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Classifying spaces for 1–truncated compact Lie groups

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A 1–truncated compact Lie group is any extension of a finite group by a torus. In this note we compute the homotopy types of $\mathrm{Map}_*(BG, BH)$, $\mathrm{Map}(BG, BH)$, and $\mathrm{Map}(EG, B_G H)$ for compact Lie groups G and H with H 1–truncated, showing that they are computed entirely in terms of spaces of homomorphisms from G to H. These results generalize the well-known case when H is finite, and the case when H is compact abelian due to Lashof, May, and Segal.

55R91; 55P92, 55R35, 55R37

1 Introduction

By a 1–*truncated* compact Lie group H, we mean one whose homotopy groups vanish in dimensions 2 and greater. Equivalently, H is a compact Lie group with identity component H_0 a torus (isomorphic to some $U(1)^d$), ie an extension of a finite group by a torus.

The class of 1–truncated compact Lie groups includes (i) all finite groups, and (ii) all compact abelian Lie groups, both of which are included in the class of (iii) all groups which are isomorphic to a product of a compact abelian Lie group with a finite group, or equivalently, a product of a torus with a finite group.

The goal of this paper is to extend certain results, which were already known for finite groups, compact abelian Lie groups, or products thereof, to all 1–truncated compact Lie groups.

We write $\mathrm{Hom}(G, H)$ for the space of continuous homomorphisms, equipped with the compact-open topology. Our first theorem relates this to the space of based maps between classifying spaces.

1.1 Theorem *For G, H compact Lie groups with H 1–truncated, the evident map*

$$B\colon \mathrm{Hom}(G, H) \to \mathrm{Map}_*(BG, BH)$$

is a weak equivalence.

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Using this, we will derive an unbased variant.

1.2 Theorem *For G, H compact Lie groups with H 1–truncated, there is a weak equivalence*

$$\mathrm{Hom}(G,H)\times_H EH \to \mathrm{Map}(BG,BH).$$

Here H acts on $\mathrm{Hom}(G,H)$ by conjugation: $h\cdot\phi = h\phi h^{-1}$.

When H is discrete, these are well known and classical results. The case of H an abelian compact Lie group is proved by Lashof, May and Segal [7]; both the finite and compact abelian Lie cases are discussed by May in [10, Theorems 5 and 10].

1.3 Remark For G and H compact, there is a homeomorphism (see Proposition 7.1)

$$\mathrm{Hom}(G,H)\approx \coprod_{[\phi\colon G\to H]} H/C_H(\phi),$$

where the coproduct is over conjugacy classes of homomorphisms, and $C_H(\phi)$ is the centralizer of $\phi(G)$ in H. When H is a 1–truncated compact Lie group, we see from Theorem 1.1 that $\mathrm{Map}_*(BG,BH)$ is therefore weakly equivalent to this coproduct, and from Theorem 1.2 that there is a weak equivalence

$$\mathrm{Map}(BG,BH)\approx \coprod_{[\phi\colon G\to H]} BC_H(\phi).$$

Finally, we will give a description of the fixed points of the equivariant classifying space B_GH, which represents G–equivariant H–principal bundles, in the case that G and H are compact Lie groups and H is 1–truncated.

1.4 Theorem *For G, H compact Lie groups with H 1–truncated, the map*

$$\pi^*\colon B_GH \to \mathrm{Map}(EG, B_GH)$$

induced by restriction along $\pi\colon EG\to *$ is a G–equivariant weak equivalence.

The case when H is finite or compact abelian is proved in [10].

1.5 Remark For any closed subgroup $G'\le G$, taking G' fixed points gives rise to maps

$$(B_GH)^{G'} \to \mathrm{Map}(EG,B_GH)^{G'} \xleftarrow{\sim} \mathrm{Map}(EG,BH)^{G'} \approx \mathrm{Map}(BG',BH),$$

and Theorem 1.4 amounts to saying that for any G', the first map in this sequence is a weak equivalence. (The middle map arises from a G–equivariant weak equivalence $\mathrm{Map}(EG, BH) \to \mathrm{Map}(EG, B_G H)$; see the proof of Lemma 6.4.) It is standard (see Lashof and May [6, Theorem 10]) that, for arbitrary compact G and H, $(B_G H)^{G'}$ is weakly equivalent to $\coprod_{[\phi\colon G' \to H]} BC_H(\phi)$, while if H is also 1–truncated, Theorem 1.2 and Remark 1.3 imply that $\mathrm{Map}(BG', BH)$ is also weakly equivalent to the same coproduct, thus giving an abstract weak equivalence $(B_G H)^{G'} \approx \mathrm{Map}(BG', BH)$. The point of Theorem 1.4 is to show that the map π^* exhibits this equivalence.

The map of Theorem 1.4 in a certain sense classifies the formation of the G–Borel quotient. That is, given a G–equivariant map $f\colon X \to B_G H$ classifying a G–equivariant principal H–bundle $P \to X$, the G–equivariant map $\pi^* f\colon X \to \mathrm{Map}(EG, B_G H)$ is adjoint to a nonequivariant map $X \times_G EG \to B_G H$ which classifies the bundle $P \times_G EG \to X \times_G EG$; see [10]. As a consequence of the theories of classifying spaces, we obtain the following.

1.6 Corollary *Let G and H be compact Lie groups with H 1–truncated. Then for a paracompact G–space X, formation of the G–Borel quotient gives rise to a bijection between* (i) *equivalence classes of G–equivariant principal H–bundles over X, and* (ii) *equivalence classes of principal H–bundles over $X \times_G EG$.*

1.7 Remark For comparison, there are well-known results (stemming from work of Dwyer and Zabrodsky [3] and Notbohm [12]) on spaces of maps *from* (rather than *to*) the classifying space of a p–toral group (a p–toral group is an extension of a *finite p–group* by a torus). For instance, [12, Theorem 1.3] may be interpreted as saying that for an arbitrary compact Lie group H and p–toral G, the map $B_G H \to \mathrm{Map}(EG, B_G H)$ (see Theorem 1.4) induces an isomorphism in mod p homology on fixed points for all closed subgroups of G (this interpretation is given as [10, Theorem 9]).

