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Algebraic & Geometric Topology 18 (2018) 493–524

A characterization for asymptotic dimension growth

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We give a characterization for asymptotic dimension growth. We apply it to CAT(0) cube complexes of finite dimension, giving an alternative proof of Wright's result on their finite asymptotic dimension. We also apply our new characterization to geodesic coarse median spaces of finite rank and establish that they have subexponential asymptotic dimension growth. This strengthens a recent result of Špakula and Wright.

20F65, 20F67, 20F69, 51F99

1 Introduction

The concept of asymptotic dimension was first introduced by Gromov [16] in 1992 as a coarse analogue of the classical topological covering dimension. It started to attract much attention in 1998 when Yu [31] proved that the Novikov higher signature conjecture holds for groups with finite asymptotic dimension (FAD). A lot of groups and spaces are known to have finite asymptotic dimension. Among those are, for instance, finitely generated abelian groups, free groups of finite rank, Gromov hyperbolic groups (see Gromov [15] and Roe [25]), mapping class groups (see Bestvina, Bromberg and Fujiwara [6]), CAT(0) cube complexes of finite dimension (see Wright [30]) — see Bell and Dranishnikov [4] for an excellent survey of these and other results. Recently Behrstock, Hagen and Sisto [2] introduced the powerful new notion of hierarchically hyperbolic spaces and showed that these have finite asymptotic dimension, recovering a number of the above results, including, notably, mapping class groups and a number of CAT(0) cube complexes.

On the other hand, there are many groups and spaces with infinite asymptotic dimension. Examples are the wreath product $\mathbb{Z} \wr \mathbb{Z}$, the Grigorchuk group (see Smith [28]), the Thompson groups, etc. Generalizing FAD, Dranishnikov [14] defined the asymptotic dimension growth for a space; if the asymptotic dimension growth function is eventually constant then the space has FAD. Dranishnikov showed that the wreath product of a finitely generated nilpotent group with a finitely generated FAD group has polynomial

Published: 19 January 2018 DOI: 10.2140/agt.2018.18.493

asymptotic dimension growth. He also showed that polynomial asymptotic dimension growth implies Yu's Property A, and, hence, the coarse Baum–Connes conjecture, provided the space has bounded geometry; see Yu [32]. Later, Ozawa [23] extended this result to spaces with subexponential growth; see also Oppenheim [22]. Bell [3] analyzed how the asymptotic dimension growth function is affected by various group-theoretical constructions.

In this paper, we give an alternative characterization for the asymptotic dimension growth function which is inspired by Brown and Ozawa's proof of Property A for Gromov's hyperbolic groups [10, Theorem 5.3.15], which is in turn inspired by Kaimanovich [18]. We use this to study two notable examples: CAT(0) cube complexes of finite dimension and coarse median spaces of finite rank.

The techniques used to study these examples are developments of those used by Špakula and Wright [29] to establish Property A for uniformly locally finite coarse median spaces of finite rank. As a byproduct, we obtain a new proof of finite asymptotic dimension for CAT(0) cube complexes which allows one to explicitly construct the required controlled covers. This compares with Wright's original proof [30], which is discussed below.

CAT(0) cube complexes are a nice class of nonpositively curved spaces, first studied by Gromov [15], who gave a purely combinatorial condition for recognizing the nonpositive curvature of cube complexes. Many well-known groups act properly on CAT(0) cube complexes. For instance, right-angled Artin groups, many small cancellation groups, and Thompson's groups admit such actions. This makes it possible to deduce properties of these groups from the corresponding properties of the CAT(0) cube complexes.

In 2010, Wright [30] proved that the asymptotic dimension of a CAT(0) cube complex X is bounded by its dimension. He proved this by constructing a family of ε–Lipschitz cobounded maps to CAT(0) cube complexes of (at most) the same dimension indexed by $\varepsilon > 0$. We use our characterization for finite asymptotic dimension to give a direct proof of this result. Namely, we construct uniformly bounded covers with suitable properties. Being more explicit, this proof loses, however, the sharp bound on the asymptotic dimension. Thus, we give an alternative proof of the following nonquantitative variant of Wright's theorem:

Theorem 1.1 *Let X be a* CAT(0) *cube complex of finite dimension; then X has finite asymptotic dimension.*

The key point in our approach is to analyze the normal cube path distance on the cube complex, introduced by Niblo and Reeves [19]. We consider the ball with respect

to the normal cube path distance rather than to the ordinary edge-path distance. We decompose such a ball into intervals and use induction on the dimension in order to construct some "separated" net satisfying a suitable consistency property. In the process, we give a detailed analysis of normal balls and normal spheres (ie balls and spheres with respect to the normal cube path distance). See Section 4 for all details.

Our second application is to coarse median spaces, a coarse variant of classical median spaces introduced by Bowditch [7]. The notion of a coarse median group leads to a unified viewpoint on several interesting classes of groups, including Gromov's hyperbolic groups, mapping class groups, and CAT(0) cubical groups. Bowditch showed that hyperbolic spaces are exactly coarse median spaces of rank 1, and mapping class groups are examples of coarse median spaces of finite rank [7]. He also established interesting properties for coarse median spaces such as rapid decay, the property of having quadratic Dehn function, etc.

Intuitively, a coarse median space is a metric space equipped with a ternary operator (called the coarse median), in which every finite subset can be approximated by a finite median algebra. In these approximations the coarse median is approximated by an actual median with the distortion being controlled by the metric. This extends Gromov's observation that, in a δ–hyperbolic space, finite subsets can be well approximated by finite trees.

Recently, Špakula and Wright [29] proved that a coarse median space with finite rank and at most exponential volume growth has Property A. Following their proof and using our characterization for asymptotic dimension growth, we obtain the following result:

Theorem 1.2 *Let X be a geodesic coarse median space with finite rank and at most exponential volume growth, then X has subexponential asymptotic dimension growth.*

Hierarchically hyperbolic spaces are examples of coarse median spaces — see Behrstock, Hagen and Sisto [1] — hence our theorem is broader in scope, though with a weaker conclusion, than the finite asymptotic dimension result proven in Behrstock, Hagen and Sisto [2]. We expect the following general result:

Conjecture 1.3 *Every geodesic coarse median space with finite rank has finite asymptotic dimension.*

By a result of Ozawa [23], subexponential asymptotic dimension growth implies Property A; thus, our theorem strengthens the result of [29].

The paper is organized as follows. In Section 2, we give some preliminaries on asymptotic dimension growth, CAT(0) cube complexes, and coarse median spaces. In Section 3, we provide a characterization of the asymptotic dimension growth function, and, as a special case, give a characterization of finite asymptotic dimension. Sections 4 and 5 deal with CAT(0) cube complexes: in Section 4, we study normal balls and spheres which are essential in our approach to prove Theorem 1.1 in Section 5. Section 6 deals with the coarse median case, and we prove Theorem 1.2 there.

Acknowledgements The authors would like to thank the referee for careful and detailed comments and suggestions, which much improved the paper.

Arzhantseva was partially supported by the Simons Foundation, while visiting the Isaac Newton Institute (Cambridge). Zhang was partially supported by the European Research Council (ERC) grant of Arzhantseva *ANALYTIC* no. 259527 and by the Royal Society Sino–British Fellowship Trust.

2 Preliminaries

2.1 Asymptotic dimension

The notion of asymptotic dimension was first introduced by Gromov in 1993 [16] as a coarse analogue of the classical Lebesgue topological covering dimension. See also [4].

Let (X,d) be a metric space and $r>0$. We call a family $\mathcal{U}=\{U_i\}$ of subsets in X *r–disjoint* if $d(U,U')\geqslant r$ for any $U\neq U'$ in $\mathcal{U}$, where $d(U,U')=\inf\{d(x,x'):x\in U,x'\in U'\}$. We write

$$\bigsqcup_{r\text{–disjoint}} U_i$$

for the union of $\{U_i\}$. A family $\mathcal{V}$ is said to be *uniformly bounded* if $\operatorname{mesh}(\mathcal{V})=\sup\{\operatorname{diam}(V):V\in\mathcal{V}\}$ is finite. Let $\mathcal{U}=\{U_i\}$ be a cover of X and $r>0$. We define the *r–multiplicity* of $\mathcal{U}$, denoted by $m_r(\mathcal{U})$, to be the minimal integer n such that for any $x\in X$, the ball $B(x,r)$ intersects at most n elements of $\mathcal{U}$. As usual, $m(\mathcal{U})$ denotes the *multiplicity* of a cover $\mathcal{U}$, that is, the maximal number of elements of $\mathcal{U}$ with a nonempty intersection.

The *Lebesgue number* $L(\mathcal{U})$ of the cover $\mathcal{U}$ is defined to be

$$\inf_{x\in X}\sup\{r>0\mid \exists U\in\mathcal{U} \text{ such that } B(x,r)\subseteq U\}.$$

Definition 2.1 [16] We say that the *asymptotic dimension* of a metric space X does not exceed n and we write $\operatorname{asdim} X \leqslant n$ if, for every $r > 0$, the space X can be covered by $n+1$ subspaces $X_0, X_1, \dots, X_n$ and each X_i can be further decomposed into some r–disjoint uniformly bounded subspaces:

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \bigsqcup_{\substack{r\text{–disjoint} \\ j\in\mathbb{N}}} X_{ij} \quad \text{and} \quad \sup_{i,j} \operatorname{diam} X_{ij} < \infty.$$

We say $\operatorname{asdim} X = n$ if $\operatorname{asdim} X \leqslant n$ and $\operatorname{asdim} X$ is not less than n.

Example 2.2 [21; 25; 5] (1) $\operatorname{asdim} \mathbb{Z}^n = n$ for all $n \in \mathbb{N}$, where $\mathbb{Z}$ is the group of integers.

(2) Those δ–hyperbolic spaces of bounded geometry, eg word hyperbolic groups, have finite asymptotic dimension. (The result in [5] actually shows something more general with a weakening of the bounded geometry condition.)

From the definition, it is easy to see that the asymptotic dimension of a subspace is at most that of the ambient space. There are other equivalent definitions of asymptotic dimension. We list one here for a later use, and guide the reader to [4] for others.

Proposition 2.3 [4] *Let X be a metric space; then $\operatorname{asdim} X \leqslant n$ if and only if for any $r > 0$, there exists a uniformly bounded cover $\mathcal{U}$ of X such that $m_r(\mathcal{U}) \leqslant n+1$.*

2.2 Asymptotic dimension growth

Let us consider the direct sum of infinitely many copies of the integers $G = \bigoplus_{\infty} \mathbb{Z}$. Since for any $n \in \mathbb{N}$ the group $\mathbb{Z}^n$ is contained in G, by the above-mentioned results G has infinite asymptotic dimension. In order to deal with such groups/spaces, Dranishnikov studied the following concept as a generalization of the property of having a finite asymptotic dimension:

Definition 2.4 [14] Let (X, d) be a metric space. Define a function

$$\operatorname{ad}_X(\lambda) = \min\{m(\mathcal{U}) : \mathcal{U} \text{ is a uniformly bounded cover of } X,\ L(\mathcal{U}) > \lambda\} - 1,$$

which is called the *asymptotic dimension function* of X.

Clearly ad_X is nondecreasing, and

$$\lim_{\lambda\to\infty} \operatorname{ad}_X(\lambda) = \operatorname{asdim}(X).$$

By definition, the asymptotic dimension of X is less than or equal to n for some finite n if and only if ad_X is eventually constant and $\operatorname{ad}_X \leqslant n$. If X has infinite asymptotic dimension, then both sides of the above equation are infinite.

Like in the case of the volume function, the growth type of the asymptotic dimension function is more essential than the function itself. Recall that for $f, g\colon \mathbb{R}_+ \to \mathbb{R}_+$, we write $f \preceq g$ if there exists $k \in \mathbb{N}$ such that $f(x) \leqslant k g(kx+k)+k$ for any $x > k$. We write $f \approx g$ if both $f \preceq g$ and $g \preceq f$. It is clear that $\approx$ is an equivalence relation. We define the growth type of f to be the $\approx$–equivalence class of f. Define the asymptotic dimension growth of X to be the growth type of ad_X.

By a result of Bell and Dranishnikov, the growth type of the asymptotic dimension function is a quasi-isometric invariant.

