n,N(n)}}$ is a reduced word.

Case 1 Suppose there does not exist $N \ge 1$ such that $y_i = y_N$ for all $i \ge N$. Then we can think of γ as an infinite word of the form $g_1 g_2 g_3 \dots$, where $g_i = x_i^{e_i}$ for some $e_i > 0$ and $x_i \in \{s_1^{\pm 1}, \dots, s_k^{\pm 1}\}$ satisfying $x_i \ne x_{i+1}$, and $x_1^{e_1} \dots x_i^{e_i}$ is a reduced word for all i. Define Y_i to be the subsurface that fully supports $\phi(x_i)$. For short, we let $\overline{Y}_i$ denote $\phi(g_1 \dots g_{i-1})Y_i$.

Because (h_n) and $(y_1 \dots y_n)$ converge to the same point in $\partial_G X$ and X is a tree, h_n and $y_1 \dots y_n$ must agree on longer and longer initial segments as $n \to \infty$. In particular, given $L \geq 1$, there exists M such that, for all $n \geq M$, we have $g_{n,1} \dots g_{n,L} = g_1 \dots g_L$. Consider $n \geq M$ and $k \geq e_1 + \dots + e_L$. Choose a curve $\beta \in \mathrm{base}(\mu)$. Given $\sigma \in A(\Gamma)$, let $\mathcal{G}(\sigma)$ denote some choice of geodesic in $\mathcal{C}(S)$ with endpoints β and $\phi(\sigma)\beta$. By Lemma 6.3, for all $1 \leq i \leq L$ there exist curves γ_i and γ_i' on $\mathcal{G}(y_1 \dots y_k)$ and $\mathcal{G}(h_n)$, respectively, such that $\pi_{\overline{Y}_i}(\gamma_i) = \varnothing$ and $\pi_{\overline{Y}_i}(\gamma_i') = \varnothing$. Observe that

$$d_S(\gamma_i, \partial \overline{Y}_i) \leq 1 \quad \text{and} \quad d_S(\gamma_i', \partial \overline{Y}_i) \leq 1.$$

Choose γ_r to be the curve in $\{\gamma_i : 1 \leq i \leq L\}$ closest to $\phi(y_1 \dots y_k)\beta$. Lemma 6.3 tells us that if $|i - j| \geq 3$, then $\gamma_i \neq \gamma_j$. So necessarily $d_S(\beta, \gamma_r) \geq \frac{1}{3}L$. Thus, the Gromov product, computed in $\mathcal{C}(S)$, is

$$\begin{aligned}
&(\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta)_\beta \\
&\quad = \tfrac{1}{2}[d_S(\beta, \phi(y_1 \dots y_k)\beta) + d_S(\beta, \phi(h_n)\beta) - d_S(\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta)] \\
&\quad \geq \tfrac{1}{2}\big[d_S(\beta, \gamma_r) + d_S(\gamma_r, \phi(y_1 \dots y_k)\beta) + d_S(\beta, \gamma_r') + d_S(\gamma_r', \phi(h_n)\beta) \\
&\qquad\quad - \big(d_S(\phi(y_1 \dots y_k)\beta, \gamma_r) + d_S(\gamma_r, \partial \overline{Y}_r) + d_S(\partial \overline{Y}_r, \gamma_r') + d_S(\gamma_r', \phi(h_n)\beta)\big)\big] \\
&\quad \geq \tfrac{1}{2}[d_S(\beta, \gamma_r) + d_S(\beta, \gamma_r') - 2] \\
&\quad \geq \tfrac{1}{2}\big(\tfrac{1}{3}L - 2\big).
\end{aligned}$$

It follows that

$$\liminf_{k,n \to \infty} (\phi(y_1 \dots y_k)\beta, \phi(h_n)\beta)_\beta = \infty. \tag{10}$$

Because (h_n) is an arbitrary sequence converging to x, we could have taken it to be $(y_1 \dots y_n)$. Thus, (10) tells us two things: (1) $(\phi(y_1 \dots y_n)\mu)$ converges to a point in $\partial\mathcal{C}(S)$, and (2) $(\phi(y_1 \dots y_n)\mu)$ and $(\phi(h_n)\mu)$ converge to the same point in $\partial\mathcal{C}(S)$. By definition of the topology on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$, this tells us that $(\phi(y_1 \dots y_n))$ and $(\phi(h_n))$ converge to the same point in $\partial\mathrm{Mod}(S)$.

