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Finite Dehn surgeries on knots in S^3

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We show that on a hyperbolic knot K in S^3, the distance between any two finite surgery slopes is at most 2, and consequently, there are at most three nontrivial finite surgeries. Moreover, in the case where K admits three nontrivial finite surgeries, K must be the pretzel knot $P(-2,3,7)$. In the case where K admits two noncyclic finite surgeries or two finite surgeries at distance 2, the two surgery slopes must be one of ten or seventeen specific pairs, respectively. For D–type finite surgeries, we improve a finiteness theorem due to Doig by giving an explicit bound on the possible resulting prism manifolds, and also prove that $4m$ and $4m+4$ are characterizing slopes for the torus knot $T(2m+1,2)$ for each $m \geq 1$.

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1 Introduction

For a knot K in S^3, its set of slopes $\{p/q : p, q \in \mathbb{Z},\ (p,q)=1\}$ will be parametrized by the standard meridian/longitude coordinates of K. Recall that the distance between two slopes p_1/q_1, p_2/q_2 of K is the number $|p_1q_2-p_2q_1|$. A Dehn surgery on a knot $K \subset S^3$ with slope p/q is called a cyclic or finite surgery if the resulting manifold, which we denote by $S^3_K(p/q)$, has cyclic or finite fundamental group, respectively. By Gabai [23], $S^3_K(0)$ has cyclic fundamental group only when K is the trivial knot. It follows that cyclic surgery on nontrivial knots in S^3 is equivalent to finite cyclic surgery. Due to Perelman's resolution of Thurston's geometrization conjecture [47; 48], a connected closed 3–manifold has finite fundamental group if and only if it is a spherical space form. For a spherical space form Y, it has cyclic fundamental group if and only if it is a lens space, and it has noncyclic fundamental group if and only if it has a Seifert fibered structure whose base orbifold is $S^2(a,b,c)$, a 2–sphere with three cone points of orders $a \leq b \leq c$, satisfying $1/a+1/b+1/c>1$; ie (a,b,c) is either $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, or $(2,2,n)$ for some integer $n>1$. Correspondingly, we say that a spherical space form Y or its fundamental group is C–type, T–type, O–type, I–type, or D–type if Y is a lens space or a Seifert-fibered space with base

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orbifold $S^2(2,3,3)$, $S^2(2,3,4)$, $S^2(2,3,5)$, or $S^2(2,2,n)$, respectively. We shall also refine finite surgeries (slopes) into C–type (often called cyclic), T–type, O–type, I–type and D–type accordingly.

If a nonhyperbolic knot in S^3 admits a nontrivial finite surgery, then the knot is either a torus knot or a cable over a torus knot (see Boyer and Zhang [9]), and finite surgeries on torus knots and on cables over torus knots are classified in Moser [39] and Bleiler and Hodgson [6], respectively. Concerning finite surgeries on hyperbolic knots in S^3, we recall the following:

Known Facts 1.1 Let $K \subset S^3$ be any fixed hyperbolic knot.

(1) Any nontrivial cyclic surgery slope of K must be an integer, K has at most two nontrivial cyclic surgery slopes, and if two, they are consecutive integers; see Culler, Gordon, Luecke and Shalen [16].

(2) Any finite surgery slope of K must be either an integer or a half-integer [9], the distance between any two finite surgery slopes of K is at most 3, and K has at most four nontrivial finite surgery slopes (see Boyer and Zhang [13]); consequently, K has at most one half-integer finite surgery slope.

(3) The distance between a finite surgery slope and a cyclic surgery slope on K is at most 2 [9].

(4) Any D–type finite surgery slope of K must be an integer. There is at most one D–type finite surgery slope on K. If there is a D–type finite surgery on K, then there is at most one nontrivial cyclic surgery on K, and the D–type finite surgery slope and the cyclic surgery slope are consecutive integers [9].

(5) Any O–type finite surgery slope of K must be an integer. If there is an O–type finite surgery on K, then there is at most one nontrivial cyclic surgery on K, and the O–type finite surgery slope and the cyclic surgery slope are consecutive integers [9].

(6) There are at most two T–type finite surgery slopes on K, and if two, one is integral, the other is half-integral, and their distance is 3 [9].

The above results were obtained by using classical techniques, mostly those derived from $\mathrm{PSL}_2(\mathbb{C})$–representations of 3–manifold groups, hyperbolic geometry and geometric and combinatorial topology in dimension 3. Recently, new progress on finite surgeries on knots in S^3 has been made through applications of Floer homology theory. Note that up to replacing a knot K by its mirror image $-K$, we may and shall assume that any finite surgery slope on a nontrivial knot in S^3 has positive sign. We recall the following:

Known Facts 1.2 (1) If a knot K in S^3 admits a nontrivial finite surgery, then K is a fibered knot; see Ni [40].

(2) If a nontrivial knot K in S^3 admits a nontrivial finite surgery slope p/q, then $p/q \geq 2g(K)-1$, where $g(K)$ is the Seifert genus of K (see Ozsváth and Szabó [46]), and the nonzero coefficients of the Alexander polynomial of K are alternating ± 1 (see Ozsváth and Szabó [44]).

(3) If a knot K in S^3 admits a T–type, O–type or I–type finite surgery with an integer slope, then the surgery slope is one of the finitely many integers listed in Tables 1–3 in Section 2, there is a sample knot K_0 (also listed in these tables) on which the same surgery slope yields the same spherical space form, and K and K_0 have the same knot Floer homology; see Gu [27].

(4) If a knot K in S^3 admits a T–type or I–type finite surgery with a half-integer slope, then the surgery slope is one of the ten slopes listed in Table 4 in Section 2 (with the two on the trefoil knot omitted as they can only be realized on the trefoil knot), there is a sample knot K_0 (also listed in the table) on which the same surgery slope yields the same spherical space form, and K and K_0 have the same knot Floer homology; see Li and Ni [36].

(5) If a knot K in S^3 admits an integer D–type finite surgery slope $p \leq 32$, then p is one of the slopes listed in Table 5 in Section 2, there is a sample knot K_0 (also listed in the table) on which the same surgery slope yields the same D–type spherical space form, and K and K_0 have the same knot Floer homology; see Doig [18]. (Also see Ballinger, Hsu, Mackey, Ni, Ochse and Vafaee [2] for more recent progress on D–type realization problem.)

(6) If a knot K in S^3 admits a cyclic surgery with an integer slope p, then there is a Berge knot K_0 (given in Berge [4]) such that $S^3_K(p) = S^3_{K_0}(p)$, and K and K_0 have the same knot Floer homology; see Greene [26].

(7) If p is a cyclic surgery slope for a hyperbolic knot K in S^3, then $p = 14$ or $p \geq 18$. Moreover, if $p \geq 4g(K)-1$, then K is a Berge knot; see Baker [1].

The purpose of this paper is to update and improve results on finite surgeries on hyperbolic knots in S^3, applying various techniques and results combined together.

As recalled in Known Facts 1.2(4), there are only ten specific half-integer slopes, listed in Table 4, each of which could possibly be a finite surgery slope for some hyperbolic knot in S^3. Our first result excludes two of them.

Theorem 1.3 *Neither* $17/2$ *nor* $23/2$ *can be a finite surgery slope for a hyperbolic knot in* S^3*.*

A main result of this paper is the following

Theorem 1.4 *Let* K *be any fixed hyperbolic knot in* S^3*.*

(1) *The distance between any two finite surgery slopes on* K *is at most* 2*. Consequently, there are at most three nontrivial finite surgeries on* K*.*

(2) *If* K *admits three nontrivial finite surgeries, then* K *must be the pretzel knot* $P(-2,3,7)$*.*

(3) *If* K *admits two noncyclic finite surgeries, the surgery slopes are one of the following ten pairs, the knot* K *has the same knot Floer homology as the sample knot* K_0 *given along with the pair, and the pair of slopes yields the same pair of spherical space forms on* K_0*:*

$$\begin{array}{lll}
\{43/2, 21, [11,2;3,2]\}, & \{53/2, 27, [13,2;3,2]\}, & \{103/2, 52, [17,3;3,2]\}, \\
\{113/2, 56, [19,3;3,2]\}, & \{22, 23, P(-2,3,9)\}, & \{28, 29, -K(1,1,0)\}, \\
\{50, 52, [17,3;3,2]\}, & \{56, 58, [19,3;3,2]\}, & \{91, 93, [23,4;3,2]\}, \\
\{99, 101, [25,4;3,2]\}. & &
\end{array}$$

(4) *If* K *admits two finite surgery slopes which are distance* 2 *apart, then the two slopes are one of the following seventeen pairs, the knot* K *has the same knot Floer homology as the sample knot* K_0 *given along with the pair, and the pair of slopes yields the same pair of spherical space forms on* K_0*:*

$$\begin{array}{lll}
\{43/2, 1/0, [11,2;3,2]\}, & \{45/2, 1/0, [11,2;3,2]\}, & \{51/2, 1/0, [13,2;3,2]\}, \\
\{53/2, 1/0, [13,2;3,2]\}, & \{77/2, 1/0, [19,2;5,2]\}, & \{83/2, 1/0, [21,2;5,2]\}, \\
\{103/2, 1/0, [17,3;3,2]\}, & \{113/2, 1/0, [19,3;3,2]\}, & \{17, 19, P(-2,3,7)\}, \\
\{21, 23, [11,2;3,2]\}, & \{27, 25, [13,2;3,2]\}, & \{37, 39, [19,2;5,2]\}, \\
\{43, 41, [21,2;5,2]\}, & \{50, 52, [17,3;3,2]\}, & \{56, 58, [19,3;3,2]\}, \\
\{91, 93, [23,4;3,2]\}, & \{99, 101, [25,4;3,2]\}. &
\end{array}$$

The notation for the sample knots will be explained in Section 2. Note that a sample knot is not necessarily hyperbolic. When a sample knot is nonhyperbolic, the corresponding case of the described finite surgeries on a hyperbolic knot may never happen.

Parts of the theorem are sharp; on the pretzel knot $P(-2,3,7)$ (which is hyperbolic), 17, 18, 19 are three finite surgery slopes, 17 being I–type and $18, 19$ being cyclic, and on the pretzel knot $P(-2,3,9)$ (which is also hyperbolic), 22 and 23 are two noncyclic finite surgery slopes, 22 being O–type and 23 being I–type.

A D–type spherical space form is also called a prism manifold. Let $P(n,m)$ be the prism manifold with Seifert invariants

$$(-1;(2,1),(2,1),(n,m)),$$

where the base orbifold has genus 0, $n>1$, and $\gcd(n,m)=1$. Every prism manifold can be expressed in this form. As a byproduct of the proof of Theorem 1.4, we have the following:

Theorem 1.5 *If $P(n,m)$ can be obtained by Dehn surgery on a knot K in S^3, then $n<|4m|$.*

Theorem 1.5 improves [19, Theorem 2] of Doig, where no explicit bound on n was given. (In an earlier preprint of [19], the author claimed a bound of $n<|16m|$ without proof.)

On a torus knot $T(2m+1,2)$, recall that $4m$ and $4m+4$ are D–type finite surgery slopes. Our next main result implies that neither of the prism manifolds $S^3_{T(2m+1,2)}(4m)$ nor $S^3_{T(2m+1,2)}(4m+4)$ can be obtained by surgery on any other knot in S^3 besides $\pm T(2m+1,2)$.

Theorem 1.6 *Suppose that $S^3_K(4n)\cong \varepsilon S^3_{T(2m+1,2)}(4n)$ for some $\varepsilon\in\{\pm\}$ and $n=m$ or $m+1$, where $\varepsilon\in\{\pm\}$ stands for an orientation. Then $\varepsilon=+$ and $K=T(2m+1,2)$.*

In the terminology of Ni and Zhang [41], the above theorem implies that $4m$ and $4m+4$ are characterizing slopes for $T(2m+1,2)$; that is, whenever $S^3_K(4n)\cong S^3_{T(2m+1,2)}(4n)$ for $n=m$ or $m+1$, then $K=T(2m+1,2)$.

Combining Theorem 1.6 with Known Facts 1.2(5) and Known Facts 1.1(4), we have:

Corollary 1.7 *Any D–type finite surgery slope of a hyperbolic knot in S^3 is an integer greater than or equal to 28.*

The bound 28 can be realized as a D–type finite surgery slope on two hyperbolic knots in S^3; see Table 5 in Section 2.

The results described above suggest the following updated conjectural picture concerning finite surgeries on hyperbolic knots in S^3.

Conjecture 1.8 *Let K be a hyperbolic knot in S^3.*

(1) **Berge conjecture** *If K admits a nontrivial cyclic surgery, then K is a primitive/primitive knot as defined in* [4] (*ie a Berge knot*).

(2) *If K admits a T–type, O–type or I–type finite surgery, then K is one of the twenty-three hyperbolic sample knots listed in Tables 1–3.* (This was raised in [27].)

(3) *K does not have any half-integral finite surgery slope.*

(4) *If K admits two noncyclic finite surgeries, then K is either $P(-2,3,9)$ or $-K(1,1,0)$.*

(5) *If K admits two finite surgeries at distance 2, then K is $P(-2,3,7)$.*

(6) *If K admits a noncyclic finite surgery, then K is a primitive/Seifert-fibered knot as defined in Dean* [17].

(7) *If the prism manifold $P(n,m)$ can be obtained by surgery on K, then $n < 2|m|-2$.* (This was improved from [19, Conjecture 12].)

The proofs of the above theorems are completed mainly using $\mathrm{PSL}_2(\mathbb{C})$–representation techniques, the correction terms from Heegaard Floer homology and the Casson–Walker invariant besides Known Facts 1.1 and 1.2. In Section 2, we give a more detailed explanation of Known Facts 1.2(3)–(5), which will be convenient to apply in later sections. In Section 3, we recall briefly some machinery for using $\mathrm{PSL}_2(\mathbb{C})$–representations for studying finite surgeries, specialized to the case for hyperbolic knots in S^3. We prove Theorem 1.3 in Section 4, where an outline of the proof will be indicated at the beginning. The method of proof is completed mainly using $\mathrm{PSL}_2(\mathbb{C})$–representation techniques combined with Known Facts 1.2(4) as well as various other results. We then present the proof of Theorem 1.4, which we split into two parts corresponding to the two cases whether a half-integer finite surgery slope is involved or not. Part I of the proof, given in Section 5, is completed mainly using $\mathrm{PSL}_2(\mathbb{C})$–representation techniques combined with Known Facts 1.2 as well as various other results, and part II of the proof, given in Section 6, is completed mainly using the Casson–Walker invariant combined with Known Facts 1.2. Section 6 also contains the proof of Theorem 1.5. In Section 7, we prove Theorem 1.6 by applying Heegaard Floer homology and some topological arguments.

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2 Tables of finite surgeries

Given a rational homology sphere Y and a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, Ozsváth and Szabó defined a rational number $d(Y, \mathfrak{s})$ called the *correction term* [42]. Recall that a rational homology sphere Y is an *L-space* if its Heegaard Floer homology $\widehat{\mathrm{HF}}(Y, \mathfrak{s})$ is isomorphic to $\mathbb{Z}$ for each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$. Lens spaces and, more generally, spherical 3–manifolds are L-spaces. The correction terms are the only informative Heegaard Floer invariants for L-spaces.

Let $L(p,q)$ be the lens space obtained by p/q–surgery on the unknot in S^3, and fix a particular identification $\mathrm{Spin}^c(L(p,q)) \cong \mathbb{Z}/p\mathbb{Z}$. The correction terms for lens spaces can be computed inductively as in [42]:

$$d(S^3,0)=0, \quad d(L(p,q),i) = -\frac{1}{4} + \frac{(2i+1-p-q)^2}{4pq} - d(L(q,r),j), \tag{2-1}$$

where $0 \le i < p+q$, and r and j are the reductions modulo q of p and i, respectively.

For a knot K in S^3, suppose its Alexander polynomial normalized by the conditions $\Delta_K(t) = \Delta_K(t^{-1})$ and $\Delta_K(1) = 1$ is

$$\Delta_K(t) = \sum_i a_i t^i.$$

Define a sequence of integers

$$t_i = \sum_{j=1}^{\infty} j a_{i+j}, \quad i \ge 0.$$

Note that $\Delta_K(t)$ can be recovered from the t_i. If K admits a L-space surgery, then one can prove [46; 49] that

$$t_i \ge 0, \quad t_i \ge t_{i+1} \ge t_i - 1, \quad t_{g(K)} = 0, \tag{2-2}$$

and the correction terms of $S^3_K(p/q)$ can be computed in terms of the t_i by the formula

$$d(S^3_K(p/q), i) = d(L(p,q), i) - 2t_{\min\{\lfloor i/q \rfloor, \lfloor (p+q-i-1)/q \rfloor\}}.$$

The above results give us a necessary condition for a spherical space form Y with $H_1(Y) \cong \mathbb{Z}/p\mathbb{Z}$ to be the p/q–surgery on any knot $K \subset S^3$.

Condition 2.1 There exist a sequence of integers $\{t_i\}_{i\geq 0}$ satisfying

$$t_i \geq 0, \quad t_i \geq t_{i+1} \geq t_i - 1 \quad \text{and} \quad t_i = 0 \text{ when } i \gg 0,$$

and a symmetric affine isomorphism $\phi\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ such that

$$d(Y, \phi(i)) = d(L(p,q), i) - 2t_{\min\{\lfloor i/q\rfloor, \lfloor (p+q-i-1)/q\rfloor\}}.$$

Here the fact that ϕ is symmetric means that ϕ commutes with the conjugation of Spin^c structures once we identify $\mathbb{Z}/p\mathbb{Z}$ with the corresponding sets of Spin^c structures.

Condition 2.1 is easy to check using a simple computer program. When it is satisfied, we can recover the Alexander polynomial of the possible knot K admitting the surgery to Y. It is surprising that Condition 2.1 is also sufficient in all known cases. For example, Ozsváth and Szabó [44, Proposition 1.13] confirmed that a lens space $L(p,q)$ with $p \leq 1500$ can be obtained by p–surgery on a knot in S^3 if and only if Condition 2.1 holds, and Rasmussen [50, Section 6] has further extended this confirmation to $p < 100\,000$. Doig [18] proved similar results for prism spaces with $|H_1| \leq 32$, and Gu [27] and Li and Ni [36] proved similar results for T–type, O–type and I–type spherical space forms.