Proposition 2.5 [3; 14] *Let X and Y be two discrete metric spaces with bounded geometry. If X and Y are quasi-isometric, then $\mathrm{ad}_X \approx \mathrm{ad}_Y$. In particular, the asymptotic dimension growth is well-defined for finitely generated groups.*

We give an alternative (equivalent) definition of the asymptotic dimension growth that is used in our characterization.

Lemma 2.6 *Let X be a metric space, and define*

$$\widetilde{\mathrm{ad}}_X(\lambda) = \min\{m_\lambda(\mathcal{U}) : \mathcal{U} \text{ is a uniformly bounded cover of } X\} - 1.$$

Then $\widetilde{\mathrm{ad}}_X \approx \mathrm{ad}_X$.

Proof Given $\lambda > 0$, suppose $\mathcal{U}$ is a uniformly bounded cover of X with $L(\mathcal{U}) > \lambda$. For any $U \in \mathcal{U}$, define the *inner λ–neighbourhood* of U to be

$$N_{-\lambda}(U) = X \setminus N_\lambda(X \setminus U),$$

where N_λ denotes the usual λ–neighbourhood of the set, and we define

$$N_{-\lambda}(\mathcal{U}) = \{N_{-\lambda}(U) : U \in \mathcal{U}\}.$$

Since $L(\mathcal{U}) > \lambda$, the set $N_{-\lambda}(\mathcal{U})$ is still a cover of X and it is uniformly bounded. By definition, it is obvious that $m_\lambda(N_{-\lambda}(\mathcal{U})) \leqslant m(\mathcal{U})$, which yields $\widetilde{\mathrm{ad}}_X \preceq \mathrm{ad}_X$.

Conversely, suppose $\mathcal{U}$ is a uniformly bounded cover of X. Consider $N_\lambda(\mathcal{U})$, which has Lebesgue number not less than λ. It is easy to show $m(N_\lambda(\mathcal{U})) \leqslant m_\lambda(\mathcal{U})$, which implies $\mathrm{ad}_X \preceq \widetilde{\mathrm{ad}}_X$. □

By the preceding lemma, we can use either ad_X or $\widetilde{\mathrm{ad}}_X$ to define the asymptotic dimension growth. Recall that if there exists a polynomial (subexponential) function f such that $\mathrm{ad}_X \preceq f$, then X is said to have polynomial (subexponential) asymptotic dimension growth.

Dranishnikov has shown that polynomial asymptotic dimension growth implies Yu's Property A, and he gave a class of groups having this property.

Proposition 2.7 [14] *Let N be a finitely generated nilpotent group and G be a finitely generated group with finite asymptotic dimension. Then the wreath product $N \wr G$ has polynomial asymptotic dimension growth. In particular, $\mathbb{Z} \wr \mathbb{Z}$ has polynomial asymptotic dimension growth.*

2.3 CAT(0) cube complexes

We recall basic notions and results on the structure of CAT(0) cube complexes. We omit some details and most of the proofs but direct the readers to [9; 13; 15; 19; 27] for more information.

A *cube complex* is a polyhedral complex in which each cell is isometric to a Euclidean cube and the gluing maps are isometries. The *dimension* of the complex is the maximum of the dimensions of the cubes. For a cube complex X, we can associate it with the intrinsic pseudometric d_{int}, which is the minimal pseudometric on X such that each cube embeds isometrically. When X has finite dimension, d_{int} is a complete geodesic metric on X. See [9] for a general discussion on polyhedral complex and the associated intrinsic metric.

There is also another metric associated with X. Let $X^{(1)}$ be the 1–skeleton of X, that is a graph with the vertex set $V = X^{(0)}$. We equip V with the edge-path metric d, which is the minimal number of edges in a path connecting two given vertices. Clearly, when $X^{(1)}$ is connected, d is a geodesic metric on V. For $x, y \in V$, the *interval* is defined by $[x, y] = \{z \in V : d(x, y) = d(x, z) + d(x, y)\}$, that is, it consists of all points on any geodesic between x and y.

A geodesic metric space (X, d) is CAT(0) if all geodesic triangles in X are slimmer than the comparative triangle in the Euclidean space. For a cube complex (X, d_{int}), Gromov has given a combinatorial characterization of the CAT(0) condition [15]: X is CAT(0) if and only if it is simply connected and the link of each vertex is a flag complex (see also [9]).

Another characterization of the CAT(0) condition was obtained by Chepoi [13] (see also [26]): a cube complex X is CAT(0) if and only if for any $x, y, z \in V$, the intersection $[x, y] \cap [y, z] \cap [z, x]$ consists of a single point $\mu(x, y, z)$, which is called the *median* of x, y and z. In this case, we call the graph $X^{(1)}$ a *median graph*, and V equipped with the ternary operator m is indeed a median algebra [17]. In particular, the following equations hold for all $x, y, z, u, v \in V$:

M1. $\mu(x, x, y) = x$;

M2. $\mu(\sigma(x), \sigma(y), \sigma(z)) = \mu(x, y, z)$, where σ is any permutation of $\{x, y, z\}$;

M3. $\mu(\mu(x, y, z), u, v) = \mu(\mu(x, u, v), \mu(y, u, v), z)$.

Obviously, $\mu(x, y, z) \in [x, y]$ and $[x, y] = \{z \in V : \mu(x, y, z) = z\}$.

Lemma 2.8 *Let $x, y, z, w \in V$ be such that $z, w \in [x, y]$. Then $z \in [x, w]$ implies $w \in [z, y]$.*

Proof Since $z \in [x, w]$ and $w \in [x, y]$, we have $\mu(z, x, w) = z$ and $\mu(x, w, y) = w$. So

$$\begin{aligned}\mu(z, w, y) &= \mu(\mu(z, x, w), w, y) = \mu(\mu(z, w, y), \mu(x, w, y), w)\\ &= \mu(\mu(z, w, y), w, w) = w,\end{aligned}$$

which implies $w \in [z, y]$. □

Lemma 2.9 *For $x, y, z \in V$ and $d(z, y) = 1$, either $[x, z] \subseteq [x, y]$ or $[x, y] \subseteq [x, z]$.*

Proof By Chepoi's result [13], $X^{(1)}$ is a median graph, hence it is weakly modular (see [13]). So $d(x, y) \neq d(x, z)$, which implies $d(x, y) = d(x, z) + 1$ or $d(x, z) = d(x, y) + 1$, ie $[x, z] \subseteq [x, y]$ or $[x, y] \subseteq [x, z]$. □

We can equip a CAT(0) cubical complex X with a set of *hyperplanes* [12; 19; 20; 27]. Each hyperplane does not intersect itself, and divides the space into two halfspaces. Given two hyperplanes h and k, if the four possible intersections of halfspaces are all nonempty, then we say h crosses k, denoted by $h \pitchfork k$. This occurs if and only if h and k cross a common cube C (also denoted by $h \pitchfork C$). Furthermore [27], given any finite collection of pairwise intersecting hyperplanes, there exists a cube which all of them cross. Thus, the dimension of X, if it is finite, is the maximal number of pairwise intersecting hyperplanes. We can also define intervals in the language of hyperplanes: $[x, y]$ consists of points which lie in all halfspaces containing both x and y.

We call a subset $Y \subseteq V$ *convex* if $[x, y] \subseteq Y$ for any $x, y \in Y$. Obviously, halfspaces are convex since any geodesic crosses a hyperplane at most once [19; 27]. This also implies

$$d(x, y) = \#\{\text{ hyperplane } h : h \text{ separates } x \text{ from } y\}.$$

2.4 Coarse median spaces

According to Gromov, hyperbolic spaces can be considered locally as a coarse version of trees, in the sense that every finite subset can be approximated by a finite tree in a controlled way [15]. If one wants to approximate a space locally by finite median algebras (graphs), this would turn to the definition of coarse median spaces introduced by Bowditch. See [7; 8; 33] for details.

Definition 2.10 [7] Let (X, ρ) be a metric space, and $\mu\colon X^3 \to X$ be a ternary operation. We say that (X, ρ, μ) is a *coarse median space* and μ is a *coarse median* on X if the following conditions hold:

C1. There exist constants $K, H(0) > 0$ such that for all $a, b, c, a', b', c' \in X$,

$$\rho(\mu(a,b,c), \mu(a',b',c')) \leqslant K(\rho(a,a') + \rho(b,b') + \rho(c,c')) + H(0).$$

C2. There exists a function $H\colon \mathbb{N} \to [0, +\infty)$ with the following property: for a finite subset $A \subseteq X$ with $1 \leqslant |A| \leqslant p$, there exists a finite median algebra (Π, ρ_Π, μ_Π) and maps $\pi\colon A \to \Pi$, $\lambda\colon \Pi \to X$ such that for all $x, y, z \in \Pi$ and $a \in A$,

$$\rho(\lambda\mu_\Pi(x,y,z), \mu(\lambda x, \lambda y, \lambda z)) \leqslant h(p)$$

and

$$\rho(a, \lambda\pi a) \leqslant h(p).$$

We refer to K and H as the *parameters* of (X, ρ, μ). Furthermore, if there exists $d \in \mathbb{N}$ such that we can always choose the median algebra Π in condition (C2) above of rank at most η, then we say X has (coarse) *rank* at most η.

A finitely generated group is said to be *coarse median* if some Cayley graph is a coarse median space.

Note that, by definition, a coarse median on a group is not required to be equivariant under the group action.

Remark 2.11 According to Bowditch, without loss of generality, we may always assume that μ satisfies the median axioms (M1) and (M2): for all $a, b, c \in X$,

M1. $\mu(a,a,b) = a$;

M2. $\mu(a,b,c) = \mu(b,c,a) = \mu(b,a,c)$.

A large class of groups and spaces have been shown to be coarse median, including Gromov's hyperbolic groups, right-angled Artin groups, mapping class groups, CAT(0) cube complexes, etc [7]. Bowditch has proved that coarse median groups have the property of rapid decay [8], quadratic Dehn's function [7], etc. This yielded a unified way to prove these properties for the above-listed groups. Recently, Špakula and Wright have proved that coarse median spaces of finite rank and of at most exponential volume growth have Yu's Property A [29].

3 Characterization for asymptotic dimension growth

In this section, we establish a characterization for asymptotic dimension growth and obtain several interesting consequences of this main result. For instance, we get a characterization for a group to have finite asymptotic dimension.

Recall a metric space (X,d) is *uniformly discrete* if there exists some constant $C>0$ such that $d(x,y)>C$ for $x\neq y\in X$. Without loss of generality, we can always assume the metric only takes integer values.

Theorem 3.1 *Let (X,d) be a uniformly discrete metric space, and $f\colon \mathbb{R}_+\to\mathbb{R}_+$ be a function. Then the following are equivalent:*

(1) $\mathrm{ad}_X \preceq f$.

(2) *There exists a function $g\colon \mathbb{R}_+\to\mathbb{R}_+$ which has the same growth type as f such that $\forall l\in\mathbb{N}$, $\forall k=1,2,\dots,3l$, $\forall x\in X$, we can assign a subset $S(x,k,l)\subseteq X$ satisfying:*

(i) *$\forall l\in\mathbb{N}$, $\exists R_l>0$ such that $S(x,k,l)\subseteq B(x,R_l)$ for all $k=1,\dots,3l$.*

(ii) *$\forall l\in\mathbb{N}$, $\forall k,k'$ with $1\leqslant k\leqslant k'\leqslant 3l$, $\forall x\in X$, we have $S(x,k,l)\subseteq S(x,k',l)$.*

(iii) *$\forall x,y\in X$ with $d(x,y)\leqslant l$, we have:*

- *$S(x,k-d(x,y),l)\subset S(x,k,l)\cap S(y,k,l)$ for $k=d(x,y)+1,\dots,3l$.*
- *$S(x,k+d(x,y),l)\supset S(x,k,l)\cup S(y,k,l)$ for $k=1,\dots,3l-d(x,y)$.*

(iv) *$\forall l\in\mathbb{N}$, $\forall k=1,2,\dots,3l$, $\forall x\in X$, we have $\#S(x,k,l)\leqslant g(l)$.*

Proof (1) $\Longrightarrow$ (2) By Lemma 2.6, we can assume there exists a function $g\colon\mathbb{R}_+\to\mathbb{R}_+$ with $g\approx f$ such that for all $l\in\mathbb{N}$, there exists a uniformly bounded cover $\mathcal{U}_l$ of X with $m_{3l}(\mathcal{U}_l)\leqslant g(l)$. Given such a cover $\mathcal{U}_l=\{U_i : i\in I\}$ we will construct the sets $S(x,k,l)$ as follows. For each $i\in I$, choose $x_i\in U_i$ and, for $k=1,2,\dots,3l$ and $x\in X$, define

$$S(x,k,l)=\{x_i : B(x,k)\cap U_i\neq\varnothing\}.$$

Now let us check the four properties in condition (2):

(i) If $B(x,k)\cap U_i\neq\varnothing$, we can choose $w\in B(x,k)\cap U_i$. Now $d(w,x_i)\leqslant \mathrm{mesh}(\mathcal{U}_l)$, so $d(x,x_i)\leqslant k+\mathrm{mesh}(\mathcal{U}_l)\leqslant 3l+\mathrm{mesh}(\mathcal{U}_l)$. In other words,

$$S(x,k,l)\subseteq B(x,R_l)$$

where $R_l=3l+\mathrm{mesh}(\mathcal{U}_l)$.

