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On the third homotopy group of Orr's space

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K Orr defined a Milnor-type invariant of links that lies in the third homotopy group of a certain space K_ω. The problem of nontriviality of this third homotopy group has been open. We show that it is an infinitely generated group. The question of realization of its elements as links remains open.

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1 Introduction

In [12] John Milnor defines his $\overline{\mu}$–invariants of links, which extract numerical values from the lower central series quotients of the link group and the homotopy classes of the link longitudes. His questions motivated generations of research concerning these invariants. While most of Milnor's original questions have been answered (see, for instance, K Orr [13; 14], Igusa and Orr [9], Cochran [5], Habegger and Lin [8]), one key question remains open. Milnor asked if we can extract similar invariants from π/π_ω. The first candidate for a transfinite Milnor invariant was given by Orr in [13], where he suggested a possible domain for these invariants, a space he denoted K_ω. A transfinite Milnor invariant is then an element in the third homotopy group, π_3, of this space. Extensions and refinements were introduced by J P Levine in [11], as mentioned toward the end of this introduction.

The space defined by Orr, denoted by K_ω, is the mapping cone of the natural map

$$K(F,1) \to K(\widehat{F},1).$$

Here $\widehat{F}$ is the pronilpotent completion of F, ie $\widehat{F} = \lim F/\gamma_i(F)$, where $\{\gamma_i(F)\}_{i\geq 1}$ is the lower central series of F. Let $L \subset S^3$ be a link, $G = \pi_1(S^3 \setminus L)$ its group and $f\colon F \to G$ a meridian homomorphism, where F is the free group with rank equal to the number of components of L. Assume that all Milnor's $\overline{\mu}$–invariants of L are zero. This means in fact that the homomorphism f induces isomorphisms of all finite lower central quotients and therefore an isomorphism $\widehat{F} \simeq \widehat{G}$. Thus we have a

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map $S^3 \setminus L \to K(\widehat{F}, 1)$, which extends to a map $S^3 \to K_\omega$. The homotopy class of this map in $\pi_3(K_\omega)$ defines an invariant θ_ω; see Orr [14] for details and discussion. The main aim of the present note is to show that the third homotopy group $\pi_3(K_\omega)$ is nontrivial, in fact that it is infinitely generated.

The space K_ω is simply connected. This follows from the fact that, for a finitely generated group G, the pronilpotent completion $\widehat{G}$ is normally generated by the images of G under the natural injection $G \to \widehat{G}$; see Section 2.2 and observe that this is not true for a free group of infinite rank; see Bousfield [2]. For the case $G = F$, the free group with at least two generators, it is shown in [2] that $H_2(\widehat{F})$ is uncountable; hence the same is true for $\pi_2(K_\omega) \simeq H_2(K_\omega) \simeq H_2(\widehat{F})$. Cochran and Orr conjectured that the relevant third homotopy group is nonvanishing: Cochran [4] wrote: "Bousfield has shown that $\pi_2(K_\omega)$ is infinitely generated and it appears likely that the same will hold for all the homotopy groups of K_ω."

In light of Orr's Milnor-type invariant in $\pi_3(K_\omega)$ the question about nontriviality of higher homotopy groups of K_ω is of interest, and is open. The following is the main result of the present note.

Theorem 1.1 *The third homotopy group $\pi_3(K_\omega)$ is infinitely generated.*

Since we cannot see why $H_3(K_\omega)$ is not zero, we concentrate attention on the kernel of the Hurewicz map in dimension three in order to carry out the argument. In fact, we prove that the kernel of the (surjective) Hurewicz homomorphism $\pi_3(K_\omega) \to H_3(K_\omega)$ is infinitely generated.