Case 2 Assume there exists $N \geq 1$ such that $y_i = y_N$ for all $i \geq N$. Corollary 6.2 in [7] tells us that the action of $\mathrm{Mod}(S)$ by left multiplication extends to an action of $\mathrm{Mod}(S)$ on $\mathrm{Mod}(S) \cup \partial\mathrm{Mod}(S)$ by homeomorphisms. Consequently, if we can show that $(\phi((y_1 \dots y_{N-1})^{-1}h_n))_{n\in\mathbb{N}}$ and $(\phi(y_N \dots y_n))_{n\in\mathbb{N}}$ converge to the same point in $\partial\mathrm{Mod}(S)$, then $(\phi(h_n))_{n\in\mathbb{N}}$ and $(\phi(y_1\dots y_n))_{n\in\mathbb{N}}$ must converge to the same point in $\partial\mathrm{Mod}(S)$. Furthermore, $((y_1\dots y_{N-1})^{-1}h_n)_{n\in\mathbb{N}}$ and $(y_N\dots y_n)_{n\in\mathbb{N}}$ converge to the same point in $\partial_G X$. Thus, without loss of generality we assume $N=1$. By our assumption, $y_1\dots y_n = y_1^n$ for all n.

Let Y be the subsurface that fully supports $\phi(y_1)$, and let $\partial Y = \{\beta_1,\dots,\beta_l\}$. Then

$$\text{(11)}\quad \lim_{n\to\infty}\frac{d_Y(\mu,\phi(y_1^n)\mu)}{n}>0 \quad\text{and}\quad \pi_Y(\phi(y_1^n)\mu)\to\lambda_Y \quad\text{for some } \lambda_Y\in\partial\mathcal{C}(Y).$$

Further observe that, for all i,

$$\text{(12)}\qquad \lim_{n\to\infty}\frac{d_{\beta_i}(\mu,\phi(y_1^n)\mu)}{n}\ge 0.$$

If (12) is an equality, let λ_i be any point in $\partial\mathcal{C}(\beta_i)$. Otherwise, define $\lambda_i\in\partial\mathcal{C}(\beta_i)$ to be $\lim_{n\to\infty}\pi_{\beta_i}(\phi(y_1^n)\mu)$. For all subsurfaces W disjoint from Y and not an annulus with core curve in ∂Y, Lemma 3.1 and Theorem 2.1 imply that $d_W(\mu,\phi(y_1^n)\mu)\le d_W(\mu,\mu)+4\le 6$. Consequently,

$$\lim_{n\to\infty}\phi(y_1^n)=c_Y\lambda_Y+\sum_{i=1}^{l}c_i\lambda_i,$$

where

$$c_Y+\sum_{i=1}^{l}c_i=1 \quad\text{and}\quad \frac{c_i}{c_Y}=\lim_{n\to\infty}\frac{d_{\beta_i}(\mu,\phi(y_1^n)\mu)}{d_Y(\mu,\phi(y_1^n)\mu)}.$$