Organization of the paper The proof of Theorem 1.1 is the probably the most interesting part of the paper. It is carried out in Sections 2–4. The key ingredient is the use of the nerve $N(H, V)$ of the "exponential crossed module" (see Example 2.3) of the 1–truncated compact Lie group H. We first show that the simplicial space $N(H, V)$ is a Reedy fibrant model for the usual simplicial nerve NH of H (see Corollary 3.8), and so can be used to compute maps from BG to BH in terms of maps of simplicial spaces from NG to $N(H, V)$. The proof is completed in Section 4 by showing that, in a certain sense, the difference between $\mathrm{Hom}(G, H)$ and the space $\mathrm{Map}_{s\mathrm{Top}}(NG, N(H, V))$ of maps between simplicial spaces is measured precisely by the continuous 2–cocycles on G with values in V, modulo boundaries of 1–cocycles. Because G is compact, Haar measure gives a contracting homotopy (see Proposition 4.2) on the complex of continuous chains on G. A sketch by the author of this proof originally appeared as an answer to a question on the site MathOverflow.[1]

Our approach gives a uniform proof of Theorem 1.1 for all 1–truncated compact Lie groups H; furthermore, even in the case of abelian H, it is somewhat more direct than the one given in [7].

We derived the unbased Theorem 1.2 from the based version Theorem 1.1 in Section 5 by comparing associated fibrations over BH.

The result on equivariant classifying spaces, Theorem 1.4, is proved in Section 6. The proof relies on an explicit model, built as the nerve of a certain topological category, of the restriction of the universal (G, H)–bundle to the fixed-point subspace $(B_G H)^G \subseteq B_G H$. The explicit model we use appears to be essentially of the type described by Guillou, May and Merling [4].

In Section 7, we give for the convenience of the reader a proof of the identification of $\mathrm{Hom}(G, H)$ as mentioned in Remark 1.3.

Conventions In this paper, we write Top for the category of *compactly generated weak Hausdorff spaces* (CGWH), the standard convenient category of spaces. This category is cartesian closed, and we write $\mathrm{Map}(X, Y)$ for the internal function object, ie continuous maps with the k–ification of the compact-open topology. We make use of the "usual" model structure on Top, in which weak equivalences are weak equivalences on homotopy groups, and fibrations are Serre fibrations.

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2 Nerve of a topological crossed module

2.1 Crossed modules

Recall that a *crossed module* consists of

[1] "Equivariant classifying spaces from classifying spaces", `http://mathoverflow.net/q/223546`.

- groups H and V,
- a homomorphism $\epsilon\colon V \to H$,
- a homomorphism $\alpha\colon H \to \mathrm{Aut}(V)$ such that
  (i) $\epsilon(\alpha(h)(v)) = h\epsilon(v)h^{-1}$, and
  (ii) $\alpha(\epsilon(v))(v') = vv'v^{-1}$ for $h \in H$ and $v, v' \in V$.

A *topological crossed module* is one in which V and H are topological groups and ϵ and α are continuous. We typically write (H, V) for the crossed module, leaving ϵ and α understood. Note that we will often consider crossed modules in which V is an abelian group, in which case we will switch to additive notation for V, though not for H.

2.2 Example Given any group H, there is a unique crossed module, denoted $(H, 0)$, for which V is the trivial group.

2.3 Example (exponential crossed module) The following example is the crucial one for this paper. Suppose H is a 1–truncated compact Lie group. We set:

- $V := T_e H$, the Lie algebra of H, which is a group under addition of vectors.
- $\epsilon := \exp\colon V \to H$, the exponential map; this is a homomorphism since H_0 is abelian.
- $\alpha := \mathrm{ad}\colon H \to \mathrm{GL}(V)$, the adjoint action.

We typically write the group law of V additively, so the identities for the crossed module structure become

$$\exp(\mathrm{ad}(h)(v)) = h\exp(v)h^{-1}, \quad \mathrm{ad}(\exp(v))(v') = v + v' - v = v'.$$

The following features of this case will be significant:

(1) $\alpha = \mathrm{ad}\colon H \to \mathrm{GL}(V)$ factors through the quotient group H/H_0;
(2) $\epsilon = \exp\colon V \to H$ is a covering map;
(3) the underlying space of V is contractible.

2.4 Nerve of a crossed module

The *nerve* of a topological crossed module $N(H, V)$ is the simplicial space defined as follows; except for the topology, this is as in [1, Section 3.1]. The space $N(H, V)_n$ in degree n is the space of tuples

$$\big((h_{ij})_{0\le i\le j\le n}, (v_{ijk})_{0\le i\le j\le k\le n}\big), \quad h_{ij} \in H,\ v_{ijk} \in V$$

satisfying the identities

(1) $h_{ii} = e$ and $v_{iij} = v_{ijj} = e$ for all $i \leq j$,
(2) $h_{ik} = \epsilon(v_{ijk}) h_{ij} h_{jk}$ for all $i \leq j \leq k$,
(3) $v_{ik\ell} v_{ijk} = v_{ij\ell}\, \alpha(h_{ij})(v_{jk\ell})$ for all $i \leq j \leq k \leq \ell$.

The action of simplicial operators $\delta\colon [n] \to [m]$ is the evident one: $(\delta h)_{ij} = h_{\delta(i),\delta(j)}$ and $(\delta v)_{ijk} = v_{\delta(i),\delta(j),\delta(k)}$. A standard argument shows that, as a space, $N(H,V)_n \approx H^n \times V^{\binom{n}{2}}$, eg via the projection to coordinates h_{0i}, $1 \leq i \leq n$, and v_{0ij}, $1 \leq i < j \leq n$. Note that $N(H,V)_0 = *$; ie $N(H,V)$ is a *reduced* simplicial space.

2.5 Example The nerve of $N(H,0)$ is precisely the usual nerve of the group H; we write $N(H) := N(H,0)$.

2.6 Simplicial spaces and the Reedy model structure

We write sTop for the category of simplicial spaces, ie functors $\Delta^{\mathrm{op}} \to \mathrm{Top}$. We are going to use the *Reedy model structure* on sTop. We will need to use the following features of this model structure:

(1) Weak equivalences $f\colon X \to Y$ in sTop are precisely the levelwise weak equivalences; ie $f_n\colon X_n \to Y_n$ is a weak equivalence for all $n \geq 0$.
(2) An object X is cofibrant (*Reedy cofibrant*) if and only if the latching space inclusions $\gamma_n\colon L_n X \to X_n$ are cofibrations in Top.
(3) An object Y is fibrant (*Reedy fibrant*) if and only if the matching space projections $\delta_n\colon Y_n \to M_n Y$ are fibrations in Top.
(4) The model structure is topological. In particular, if X is a cofibrant simplicial space and $Y \to Y'$ is a weak equivalence between fibrant simplicial spaces, then $\mathrm{Map}_{s\mathrm{Top}}(X,Y) \to \mathrm{Map}_{s\mathrm{Top}}(X,Y')$ is a weak equivalence of spaces.

We will need to examine latching and matching spaces in a bit more detail.

