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Loop homology of some global quotient orbifolds

YASUHIKO ASAO

We determine the ring structure of the loop homology of some global quotient orbifolds. We can compute by our theorem the loop homology ring with suitable coefficients of the global quotient orbifolds of the form $[M/G]$ for M being some kinds of homogeneous manifolds, and G being a finite subgroup of a path-connected topological group $\mathcal{G}$ acting on M. It is shown that these homology rings split into the tensor product of the loop homology ring $\mathbb{H}_*(LM)$ of the manifold M and that of the classifying space of the finite group, which coincides with the center of the group ring $Z(k[G])$.

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1 Introduction

The free loop space of a topological space X is a space of the continuous maps from the circle S^1 to X,

$$LX = \mathrm{Map}(S^1, X), \tag{1}$$

with the compact-open topology. The loop homology of X is the homology of the free loop space, $H_*(LX)$. In the 1990s Moira Chas and Dennis Sullivan [3] discovered a product $\circ$ on the loop homology of a closed oriented smooth manifold $H_*(LM)$,

$$\circ\colon H_p(LM) \otimes H_q(LM) \to H_{p+q-\dim M}(LM), \tag{2}$$

called the loop product, which is a mixture of the intersection product of a manifold and the concatenation operation identifying $S^1 \vee S^1$ with the image of a map from S^1. They also showed in [3] that the product defines a ring structure and a kind of Lie algebra structure called the Batalin–Vilkovisky algebra (BV–algebra) structure on the homology. These rich algebraic structures are called *string topology* of closed oriented manifolds. The string topology of a manifold is related to several areas of mathematics including mathematical physics through the tools of algebraic topology.

The string topology of a wider class of spaces has been developed by several authors:

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- For classifying spaces of connected compact Lie groups, the existence of the loop product and the BV–structure on its homology is proved by Chataur and Menichi in [4].
- For fiberwise monoids including the adjoint bundle of principal bundles, the loop product is constructed by Gruher and Salvatore in [8].
- And more generally, for the Borel construction of a smooth manifold with a smooth action of a compact Lie group, the loop product is constructed by Kaji and Tene in [11].
- In the 2000s, Lupercio, Uribe, and Xicoténcatl defined the loop homology of a global quotient orbifold, which is an orbifold of the form $[M/G]$ for M being a smooth manifold and G being a finite group acting smoothly on M, as the loop homology of the Borel construction $M \times_G EG$. They discovered a product

$$\circ\colon H_p(L(M \times_G EG)) \otimes H_q(L(M \times_G EG)) \to H_{p+q-\dim M}(L(M \times_G EG)) \tag{3}$$

  on the loop homology of a global quotient orbifold $H_*(L(M \times_G EG))$, and showed that the product defines a BV–algebra structure on $H_*(L(M \times_G EG))$; see [14]. They coined the name *orbifold string topology* for this structure.

In spite of these interesting structural discoveries, concrete computations of loop homology have been achieved for only a few kinds of classes of manifolds. In order to accommodate the change in grading, we define $\mathbb{H}_*(LM) = H_{*+\dim M}(LM)$.

- For a compact Lie group Γ, there is a homeomorphism $L\Gamma \cong \Omega\Gamma \times \Gamma$, and a loop homology ring isomorphism $\mathbb{H}_*(L\Gamma) \cong H_*(\Omega\Gamma) \otimes \mathbb{H}_*(\Gamma)$. Furthermore, the BV–algebra structure with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_2$ of any compact Lie group G is determined by Hepworth in [10].
- For spheres S^n and complex projective spaces $\mathbb{C}\mathrm{P}^n$, the ring structure with coefficients in $\mathbb{Z}$ is determined by Cohen, Jones, and Yan in [6] by constructing the spectral sequence converging to the loop homology.
- The BV–algebra structure for $\mathbb{C}\mathrm{P}^n$ with coefficients in $\mathbb{Z}$ is determined by Hepworth in [9].
- For complex Stiefel manifolds $\mathrm{SU}(n)/\mathrm{SU}(k)$ including odd-dimensional spheres S^{2n+1}, the BV–structure is determined by Tamanoi in [18].
- For arbitrary spheres, Menichi determines the BV–structure for it in [15] by using the Hochschild cohomology.

- For the aspherical manifold $K(\pi, 1)$, the BV–structure is determined by Vaintrob in [20] by establishing an isomorphism between the loop homology $\mathbb{H}_*(K(\pi, 1))$ and the Hochschild cohomology $\mathrm{HH}^*(\mathbb{Z}[\pi]; \mathbb{Z}[\pi])$, and another proof is obtained by Kupers in [12].