Here we list the T–type, O–type, I–type and some D–type finite surgery slopes and the sample knots mentioned in Known Facts 1.2(3)–(5), which are reproduced from [27; 36; 18] for the reader's convenience. For each of these finite surgeries, we actually list four to six relevant pieces of data: p $\bigl(\text{or } \frac{1}{2}p\bigr)$, Y, K_0, g, $\det(K)$, and $\Delta''_K(1)$, which are useful in this and later sections. Here $p > 0$ $\bigl(\text{or } \frac{1}{2}p > 0\bigr)$ is the finite surgery slope, Y is the resulting manifold, K_0 is a sample knot which admits the finite surgery, and g, $\det(K)$ and $\Delta_K(t)$ are respectively the Seifert genus, the determinant and the normalized Alexander polynomial for any knot $K \subset S^3$ which admits the finite surgery.

Now we explain the notation for the manifolds $\mathbb{T}(p/q)$ and for the sample knots K_0 we use in these tables. Let $-K$ be the mirror image of K.

- It is not hard to see that every T-, O- or I–type spherical space form, up to orientation reversal, can be obtained by Dehn filling on $\mathbb{T}$, the exterior of the right-hand trefoil knot in S^3. Thus we represent the corresponding surgery manifold Y by Dehn filling on $\mathbb{T}$ and specify the orientation.

p	Y	K_0	g	$\det(K)$	$\Delta''_K(1)$
3	$\mathbb{T}(3)$	$T(3,2)$	1	3	2
9	$\mathbb{T}(9)$	$T(3,2)$	1	3	2
21	$\mathbb{T}(21/4)$	$[11,2;3,2]$	7	11	38
27	$\mathbb{T}(27/4)$	$[13,2;3,2]$	8	13	50
51	$-\mathbb{T}(51/8)$	$K_2^\#$	18	1	200
69	$-\mathbb{T}(69/11)$	$-K(2,3,5,-3)$	25	1	368
81	$\mathbb{T}(81/13)$	$K(2,3,5,3)$	31	1	536
93	$\mathbb{T}(93/16)$	$[23,4;3,2]$	37	23	692
99	$\mathbb{T}(99/16)$	$[25,4;3,2]$	40	25	812

Table 1: Integral T–type surgeries [27]

- Many of the knots in the tables are torus knots or iterated torus knots. As in [6], we use $[p,q;r,s]$ to denote the (p,q)–cable of the torus knot $T(r,s)$.
- There are two hyperbolic pretzel knots in the tables: $P(-2,3,7)$ and $P(-2,3,9)$.
- Following [38], let $K(p,q,r,n)$ be the twist torus knot obtained by applying n full twists to r parallel strings in $T(p,q)$, where p,q are coprime integers, $q>|p|\ge 2$, $0\le r\le p+q$, and $n\in\mathbb{Z}$. It is proved in [38] that the $pq+n(p+q)^2$–surgery on $K(p,q,p+q,n)$ yields a Seifert-fibered manifold with base orbifold $S^2(|p|,q,|n|)$.
- Let $B(p,q;a)$ be the Berge knot [4] whose dual is the simple knot [50] in the homology class a in $L(p,q)$.
- Three knots $K_2^\#$, K_3^*, K_3 are from [9, Section 10].
- One knot $K(1,1,0)$ is from [21, Section 4], which is also the knot K_1 given in [6, Proposition 18].
- Three knots are primitive/Seifert knots from [5]. We will use the notation there, starting with "P/SF$_d$ KIST".

Remark 2.2 In [6], the authors enumerated all finite surgeries on iterated torus knots. However, one case was missed in their list: the 58–surgery on $[19,3;3,2]$ yields the O–type manifold $\mathbb{T}(58/9)$. This mistake was inherited in [27], where the author found a knot on which 58–surgery yields $\mathbb{T}(58/9)$, but she thought the knot was hyperbolic because this case was not listed in [6].

p	Y	K_0	g	$\det(K)$	$\Delta''_K(1)$
2	$\mathbb{T}(2)$	$T(3,2)$	1	3	2
10	$\mathbb{T}(10)$	$T(3,2)$	1	3	2
10	$-\mathbb{T}(10)$	$T(4,3)$	3	3	10
14	$-\mathbb{T}(14/3)$	$T(4,3)$	3	3	10
22	$\mathbb{T}(22/3)$	$P(-2,3,9)$	6	3	34
38	$\mathbb{T}(38/7)$	$B(39,16;16)$	13	3	114
46	$-\mathbb{T}(46/7)$	$B(45,19;8)$	16	3	162
50	$\mathbb{T}(50/9)$	$[17,3;3,2]$	19	3	210
58	$\mathbb{T}(58/9)$	$[19,3;3,2]$	21	3	258
62	$-\mathbb{T}(62/11)$	P/SF$_d$ KIST III $(-5,-3,-2,-1,1)$	23	3	306
70	$\mathbb{T}(70/11)$	$B(71,27;11)$	27	3	402
86	$\mathbb{T}(86/15)$	$-K(3,4,7,-2)$	33	3	586
94	$-\mathbb{T}(94/15)$	$-K(2,3,5,-4)$	35	3	690
106	$\mathbb{T}(106/17)$	$K(2,3,5,4)$	41	3	914
106	$-\mathbb{T}(106/17)$	$[35,3;4,3]$	43	3	906
110	$-\mathbb{T}(110/19)$	$K(3,4,7,2)$	45	3	1002
110	$-\mathbb{T}(110/19)$	$[37,3;4,3]$	45	3	1002
146	$\mathbb{T}(146/25)$	$[29,5;3,2]$	61	3	1730
154	$\mathbb{T}(154/25)$	$[31,5;3,2]$	65	3	1970

Table 2: Integral O–type surgeries [27]

In Table 4, we omit the two half-integer finite surgeries on $T(3,2)$ as it was already known that only $T(3,2)$ can have such surgeries [45].

Lemma 2.3 *In Tables 1–5, any two sample knots expressed in different notation are different knots with different Alexander polynomials.*

Proof By Known Facts 1.2(1), all the knots in S^3 with finite surgeries are fibered. So if two sample knots in the tables have different genera, they must have different Alexander polynomials. Below we will compare the Alexander polynomials of sample knots with the same genera. Since $\det(K)=|\Delta_K(-1)|$, we often just compare $\det(K)$ and $\Delta''_K(1)$.

- $\boldsymbol{g=3}$ There are two knots $T(4,3)$ and $T(7,2)$. They have different $\det(K)$.
- $\boldsymbol{g=4}$ There are two knots $T(5,3)$ and $T(9,2)$. They have different $\det(K)$.

p	Y	K_0	g	$\det(K)$	$\Delta''_K(1)$
1	$\mathbb{T}(1)$	$T(3,2)$	1	3	2
7	$\mathbb{T}(7/2)$	$T(5,2)$	2	5	6
11	$\mathbb{T}(11)$	$T(3,2)$	1	3	2
13	$-\mathbb{T}(13/3)$	$T(5,2)$	2	5	6
13	$\mathbb{T}(13/3)$	$T(5,3)$	4	1	16
17	$\mathbb{T}(17/2)$	$T(5,3)$	4	1	16
17	$-\mathbb{T}(17/2)$	$P(-2,3,7)$	5	1	24
19	$\mathbb{T}(19/4)$	$[9,2;3,2]$	6	9	28
23	$\mathbb{T}(23/3)$	$P(-2,3,9)$	6	3	34
29	$\mathbb{T}(29/4)$	$[15,2;3,2]$	9	15	64
29	$-\mathbb{T}(29/4)$	$-K(1,1,0)$	9	11	62
37	$-\mathbb{T}(37/7)$	$-K(-2,5,3,-3)$	11	1	96
37	$\mathbb{T}(37/7)$	$[19,2;5,2]$	13	19	114
43	$-\mathbb{T}(43/8)$	$[21,2;5,2]$	14	21	134
47	$\mathbb{T}(47/7)$	K_3^*	16	1	168
49	$\mathbb{T}(49/9)$	$[16,3;3,2]$	18	9	188
59	$\mathbb{T}(59/9)$	$[20,3;3,2]$	22	9	284
83	$-\mathbb{T}(83/13)$	P/SF$_d$ KIST V$(1,-2,-1,2,2)$	32	1	552
91	$\mathbb{T}(91/16)$	$[23,4;3,2]$	37	23	692
101	$\mathbb{T}(101/16)$	$[25,4;3,2]$	40	25	812
113	$-\mathbb{T}(113/18)$	$-K(3,5,8,-2)$	45	1	1024
113	$\mathbb{T}(113/18)$	$-$P/SF$_d$ KIST V$(-3,-2,-1,2,2)$	46	1	1048
119	$-\mathbb{T}(119/19)$	$-K(2,3,5,-5)$	45	1	1112
131	$\mathbb{T}(131/21)$	$K(2,3,5,5)$	51	1	1392
133	$\mathbb{T}(133/23)$	$[44,3;5,3]$	55	3	1434
137	$-\mathbb{T}(137/22)$	$-K(2,5,7,-3)$	55	3	1506
137	$\mathbb{T}(137/22)$	$[46,3;5,3]$	57	3	1554
143	$\mathbb{T}(143/23)$	$K(3,5,8,2)$	60	1	1696
157	$\mathbb{T}(157/27)$	$[39,4;5,2]$	65	39	1996
157	$-\mathbb{T}(157/27)$	$K(2,5,7,3)$	65	3	2034
163	$-\mathbb{T}(163/28)$	$[41,4;5,2]$	68	41	2196
211	$\mathbb{T}(211/36)$	$[35,6;3,2]$	91	35	3642
221	$\mathbb{T}(221/36)$	$[37,6;3,2]$	96	37	4062

Table 3: Integral I–type surgeries [27]

$\frac{1}{2}p$	Y	K_0	g	$\frac{1}{2}p$	Y	K_0	g
$17/2$	$-\mathbb{T}(17/2)$	$T(5,2)$	2	$53/2$	$\mathbb{T}(53/8)$	$[13,2;3,2]$	8
$23/2$	$\mathbb{T}(23/3)$	$T(5,2)$	2	$77/2$	$-\mathbb{T}(77/12)$	$[19,2;5,2]$	13
$43/2$	$\mathbb{T}(43/8)$	$[11,2;3,2]$	7	$83/2$	$\mathbb{T}(83/13)$	$[21,2;5,2]$	14
$45/2$	$\mathbb{T}(45/8)$	$[11,2;3,2]$	7	$103/2$	$\mathbb{T}(103/18)$	$[17,3;3,2]$	19
$51/2$	$\mathbb{T}(51/8)$	$[13,2;3,2]$	8	$113/2$	$\mathbb{T}(113/18)$	$[19,3;3,2]$	21

Table 4: Half-integral surgeries [36]

p	K_0	g	$\det(K)$	$\Delta''_K(1)$
4	$T(3,2)$	1	3	2
8	$T(3,2)$	1	3	2
8	$T(5,2)$	2	5	6
12	$T(5,2)$	2	5	6
12	$T(7,2)$	3	7	12
16	$T(7,2)$	3	7	12
16	$T(9,2)$	4	9	20
20	$T(9,2)$	4	9	20
20	$T(11,2)$	5	11	30
24	$T(11,2)$	5	11	30
24	$T(13,2)$	6	13	42
28	$T(13,2)$	6	13	42
28	$-K(1,1,0)$	9	11	62
28	K_3	8	5	54
28	$T(15,2)$	7	15	56
32	$T(15,2)$	7	15	56
32	$T(17,2)$	8	17	72

Table 5: Integral D–type p–surgeries with $p \le 32$ [18]

- $\boldsymbol{g = 5}$ There are two knots $P(-2,3,7)$ and $T(11,2)$. They have different $\det(K)$.
- $\boldsymbol{g = 6}$ There are three knots $[9,2;3,2]$, $P(-2,3,9)$ and $T(13,2)$. They have different $\det(K)$.
- $\boldsymbol{g = 7}$ There are two knots $[11,2;3,2]$ and $T(15,2)$. They have different $\det(K)$.

- $\boldsymbol{g=8}$ There are three knots $[13,2;3,2]$, $T(17,2)$ and K_3. They have different $\det(K)$.
- $\boldsymbol{g=9}$ There are two knots $[15,2;3,2]$ and $-K(1,1,0)$. They have different $\det(K)$.
- $\boldsymbol{g=13}$ There are two knots $B(39,16;16)$ and $[19,2;5,2]$. They have different $\det(K)$.
- $\boldsymbol{g=16}$ There are two knots $B(45,19;8)$ and K_3^*. They have different $\det(K)$.
- $\boldsymbol{g=18}$ There are two knots $K_2^\#$ and $[16,3;3,2]$. They have different $\det(K)$.
- $\boldsymbol{g=45}$ There are four knots. The two knots with O–type surgery can be distinguished from the two with I–type surgery via $\det(K)$. The two knots with I–type surgery have different $\Delta''_K(1)$. The two knots with O–type surgery have different Alexander polynomials
$$\Delta_{K(3,4,7,2)}(t) = 1-(t^2+t^{-2})+(t^3+t^{-3)}-(t^5+t^{-5})+(t^7+t^{-7})+\cdots,$$
$$\Delta_{[37,3;4,3]}(t) = 1-(t^2+t^{-2})+(t^3+t^{-3)}-(t^5+t^{-5})+(t^6+t^{-6})+\cdots.$$
- $\boldsymbol{g=55}$ There are two knots $[44,3;5,3]$ and $K(2,5,7,-3)$. They have different $\Delta''_K(1)$.
- $\boldsymbol{g=65}$ There are three knots $[31,5;3,2]$, $[39,4;5,2]$ and $K(2,5,7,3)$. They have different $\Delta''_K(1)$.

This finishes the proof. □

Remark 2.4 In Tables 1–5, any sample knot expressed in a notation different from that of a torus knot or a cable over a torus knot is a hyperbolic knot. This is because that those torus knots and cables over torus knots appearing in Tables 1–5 constitute the set of all nonhyperbolic knots in S^3 which admit T–type, O–type or I–type finite surgeries, or D–type integer p–surgeries with $p \le 32$. By Lemma 2.3, all other sample knots are different from these nonhyperbolic ones.

3 $\mathrm{PSL}_2(\mathbb{C})$–character variety, Culler–Shalen norm or semi-norm, and finite surgery

In this section, we briefly review some machinery and results from [16; 9; 11; 12] used in studying cyclic and finite surgeries but specialized to the case of knots in S^3. In fact, for simplicity, we shall mainly restrict our discussion to the following very special situation: any hyperbolic knot in S^3 which is assumed to have a half-integer finite surgery. This is sufficient for our purposes in this paper.

For a finitely generated group Γ, we use $R(\Gamma)$ to denote the $\mathrm{PSL}_2(\mathbb{C})$–representation variety of Γ. (The term variety used here means complex affine algebraic set). Let $\Phi\colon \mathrm{SL}_2(\mathbb{C})\to \mathrm{PSL}_2(\mathbb{C})$ be the canonical quotient homomorphism. Given an element Υ in $\mathrm{PSL}_2(\mathbb{C})$, we have that $\Phi^{-1}(\Upsilon)=\{A,-A\}$ for some $A\in \mathrm{SL}_2(\mathbb{C})$, and we often simply write $\Upsilon=\pm A$. In particular, we may define $\mathrm{tr}^2(\Upsilon) := [\mathrm{trace}(A)]^2$, which is obviously well defined on Υ. An element $\Upsilon\in \mathrm{PSL}_2(\mathbb{C})$ is said to be parabolic if it is not the identity element $\pm I$ and satisfies $\mathrm{tr}^2(\Upsilon)=4$.

A representation $\rho\in R(\Gamma)$ is said to be irreducible if it is not conjugate to a representation whose image lies in

$$\left\{\pm\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} : \ a,b\in\mathbb{C},\ a\neq 0\right\}.$$

A representation $\rho\in R(\Gamma)$ is said to be strictly irreducible if it is irreducible and is not conjugate to a representation whose image lies in

$$\left\{\pm\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, \pm\begin{pmatrix} 0 & b \\ -b^{-1} & 0\end{pmatrix} : \ a,b\in\mathbb{C},\ a\neq 0,\ b\neq 0\right\}.$$

For a representation $\rho\in R(\Gamma)$, its character χ_ρ is the function $\chi_\rho\colon \Gamma\to\mathbb{C}$ defined by $\chi_\rho(\gamma)=\mathrm{tr}^2(\rho(\gamma))$ for each $\gamma\in\Gamma$. Let $X(\Gamma)=\{\chi_\rho : \rho\in R(\Gamma)\}$ denote the set of characters of representations of Γ. Then $X(\Gamma)$ is also a complex affine algebraic set, usually referred as the $\mathrm{PSL}_2(\mathbb{C})$–character variety of Γ.

A character $\chi_\rho\in X(\Gamma)$ is said to be irreducible or strictly irreducible or discretely faithful or dihedral if the representation ρ has the corresponding property.

Let $t\colon R(\Gamma)\to X(\Gamma)$ denote the natural onto map defined by $t(\rho)=\chi_\rho$. Then t is a regular map between the two algebraic sets. For an element $\gamma\in\Gamma$, the function $f_\gamma\colon X(\Gamma)\to\mathbb{C}$ is defined by $f_\gamma(\chi_\rho)=\chi_\rho(\gamma)-4=\mathrm{tr}^2(\rho(\gamma))-4$ for each $\chi_\rho\in X(\Gamma)$. Each f_γ is a regular function on $X(\Gamma)$. Obviously, $\chi_\rho\in X(\Gamma)$ is a zero point of f_γ if and only if either $\rho(\gamma)=\pm I$ or $\rho(\gamma)$ is a parabolic element. It is also evident that f_γ is invariant when γ is replaced by a conjugate of γ or by the inverse of γ.

If $h\colon \Gamma\to\Gamma'$ is a surjective homomorphism between two finitely generated groups, it naturally induces an embedding of $R(\Gamma')$ into $R(\Gamma)$ and an embedding of $X(\Gamma')$ into $X(\Gamma)$. So we may simply consider $R(\Gamma')$ and $X(\Gamma')$ as subsets of $R(\Gamma)$ and $X(\Gamma)$, respectively, and write $R(\Gamma')\subset R(\Gamma)$ and $X(\Gamma')\subset X(\Gamma)$.

For a connected compact manifold Y, let $R(Y)$ and $X(Y)$ denote $R(\pi_1(Y))$ and $X(\pi_1(Y))$, respectively.