2.7 Latching and matching spaces

We recall the notion of latching and matching spaces. For simplicial spaces $X\colon \Delta^{\mathrm{op}} \to \mathrm{Top}$ and all $n \geq 0$, we have natural maps of spaces

$$L_n X \xrightarrow{\gamma_n} X_n \xrightarrow{\delta_n} M_n X,$$

where

$$L_n X = \mathrm{colim}_{(\Delta^{\mathrm{op}}_{/[n]})_{<n}} X, \quad M_n = \lim_{(\Delta^{\mathrm{op}}_{[n]/})_{<n}} X,$$

called the *latching* and *matching* spaces of X.

2.8 Latching spaces for the nerve of a group

2.9 Proposition *Let G be a topological group, and $NG \in s\mathrm{Top}$ its nerve. Then for each $n \geq 0$, the latching inclusion $\gamma_n\colon L_n(NG) \to (NG)_n$ is isomorphic to the inclusion*

$$\{(g_1, \dots, g_n) \mid g_i = e \text{ for some } i\} \to G^n.$$

In particular, NG is Reedy cofibrant if $\{e\} \to G$ is a cofibration in Top.

The proof of this is standard.

2.10 Matching spaces for the nerve of a crossed module

We describe the matching projections for the nerve of a topological crossed module.

2.11 Proposition *Consider $N := N(H, V)$ the nerve of a topological crossed module. We write $M_n := M_n N$ for its matching spaces.*

(0) *$\delta_0\colon N_0 \to M_0$ is the isomorphism of 1–point spaces.*

(1) *$\delta_1\colon N_1 \to M_1$ is the projection $H \to *$.*

(2) *$M_2 \approx H^{\times 3}$, and there is a pullback square*

$$\begin{array}{ccc} N_2 & \longrightarrow & V \\ {\scriptstyle \delta_2}\downarrow & & \downarrow{\scriptstyle \epsilon} \\ M_2 & \xrightarrow[(h_{01},h_{02},h_{12}) \mapsto h_{02}h_{12}^{-1}h_{02}^{-1}]{} & H \end{array}$$

(3) *There is a commutative diagram*

$$\begin{array}{ccccc} N_3 & \longrightarrow & \{e\} & & \\ {\scriptstyle \delta_3}\downarrow & & \downarrow & & \\ M_3 & \longrightarrow & \operatorname{Ker}\epsilon & \longrightarrow & \{e\} \\ \downarrow & & \downarrow & & \downarrow \\ H^{\times 3} \times V^{\times 3} & \xrightarrow[\pi]{} & V & \xrightarrow[\epsilon]{} & H \end{array}$$

in which all squares are pullback squares, and π is given by

$$(h_{01}, h_{12}, h_{23}, v_{012}, v_{013}, v_{023}, v_{123}) \mapsto v_{023}v_{012}\alpha(h_{01})(v_{123})^{-1}v_{013}^{-1}.$$

(≥ 4) *$\delta_n\colon N_n \to M_n$ is an isomorphism for $n \geq 4$.*

Proof This is straightforward. In (3), one shows directly that the right-hand lower square, bottom rectangle, and left rectangle are pullbacks. □

Recall that a simplicial space X is *Reedy fibrant* if each of the maps $\delta_n\colon X_n \to M_n X$ is a fibration of spaces.

2.12 Corollary *If (H, V) is a topological crossed module such that ϵ is a covering map, then $N(H, V)$ is Reedy fibrant.*

Proof This is immediate using Proposition 2.11. Note that the condition that ϵ be a covering map in Proposition 2.11(3) implies that δ_3 is an open and closed embedding. □

In particular, Corollary 2.12 applies to our main Example 2.3.

3 Maps between reduced simplicial spaces

A simplicial space $X \in s\mathrm{Top}$ is said to be *reduced* if $X_0 \approx *$. We write $s\mathrm{Top}^{\mathrm{red}} \subset s\mathrm{Top}$ for the full subcategory of reduced simplicial spaces. Note that reduced simplicial spaces are canonically based, so we may in fact regard $s\mathrm{Top}^{\mathrm{red}}$ as a full subcategory of simplicial based spaces $s\mathrm{Top}_*$.

3.1 Realization of reduced simplicial spaces

We recall the geometric realization functor $\|-\|\colon s\mathrm{Top} \to \mathrm{Top}$, defined so that $\|X\|$ is the coend of the functor $\Delta^{\mathrm{op}} \times \Delta \to \mathrm{Top}$ given by $([m],[n]) \mapsto X_m \times \Delta^n$, where Δ^n is the topological n–simplex.

3.2 Proposition *The restriction of the geometric realization functor to a functor $\|-\|\colon s\mathrm{Top}^{\mathrm{red}} \to \mathrm{Top}_*$ admits a right adjoint $\nabla\colon \mathrm{Top}_* \to s\mathrm{Top}^{\mathrm{red}}$ defined by*

$$(\nabla Y)_n := \mathrm{Map}_*(\Delta^n / \mathrm{Sk}_0\, \Delta^n, Y),$$

where $\mathrm{Sk}_0\, \Delta^n \subseteq \Delta^n$ is the set of vertices of the simplex. The adjunction is compatible with the topological enrichment, and so gives a natural homeomorphism

$$\mathrm{Map}_{s\mathrm{Top}}(X, \nabla Y) \approx \mathrm{Map}_*(\|X\|, Y)$$

for $X \in s\mathrm{Top}^{\mathrm{red}}$ and $Y \in \mathrm{Top}_*$.

Proof This is a straightforward consequence of the observation that, for reduced simplicial spaces X, we see that $\|X\|$ is isomorphic to the coend (in Top_*) of $([m],[n]) \mapsto X_m \wedge (\Delta^n / \mathrm{Sk}_0\, \Delta^n)$. □

3.3 Proposition *For any* $Y \in \mathrm{Top}_*$, *the simplicial space* ∇Y *is Reedy fibrant.*

Proof The matching space projection has the form

$$\mathrm{Map}_*(\Delta^n/\operatorname{Sk}_0 \Delta^n, Y) \to \mathrm{Map}_*(\partial\Delta^n/\operatorname{Sk}_0 \Delta^n, Y),$$

which is clearly a fibration. □

For a topological group H, we consider the classifying space $BH := \|NH\|$.

3.4 Proposition *If* H *is a topological group with identity element a nondegenerate basepoint (ie* $\{e\} \to H$ *has the HEP), then the map*

$$\eta\colon NH \to \nabla\|NH\| = \nabla BH$$

given by the unit map of the adjunction of Proposition 3.2 is a levelwise weak equivalence of simplicial spaces.

Proof In degree 0, the map η is the isomorphism of one-point spaces. In degree 1, it has the form

$$H \to \mathrm{Map}_*(\Delta^1/\{0,1\}, \|NH\|) \approx \Omega BH.$$

A standard argument (eg using the usual simplicial model for the universal fibration [9]) shows that this is a weak equivalence.