The purpose of this paper is to determine the ring structure of some global quotient orbifolds by using the method of the orbifold string topology. We can compute by our theorem the loop homology ring with suitable coefficients of the global quotient orbifolds of the form $[M/G]$ for M a homogeneous manifold of a connected Lie group $\mathcal{G}$, and G a finite subgroup of $\mathcal{G}$.

Now we briefly review a part of the work of Lupercio, Uribe and Xicoténcatl in [14]. For simplicity, we denote $\mathrm{Map}(S^1, M \times_G EG)$ as $L[M/G]$. In [14], the *loop orbifold* of the global quotient orbifold $[M/G]$ is defined as the groupoid $P_G M \rtimes G$, and they show that its Borel construction $P_G M \times_G EG$ is weak homotopy equivalent to the free loop space $L(M \times_G EG)$. To determine the loop homology ring of the lens space $S^{2n+1}/\mathbb{Z}_p$, they constructed a *non-G–equivariant* homotopy equivalence

$$P_{\mathbb{Z}_p} S^{2n+1} \simeq \coprod_{g \in \mathbb{Z}_p} L S^{2n+1} \tag{4}$$

by using the fact that the action of $\mathbb{Z}_p$ extends to an S^1 action. We prove:

Proposition 3.1 [14] *Let M be a closed oriented manifold, and G be a finite subgroup of a path-connected group $\mathcal{G}$ acting continuously on M. Then for each $g \in G$, there exists a homotopy equivalence*

$$P_g X \simeq LM. \tag{5}$$

We prove that this homotopy equivalence can be extended to a wider class of spaces. Furthermore, we find a condition so that the above homotopy equivalence is *G–equivariant at homology level.*

Now we state our main theorem. Let $\mathcal{G}$ be a path-connected group acting continuously on M, and G a finite subgroup of $\mathcal{G}$ acting smoothly on M. Then there is the map $\Phi\colon \Omega\mathcal{G} \times LM \to LM$, with $\Phi(a, l) = (t \mapsto a(t) \cdot l(t))$, and this map induces an action of the Pontrjagin ring $H_*(\Omega\mathcal{G})$, namely $\Phi_*\colon H_*(\Omega\mathcal{G}) \otimes H_*(LM) \to H_*(LM)$. If the action Φ_* of $\pi_0(\Omega\mathcal{G})$ on $H_*(LM; k)$ satisfies the equation $\Phi_*(x) = x$ for any $x \in H_*(LM)$, we call the action Φ_* *trivial.* Then we prove the following.

Proposition 3.2 *Assume that the action Φ_* is trivial with coefficient in k. Then the direct sum of the homotopy equivalence of the above proposition,*

$$\coprod_{g\in G} P_g M \simeq \coprod_{g\in G} LM,$$

is G–equivariant at homology level with coefficients in k.

By using this homotopy equivalence, we can compute the loop homology of a certain class of orbifolds. Our main theorem is the following; we denote the order of a group G by $|G|$.

Theorem 4.1 *Let $\mathcal{G}$ be a path-connected topological group acting continuously on an oriented closed manifold M, let G be its finite subgroup, and k be a field whose characteristic is coprime to $|G|$. If the action Φ_* is trivial with coefficients in k, then there exists an isomorphism as k–algebras*

$$\mathbb{H}_*(L[M/G];k) \cong \mathbb{H}_*(LM;k) \otimes Z(k[G]), \tag{6}$$

where $Z(k[G])$ denotes the center of the group ring $k[G]$.

We show that the following are the necessary conditions for the action Φ_* to be trivial.

Proposition 6.1 *If $\mathcal{G}$ is simply connected, then the action Φ_* is trivial for any field k.*

Proposition 6.2 *If $H_{\dim M}(LM;k) = k$, then the action Φ_* is trivial for any pair $(\mathcal{G}, G)$.*

Proposition 6.4 *If the following conditions are satisfied, then the action Φ_* is trivial:*

- (i) *M is simply connected.*
- (ii) $|\pi_1\mathcal{G}| < \infty$.
- (iii) *The homomorphism $H_*(\Omega M;k) \to H_*(LM;k)$ induced by the inclusion map $\Omega M \to LM$ is injective, that is, the free loop fibration $\Omega M \to LM \to M$ is* ***totally noncohomologous to zero*** *(TNCZ) with coefficients in the field k.*
- (iv) *The characteristic of k is coprime to $|\pi_1\mathcal{G}|$.*

The organization of this paper is as follows. After this introduction in Section 1, we briefly review the string topology first developed in [3] and the orbifold string topology constructed in [14] in Section 2. In Section 3, we show some propositions necessary for the proof of the main theorem. In Section 4, we prove the theorem first for vector spaces and second for algebras. In Section 5, we remark on the relation to Hochschild cohomology. Finally in Section 6, we compute concrete examples by applying our theorem.