Because (h_n) and (y_1^n) converge to the same point in $\partial_G X$, given any $L\ge 1$, for all sufficiently large n we have $x_{n,1}=y_1$ and $e_{n,1}\ge L$. So, by removing finitely many initial terms from (h_n), for convenience we may assume that $g_{n,1}=y_1^{e_{n,1}}$ for all n. Observe that $e_{n,1}\to\infty$ as $n\to\infty$. It is immediate from this and the definition of the topology of $\mathrm{Mod}(S)\cup\partial\mathrm{Mod}(S)$ that $\lim_{n\to\infty}\phi(g_{n,1})=\lim_{n\to\infty}\phi(y_1^n)$. Thus, to finish the proof, we must show $\lim_{n\to\infty}\phi(g_{n,1})=\lim_{n\to\infty}\phi(h_n)$. By passing to subsequences, we may assume that either $N(n)=1$ for all n or $N(n)\ge 2$ for all n. If the former holds, then $h_n=g_{n,1}$, and we are done. Assume then that $N(n)\ge 2$ for all n. To proceed, we require the following claims:

Claim 1 *$d_Y(\phi(g_{n,1})\mu,\phi(h_n)\mu)$ is bounded above, independent of n.*

Claim 2 *Let W be a subsurface that is disjoint from Y. Then $d_W(\phi(g_{n,1})\mu,\phi(h_n)\mu)$ is bounded above, independent of n.*

We postpone the proofs of these claims and for now assume they are true. First, observe that Claim 1 and (11) imply that $\pi_Y(\phi(h_n)\mu) \to \lambda_Y$. If Inequality (12) is strict, then Claim 2 implies that $\pi_{\beta_i}(\phi(h_n)\mu) \to \lambda_i$. Further observe that Claims 1 and 2 imply that, for all W disjoint from Y,

$$\lim_{n\to\infty} \frac{d_W(\mu, \phi(g_{n,1})\mu)}{d_Y(\mu, \phi(g_{n,1})\mu)} = \frac{\lim_{n\to\infty} d_W(\mu, \phi(g_{n,1})\mu)/e_{n,1}}{\lim_{n\to\infty} d_Y(\mu, \phi(g_{n,1})\mu)/e_{n,1}} = \frac{\lim_{n\to\infty} d_W(\mu, \phi(h_n)\mu)/e_{n,1}}{\lim_{n\to\infty} d_Y(\mu, \phi(h_n)\mu)/e_{n,1}} = \lim_{n\to\infty} \frac{d_W(\mu, \phi(h_n)\mu)}{d_Y(\mu, \phi(h_n)\mu)}.$$

It follows that $\lim_{n\to\infty} \phi(g_{n,1}) = \lim_{n\to\infty} \phi(h_n)$, as desired.

To finish the proof, we will now prove Claims 1 and 2. For each n, let Z_n denote the subsurface that fully supports $\phi(x_{n,2})$.

Proof of Claim 1 Fix $n \geq 1$. Because Y fully supports $\phi(x_{n,1})$, by Proposition 6.2, we know $Y \prec \phi(g_{n,1})Z_n$, where $\prec$ denotes the partial order on $\Omega(K, \mu, \phi(h_n)\mu)$. Thus, $d_Y(\partial\phi(g_{n,1})Z_n, \phi(h_n)\mu) \leq 4$. Therefore,

$$\begin{aligned} d_Y(\phi(g_{n,1})\mu, \phi(h_n)\mu) &\leq d_Y(\phi(g_{n,1})\mu, \partial\phi(g_{n,1})Z_n) + d_Y(\partial\phi(g_{n,1})Z_n, \phi(h_n)\mu) \\ &\leq d_Y(\mu, \partial Z_n) + 4. \end{aligned}$$

There are finitely many possibilities for Z_n, so this completes the proof of Claim 1. □

Proof of Claim 2 Fix $n \geq 1$. Because Y and Z_n fill S, and Y and W are disjoint, it must be that $\pi_{Z_n}(\partial W) \neq \varnothing$. There are two cases to consider: (1) $W \pitchfork Z_n$, and (2) $W \subsetneq Z_n$. First, suppose that $W \pitchfork Z_n$. It then follows from Proposition 6.2, Theorem 2.1 and the definition of K that

$$\begin{aligned} &d_{Z_n}(\partial W, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\qquad \geq d_{Z_n}(\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu) - d_{Z_n}(\partial Y, \partial W) - d_{Z_n}(\mu, \partial Y) \\ &\qquad \geq K - 2 - \tfrac{1}{2}K \geq 10. \end{aligned}$$