For $n \geq 2$, we reduce to the $n = 1$ case using the fact that $I_n/\operatorname{Sk}_0 \Delta^n \to \Delta^n/\operatorname{Sk}_0 \Delta^n$ is a homotopy equivalence of pointed spaces, and thus

$$\mathrm{Map}_*(\Delta^n/\operatorname{Sk}_0 \Delta^n,\ \|NH\|) \to \mathrm{Map}_*(I_n/\operatorname{Sk}_0 \Delta^n,\ \|NH\|) \approx (\Omega BG)^{\times n}$$

is a weak equivalence, where $I_n \subseteq \Delta^n$ is the union of the edges with vertices $\{k-1, k\}$ for all $k = 1, \dots, n$. □

3.5 Map($X, N(H, V)$) computes Map$_*$($\|X\|, BH$)

Now we fix a 1–truncated compact Lie group H and the corresponding exponential crossed module (H, V) of Example 2.3. We have a map of reduced simplicial spaces

$$NH \xrightarrow{(\iota,\eta)} N(H,V) \times \nabla\|NH\|$$

in which ι is the evident inclusion $NH = N(H,0) \subseteq N(H,V)$ and η the unit map of the adjunction of Proposition 3.2. Observe that both ι and η are levelwise weak equivalences (ι because V is contractible, η by Proposition 3.4). Furthermore, both $N(H,V)$ (see Corollary 2.12) and $\nabla\|NH\|$ (see Proposition 3.3) are Reedy fibrant.

Using the Reedy model structure on simplicial spaces, we can factor the above map as

$$NH \xrightarrow{j} (NH)^f \xrightarrow{(\iota',\eta')} N(H,V)\times\nabla\|NH\| \tag{3.6}$$

so that $(NH)^f$ is Reedy fibrant and j is a levelwise weak equivalence, whence ι' and η' are also levelwise weak equivalences.

3.7 Proposition *For X a Reedy cofibrant simplicial space with $X_0=*$, and (H,V) the exponential crossed module of a 1–truncated compact Lie group H, we have that $\mathrm{Map}_{s\mathrm{Top}}(X,N(H,V))$ is weakly equivalent to $\mathrm{Map}_*(\|X\|,\|NH\|)$. Furthermore, $\iota_*\colon \mathrm{Map}_{s\mathrm{Top}}(X,NH)\to\mathrm{Map}_{s\mathrm{Top}}(X,N(H,V))$ is a weak equivalence of spaces if and only if $\eta_*\colon \mathrm{Map}_{s\mathrm{Top}}(X,NH)\to\mathrm{Map}_{s\mathrm{Top}}(X,\nabla\|NH\|)$ is.*

Proof This is straightforward using the factorization (3.6), the fact that Reedy model structure is compatible with the topological enrichment, and the adjunction from Proposition 3.2. □

3.8 Corollary *If (H,V) is as above, and G is a topological group such that $\{e\}\to G$ is a cofibration, then $\mathrm{Map}_*(BG,BH)$ is weakly equivalent to $\mathrm{Map}_{s\mathrm{Top}}(NG,N(H,V))$, and*

$$B\colon \mathrm{Hom}(G,H)\to\mathrm{Map}_*(BG,BH)$$

is a weak equivalence if and only if

$$\iota_*\colon \mathrm{Map}_{s\mathrm{Top}}(NG,NH)\to\mathrm{Map}_{s\mathrm{Top}}(NG,N(H,V))$$

is a weak equivalence.

Proof Use Proposition 3.7 with $X=NG$, which is Reedy cofibrant by Proposition 2.9. It is straightforward to see that $\mathrm{Map}_{s\mathrm{Top}}(NG,NH)\to\mathrm{Hom}(G,H)$ (evaluation at spaces in degree 1) is a homeomorphism, and so the map B coincides with ι_*. □

3.9 Remark If H is a discrete group, then NH is already Reedy fibrant, in which case we can immediately derive the well-known fact that $B\colon \mathrm{Hom}(G,H)\to\mathrm{Map}_*(BG,BH)$ is a weak equivalence for any such topological group G.

4 Proof of Theorem 1.1 Based mapping space

As above, we assume that H is a 1–truncated compact Lie group. We will now also assume that G is a compact Lie group. By Corollary 3.8, we have reduced Theorem 1.1 to showing that $\mathrm{Map}_{s\mathrm{Top}}(NG, NH) \to \mathrm{Map}_{s\mathrm{Top}}(NG, N(H,V))$ is a weak equivalence.

Let $E := \mathrm{Map}_{s\mathrm{Top}}(NG, N(H,V))$. Using Proposition 2.11 and the identification of the latching inclusions $L_n NG \to G^n$ (see Proposition 2.9), we see that E is precisely the space of pairs

$$(\zeta, \nu) \in \mathrm{Map}(G, H) \times \mathrm{Map}(G \times G, V)$$

such that

(1) $\zeta(e) = e$ and $\nu(g,e) = 0 = \nu(e,g)$ for $g \in G$,

(2) $\zeta(g_1 g_2) = \exp[\nu(g_1,g_2)]\zeta(g_1)\zeta(g_2)$ for $g_1, g_2 \in G$,

(3) $\nu(g_1g_2,g_3)+\nu(g_1,g_2) = \nu(g_1,g_2g_3)+\mathrm{ad}(\zeta(g_1))[\nu(g_2,g_3)]$ for $g_1,g_2,g_3 \in G$.

Explicitly, this corresponds to the map $NG \to N(H,V)$ which (in the notation of Section 2.1) sends $(g_{ij}) \in (NG)_n$ to $(h_{ij}, v_{ijk}) \in N(H,V)_n$ with $h_{ij} = \zeta(g_{ij})$ and $v_{ijk} = \nu(g_{ij}, g_{jk})$.

Let $E^0 := \mathrm{Map}_{s\mathrm{Top}}(NG, NH)$. The map $E^0 \to E$ is precisely inclusion into the subspace consisting of points of the form $(\zeta, 0)$.

For a continuous map $\zeta\colon G \to H$, we write $\bar{\zeta}\colon G \to H/H_0$ for the composite with the quotient map $H \to H/H_0$. Note that if $(\zeta,\nu) \in E$, then $\bar{\zeta}$ is a continuous homomorphism of groups. Since H/H_0 is discrete, we obtain coproduct decompositions

$$E = \coprod_{\gamma} E_\gamma, \quad E^0 = \coprod_{\gamma} E^0_\gamma, \quad \gamma \in \mathrm{Hom}(G, H/H_0).$$

Thus, we must show that for each such γ, the inclusion $E^0_\gamma \subseteq E_\gamma$ is a weak equivalence. In fact, we can give an explicit (strong) deformation retraction of E_γ to E^0_γ, which relies on the existence of a contracting homotopy of the complex $C^*(G, V_{\mathrm{ad}\,\gamma})$ of normalized continuous cochains on G with values in the representation $\mathrm{ad}\,\gamma\colon G \to \mathrm{Aut}(V)$, which may be constructed explicitly using an invariant measure on the compact group G. We spell out the details we need below.