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2 Preliminaries for string topology

In this section, we briefly review the loop product in string topology.

Loop product

Let M be a smooth closed oriented manifold, and $LM = \mathrm{Map}(S^1, M)$ be the free loop space of M, the space of piecewise smooth maps from S^1 to M with compact-open topology. Then we have the following pullback diagram:

(7)
$$\begin{tikzcd}
\mathrm{Map}(S^1 \vee S^1, M) \arrow[r, equal] & LM \times_M LM \arrow[r, "\tilde{\iota}"] \arrow[d, "\mathrm{ev}_{1/2}"] & LM \times LM \arrow[d, "\mathrm{ev} \times \mathrm{ev}"] \\
 & M \arrow[r, "\iota"] & M \times M
\end{tikzcd}$$

where ι denotes the diagonal embedding, and ev_t denotes the evaluation map with $\mathrm{ev}_t(l) = l(t)$. Then we can consider above $\tilde{\iota}$ as codimension n embedding of the infinite-dimensional manifold, and we have the generalized Pontrjagin–Thom map

(8)
$$\tilde{\iota}_{!*}\colon H_*(LM \times LM) \to H_*((LM \times_M LM)^{\mathrm{ev}^*_{1/2}\nu_\iota})$$

due to Cohen and Klein [7], where ν_ι denotes the normal bundle of the embedding ι, and $(LM \times_M LM)^{\mathrm{ev}^*_{1/2}\nu_\iota}$ denotes the Thom space of the vector bundle $\mathrm{ev}^*_{1/2}\nu_\iota$. The similar but more homotopy-theoretic construction of this umkehr map is considered in [11]. The loop product is formulated in [5] as the composition of maps $\tilde{\iota}_{!*}$, the Thom isomorphism

(9)
$$H_*((LM \times_M LM)^{\mathrm{ev}^*_{1/2}\nu_\iota}) \xrightarrow{\cong} H_{*\,\dim M}(LM \times_M LM),$$

and the concatenating map $\gamma\colon LM \times_M LM \to LM$ with

(10)
$$\gamma(l_1, l_2) = l_1 * l_2 := \begin{cases} l_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ l_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Definition 2.1 [5] The following sequence of compositions defines the loop product:

$$H_p(LM) \otimes H_q(LM) \xrightarrow{\times} H_{p+q}(LM \times LM) \xrightarrow{e_{!*}} H_{p+q}((LM \times_M LM)^{\mathrm{ev}^*_{1/2}\nu_\iota})$$
$$\xrightarrow{\text{Thom isom.}} H_{p+q-n}(LM \times_M LM) \xrightarrow{\gamma_*} H_{p+q-n}(LM).$$

In order to accommodate the change in grading, we define $\mathbb{H}_*(LM) = H_{*+\dim M}(LM)$. Chas and Sullivan prove the following in [3].

Theorem 2.2 *The loop product makes $\mathbb{H}_*(LM)$ an associative graded commutative algebra.*

Remark Cohen and Jones give in [5] an operadic proof of Theorem 2.2, and Tamanoi gives in [19] a more homotopy-theoretic one.

Orbifolds

In this section, we review the basic definitions and properties of orbifolds that we will use in this paper. Following Moerdijk [17], we use the groupoid notion of an orbifold. For more detail, see [1; 17].

Definition 2.3 A *groupoid* is a category whose morphisms are all invertible. In other words, a groupoid G is a pair of sets $(\mathsf{G}_0, \mathsf{G}_1)$ with the structure maps

- **source and target** $s, t\colon \mathsf{G}_1 \to \mathsf{G}_0$,
- **identity** $e\colon \mathsf{G}_0 \to \mathsf{G}_1$,
- **inverse** $i\colon \mathsf{G}_1 \to \mathsf{G}_1$,
- **composition** $m\colon \mathsf{G}_1 \times_{\mathsf{G}_0} \mathsf{G}_1 = \{(a,b) \in \mathsf{G}_1 \times \mathsf{G}_1 | t(a) = s(b)\} \to \mathsf{G}_1$,

satisfying suitable compatibility conditions.