Observe that the nontriviality of $\pi_3(K_\omega)$ does not imply the existence of links with nonzero invariants θ_ω. In order to solve that realization problem, another space K_∞ was defined by Levine [11]. The definition of K_∞ is similar to K_ω, the difference being that the algebraic closure is used instead of the pronilpotent completion. Any element from $\pi_3(K_\infty)$ can be realized as an invariant for a certain link; however, the problem of whether the Levine group $\pi_3(K_\infty)$ is nonzero is still very much open. The claim, without proof, in Orr's thesis [13] that all elements in his group $\pi_3(K_\omega)$ are realizable as links is still unsubstantiated. In fact, there is a well-defined map relating Levine's group and Orr's: $l\colon \pi_3(K_\infty) \to \pi_3(K_\omega)$. It is not hard to see from Levine's arguments that the elements in the image of this map are exactly the realizable elements. We do not know whether this map l is surjective. In light of our results below it is probably not surjective.

2 Preliminaries

2.1 Whitehead exact sequence

In order to get a hold on $\pi_3(K_\omega)$ we use an approach due to J H C Whitehead, who defined a certain exact sequence relating the homology and homotopy groups of a space which has a transparent form for 1–connected spaces such as K_ω. Recall the definition of Whitehead's *universal quadratic functor*, Γ^2, introduced in [15, Chapter II]. For an abelian group A, the group $\Gamma^2(A)$ is generated by the symbols $\gamma(x)$, one for each $x \in A$, subject to the defining relations

(1) $\gamma(-x) = \gamma(x)$;

(2) $\gamma(x+y+z) - \gamma(x+y) - \gamma(y+z) - \gamma(x+z) + \gamma(x) + \gamma(y) + \gamma(z) = 0$.

It follows from [7] that this group is naturally isomorphic to the fourth homology group

$$\Gamma^2(A) \cong H_4\, K(A,2).$$

It follows directly form the generator-relation definition that a surjection of abelian groups induces a surjection on the Γ^2 construction. This is used repeatedly in what follows, for example, in the proof in Section 3 of the main theorem.

Let us now recall a natural exact sequence associated to a connected pointed space X, due to Whitehead. Let X be a pointed connected CW-complex with skeletal filtration

$$\mathrm{sk}_1(X) \subset \mathrm{sk}_2(X) \subset \cdots.$$

In [15], Whitehead constructs the long exact sequence

$$\cdots \to \pi_4(X) \xrightarrow{\mathrm{Hur}_4} H_4(X) \to \Gamma_3(X) \to \pi_3(X) \xrightarrow{\mathrm{Hur}_3} H_3(X) \to 0, \tag{2-1}$$

where

$$\Gamma_3(X) := \mathrm{im}\big(\pi_3(\mathrm{sk}_2(X)) \to \pi_3(\mathrm{sk}_3(X))\big)$$

and Hur_i is the i^{th} Hurewicz homomorphism.

There is a natural map $\Gamma^2(\pi_2(X)) \to \Gamma_3(X)$, constructed as follows. Let $\eta\colon S^3 \to S^2$ be the Hopf map and let $x \in \pi_2(X)$ be expressed as the composition

$$S^2 \to \mathrm{sk}_2(X) \hookrightarrow \mathrm{sk}_3(X).$$

Then the composition

$$S^3 \xrightarrow{\eta} S^2 \to \mathrm{sk}_2(X) \hookrightarrow \mathrm{sk}_3(X)$$

defines an element $\eta^*(x) \in \Gamma_3(X)$. According to [15, Section 13], for a simply connected space X, the natural homomorphism

$$\eta_1\colon \Gamma^2(\pi_2(X)) \to \Gamma_3(X), \quad \gamma(x) \mapsto \eta^*(x) \tag{2-2}$$

is an isomorphism of groups.

In more direct terms, for a simply connected space X, the portion

$$\pi_4(X) \to H_4(X) \to H_4K(\pi_2(X),2) \to \pi_3(X) \xrightarrow{\mathrm{Hur}_3} H_3(X) \to 0 \tag{2-3}$$

of Whitehead's exact sequence is a portion of the Leray–Serre spectral sequence of the integral homology of the (2–connected cover) fiber sequence $X[2] \to X \to K(\pi_2(X),2)$. In particular, we have an isomorphism $\Gamma^2(\pi_2(X)) \cong \Gamma_3(X) \cong H_4K(\pi_2(X),2)$ for any 1–connected space X.