Fix $\gamma \in \mathrm{Hom}(G, H/H_0)$. Let $C^1_\gamma \subseteq \mathrm{Map}(G,V)$ be the subspace of functions $\mu\colon G \to V$ such that

$$\mu(e) = 0.$$

Let $Z^2_\gamma \subseteq \mathrm{Map}(G\times G, V)$ be the subspace of functions $\nu\colon G\times G\to V$ such that

$$\nu(g,e)=0=\nu(e,g),\quad g\in G,$$

and

$$\nu(g_1g_2,g_3)+\nu(g_1,g_2)=\nu(g_1,g_2g_3)+\operatorname{ad}\gamma(g_1)[\nu(g_2,g_3)],\quad g_1,g_2,g_3\in G.$$

Both Z^2_γ and C^1_γ are topological real vector spaces. Define continuous and linear maps

$$d\colon C^1_\gamma\to Z^2_\gamma,\quad H\colon Z^2_\gamma\to C^1_\gamma$$

by

$$\begin{aligned} d\mu(g_1,g_2)&:=\mu(g_1)-\mu(g_1g_2)+\operatorname{ad}\gamma(g_1)\mu(g_2),\\ H\nu(g)&:=\int_G x^{-1}\nu(x,g)\,dx,\end{aligned}$$

where we use right-invariant Haar measure on G normalized so that $\int_G dx=1$.

4.1 Lemma *The composite $dH\colon Z^2_\gamma\to Z^2_\gamma$ is the identity map.*

Proof For $g\in G$ and $v\in V$, write gv for $\operatorname{ad}\gamma(g)(v)$ below. Given $\nu\in Z^2_\gamma$, we have

$$\begin{aligned} dH\nu(g_1,g_2)&=\int_G\bigl(x^{-1}\nu(x,g_1)-x^{-1}\nu(x,g_1g_2)+g_1x^{-1}\nu(x,g_2)\bigr)\,dx\\ &=\int_G\bigl(x^{-1}\nu(x,g_1)-x^{-1}[\nu(xg_1,g_2)+\nu(x,g_1)-x\nu(g_1,g_2)]\\ &\qquad\qquad+g_1x^{-1}\nu(x,g_2)\bigr)\,dx\\ &=\nu(g_1,g_2)-\int_G g_1(xg_1)^{-1}\nu(xg_1,g_2)\,dx+\int_G g_1x^{-1}\nu(x,g_2)\,dx\\ &=\nu(g_1,g_2),\end{aligned}$$

where the last cancellation is by right invariance of the measure. □

4.2 Proposition *The inclusion $E^0_\gamma\subseteq E_\gamma$ admits a strong deformation retraction.*

Proof Define $K_t\colon E_\gamma\to E_\gamma$ for $0\le t\le 1$ by $K_t(\zeta,\nu):=(\zeta_t,\nu_t)$, with

$$\begin{aligned}\zeta_t(g)&:=\exp[tH\nu(g)]\zeta(g),\\ \nu_t(g_1,g_2)&:=\nu(g_1,g_2)-t\,dH\nu(g_1,g_2).\end{aligned}$$

We have $K_0=\mathrm{id}_{E_\gamma}$, $K_t|E^0_\gamma=\mathrm{id}_{E^0_\gamma}$, and $K_1(E_\gamma)\subseteq E^0_\gamma$ as desired, the last using Lemma 4.1. □

The proof of Theorem 1.1 follows using Proposition 3.7 and the remarks above.

4.3 Remark If H is an abelian group, then ad: $H \to \mathrm{Aut}(V)$ is trivial. In this case, the proof of Proposition 4.2 directly gives a deformation retraction of $E^0 \subseteq E$.

5 Proof of Theorem 1.2 Unbased mapping space

Given simplicial spaces X and Y, we have an internal function object $\underline{\mathrm{Map}}(X,Y) \in s\mathrm{Top}$, characterized so that $\underline{\mathrm{Map}}(X,-)$ is the right adjoint to $(-)\times X$. We have that

$$\underline{\mathrm{Map}}(X,Y)_n = \mathrm{Map}_{s\mathrm{Top}}(X \times N[n], Y),$$

where $[n]$ is the n–arrow category. In particular, $\underline{\mathrm{Map}}(X,Y)_0 \approx \mathrm{Map}_{s\mathrm{Top}}(X,Y)$.

Formation of the internal function object is compatible with realization: there are canonical maps

$$\rho\colon \|\underline{\mathrm{Map}}(X,Y)\| \to \mathrm{Map}(\|X\|,\|Y\|), \tag{5.1}$$

natural in X and Y. This map exists exactly because $\|-\|\colon s\mathrm{Top}\to\mathrm{Top}$, the realization functor, preserves finite products and is characterized as the map adjoint to

$$\|\underline{\mathrm{Map}}(X,Y)\| \times \|X\| \xleftarrow{\sim} \|\underline{\mathrm{Map}}(X,Y)\times X\| \xrightarrow{|\mathrm{eval}|} \|Y\|.$$

Given topological groups G and H, we consider the function object $\underline{\mathrm{Map}}(NG,NH)$. We have an evident isomorphism

$$\underline{\mathrm{Map}}(NG,NH) \approx N\mathrm{Fun}(G,H),$$

where $\mathrm{Fun}(G,H)$ is the internal category in Top of functors and natural transformations from G to H. Explicitly, this has

- objects $\phi \in \mathrm{Hom}(G,H)$, and
- morphisms $\phi_0 \xrightarrow{h} \phi_1$, where $h \in H$ and $\phi_1 = h\phi_0 h^{-1}$,

and thus homeomorphisms $N\mathrm{Fun}(G,H)_n = \underline{\mathrm{Map}}(NG,NH)_n = \mathrm{Hom}(G,H)\times H^{\times n}$.

Write $(H \curvearrowright H)$ for the translation category of the left action of H on itself, viewed as a category object in Top. This has

- objects $h_0 \in H$, and
- morphisms $h_0 \xrightarrow{h} h_1$, where $h \in H$ and $h_1 = hh_0$.

We have homeomorphisms $N(H \curvearrowright H)_n = H^{\times(n+1)}$. The group H acts on the category $(H \curvearrowright H)$ by $\delta\cdot h_0 = h_0\delta^{-1}$ (on objects) and $\delta\cdot(h_0 \xrightarrow{h} h_1) = h_0\delta^{-1} \xrightarrow{h} h_1\delta^{-1}$ (on morphisms), where $\delta \in H$.