(ii) This is immediate by our definition of the sets $S(x,k,l)$.

(iii) For all $x,y\in X$ with $d(x,y)\leqslant l$ and $k=d(x,y)+1,\dots,3l$, we have

$$S(x,k-d(x,y),l)=\{x_i : B(x,k-d(x,y))\cap U_i\neq\varnothing\}.$$

Now if $B(x,k-d(x,y))\cap U_i\neq\varnothing$, we can choose $z\in B(x,k-d(x,y))\cap U_i$, ie $z\in U_i$ and $d(z,x)\leqslant k-d(x,y)$. So $d(z,y)\leqslant k$, ie $z\in B(y,k)\cap U_i$. So

$B(y,k)\cap U_i\neq\varnothing$, which implies

$$S(x,k-d(x,y),l)\subset S(x,k,l)\cap S(y,k,l).$$

On the other hand, suppose that $x_j\in S(x,k',l)\cup S(y,k',l)$ for some $k'=1,\dots,3l-d(x,y)$. We can assume that $x_j\in S(y,k',l)$, ie $B(y,k')\cap U_j\neq\varnothing$, which implies $B(x,k'+d(x,y))\cap U_j\neq\varnothing$. So we have

$$S(x,k'+d(x,y),l)\supset S(x,k',l)\cup S(y,k',l).$$

(iv) For all $l\in\mathbb{N}$ $k=1,2,\dots,3l$ and $x\in X$, we have

$$\#S(x,k,l)=\#\{x_i : B(x,k)\cap U_i\neq\varnothing\}\leqslant m_{3l}(\mathcal{U}_l)\leqslant g(l).$$

(2) $\Longrightarrow$ (1) Now, given sets $S(x,k,l)$, we construct a sequence of covers $\mathcal{U}_l$ with the required properties. For $l\in\mathbb{N}$, let $H=\bigcup_{x\in X}S(x,l,l)$. For $h\in H$, we define $A_h=\{y : h\in S(y,l,l)\}$. We define $\mathcal{U}_l=\{A_h : h\in H\}$. This is a cover of X since $\forall x\in X$, if we take $h\in S(x,l,l)$, then $x\in A_h$. Since $\exists R_l>0$ such that $S(x,l,l)\subseteq B(x,R_l)$, we know that $d(h,y)\leqslant R_l$ for all $y\in A_h$, which implies $\operatorname{mesh}(\mathcal{U}_l)\leqslant R_l$. Finally, let us analyze $m_l(\mathcal{U}_l)$. For $x\in X$, consider $h\in H$ with $B(x,l)\cap A_h\neq\varnothing$. Take $y\in B(x,l)\cap A_h$, ie $d(y,x)\leqslant l$ and $h\in S(y,l,l)$. Now, by the assumptions in condition (2), we have

$$S(y,l,l)\subseteq S(x,l+d(x,y),l)\subseteq S(x,2l,l).$$

So $m_l(\mathcal{U}_l)\leqslant\#S(x,2l,l)\leqslant g(l)$. Finally, by Lemma 2.6, we have

$$\operatorname{ad}_X\approx\widetilde{\operatorname{ad}}_X\leqslant g\approx f.$$ □

Taking in the preceding theorem a constant function f, we obtain a characterization for finite asymptotic dimension.

Corollary 3.2 *Let (X,d) be a uniformly discrete metric space and $n\in\mathbb{N}$. Then the following are equivalent:*

(1) $\operatorname{asdim}X\leqslant n$.

(2) *$\forall l\in\mathbb{N}$, $\forall k=1,2,\dots,3l$, $\forall x\in X$, we can assign a subset $S(x,k,l)\subseteq X$, satisfying:*
   - (i) *$\forall l\in\mathbb{N}$, $\exists R_l>0$ such that $S(x,k,l)\subseteq B(x,R_l)$ for all $k=1,\dots,3l$.*
   - (ii) *$\forall l\in\mathbb{N}$, $\forall k,k'$ with $1\leqslant k\leqslant k'\leqslant 3l$, $\forall x\in X$, we have $S(x,k,l)\subseteq S(x,k',l)$.*
   - (iii) *$\forall x,y\in X$ with $d(x,y)\leqslant l$, we have:*
     - *$S(x,k-d(x,y),l)\subset S(x,k,l)\cap S(y,k,l)$ for $k=d(x,y)+1,\dots,3l$.*
     - *$S(x,k+d(x,y),l)\supset S(x,k,l)\cup S(y,k,l)$ for $k=1,\dots,3l-d(x,y)$.*
   - (iv) *$\forall l\in\mathbb{N}$, $\forall k=1,2,\dots,3l$, $\forall x\in X$, we have $\#S(x,k,l)\leqslant n+1$.*

Now we turn to the case when X is a graph, and obtain a characterization for finite asymptotic dimension which is easier to check.

Corollary 3.3 *Given a graph $X=(V,E)$ with vertices V and edges E, and equipped with the edge-path length metric d, the following are equivalent:*

(1) $\operatorname{asdim} X \leqslant n$.

(2) $\forall l \in \mathbb{N}$, $\forall k = 1,2,\dots,3l$, $\forall x \in X$, *we can assign a subset $S(x,k,l) \subseteq X$, satisfying:*
   (i) $\forall l \in \mathbb{N}$, $\exists R_l > 0$ *such that $S(x,k,l) \subseteq B(x,R_l)$ for all $k = 1,\dots,3l$.*
   (ii) $\forall l \in \mathbb{N}$, $\forall k,k'$ *with $1 \leqslant k \leqslant k' \leqslant 3l$, $\forall x \in X$, we have $S(x,k,l) \subseteq S(x,k',l)$.*
   (iii) $\forall x,y \in X$ *with $d(x,y)=1$* (ie with x and y connected by an edge), *we have $S(y,k,l) \subseteq S(x,k+1,l)$ for all $k = 1,2,\dots,3l-1$.*
   (iv) $\forall l \in \mathbb{N}$, $\#S(x,2l,l) \leqslant n+1$.

Remark 3.4 The only distinction between the above two corollaries is that in Corollary 3.3, assumption (2)(iii) is required only for endpoints of an edge, rather than for an arbitrary pair of points as in Corollary 3.2. We point out that the preceding corollaries can be generalized to the case of arbitrary asymptotic dimension growth. We will not use such a generalization, so we omit it.

Proof of Corollary 3.3 (1) $\Longrightarrow$ (2) is implied directly by Corollary 3.2, so we focus on (2) $\Longrightarrow$ (1).

Following the proof of (2) $\Longrightarrow$ (1) in Theorem 3.1, let $H = \bigcup_{x\in X} S(x,l,l)$ for all $l \in \mathbb{N}$. And for all $h \in H$, define $A_h = \{y : h \in S(y,l,l)\}$. Define $\mathcal{U}_l = \{A_h : h \in H\}$. For all $x \in X$, if we take $h \in S(x,l,l)$ then $x \in A_h$. So $\mathcal{U}_l$ is a cover of X. By condition (2)(i), there is $R_l > 0$ such that $S(x,l,l) \subseteq B(x,R_l)$, we know $d(h,y) \leqslant R_l$ for all $y \in A_h$, which implies $\operatorname{mesh}(\mathcal{U}_l) \leqslant R_l$. Finally, let us analyze $m_l(\mathcal{U}_l)$. For all $x \in X$, consider $h \in H$ with $B(x,l) \cap A_h \neq \varnothing$. Take $y \in B(x,l) \cap A_h$, ie $d(y,x) \leqslant l$ and $h \in S(y,l,l)$. By the definition of the edge-path length metric d, we know that there exists a sequence of vertices $y = y_0, y_1, \dots, y_k = x$ such that $y_i \in V$ for all $i = 0,1,\dots,k$, $d(y_i, y_{i+1}) = 1$ for all $i = 0,1,\dots,k-1$ and $k \leqslant l$. Now, by the hypothesis, we know

$$S(y,l,l) \subseteq S(y_1,l+1,l) \subseteq S(y_2,l+2,l) \subseteq \cdots \subseteq S(y_k,l+k,l) = S(x,k+l,l) \subseteq S(x,2l,l).$$

So $\{h \in H : B(x,l) \cap A_h \neq \varnothing\} \subseteq S(x,2l,l)$, which implies $m_l(\mathcal{U}_l) \leqslant \#S(x,2l,l) \leqslant n+1$. □

4 Normal cube path and normal distance

In the next two sections, we focus on CAT(0) cube complexes, and prove Theorem 1.1. We prove it by constructing a uniformly bounded cover with suitable properties. Such a construction relies deeply on the analysis of normal balls and spheres, which we give in this section.

Normal cube paths, which were introduced by Niblo and Reeves [19] play a key role in the construction of the cover. They determine a distance function on the vertices and the balls and spheres defined in terms of this distance are essential in our proof of Theorem 1.1.

Throughout this section we fix a CAT(0) cube complex X with a fixed vertex x_0. The 1–skeleton $X^{(1)}$ of X is a graph with vertex set $V = X^{(0)}$ and edge set E, which give us the edge metric d on V. This is the restriction of the ℓ^1 metric to the 0–skeleton.

4.1 Normal cube paths

Given a cube $C \in X$, we denote by $\mathrm{St}(C)$ the union of all cubes which contain C as a subcube.

Definition 4.1 [19] Let $\{C_i\}_{i=0}^n$ be a sequence of cubes such that each cube has dimension at least 1, and $C_{i-1} \cap C_i$ consists of a single point, denoted by v_i.

- Call $\{C_i\}_{i=0}^n$ a *cube path* if C_i is the (unique) cube of minimal dimension containing v_i and v_{i+1}, ie v_i and v_{i+1} are diagonally opposite vertices of C_i. Define v_0 to be the vertex of C_0 diagonally opposite to v_1, and v_{n+1} to be the vertex of C_n diagonally opposite to v_n. The so-defined vertices $\{v_i\}_{i=0}^{n+1}$ are called the vertices of the cube path, and we say the cube path is from v_0 to v_{n+1}.
- The *length* of a cube path is the number of the cubes in the sequence.
- A cube path is called *normal* if $C_i \cap \mathrm{St}(C_{i-1}) = v_i$.

Normal cube paths in CAT(0) cube complexes behave like geodesics in trees. More precisely, in [19], the existence and uniqueness of normal cube paths connecting any pair of vertices is established. See also [24].

Proposition 4.2 [19] *For any two vertices $x, y \in V$, there exists a unique normal cube path from x to y.* (Note that the order is important here since in general normal cube paths are not reversible).