We note here some technical terms on groupoids. A *Lie groupoid* is a groupoid $(\mathsf{G}_0, \mathsf{G}_1)$ such that G_0 and G_1 both have the structure of a smooth manifold, and the structure maps are all smooth. We will also require that s, t are submersions. An *isotropy group* of a groupoid $(\mathsf{G}_0, \mathsf{G}_1)$ at $x \in \mathsf{G}_0$ is the group $G_x = \{f \in \mathsf{G}_1 \mid s(f) = t(f) = x\}$. A Lie groupoid $(\mathsf{G}_0, \mathsf{G}_1)$ is said to be a *proper groupoid* if the map $(s,t)\colon \mathsf{G}_1 \to \mathsf{G}_0 \times \mathsf{G}_0$ is proper. A Lie groupoid $(\mathsf{G}_0, \mathsf{G}_1)$ is said to be a *foliation groupoid* if the isotropy group G_x is discrete for each $x \in \mathsf{G}_0$.

Definition 2.4 A groupoid is said to be an *orbifold groupoid* if it is a proper foliation Lie groupoid.

Example 2.5 Let S be a set, and $\mathcal{G}$ be a group acting on S. Then $(S, S \times \mathcal{G})$ is a groupoid with structure maps

- $s(x,g) = x,\ t(x,g) = gx$,
- $e(x) = (x, \mathrm{id}_{\mathcal{G}})$,

- $i(x,g) = (gx, g^{-1})$,
- $m((x,g),(gx,h)) = (x,hg)$,

for any $x, y \in S$ and $g, h \in \mathcal{G}$. We call this groupoid the *action groupoid*, and denote it by $S \rtimes \mathcal{G}$.

Example 2.6 Let M be a smooth manifold, and Γ be a compact Lie group acting smoothly on M. If the isotropy group Γ_x is finite for each $x \in M$, then the action groupoid $M \rtimes \Gamma$ is an orbifold groupoid.

Orbifolds are defined using the notion of groupoids, as follows.

Definition 2.7 A Morita equivalent class [G] of orbifold groupoids is called an *orbifold*.

Definition 2.8 An orbifold X is called a *global quotient orbifold* if X has a representation of an action groupoid $M \rtimes G$, where M is a smooth manifold, and G is a finite group. We write $\mathsf{X} = [M/G]$.

Remark The following are fundamental properties of orbifolds; see for example [1] for proof.

(i) Any orbifold groupoid is Morita equivalent to an action groupoid $M \rtimes \Gamma$, where M is a smooth manifold, and Γ is a compact Lie group acting smoothly on M with its isotropy groups being finite.

(ii) Let $\mathsf{X} = [M \rtimes \Gamma]$ be an orbifold. Then the homotopy type of the Borel construction $M \times_\Gamma E\Gamma$ is an orbifold invariant; that is, it is invariant under Morita equivalences.

Orbifold loop product

In this section, we review the construction of orbifold loop product defined in [14].

The following notion of loop orbifold is defined also in [14].

Definition 2.9 [14] Let $[M/G]$ be a global quotient orbifold. The *loop orbifold* of $[M/G]$ is the action groupoid $P_G M \rtimes G$, where

$$P_G M = \coprod_{g \in G} P_g M, \quad P_g M = \{\sigma\colon [0,1] \to M \mid \sigma(1) = \sigma(0)g\} \subset LM,$$

and G acts on $P_G M$ via the map

$$P_g M \times G \to P_{h^{-1}gh} M, \quad (\sigma, h) \mapsto \sigma h.$$

In the same paper, they prove the weak homotopy equivalence $L(X \times_G EG) \simeq P_G X \times_G EG$. Hence we can consider $P_G M \times_G EG$ instead of $L(M \times_G EG)$ for studying the string topology of the orbifold $[M/G]$ because of the Whitehead theorem. They construct the orbifold loop product as follows.

Construction of the orbifold loop product We consider the pullback diagram

$$\begin{tikzcd} P_g M \times_M P_h M \arrow[r, "\tilde{\iota}"] \arrow[d, "\mathrm{ev}_{1/2}"] & P_g M \times P_h M \arrow[d, "\mathrm{ev}_1 \times \mathrm{ev}_0"] \\ M \arrow[r, "\iota"] & M \times M \end{tikzcd} \tag{11}$$

for any $g, h \in G$, where ev_t denotes the evaluation map. Then we have the generalized Pontryagin–Thom map

$$\tilde{\iota}_!\colon P_g M \times P_h M \to (P_g M \times_M P_h M)^{\mathrm{ev}_{1/2}^* \nu_\iota},$$

where ν_ι denotes the normal bundle of the embedding ι, and $(P_g M \times_M P_h M)^{\mathrm{ev}_{1/2}^* \nu_\iota}$ denotes the Thom space of the vector bundle $\mathrm{ev}_{1/2}^* \nu_\iota$. We also have the concatenation map $\gamma\colon P_g M \times_M P_h M \to P_{gh} M$ with

$$\gamma(\sigma_g, \sigma_h) = \begin{cases} \sigma_g(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \sigma_h(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases} \tag{12}$$