It follows directly from Whitehead's exact sequence that in order to estimate $\pi_3(X)$ for a 1–connected space, we might consider the cokernel of the natural map of Whitehead

$$w_X \equiv w\colon H_4(X) \to \Gamma^2(\pi_2(X)) \cong H_4K(H_2(X),2).$$

The evaluation of this cokernel for $X = K_\omega$ is thus our main concern from now on.

2.2 Completions

The proof of the following lemma follows the argument of Bousfield given in detail in [1]. We include it here for completeness of exposition, in a more general form.

Lemma 2.1 *Let H be a group and $\{x_1,\dots,x_n\}$ a finite set of elements of H. Let*

$$H \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots$$

be a central series of H such that $[H,N_i] = N_{i+1}$ for $i = 1,2,\dots$. Suppose that, for every $i \geq 1$, the quotient H/N_i is normally generated by $X \cdot N_i = \{x_1N_i,\dots,x_nN_i\}$. Consider the inverse limit

$$\Gamma := \lim H/N_i$$

and the associated natural map $h\colon H \to \Gamma$. The group Γ is normally generated by the elements $h(X) = \{h(x_1),\dots,h(x_n)\}$.

Proof The normal generation condition implies that, for every $u \in N_i$, where $i \geq 2$, there exist elements $u_1,\dots,u_n \in N_{i-1}$ such that

$$u \equiv [u_1,x_1]\cdots[u_n,x_1] \mod N_{i+1}.$$

Let $a = (a_1 N_1, a_2 N_2, \dots) \in \Gamma$, where $a_{i+1} \equiv a_i \mod N_i,\ a_i \in H,\ i \geq 1$. Suppose that $a_1 \in N_1$. We claim that there exist elements $g_1, \dots, g_n \in \Gamma$ such that

$$a = [g_1, h(x_1)] \cdots [g_n, h(x_n)]. \tag{2-4}$$

We construct n elements $(g_i)_{1 \leq i \leq n}$ in the inverse limit as a sequence of elements in the corresponding quotient groups:

$$g_i = (y_1^i N_1, y_2^i N_2, \dots) \in \Gamma \quad \text{for } 1 \leq i \leq n,$$

with the elements y_j^i constructed by induction as follows. Since $a_1 \in N_1$, the element a_2 also lies in N_1. There exist elements $u_1, \dots, u_n \in N_1$ such that

$$a_2 \equiv [u_1, x_1] \cdots [u_n, x_n] \mod N_3.$$

We set $y_1^i = y_2^i := u_i$. Then

$$a_2 \equiv [y_2^1, x_1] \cdots [y_2^n, x_n] \mod N_3.$$

Suppose we have constructed elements $y_1^i, \dots, y_k^i$ for $1 \leq i \leq n$ such that

$$\begin{aligned} y_{j+1}^i &\equiv y_j^i \mod N_j && \text{for } 1 \leq j \leq k-1, \\ a_j &\equiv [y_j^1, x_1] \cdots [y_j^n, x_n] \mod N_{j+1} && \text{for } 1 \leq j \leq k, \end{aligned}$$

and

$$a_{k+1} \equiv [y_k^1, x_1] \cdots [y_k^n, x_n] \mod N_{k+1}.$$

There exists $r_{k+2} \in N_{k+1}$ such that

$$a_{k+1} = [y_k^1, x_1] \cdots [y_k^n, x_n] r_{k+2}.$$

Now we find elements $v_i \in N_k$ such that

$$r_{k+2} \equiv [v_1, x_1] \cdots [v_n, x_n] \mod N_{k+2}.$$

We set $y_{k+1}^i := y_k^i v_i$ for $1 \leq i \leq n$. It follows that

$$a_{k+1} \equiv [y_{k+1}^1, x_1] \cdots [y_{k+1}^n, x_n] \mod N_{k+2}.$$

Now the result follows from the condition that H/N_1 is normally generated by elements $x_1 N_1, \dots, x_n N_n$ and the presentation (2-4). □

Remark Let G be a finitely generated group and $\{\gamma_i(G)\}_{i \geq 1}$ its lower central series. The completion map $G \to \widehat{G} := \lim G/\gamma_i(G)$ induces isomorphisms of lower central quotients $G/\gamma_i(G) \xrightarrow{\sim} \widehat{G}/\gamma_i(\widehat{G})$ for $i \geq 1$; see [1]. Hence, by Lemma 2.1, $\widehat{G}$ is normally generated by images of G.