Proposition 4.3 [19] *The intersection of a normal cube path and a hyperplane is connected. In other words, a normal cube path crosses a hyperplane at most once.*

Proposition 4.4 [19; 11] *Let $\{C_i\}_{i=0}^n$ and $\{D_j\}_{j=0}^m$ be two normal cube paths in X, and let $\{v_i\}_{i=0}^{n+1}$ and $\{w_j\}_{j=0}^{m+1}$ be the vertices of these normal cube paths. If $d(v_0, w_0) \leqslant 1$ and $d(v_{n+1}, w_{m+1}) \leqslant 1$, then for all k, we have $d(v_k, w_k) \leqslant 1$.*

We omit the proofs for the above three propositions; the readers can find them in the original paper. However, let us recall the construction of the normal cube path from x to y as follows: Consider all the hyperplanes separating x from y and adjacent to x. The key fact is that these hyperplanes all cross a unique cube adjacent to x lying in the interval from x to y. This cube is defined to be the first cube on the normal cube path; then one proceeds inductively to construct the required normal cube path.

We will also need the following lemma, abstracted from [19]. Recall that $h \pitchfork C$ for a hyperplane h and a cube C means the intersection of h and C is nonempty (see Section 2.3).

Lemma 4.5 *Let $\{C_i\}_{i=0}^n$ be the normal cube path and h be a hyperplane. If $h \pitchfork C_i$, then there exists a hyperplane k such that $k \pitchfork C_{i-1}$ and h does not intersect with k.*

Proof Otherwise, $\forall k \pitchfork C_{i-1}$, we have $h \pitchfork k$. Now by Lemma 2.15 in [19], we know that there exists a cube $C \in X$, such that all such $k \pitchfork C$ and $h \pitchfork C$, and C_{i-1} is a face of C. Moreover, C contains an edge e of C_i since $h \pitchfork C$. So $\mathrm{St}(C_{i-1}) \cap C_i$ contains e, which is a contradiction to the definition of normal cube path. □

Now for any two vertices of X, we consider all the hyperplanes separating them, with a partial order by inclusion. More explicitly, for any $x, y \in V$, let $H(x, y)$ be the set of hyperplanes separating x and y. For any $h \in H(x, y)$, let h^- be the halfspace containing x. Define $h \leqslant k$ if $h^- \subseteq k^-$. Note that the definition depends on the vertices we choose, and we may change them under some circumstances, but still write h^- for abbreviation. To avoid ambiguity, we point out the vertices if necessary. We write $h < k$ to mean a strict containment $h^- \subsetneq k^-$.

Lemma 4.6 *For any $h, k \in H(x, y)$, the hyperplanes h and k do not intersect if and only if $h \leqslant k$ or $k \leqslant h$.*

Proof We only need to show the necessity. Let $\{C_i\}_{i=0}^n$ be the normal cube path from x to y, and choose i and j so that $h \pitchfork C_i$ and $k \pitchfork C_j$. Since h and k do not intersect, $i \neq j$. Assume $i < j$. Obviously, $x \in h^- \cap k^-$ and $y \in h^+ \cap k^+$. Since $h \pitchfork C_i$ and $k \pitchfork C_j$, by Proposition 4.3 we have $v_{i+1} \in h^+ \cap k^-$. Since h does not intersect with k, we have $h^- \cap k^+ = \varnothing$, which implies $h^- \subseteq k^-$. □

Combining the above two lemmas, we have the following result on the existence of chains in $H(x, y)$.

Proposition 4.7 *Let $\{C_i\}_{i=0}^n$ be the normal cube path from x to y, and h be a hyperplane such that $h \pitchfork C_l$. Then there exists a chain of hyperplanes $h_0 < h_1 < \cdots < h_{l-1} < h_l = h$ such that $h_i \pitchfork C_i$.*

Proof By Lemma 4.5, there exists a hyperplane k such that $k \pitchfork C_{l-1}$ and h does not intersect with k. Define $h_{l-1} = k$. Inductively, we can define a sequence of hyperplanes as required. Then the conclusion follows by Lemma 4.6. □

Finally, we give a lemma used in the proof of the consistency part of our main theorem.

Lemma 4.8 *Let $x_0, x, y \in V$ with $[x_0, y] \subseteq [x_0, x]$, and let x' and y' be the n^{th} vertex on the normal cube path from x_0 to x, and to y. If $x' \neq y'$, then $x' \notin [x_0, y]$.*

Proof Otherwise, $x' \in [x_0, y]$. By the construction of the normal cube path, we know x' is also the n^{th} vertex on the normal cube path from x_0 to y, since $y \in [x_0, x]$. In other words, $x' = y'$, which is a contradiction to the assumption. □

4.2 Normal metric

We define a new metric on $V = X^{(0)}$ using normal cube paths [19; 24].

Definition 4.9 For any $x, y \in V$, define $d_{\text{nor}}(x, y)$ to be the length of the normal cube path from x to y. We call d_{nor} the *normal metric* on V.

One needs to verify that d_{nor} is indeed a metric. It is easy to see that $d_{\text{nor}}(x, y) \geqslant 0$, and $d_{\text{nor}}(x, y) = 0$ if and only if $x = y$. Note that the normal cube path from x to y is not the one from y to x in general, so the symmetric relation is not that obvious. In order to show the symmetric relation and the triangle inequality, we give the following characterization:

Lemma 4.10 *For $x, y \in V$, let $<$ be the relation defined as above. Then*

$$d_{\text{nor}}(x, y) = \sup\{m + 1 : h_0 < h_1 < \cdots < h_m,\ h_i \in H(x, y)\}.$$

Proof Suppose $\{C_i\}_{i=0}^n$ is the normal cube path from x to y, so $d_{\text{nor}}(x, y) = n + 1$. Denote the right-hand side of the equality in the lemma by n'. Now for any chain $h_0 < h_1 < \cdots < h_m$ in $H(x, y)$, by Proposition 4.3, h_i intersects with just one cube,

denoted by $C_{k(i)}$. Obviously, if $h, k \pitchfork C_i$, then $h \pitchfork k$. So $k(i) \neq k(j)$ if $i \neq j$, which implies $m \leqslant n$, so $n' \leqslant n$.

On the other hand, for any $h \pitchfork C_n$, by Proposition 4.7, we have a chain of hyperplanes $h_0 < h_1 < \cdots < h_{n-1} < h_n = h$ such that $h_i \pitchfork C_i$, which implies $n \leqslant n'$. □

Proposition 4.11 *d_{nor} is indeed a metric on V.*

Proof By Lemma 4.10 $d_{\mathrm{nor}}(x, y)$ is the maximal length of a chain in the poset $H(x, y)$. Since $H(y, x)$ is the opposite poset of $H(x, y)$ we deduce that $d_{\mathrm{nor}}(x, y) = d_{\mathrm{nor}}(y, x)$. For $x, y, z \in V$, $H(x, y) \triangle H(y, z) = H(x, z)$, where $\triangle$ is the symmetric difference operation. The inclusions of $H(x, y) \cap H(x, z)$ into $H(x, y)$ and $H(y, z) \cap H(x, z)$ into $H(y, z)$ are both order-preserving, and therefore, by Lemma 4.10, we have $d_{\mathrm{nor}}(x, z) \leqslant d_{\mathrm{nor}}(x, y) + d_{\mathrm{nor}}(y, z)$. □

4.3 Normal balls and normal spheres

Recall that for any two points x and y in $V = X^{(0)}$, the interval between them is

$$[x, y] = \{z \in V : d(x, y) = d(x, z) + d(z, y)\}.$$

In other words, $[x, y]$ is the set of vertices on the union of all the edge geodesics from x to y. A subset $Y \subseteq V$ is called *convex* if $[x, y] \subseteq Y$ for any $x, y \in Y$.

Now let $B(x, n)$ be the closed ball in the edge metric with centre $x \in V$ and radius n. Generally, $B(x, n)$ is not convex (for example, take $X = \mathbb{Z}^2$). However, as we will see, for the normal metric balls are convex. More precisely, we define the *normal ball* with centre $x \in V$ and radius n to be

$$B_{\mathrm{nor}}(x, n) = \{y \in V : d_{\mathrm{nor}}(x, y) \leqslant n\}$$

and the *normal sphere* with centre $x \in V$ and radius n to be

$$S_{\mathrm{nor}}(x, n) = \{y \in V : d_{\mathrm{nor}}(x, y) = n\}.$$

Lemma 4.12 *$B_{\mathrm{nor}}(x, n)$ is convex for all $x \in V$ and $n \in \mathbb{N}$.*

Proof Given $z, w \in B_{\mathrm{nor}}(x, n)$, and a geodesic γ from z to w, if $\gamma \not\subseteq B_{\mathrm{nor}}(x, n)$, we can assume u is the first vertex on γ which is not in $B_{\mathrm{nor}}(x, n)$, which implies $d_{\mathrm{nor}}(x, u) = n + 1$. Let z' be the vertex preceding u on γ, so $d_{\mathrm{nor}}(x, z') = n$ (since $d_{\mathrm{nor}}(z', u) = 1$). Since $d(z', u) = 1$, there exists a unique hyperplane h separating z' from u, so $H(x, u) = H(x, z') \sqcup \{h\}$. Now, according to Lemma 4.10, there exists a chain $h_0 < \cdots < h_{n-1} < h$ in $H(x, u)$ with $h_i \in H(x, z')$. Since every geodesic

intersects with any hyperplane at most once (see for example [27]), $w \in h^+$, which implies $h_0 < \cdots < h_{n-1} < h$ is also a chain in $H(x, w)$. This is a contradiction to $d_{\mathrm{nor}}(x, w) \leqslant n$, by Lemma 4.10. □

Since the intersection of two convex sets is still convex, we have the following corollary:

Corollary 4.13 *For any $x_0, x \in V$ and $n \in \mathbb{N}$, the set $[x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$ is convex.*

In this section, from now on, we fix a basepoint x_0 in X.

It is well known that for a convex subset Y in a CAT(0) cube complex and a point $v \notin Y$, there is a unique point in Y which is closest to v (see for example [9]). This statement is true both for the intrinsic CAT(0) metric on the cube complex and the edge metric on the vertex set, and we have a dual statement for the normal distance:

Proposition 4.14 *There exists a unique point $v \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$ such that*

$$[x_0, x] \cap B_{\mathrm{nor}}(x_0, n) \subseteq [x_0, v].$$

The point v is characterized by

$$d(x_0, v) = \max\{d(x_0, v') : v' \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)\}.$$

Furthermore, if $d_{\mathrm{nor}}(x_0, x) \geqslant n$, then $v \in [x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$, which implies that v is also the unique point in $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$ such that

$$d(x_0, v) = \max\{d(x_0, v') : v' \in [x_0, x] \cap S_{\mathrm{nor}}(x_0, n)\}.$$

Proof If there exist $z \neq w \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$ such that $d(x_0, z) = d(x_0, w)$ attains the maximum, consider the median $m = \mu(z, w, x)$. By Corollary 4.13, $m \in [z, w] \subseteq [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$, so $d(m, x_0) = d(z, x_0) = d(w, x_0)$. On the other hand, $m \in [z, x] \cap [w, x]$, so $m = z = w$, which is a contradiction.

By Corollary 4.13,

$$[x_0, v] \subseteq [x_0, x] \cap B_{\mathrm{nor}}(x_0, n).$$

Conversely, for any $u \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$, let $m = \mu(u, v, x) \in [u, v]$. By Corollary 4.13, $m \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$. Also, $m \in [v, x]$, so $d(m, x_0) \geqslant d(v, x_0)$, which implies $m = v$ by the choice of v, ie $\mu(u, v, x) = v$, so $v \in [u, x]$. Now, by Lemma 2.8, $u \in [x_0, v]$.

Now for x and n satisfying $d_{\mathrm{nor}}(x_0, x) \geqslant n$, if $v \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n-1)$, take a geodesic γ from v to x, and let $v = y_0, y_1, \ldots, y_k = x$ be the vertices on γ. Since $d_{\mathrm{nor}}(x_0, x) \geqslant n$, we have $x \neq v$, which implies $k > 0$. Now, for y_1, since $y_1 \in [v, x]$,

$d(x_0, v) < d(x_0, y_1)$. By the definition of v, we know $y_1 \notin [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$, so $y_1 \notin B_{\mathrm{nor}}(x_0, n)$. However, since $d(v, y_1) = d_{\mathrm{nor}}(v, y_1) = 1$, we have

$$d_{\mathrm{nor}}(x_0, y_1) \leqslant d_{\mathrm{nor}}(x_0, v) + d_{\mathrm{nor}}(v, y_1) \leqslant n,$$

which is a contradiction. □

To use the above proposition more flexibly, we give another characterization of v, which can also be viewed as an alternative definition of v. In the rest of this subsection, we fix $x \in V$ and $n \in \mathbb{N}$ with $d_{\mathrm{nor}}(x_0, x) \geqslant n$, where x_0 is the chosen basepoint.