Let M be the exterior of a knot K in S^3. A slope on ∂M is called a boundary slope if there is an orientable properly embedded incompressible and boundary-incompressible surface F in M whose boundary ∂F is a nonempty set of parallel essential curves in ∂M of slope γ. For a slope γ on ∂M, we denote by $M(\gamma)$ the Dehn filling of M with slope γ. Throughout, we let μ denote the meridian slope and λ the canonical longitude slope on ∂M. We use $\Delta(\gamma_1, \gamma_2)$ to denote the distance between two slopes γ_1 and γ_2 on ∂M. We call K or M hyperbolic if the interior of M supports a complete hyperbolic metric of finite volume. Note that for any slope γ, there is a surjective homomorphism from $\pi_1(M)$ to $\pi_1(M(\gamma))$, and thus $R(M(\gamma)) \subset R(M)$ and $X(M(\gamma)) \subset X(M)$.

For the exterior M of a nontrivial knot in S^3, we consider $H_1(\partial M; \mathbb{Z}) \cong \pi_1(\partial M)$ as a subgroup of $\pi_1(M)$ which is well defined up to conjugation. Hence the function f_γ on $X(M)$ is well defined for each class $\gamma \in H_1(\partial M; \mathbb{Z})$. As f_γ is also invariant under the change of the orientation of γ, we see that f_γ is also well defined when γ is a slope in ∂M. For convenience, we will often not make a distinction among a primitive class of $H_1(\partial M; \mathbb{Z})$, the corresponding element of $\pi_1(\partial M)$ and the corresponding slope in ∂M; that is, we shall often use these terms exchangeably under the same notation.

Lemma 3.1 [9, Lemma 5.3] *Let Y be a spherical space form.*

(1) *If Y is T–type, then $X(Y)$ has exactly one irreducible character χ_ρ, and the image of ρ is isomorphic to the tetrahedral group*

$$T_{12} = \{x, y : x^2 = y^3 = (xy)^3 = 1\}.$$

(2) *If Y is O–type, then $X(Y)$ has exactly two irreducible characters χ_{ρ_1} and χ_{ρ_2}, one of ρ_1 and ρ_2 has its image isomorphic to the octahedral group*

$$O_{24} = \{x, y : x^2 = y^3 = (xy)^4 = 1\}$$

(we name this character the O–type character of $X(Y)$)*, and the other has image isomorphic to the dihedral group*

$$D_6 = \{x, y : x^2 = y^2 = (xy)^3 = 1\}$$

(we name this character the D–type character of $X(Y)$).

(3) *If Y is I–type, then $X(Y)$ has exactly two irreducible characters χ_{ρ_1} and χ_{ρ_2}, and both ρ_1 and ρ_2 have image isomorphic to the icosahedral group*

$$I_{60} = \{x, y : x^2 = y^3 = (xy)^5 = 1\}.$$

Let M be the exterior of a knot in S^3 and suppose that β is a D–type, T–type, O–type or I–type finite surgery slope on ∂M. Let $\rho \in R(M(\beta)) \subset R(M)$ be an irreducible representation, and we require ρ to have image O_{24} when β is O–type. Let ϕ denote the composition $\pi_1(\partial M) \to \pi_1(M) \to \pi_1(M(\beta)) \xrightarrow{\rho} \mathrm{PSL}_2(\mathbb{C})$, and let $q = |\phi(\pi_1(\partial M))|$. Then q is uniquely associated to the finite surgery slope β, and following [13], we say more specifically that the finite surgery slope β is a $D(q)$–, $T(q)$–, $O(q)$–, or $I(q)$–type, respectively.

Remark 3.2 The number q possesses the following useful property which we shall often apply: for any slope γ on ∂M whose distance from β is divisible by q, the representation ρ factors through $\pi_1(M(\gamma))$; ie $\rho(\gamma) = \pm I$.

The following lemma can be extracted from specializing [13] to exteriors of hyperbolic knots in S^3.

Lemma 3.3 [13] *Let M be the exterior of a hyperbolic knot K in S^3 and β a finite noncyclic surgery slope of K.*

(1) *If β is D–type, it is actually $D(2)$–type, and β is an integer divisible by 4.*
(2) *If β is T–type, it is actually $T(3)$–type, and β is an integer or half-integer whose numerator is an odd integer divisible by 3.*
(3) *If β is I–type, it is $I(2)$–, $I(3)$–, or $I(5)$–type, and β is an integer or half-integer whose numerator is relatively prime to 30.*
(4) *If β is O–type, it is $O(2)$– or $O(4)$–type, and β is an even integer not divisible by 4.*

By a curve in an algebraic set, we mean an irreducible 1–dimensional algebraic subset. It is known (eg [11]) that any curve X_0 in $X(M)$ belongs to one of the following three mutually exclusive types:

(a) for each slope γ on ∂M, the function f_γ is constant on X_0;
(b) there is a unique slope γ_0 on ∂M such that the function f_{γ_0} is constant on X_0;
(c) for each slope γ on ∂M, the function f_γ is nonconstant on X_0.

We call a curve of $X(M)$ in case (a) a constant curve, in case (b) a seminorm curve and in case (c) a norm curve. Indeed as the names indicate, a seminorm curve or norm curve in $X(M)$ can be used to define a seminorm or norm, respectively, on the real 2–dimensional plane $H_1(\partial M;\mathbb{R})$ satisfying certain properties. In this paper, we only need to consider those curves in $X(M)$ which are irreducible components of $X(M)$ and contain irreducible characters. We call such a curve a nontrivial curve component of $X(M)$. In a nontrivial curve component of $X(M)$, all but finitely many characters are irreducible. But to see if a curve component in $X(M)$ is nontrivial, it suffices to check if the curve contains at least one irreducible character. Note that a norm curve component is automatically nontrivial. Also note that a norm curve component always exists for any hyperbolic knot exterior, but a constant curve or a nontrivial seminorm curve may not always exist.

For a curve X_0 in $X(M)$, let $\widetilde{X}_0$ be the smooth projective completion of X_0 and let $\phi\colon \widetilde{X}_0 \to X_0$ be the birational equivalence. The map ϕ is onto and is defined at all but finitely many points of $\widetilde{X}_0$. The points where ϕ is not defined are called ideal points. The map ϕ induces an isomorphism from the function field of X_0 to that of $\widetilde{X}_0$. In particular, every regular function f_γ on X_0 corresponds uniquely to its extension $\widetilde{f}_\gamma$ on $\widetilde{X}_0$ which is a rational function. If $\widetilde{f}_\gamma$ is not a constant function on $\widetilde{X}_0$, its degree, denoted by $\deg(\widetilde{f}_\gamma)$, is equal to the number of zeros of $\widetilde{f}_\gamma$ in $\widetilde{X}_0$ counted with multiplicity; ie if $Z_x(\widetilde{f}_\gamma)$ denotes the zero degree of $\widetilde{f}_\gamma$ at a point $x\in\widetilde{X}_0$, then

$$\deg(\widetilde{f}_\gamma)=\sum_{x\in\widetilde{X}_0} Z_x(\widetilde{f}_\gamma).$$

Note that if χ_ρ is a smooth point of X_0 then $\phi^{-1}(\chi_\rho)$ is a single point and the zero degree of f_γ at χ_ρ is equal to the zero degree of $\widetilde{f}_\gamma$ at $x=\phi^{-1}(\chi_\rho)$.

From now on, in this section, we make the following *special assumption*: let M be the exterior of a hyperbolic knot in S^3, and suppose M has a half-integer finite surgery slope α. It follows from [16, Theorem 2.0.3] that each of α and the meridian slope μ is not a boundary slope.

We shall identify $H_1(\partial M,\mathbb{R})$ with the real xy–plane so that $H_1(\partial M;\mathbb{Z})$ are integer lattice points (m,n) with $\mu=(1,0)$ being the meridian class and $\lambda=(0,1)$ the longitude class. So each slope p/q in ∂M corresponds to the pair of primitive elements $\pm(p,q)\in H_1(\partial M;\mathbb{Z})$.

Theorem 3.4 *Let X_1 be a norm curve component of $X(M)$. Then X_1 can be used to define a norm $\|\cdot\|_{X_1}$ on $H_1(\partial M;\mathbb{R})$, known as the Culler–Shalen norm, with the following properties:*

(1) *For each nontrivial element $\gamma=(m,n)\in H_1(\partial M;\mathbb{Z})$, we have that $\|\gamma\|_{X_1}=\deg(\widetilde{f}_\gamma)\neq 0$ (thus is a positive integer).*

(2) *The norm is symmetric around the origin; ie $\|(a,b)\|_{X_1}=\|(-a,-b)\|_{X_1}$ for all $(a,b)\in H_1(\partial M;\mathbb{R})$. Let*

$$s_1=\min\{\|\gamma\|_{X_1}:\gamma\in H_1(\partial M;\mathbb{Z}),\ \gamma\neq 0\}$$

and B_1 be the set of points in $H_1(\partial M;\mathbb{R})$ with norm less than or equal to s_1. Then B_1 is a convex finite-sided polygon symmetric around the origin whose interior does not contain any nonzero element of $H_1(\partial M;\mathbb{Z})$.

(3) *If (a,b) is a vertex of B_1, then there is a boundary slope p/q in ∂M such that $\pm(p,q)$ lie on the line passing through (a,b) and $(0,0)$.*

(4) *If we normalize the area of a parallelogram spanned by any pair of generators of $H_1(\partial M;\mathbb{Z})$ to be 1, then $\mathrm{Area}(B_1)\leq 4$.*

(5) *If β is a cyclic surgery slope but is not a boundary slope, then $\beta\in\partial B_1$ (so $\|\beta\|_{X_1}=s_1$) but is not a vertex of B_1. More precisely, for each nonzero element $\gamma\in H_1(\partial M;\mathbb{Z})$ and for every point $x\in\widetilde{X}_1$, we have $Z_x(\widetilde{f}_\beta)\leq Z_x(\widetilde{f}_\gamma)$. In particular, the meridian slope μ has this property.*

(6) *If β is a T–type finite surgery slope but is not a boundary slope, then $\|\beta\|_{X_1}=s_1+2$ or s_1 corresponding to whether the irreducible character χ_ρ of $X(M(\beta))$* (given by Lemma 3.1(1)) *is contained in X_1 or not, respectively.*

(7) *If β is an O–type finite surgery slope but is not a boundary slope, then $\|\beta\|_{X_1}$ is either s_1+3, s_1+2, s_1+1, or s_1 corresponding to whether both, only the O–type, only the D–type, or neither of the two irreducible characters of $X(M(\beta))$* (given by Lemma 3.1(2)) *is contained in X_1, respectively.*

(8) *If β is an I–type finite surgery slope but is not a boundary slope, then $\|\beta\|_{X_1}$ is either s_1+4, s_1+2, or s_1 corresponding to whether both, only one, or neither of the two irreducible characters of $X(M(\beta))$* (given by Lemma 3.1(3)) *is contained in X_1, respectively.*

(9) *The half-integral finite surgery slope α is either T–type or I–type.*

(9a) *If* α *is* T*–type, the curve* X_1 *contains the unique irreducible character of* $X(M(\alpha))$, $\|\alpha\|_{X_1} = \|\mu\|_{X_1} + 2 = s_1 + 2$, *and* $X(M)$ *has no other norm curve component.*

(9b) *If* α *is* I*–type, the curve* X_1 *contains at least one of the two irreducible characters of* $X(M(\alpha))$, *and either* $\|\alpha\|_{X_1} = \|\mu\|_{X_1} + 2 = s_1 + 2$ *if exactly one of the two irreducible characters of* $X(M(\alpha))$ *is contained in* X_1, *or* $\|\alpha\|_{X_1} = \|\mu\|_{X_1} + 4 = s_1 + 4$ *if both of the two irreducible characters of* $X(M(\alpha))$ *are contained in* X_1. *Also, if* $\|\alpha\|_{X_1} = \|\mu\|_{X_1} + 4 = s_1 + 4$, *then* $X(M)$ *does not have any other norm curve component.*

Properties (1)–(5) of Theorem 3.4 originate from [16] and properties (6)–(9) from [9; 11]. As (9) was not explicitly stated in [9; 11], we give here a brief explanation. Since each of μ and α is not a boundary slope, each of them is not contained in a line passing a vertex of B_1 and the origin by (4). In particular, each of μ and β is not a vertex of B_1. By (5), we have $\|\mu\|_{X_1} = s_1$. We claim that α is not contained in B_1; ie $\|\alpha\|_{X_1} > \|\mu\|_{X_1} = s_1$. For if $\alpha = (2p+1, 2) \in B_1$, then since $\mu = (1, 0) \in B_1$ and since B_1 is a convex set, B_1 also contains the points $(p, 1)$ and $(p+1, 1)$. This would imply that the area of B_1 is ≥ 4, which by (4) would imply that B_1 is a parallelogram with $\pm\mu$ and $\pm\alpha$ as vertices, contradicting to our early conclusion. Now the conclusion that α is either T–type or I–type follows from Lemma 3.3 and all the conclusions of (9a) and (9b) follow directly from [9; 11], due essentially to the facts that each irreducible character in $X(M(\alpha))$ is a smooth point of $X(M)$ and that the zero degree of f_α at such a character is 2 while the zero degree of f_μ at such a point is 0.

Remark 3.5 Properties (6)–(8) of Theorem 3.4 are also due to similar facts: when β is a finite noncyclic slope of M, each irreducible character of $X(M(\beta))$ is a smooth point of $X(M)$ [9; 11] and thus is contained a unique component of $X(M)$, and when such a character is contained in X_1, then the zero degree of f_β at the character is 2 (except when the character is dihedral in which case the zero degree is 1) while the zero degree of f_μ at the character is 0. Moreover, if the character factors through $M(\gamma)$ for some slope γ then the zero degree of f_γ at this point is also 2 (or 1 when the character is dihedral). We shall say that the character contributes to the norm of γ by 2 (or 1) beyond the minimum norm s_1. This extended property shall also be applied later in this paper.

As M is hyperbolic, any component X_1 of $X(M)$ which contains the character of a discretely faithful representation of $\pi(M)$ is a norm curve component of $X(M)$. To apply Theorem 3.4 more effectively we consider the set C of all (mutually distinct) norm curve components $X_1, \dots, X_k$ in $X(M)$ and let $\|\cdot\| = \|\cdot\|_{X_1} + \dots + \|\cdot\|_{X_k}$ be the norm defined by C. In particular, C contains the orbit of X_1 under the $\mathrm{Aut}(\mathbb{C})$–action on $X(M)$; see [13, Section 5] for the $\mathrm{Aut}(\mathbb{C})$–action. In fact, under the special assumption that M has a half-integer finite surgery slope, C has at most two components. Let

$$s = \min\{\|\gamma\| : \gamma \in H_1(M, \partial M),\ \gamma \neq 0\},$$

and let B be the disk in the plane $H_1(\partial M; \mathbb{R})$ centered at origin with radius s with respect to the norm $\|\cdot\|$. Obviously, C, $\|\cdot\|$, s and B are uniquely associated to M. The following theorem follows directly from Theorem 3.4.

Theorem 3.6 *With C, $\|\cdot\|$, s and B defined above for M, we have:*

(1) *The value $s > 0$ is an integer, and B is a convex finite-sided polygon symmetric around the origin.*

(2) *If β is a cyclic slope but is not a boundary slope, then $\beta \in \partial B$ (so $\|\beta\| = s$) but is not a vertex of B. In particular, μ is such a slope.*

(3) *If β is a T–type finite surgery slope but is not a boundary slope, then $\|\beta\| = s+2$ or s corresponding to whether the irreducible character χ_ρ of $X(M(\beta))$* (given by Lemma 3.1(1)) *is contained in C or not, respectively.*

(4) *If β is an O–type finite surgery slope, then $\|\beta\|$ is either $s+3$, $s+2$, $s+1$ or s corresponding to whether both, only the O–type, only the D–type, or neither of the two irreducible characters of $X(M(\beta))$* (given by Lemma 3.1(2)) *is contained in C, respectively.*

(5) *If β is an I–type finite surgery slope but is not a boundary slope, then $\|\beta\| = s+4$ or s corresponding to whether both or neither of the two irreducible characters of $X(M(\beta))$* (given by Lemma 3.1(3)) *is contained in C, respectively.*

(6) *The half-integral finite surgery slope α is either T–type or I–type.*

(6a) *If α is T–type, then $\|\alpha\| = \|\mu\| + 2 = s + 2$, and the irreducible character of $X(M(\alpha))$ is contained in C.*

(6b) *If α is I–type, then $\|\alpha\| = \|\mu\| + 4 = s + 4$, and both irreducible characters of $X(M(\alpha))$ are contained in C.*

We only need to note that Theorem 3.6(4) and (6b) hold because of the $\mathrm{Aut}(\mathbb{C})$–action; see [13, Remark 9.4].

Theorem 3.7 *Suppose that* X_0 *is a nontrivial seminorm curve component of* $X(M)$ (such a curve may not always exist), *and let* γ_0 *be the unique slope such that* f_{γ_0} *is constant on* X_0 (we call γ_0 the associated slope to X_0). *The curve* X_0 *can be used to define a seminorm* $\|\cdot\|_{X_0}$ *on* $H_1(\partial M;\mathbb{R})$, *called the Culler–Shalen seminorm, with the following properties:*

(1) $\|\gamma_0\|_{X_0} = 0$, *and* γ_0 *is a boundary slope of* M.

(2) *For each slope* $\gamma \neq \gamma_0$, *we have* $\|\gamma\|_{X_0} = \deg(\widetilde{f}_\gamma) \neq 0$ (*so is a positive integer*).

(3) *For the meridian slope* $\mu = (1,0)$, *we have that* $\mu \neq \gamma_0$ *and that* $\|\mu\|_{X_0} > 0$ *is minimal among all slopes* $\gamma \neq \gamma_0$. *More precisely, for every point* $x \in \widetilde{X}_0$, $Z_x(\widetilde{f}_\mu) \leq Z_x(\widetilde{f}_\gamma)$ *for each slope* $\gamma \neq \gamma_0$. *Furthermore, for every slope* γ, *we have* $\|\gamma\|_{X_0} = \Delta(\gamma,\gamma_0)\|\mu\|_{X_0}$. *In particular,* $\Delta(\mu,\gamma_0) = 1$; *ie* γ_0 *is an integer slope.*

(4) *For the half-integer finite surgery slope* α, *we have* $\|\alpha\|_{X_0} = \|\mu\|_{X_0}$ *and thus* $\Delta(\alpha,\gamma_0) = 1$.