2.3 The second homology of nilpotent completions

We need certain information on the second homology groups, to be used in the proof of the main theorem. In this paragraph we recall a useful relation between the second homology $H_2(G)$ of a finitely generated group and that of its nilpotent completion $H_2(\widehat{G})$. Let F be as above a finitely generated free group. Bousfield shows in [2] that $H_2(\widehat{F})$ is uncountable. In fact, $H_2(\widehat{F})$ maps onto the exterior square of 2–adic integers. An exposition of Bousfield's method of study, the homology of pronilpotent completions, is given in [10]. Let G be a finitely presented group, F a free group and $F \to G$ an epimorphism, and let $R = \ker\{F \to G\}$. The quotients

$$\mathcal{B}_k(G) := \frac{R \cap \gamma_k(F)}{[R, \underbrace{F, \dots, F}_{k-1}]}$$

are known as *Baer invariants* of G and are naturally isomorphic to the first (nonadditive) derived functors of the k^{th} lower central quotients $G/\gamma_k(G)$. They form a tower of groups.

We will need the following property (see [10, Lemma 6.1]):

Proposition 2.2 *Assume that*

$$\mathsf{lim}^1 \mathcal{B}_k(G) = 0.$$

Then the cokernel of the natural map $H_2(\widehat{F}) \to H_2(\widehat{G})$ is isomorphic to a quotient of the homology group $H_2(G)$, as in the diagram:

$$\begin{array}{ccccc} & & H_2(G) & & \\ & & \downarrow & \searrow & \\ H_2(\widehat{F}) & \longrightarrow & H_2(\widehat{G}) & \longrightarrow & \text{coker} \end{array}$$

The condition holds, for example, for all finitely presented groups G with finite $H_2(G)$.

3 Proof of Theorem 1.1

We first prove a statement similar to the claim of the main theorem but regarding a simpler, virtually nilpotent group G, rather than the free group F above. We will then compare K_ω to the following space cof_G and complete the proof of the main theorem.

Given any group G we consider the space cof_G defined as the cofiber of

$$K(G,1) \to K(\widehat{G},1).$$

Thus cof_F is just Orr's space K_ω.

In all our cases this cofiber space will be simply connected, since we only consider groups G for which $\widehat{G}$ is normally generated by the images of G. Notice that it is the case for any finitely generated group (see Remark).

Let G be given by

$$G = \langle a, b \mid b^a = b^{-1},\, a^2 = 1\rangle = \mathbb{Z} \rtimes C_2,$$

where $C_2 = \langle a \mid a^2 = 1\rangle$ is the two-element cyclic group acting nontrivially on the integers $\mathbb{Z}$.

We are interested in the third homotopy group of cof_G. Note that by an easy computation below, the third homology $H_3(\mathrm{cof}_G)$ is infinitely generated, and thus the third homotopy group of this space does not vanish, since the space is simply connected. But we need a more precise statement as follows.

Proposition 3.1 *Let G be the semidirect product as above. The kernel of the Hurewicz map*

$$\pi_3(\mathrm{cof}_G) \twoheadrightarrow H_3(\mathrm{cof}_G)$$

is infinitely generated.

Proof Our proof is based on the naturality of Whitehead's map $w\colon H_4(X) \to \Gamma^2(H_2(X))$ with respect to the following zigzagging sequence of maps involving rational completion, denoted by $\mathbb{Q}_\infty$:

$$\mathrm{cof}_G \to \mathrm{cof} \leftarrow \mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) \to \mathbb{Q}_\infty K(\mathbb{Q}^{\oplus n} \rtimes C_2, 1).$$

Spaces and maps including cof and cof_G in this sequence are discussed below. The functor $\mathbb{Q}_\infty$ is the $\mathbb{Q}$–completion in the sense of Bousfield and Kan [3].

Our interest of course is in the value of the cokernel of w for the space on the left. We estimate the cokernel of w for the left-hand space cof_G by reducing the calculations to an estimate the cokernel of w for the rightmost space.