Proposition 4.15 *Let v be the unique point in $[x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$ such that*

$$[x_0, x] \cap B_{\mathrm{nor}}(x_0, n) \subseteq [x_0, v];$$

then v is the n^{th} vertex on the normal cube path from x_0 to x.

To prove this result, let us focus on subsets in $H(x_0, x)$. Recall that $H(x_0, x)$ is endowed with the relation $\leqslant$, as defined prior to Lemma 4.6.

Definition 4.16 A subset $A \subseteq H(x_0, x)$ is *closed* (under $<$) if $k \in A$ for all $h \in A$ and $k < h$.

Lemma 4.17 *Let $\widetilde{v}$ be the n^{th} vertex on the normal cube path from x_0 to x; then $H(x_0, \widetilde{v})$ is maximal in the following sense: for any closed $A \subseteq H(x_0, x)$ which contains chains only with lengths at most n, $A \subseteq H(x_0, \widetilde{v})$.*

Proof We proceed by induction on n. Suppose that the lemma holds for $n-1$, and let v' be the $(n-1)^{\mathrm{st}}$ vertex on the normal cube path from x_0 to x. Given a closed $A \subseteq H(x_0, x)$ containing chains only with lengths at most n, and a maximal chain $h_0 < h_1 < \cdots < h_m$ in A, if $m \leqslant n-2$, then the closed set $\{h \in A : h \leqslant h_m\}$ contains chains only with lengths at most $n-1$; by induction, it is contained in $H(x_0, v') \subseteq H(x_0, \widetilde{v})$. Now, for $m = n-1$, similarly, $\{h \in A : h \leqslant h_{n-2}\} \subseteq H(x_0, v')$, which implies $h_i \pitchfork C_i$ for $i = 0, 1, \ldots, n-2$. So $h_{n-1} \pitchfork C_k$ for some $k \geqslant n-1$. If $k \neq n-1$, then by Proposition 4.7 and the closeness of A we get a chain in A with length greater than n, which is a contradiction. So $h_{n-1} \pitchfork C_{n-1}$, ie $h_{n-1} \in H(x_0, \widetilde{v})$. □

Proof of Proposition 4.15 Let $\widetilde{v}$ denote the n^{th} vertex on the normal cube path from x_0 to x. By Proposition 4.14, $\widetilde{v} \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n) \subseteq [x_0, v]$, which implies $H(x_0, \widetilde{v}) \subseteq H(x_0, v)$. However, $H(x_0, v)$ is closed and contains chains only with lengths at most n according to Lemma 4.10, so $H(x_0, v) \subseteq H(x_0, \widetilde{v})$ by Lemma 4.17, which implies $H(x_0, v) = H(x_0, \widetilde{v})$. So $H(v, \widetilde{v}) = H(x_0, v) \triangle H(x_0, \widetilde{v}) = \varnothing$, which implies $v = \widetilde{v}$. □

Finally, we characterize those points in $[x_0, x]$ which lie in $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$. This will be used in the next subsection to decompose the intersection into a union of intervals.

Let C_{n-1} be the n^{th} cube on the normal cube path from x_0 to x, and let v be the n^{th} vertex on the cube path as above. Let H_n be the set of all hyperplanes intersecting with C_{n-1}.

Proposition 4.18 *For $w \in [x_0, x]$, the following are equivalent:*

(1) $w \in [x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$.

(2) *$\exists h \in H_n$ such that h crosses the last cube on the normal cube path from x_0 to w.*

(3) *$\exists h \in H_n$ such that h separates w from x_0 and $w \in [x_0, v]$.*

Proof (1) $\Longrightarrow$ (3) By Proposition 4.14, $w \in [x_0, v]$. Since $d_{\mathrm{nor}}(x_0, w) = n$, by Lemma 4.10, the maximum length of chains in $H(x_0, w)$ is n. Take such a chain $h_0 < h_1 < \cdots < h_{n-1}$ in $H(x_0, w) \subseteq H(x_0, v)$. Obviously, h_i intersects with different cubes, which implies $h_i \pitchfork C_i$. So $h_{n-1} \in H_n$, and it separates w from x_0.

(3) $\Longrightarrow$ (2) Since h separates w from x_0, h must cross some cube C on the normal cube path from x_0 to w. Since $h \in H_n$, we know there is a chain $h_0 < h_1 < \cdots < h_{n-1} = h$ in $H(x_0, v)$, which is also a chain in $H(x_0, w)$. So h cannot cross the first $n-1$ cubes of the normal cube path from x_0 to w. If h does not cross the last cube, then $d_{\mathrm{nor}}(x_0, w) > n$. However, $w \in [x_0, v]$ implies $H(x_0, w) \subseteq H(x_0, v)$, so by Lemma 4.10, $d_{\mathrm{nor}}(x_0, v) > n$, which is a contradiction.

(2) $\Longrightarrow$ (1) This is immediate, by Lemma 4.10. □

We have another description for H_n, which is implied by Proposition 4.7 directly.

Lemma 4.19 *For $h \in H(x_0, x)$, $h \in H_n$ if and only if the maximal length of chains in $\{k \in H(x_0, x) : k \leqslant h\}$ is n.*

4.4 Decomposition of $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$

We want to decompose the set $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$, so that we can proceed by the induction on dimension in the proof of Theorem 1.1.

Throughout this subsection, we fix $x \in V$ and $n \in \mathbb{N}$ with $d_{\mathrm{nor}}(x_0, x) \geqslant n$, and let v be as defined in Proposition 4.14. At the end of the preceding subsection, we have

defined H_n to be the set of all hyperplanes intersecting with C_{n-1}, where $\{C_i\}$ is the normal cube path from x_0 to x.

Now we decompose $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n)$ into a union of intervals with dimensions lower than $[x_0, x]$, and the number of these intervals can be controlled by the dimension of $[x_0, x]$. This will make it possible to do induction on the dimension.

Definition 4.20 For $h \in H_n$, we define

$$F_h = \{w \in [x_0, x] \cap S_{\mathrm{nor}}(x_0, n) : h \text{ separates } w \text{ from } x_0\}.$$

By Proposition 4.18, we immediately obtain the following two lemmas:

Lemma 4.21 *We have*

$$\begin{aligned} F_h &= \{w \in [x_0, x] : h \text{ crosses the last cube on the normal cube path from } x_0 \text{ to } w\} \\ &= \{w \in [x_0, v] : h \text{ separates } w \text{ from } x_0\}. \end{aligned}$$

Lemma 4.22 $$[x_0, x] \cap S_{\mathrm{nor}}(x_0, n) = \bigcup_{h \in H_n} F_h.$$

By definition, we know

$$F_h = [x_0, x] \cap B_{\mathrm{nor}}(x_0, n) \cap \{v' : h \text{ separates } v' \text{ from } x_0\},$$

which implies that F_h is convex. Moreover, we will show that F_h is actually an interval.

Lemma 4.23 *Let $x_h \in F_h$ be the point minimizing $d(x_0, x_h)$. Then $F_h = [x_h, v]$.*

Proof Since F_h is convex and $x_h, v \in F_h$, so $[x_h, v] \subseteq F_h$. On the other hand, for all $z \in F_h$, let $m = \mu(x_0, z, x_h)$. So, $m \in F_h$ and $d(x_0, m) \leqslant d(x_0, x_h)$. By the choice of x_h, we know that $d(x_0, m) = d(x_0, x_h)$, which implies $m = x_h$, so $x_h \in [x_0, z]$. By Proposition 4.14, $x_h, z \in [x_0, v]$. Thus, by Lemma 2.8, $z \in [x_h, v]$. □

Proposition 4.24 $[x_0, x] \cap S_{\mathrm{nor}}(x_0, n) = \bigcup_{h \in H_n} [x_h, v]$ *and* $\dim[x_h, v] < \dim[x_0, x]$.

Proof We only need to show $\dim[x_h, v] < \dim[x_0, x]$. For any hyperplane k crossing $[x_h, v]$, by Proposition 4.18, $k \pitchfork h$. So $\dim[x_h, v] < \dim[x_0, x]$. □

Now we give another characterization for x_h, which is useful in the proof of the consistency condition of Theorem 1.1.

Lemma 4.25 *Let x_h be the closest point to x_0 on F_h; then x_h is the unique point in $B_{\mathrm{nor}}(x_0, n)$ such that h separates x_0 from x_h, and for any hyperplane $k \pitchfork h$, k does not separate x_h from x_0.*

Proof Since $x_h \in F_h$, we have $x_h \in [x_0, x] \cap B_{\mathrm{nor}}(x_0, n)$ and h separates x_0 from x_h. Now, for any hyperplane $k \pitchfork h$, if k separates x_h from x_0, we have $x_h \in h^+ \cap k^+$ and $x_0 \in h^- \cap k^-$. Choose $\widetilde{x}_h \in [x_0, x_h]$ such that $\widetilde{x}_h \in h^+ \cap k^-$. Since k does not separate $\widetilde{x}_h$ from x_0, we have $d(x_0, \widetilde{x}_h) < d(x_0, x_h)$. However, by Lemma 4.21, $\widetilde{x}_h \in F_h$, which is a contradiction.

It remains to show that x_h is the unique point satisfying these conditions. Otherwise, let $\hat{x}_h$ be another point satisfying the hypothesis in the lemma and $\hat{x}_h \neq x_h$. Let k be a hyperplane separating $\hat{x}_h$ from x_h, and assume $x_h \in k^-$. Obviously, $k \neq h$. If $k \pitchfork h$, by hypothesis, k does not separate x_h from x_0, as well as $\hat{x}_h$ from x_0, which is a contradiction since k separates $\hat{x}_h$ from x_h. So k does not cross h, which implies $h^- \subsetneq k^-$ by Lemma 4.6. However, $\hat{x}_h \in k^+$, so by Lemma 4.10, $d_{\mathrm{nor}}(x_0, x_h) < d_{\mathrm{nor}}(x_0, \hat{x}_h)$. This is a contradiction since $d_{\mathrm{nor}}(x_0, x_h) \geqslant n$ as h separates x_0 from x_h. □

4.5 Špakula and Wright's construction

We conclude this section with a recent application of normal cube paths, which were invoked by Špakula and Wright [29] in order to provide a new proof that finite-dimensional CAT(0) cube complexes have Yu's Property A. The key to their proof was the construction of a family of maps h_l with the property that, for any interval and any neighbourhood of an endpoint of the interval, the maps push that neighbourhood into the interval itself. These maps were defined in terms of the normal cube paths as follows:

Definition 4.26 (the h maps) Given $l \in \mathbb{N}$, we define $h_l \colon X \to X$ as follows. For $x \in X$, let $h_l(x)$ be the $3l^{\text{th}}$ vertex on the normal cube path from x to x_0 if $d_{\mathrm{nor}}(x, x_0) \geqslant 3l$, and let it be x_0 if $d_{\mathrm{nor}}(x, x_0) < 3l$.

Lemma 4.27 [29] *Let h_l be defined as above and $y \in B(x, 3l)$. Then $h_l(y) \in [x_0, x]$.*

Proof We only need to show that every halfspace containing x and x_0 contains also $z = h_l(y)$. For any hyperplane h such that one of the associated halfspaces, say h^+, contains x and x_0, either $y \in h^+$ or $y \in h^-$. In the former case, $z \in h^+$, so we only need to check the case that h separates x and x_0 from y.

Denote by $C_0, C_1, \ldots, C_m$ the normal cube path from y to x_0, and denote by $y = v_0, v_1, \ldots, v_m = x_0$ the vertices on this cube path. We shall argue that any hyperplane separating y from x and x_0 is "used" within the first $d(x, y)$ steps on the cube path. Suppose that the cube C_i does not cross any hyperplane h with h separating y from x and x_0. Hence every hyperplane $k \pitchfork C_i$ separates y and x from x_0 and v_{i+1}. If there was a hyperplane l separating y from x and x_0 before C_i, then necessarily l separates y and v_{i+1} from x and x_0, hence l crosses all the hyperplanes k crossing C_i. This contradicts the maximality of this step on the normal cube path. Thus, there is no such l, and so all the hyperplanes h separating y from x and x_0 must be crossed within the first $d(x, y)$ steps.