Thus Theorem 2.3 implies that $d_W(\partial Z_n, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \leq 4$. From this and Theorem 2.2 we find that

$$\begin{aligned} d_W(\phi(g_{n,1})\mu, \phi(h_n)\mu) &= d_W(\mu, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\leq d_W(\mu, \partial Z_n) + d_W(\partial Z_n, \phi(g_{n,2} \dots g_{n,N(n)})\mu) \\ &\leq 4 \max\{d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu_i) : 1 \leq i \leq k\} + 4, \end{aligned} \tag{13}$$

where μ_i is a fixed choice of marking with $\partial X_i \subseteq \mathrm{base}(\mu_i)$ for each $1 \le i \le k$. This provides a uniform bound in the case that $W \pitchfork Z_n$.

Now suppose that $W \subsetneq Z_n$. First, observe that because Z_n fully supports $\phi(x_{n,2})$, the sequence $(\pi_{Z_n}(\phi(x_{n,2})^m \mu))_{m\in\mathbb{N}}$ converges to a point in $\partial\mathcal{C}(Z_n)$. Thus, by Corollary 2.7 there exists a constant M, which depends on W and $x_{n,2}$, such that $d_W(\mu, \phi(g_{n,2})\mu) \le M$ for all n. Note that there are only finitely many possibilities for $x_{n,2}$, so M can be chosen to be independent of n. This implies that

$$\begin{aligned} d_W(\phi(g_{n,1})\mu, \phi(h_n)\mu) &\le d_W(\mu, \phi(g_{n,2}\dots g_{n,N(n)})\mu) \\ &\le d_W(\mu, \phi(g_{n,2})\mu) + d_W(\phi(g_{n,2})\mu, \phi(g_{n,2}\dots g_{n,N(n)})\mu) \\ &\le M + d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3}\dots g_{n,N(n)})\mu). \end{aligned}$$

Now, if $N(n) = 2$, then we can apply Theorem 2.1 to see that

$$d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3}\dots g_{n,N(n)})\mu) = d_{\phi(g_{n,2})^{-1}W}(\mu, \mu) \le 2,$$

and Claim 2 is established. Suppose then that $N(n) \ge 3$. Let V_n denote the subsurface that fully supports $\phi(x_{n,3})$. Observe that, because $\tau_{Z_n}(\phi(x_{n,2})) \ge 2K$ and ∂Y and ∂W form a multicurve, we have

$$\begin{aligned} &d_{Z_n}(\partial\phi(g_{n,2})^{-1}W, \partial V_n) \\ &\qquad \ge d_{Z_n}(\partial W, \partial\phi(g_{n,2})^{-1}W) - d_{Z_n}(\partial W, \partial Y) - d_{Z_n}(\mu, \partial Y) - d_{Z_n}(\mu, \partial V_n) \\ &\qquad \ge 2K - 2 - \tfrac{1}{2}K - \tfrac{1}{2}K > 2. \end{aligned}$$

This together with Theorem 2.2 establishes that $\partial\phi(g_{n,2})^{-1}W$ and ∂V_n do not form a multicurve. Thus, $\phi(g_{n,2})^{-1}W \pitchfork V_n$. So, to bound

$$d_{\phi(g_{n,2})^{-1}W}(\mu, \phi(g_{n,3}\dots g_{n,N(n)})\mu)$$

from above independent of n, we can use the same techniques used above to bound $d_W(\mu, \phi(g_{n,2}\dots g_{n,N(n)})\mu)$ when $W \pitchfork Z_n$. This completes the proof of Claim 2, and thus the proof of Theorem 6.1. □

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Received: 17 November 2016 Revised: 17 July 2017