We let $EH := \|N(H \curvearrowright H)\|$, a contractible H space with free H–action.

5.2 Lemma *There is a homeomorphism* $\|\underline{\mathrm{Map}}(NG, NH)\| \approx (\mathrm{Hom}(G,H)\times EH)/H$*, where* H *acts on* $\mathrm{Hom}(G,H)$ *by conjugation.*

Proof of Theorem 1.2 We have a commutative diagram

$$\begin{array}{ccc} \|\underline{\mathrm{Map}}(NG,NH)\| & \xrightarrow{\rho} & \mathrm{Map}(\|NG\|,\|NH\|) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \|\underline{\mathrm{Map}}(*,NH)\| & \xrightarrow[\approx]{} & \mathrm{Map}(\|*\|,\|NH\|) \end{array}$$

where the vertical maps are induced by restriction along $* \to NG$, and the lower horizontal map is the evident homeomorphism (both source and target are homeomorphic to BH). We claim that ρ is a weak equivalence.

By Lemma 5.2 we see that $\alpha\colon (\mathrm{Hom}(G,H)\times EH)/H \to BH$ is a fiber bundle with fiber $\mathrm{Hom}(G,H)$. Since $\beta\colon \mathrm{Map}(BG,BH) \to BH$ is also a fibration, and the base space BH is path connected, ρ is a weak equivalence if and only if its restriction to the fiber over the base point is, which is precisely the weak equivalence $\mathrm{Hom}(G,H) \to \mathrm{Map}_*(BG,BH)$ of Theorem 1.1. □

6 Proof of Theorem 1.4 Equivariant classifying space

6.1 Recollections on equivariant bundles

A *G–equivariant principal H–bundle* (or *(G,H)–bundle*), is a principal H–bundle $\pi\colon P \to X$, together with actions of G on P and X, compatible with π, such that G acts via maps of principal H–bundles. We will always assume that both G and H are compact Lie groups.

This definition is somewhat anomalous in that (G,H)–bundles are not characterized by a property which is local in X. Thus, we say that a (G,H)–bundle is *locally trivial* if it looks locally like

$$(G\times H)\times_{\Lambda_\phi} U \to G\times_{G'} U,$$

where $G' \leq G$ is a closed subgroup, $\Lambda_\phi := \{(g, \phi(g)) \mid g \in G'\}$ is the graph of some homomorphism $\phi\colon G' \to H$, and $\Lambda \xrightarrow{\sim} G'$ acts on a space U. The key result is that if G and H are compact and X is completely regular,[2] then any (G, H)–bundle over X is locally trivial; see [5, Corollary 1.5].

A (G, H)–bundle $P \to X$ is *numerable* if it admits a locally trivializing cover which itself admits a subordinate partition of unity by G–invariant functions. Over a paracompact base X, every locally trivial bundle is numerable; see [5, Corollary 1.13]. There is a universal (G, H)–bundle $E_G H \to B_G H$, which classifies equivalence classes of numerable bundles; see [5], [6], and also [8] for a recent and more general treatment.

We will be mainly concerned with the case of (G, H)–bundles $\pi\colon P \to X$ such that G acts trivially on X. In such a case, there is a natural function

$$\tau\colon P \to \mathrm{Hom}(G, H)$$

defined so that $\tau(p)(\gamma) \in H$ is the unique $\delta \in H$ such that $(\gamma, \delta) \cdot p = p$. When $P \to X$ is locally trivial, the map τ is seen to be continuous. Observe that τ is $G \times H$–equivariant, where this group acts on $\mathrm{Hom}(G, H)$ by conjugation: $(\gamma, \delta) \cdot \phi = \delta\phi(\gamma)^{-1}\phi\phi(\gamma)\delta^{-1}$.

6.2 Lemma *For any locally trivial (G, H)–bundle $\pi\colon P \to X$ over a G–fixed base X, the map $\tau\colon P \to \mathrm{Hom}(G, H)$ is a Serre fibration.*

Proof This will follow by showing that $(\tau, \pi)\colon P \to \mathrm{Hom}(G, H) \times X$ is actually a fiber bundle. Since π is locally trivial, we can reduce to the case when π has the form $\pi\colon (G \times H)/\Lambda_\phi \times U \to U$, where $\Lambda_\phi \leq G \times H$ is the graph of some homomorphism $\phi\colon G \to H$. Then

$$(\tau, \pi) = \rho \times \mathrm{id}_U\colon (G \times H)/\Lambda_\phi \times U \to \mathrm{Hom}(G, H) \times U,$$

where $\rho\colon (G \times H)/\Lambda_\phi \to \mathrm{Hom}(G, H)$ sends $[\gamma, \delta] \mapsto \delta\phi(\gamma)^{-1}\phi(\delta\phi(\gamma)^{-1})^{-1}$. Because $\mathrm{Hom}(G, H)$ is topologically a coproduct of orbits under H–conjugation (see Remark 1.3 and Proposition 7.1), we see that ρ is isomorphic to the composite of a projection map $(G \times H)/\Lambda_\phi \to H/C_H(\phi)$ (induced by $(\gamma, \delta) \mapsto \delta\phi(\gamma)^{-1}$) with an open and closed immersion, and thus is a fibration. □

[2] Completely regular means that points are closed, and any point and disjoint closed subset are separated by a real valued function.

6.3 Outline of the proof

To prove that the map $B_GH \to \mathrm{Map}(EG, B_GH)$ (induced by restriction along $EG \to *$) is a G–equivariant weak equivalence, it suffices to show that it induces a weak equivalence of spaces $(B_GH)^{G'} \to \mathrm{Map}(EG, B_GH)^{G'}$ for all closed subgroups $G' \le G$. Without loss of generality, we may assume $G' = G$ since, when the group action is restricted to the subgroup G', we have that B_GH is a $B_{G'}H$ and EG is an EG'. Thus, we will show that $(B_GH)^G \to \mathrm{Map}(EG, B_GH)^G$ is an equivalence, using the following.

6.4 Lemma *Suppose we are given a (G, H)–bundle $P \to X$ over a space X with trivial G–action, together with maps*

(1) α: $X \to (B_GH)^G \subseteq B_GH$ *classifying the G–equivariant H–bundle $P \to X$, ie covered by a (G, H)–bundle map $P \to E_GH$, and*

(2) ρ: $X \to \mathrm{Map}(BG, BH)$, *whose adjoint* $\tilde{\rho}$: $X \times BG = (X \times EG)/G \to BH$ *classifies the H–bundle $(P \times EG)/G \to (X \times EG)/G$, ie is covered by an H–bundle map $(P \times EG)/G \to EH$.*

Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \alpha\ } & (B_GH)^G \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ \mathrm{Map}(BG, BH) & \xrightarrow[\sim]{} & \mathrm{Map}(EG, B_GH)^G \end{array} \tag{6.5}$$

commutes up to homotopy, where the bottom map is induced by a G–equivariant map υ: $BH \to B_GH$ (with G acting trivially on BH) which classifies the universal H–bundle viewed as a G–equivariant H–bundle with trivial G–action.