Since z is the $3l^{\text{th}}$ vertex on the cube path and $d(x, y) \leqslant 3l$, all the hyperplanes h separating y from x and x_0 must have been crossed before z. Thus, any such h actually also separates y from x, x_0 and z. □

We will use the remarkable properties of the h maps to construct the S sets defined in our characterization of finite asymptotic dimension in the next section.

5 Finite-dimensional CAT(0) cube complexes

Throughout this section, we fix a CAT(0) cube complex X of finite dimension η and equipped with a basepoint $x_0 \in X$. We will make use of the characterization obtained in Corollary 3.3 in order to prove Theorem 1.1.

5.1 Constructing the sets $S(x, k, l)$

By Corollary 3.3, in order to prove X has finite asymptotic dimension, we need to find a constant $N \in \mathbb{N}$ such that for all $l \in \mathbb{N}$, $k = 1, 2, \ldots, 3l$ and $x \in X$, we can assign a subset $S(x, k, l) \subseteq X$ satisfying:

(i) $\forall l \in \mathbb{N}$, $\exists R_l > 0$ such that $S(x, k, l) \subseteq B(x, R_l)$ for $k = 1, \ldots, 3l$.

(ii) $\forall l \in \mathbb{N}$, $\forall k, k'$ with $1 \leqslant k \leqslant k' \leqslant 3l$, $\forall x \in X$, $S(x, k, l) \subseteq S(x, k', l)$.

(iii) $\forall x, y \in X$ with $d(x, y) = 1$, $S(y, k, l) \subseteq S(x, k+1, l)$ for $k = 1, 2, \ldots, 3l-1$.

(iv) $\forall l \in \mathbb{N}$, $\#S(x, 2l, l) \leqslant N + 1$.

Now, for $l \in \mathbb{N}$, $k = 1, 2, \ldots, 3l$ and $x \in X$, we define

$$\widetilde{S}(x, k, l) = h_l(B(x, k)).$$

It is easy to show that $\{\widetilde{S}(x, k, l)\}$ satisfies (i)–(iii), but it does not satisfy (iv) above, so we need some modification. Intuitively, we construct $S(x, k, l)$ as a uniformly separated net in $\widetilde{S}(x, k, l)$. To be more precise, we require the following lemma:

Lemma 5.1 *There exist two constants N and K, only depending on the dimension η, such that for all $l \in \mathbb{N}$ and $x \in V$, there are subsets $C_x \subseteq [x_0, x]$ and maps $p_x\colon [x_0, x] \to \mathcal{P}(C_x)$, where $\mathcal{P}(C_x)$ denotes the power set of C_x, satisfying:*

- *If $d(x, y) = 1$ and $y \in [x_0, x]$, then $C_x \cap [x_0, y] = C_y$, and $p_x|_{[x_0,y]} = p_y$.*
- *For $z \in [x_0, x]$ and $w \in p_x(z)$, we have $d(z, w) \leqslant Kl$.*
- *$\forall z \in [x_0, x]$, $\#(B(z, Ml) \cap C_x) \leqslant N$, where $M = 3\eta + 3 + K$.*

We postpone the proof of the above lemma and first show how to use it to construct $S(x, k, l)$ (and, hence, to conclude the proof of Theorem 1.1).

Proof of Theorem 1.1 Let N and K be the constants in Lemma 5.1. For all $l \in \mathbb{N}$, $k = 1, 2, \ldots, 3l$ and $x \in X$, let $\widetilde{S}(x, k, l) = h_l(B(x, k))$ be as above, and by Lemma 4.27, we know $\widetilde{S}(x, k, l) \subseteq [x_0, x]$. Recall that $p_x\colon [x_0, x] \to \mathcal{P}(C_x)$ assigns a subset of C_x to each point y of the interval $[x_0, x]$. Now given a *subset* Y of the interval, $p_x(Y)$ is a collection of subsets of C_x and we denote by $\bigcup p_x(Y)$ the union of these subsets $p_x(y)$ for $y \in Y$.

Now we define

$$S(x, k, l) = \bigcup p_x(\widetilde{S}(x, k, l)).$$

Now the only thing left to complete the proof is to verify the conditions in Corollary 3.3:

(i) By the definition of h_l, we know $d(y, h_l(y)) \leqslant 3\eta l$ for all $y \in B(x, k)$. So $d(z, x) \leqslant (3\eta + 3)l$ for any $z \in \widetilde{S}(x, k, l)$. For such z and any $w \in p_x(z)$, by Lemma 5.1, we know $d(z, w) \leqslant Kl$, which implies

$$S(x, k, l) \subseteq B(x, (3\eta + 3 + K)l) = B(x, Ml).$$

(ii) For all $l \in \mathbb{N}$, k and k' with $1 \leqslant k \leqslant k' \leqslant 3l$, and $x \in X$, we have $\widetilde{S}(x, k, l) \subseteq \widetilde{S}(x, k', l)$. Now, immediately from the definition, $S(x, k, l) \subseteq S(x, k', l)$.

(iii) For all $x, y \in X$ with $d(x, y) = 1$, by Lemma 2.9, $y \in [x_0, x]$ or $x \in [x_0, y]$. Assume the former. Let $k = 1, 2, \ldots, 3l-1$. Obviously, $\widetilde{S}(y, k, l) \subseteq \widetilde{S}(x, k+1, l)$, so we have

$$\begin{aligned} S(y, k, l) &= \bigcup p_y(\widetilde{S}(y, k, l)) = \bigcup p_x|_{[x_0,y]}(\widetilde{S}(y, k, l)) = \bigcup p_x(\widetilde{S}(y, k, l)) \\ &\subseteq \bigcup p_x(\widetilde{S}(x, k+1, l)) = S(x, k+1, l). \end{aligned}$$

Here we use the first part of Lemma 5.1 in the second equation. On the other hand, $\widetilde{S}(x, k, l) \subseteq \widetilde{S}(y, k+1, l)$, so we have

$$\begin{aligned} S(x,k,l) &= \bigcup p_x(\widetilde{S}(x,k,l)) \subseteq \bigcup p_x(\widetilde{S}(y,k+1,l)) \\ &= \bigcup p_x|_{[x_0,y]}(\widetilde{S}(y,k+1,l)) \\ &= \bigcup p_y(\widetilde{S}(y,k+1,l)) = S(y,k+1,l). \end{aligned}$$

Here we use the first part of Lemma 5.1 in the fourth equality.

(iv) By (i), we know that $S(x,k,l) \subseteq B(x,Ml)$ for all $k = 1,2,\dots,3l$. Hence, by definition, $S(x,k,l) \subseteq B(x,Ml) \cap C_x$. Now, by the third part of Lemma 5.1, we have $\#S(x,k,l) \leqslant N$. □

The last thing is to prove Lemma 5.1. We use the analysis in Section 4 to construct C_x and p_x inductively. Recall that in Section 4 (Proposition 4.24), for any $l, n \in \mathbb{N}$ and any $x \in X$, we have

$$[x_0,x] \cap S_{\mathrm{nor}}(x_0,nl) = \bigcup_{h \in H_{nl}} [x_h,v],$$

with $\#H_{nl} \leqslant \eta$ and $\dim[x_h,v] < \dim[x_0,x]$. In order to carry out induction on the dimension of $[x_0,x]$, we require a stronger version of Lemma 5.1, which is more flexible on the choice of endpoints of intervals. More explicitly, we have:

Lemma 5.2 *There exist two constants N and K, only depending on the dimension η, such that $\forall l \in \mathbb{N}$, $\forall \overline{x}, x \in V$, $\exists C_{\overline{x},x} \subseteq [\overline{x},x]$ and a map $p_{\overline{x},x}\colon [\overline{x},x] \to \mathcal{P}(C_{\overline{x},x})$ satisfying:*

- *If $d(x,y) = 1$ and $y \in [\overline{x},x]$, then $C_{\overline{x},x} \cap [\overline{x},y] = C_{\overline{x},y}$, and $p_{\overline{x},x}|_{[\overline{x},y]} = p_{\overline{x},y}$.*
- *For $z \in [\overline{x},x]$ and $w \in p_{\overline{x},x}(z)$, we have $d(z,w) \leqslant Kl$.*
- *$\forall z \in [\overline{x},x]$, $\#(B(z,Ml) \cap C_{\overline{x},x}) \leqslant N$, where $M = 3\eta + 3 + K$.*

It is obvious that Lemma 5.1 is implied by Lemma 5.2 (one just needs to take $\overline{x} = x_0$). Now we prove Lemma 5.2.

Proof of Lemma 5.2 Fix an $l \in \mathbb{N}$. We will carry out induction on $\dim[\overline{x},x]$.

Given any $\overline{x}, x \in V$ with $\dim[\overline{x},x] = 1$, we define

$$C_{\overline{x},x} = \{y \in [\overline{x},x] : d_{\mathrm{nor}}(\overline{x},y) \in l\mathbb{N}\},$$

where $l\mathbb{N} = \{0,l,2l,3l,\dots\}$. Since $\dim[\overline{x},x] = 1$, $[\overline{x},x]$ is indeed isometric to an interval in $\mathbb{R}$. We define $p_{\overline{x},x}\colon [\overline{x},x] \to \mathcal{P}(C_{\overline{x},x})$ as follows: for any $y \in [\overline{x},x]$, $p_{\overline{x},x}(y)$ consists of a single point which is at distance $l\lfloor d_{\mathrm{nor}}(\overline{x},y)/l \rfloor$ from $\overline{x}$ in $[\overline{x},y]$, where $\lfloor\cdot\rfloor$ is the function of taking integer part. Now it is obvious that:

- If $d(x,y)=1$ and $y\in[\overline{x},x]$, then $C_{\overline{x},x}\cap[\overline{x},y]=C_{\overline{x},y}$ and $p_{\overline{x},x}|_{[\overline{x},y]}=p_{\overline{x},y}$.
- For $z\in[\overline{x},x]$ and $w\in p_{\overline{x},x}(z)$, we have $d(z,w)\leqslant l$.
- $\forall z\in[\overline{x},x]$, $\#(B(z,Ml)\cap C_{\overline{x},x})\leqslant 3M$.

Suppose for any $\overline{x},x\in V$ with $\dim[\overline{x},x]\leqslant\eta-1$, we have defined $C_{\overline{x},x}\subseteq[\overline{x},x]$ and a map $p_{\overline{x},x}\colon[\overline{x},x]\to\mathcal{P}(C_{\overline{x},x})$ satisfying:

- If $d(x,y)=1$ and $y\in[\overline{x},x]$, then $C_{\overline{x},x}\cap[\overline{x},y]=C_{\overline{x},y}$ and $p_{\overline{x},x}|_{[\overline{x},y]}=p_{\overline{x},y}$.
- For $z\in[\overline{x},x]$ and $w\in p_{\overline{x},x}(z)$, we have $d(z,w)\leqslant\frac{1}{2}(\eta-1)\eta l$.
- $\forall z\in[\overline{x},x]$, $\#(B(z,Ml)\cap C_{\overline{x},x})\leqslant(3M)^{\eta-1}(\eta-1)!$.