Then we obtain a sequence of compositions

$$H_p(P_g M) \otimes H_q(P_h M) \xrightarrow{\times} H_{p+q}(P_g M \times P_h M) \xrightarrow{\tilde{\iota}_{!*}} H_{p+q}((P_g M \times_M P_h M)^{\mathrm{ev}_{1/2}^* \nu_\iota})$$
$$\xrightarrow{\text{Thom isom.}} H_{p+q-\dim M}(P_g M \times_M P_h M) \xrightarrow{\gamma_*} H_{p+q-\dim M}(P_{gh} M),$$

and we obtain a map

$$\bullet\colon H_p(P_g M) \otimes H_q(P_h M) \to H_{p+q-\dim M}(P_{gh} M). \tag{13}$$

By summing over $g, h \in G$, we obtain a map, which we also denote by $\bullet$,

$$\bullet\colon H_p(P_G M) \otimes H_q(P_G M) \to H_{p+q-\dim M}(P_G M). \tag{14}$$

To lift the map $\bullet$ to a map

$$\circ\colon H_p(P_G M \times_G EG) \otimes H_q(P_G M \times_G EG) \to H_{p+q-\dim M}(P_G M \times_G EG), \tag{15}$$

we use the following fundamental lemma in algebraic topology.

Lemma 2.10 *Let $p\colon \widetilde{X} \to X$ be a finite Galois covering and G its Galois group. If k is a field whose characteristic is coprime to $|G|$, then there exists an injective homomorphism μ_* called the transfer map,*

$$\mu_*\colon H_*(X;k) \to H_*(\widetilde{X};k), \tag{16}$$

such that

$$\operatorname{Im}\mu_* \cong H_*(\widetilde{X};k)^G, \tag{17}$$

where the right-hand side is the G–invariant subspace of $H_*(X;k)$.

Proof We first define the transfer map $\mu_*\colon H_*(X) \to H_*(\widetilde{X})$. Let μ be a homomorphism between singular chain complexes, $\mu\colon C_*(X) \to C_*(\widetilde{X})$ with

$$\mu(\Delta) = \sum_{\text{lifts of } \Delta} \widetilde{\Delta}.$$

The summation runs over all lifts of the singular simplex Δ. We can see that μ commutes with the differentials, hence we can define $\mu_*\colon H_*(X) \to H_*(\widetilde{X})$ with

$$\mu_*(x) = \sum_{g\in G} g_* x. \tag{18}$$

Then both the maps $p_*\circ\mu_*\colon H_*(X) \to H_*(X)$ and $\mu_*\circ p_*\colon H_*(\widetilde{X})^G \to H_*(\widetilde{X})^G$ are multiplication by $|G| \in k$. As the characteristic of k is coprime to $|G|$, multiplication by $|G|$ is an isomorphism on a k–vector space, which implies μ_* and p_* are also isomorphisms. Thus (16) holds. □

By using this transfer map, we obtain a sequence of compositions

$$\begin{aligned} H_p(P_G M \times_G EG) \otimes H_q(P_G M \times_G EG) &\xrightarrow{\mu_*\otimes\mu_*} H_p(P_G M) \otimes H_q(P_G M) \\ &\xrightarrow{\bullet} H_{p+q-n}(P_G M) \xrightarrow{p_*} H_{p+q-\dim M}(P_G M \times_G EG), \end{aligned}$$

where p_* is the map induced from the covering map $p\colon P_G M \times EG \to P_G M \times_G EG$. The obtained map

$$\circ\colon H_p(P_G M \times_G EG) \otimes H_q(P_G M \times_G EG) \to H_{p+q-\dim M}(P_G M \times_G EG), \tag{19}$$

is the desired orbifold loop product, which coincides with the ordinary loop product when G is the trivial group by the above construction. □

Remark It is shown in [14] that the orbifold loop product $\circ$ is indeed an orbifold invariant.

3 Propositions for the proof of the main theorem

In this section, we prove some propositions that we use for the proof of the main theorem. Unless otherwise stated, we denote by M a smooth closed oriented manifold, by $\mathcal{G}$ a path-connected group acting continuously on M, and by G a finite subgroup of $\mathcal{G}$ acting smoothly on M.

The following proposition is proved in [14] for computing the loop product of some lens spaces. We prove it below because we use the proof of it in this paper.