Theorem 3.7 is contained in [11]. We only need to note that Theorem 3.7(4) holds because X_0 cannot contain any irreducible character of $X(M(\alpha))$ due to Theorem 3.6(6).

Lemma 3.8 *Let* η *be any one of the two integer slopes which are distance* 1 *from the half-integer finite surgery slope* α. *Then* $\|\eta\| \leq s+1$ *if* α *is* T*–type, and* $\|\eta\| \leq s+2$ *if* α *is* I*–type.*

Proof Write $\alpha = (2p+1,2)$. Then $\eta = (p,1)$ or $(p+1,1)$. We prove the case when $\alpha = (2p+1,2)$ is I–type and $\eta = (p+1,1)$. The other three cases can be treated similarly. So we have $\|\alpha\| = s+4$ by Theorem 3.6(6b). Let $B(r)$ be the norm disk in the plane $H_1(\partial M;\mathbb{R})$ centered at the origin with radius r. Then $B(s) = B$. The point $\mu = (1,0)$ lies in $\partial B(s)$. There is a positive real number a such that the point $(1+2a,0)$ has norm $\|\mu\|+4 = s+4$. By the convexity of $B(r)$ with any radius r, the line segment in the plane $H_1(\partial M;\mathbb{R})$ with endpoints $(1+2a,0)$ and $(2p+1,2)$ is contained in the norm disk $B(s+4)$. It follows that the line segment with endpoints $(1+a,0)$ and $(p+1,1)$ is contained in the norm disk $B(s+2)$. □

4 Proof of Theorem 1.3

Proof of Theorem 1.3 Suppose otherwise that K is a hyperbolic knot in S^3 on which $17/2$ or $23/2$ is a finite surgery slope. We will get a contradiction from this assumption. Here is an outline of our strategy. Let α be the finite surgery slope $17/2$ or $23/2$ on K, and let δ be the slope

$$\delta = \begin{cases} 9 & \text{if } \alpha = 17/2, \\ 11 & \text{if } \alpha = 23/2. \end{cases}$$

Our first task is to show:

Proposition 4.1 *Dehn surgery on the given hyperbolic knot K with slope δ does not yield a hyperbolic 3–manifold.*

Note that by Known Facts 1.2(4) and Table 4, the knot K has genus 2. It is known that Dehn surgery with the slope δ on any hyperbolic knot in S^3 of genus 2 can never produce a lens space (Known Facts 1.2(7)) or a reducible manifold [37] or a toroidal manifold [34]. It also follows from Lemma 3.3, Known Facts 1.2(3), and Tables 1 and 3 that the δ–surgery on K cannot yield a spherical space form. Therefore, by Proposition 4.1, the δ–surgery on K must produce an irreducible Seifert-fibered space which has infinite fundamental group but does not contain incompressible tori. But this will contradict our next assertion:

Proposition 4.2 *For the given knot K, Dehn surgery with the slope δ cannot yield an irreducible Seifert-fibered space with infinite fundamental group but containing no incompressible tori.*

The rest of this section is devoted to the proofs of the above two propositions. The main tool is the character variety method.

Recall that the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ of $\mathrm{SL}_2(\mathbb{C})$ consists of all 2×2 complex matrices with zero trace. The group $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathrm{sl}_2(\mathbb{C})$ through the adjoint homomorphism Ad: $\mathrm{SL}_2(\mathbb{C})\to \mathrm{Aut}(\mathrm{sl}_2(\mathbb{C}))$ given by matrix conjugation. As $-I$ acts trivially on $\mathrm{sl}_2(\mathbb{C})$, the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{sl}_2(\mathbb{C})$ factors through $\mathrm{PSL}_2(\mathbb{C})$. If $\rho \in R(\Gamma)$ is a $\mathrm{PSL}_2(\mathbb{C})$–representation of a group Γ, we use $\mathrm{Ad}\circ\rho$ to denote the induced action of Γ on $\mathrm{sl}_2(\mathbb{C})$. Let $H^1(\Gamma; \mathrm{Ad}\circ\rho)$ denote the group cohomology with respect to the module $\mathrm{Ad}\circ\rho\colon \Gamma\to \mathrm{Aut}(\mathrm{sl}_2(\mathbb{C}))$. Note that for a connected compact manifold Y and $\rho\in R(Y)$, we have $H^1(Y; \mathrm{Ad}\circ\rho) \cong H^1(\pi_1(Y); \mathrm{Ad}\circ\rho)$.

Lemma 4.3 *Let M be the exterior of a knot in S^3. Suppose that for some slope β, $X(M(\beta))$ has a character χ_{ρ_0} such that*

(i) *χ_{ρ_0} is strictly irreducible,*

(ii) $H^1(M(\beta); \mathrm{Ad} \circ \rho_0) = 0$,

(iii) *the image of ρ_0 does not contain parabolic elements.*

Then the following conclusions hold:

(1) *Viewing χ_{ρ_0} as a point in $X(M(\beta))$, it is a 0–dimensional algebraic component of $X(M(\beta))$, and as a point in $X(M)$, it is a smooth point contained in a unique curve component X_0 of $X(M)$.*

(2) *For the curve component X_0 given in* (1), *the function f_β is not constant on X_0. So in particular, X_0 is not a constant curve.*

(3) *The point χ_{ρ_0} is a zero point of f_β but is not a zero point of f_μ, and moreover, the zero degree of f_β at χ_{ρ_0} is 2.*

Proof The conclusion of part (1) follows from conditions (i) and (ii) and is a special case of [12, Theorem 3] (although the theorem there was stated for $\mathrm{SL}_2(\mathbb{C})$–representations, the same proof applies to $\mathrm{PSL}_2(\mathbb{C})$–representations).

The idea of proof for part (2) is essentially contained in [10] for a similar situation in the $\mathrm{SL}_2(\mathbb{C})$–setting. For the reader's convenience, we give a proof for our current situation. Suppose otherwise that f_β is constant on X_0. Then it is constantly zero on X_0 since χ_{ρ_0} is obviously a zero point of f_β. So $\rho(\beta)$ is either $\pm I$ or a parabolic element for every $\chi_\rho \in X_0$. Note that $\rho(\beta)$ cannot be $\pm I$ for all $\chi_\rho \in X_0$, for otherwise X_0 becomes a curve in $X(M(\beta))$ containing χ_{ρ_0}, which contradicts the fact that χ_{ρ_0} is an isolated point in $X(M(\beta))$. Therefore, $\rho(\beta)$ is parabolic for all but finitely many points χ_ρ in X_0. As the meridian μ commutes with β in $\pi_1(M)$, we have that $\rho(\mu)$ is either $\pm I$ or parabolic for all but finitely many points $\chi_\rho \in X_0$. Hence f_μ is also constantly zero. In particular, χ_{ρ_0} is a zero point of f_μ. But $\rho_0(\mu)$ cannot be $\pm I$, so $\rho_0(\mu)$ is parabolic. This violates condition (iii).

As we have seen in the proof of part (2), χ_{ρ_0} is a zero point of f_β but cannot be a zero point of f_μ. Combined with condition (i), the conclusion of part (3) now follows from [3, Theorem 2.1(2)]. □

Let M be the exterior of the given hyperbolic knot K and $\|\cdot\|$ be the total Culler–Shalen norm on $H_1(\partial M;\mathbb{R})$ defined in Section 3. Recall

$$s=\min\{\|\gamma\|:\gamma\in H_1(\partial M;\mathbb{Z}),\ \gamma\neq 0\}.$$

Let $B(r)$ be the norm disk in the plane $H_1(\partial M;\mathbb{R})$ centered at the origin of radius r. We already knew that each of μ and α is not a boundary slope. By Lemma 3.3, α is an I–type finite surgery slope, and by Theorem 3.6,

$$\|\mu\|=s,\quad \|\alpha\|=s+4.$$

By Lemma 3.8, we have

$$\|\delta\|\leq s+2. \tag{4-1}$$

Proposition 4.1 follows from (4-1) and the following proposition.

Proposition 4.4 *If $M(\delta)$ is a hyperbolic 3–manifold, then*

$$\|\delta\|\geq s+4.$$

Proof Note that by Mostow rigidity the closed hyperbolic 3–manifold $M(\delta)$ has exactly two discretely faithful $\mathrm{PSL}_2(\mathbb{C})$–representations ρ_1 and ρ_2, up to conjugation. Obviously, the two distinct characters χ_{ρ_1} and χ_{ρ_2} satisfy conditions (i) and (iii) of Lemma 4.3. Both of them also satisfy condition (ii) of Lemma 4.3, which is proved in [53]; compare [12, Corollary 5]. Hence Lemma 4.3 applies: each χ_{ρ_i} lies in a unique curve component X_i of $X(M)$ (note that $X_1=X_2$ is possible) as a smooth point, X_i is not a constant curve, and the zero degree of f_δ at χ_{ρ_i} is 2 but f_μ is not zero-valued at χ_{ρ_i}.

Claim *Each X_i is a norm curve component of $X(M)$.*

Proof of claim We just need to show that each X_i is not a seminorm curve. Suppose otherwise that some X_i is a seminorm curve. Let γ_i be the associated slope and $\|\cdot\|_{X_i}$ the corresponding Culler–Shalen seminorm.

By Lemma 4.3(2), $\gamma_i\neq\delta$. By Theorem 3.7(3), $\mu\neq\gamma_i$, μ has the minimal Culler–Shalen seminorm among all slopes $\gamma\neq\gamma_i$ and γ_i is an integer slope. Moreover, by Theorem 3.7(4), γ_i must be slope 8 if $\alpha=17/2$ or slope 10 if $\alpha=23/2$. So we have $\Delta(\gamma_i,\delta)=1$, which implies that $\|\delta\|_{X_i}=\|\mu\|_{X_i}$ by Theorem 3.7(3). But at χ_{ρ_i}, we have that f_δ is zero and f_μ is nonzero. Hence it follows from Theorem 3.7(3) that $\|\delta\|_{X_i}$ is strictly larger than $\|\mu\|_{X_i}$. We get a contradiction and the claim is proved. □

Hence each X_i is a norm curve component of $X(M)$ and thus is a member of the set C, which is the union of all norm curve components of $X(M)$. In particular, both χ_{ρ_1} and χ_{ρ_2} are contained in C which, by Lemma 4.3(3) and Theorem 3.4(5), implies that $\|\delta\| \geq \|\mu\| + 4 = s + 4$. This completes the proof of the proposition. □

Proposition 4.2 follows from (4-1) and the following proposition.

Proposition 4.5 *If* $M(\delta)$ *is an irreducible Seifert-fibered space with infinite fundamental group but containing no incompressible tori, then*

$$\|\delta\| \geq s + 4.$$

Proof Since $M(\delta)$ is an irreducible Seifert-fibered space with infinite fundamental group but containing no incompressible tori, its base orbifold is a 2–sphere with three cone points whose cone orders do not form an elliptic triple. So the base orbifold is $S^2(a,b,c)$ and $1/a + 1/b + 1/c \leq 1$. Note that the fundamental group of $\pi_1(M(\delta))$ surjects onto the orbifold fundamental group of $S^2(a,b,c)$, which is the triangle group

$$\triangle(a,b,c) = \langle x, y : x^a = y^b = (xy)^c = 1 \rangle.$$

We may assume that $a \geq b \geq c \geq 2$. Note that $M(\delta)$ has cyclic first homology of odd order. It follows that $\gcd(a,b,c) = 1$, $b \geq 3$, $a \geq 5$, and at most one of a, b, c is even. In particular, $S^2(a,b,c)$ must be a hyperbolic 2–orbifold and thus $\triangle(a,b,c)$ has a discretely faithful representation ρ_1 into $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{PSL}_2(\mathbb{C})$. Therefore, ρ_1 is strictly irreducible and its image group does not contain parabolic elements.

On the other hand, applying [7, Addendum on page 224], we see that the triangle group $\triangle(a,b,c)$ has a nonabelian representation ρ_2 into $\mathrm{SO}(3) \subset \mathrm{PSL}_2(\mathbb{C})$. Thus ρ_2 is irreducible and its image does not contain parabolic elements. It is easy to check that ρ_2 is also strictly irreducible for otherwise ρ_2 would be a dihedral representation, contradicting the fact that $M(\delta)$ has odd order first homology.

Evidently ρ_1 is not conjugate to ρ_2. So we have two distinct characters χ_{ρ_1} and χ_{ρ_2} in $X(\triangle(a,b,c)) \subset X(M(\delta)) \subset X(M)$. As points in $X(M(\delta))$, the two characters χ_{ρ_1} and χ_{ρ_2} both satisfy conditions (i) and (iii) of Lemma 4.3. They also both meet condition (ii) by [12, Proposition 7] (although the result there is stated for $\mathrm{SL}_2(\mathbb{C})$–representations, a similar argument works for $\mathrm{PSL}_2(\mathbb{C})$–representations). Hence Lemma 4.3 applies: each χ_{ρ_i} lies in a unique curve component X_i of $X(M)$ ($X_1 = X_2$ is possible) as a smooth point, X_i is not a constant curve and the zero degree of f_δ at χ_{ρ_i} is 2 but f_μ is not zero valued at χ_{ρ_i}.

We can now argue exactly as in the proof of Proposition 4.4 to show that each X_i is a norm curve component of $X(M)$, which leads to the conclusion that $\|\delta\| \geq s+4$. $\square$

The proof of Theorem 1.3 is now completed. $\square$

5 Proof of Theorem 1.4, part I

In this section, we prove Theorem 1.4 in case there is a half-integer finite surgery slope on the given knot K. We actually show the following theorem, which provides more information in this case.

Theorem 5.1 *Suppose that K is a hyperbolic knot in S^3 which admits a half-integer finite surgery slope α.*

(1) *α is one of the following slopes:*

$\{43/2, [11,2;3,2]\}$, $\{45/2, [11,2;3,2]\}$, $\{51/2, [13,3;3,2]\}$, $\{53/2, [13,3;3,2]\}$, $\{77/2, [19,2;5,2]\}$, $\{83/2, [21,2;5,2]\}$, $\{103/2, [17,3;3,2]\}$, $\{113/2, [19,3;3,2]\}$.

Here each sample knot attached to a slope in the list plays the same role as before: the same surgery slope on the sample knot yields the same spherical space form, and K has the same knot Floer homology as the sample knot.

(2) *There is at most one other nontrivial finite surgery slope β, and if there is one, it is an integer slope distance 1 from α. The only possible pairs for such α and β are*

$$\{43/2, 21, [11,2;3,2]\}, \quad \{53/2, 27, [13,2;3,2]\},$$
$$\{103/2, 52, [17,3;3,2]\}, \quad \{113/2, 56, [19,3;3,2]\}$$

when β is noncyclic, and

$$\{45/2, 23, [11,2;3,2]\}, \quad \{51/2, 25, [13,2;3,2]\},$$
$$\{77/2, 39, [19,2;5,2]\}, \quad \{83/2, 41, [21,2;5,2]\}$$

when β is cyclic. Here each sample knot attached to a pair plays the same role as before: the same surgery slopes on the sample knot yield the same spherical space forms, and K has the same knot Floer homology as the sample knot.

The proof uses mainly character variety techniques based on Known Facts 1.1 and 1.2. First we need to prepare a few more lemmas.

Lemma 5.2 *Let K be a knot in S^3. Suppose the Alexander polynomial $\Delta_K(t)$ of K has a simple root $\xi = e^{i\theta}$ on the unit circle in the complex plane of order n. Then the knot exterior M of K has a reducible nonabelian $\mathrm{PSL}_2(\mathbb{C})$–representation ρ such that:*

(1) *$\rho(\lambda) = \pm I$, and $\rho(\mu)$ has order n.*

(2) *The character χ_ρ of ρ is contained in a unique nontrivial curve component X_0 of $X(M)$ and is a smooth point of X_0. Moreover, X_0 is either a seminorm curve or a norm curve.*

(3) *For any slope γ on ∂M, if f_γ is nonconstant on X_0 and if the reducible nonabelian character χ_ρ is a zero point of f_γ, then the zero degree of f_γ at χ_ρ is at least 2.*

Proof (1) It was known a long time ago [51; 14] that the exterior of a knot K in S^3 has a reducible, nonabelian $\mathrm{PSL}_2(\mathbb{C})$–representation ρ with $\rho(\lambda) = \pm I$ and $\rho(\mu) = \pm\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ if and only if $\Delta_K(a^2) = 0$. Hence the conclusions of (1) follow from the given conditions.

(2) It was shown in [22] that the given reducible character χ_ρ is an endpoint of a (real) curve of irreducible SO(3) characters and also an endpoint of a (real) curve of irreducible $\mathrm{PSL}_2(\mathbb{R})$ characters. Later [30], it was shown that for the given reducible nonabelian representation ρ, the space of group 1–cocycles $Z^1(\pi_1(M), \mathrm{Ad}\circ\rho)$ is 4–dimensional, ρ is contained in a unique 4–dimensional component R_0 of $R(M)$ as a smooth point, and χ_ρ is contained in a unique 1–dimensional nontrivial component X_0 of $X(M)$ and is a smooth point of X_0 (although in [30], the above conclusions are given for SL_2–representation and character varieties, the same conclusions also hold in PSL_2 setting; see [8, Theorem 4.1; 29]). The fact that X_0 is not a constant curve is proved in [15, Lemma 7.3(d)]. So X_0 is either a seminorm or a norm curve component of $X(M)$.

(3) First note that since

$$\dim_{\mathbb{C}} Z^1(\pi_1(M), \mathrm{Ad}\circ\rho) = \dim_{\mathbb{C}} R_0 = 4,$$

the Zariski tangent space of R_0 at ρ can be identified with $Z^1(\pi_1(M), \mathrm{Ad}\circ\rho)$. By [8, Theorem 4.1], there is an analytic 2–disk D smoothly embedded in R_0 containing ρ such that $D \cap t^{-1}(\chi_\rho) = \{\rho\}$ and $t|_D$ is an analytic isomorphism onto a smooth 2–disk neighborhood of χ_ρ in X_0. One can choose a smooth path ρ_s in $D \subset R_0$

depending differentiably on a real parameter s close to 0, passing through ρ at $s=0$, of the form

$$\rho_s = \pm\exp(su + O(s^2))\rho$$

for some $u \in Z^1(\pi_1(M), \mathrm{Ad}\circ\rho)$; see [25]. Now letting $\sigma(s) = t(\rho_s) \subset X_0$ and calculating as in [9, Section 4], we get

$$f_\gamma(\sigma(s)) = [\mathrm{trace}(\rho_s(\gamma))]^2 - 4 = 2\,\mathrm{trace}(u(\gamma)^2)s^2 + O(s^3),$$

which implies that the zero degree of f_γ at χ_ρ is at least 2 (applying [9, Lemma 4.8]). Here we have used the fact that $\rho(\pi_1(\partial M))$ is a cyclic group of finite order, and thus $f_\gamma(\chi_\rho) = 0$ means $\rho(\gamma) = \pm I$. □

Remark 5.3 Distinct roots of $\Delta_K(t)$ lying on the upper-half unit circle of the complex plane give rise to distinct reducible nonabelian characters because these characters have distinct real values in $(0,1)$ when valued on the meridian μ of the knot.