Now we focus on $\overline{x},x\in V$ with $\dim[\overline{x},x]=\eta$. For any $n\in\mathbb{N}$ with $nl\leqslant d_{\mathrm{nor}}(\overline{x},x)$, by Proposition 4.24,

$$[\overline{x},x]\cap S_{\mathrm{nor}}(\overline{x},nl)=\bigcup_{h\in H_{nl}^x}F_h^x=\bigcup_{h\in H_{nl}^x}[x_h,v_{nl}^x],$$

where v_{nl}^x is the farthest point from $\overline{x}$ in $[\overline{x},x]\cap S_{\mathrm{nor}}(\overline{x},nl)$, H_{nl}^x is the set of hyperplanes crossing the nl^{th} cube of the normal cube path from $\overline{x}$ to x, and we also have $\dim[x_h,v_{nl}^x]<\dim[\overline{x},x]$. By induction, C_{x_h,v_{nl}^x} and p_{x_h,v_{nl}^x} have already been defined. Now we define

$$C_{\overline{x},x}^n=\bigcup_{h\in H_{nl}^x}C_{x_h,v_{nl}^x}$$

and

$$C_{\overline{x},x}=\bigcup_{n=0}^{\lfloor d_{\mathrm{nor}}(\overline{x},x)/l\rfloor}C_{\overline{x},x}^n.$$

For any $z\in[\overline{x},x]$, let $\widetilde{z}$ be the nl^{th} vertex on the normal cube path from $\overline{x}$ to z, where $n=\lfloor d_{\mathrm{nor}}(\overline{x},z)/l\rfloor$, so $d_{\mathrm{nor}}(\widetilde{z},z)\leqslant l$, which implies $d(\widetilde{z},z)\leqslant\eta l$ and

$$\widetilde{z}\in[\overline{x},x]\cap S_{\mathrm{nor}}(\overline{x},nl)=\bigcup_{h\in H_{nl}^x}[x_h,v_{nl}^x].$$

Now we define

$$p_{\overline{x},x}(z)=\bigcup\{p_{x_h,v_{nl}^x}(\widetilde{z}):h\in H_{nl}^x\text{ and }\widetilde{z}\in[x_h,v_{nl}^x]\},$$

and we need to verify the requirements hold for $C_{\overline{x},x}$ and $p_{\overline{x},x}$.

First, suppose $d(x,y)=1$ and $y\in[\overline{x},x]$, and let h' be the hyperplane separating x from y. Given $n\in\mathbb{N}$ such that $[\overline{x},y]\cap S_{\mathrm{nor}}(\overline{x},nl)\neq\varnothing$, by Proposition 4.15, v_{nl}^x is the nl^{th} vertex on the normal cube path from $\overline{x}$ to x, and v_{nl}^y is the nl^{th} vertex on the

normal cube path from $\overline{x}$ to y. Due to the fellow-traveller property, Proposition 4.4, $d(v_{nl}^x, v_{nl}^y) \leqslant 1$. By Proposition 4.14, we have

$$v_{nl}^y \in S_{\mathrm{nor}}(\overline{x}, nl) \cap [\overline{x}, y] \subseteq S_{\mathrm{nor}}(\overline{x}, nl) \cap [\overline{x}, x] \subseteq [\overline{x}, v_{nl}^x].$$

Recall that $H(z, w)$ denotes the set of all hyperplanes separating z from w. Obviously,

$$H(\overline{x}, x) = H(\overline{x}, y) \cup \{h'\},$$

which implies $H_{nl}^y \subseteq H_{nl}^x \subseteq H_{nl}^y \cup \{h'\}$, by Lemmas 4.10 and 4.19.

If $h' \in H_{nl}^x$ then $F_{h'}^x \cap [\overline{x}, y] = \varnothing$ by Proposition 4.18. On the other hand, for all $h \in H_{nl}^y$, by Lemma 4.25, y_h is the unique point in $B_{\mathrm{nor}}(\overline{x}, nl)$ such that h separates $\overline{x}$ from y_h, and for any hyperplane $k \pitchfork h$, k does not separate y from $\overline{x}$. This implies $y_h = x_h$ since $H_{nl}^y \subseteq H_{nl}^x$, so we can do induction for the new "basepoint" $y_h = x_h$ and v_{nl}^y, v_{nl}^x, since $d(v_{nl}^x, v_{nl}^y) \leqslant 1$ and $v_{nl}^y \in [x_h, v_{nl}^x]$. This implies

$$C_{x_h, v_{nl}^x} \cap [x_h, v_{nl}^y] = C_{x_h, v_{nl}^y}.$$

Since $C_{x_h, v_{nl}^x} \subseteq [x_h, v_{nl}^x]$, we have

$$C_{x_h, v_{nl}^x} \cap [\overline{x}, y] = C_{x_h, v_{nl}^x} \cap [x_h, v_{nl}^x] \cap [\overline{x}, y] = C_{x_h, v_{nl}^x} \cap [x_h, \mu(x_h, v_{nl}^x, y)].$$

Claim $$\mu(x_h, v_{nl}^x, y) = v_{nl}^y.$$

Indeed, if $v_{nl}^x = v_{nl}^y$, then it holds naturally. If $v_{nl}^x \neq v_{nl}^y$, then by Lemma 4.8, $v_{nl}^x \notin [\overline{x}, y]$. Since $d(v_{nl}^x, v_{nl}^y) = 1$, so $v_{nl}^x \in [v_{nl}^y, y]$ or $v_{nl}^y \in [v_{nl}^x, y]$. But the former cannot hold since $[v_{nl}^y, y] \subseteq [\overline{x}, y]$, so $v_{nl}^y \in [v_{nl}^x, y]$, which implies

$$v_{nl}^y \in [x_h, y] \cap [x_h, v_{nl}^x] \cap [v_{nl}^x, y],$$

ie $v_{nl}^y = \mu(x_h, v_{nl}^x, y)$.

By the claim,

$$C_{x_h, v_{nl}^x} \cap [\overline{x}, y] = C_{x_h, v_{nl}^x} \cap [x_h, v_{nl}^y] = C_{x_h, v_{nl}^y}.$$

Now, for the above n, we have

$$\begin{aligned} C_{\overline{x},x}^n \cap [\overline{x}, y] &= \bigcup_{h \in H_{nl}^x} (C_{x_h, v_{nl}^x} \cap [\overline{x}, y]) = \bigcup_{h \in H_{nl}^y} (C_{x_h, v_{nl}^x} \cap [\overline{x}, y]) \\ &= \bigcup_{h \in H_{nl}^y} C_{x_h, v_{nl}^y} = \bigcup_{h \in H_{nl}^y} C_{y_h, v_{nl}^y} = C_{\overline{x},y}^n. \end{aligned}$$

Since $C_{\overline{x},x}^n \subseteq [\overline{x}, x] \cap S_{\mathrm{nor}}(\overline{x}, nl)$, we have

$$C_{\bar{x},x}\cap[\bar{x},y]=\bigcup_{n=0}^{\lfloor d_{\mathrm{nor}}(\bar{x},x)/l\rfloor}C^n_{\bar{x},x}\cap[\bar{x},y]=\bigcup_{n\in J_x}C^n_{\bar{x},x}\cap[\bar{x},y]$$
$$=\bigcup_{n\in J_y}C^n_{\bar{x},x}\cap[\bar{x},y]=\bigcup_{n\in J_y}C^n_{\bar{x},y}=C_{\bar{x},y},$$

where $J_x=\{n:[\bar{x},x]\cap S_{\mathrm{nor}}(\bar{x},nl)\neq\varnothing\}$ and $J_y=\{n:[\bar{x},y]\cap S_{\mathrm{nor}}(\bar{x},nl)\neq\varnothing\}$. For all $z\in[\bar{x},y]$, one needs to show that $p_{\bar{x},x}(z)=p_{\bar{x},y}(z)$. Let $\widetilde{z}$ be the nl^{th} vertex on the normal cube path from $\bar{x}$ to z, where $n=\lfloor d_{\mathrm{nor}}(\bar{x},z)/l\rfloor$. By the analysis above, we know

$$d(v^x_{nl},v^y_{nl})\leqslant 1,\quad v^y_{nl}\in[v^x_{nl},y],\quad x_h=y_h,\quad H^y_{nl}\subseteq H^x_{nl}\subseteq H^y_{nl}\cup\{h'\}.$$

For $h\in H^x_{nl}$ with $\widetilde{z}\in[x_h,v^x_{nl}]$, we have $h\in H^y_{nl}$, ie $h\neq h'$ since $\widetilde{z}\in[\bar{x},y]$. Now, for such h,

$$\widetilde{z}\in[\bar{x},y]\cap[x_h,v^x_{nl}]=[x_h,v^y_{nl}]=[y_h,v^y_{nl}],$$

where the first equality comes from the claim above. Inductively, we know, for such h,

$$p_{x_h,v^x_{nl}}(\widetilde{z})=p_{y_h,v^y_{nl}}(\widetilde{z}).$$

Now, by definition,

$$\begin{aligned}p_{\bar{x},x}(z)&=\bigcup\{p_{x_h,v^x_{nl}}(\widetilde{z}):h\in H^x_{nl}\text{ and }\widetilde{z}\in[x_h,v^x_{nl}]\}\\&=\bigcup\{p_{x_h,v^x_{nl}}(\widetilde{z}):h\in H^y_{nl}\text{ and }\widetilde{z}\in[x_h,v^x_{nl}]\}\\&=\bigcup\{p_{y_h,v^y_{nl}}(\widetilde{z}):h\in H^y_{nl}\text{ and }\widetilde{z}\in[y_h,v^y_{nl}]\}\\&=p_{\bar{x},y}(z).\end{aligned}$$

Second, for any $z\in[\bar{x},x]$ and $w\in p_{\bar{x},x}(z)$, assume that $w\in p_{x_h,v^x_{nl}}(\widetilde{z})$ for some $h\in H^x_{nl}$ and $\widetilde{z}\in[x_h,v^x_{nl}]$ as in the definition. By induction, we know $d(\widetilde{z},w)\leqslant\frac{1}{2}(\eta-1)\eta l$ since $\dim[x_h,v^x_{nl}]\leqslant\eta-1$. So

$$d(z,w)\leqslant d(z,\widetilde{z})+d(\widetilde{z},w)\leqslant\eta l+\tfrac{1}{2}(\eta-1)\eta l=\tfrac{1}{2}\eta(\eta+1)l.$$

Third, for any $z\in[\bar{x},x]$, consider $B(z,Ml)\cap C_{\bar{x},x}$. Suppose $n\in\mathbb{N}$ satisfying $B(z,Ml)\cap C_{\bar{x},x}\cap S_{\mathrm{nor}}(\bar{x},nl)\neq\varnothing$, so $B(z,Ml)\cap\big(\bigcup_{h\in H^x_{nl}}[x_h,v^x_{nl}]\big)\neq\varnothing$, which means there exists some $h\in H^x_{nl}$ such that $B(z,Ml)\cap[x_h,v^x_{nl}]\neq\varnothing$. For such n and h, let $z'=\mu(z,x_h,v^x_{nl})\in[x_h,v^x_{nl}]$. Obviously,

$$B(z,Ml)\cap[x_h,v^x_{nl}]\subseteq B(z',Ml)\cap[x_h,v^x_{nl}].$$

By induction, we have

$$\#(B(z,Ml)\cap C_{x_h,v^x_{nl}})\leqslant\#(B(z',Ml)\cap C_{x_h,v^x_{nl}})\leqslant(3M)^{\eta-1}(\eta-1)!.$$

Now, for the above z, there exist at most $3M$ values of n such that

$$B(z, Ml) \cap [\overline{x}, x] \cap S_{\mathrm{nor}}(\overline{x}, nl) \neq \varnothing;$$

and for such n, since $\#H^x_{nl} \leqslant \eta$, there exist at most η hyperplanes h such that $B(z, Ml) \cap [\overline{x}, x] \cap [x_h, v^x_{nl}] \neq \varnothing$. So we have

$$\begin{aligned}\#(B(z, Ml) \cap C_{\overline{x},x}) &\leqslant \sum_{\substack{n \\ B(z,Ml)\cap[\overline{x},x]\cap S_{\mathrm{nor}}(\overline{x},nl)\neq\varnothing}} \#(B(z, Ml) \cap C^n_{\overline{x},x}) \\ &\leqslant \sum_{n \text{ as above}} \sum_{h \in H^x_{nl}} \#(B(z, Ml) \cap C_{x_h, v^x_{nl}}) \\ &\leqslant 3M \cdot \eta \cdot (3M)^{\eta-1}(\eta-1)! = (3M)^\eta \eta!.\end{aligned}$$

Now we take $K = \frac{1}{2}(\eta-1)\eta$ and $N = (3M)^\eta \eta! = (3K + 9\eta + 9)^\eta \eta!$; then the lemma holds for these constants. □

6 Coarse median spaces

In this section, we discuss the coarse median case, and prove Theorem 1.2. We fix a coarse median space X with geodesic metric ρ and coarse median μ with parameters K and H and finite rank η. The definitions and notations are the same as in Section 2.4. According to Remark 2.11, we also assume that the coarse median μ satisfies (M1) and (M2). We recall:

Theorem 6.1 [29] *Any geodesic uniformly locally finite coarse median space of finite rank and at most exponential growth has Property A.*

Our result, Theorem 1.2, says that any coarse median space as above has subexponential asymptotic dimension growth. Thus, combining with Ozawa's result [23], our theorem yields a strengthening of Theorem 6.1.