We will show shortly that in the above sequence of spaces, the two maps on the left induce isomorphisms on the said cokernel of the Whitehead map w while the map on the right induces (again, on the said cokernels) a *surjection* onto an infinitely generated group (= the cokernel for the space on the right-hand side), thus completing the proof.

Recall that to prove the current proposition, using the above exact sequence of Whitehead for the case of 1–connected spaces it is enough to show that the cokernel of the natural map $H_4(\mathrm{cof}_G) \to \Gamma^2(\pi_2(\mathrm{cof}_G))$ is infinitely generated. However, since the space cof_G is simply connected we have $H_2(\mathrm{cof}_G) \cong \pi_2(\mathrm{cof}_G)$; thus we shall consider the second homology in what follows.

First notice that the nilpotent completion of G is just a semidirect product of the 2–adic integers with the cyclic group C_2, namely, $\widehat{G} \cong \mathbb{Z}_2 \rtimes C_2$. This follows from a direct computation of the nilpotent quotients as a semidirect product.

Now we consider the map on the right in the above zigzag of maps. The spaces and map are defined using the following cofiber sequence. Its usefulness follows from the fact that integral homology groups of cof_G turns out to be rational vector spaces, by direct computation as in the lemma below. This allows us to pass to rational completions and work with vector spaces over the rational numbers:

The first map on the left in the zigzag and its domain and codomain are defined in the following cofibration sequences diagram where the commutative square on the left is induced by the maps of C_2–modules $\mathbb{Z} \to \mathbb{Q}$ and $\mathbb{Z}_2 \to \mathbb{Z}_2 \otimes \mathbb{Q}$:

$$\begin{array}{ccccc} K(\mathbb{Z} \rtimes C_2, 1) & \longrightarrow & K(\mathbb{Z}_2 \rtimes C_2, 1) & \longrightarrow & \mathrm{cof}_G \\ \downarrow & & \downarrow & & \downarrow \\ K(\mathbb{Q} \rtimes C_2, 1) & \longrightarrow & K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) & \longrightarrow & \mathrm{cof} \end{array}$$

Lemma 3.2 *The cofiber spaces* cof_G *and* cof *defined above are simply connected and their homotopy and homology groups are naturally vector spaces over the field of rational numbers* $\mathbb{Q}$. *In addition, the above induced map* $\mathrm{cof}_G \to \mathrm{cof}$ *is a homotopy equivalence.*

Proof Both spaces cof_G and cof are simply connected: this is because their fundamental groups are clearly abelian and their first homology groups vanish, for example by the calculation below.

The homology groups $H_i(\mathbb{Z} \rtimes C_2)$ can be computed using the usual spectral sequence for the group extension $\mathbb{Z} \hookrightarrow \mathbb{Z} \rtimes C_2 \twoheadrightarrow C_2$. The second stage of this spectral sequence stabilizes by the existence of the splitting map $C_2 \hookrightarrow (\mathbb{Z} \rtimes C_2)$:

$$E^2_{i,j} = \begin{cases} H_i(C_2, \langle b \rangle) = \mathbb{Z}/2 & \text{if } (i,j) = (2k,1),\ k \geq 0 \\ H_i(C_2) = \mathbb{Z}/2 & \text{if } (i,j) = (2k+1,0),\ k \geq 0, \\ \mathbb{Z} & \text{if } (i,j) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we get

$$H_i(\mathbb{Z} \rtimes C_2) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } i \text{ odd,} \\ 0 & \text{for } i \text{ even} > 0. \end{cases}$$

In the same way we compute the homology groups $H_i(\mathbb{Z}_2 \rtimes C_2)$. In this case, the spectral sequence has the form

$$E^2_{i,j} = \begin{cases} \Lambda^{2k}(\mathbb{Z}_2) & \text{if } (i,j) = (0,2k),\ k \geq 1, \\ H_i(C_2, \langle b \rangle) = \mathbb{Z}/2 & \text{if } (i,j) = (2k,1),\ k \geq 0, \\ H_i(C_2) = \mathbb{Z}/2 & \text{if } (i,j) = (2k+1,0),\ k \geq 0, \\ \mathbb{Z} & \text{if } (i,j) = (0,0), \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$H_i(\mathbb{Z}_2 \rtimes C_2) = \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } i \text{ odd,} \\ \Lambda^{2k}(\mathbb{Z}_2) & \text{for } i = 2k,\ k \geq 1. \end{cases}$$