Proposition 3.1 [14] *Let $\mathcal{G}$ be a path-connected group acting continuously on M, and G a finite subgroup of $\mathcal{G}$ acting smoothly on M. Then for each $g \in G$, there exists a homotopy equivalence*

$$P_g X \simeq LM. \tag{20}$$

Proof By the assumption on G, for each $g \in G$ there exists a path θ_g in $\mathcal{G}$ which starts at the unit in G and ends at g. We fix such paths $\{\theta_g\}_{g \in G}$.

For each $g \in G$ we consider the maps $\tau_g\colon P_g M \to LM$ and $\eta_g\colon LM \to P_g M$ with

$$\tau_g(t) = \begin{cases} \sigma(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \sigma(1)\check{\theta}_g(2t-1) & \text{for } \frac{1}{2} \le t \le 1, \end{cases} \tag{21}$$

$$\eta_g(l) = \begin{cases} l(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ l(1)\theta_g(2t-1) & \text{for } \frac{1}{2} \le t \le 1, \end{cases} \tag{22}$$

where $\check{\theta}_g$ denotes the image of θ_g by the map $\mathcal{G} \to \mathcal{G}$ which sends an element to its inverse. Then we can see $\tau_g \circ \eta_g \simeq \mathrm{id}_{LM}$ and $\eta_g \circ \tau_g \simeq \mathrm{id}_{P_g M}$, hence $P_g M \simeq LM$ for each $g \in G$. □

We consider the map $\Phi\colon \Omega\mathcal{G} \times LM \to LM$ with

$$\Phi(a, l) = (t \mapsto a(t) \cdot l(t)), \tag{23}$$

and the induced map $\Phi_*\colon H_*(\Omega\mathcal{G}) \otimes H_*(LM) \to H_*(LM)$, which defines an action of the Pontrjagin ring $H_*(\Omega\mathcal{G})$ on $H_*(LM)$. If the action Φ_* of $\pi_0(\Omega\mathcal{G})$ on $H_*(LM)$ satisfies $\Phi_*(x) = x$ for any $x \in H_*(LM)$, we call the action Φ_* *trivial.*

Proposition 3.2 *Assume that the action Φ_* is trivial. Then the homotopy equivalence*

$$\coprod_{g \in G} P_g M \simeq \coprod_{g \in G} LM \tag{24}$$

of Proposition 3.1 is G–equivariant at homology level with coefficients in k.

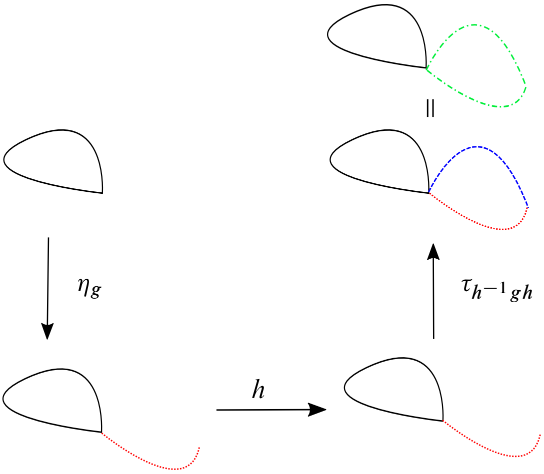

Figure 1: Picture for Proposition 3.2

We use the following lemma for the proof.

Lemma 3.3 *Let $\mathcal{G}$ be a path-connected group acting continuously on M, and G a finite subgroup of $\mathcal{G}$ acting smoothly on M. Then G acts on $H_*(LM)$ trivially.*

Proof The paths $\{\theta_g\}_{g\in G}$ in the proof of Proposition 3.1 make a homotopy between the actions of e and g on $H_*(LM)$. Because $H_*(LM)$ is discrete, they act trivially. □

Proof of Proposition 3.2 We need to show the commutativity of the following diagram for each $g, h \in G$:

$$\begin{tikzcd}
H_*(LM) \arrow[r, "h_*"] & H_*(LM) \\
H_*(P_g M) \arrow[u, "\tau_{g*}"] \arrow[r, "h_*"] & H_*(P_{h^{-1}gh} M) \arrow[u, "\tau_{h^{-1}gh*}"']
\end{tikzcd}$$

By Lemma 3.3, the upper h_* is equal to an identity. Thus we need to show

$$(\tau_{h^{-1}gh} \circ h \circ \eta_g)_* = \mathrm{id}.$$

Let a be the loop $h^{-1}\theta_g h * \check{\theta}_{h^{-1}gh}$ in $\mathcal{G}$, where $*$ denotes concatenating operation defined by (10); see Figure 1. Then the map $\tau_{h^{-1}gh} \circ h \circ \eta_g$ is equal to the map $\Phi(a, -)\colon LM \to LM$, and the induced homomorphism $(\tau_{h^{-1}gh} \circ h \circ \eta_g)_*$ is equal to the action $\Phi_*([a], -)\colon H_*(LM) \to H_*(LM)$, which is trivial by the assumption. □