Furthermore, the bottom map of the diagram is a weak equivalence.

Proof The adjoints of both composite maps $X \to \mathrm{Map}(EG, B_GH)^G$ are G–equivariant maps $(X \times EG)/G \to B_GH$, which in either case are covered by maps $(P \times EG)/G \to E_GH$ of (G, H)–bundles. The homotopy-commutativity of the diagram follows from the universal property of B_GH as the classifying space for such bundles.

To see that the bottom map of the diagram is a weak equivalence, note that it may be constructed as follows. Choose any $G \times H$ equivariant map $EH \to E_GH$ (unique up to homotopy by the defining property of E_GH), and take the quotient with respect to the free H–actions, obtaining a G–equivariant map υ: $BH \to B_GH$ which classifies the universal bundle as stated. By construction, υ is a G–equivariant map which is a weak equivalence on underlying spaces.

For any G–space X and subgroup $G' \leq G$, the fixed-point space $\mathrm{Map}(EG, X)^{G'}$ is the space of ordinary homotopy fixed points of the G'–action on X; as a consequence, $\mathrm{Map}(EG, -)$ takes a map of G–spaces which is a weak equivalence of underlying spaces, to a G–equivariant weak equivalence. In particular, we see that $\mathrm{Map}(EG, BH) \to \mathrm{Map}(EG, B_G H)$ is a G–equivariant weak equivalence. Taking G–fixed points gives an equivalence

$$\mathrm{Map}(BG, BH) \approx \mathrm{Map}(EG, BH)^G \to \mathrm{Map}(EG, B_G H)^G,$$

which is the map in the diagram. □

The strategy is as follows. Fix compact Lie groups G and H, and take $\rho\colon X \to \mathrm{Map}(BG, BH)$ in Lemma 6.4 to be isomorphic to the map $(\mathrm{Hom}(G, H) \times EH)/H \to \mathrm{Map}(BG, BH)$ described in Section 5 which, for 1–truncated H, gives the weak equivalence of Theorem 1.2. We will

(1) construct a certain (G, H)–bundle $P \to X$ (where G acts trivially on X),

(2) prove that a map $\alpha\colon X \to B_G H$ classifying $P \to X$ induces a weak equivalence $X \xrightarrow{\sim} (B_G H)^G \subseteq B_G H$, and

(3) construct a bundle map $(P \times EG)/G \to EH$ covering $\widetilde{\rho}\colon X \times BG \to BH$, the adjoint to ρ.

Thus by Lemma 6.4, both α and ρ fit in a homotopy commutative square (6.5). It follows that if H is 1–truncated, Theorem 1.2 implies that ρ is a weak equivalence, from which it follows that the right-hand vertical arrow is a weak equivalence, which is the desired result. Note: the hypothesis that H is 1–truncated is used only to show that ρ (which exists for arbitrary H) is a weak equivalence.

6.6 Step 1: Construction of $P \to X$

As in the previous section, we consider categories $\mathrm{Fun}(G, H)$ and $(H \curvearrowright H)$ (internal to Top), where G and H are compact Lie groups. Consider the topological category C defined as the fiber product

$$C := \mathrm{Fun}(G, H) \times_H (H \curvearrowright H)$$

via the evident restriction functors $\mathrm{Fun}(G,H)\to\mathrm{Fun}(\{e\},H)=H$ and $(H\curvearrowright H)\to(H\curvearrowright *)=H$. (Here H represents a topological category with one object.)

The group $G\times H$ acts on C via

$$(\gamma,\delta)\cdot(\phi_0,h_0)=(\phi_0,\phi_0(\gamma)h_0\delta^{-1})$$

on objects, and

$$(\gamma,\delta)\cdot(\phi_0\xrightarrow{h}\phi_1,h_0\xrightarrow{h}h_1)=(\phi_0\xrightarrow{h}\phi_1,\phi_0(\gamma)h_0\delta^{-1}\xrightarrow{h}\phi_1(\gamma)h_1\delta^{-1})$$

on morphisms, for $(\gamma,\delta)\in G\times H$. (This works exactly because $h\phi_0(\gamma)=\phi_1(\gamma)h$.) The evident projection functor $C\to\mathrm{Fun}(G,H)$ is invariant under the $G\times H$–action on C, with respect to the trivial action on $\mathrm{Fun}(G,H)$.

We set $P:=\|NC\|$ and $X:=\|N\mathrm{Fun}(G,H)\|$, with $P\to X$ induced by the evident projection functor. It is straightforward to show that the induced $G\times H$–action on P is compatible with the projection map to X and that H acts freely on P with $P/H\approx X$. In particular, $P\to X$ has the structure of a G–equivariant principal H–bundle.

We note an equivalent description of C, and hence of P. Let

$$C':=\mathrm{Hom}(G,H)\times N(H\curvearrowright H),$$

where $\mathrm{Hom}(G,H)$ is viewed as a topological category with only identity maps. There is an isomorphism $C'\to C$ of topological categories, given on objects and morphisms by

$$(\phi,h_0)\mapsto(h_0\phi h_0^{-1},\,h_0),\quad(\phi,h_0\xrightarrow{h}h_1)\mapsto(h_0\phi h_0^{-1}\xrightarrow{h}h_1\phi h_1^{-1},\,h_0\xrightarrow{h}h_1).$$

The $G\times H$–action on C' induced by this isomorphism is described by

$$\begin{aligned}(\gamma,\delta)\cdot(\phi,h_0)&=(\delta\phi(\gamma)^{-1}\phi\phi(\gamma)\delta^{-1},\,h_0\phi(\gamma)\delta^{-1}),\\(\gamma,\delta)\cdot(\phi,h_0\xrightarrow{h}h_1)&=(\delta\phi(\gamma)^{-1}\phi\phi(\gamma)\delta^{-1},\,h_0\phi(\gamma)\delta^{-1}\xrightarrow{h}h_1\phi(\gamma)\delta^{-1}).\end{aligned}$$

In particular, the projection functor $C'\to\mathrm{Hom}(G,H)$ induces a $G\times H$–equivariant map $P\to\mathrm{Hom}(G,H)$ (using the conjugation $G\times H$–action on $\mathrm{Hom}(G,H)$), and this map is a nonequivariant weak equivalence since $P\approx\|NC'\|\approx\mathrm{Hom}(G,H)\times EH$.