The monomorphism $\langle b \rangle = \mathbb{Z} \hookrightarrow \mathbb{Z}_2$ induces isomorphisms of homology groups

$$H_i(C_2, \langle b \rangle) \simeq H_i(C_2, \mathbb{Z}_2).$$

Hence, the homology groups of cof_G are

$$H_i(\mathrm{cof}_G) = \begin{cases} \Lambda^{2k}(\mathbb{Z}_2) & \text{for } i = 2k,\ k \geq 1, \\ 0 & \text{for } i = 2k+1. \end{cases}$$

Similarly, the homology groups of cof are

$$H_i(\mathrm{cof}) = \begin{cases} \Lambda^{2k}(\mathbb{Z}_2 \otimes \mathbb{Q}) & \text{for } i = 2k,\ k \geq 1, \\ 0 & \text{for } i = 2k+1. \end{cases}$$

Note that all exterior powers ≥ 2 of $\mathbb{Z}_2$ are $\mathbb{Q}$–vector spaces. To see this, we present the exterior power as a natural quotient of the tensor power, and after tensoring it with $\mathbb{Z}/l$, where $l \geq 2$, one obtains a finite cyclic group, whose image in the exterior power is zero. That is, the exterior powers ≥ 2 of $\mathbb{Z}_2$ are divisible. Since these groups are torsion-free, we conclude that they are $\mathbb{Q}$–vector spaces. Thus the map $\mathrm{cof}_G \to \mathrm{cof}$ is a homology equivalence. Since cof_G and cof are simply connected, $\mathrm{cof}_G \to \mathrm{cof}$ is a homotopy equivalence. □

This completes our discussion of the first map in the above zigzag of maps.

We now turn our attention to the second map. This map arises from the fact proven in Lemma 3.2. As we saw in Lemma 3.2 the homology of cof is a vector space over the field $\mathbb{Q}$.

Spaces here are virtually nilpotent, hence their $\mathbb{Q}$–completions are the same as $\mathbb{Q}$–localizations, in particular, the $\mathbb{Q}$–homology groups of their $\mathbb{Q}$–completions are the same as those of the spaces (see [6, Proposition 3.4]). Thus the map above that defines the space cof as homotopy quotient, namely,

$$K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) \to \mathrm{cof},$$

factors through the rational completion-localization of the domain. The second map is this canonical factorization.

Lemma 3.3 *The second map* $K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) \to \mathrm{cof}$ *above induces an isomorphism homology with rational coefficients.*

Proof This follows directly from the fact that the cobase in the cofibration that defines cof, namely, the space $K(\mathbb{Q} \rtimes C_2, 1)$, has trivial homology with coefficients in $\mathbb{Q}$ by a direct computation via the usual $H(-, \mathbb{Q})$ spectral sequence whose $E^2_{p,q}$ term vanishes anywhere except at $(p,q) = (0,0)$. Put otherwise, the group G is isomorphic to the free product $C_2 * C_2$ of two cyclic groups whose $\mathbb{Q}$–homology vanishes by the usual pushout Mayer–Vietoris argument. □

It follows from the lemma above that to estimate the cokernel of the Whitehead natural transformation $H_4 \to \Gamma^2 H_2$ for the 1–connected space cof_G we work entirely with vector spaces over $\mathbb{Q}$ and we need to estimate the following cokernel of Whitehead's map w for the third space in the above zigzag. Using the computation in the proof of Lemma 3.2 we need to estimate

$$\mathrm{coker}(\Lambda^4(\mathbb{Z}_2 \otimes \mathbb{Q}) \to \Gamma^2 \Lambda^2(\mathbb{Z}_2 \otimes \mathbb{Q})).$$

We show that this cokernel maps *onto* an infinitely generated group by comparing it to the cokernel of the Whitehead map of another space, the fourth in the zigzag above, as follows:

To build the map on the right of our zigzag we note that for any n, there is an epimorphism $\mathbb{Z}_2 \otimes \mathbb{Q} \to \mathbb{Q}^{\oplus n}$ which induces a map between groups

$$(\mathbb{Z}_2 \otimes \mathbb{Q}) \rtimes C_2 \to \mathbb{Q}^{\oplus n} \rtimes C_2,$$

where the action by C_2 in both semidirect products is given by negation. The group $\mathbb{Q}^{\oplus n} \rtimes C_2$ is $\mathbb{Q}$–perfect, ie its abelianization is a torsion group. We have a map between $\mathbb{Q}$–completions

$$\mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) \to \mathbb{Q}_\infty K(\mathbb{Q}^{\oplus n} \rtimes C_2, 1).$$

As noted above, both spaces $K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1)$ and $K(\mathbb{Q}^{\oplus n} \rtimes C_2, 1)$ are virtually nilpotent, hence their $\mathbb{Q}$–completions are the same as $\mathbb{Q}$–localizations, in particular, the $\mathbb{Q}$–homology groups of their $\mathbb{Q}$–completions are the same as those of the spaces (see [6, Proposition 3.4]).

Recall that the Γ^2 functor sends epimorphisms to epimorphisms. The natural maps between Whitehead sequences

$$\begin{array}{ccc} H_4\mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1) & \longrightarrow & \Gamma^2(H_2\mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1)) \\ \downarrow & & \downarrow \\ H_4\mathbb{Q}_\infty K(\mathbb{Q}^{\oplus n} \rtimes C_2, 1) & \longrightarrow & \Gamma^2(H_2\mathbb{Q}_\infty K(\mathbb{Q}^{\oplus n} \rtimes C_2, 1)) \end{array}$$

are the following:

$$\begin{array}{ccc} \Lambda^4(\mathbb{Z}_2 \otimes \mathbb{Q}) & \longrightarrow & \Gamma^2(\Lambda^2(\mathbb{Z}_2 \otimes \mathbb{Q})) \\ \twoheaddownarrow & & \twoheaddownarrow \\ \Lambda^4(\mathbb{Q}^{\oplus n}) & \longrightarrow & \Gamma^2(\Lambda^2(\mathbb{Q}^{\oplus n})) \end{array}$$

Now observe that for $n = 2, 3$, the group $\Lambda^4(\mathbb{Q}^{\oplus n})$ is zero, but the group $\Gamma^2(\Lambda^2(\mathbb{Q}^{\oplus n}))$ is infinitely generated. Therefore, the cokernel of the upper horizontal map in the last diagram must be an infinitely generated group.

We conclude that the cokernel of the map

$$H_4(\mathrm{cof}_G) \to \Gamma^2(\pi_2(\mathrm{cof}_G))$$

is an infinite divisible group as claimed. □

Proof of Theorem 1.1 We use the statement in Proposition 3.1 about cof_G for the semidirect product G as above to deduce a similar statement about the free group F. Consider a free group F of rank 2 and an epimorphism $F \to G$, and construct the map between cofibers:

$$\begin{array}{ccccc} K(F,1) & \longrightarrow & K(\widehat{F},1) & \longrightarrow & K_\omega \\ \downarrow & & \downarrow & & \downarrow \\ K(G,1) & \longrightarrow & K(\widehat{G},1) & \longrightarrow & \mathrm{cof}_G \end{array}$$

In light of $H_1(F) \cong H_1(\widehat{F})$ above we have an isomorphism $H_2(\widehat{F}) \simeq H_2(K_\omega)$. The cokernel of the natural map

$$H_2(\widehat{F}) \simeq H_2(K_\omega) \to H_2(\mathrm{cof}_G) = \Lambda^2(\mathbb{Z}_2)$$

is isomorphic, by Proposition 2.2, to a quotient of $H_2(G) = 0$ (see Section 2); hence we conclude that $H_2(\widehat{F}) \to H_2(\mathrm{cof}_G)$ is an epimorphism. Hence the middle vertical map in the natural map between Whitehead exact sequences is a surjection:

$$\begin{array}{ccccccc} H_4(\widehat{F}) & \longrightarrow & \Gamma^2(H_2(\widehat{F})) & \twoheadrightarrow & \mathrm{coker} & \xrightarrow{1\text{-}1} & \pi_3(K_\omega) \\ \downarrow & & \twoheaddownarrow & & \twoheaddownarrow ? & & \downarrow \\ H_4(\mathrm{cof}_G) & \longrightarrow & \Gamma^2(H_2(\mathrm{cof}_G)) & \twoheadrightarrow & \mathrm{coker} & \xrightarrow{1\text{-}1} & \pi_3(\mathrm{cof}_G) \end{array}$$