To prove Theorem 1.2, we use several notations and lemmas from [29]. We use the notation $x \sim_s y$ for $\rho(x, y) \leqslant s$. Given $r > 0$ and $a, b \in X$, the coarse interval $[a, b]_r$ is defined to be

$$[a, b]_r = \{z \in X : \mu(a, b, z) \sim_r z\}.$$

By a result of Bowditch [8], there exists a constant $\lambda > 0$, depending only on the parameters K and H, such that $\mu(x, y, z) \in [x, y]_\lambda$ for all $x, y, z \in X$.

Also recall that the median axiom (M3) holds in the coarse median case up to a constant $\gamma \geqslant 0$ depending only on the parameters K and H: for all $x, y, z, u, v \in X$, we have

$$\mu(\mu(x, y, z), u, v) \sim_\gamma \mu(\mu(x, u, v), \mu(y, u, v), z).$$

Actually we can take $\gamma = 3K(3K+2)H(5) + (3K+2)H(0)$.

Given $r, t, \kappa \geqslant 0$, denote

$$\begin{aligned} L_1(r) &= (K+1)r + K\lambda + \gamma + 2H(0), \\ L_2(r,\kappa) &= (K+1)r + \kappa + H(0), \\ L_3(r,t) &= 3^\eta K^\eta r t + r. \end{aligned}$$

We need the following lemmas from [29]:

Lemma 6.2 [29] *Let X be a coarse median space, $r \geqslant 0$, and let $a, b \in X$ and $x \in [a,b]_\lambda$. Then $[a,x]_r \subset [a,b]_{L_1(r)}$.*

Lemma 6.3 [29] *Let X be a geodesic coarse median space of rank at most η. For every $\kappa > 0$ and $t > 0$, there exists $r_t > 0$, such that for all $r \geqslant r_t$ and $a, b \in X$, there exists $h \in [a,b]_{L_1(r)}$ such that*

- *$\rho(a,h) \leqslant L_3(r,t)$, and*
- *$B(a, rt) \cap [a,b]_\kappa \subset [a,h]_{L_2(r,\kappa)}$.*

Lemma 6.4 [29] *Let X be a coarse median space. Fix $\kappa > 0$. There exist constants $\alpha, \beta \geqslant 0$, depending only on the parameters of the coarse median structure and κ, such that the following holds: Let $a, b, h, m \in X$ and $r \geqslant 0$ satisfy $m \in [a,h]_{L_2(r,\kappa)}$ and $h \in [a,b]_{L_1(r)}$. Then $p = \mu(m,b,h)$ satisfies $\rho(h,p) \leqslant \alpha r + \beta$.*

Proof of Theorem 1.2 The proof is based on the construction used in [29] to prove Property A, and for the reader's convenience, we give a sketch of their proof. In fact we will verify the stronger conditions on the S sets required to apply Theorem 3.1. Fix a basepoint $x_0 \in X$, and let α and β be the constants from Lemma 6.4. First apply Lemma 6.3 for $\kappa = \lambda$ and all $t \in \mathbb{N}$ to obtain a sequence $r_t \in \mathbb{N}$ such that the conclusion of the lemma holds. Furthermore, we can choose the r_t inductively to arrange the sequence $t \mapsto l_t = (t r_t - H(0))/(3K)$ for $t \in \mathbb{N}$ is increasing.

Now fix $x \in X$, $t \in \mathbb{N}$ and $k \in \{1, 2, \ldots, 3l_t\}$. For any $y \in B(x,k)$, Lemma 6.3 applied for $a = y$, $b = x_0$ and $r = r_t$ produces a point $h_y \in [y, x_0]_{L_1(r_t)}$. We define

$$S(x,k,l_t) = \{h_y \in X : y \in B(x,k)\}.$$

We need to verify these sets satisfy condition (2) in the statement of Theorem 3.1, ie we need to show there exists a subexponential function $f\colon \mathbb{R} \to \mathbb{R}$, satisfying:

(i) $\forall t \in \mathbb{N}$, $\exists R_t > 0$ such that $S(x,k,l_t) \subseteq B(x, R_t)$ for all $k = 1, \ldots, 3l_t$.

(ii) $\forall t \in \mathbb{N}$, $\forall k, k'$ with $1 \leqslant k \leqslant k' \leqslant 3l_t$, $\forall x \in X$, we have $S(x,k,l_t) \subseteq S(x,k',l_t)$.

(iii) $\forall x, y \in X$ with $\rho(x,y) \leqslant l_t$, we have:
- $S(x,k-\rho(x,y),l_t) \subset S(x,k,l_t) \cap S(y,k,l_t)$, for $k = \rho(x,y)+1, \ldots, 3l_t$.
- $S(x,k+\rho(x,y),l_t) \supset S(x,k,l_t) \cap S(y,k,l_t)$, for $k = 1, \ldots, 3l_t - \rho(x,y)$.

(iv) $\forall t \in \mathbb{N}$, $\forall k = 1,2,\ldots,3l_t$, $\forall x \in X$, we have $\#S(x,k,l_t) \leqslant f(l_t)$.

By the construction, (ii)–(iii) hold naturally. For (i), by Lemma 6.3, we know

$$S(x,k,l_t) \subseteq B(x, l_t + L_3(r_t,t)).$$

The only thing left is to find a subexponential function f such that condition (iv) holds. The following argument follows totally from the proof in [29], and we omit some calculation. The readers can turn to their original paper for more details.

Take $y \in B(x,k)$, with the notation as above. Let $m_y = \mu(x,y,x_0)$. Then, by Lemma 6.3, one can deduce that $m_y \in [y,h_y]_{L_2(r_t,\lambda)}$. Now, since $h_y \in [y,x_0]_{L_1(r_t)}$, Lemma 6.4 implies the point $p_y = \mu(m_y,x_0,h_y) \in [m_y,x_0]_\lambda$ satisfies $\rho(h_y,p_y) \leqslant \alpha r_t + \beta$. As $m_y = \mu(x,y,x_0) \in [x,x_0]_\lambda$, Lemma 6.2 now implies $p_y \in [x,x_0]_{L_1(\lambda)}$. Consequently, we have $\rho(x,p_y) \leqslant 3l_t + 3^\eta K^\eta t r_t + r_t + \alpha r_t + \beta$, which depends linearly on l_t. Now, by Proposition 9.8 in [8], the number of possible points p_y is bounded by $P(l_t)$ for some polynomial P depending only on H, K, η and the uniform local finiteness of X. Since X has at most exponential growth, it follows that $\#S(x,k,l_t)$ is at most $P(l_t)c'c^{r_t}$ for some constants $c, c' \geqslant 1$. Take $f(l_t) = P(l_t)c'c^{r_t}$ and recall that, in the limit, $r_t/l_t \to 0$. We extend f to a function on $\mathbb{R}^+$ by setting $f(r) := f(l_t)$ for $r \in (l_{t-1}, l_t]$. This completes the proof. □

References

[1] **J Behrstock**, **M F Hagen**, **A Sisto**, *Hierarchically hyperbolic spaces, I: Curve complexes for cubical groups*, preprint (2014) arXiv

[2] **J Behrstock**, **M F Hagen**, **A Sisto**, *Asymptotic dimension and small-cancellation for hierarchically hyperbolic spaces and groups*, Proc. Lond. Math. Soc. 114 (2017) 890–926 MR

[3] **G C Bell**, *Growth of the asymptotic dimension function for groups*, preprint (2005) arXiv

[4] **G Bell**, **A Dranishnikov**, *Asymptotic dimension in Będlewo*, Topology Proc. 38 (2011) 209–236 MR

[5] **G C Bell**, **K Fujiwara**, *The asymptotic dimension of a curve graph is finite*, J. Lond. Math. Soc. 77 (2008) 33–50 MR

[6] **M Bestvina**, **K Bromberg**, **K Fujiwara**, *Constructing group actions on quasi-trees and applications to mapping class groups*, preprint (2010) arXiv

[7] **B H Bowditch**, *Coarse median spaces and groups*, Pacific J. Math. 261 (2013) 53–93 MR

[8] **B H Bowditch**, *Embedding median algebras in products of trees*, Geom. Dedicata 170 (2014) 157–176 MR

[9] **M R Bridson**, **A Haefliger**, *Metric spaces of non-positive curvature*, Grundl. Math. Wissen. 319, Springer (1999) MR

[10] **N P Brown**, **N Ozawa**, *C^*–algebras and finite-dimensional approximations*, Graduate Studies in Mathematics 88, Amer. Math. Soc., Providence, RI (2008) MR

[11] **S Campbell**, **G A Niblo**, *Hilbert space compression and exactness of discrete groups*, J. Funct. Anal. 222 (2005) 292–305 MR

[12] **I Chatterji**, **G Niblo**, *From wall spaces to* CAT(0) *cube complexes*, Internat. J. Algebra Comput. 15 (2005) 875–885 MR

[13] **V Chepoi**, *Graphs of some* CAT(0) *complexes*, Adv. in Appl. Math. 24 (2000) 125–179 MR

[14] **A N Dranishnikov**, *Groups with a polynomial dimension growth*, Geom. Dedicata 119 (2006) 1–15 MR

[15] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR

[16] **M Gromov**, *Asymptotic invariants of infinite groups*, from "Geometric group theory, II" (G A Niblo, M A Roller, editors), London Math. Soc. Lecture Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR

[17] **J R Isbell**, *Median algebra*, Trans. Amer. Math. Soc. 260 (1980) 319–362 MR

[18] **V A Kaimanovich**, *Boundary amenability of hyperbolic spaces*, from "Discrete geometric analysis" (M Kotani, T Shirai, T Sunada, editors), Contemp. Math. 347, Amer. Math. Soc., Providence, RI (2004) 83–111 MR

[19] **G A Niblo**, **L D Reeves**, *The geometry of cube complexes and the complexity of their fundamental groups*, Topology 37 (1998) 621–633 MR

[20] **B Nica**, *Cubulating spaces with walls*, Algebr. Geom. Topol. 4 (2004) 297–309 MR

[21] **P W Nowak**, **G Yu**, *Large scale geometry*, Eur. Math. Soc., Zürich (2012) MR

[22] **I Oppenheim**, *An intermediate quasi-isometric invariant between subexponential asymptotic dimension growth and Yu's property A*, Internat. J. Algebra Comput. 24 (2014) 909–922 MR

[23] **N Ozawa**, *Metric spaces with subexponential asymptotic dimension growth*, Internat. J. Algebra Comput. 22 (2012) art. id. 1250011 MR

[24] **L D Reeves**, *Biautomatic structures and combinatorics for cube complexes*, PhD thesis, University of Melbourne (1995)

[25] **J Roe**, *Hyperbolic groups have finite asymptotic dimension*, Proc. Amer. Math. Soc. 133 (2005) 2489–2490 MR

[26] **M Roller**, *Poc sets, median algebras and group actions: an extended study of Dunwoody's construction and Sageev's theorem*, preprint (2016) arXiv

[27] **M Sageev**, *Ends of group pairs and non-positively curved cube complexes*, Proc. London Math. Soc. 71 (1995) 585–617 MR

[28] **J Smith**, *The asymptotic dimension of the first Grigorchuk group is infinity*, Rev. Mat. Complut. 20 (2007) 119–121 MR

[29] **J Špakula**, **N Wright**, *Coarse medians and Property A*, preprint (2016) arXiv

[30] **N Wright**, *Finite asymptotic dimension for* CAT(0) *cube complexes*, Geom. Topol. 16 (2012) 527–554 MR

[31] **G Yu**, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. 147 (1998) 325–355 MR

[32] **G Yu**, *The coarse Baum–Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. 139 (2000) 201–240 MR

[33] **R Zeidler**, *Coarse median structures on groups*, Master's thesis, University of Vienna (2013) Available at `http://othes.univie.ac.at/28999/`

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Received: 6 January 2017 Revised: 30 May 2017