Remark 5.4 The curve X_0 given in Lemma 5.2(2) is either a norm or seminorm curve on which f_μ is nonconstant. Note that the reducible nonabelian character χ_ρ given in Lemma 5.2 is not a zero point of f_μ. Now suppose that μ is not a boundary slope, so it has the minimal norm or seminorm. Then if f_γ is nonconstant on X_0 and $f_\gamma(\chi_\rho) = 0$ for some slope γ, then the point χ_ρ contributes to the norm or seminorm of γ at least by 2 beyond the norm or seminorm of μ. Also note that χ_ρ is a zero of f_γ if and only if the numerator of γ is divisible by n (which is the order of $\rho(\mu)$).

Lemma 5.5 *Let M be the exterior of a hyperbolic knot K in S^3. Suppose that M admits two I–type surgery slopes β_1 and β_2. Then as points in $X(M)$, the set of two irreducible characters of $X(M(\beta_1))$ is equal to the set of two irreducible characters of $X(M(\beta_2))$* (see Lemma 3.1(3)). *Hence in particular, β_1 and β_2 are of the same $I(q)$–type, q divides $\Delta(\beta_1,\beta_2)$ and $q=2$ or 3. Also $q=3$ if and only if one of β_1 and β_2 is half-integral.*

Proof Because the fundamental group of any I–type spherical space form is of the form $I_{120}\times\mathbb{Z}_j$ and because any irreducible $\mathrm{PSL}_2(\mathbb{C})$–representation of $I_{120}\times\mathbb{Z}_j$ kills the factor $\mathbb{Z}_j$ and sends the factor I_{120} (the binary icosahedral group) onto I_{60} (the icosahedral group), the first conclusion of the lemma follows. The rest of the conclusions of the lemma follow from Lemma 3.3(3) and the distance bound 3 for finite surgery slopes on hyperbolic knots; see Known Facts 1.1(2). □

Lemma 5.6 (1) *If a knot K in S^3 admits a D–type finite surgery, then* $\det(K) > 1$.

(2) *If a knot K in S^3 admits an O–type finite surgery, then* $\det(K) = 3$.

(3) *If a knot K in S^3 admits a cyclic surgery slope with an even numerator, then* $\det(K) = 1$.

Proof The lemma follows from [33, Theorem 10] which states that for any knot K in S^3, its knot group has precisely $\frac{1}{2}(\det(K) - 1)$ distinct $\mathrm{PSL}_2(\mathbb{C})$ dihedral representations, modulo conjugation, and moreover any such representation will kill any slope with even numerator. □

Now we proceed to prove Theorem 5.1. Part (1) of the theorem is just the combination of Known Facts 1.2(4) and Theorem 1.3. So the slope α is one of the eight elements in

$$\{43/2,\ 45/2,\ 51/2,\ 53/2,\ 77/2,\ 83/2,\ 103/2,\ 113/2\}.$$

We divide the proof of part (2) correspondingly into eight cases. If β is another nontrivial finite surgery slope on K, then β must be an integer slope by Known Facts 1.1(2). We shall show, in each of the eight cases, that

(1) the assumption $\Delta(\alpha, \beta) > 1$ will lead to a contradiction,

(2) there is at most one such β and $\{\alpha, \beta\}$ is one of the pairs listed in Theorem 5.1.

That each attached sample knot has the said properties will also be checked.

Before we get into the cases, we make some general notes that apply to every case. If $\Delta(\alpha, \beta) > 1$, then $\Delta(\alpha, \beta) \geq 3$, and thus $\Delta(\alpha, \beta) = 3$ and β is noncyclic by Known Facts 1.1(2)–(3). Also if $\Delta(\alpha, \beta) > 1$, then β is not a boundary slope by [16, Theorem 2.0.3]. Recall that by the same reason, each of α and μ is not a boundary slope. By Lemma 3.3, α is either a T–type or I–type slope. In fact, α is T–type if and only if its numerator is divisible by 3. Let $\|\cdot\|$ be the total Culler–Shalen norm defined by the norm curve set C and $B(r)$ the norm disk of radius r. By Theorem 3.6, $\|\mu\| = s$ has minimal norm among all slopes, μ is in $\partial B(s)$ but is not a vertex of $B(s)$, $\|\alpha\| = s + 2$ if α is T–type and $\|\alpha\| = s + 4$ if α is I–type.

Case 1 $\boldsymbol{\alpha = 43/2}$ By Known Facts 1.2(4) and Table 4, K has the same Alexander polynomial as $[11, 2; 3, 2]$, which is

$$\Delta_K(t) = \Delta_{T(11,2)}(t)\Delta_{T(3,2)}(t^2).$$

In particular, $\det(K) = \det(T(11, 2)) = 11$.

If $\Delta(\alpha,\beta)=3$, then β is either 20 or 23. If $\beta=23$, then by Lemma 3.3, β is I–type. But this is impossible by Known Facts 1.2(3), Table 3 and Lemma 2.3. If $\beta=20$, then β is D–type by Lemma 3.3. But this is impossible by Known Facts 1.2(5), Table 5 and Lemma 2.3.

So $\Delta(\alpha,\beta)=1$ and β is either 21 or 22. By Lemma 3.3 and Lemma 5.6, 22 cannot be a finite surgery slope for K. So the only possible value for β is 21.

Claim $\beta=21$ *cannot be a cyclic slope on* K.

Proof of claim By Known Facts 1.2(6), if 21 is a cyclic slope for K, then there is a Berge knot K_0 on which 21 is also a cyclic slope and K_0 has the same Alexander polynomial as K and thus as $[11,2;3,2]$. By [4, Table 1], K_0 is not hyperbolic and thus is a torus knot or a cable over torus knot. From the Alexander polynomial, we see that K_0 has to be $[11,2;3,2]$. But then 21 is not a cyclic slope for $[11,2;3,2]$ (by eg [6, Table 1]). The claim is proved. □

So $\beta=21$ can only possibly be a T–type slope by Lemma 3.3. Finally we note that 21 is a T–type slope for $[11,2;3,2]$; see Table 1. Thus in this case we arrive at $\{43/2,21,[11,2;3,2]\}$. Theorem 5.1(2) is proved in this case.

Case 2 $\boldsymbol{\alpha=45/2}$ Then α is a T–type finite surgery slope and K has the same Alexander polynomial as $[11,2;3,2]$, which is

$$\Delta_K(t)=\Delta_{T(11,2)}(t)\Delta_{T(3,2)}(t^2).$$

If $\Delta(\alpha,\beta)=3$, then β is either 21 or 24. If $\beta=24$, then by Lemma 3.3, β is D–type. But this is impossible by Table 5 and Lemma 2.3.

When $\beta=21$, it is a T–type slope and $\|\alpha\|=s+2$. As $\Delta(\alpha,\beta)=3$, we have $\|\beta\|=\|\alpha\|$ by Lemma 3.3 and Theorem 3.6(3). By the convexity of the norm disk of any radius, the line segment with endpoints $(21,1)$ and $(45,2)$ is contained in the norm disk of radius $s+2$ and in particular the midpoint $\left(33,\frac{3}{2}\right)$ of the segment has norm less than or equal to $s+2$. As $\left\|\left(33,\frac{3}{2}\right)\right\|=\frac{3}{2}\|(22,1)\|\geq\frac{3}{2}s$, we have $\frac{3}{2}s\leq s+2$, from which we get

$$s\leq 4. \tag{5-1}$$

The integer slope 23 is distance 1 from α and thus $\|(23,1)\|\leq s+1$ by Lemma 3.8. As $s\leq 4$, the norm of the point $(1.25,0)$ is at most $s+1$. Hence the line segment with endpoints $(23, 1)$ and $(1.25, 0)$ is contained in $B(s+1)$. It follows that the line segment with endpoints $(24, 1)$ and $(2.25, 0)$ is contained in $B(2s+1)$. In particular,

$$\|(24, 1)\| \le 2s + 1. \tag{5-2}$$

On the other hand, the roots of the Alexander polynomial $\Delta_K(t)$ of K are all simple and are all roots of unity. The factor $\Delta_{T(3,2)}(t^2)$ of $\Delta_K(t)$ contains three roots on the upper-half unit circle in the complex plane: $e^{\pi i/3}$, $e^{\pi i/6}$ and $e^{5\pi i/6}$, of order 6, 12 and 12, respectively, so the corresponding three distinct reducible nonabelian $\mathrm{PSL}_2(\mathbb{C})$ characters of $X(M)$ factor through $M(24)$; see Lemma 5.2 and Remarks 5.3 and 5.4. If they are all contained in the norm curve set C then they will contribute to the norm of $(24, 1)$ by at least 6 beyond $s = \|\mu\|$; see Lemma 5.2(3). Hence we have

$$\|(24, 1)\| \ge s + 6. \tag{5-3}$$

Combining (5-2) and (5-3), we have $s + 6 \le 2s + 1$, ie $s \ge 5$, which contradicts (5-1).

Hence at least one of the above three reducible nonabelian characters, which we denote by χ_{ρ_0}, is not contained in C. By Lemma 5.2(2) and the definition of C we see that χ_{ρ_0} is contained in a nontrivial seminorm curve component X_0. Now by Theorem 3.7(3)–(4) we have that the associated boundary slope γ_0 of X_0 is an integer and $\Delta(\alpha, \gamma_0) = 1$. Hence $\gamma_0 = 22$ or 23. But f_{γ_0} must be constantly zero on X_0 (otherwise $\widetilde{f}_\mu$ would have larger zero degree than $\widetilde{f}_{\gamma_0}$ at some point of $\widetilde{X}_0$ which is impossible by [16, Proposition 1.1.3]) and in particular is zero valued at the nonabelian reducible character χ_{ρ_0}, which implies that γ_0 is divisible by 6; see Remark 5.4. We arrive at a contradiction.

So $\Delta(\alpha, \beta) = 1$ and β is either 22 or 23. For the same reasons as given in Case 1, we see that 22 cannot be a finite surgery slope and 23 cannot be a noncyclic finite surgery slope. So $\beta = 23$ is possibly a cyclic slope for K. In fact, 23 is a cyclic slope for $[11, 2; 3, 2]$. Hence in Case 2, we arrive at the pair $\{45/2, 23\}$ with the sample knot $[11, 2; 3, 2]$.

Case 3 $\boldsymbol{\alpha = 51/2}$ This case can be handled very similarly to Case 2, and we get the pair $\{51/2, 25\}$, where 25 is a possible cyclic slope, with $[13, 2; 3, 2]$ as a sample knot.

Case 4 $\boldsymbol{\alpha = 53/2}$ In this case, K has the same Alexander polynomial as the sample knot $[13, 2; 3, 2]$. If $\Delta(\alpha, \beta) = 3$, then β is either 25 or 28. By Lemma 3.3, 25 cannot be a finite surgery slope. Using Lemma 3.3, Table 5 and Lemma 2.3, we can easily rule out $\beta = 28$. So $\beta = 26$ or 27. By Lemma 3.3 and Lemma 5.6(2), $\beta = 26$ cannot be a noncyclic finite surgery slope and by Lemma 5.6(3), $\beta = 26$ cannot be a cyclic surgery slope. Hence $\beta = 27$ which is a T–type slope for the sample knot $[13, 2; 3, 2]$. So we just need to rule out the possibility that $\beta = 27$ is a cyclic slope for K. This can be done by checking that there is no Berge knot which has 27 as cyclic slope and has the same Alexander polynomial as $[13, 2; 3, 2]$. So in this case, we get the member $\{53/2, 27, [13, 2; 3, 2]\}$.

Case 5 $\boldsymbol{\alpha = 77/2}$ The argument is pretty much similar to that for Case 2. The knot K has the same Alexander polynomial as $[19, 2; 5, 2]$, which is

$$\Delta_K(t) = \Delta_{T(19,2)}(t)\Delta_{T(5,2)}(t^2).$$

In particular, $\det(K) = \det(T(19, 2)) = 19$.

If $\Delta(\alpha, \beta) = 3$, then β is either 37 or 40. The Alexander polynomial of K has six simple roots provided by the factor $\Delta_{T(5,2)}(t^2)$ on the upper-half unit circle: $e^{k\pi i/10}$ for $k = 1, 2, 3, 6, 7, 9$; these are of order 10 or 20, and they give rise to six reducible nonabelian $\mathrm{PSL}_2(\mathbb{C})$ characters, all of which factor through $M(40)$. Hence 40 cannot be a finite surgery slope for K.

So suppose $\beta = 37$. Both α and β are of I–type. As $\|\alpha\| = s+4$, we have $\|\beta\| = s+4$ by Lemma 5.5. So the line segment with endpoints $(37, 1)$ and $(77, 2)$ is contained in $B(s + 4)$; in particular, the midpoint $\left(57, \frac{3}{2}\right)$ of the segment has norm less than or equal to $s + 4$. As $\left\|\left(57, \frac{3}{2}\right)\right\| = \frac{3}{2}\|(38, 1)\| \geq \frac{3}{2}s$, we have $\frac{3}{2}s \leq s + 4$, whence $s \leq 8$.

The integer slope 39 is distance 1 from α, and thus $\|(39, 1)\| \leq s + 2$ by Lemma 3.8. Hence the line segment with endpoints $(39, 1)$ and $(1.25, 0)$ is contained in the norm disk of radius $s + 2$. It follows that the line segment with endpoints $(40, 1)$ and $(2.25, 0)$ is contained in $B(2s + 2)$. In particular, $\|(40, 1)\| \leq 2s + 2$.

On the other hand, if the above six reducible nonabelian characters are all contained in the norm curve set C, then they would contribute to the norm of $(40, 1)$ by at least 12 beyond s; ie $\|(40, 1)\| \geq s + 12$. Hence combining the last two inequalities, we have $s + 12 \leq 2s + 2$; ie $s \geq 10$. We arrive at a contradiction with the earlier inequality $s \leq 8$.

So at least one of the above six reducible nonabelian characters is not contained in C in which case we can get a contradiction exactly as in Case 2.

Hence if β is another nontrivial finite surgery slope, then $\Delta(\alpha, \beta) = 1$ and β is either 38 or 39. By Lemma 3.3 and Lemma 5.6(2)–(3), 38 cannot be a finite surgery slope for K. If 39 is a finite surgery slope, it cannot be noncyclic by Lemma 3.3 and Table 1.

It could be a cyclic slope for K. In fact, it is a cyclic slope of $[19, 2; 5, 2]$ by [6, Table 1]. So in this case, $\alpha = 77/2$ and $\beta = 39$ are the only possible finite surgery slopes for K (the former an I–type and the latter a C–type), with $[19, 2; 5, 2]$ as a sample knot.

Case 6 $\boldsymbol{\alpha = 83/2}$ This case can be treated very similarly to Case 5, and $\alpha = 83/2$ and $\beta = 41$ are the only possible finite surgery slopes for K (with α an I–type and β a C–type), with $[21, 2; 5, 2]$ as a sample knot.

Case 7 $\boldsymbol{\alpha = 103/2}$ This is perhaps the hardest case. We know that α is I–type, and K has the same Alexander polynomial as $[17, 3; 3, 2]$, which is

$$\Delta_K(t) = \Delta_{T(17,3)}(t)\Delta_{T(3,2)}(t^3).$$

If $\Delta(\alpha, \beta) = 3$, then β is either 50 or 53. By Lemma 3.3 and Table 3, 53 cannot be a finite surgery slope for K. If $\beta = 50$ is a finite surgery slope, it is O–type by Lemma 3.3. We have $\|\alpha\| = s + 4$ and $\|\beta\| \le s + 3$ by Theorem 3.6(4) and (6). So the line segment with endpoints $(50, 1)$ and $(103, 2)$ is contained in the norm disk of radius $s + 4$, and moreover, the midpoint $\frac{3}{2}(51, 1)$ of the segment is contained in the interior of $B(s + 4)$ and thus has norm less than $s + 4$. But $\frac{3}{2}\|(51, 1)\| \ge \frac{3}{2}s$. So we have $\frac{3}{2}s < s + 4$, from which we get $s \le 7$.

As K is fibered by Known Facts 1.2(1), we may apply [24, Theorem 5.3], which asserts that there is an essential lamination in M with a degenerate slope γ_0 such that $M(\gamma)$ has an essential lamination and thus has infinite fundamental group if $\Delta(\gamma, \gamma_0) > 1$. Hence γ_0 must be the slope 51. Furthermore, by [54, Theorem 2.5] combined with the geometrization theorem of Perelman, $M(\gamma)$ is hyperbolic if $\Delta(\gamma, \gamma_0) > 2$. Hence $M(54)$ is hyperbolic. In particular, $M(54)$ has two discretely faithful characters corresponding to the hyperbolic structure which must be contained in C (by the proof of Proposition 4.4). So these two points of C contribute to the norm $\|(54, 1)\|$ by 4 beyond s. Since $\Delta(\alpha, \beta) = 3$ and β is O–type, α cannot be $I(3)$–type; see Remark 3.2. Similarly, α cannot be $I(2)$–type since $\Delta(\alpha, \mu) = 2$. Thus α is $I(5)$–type by Lemma 3.3. Hence the two irreducible characters of $M(\alpha)$ factor through $M(54)$ (see Remark 3.2 again), and these two characters are contained in C by Theorem 3.6(6b). Hence these two points of C contribute another 4 to the norm $\|(54, 1)\|$ beyond s; see Remark 3.5.