It is clear now that the arrow with a question mark must be a surjection: since the map $H_2(\widehat{F}) \to H_2(\mathrm{cof}_F) = H_2 K_\omega$ is an epimorphism, the same is true for $\Gamma^2(H_2(\widehat{F})) \to \Gamma^2(H_2(\mathrm{cof}_G))$, and we conclude that the map

$$H_4(\widehat{F}) \to \Gamma^2(H_2(\widehat{F}))$$

has a cokernel which maps onto an infinite divisible group. Theorem 1.1 now follows from the Whitehead exact sequence for the simply connected space K_ω:

$$H_4(\widehat{F}) \to \Gamma^2(H_2(\widehat{F})) \to \pi_3(K_\omega) \to H_3(\widehat{F}) \to 0. \qquad \square$$

4 Localization of virtually nilpotent spaces

Here we deduce from the above results that, for a virtually nilpotent group G, the $H\mathbb{Z}$–localization of $K(G,1)$ is not, in general, a $K(-,1)$. In fact we consider the simplest possible nontrivial example of such a group, namely, the semidirect product G above. Recall from [6] that, for a virtually nilpotent space X, the arithmetic square

$$\begin{array}{ccc} X_{\mathbb{Z}} & \longrightarrow & X_P \\ \downarrow & & \downarrow \\ X_{\mathbb{Q}} & \longrightarrow & X_{P,\mathbb{Q}} \end{array}$$

is, up to homotopy, a fiber square. Here $P = \bigoplus \mathbb{Z}/p$ over all primes and, for a ring R, X_R is the HR–localization.

Consider $X = K(G,1)$, where G is the group from the previous proof, ie $G = \mathbb{Z} \rtimes C_2$. Clearly, G is virtually nilpotent, and it has an infinite cyclic subgroup of index two. Thus the space X is virtually nilpotent and therefore the arithmetic square for X is a fiber square. Since the group G is $\mathbb{Q}$–acyclic, $X_{\mathbb{Q}}$ is contractible and the arithmetic square degenerates to the fiber sequence

$$X_{\mathbb{Z}} \to X_{\mathbb{Z}/2} \to X_{\mathbb{Z}/2,\mathbb{Q}}.$$

Clearly we can ignore all primes from P except for $p=2$.

The $\mathbb{Z}/2$–localization $X_{\mathbb{Z}/2}$ coincides with the $\mathbb{Z}/2$–completion and is equivalent to $K(\mathbb{Z}_2 \rtimes C_2, 1)$. The space $K(\mathbb{Z}_2 \rtimes C_2, 1)$ is virtually nilpotent; hence its $\mathbb{Q}$–localization coincides with its $\mathbb{Q}$–completion. Therefore

$$\pi_i(X_{\mathbb{Z}}) \simeq \pi_{i+1}(\mathbb{Q}_\infty K(\mathbb{Z}_2 \rtimes C_2, 1)),$$

for all $i \geq 2$. The natural map $\mathbb{Z}_2 \rtimes C_2 \to \mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2$ induces an isomorphism of $\mathbb{Q}$–homology groups; hence

$$\pi_i(X_{\mathbb{Z}}) \simeq \pi_{i+1}(\mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1)).$$

It follows from the previous section that the kernel of the third Hurewicz homomorphism for the space $\mathbb{Q}_\infty K(\mathbb{Z}_2 \otimes \mathbb{Q} \rtimes C_2, 1)$ contains an infinitely generated subgroup; hence $\pi_2(X_{\mathbb{Z}}) \neq 0$.

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