6.7 Step 2: The weak equivalence $\alpha\colon X\to(B_GH)^G$

Choose any $X\to B_GH$ classifying the $P\to X$ constructed above (this exists because X is paracompact and completely regular, so numerable), and so covered by a $G\times H$–equivariant map $P\to E_GH$. Since the action of G on X is trivial, these factor through $\alpha\colon X \to (B_G H)^G$ and $\alpha'\colon P \to p^{-1}((B_G H)^G)$, where $p\colon E_G H \to B_G H$ is the universal bundle.

6.8 Lemma *The map $\alpha'\colon P \to p^{-1}((B_G H)^G)$ is a weak equivalence of underlying spaces.*

Proof The map α' fits in the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha'} & p^{-1}((B_G H)^G) \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle\tau} & \\ & \mathrm{Hom}(G,H) & \end{array}$$

where by Lemma 6.2, both maps marked τ are Serre fibrations. The fibers of these τ over $\phi \in \mathrm{Hom}(G,H)$ are EH and $(E_G H)^{\Lambda_\phi}$, respectively, both of which are contractible spaces. Thus α' is a weak equivalence (as are both τ). □

It follows that $\alpha\colon X \to (B_G H)^G$ is a weak equivalence as it is obtained by the quotient of α' by free H–actions.

6.9 Step 3: The bundle map covering ρ: $X \to \mathrm{Map}(BG, BH)$

We have a commutative square of functors

$$\begin{array}{ccc} C \times (G \curvearrowright G) = \mathrm{Fun}(G,H) \times_H (H \curvearrowright H) \times (G \curvearrowright G) & \longrightarrow & (H \curvearrowright H) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(G,H) \times (G \curvearrowright G) & \longrightarrow & H \end{array}$$

where the vertical arrows are the evident projections, the top horizontal arrow is given by

$$(\phi_0, h_0, g_0) \mapsto (\phi_0(g_0)h_0),$$
$$(\phi_0 \xrightarrow{h} \phi_1, h_0 \xrightarrow{h} h_1, g_0 \xrightarrow{g} g_1) \mapsto (\phi_0(g_0)h_0 \xrightarrow{h\phi_0(g)=\phi_1(g)h} \phi_1(g_1)h_1).$$

on objects and morphisms, and the bottom horizontal arrow is given by

$$(\phi_0, g_0) \mapsto *,$$
$$(\phi_0 \xrightarrow{h} \phi_1, g_0 \xrightarrow{g} g_1) \mapsto (* \xrightarrow{h\phi_0(g)=\phi_1(g)h} *).$$

The group G acts on the objects on the left-hand side of the square, where G acts on C as described above, by the tautological right action on $(G \curvearrowright G)$, and trivially on $\mathrm{Fun}(G,H)$. The horizontal arrows are invariant under this G–action.

Thus, taking geometric realizations of nerves and passing to quotients by G–actions, we obtain a commutative square

$$\begin{array}{ccc} (P\times EG)/G & \longrightarrow & EH \\ \downarrow & & \downarrow \\ (X\times EG)/G & \longrightarrow & BH \end{array}$$

which is evidently a map of H–bundles. Under the identification $(X\times EG)/G\approx X\times BG\approx N(\mathrm{Fun}(G,H)\times G)$, we see that the bottom arrow is isomorphic to that obtained from the evaluation functor $\mathrm{Fun}(G,H)\times G\to H$, and thus is adjoint to the map $\rho\colon X\to \mathrm{Map}(BG,BH)$ described earlier.

7 The space of homomorphisms between compact Lie groups

Recall that $\mathrm{Hom}(G,H)$ denotes the space of homomorphisms equipped with the compact-open topology. We give a proof of the following fact, which is standard but not easily read from the literature with which the author is familiar.

7.1 Proposition *Let G and H be Lie groups, with G compact. The map*

$$(\phi,hC(\phi))\mapsto h\phi h^{-1}\colon \coprod_{[\phi]} H/C(\phi)\to \mathrm{Hom}(G,H),$$

where $C(\phi)=\{h\in H\mid \phi(g)h=h\phi(g)\text{ for all }g\in G\}$ and $[\phi]$ runs over a set of H–conjugacy classes in $\mathrm{Hom}(G,H)$, is a homeomorphism. In particular, $\mathrm{Hom}(G,H)$ is locally compact, and thus a CGWH space.

Proof We quote a classical theorem of Montgomery and Zippin [11, page 216]: for every compact subgroup K of a Lie group L, there exists a neighborhood U of K such that every closed subgroup of L in U is L–conjugate to a subgroup of K. Applied to $L=G\times H$ and $K=\Lambda_\phi=\{(g,\phi(g))\mid g\in G\}$, the graph of a continuous homomorphism $\phi\colon G\to H$, we obtain a neighborhood $U\subseteq G\times H$ of Λ_ϕ such that if $\Lambda_{\phi'}\in U$ for $\phi'\in\mathrm{Hom}(G,H)$, then ϕ' is H–conjugate to ϕ; see [2, Lemma 38.1].

There exists a neighborhood V of $e\in H$ such that $\Lambda_\phi\subseteq\{(g,h)\mid h\phi(g)^{-1}\in V\}\subseteq U$. To see this, use the homeomorphism $\alpha\colon G\times H\to G\times H$, $\alpha(g,h)=(g,h\phi(g)^{-1})$, together with the tube lemma applied to $G\times\{e\}\subseteq\alpha(U)$.

By definition, the set $V' := \{f\colon G \to H \mid f(G) \subseteq V\}$ is an open subset of $C(G, H)$, the space of continuous maps $G \to H$ equipped with the compact-open topology. The space $C(G, H)$ is a topological group under pointwise multiplication in H; to prove this, use the fact that G, H, and finite products thereof are locally compact, so the relevant evaluation maps are continuous. Therefore, the translated subset $V'\phi$ is open in $C(G, H)$. Tracing through the definitions, we see that any continuous homomorphism $G \to H$ in $V'\phi$ must be conjugate to ϕ.

Thus, we have shown that conjugacy classes are open subsets of $\mathrm{Hom}(G, H)$.

Now consider the map of the proposition. Each $H/C(\phi)$ maps bijectively to a conjugacy class in $\mathrm{Hom}(G, H)$. As H is Hausdorff, so is $C(G, H)$ and hence so is the subspace $\mathrm{Hom}(G, H)$. Therefore, each $H/C(\phi) \to \mathrm{Hom}(G, H)$ gives a homeomorphism to its image, since $H/C(\phi)$ is compact. Because the image is also open, the homeomorphism of the proposition follows.

As an immediate consequence, we see that $\mathrm{Hom}(G, H)$ is a coproduct of compact Hausdorff spaces, and thus locally compact. □

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