The Alexander polynomial of K has four simple roots of orders divisible by 6 (they are roots of the factor $\Delta_{T(3,2)}(t^3)$) which provide four reducible nonabelian characters which factor through $M(54)$. Let χ_{ρ_0} be the irreducible character of $M(\beta)$ such that the image of ρ_0 is the octahedral group. By Lemma 3.3, ρ_0 also factors through $M(54)$. If all the four reducible nonabelian characters and the O–type character χ_{ρ_0} are contained in C, then $\|(54,1)\| \geq s+4+4+8+2 = s+18$. On the other hand, by Lemma 3.8, $\|(52,1)\| \leq s+2$, from which we see that $\|(54,1)\| \leq 3s+2$. So $s+18 \leq 3s+2$, ie $s \geq 8$, yielding a contradiction with the early conclusion that $s \leq 7$.

So some of the four reducible nonabelian characters or the O–type character χ_{ρ_0} are not contained in C. If some of the four reducible nonabelian characters are not contained in C, then we can get a contradiction similar to that in Case 2. Thus we may suppose that the O–type character χ_{ρ_0} is not contained in C and all the four reducible nonabelian characters are contained in C. Then the same argument as above yields $\|(54,1)\| \geq s+16$ and $s+16 \leq 3s+2$; ie $s \geq 7$. Hence $s = 7$. Since χ_{ρ_0} is not contained in C, we have $\|\beta\| \leq s+1 = 8$ by Theorem 3.6(4). As $\|\alpha\| = \|(103,2)\| = s+4 = 11$, the point $(824/11, 16/11)$ (which lies in the line segment with end points $(0,0)$ and $(103,2)$) has norm 8. So the line segment with endpoints $(50,1)$ and $(824/11, 16/11)$ is contained in $B(8)$. The intersection point of this line segment with the line passing through $(0,0)$ and $(51,1)$ is $(1224/19, 24/19)$. So $\|(1224/19, 24/19)\| \leq 8$. But $\|(1224/19, 24/19)\| = \left\|\frac{24}{19}(51,1)\right\| = \frac{24}{19}\|(51,1)\| \geq \frac{24}{19}s = 168/19$. So we would have $168/19 \leq 8$, which is absurd. This final contradiction shows that 50 cannot be a finite surgery slope for K.

So β is possibly 51 or 52. If 51 is a finite noncyclic surgery slope for K, then it is T–type. But this cannot happen from Table 1 and Lemma 2.3. With a similar argument as that of the claim in Case 1, 51 cannot be a cyclic surgery slope. So β is possibly 52. In fact, 52 is a D–type surgery slope for [17, 3; 3, 2]; see [6, Table 1]. From Lemma 5.6(3), 52 cannot be a cyclic surgery slope. Hence in this case, we have possibly $\alpha = 103/2$ an I–type and $\beta = 52$ a D–type, with [17, 3; 3, 2] as an sample knot.

Case 8 $\boldsymbol{\alpha = 113/2}$ This case can be handled entirely as Case 7, and the only possibility is $\alpha = 113/2$ an I–type and $\beta = 56$ a D–type, with [19, 3; 3, 2] as a sample knot.

The proof of Theorem 5.1 is finished. □

6 Proof of Theorem 1.4, part II, and proof of Theorem 1.5

Theorem 1.4 is included in the combination of Theorem 5.1 and the following theorem, which we prove in this section.

Theorem 6.1 *Let* K *be a hyperbolic knot in* S^3 *which does not admit a half-integer finite surgery.*

(1) *The distance between any two integer finite surgery slopes is at most* 2. *Consequently, there are at most three nontrivial finite surgery slopes, and if three, they are consecutive integers.*

(2) *There are at most two integer noncyclic finite surgery slopes, and all possible such pairs of slopes are*

$$\{22, 23, P(-2,3,9)\}, \quad \{28, 29, -K(1,1,0)\}, \quad \{50, 52, [17,3;3,2]\},$$
$$\{56, 58, [19,3;3,2]\}, \quad \{91, 93, [23,4;3,2]\}, \quad \{99, 101, [25,4;3,2]\}.$$

Also included with each pair is a sample knot which has identical knot Floer homology and the same pair of finite surgeries as K.

(3) *If there are three integer finite surgery slopes on* K, *they must be the triple* $(17,18,19)$, *and they produce the same spherical space forms as those on the pretzel knot* $P(-2,3,7)$. *Also,* K *has the same knot Floer homology as* $P(-2,3,7)$.

(4) *If there are three integer finite surgery slopes on* K, *then* K *is the knot* $P(-2,3,7)$.

(5) *If there are two finite surgery slopes on* K *realizing distance* 2, *they must be one of the following pairs:*

$$\{17, 19, P(-2,3,7)\}, \quad \{21, 23, [11,2;3,2]\}, \quad \{27, 25, [13,2;3,2]\},$$
$$\{37, 39, [19,2;5,2]\}, \quad \{43, 41, [21,2;5,2]\}, \quad \{50, 52, [17,3;3,2]\},$$
$$\{56, 58, [19,3;3,2]\}, \quad \{91, 93, [23,4;3,2]\}, \quad \{99, 101, [25,4;3,2]\}.$$

Also included with each pair is a sample knot which has identical knot Floer homology and the same pair of finite surgeries as K.

Of course, part (4) of the theorem supersedes part (3), but to get part (4) we need to get part (3) first.

We assumed that K has no half-integer finite surgery slope, so by Known Facts 1.1(2), all nontrivial finite surgery slopes of K are integers, and their mutual distance is at most 3. Also, Known Facts 1.2(3) puts significant restrictions on possible T-, O- and I–type finite surgeries, and Known Facts 1.2(6) on cyclic surgeries. The main issue arises when a D–type finite surgery is involved, in which case our method is to apply the Casson–Walker invariant. We first make some preparations accordingly. Along the way, we shall also give a proof of Theorem 1.5.

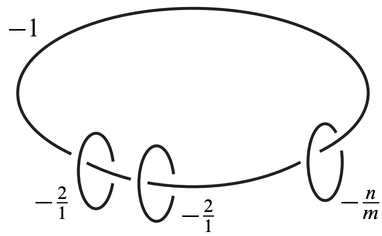

Figure 1: A surgery diagram of $P(n,m)$

Let $P(n,m)$ be the prism manifold with Seifert invariants

$$(-1;(2,1),(2,1),(n,m)),$$

where $n>1$ and $\gcd(n,m)=1$. It is easy to see that $P(n,-m)=-P(n,m)$ and $|H_1(P(n,m))|=|4m|$. As noted earlier, every D–type spherical space form is homeomorphic to some $P(n,m)$.

Given a real number x, let $\{x\}=x-\lfloor x\rfloor$ be the fractional part of x. Given a pair of coprime integers p,q with $p>0$, let $\boldsymbol{s}(q,p)$ be the Dedekind sum

$$\boldsymbol{s}(q,p)=\sum_{i=1}^{p-1}\left(\left(\frac{i}{p}\right)\right)\left(\left(\frac{iq}{p}\right)\right),$$

where

$$((x))=\begin{cases}\{x\}-\frac{1}{2} & \text{if } x\in\mathbb{R}\setminus\mathbb{Z},\\ 0 & \text{if } x\in\mathbb{Z}.\end{cases}$$

By [35, Proposition 6.1.1], the Casson–Walker invariant of $-P(n,m)$, when $m>0$, can be computed by the formula

$$\lambda(-P(n,m))=-\frac{1}{12}\Big(-\frac{n}{m}\Big(\frac{1}{n^2}-\frac{1}{2}\Big)-\frac{m}{n}+3+12\boldsymbol{s}(m,n)\Big). \tag{6-1}$$

The Casson–Walker invariant has the following surgery formula for knots in S^3 [52, Theorem 4.2]:

$$\lambda(S^3_K(p/q))=-\boldsymbol{s}(q,p)+\frac{q}{p}\Delta''_K(1). \tag{6-2}$$

Here, the Alexander polynomial $\Delta_K(t)$ is normalized to be symmetric and to obey $\Delta_K(1)=1$.

Note that the Casson–Walker invariant has the property $\lambda(-Y)=-\lambda(Y)$. In our application, it is sufficient to use only $|\lambda(Y)|$, so we do not have to worry about the orientation of the manifold involved.

Lemma 6.2 *If a knot K in S^3 admits a D–type finite surgery slope β, then the numerator of β is an integer $4m$, and the resulting prism manifold is $\varepsilon P(n,|m|)$ for some $\varepsilon \in \{\pm\}$, where n is the determinant of the knot K. In particular, if $\det(K)=1$, then K does not admit D–type finite surgery.*

Proof This lemma is just a refinement of Lemma 3.3(1) and Lemma 5.6(1). Note that $|H_1(\varepsilon P(n,m))| = |4m|$. So we just need to show that $n=\det(K)$. Theorem 10 of [33] says that for any knot K in S^3, its knot group has precisely $\frac{1}{2}(\det(K)-1)$ distinct $\mathrm{PSL}_2(\mathbb{C})$ dihedral representations, modulo conjugation, and moreover, any such representation will kill any slope with even numerator. On the other hand, for a prism manifold $\varepsilon P(n,m)$, it has precisely $\frac{1}{2}(n-1)$ distinct $\mathrm{PSL}_2(\mathbb{C})$ dihedral representations, modulo conjugation [3, Proposition D]. As a D–type finite surgery on a knot in S^3 is actually $D(2)$–type (see Lemma 3.3(1)), the set of dihedral representations of any D–type surgery manifold on a knot in S^3 is precisely the set of dihedral representations of the knot group. The conclusion of the lemma follows. □

Proof of Theorem 1.5 By Lemma 6.2, the surgery slope is $4m/q$, $(4m,q)=1$, and $n=\det(K)$. Up to reversing the orientation of $P(n,m)$, we may assume m is positive and up to replacing K by its mirror image, we may assume that $q>0$. So by Known Facts 1.2(2), $4m/q \geq 2g(K)-1$. Also by Known Facts 1.2(2), the nonzero coefficients of the Alexander polynomial of K are all ± 1, and by Known Facts 1.2(1), the knot K is fibered, and thus the degree of the Alexander polynomial is $2g(K)$, which together imply that

$$n=\det(K)=|\Delta_K(-1)| \leq 2g(K)+1 \leq \frac{4m}{q}+2 \leq 4m+2.$$

Since $n=\det(K)$ must be an odd integer, we just need to rule out the possibility of $n=4m+1$. If $n=4m+1$, then $g(K)=2m$, $q=1$ and

$$\Delta_K(t)=1+\sum_{i=1}^{2m}(-1)^i(t^i+t^{-i}).$$

It follows that $\Delta_K''(1)=4m^2+2m$. Since

$$s(1,4m)=\sum_{i=1}^{4m-1}\Big(\frac{i}{4m}-\frac{1}{2}\Big)^2=\frac{8m^2-6m+1}{24m}, \tag{6-3}$$

using (6-2), we get

$$\lambda(S^3_K(4m))=\frac{16m^2+18m-1}{24m}.$$

On the other hand, we have

$$\begin{aligned}
s(m,4m+1) &= \sum_{i=1}^{4m}\Big(\Big(\frac{i}{4m+1}\Big)\Big)\Big(\Big(\frac{im}{4m+1}\Big)\Big)\\
&= \sum_{i=1}^{4m}\Big(\Big(\frac{4i}{4m+1}\Big)\Big)\Big(\Big(\frac{-i}{4m+1}\Big)\Big)\\
&= \sum_{k=0}^{3}\sum_{j=1}^{m}\Big(\Big(\frac{4(km+j)}{4m+1}\Big)\Big)\Big(\Big(\frac{-km-j}{4m+1}\Big)\Big)\\
&= \sum_{k=0}^{3}\sum_{j=1}^{m}\Big(\frac{4j-k}{4m+1}-\frac{1}{2}\Big)\Big(\frac{(4-k)m+1-j}{4m+1}-\frac{1}{2}\Big)\\
&= \frac{4m-m^2}{12m+3}.
\end{aligned}$$

So it follows from (6-1) that

$$\lambda(-P(4m+1,m)) = \frac{2m^2-18m+1}{24m}.$$

Thus $\lambda(S^3_K(4m)) \neq \pm\lambda(-P(4m+1,m))$ for any positive integer m. We get a contradiction. This shows $n \neq 4m+1$. □

Proof of Theorem 6.1 Suppose α and β are two integer finite surgery slopes on K. To prove part (1) of the theorem we only need to rule out the possibility of $\Delta(\alpha,\beta)=3$. So suppose that $\Delta(\alpha,\beta)=3$. Then one of α and β, say α, is an odd integer and the other, β, is an even integer. We know from Known Facts 1.1(3) that neither α nor β can be C–type. So by Lemma 3.3, α is I–type or T–type and β is O–type or D–type. In fact, α cannot be T–type for otherwise by Lemma 3.3(2) α is $T(3)$–type which means, since $\Delta(\alpha,\beta)=3$, that the irreducible representation of $M(\alpha)$ with image T_{12} also factors through $M(\beta)$; ie $M(\beta)$ has an irreducible representation with image T_{12}. But this is impossible since any O–type or D–type spherical space form does not have such representation. So α is I–type. Now from Tables 2 and 3, one can check quickly that there is no sample knot which admits an integer I–type surgery and an integer O–type surgery, distance 3 apart (one just need to check for those sample knots in Table 3 with $\det(K)=3$ and there are only 7 of them). Hence by Lemma 2.3 and Known Facts 1.2(3), there is no knot in S^3 which admits an integer I–type surgery and an integer O–type surgery, distance 3 apart. So β is a D–type slope.

So we have that α is I–type, β is D–type and they are distance 3 apart. To rule out this case, we shall apply the Casson–Walker invariant. By Known Facts 1.2(3), α is one of the slopes given in Table 3 and K has the same Alexander polynomial (in particular the same determinant) as the corresponding sample knot. We only need to consider those slopes in the table whose sample knots have determinant larger than 1 (by Lemma 6.2). We may express such a slope as $\alpha = 4m + 3$ or $4m - 3$ for some integer $m > 0$. So we just need to show that $S^3_K(4m)$ is not a prism manifold. To do this, we compute $\lambda(S^3_K(4m))$ using (6-2) and compute $\lambda(-P(n,m))$ for $n = \det(K)$ using (6-1), and check whether $|\lambda(S^3_K(4m))|$ is equal to $|\lambda(-P(n,m))|$. Also note that by Known Facts 1.2(5), if $4m \le 32$, then K must have the same Alexander polynomial as a corresponding sample knot given in Table 5. This finite process of computation shows that the only possible case is when $m = 1$ and the corresponding sample knot is $T(3,2)$. It follows from [45] that K must be $T(3,2)$, contradicting the assumption that K is hyperbolic. Part (1) of the theorem is proved.

Part (2) is treated with a similar strategy. Suppose α and β are two distinct integer noncyclic finite surgery slopes of K. We are going to show that (α,β) must be one of the pairs listed in part (3) with the corresponding sample knot playing the said role, and that there cannot be a third noncyclic finite surgery on K. By part (1), $\Delta(\alpha,\beta) = 2$ or 1.

Let us first consider the case when $\Delta(\alpha,\beta) = 2$. Then α and β are both odd or both even integers. If they are both odd, then each of α and β is T–type or I–type by Lemma 3.3. Then by Known Facts 1.2(3) and Lemma 2.3, we just need to check which sample knots in Tables 1 and 3 have two slopes listed in these tables distance 2 apart. There are only four such instances:

$$\{1,3,T(3,2)\},\quad \{9,11,T(3,2)\},\quad \{91,93,[23,4;3,2]\},\quad \{99,101,[25,4;3,2]\}.$$

The first two instances can be excluded due to [45]. In the third instance, we just need to show that 92 cannot be a finite noncyclic surgery slope for the same knot K. Suppose otherwise that 92 is a noncyclic finite surgery slope for K. Then it must be a D–type surgery slope by Lemma 3.3. But by Lemma 6.2, the resulting prism manifold would be $\varepsilon P(23,23)$, which does not make sense since 23 and 23 are not relative prime integers. Thus 92 cannot be a D–type slope for K. The fourth instance can be treated exactly as the third one.

If both α and β are even, then each of α and β is O–type or D–type by Lemma 3.3. Note that α and β cannot both be D–type by Known Facts 1.1(4). We see that α and β cannot both be O–type from Known Facts 1.2(3) and Table 2. So we may assume that α is an O–type slope and β a D–type slope. Each slope α in Table 2 can be expressed as $4m+2$. If $4m$ or $4m+4$ is a D–type slope for K, then we should have $|\lambda(P(3,m))| = |\lambda(S^3_K(4m))|$ or $|\lambda(P(3,m+1))| = |\lambda(S^3_K(4m+4))|$, respectively. Calculation using (6-1) and (6-2) shows that this happens only in four instances:

$$\{2,4,T(3,2)\}, \quad \{8,10,T(3,2)\}, \quad \{50,52,[17,3;3,2]\}, \quad \{56,58,[19,3;3,2]\}.$$

Again the first two instances cannot happen for a hyperbolic knot due to [45]. In the third instance, 52 is indeed a D–type slope for $[17,3;3,2]$, and one can easily rule out the possibility for 51 to be a noncyclic finite surgery slope. The fourth instance can be treated exactly as the third one.

Next we consider the case when $\Delta(\alpha,\beta)=1$. As in part (1), we may assume α is a T–type or I–type slope and β an O–type or D–type slope. With a similar process as used in part (1), we only obtain the following instances (with the trefoil case excluded):

$$\{7,8,T(5,2)\}, \quad \{12,13,T(5,2)\}, \quad \{22,23,P(-2,3,9)\}, \quad \{28,29,-K(1,1,0)\}.$$

We note that each pair of slopes are realized on the sample knot as noncyclic finite surgery slopes, and, by the result obtained in the preceding two paragraphs, there is no third noncyclic finite surgery in each instance for the same hyperbolic knot K. The first two instances with the sample knot $T(5,2)$ can be ruled out by Theorem 1.6. Part (2) of the theorem is proved.

To prove part (3), suppose that K has three integer finite surgery slopes. They are consecutive integers by part (1). At least one of them is noncyclic by Known Facts 1.1(1), and at most two of them are noncyclic by part (2).

If two of them are noncyclic, then the two slopes must be one the pairs listed in part (2) of the theorem with the corresponding sample knot. The case of $\{7,8,T(5,2)\}$ cannot happen since any hyperbolic knot cannot have a cyclic surgery slope 6 or 9 by Known Facts 1.2(7). In each of other cases, the same hyperbolic knot K can no longer have a cyclic surgery slope. For if it does, then by Known Facts 1.2(6) there will be a Berge knot having the same cyclic slope and same Alexander polynomial as the corresponding sample knot attached to the pair of noncyclic finite surgery slopes. But one can check (which is a finite process) that there does not exist such Berge knot.