4 Main theorem

Let τ be a direct sum

$$\tau = \bigoplus_g \tau_g \colon \coprod_{g\in G} P_g M \to \coprod_{g\in G} LM.$$

Then by Proposition 3.2 and Lemma 3.3, τ induces an isomorphism

$$H_*(L[M/G]) = H_*\Big(\coprod_{g\in G} P_g M\Big)^G \tag{25}$$

$$\overset{\tau_*}{\cong} H_*\Big(\coprod_{g\in G} LM\Big)^G \tag{26}$$

$$\cong H_*(LM) \otimes Z(k[G]). \tag{27}$$

The last isomorphism follows from Lemma 3.3.

Our main theorem, asserting that τ_* preserves loop products, is the following.

Theorem 4.1 *Let $\mathcal{G}$ be a path-connected group acting continuously on M, and G be a finite subgroup of $\mathcal{G}$ acting smoothly on M, and k be a field whose characteristic is coprime to $|G|$. If the action Φ_* is trivial with coefficients in k, then there exists an isomorphism as k–algebras,*

$$\mathbb{H}_*(L[M/G];k) \cong \mathbb{H}_*(LM;k) \otimes Z(k[G]), \tag{28}$$

where $Z(k[G])$ denotes the center of the group ring $k[G]$.

For the proof, we need to show that the diagram

$$\begin{tikzcd}
H_p(\coprod_g LM)^G \otimes H_q(\coprod_g LM)^G \arrow[r, "\circ"] & H_{p+q-\dim M}(\coprod_g LM)^G \\
H_p(\coprod_g P_g M)^G \otimes H_q(\coprod_g P_g M)^G \arrow[r, "\circ"] \arrow[u, "\tau_*\otimes\tau_*"] & H_{p+q-\dim M}(\coprod_g P_g M)^G \arrow[u, "\tau_*"']
\end{tikzcd} \tag{29}$$

is commutative. To prove this we need some lemmas. We consider the following commutative diagram of pullbacks, where $\tau_{1/2}$ is a map induced from the universality of the pullback:

(30)
$$\begin{tikzcd}
LM \times_M LM \arrow[r] \arrow[dd, bend left=60, "\mathrm{ev}_{1/2}"] & LM \times LM \arrow[dd, bend left=60, "\mathrm{ev}_{1/2} \times \mathrm{ev}_0"] \\
P_g M \times_M P_h M \arrow[u, dashed, "\tau_{1/2}"] \arrow[r] \arrow[d, "\mathrm{ev}_{1/2}"] & P_g M \times P_h M \arrow[u, "\tau_g \times \tau_h"'] \arrow[d, "\mathrm{ev}_1 \times \mathrm{ev}_0"] \\
M \arrow[r, "\Delta"] & M \times M
\end{tikzcd}$$

Lemma 4.2 *The diagram*

(31)
$$\begin{tikzcd}
LM & LM \times_M LM \arrow[l, "\gamma'"'] \\
P_{gh} M \arrow[u, "\tau_{gh}"] & P_g M \times_M P_h M \arrow[l, "\gamma"'] \arrow[u, dashed, "\tau_{1/2}"]
\end{tikzcd}$$

is commutative at homology level, where γ' and γ are concatenating maps defined by

$$\gamma'((l_1, l_2))(t) = \begin{cases} l_1(2t) & \text{for } 0 \leq t \leq \frac{1}{4}, \\ l_2\big(2t - \frac{1}{2}\big) & \text{for } \frac{1}{4} \leq t \leq \frac{3}{4}, \\ l_1(2t-1) & \text{for } \frac{3}{4} \leq t \leq 1, \end{cases} \tag{32}$$

$$\gamma((\sigma_1, \sigma_2))(t) = \begin{cases} \sigma_1(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\ \sigma_2(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \tag{33}$$

Proof Let b be the loop $h\check{\theta}_h * \check{\theta}_g * g^{-1}\theta_{gh}$ in $\mathcal{G}$ (see Figure 2). Then we have the commutative diagram

(34)
$$\begin{tikzcd}
LM \arrow[d, "\eta_{gh}"] & LM \times_M LM \arrow[l, "\gamma'"] \\
P_{gh} M & P_g M \times_M P_h M \arrow[u, dashed, "\tau_{1/2}"] \arrow[dl, "\gamma"] \\
P_{gh} M \arrow[u, "*b"] &
\end{tikzcd}$$

where $*b$ denotes the map $\sigma \mapsto \sigma * \sigma(1)b$. Hence it reduces to showing that $(*b)_*\colon H_*(P_{gh}M) \to H_*(P_{gh}M)$ is equal to an identity, which follows from the similar argument in the proof of Proposition 3.2. □