So we may assume that there are exactly one noncyclic finite slope and two cyclic surgery slopes on K. By Known Facts 1.1(4)–(5), the noncyclic finite surgery slope α cannot be D–type or O–type. So α is a T–type or I–type surgery slope belonging to

Table 1 or Table 3. In particular, it is a positive integer less than or equal to 221. Also the determinant of K is 1 by Lemma 5.6(3) and so there are only 14 possible values for α. In each of the 14 cases, we check that there is no Berge knot K_0 such that K_0 admits two integer cyclic surgery slopes which form consecutive integers with α and that K_0 has the same Alexander polynomial as the sample knot attached to the slope α in Table 1 or Table 3, except for the case when $\alpha = 17$. In fact, we have:

Lemma 6.3 *If for some positive integer $p \le 222$, we have that p and $p+1$ are cyclic surgery slopes for a nontrivial Berge knot, then p is one of the ten values* 18, 30, 31, 67, 79, 116, 128, 165, 177, 214. *If for such p, we have that p, $p+1$ and a slope α from Tables 1 or 3 form consecutive integers, then the corresponding Berge knot and the corresponding sample knot associated to α have different Alexander polynomials, except when $\alpha = 17$ ($p = 18$).*

Proof We use a Mathematica program to check the following fact: if for some positive integer $p \le 222$ and integers q_1, q_2, both $L(p, q_1)$ and $L(p+1, q_2)$ satisfy Condition 2.1, then p is one of the ten values in the lemma. For each of these values of p, there is only one possible Alexander polynomial for the corresponding knot, which can be realized by a Berge knot. In fact, for each integer n, the Eudave-Muñoz knot $k(2, 2, n, 0)$ has two lens space surgeries with slopes $49n - 18$ and $49n - 19$, and the knot $k(2, -1, 2, 0)$ has two lens space surgeries with slopes 30 and 31; see [20]. Moreover, the Alexander polynomials of these knots are different from those of the knots in Tables 1 and 3 except when $p = 18$. □

When $\alpha = 17$, the sample knot is $P(-2, 3, 7)$, which does have 18 and 19 as cyclic surgery slopes. Part (3) of the theorem is proved.

Based on part (3), we can quickly prove part (4). If K admits three nontrivial finite surgery, then by part (3), the surgery slopes are the triple 17, 18, 19, with 17 an I–type and 18, 19 C–type slopes, and K has the same knot Floer homology as $P(-2, 3, 7)$. In particular, K has genus 5. Now applying Known Facts 1.2(7) to the cyclic slope 19, we see that K is a Berge knot. But among all hyperbolic Berge knots, $P(-2, 3, 7)$ is the only one which admits cyclic slope 18 or 19. This last assertion follows from Lemma 1, Theorem 3 and the table of lens spaces of [4].

To prove part (5), assume that α and β are two integer finite surgery slopes for K with $\Delta(\alpha, \beta) = 2$. If both α and β are noncyclic, then by part (2), they are one of the pairs

$$\{50, 52, [17, 3; 3, 2]\}, \ \{56, 58, [19, 3; 3, 2]\}, \ \{91, 93, [23, 4; 3, 2]\}, \ \{99, 101, [25, 4; 3, 2]\}.$$

So we may assume that exactly one of them, say α, is noncyclic by Known Facts 1.1(1), and α must be a T–type or I–type slope by Known Facts 1.1(4)–(5). So α is one of the slopes in Tables 1 and 3, and K has the same Alexander polynomial as the corresponding sample knot associated to α. Again, in this situation, we only need to check the following lemma.

Lemma 6.4 *If there exists a hyperbolic knot K such that K admits a cyclic surgery slope $p \le 223$ which is distance 2 from a slope α in Table 1 or Table 3, and that K has the same Alexander polynomial as the sample knot attached to α in Table 1 or Table 3, then α, p are one of the pairs*

$$\{17, 19, P(-2,3,7)\}, \quad \{21, 23, [11,2;3,2]\}, \quad \{27, 25, [13,2;3,2]\},$$
$$\{37, 39, [19,2;5,2]\}, \quad \{43, 41, [21,2;5,2]\},$$

and each pair is realized on the attached sample knot.

Proof Again, this is proved by using a Mathematica program to check Condition 2.1 for each α–surgery in Tables 1 and 3 and a lens space $L(p,q)$ with $\Delta(\alpha, p) = 2$. We get all such pairs α, p satisfying Condition 2.1 along with the recovered Alexander polynomials, which yield corresponding sample knots in Tables 1 and 3. Such a sample knot is either a torus knot ($T(3,2)$ or $T(5,2)$), an iterated torus knot listed in the lemma or $P(-2,3,7)$. The case of torus knots can be ruled out by [45] and Known Facts 1.2(7). □

Part (5) of the theorem is proved. □

7 Proof of Theorem 1.6

We first consider the case of Theorem 1.6 when K is nonhyperbolic. This case follows easily from existing results. In fact, by Thurston's geometrization theorem for Haken manifolds, if K is not hyperbolic, then K is either a torus knot or a satellite knot. The classification of surgeries on torus knots is carried out in [39]. By [9, Corollary 1.4], if a satellite knot admits a finite surgery, then this knot is a cable of a torus knot. Finite surgeries on such cable knots are classified in [6, Theorem 7]. From these classification results, one can readily check that $T(2m+1,2)$ is the only nonhyperbolic knot in S^3 admitting a surgery to $\varepsilon S^3_{T(2m+1,2)}(4n)$ with slope $4n$ (it is easy to see this for torus knots and cables of torus knots; see [6, Table 1]).

From now on, we assume K is hyperbolic. We first get an estimate on the genus of K applying the correction terms from Heegaard Floer homology. In our current situation, the correction terms of $S^3_K(4n)$ are given by the formula

$$d(S^3_K(4n),i) = -\frac{1}{4} + \frac{(2n-i)^2}{4n} - 2t_{\min\{i,4n-i\}}(K), \quad i = 0,1,2,\dots,4n-1. \tag{7-1}$$

For the torus knot $T(2m+1,2)$, the coefficients of its normalized Alexander polynomial are

$$a_i = \begin{cases} (-1)^{m-i} & \text{if } |i| \le m, \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned} t_i(T(2m+1,2)) &= \begin{cases} \sum_{j=1}^{m-i} (-1)^{m-i-j} j & \text{for } 0 \le i < m, \\ 0 & \text{for } i \ge m, \end{cases} \\ &= \begin{cases} \lceil \frac{1}{2}(m-i) \rceil & \text{for } 0 \le i < m, \\ 0 & \text{for } i \ge m. \end{cases} \end{aligned} \tag{7-2}$$

For any $k \in \mathbb{Z}$, let $\theta(k) \in \{0,1\}$ be the reduction of k modulo 2. Let $\zeta = n-m \in \{0,1\}$. Applying (7-1) to $T(2m+1,2)$, we can compute that

$$d(S^3_{T(2m+1,2)}(4n),i) = \begin{cases} -\frac{1}{4} + i^2/(4n) + \zeta - \theta(n-\zeta-i) & \text{if } 0 \le i < n, \\ -\frac{1}{4} + (2n-i)^2/(4n) & \text{if } n \le i \le 2n, \\ d(S^3_{T(2m+1,2)}(4n), 4n-i) & \text{if } 2n < i < 4n. \end{cases} \tag{7-3}$$

Proposition 7.1 *If $S^3_K(4n) \cong \varepsilon S^3_{T(2m+1,2)}(4n)$, then $\varepsilon = +$, and $g(K) \le n$.*

Proof Since $S^3_K(4n) \cong \varepsilon S^3_{T(2m+1,2)}(4n)$, we know there exists an affine isomorphism $\phi\colon \mathbb{Z}/4n\mathbb{Z} \to \mathbb{Z}/4n\mathbb{Z}$ such that

$$d(S^3_K(4n),i) = \varepsilon d(S^3_{T(2m+1,2)}(4n), \phi(i)). \tag{7-4}$$

The map ϕ sends each Spin Spin^c structure of $S^3_K(4n)$ to a Spin Spin^c structure of $S^3_{T(2m+1,2)}(4n)$, namely, $\phi(\{0,2n\}) = \{0,2n\}$. Using (7-4), (7-1) and (7-3) for $i = 0$ and $i = 2n$, one easily sees that $\varepsilon = +$ since otherwise $t_{2n}(K)$ would not be integer valued. Since ϕ is an affine isomorphism of $\mathbb{Z}/4n\mathbb{Z}$, it must send $\{n,3n\}$ to $\{n,3n\}$. Using (7-4), (7-1) and (7-3) for $i = n$ and $i = 3n$, together with $\varepsilon = +$, one sees directly that $t_n(K) = 0$. It follows from (2-2) that $g(K) \le n$. □

Corollary 7.2 *Suppose that $S^3_K(4n) \cong \varepsilon S^3_{T(2m+1,2)}(4n)$ and that K is hyperbolic. Then $n = m = g(K)$, and K has the same Alexander polynomial as $T(2m+1,2)$. Moreover, there is a once-punctured Klein bottle properly embedded in the exterior of K of boundary slope $4m$.*

Proof By Proposition 7.1, $g(K) \le n$. Since $S^3_K(4n)$, being a prism space, contains a Klein bottle, it follows from [31, Corollary 1.3] that $4n \le 4g(K)$. So $n = g(K)$. Again by [31, Corollary 1.3], $4n$ is the boundary slope of a once-punctured Klein bottle in the exterior of K.

Next we prove that $n = m$. If not, we have $n = m + 1$. By Known Facts 1.2(2),

$$\Delta_K(t) = (-1)^r + \sum_{i=1}^{r} (-1)^{i-1}(t^{n_i} + t^{-n_i}),$$

where

$$m + 1 = n_1 > n_2 > \cdots > n_r > 0.$$

Lemma 6.2 implies that $|\Delta_K(-1)| = \det(K) = \det(T(2m+1,2)) = 2m+1$, so $r = m$ or $m + 1$.

If $r = m$, since $|\Delta_K(-1)| = \det(K) = 2m + 1$, we see that $n_i + i - 1$ has the same parity as m for any $i \in \{1, 2, \ldots, m\}$. This contradicts the assumption that $n_1 = m+1$.

If $r = m + 1$, then $\Delta_K(t) = (-1)^{m+1} + \sum_{i=0}^{m} (-1)^i (t^{m+1-i} + t^{-m-1+i})$, and $\det(K) = 2m + 3 \ne 2m + 1$; we also get a contradiction.

So we have proved $n = m$. Since $|\Delta_K(-1)| = \det(K) = 2m + 1$, we must have $\Delta_K(t) = \Delta_{T(2m+1,2)}(t)$. □

Let M be the exterior of K. Let P be a once-punctured Klein bottle in M with boundary slope $4m$, provided by Corollary 7.2. Let H be a regular neighborhood of P in M, then H is a handlebody of genus 2. Let $H' = M \setminus H$. Then $F = H \cap H' = \partial H \cap \partial H'$ is a twice-punctured genus-1 surface properly embedded in M. Each component of ∂F is a simple closed curve in ∂M parallel to ∂P and thus is of slope $4m$. Note that ∂F separates ∂M into two annuli A and A' such that $\partial H = F \cup A$ and $\partial H' = F \cup A'$.

Lemma 7.3 *F is compressible in H' and is incompressible in H.*

Proof Let Q be the Seifert surface for K of genus $g(K) = m$ from Corollary 7.2. By the incompressibility of the surfaces P and Q, we may assume that P and Q intersect transversely, that $P \cap Q$ contains no circle component which bounds a disk in P or Q, and that ∂P intersects ∂Q in exactly $4m$ points. Hence $P \cap Q$ has precisely $2m$ arc components, each of which is essential in P and Q (again because the incompressibility of P and Q).

Now consider the intersection graphs G_P and G_Q determined by the surfaces P and Q as usual (see eg [31]); that is, if $\widehat{P}$ (resp. $\widehat{Q}$) is the closed surface in $M(4m)$ (resp. in $M(0)$) obtained from P (resp. Q) by capping off its boundary by a disk, then G_P (resp. G_Q) is a graph in $\widehat{P}$ (resp. $\widehat{Q}$) obtained by taking the disk $\widehat{P} \setminus P$ (resp. $\widehat{Q} \setminus Q$) as a fat vertex and taking the arc components of $P \cap Q$ as edges. In particular, each of G_P and G_Q has precisely $2m$ edges.

A simple Euler characteristic calculation shows that the graph G_Q must have at least one disk face D. Let $D' = D \setminus H = D \cap H'$. Then D' is a properly embedded disk in H'.

We claim that $\partial D'$ is an essential curve on $\partial H'$. In fact, a component C of ∂F is an essential curve on $\partial H'$ and is also an essential curve in ∂M of slope $4m$. As C has $4m$ intersection points with ∂Q, all with the same sign, if D is a k–gon face of the graph G_Q, then $\partial D'$ has k intersection points with C, all with the same sign. So $\partial D'$ is an essential curve on $\partial H'$.

The claim proved in the last paragraph implies that D' is a compressing disk for $\partial H'$.

If F is incompressible in H', then by the handle addition lemma [32], the manifold obtained by attaching a 2–handle to H' along A' will give a manifold Y' with incompressible boundary (which is a torus). The manifold Y obtained by attaching a 2–handle to H along A gives a twisted I-bundle over Klein bottle, whose boundary is incompressible. Then $M(4m)$ is the union of Y and Y' along their torus boundary and thus is a Haken manifold, a contradiction. Therefore, F is compressible in H'.

Note that H is an I-bundle over P and that F is the horizontal boundary of H with respect to the I-bundle structure. It follows that the composition of the inclusion map $F \hookrightarrow H$, and the projection map $H \to P$ with respect to the I-bundle structure is a 2–fold covering map and is thus π_1–injective. As the fundamental group of H is isomorphic to $\pi_1(P)$, it follows that F is π_1–injective in H and thus is incompressible in H. □

Lemma 7.4 *Let $\widehat{P} \subset M(4m)$ be the Klein bottle obtained by capping off ∂P with a disk, and let $\nu(\widehat{P})$ be its tubular neighborhood. Then $\nu(\widehat{P})$ is a twisted I-bundle over $\widehat{P}$, $V = M(4m) \setminus \nu(\widehat{P})$ is a solid torus, and the dual knot $K' \subset M(4m)$ can be arranged by an isotopy to intersect $\nu(\widehat{P})$ in an I-fiber and intersect V in a boundary parallel arc.*

Proof By Lemma 7.3, $F = \partial H' \setminus A'$ is compressible in H'. Let D_* be a compressing disk for F in H'. Let $\widehat{F}$ be the closed surface in $M(4m)$ obtained from F by capping off each component of ∂F with a disk. Then $\widehat{F}$ is a torus.

If ∂D_* is an inessential curve in the torus $\widehat{F}$, ie bounds a disk B in $\widehat{F}$, then B must contain both components of ∂F since ∂D_* is an essential curve in F and ∂M is incompressible in M. So compressing F with D_* produces the disjoint union of a torus T_* and an annulus A_*. As M is hyperbolic, the torus T_* bounds a solid torus in M or is parallel to ∂M in M. From the construction of T_*, we see that T_* cannot be parallel to ∂M since otherwise P would be contained in the regular neighborhood of ∂M bounded by ∂M and T_*, which is obviously impossible. So T_* bounds a solid torus in M and in fact in H'. Similarly the annulus A_* cannot be essential in M and thus must be parallel to ∂M. In fact, A_* must be parallel to A' in H'. In particular, there exists a proper disk $D' \subset H'$ whose boundary consist of an essential arc in A_* and an essential arc in A'. It follows that the surface S which is ∂H pushed slightly into the interior of M and ∂M bound a compression body. In other words, S is a genus-2 Heegaard surface of M. Attaching a 2–handle to H' along A' will cancel the 1–handle with cocore D', hence we get a solid torus V.

If ∂D_* is an essential curve in the torus $\widehat{F}$, then compressing F with D_* gives an annulus $A_\#$. Again as M is hyperbolic, $A_\#$ must be parallel to ∂M and in fact must be parallel to A' in H'. This implies that the surface S which is ∂H pushed slightly into the interior of M and ∂M bound a compression body and thus S is a genus-2 Heegaard surface of M. Let $D' \subset H'$ be a proper disk whose boundary consist of an essential arc in $A_\#$ and an essential arc in A'. Attaching a 2–handle to H' along A' will cancel the 1–handle with cocore D'; hence we get a solid torus V.

In any case, we have shown that $M(4m)$ is the union of $\nu(\widehat{P})$ and a solid torus V. Some neighborhood of $K' \cap V$ is the 2–handle added to A'; thus D' gives a parallelism between $K' \cap V$ and an arc in ∂V. Some neighborhood of $K' \cap \nu(\widehat{P})$ is the 2–handle added to A consisting of I-fibers of $\nu(\widehat{P})$. Clearly, $K' \cap \nu(\widehat{P})$ can be considered as an I-fiber of $\nu(\widehat{P})$. □

Lemma 7.5 *Let Z be the double branched cover of S^3 with ramification locus K, and let $\widetilde{K} \subset Z$ be the preimage of K. Then $Z_{\widetilde{K}}(2m)$ is a 2–fold cover of $S^3_K(4m)$, which is a lens space. Moreover, let $\widetilde{K}' \subset Z_{\widetilde{K}}(2m)$ be the dual knot of $\widetilde{K}$. Then $\widetilde{K}'$ is a 1–bridge knot with respect to the standard genus-1 Heegaard splitting of $Z_{\widetilde{K}}(2m)$.*

Proof Let $\pi\colon Z \to S^3$ be the branched covering map. Then $\pi\colon Z \setminus \widetilde{K} \to S^3 \setminus K$ is an unramified 2–fold covering map, and π maps the simple loop with slope $2m$ on $\partial\nu(\widetilde{K})$ homeomorphically to the simple loop with slope $4m$ on $\partial\nu(K)$. Thus $\pi\colon Z \setminus \widetilde{K} \to S^3 \setminus K$ can be extended to an unramified 2–fold covering map $Z_{\widetilde{K}}(2m) \to S^3_K(4m)$.

Now we look at the double cover of $S^3_K(4m)$. Since $H_1(S^3_K(4m);\mathbb{Z}/2\mathbb{Z})\cong\mathbb{Z}/2\mathbb{Z}$, this cover is unique. Let $U=\nu(\widehat{P})$ be a twisted I-bundle over $\widehat{P}$. Then $S^3_K(4m)$ is the union of U and V, where V is the solid torus in Lemma 7.4. Let $\pi_U\colon \widetilde{U}\to U$ be the 2–fold covering map induced by the covering map $\partial U\to\widehat{P}$. Clearly, $\widetilde{U}$ is homeomorphic to $T^2\times I$. So we may construct a cover of $S^3_K(4m)$ by gluing two copies of V to $\partial\widetilde{U}$. As a result, $Z_{\widetilde{K}}(2m)$ is a lens space. Since $\widetilde{K}'$ is the preimage of K' under the covering map, it follows from Lemma 7.4 that $\widetilde{K}'$ is a 1–bridge knot. $\square$

Lemma 7.6 *Z is an L-space.*

Proof This follows from a standard fact in Heegaard Floer homology. Notice that m is also the genus of $\widetilde{K}$. By [43, Corollary 4.2 and Remark 4.3], for any Spin^c structure $\mathfrak{s}$ over Z, there exists a Spin^c structure $[\mathfrak{s},k]$ over $Z_{\widetilde{K}}(2m)$ such that $\widehat{\mathrm{HF}}(Z,\mathfrak{s})\cong\widehat{\mathrm{HF}}(Z_{\widetilde{K}}(2m),[\mathfrak{s},k])$. Since $Z_{\widetilde{K}}(2m)$ is an L-space, we have $\widehat{\mathrm{HF}}(Z,\mathfrak{s})\cong\mathbb{Z}$ for all $\mathfrak{s}$, so Z is also an L-space. $\square$

Now we will use a result due to Hedden [28] and Rasmussen [50]. We will use the form in [28, Theorem 1.4(2)]. Although the original statement is only for knots in S^3, the same proof works for null-homologous knots in L-spaces.

Theorem 7.7 [28; 50] *Let Z_1 be an L-space and $L\subset Z_1$ a null-homologous knot with genus g. Suppose that the p–surgery on L yields an L-space Z_2, and $p\ge 2g$. Then the dual knot $L'\subset Z_2$ is Floer simple; ie* $\operatorname{rank}\widehat{\mathrm{HFK}}(Z_2,L')=\operatorname{rank}\widehat{\mathrm{HF}}(Z_2)$.

Let $L(p,q)$ be a lens space, let $V_1\cup V_2$ be the standard genus-1 Heegaard splitting of $L(p,q)$, and let $D_i\subset V_i$ be a meridian disk such that $D_1\cap D_2$ consists of exactly p points. A knot L in a lens space $L(p,q)$ is *simple* if it is the union of two arcs a_1,a_2, where a_i is a boundary parallel arc in V_i that is disjoint from D_i for $i=1,2$. In each homology class in $H_1(L(p,q))$, there exists a unique (up to isotopy) oriented simple knot.

Corollary 7.8 *$\widetilde{K}'$ is a simple knot in the lens space $Z_{\widetilde{K}}(2m)$.*

Proof Lemmas 7.5 and 7.6 and Theorem 7.7 imply that $\widetilde{K}'$ is Floer simple in $Z_{\widetilde{K}}(2m)$. Lemma 7.5 also tells us that $\widetilde{K}'$ is 1–bridge. Using [28, Proposition 3.3], we see that $\widetilde{K}'$ is simple. $\square$

Lemma 7.9 *Let T' be the knot dual to $T=T(2m+1,2)$ in $S^3_T(4m)\cong S^3_K(4m)$, and let $\widetilde{T}'$ be its preimage in $Z_{\widetilde{K}}(2m)$. Then*

(1) *$[K']=\pm[T']$ or $\pm(2m-1)[T']$ in $H_1(S^3_K(4m))$,*

(2) *$[\widetilde{K}']=\pm[\widetilde{T}']$ in $H_1(Z_{\widetilde{K}}(2m))$.*

Proof (1) Recall that $\mathrm{Spin}^c(S^3_K(4m))$ is an affine space over $H^2(S^3_K(4m))$. In other words, $H_1(S^3_K(4m)) \cong H^2(S^3_K(4m))$ acts on $\mathrm{Spin}^c(S^3_K(4m))$. There is a standard way to identify $\mathrm{Spin}^c(S^3_K(4m))$ with $\mathbb{Z}/4m\mathbb{Z}$ in [43, Section 4]: Let W'_{4m} be the 2–handle cobordism from $S^3_K(4m)$ to S^3, let G be a Seifert surface for K and let $\widehat{G} \subset W'_{4m}$ be obtained by capping off ∂G with a disk. For any integer i, let $\mathfrak{t}_i \in \mathrm{Spin}^c(W'_{4m})$ be the unique Spin^c structure satisfying

$$\langle c_1(\mathfrak{t}_i), [\widehat{G}]\rangle = 2i - 4m. \tag{7-5}$$

Then we have an affine isomorphism $\sigma\colon \mathrm{Spin}^c(S^3_K(4m)) \to \mathbb{Z}/4m\mathbb{Z}$ which sends $\mathfrak{t}_i|_{S^3_K(4m)}$ to $i \pmod{4m}$.

Let μ be the meridian of K. Then μ is isotopic to K' in $S^3_K(4m)$. Using (7-5), we see that

$$\sigma(\mathfrak{s} + \mathrm{PD}[\mu]) - \sigma(\mathfrak{s}) = [\mu] \cdot [G] = 1.$$

So the action of $[K']$ on $\mathrm{Spin}^c(S^3_K(4m))$ is equivalent to adding 1 in $\mathbb{Z}/4m\mathbb{Z}$. There is a similar result when we replace K with T.

We identify $S^3_K(4m)$ with $S^3_T(4m)$ by a homeomorphism $f\colon S^3_K(4m) \to S^3_T(4m)$, which induces a symmetric affine isomorphism $\phi\colon \mathrm{Spin}^c(S^3_K(4m)) \to \mathrm{Spin}^c(S^3_T(4m))$. Clearly, ϕ is equivariant with respect to the $H_1(S^3_K(4m)) = H_1(S^3_T(4m))$ action, where we identify $H_1(S^3_K(4m))$ with $H_1(S^3_T(4m))$ using f_*. If $\phi(i) = ai + b$, consider the actions of $[K']$ and $[T']$ on the Spin^c structures, so we get that $[K'] = f_*([K'])$ acts as adding a on $\mathrm{Spin}^c(S^3_T(4m))$. Since $[T']$ acts as adding 1 on $\mathrm{Spin}^c(S^3_T(4m))$, $[K'] = a[T']$.

We should have

$$d(S^3_T(4m), i) = d(S^3_K(4m), i) = d(S^3_T(4m), \phi(i))$$

for any $i \in \mathbb{Z}/4m\mathbb{Z}$, where the first equality holds since $\Delta_K(t) = \Delta_T(t)$. We will use (7-3) to compute $d(S^3_T(4m), i)$. Note that $m = n$ and $\zeta = 0$. Recall from the proof of Proposition 7.1 that $\phi(\{0, 2n\}) = \{0, 2n\}$.

When m is even, it is straightforward to check that the minimal value of $d(S^3_T(4m), i)$ is $-\frac{1}{4} + 1/(4m) - 1$, which is attained if and only if $i = 1$ or $4m - 1$. So $\phi(1) = 1$ or $4m-1$. Since $\phi(0) = 0$ or $2m$, we have $a = \phi(1) - \phi(0) \in \{\pm 1, \pm(2m-1)\} \pmod{4m}$.

When m is odd, $d(S^3_T(4m), 0) \neq d(S^3_T(4m), 2m)$, so we must have $\phi(0) = 0$. We have $d(S^3_T(4m), 1) = -\frac{1}{4} + 1/(4m)$. Since m is odd, $4m + 1 \equiv 5 \pmod 8$, so

$-\frac{1}{4}+i^2/(4m)-1\neq d(S^3_T(4m),1)$ for any integer i. It follows that $d(S^3_T(4m),i)=d(S^3_T(4m),1)$ only when $i\in\{1,2m\pm 1,4m-1\}$. Hence $a=\phi(1)-\phi(0)=\phi(1)\in\{\pm1,\pm(2m-1)\}\pmod{4m}$.

In any case, we proved that $a\in\{\pm1,\pm(2m-1)\}\pmod{4m}$, thus our conclusion holds.

(2) Let

$$\tau_*\colon H_1(S^3_K(4m))\cong\mathbb{Z}/(4m\mathbb{Z})\to H_1(Z_{\widetilde{K}}(2m))\cong\mathbb{Z}/(2m(2m+1)\mathbb{Z})$$

be the transfer homomorphism, so then $[\widetilde{T}']=\tau_*([T'])$ and $[\widetilde{K}']=\tau_*([K'])$. Since $\gcd(4m,2m+1)=1$, the order of any element in the image of τ_* is a divisor of $2m$. It follows from (1) that $[\widetilde{K}']=\pm[\widetilde{T}']$. □

Lemma 7.10 *$\widetilde{T}'$ is a simple knot in the lens space $Z_{\widetilde{K}}(2m)$.*

Proof The knot $T=T(2m+1,2)$ is also a pretzel knot $P(2,-1,2m+3)$. Using this pretzel diagram, it is easy to find a once-punctured Klein bottle P_T in the complement of T such that the boundary slope of P_T is $4m$. Moreover, the complement of a neighborhood of P_T is a genus-2 handlebody with respect to which T is primitive. So the same argument as in the proof of Lemma 7.5 shows that $\widetilde{T}'$ is a 1–bridge knot in the lens space $Z_{\widetilde{K}}(2m)$. Then the same argument as in the proof of Corollary 7.8 shows that $\widetilde{T}'$ is simple. □

Proof of Theorem 1.6 As mentioned in the first paragraph of this section, our theorem holds when K is nonhyperbolic. So we assume K is hyperbolic. Proposition 7.1 and Corollary 7.2 imply that $\varepsilon=+$ and $n=m$. Corollary 7.8 and Lemma 7.10 say that both $\widetilde{K}'$ and $\widetilde{T}'$ are simple knots in the lens space $Z_{\widetilde{K}}(2m)$. Now Lemma 7.9 implies that $\widetilde{K}'$ and $\widetilde{T}'$ are isotopic up to orientation reversal. But this is impossible since the complement of $\widetilde{K}'$ is hyperbolic, while the complement of $\widetilde{T}'$ is Seifert fibered. □

References

[1] **K L Baker**, *Small genus knots in lens spaces have small bridge number*, Algebr. Geom. Topol. 6 (2006) 1519–1621 MR

[2] **W Ballinger**, **C C-Y Hsu**, **W Mackey**, **Y Ni**, **T Ochse**, **F Vafaee**, *The prism manifold realization problem*, preprint (2016) arXiv

[3] **L Ben Abdelghani**, **S Boyer**, *A calculation of the Culler–Shalen seminorms associated to small Seifert Dehn fillings*, Proc. London Math. Soc. 83 (2001) 235–256 MR

[4] **J Berge**, *Some knots with surgeries yielding lens spaces*, unpublished manuscript

[5] **J Berge**, **S Kang**, *The hyperbolic P/P, P/SF$_d$, and P/SF$_m$ knots in* S^3, preprint

[6] **S A Bleiler**, **C D Hodgson**, *Spherical space forms and Dehn filling*, Topology 35 (1996) 809–833 MR

[7] **S Boyer**, *On proper powers in free products and Dehn surgery*, J. Pure Appl. Algebra 51 (1988) 217–229 MR

[8] **S Boyer**, *On the local structure of* $\mathrm{SL}(2,\mathbb{C})$*–character varieties at reducible characters*, Topology Appl. 121 (2002) 383–413 MR

[9] **S Boyer**, **X Zhang**, *Finite Dehn surgery on knots*, J. Amer. Math. Soc. 9 (1996) 1005–1050 MR

[10] **S Boyer**, **X Zhang**, *Cyclic surgery and boundary slopes*, from "Geometric topology" (W H Kazez, editor), AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 62–79 MR

[11] **S Boyer**, **X Zhang**, *On Culler–Shalen seminorms and Dehn filling*, Ann. of Math. 148 (1998) 737–801 MR

[12] **S Boyer**, **X Zhang**, *On simple points of character varieties of* 3*–manifolds*, from "Knots in Hellas '98" (C M Gordon, V F R Jones, L H Kauffman, S Lambropoulou, J H Przytycki, editors), Ser. Knots Everything 24, World Sci., River Edge, NJ (2000) 27–35 MR

[13] **S Boyer**, **X Zhang**, *A proof of the finite filling conjecture*, J. Differential Geom. 59 (2001) 87–176 MR

[14] **G Burde**, *Darstellungen von Knotengruppen*, Math. Ann. 173 (1967) 24–33 MR

[15] **M Culler**, **N M Dunfield**, *Orderability and Dehn filling*, preprint (2016) arXiv

[16] **M Culler**, **C M Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. 125 (1987) 237–300 MR

[17] **J C Dean**, *Small Seifert-fibered Dehn surgery on hyperbolic knots*, Algebr. Geom. Topol. 3 (2003) 435–472 MR

[18] **M I Doig**, *Finite knot surgeries and Heegaard Floer homology*, Algebr. Geom. Topol. 15 (2015) 667–690 MR

[19] **M I Doig**, *On the number of finite* p/q*–surgeries*, Proc. Amer. Math. Soc. 144 (2016) 2205–2215 MR

[20] **M Eudave-Muñoz**, *Non-hyperbolic manifolds obtained by Dehn surgery on hyperbolic knots*, from "Geometric topology" (W H Kazez, editor), AMS/IP Stud. Adv. Math. 2, Amer. Math. Soc., Providence, RI (1997) 35–61 MR

[21] **M Eudave-Muñoz**, *On hyperbolic knots with Seifert fibered Dehn surgeries*, Topology Appl. 121 (2002) 119–141 MR

[22] **C D Frohman**, **E P Klassen**, *Deforming representations of knot groups in* SU(2), Comment. Math. Helv. 66 (1991) 340–361 MR

[23] **D Gabai**, *Foliations and the topology of* 3*–manifolds, III*, J. Differential Geom. 26 (1987) 479–536 MR

[24] **D Gabai**, **U Oertel**, *Essential laminations in* 3*–manifolds*, Ann. of Math. 130 (1989) 41–73 MR

[25] **W M Goldman**, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. 54 (1984) 200–225 MR

[26] **J E Greene**, *The lens space realization problem*, Ann. of Math. 177 (2013) 449–511 MR

[27] **L Gu**, *Integral finite surgeries on knots in* S^3, PhD thesis, California Institute of Technology (2014) MR Available at `https://search.proquest.com/docview/1550893042`

[28] **M Hedden**, *On Floer homology and the Berge conjecture on knots admitting lens space surgeries*, Trans. Amer. Math. Soc. 363 (2011) 949–968 MR

[29] **M Heusener**, **J Porti**, *Deformations of reducible representations of* 3*–manifold groups into* $\mathrm{PSL}_2(\mathbb{C})$, Algebr. Geom. Topol. 5 (2005) 965–997 MR

[30] **M Heusener**, **J Porti**, **E Suárez Peiró**, *Deformations of reducible representations of* 3*–manifold groups into* $\mathrm{SL}_2(\mathbf{C})$, J. Reine Angew. Math. 530 (2001) 191–227 MR

[31] **K Ichihara**, **M Teragaito**, *Klein bottle surgery and genera of knots*, Pacific J. Math. 210 (2003) 317–333 MR

[32] **W Jaco**, *Adding a* 2*–handle to a* 3*–manifold: an application to property* R, Proc. Amer. Math. Soc. 92 (1984) 288–292 MR

[33] **E P Klassen**, *Representations of knot groups in* SU(2), Trans. Amer. Math. Soc. 326 (1991) 795–828 MR

[34] **S Lee**, *Toroidal surgeries on genus-two knots*, J. Knot Theory Ramifications 18 (2009) 1205–1225 MR

[35] **C Lescop**, *Global surgery formula for the Casson–Walker invariant*, Annals of Mathematics Studies 140, Princeton Univ. Press (1996) MR

[36] **E Li**, **Y Ni**, *Half-integral finite surgeries on knots in* S^3, Ann. Fac. Sci. Toulouse Math. 24 (2015) 1157–1178 MR

[37] **D Matignon**, **N Sayari**, *Longitudinal slope and Dehn fillings*, Hiroshima Math. J. 33 (2003) 127–136 MR

[38] **K Miyazaki**, **K Motegi**, *On primitive/Seifert-fibered constructions*, Math. Proc. Cambridge Philos. Soc. 138 (2005) 421–435 MR

[39] **L Moser**, *Elementary surgery along a torus knot*, Pacific J. Math. 38 (1971) 737–745 MR

[40] **Y Ni**, *Knot Floer homology detects fibred knots*, Invent. Math. 170 (2007) 577–608 MR

[41] **Y Ni**, **X Zhang**, *Characterizing slopes for torus knots*, Algebr. Geom. Topol. 14 (2014) 1249–1274 MR

[42] **P Ozsváth**, **Z Szabó**, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. 173 (2003) 179–261 MR

[43] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[44] **P Ozsváth**, **Z Szabó**, *On knot Floer homology and lens space surgeries*, Topology 44 (2005) 1281–1300 MR

[45] **P Ozsváth**, **Z Szabó**, *The Dehn surgery characterization of the trefoil and the figure-eight knot*, preprint (2006) arXiv

[46] **P S Ozsváth**, **Z Szabó**, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. 11 (2011) 1–68 MR

[47] **G Perelman**, *The entropy formula for the Ricci flow and its geometric applications*, preprint (2002) arXiv

[48] **G Perelman**, *Ricci flow with surgery on three-manifolds*, preprint (2003) arXiv

[49] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR Available at `https://search.proquest.com/docview/305332635`

[50] **J Rasmussen**, *Lens space surgeries and L-space homology spheres*, preprint (2007) arXiv

[51] **G de Rham**, *Introduction aux polynômes d'un nœud*, Enseign. Math. 13 (1967) 187–194 MR

[52] **K Walker**, *An extension of Casson's invariant*, Annals of Mathematics Studies 126, Princeton Univ. Press (1992) MR

[53] **A Weil**, *On discrete subgroups of Lie groups, II*, Ann. of Math. 75 (1962) 578–602 MR

[54] **Y-Q Wu**, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, J. Differential Geom. 48 (1998) 407–437 MR

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