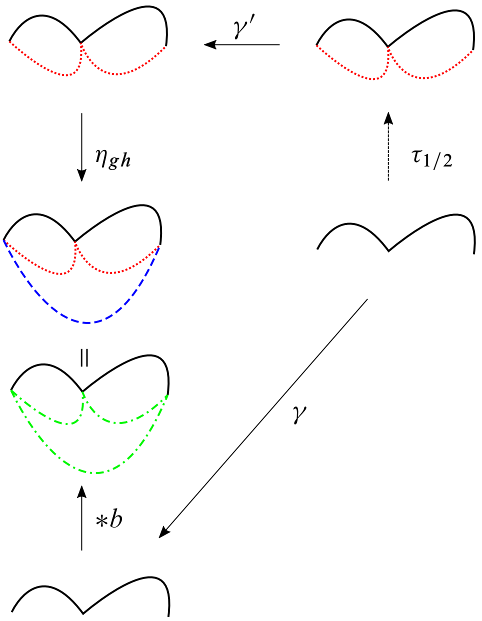

Figure 2: Picture for Lemma 4.2

Lemma 4.3 *The concatenating map γ' defines the same product as the loop product $\circ$; in other words, the following diagram commutes:*

$$\begin{tikzcd}
H_*(LM\times LM) \arrow[r, "\mathrm{PT}"] & H_*((LM\times_M LM)^{TM}) \arrow[r, "\mathrm{Thom}"] & H_{*-\dim M}(LM\times_M LM) \arrow[r, "\gamma'_*"] & H_{*-\dim M}(LM) \\
H_*(LM\times LM) \arrow[r, "\mathrm{PT}"] \arrow[u, "(\rho_{1/2}\times \mathrm{id})_*"'] & H_*((LM\times_M LM)^{TM}) \arrow[r, "\mathrm{Thom}"] \arrow[u, "\widetilde{(\rho_{1/2}\times_{1/2}\mathrm{id})}_*"'] & H_{*-\dim M}(LM\times_M LM) \arrow[r, "\gamma_*"] \arrow[u, "(\rho_{1/2}\times_{1/2}\mathrm{id})_*"'] & H_{*-\dim M}(LM) \arrow[u, "\rho_{1/4*}"']
\end{tikzcd} \tag{35}$$

Proof It follows from the commutative diagram

$$\begin{tikzcd}
LM & LM\times_M LM \arrow[l, "\gamma'"'] \arrow[r] \arrow[dd, bend left=40] & LM\times LM \arrow[dd, bend left=50, "\mathrm{ev}_{1/2}\times\mathrm{ev}_0"] \\
LM \arrow[u, "\rho_{1/4}"] & LM\times_M LM \arrow[l, "\gamma"] \arrow[r] \arrow[u, dashed, "\rho_{1/2}\times_{1/2}\mathrm{id}"] \arrow[d] & LM\times LM \arrow[u, "\rho_{1/2}\times \mathrm{id}"] \arrow[d, "\mathrm{ev}_0\times\mathrm{ev}_0"] \\
& M \arrow[r, "\Delta"] & M\times M
\end{tikzcd} \tag{36}$$

and the naturality of the Pontrjagin–Thom construction and Thom isomorphism, where ρ_a is the parameter transformation map defined by $\rho_a(l) = (t \mapsto l(t+a))$. □

Proposition 4.4 *For any $g, h \in G$, the following diagram commutes:*

$$\begin{tikzcd}
H_p(LM) \otimes H_q(LM) \arrow[r, "\circ"] & H_{p+q-\dim M}(LM) \\
H_p(P_gM) \otimes H_q(P_hM) \arrow[r, "\bullet"] \arrow[u, "\tau_{g*}\otimes\tau_{h*}"] & H_{p+q-\dim M}(P_{gh}M) \arrow[u, "\tau_{gh*}"']
\end{tikzcd} \tag{37}$$

Proof Because of the diagram (30) and Lemma 4.2, by the naturality of the Pontrjagin–Thom map and Thom isomorphism we obtain the following commutative diagram:

$$\begin{tikzcd}
H_*(LM \times LM) \arrow[r] & H_*((LM \times_M LM)^{TM}) \arrow[r] & H_{*-\dim M}(LM \times_M LM) \arrow[r] & H_{*-\dim M}(LM) \\
H_*(P_gM \times P_hM) \arrow[r] \arrow[u, "(\tau_g\times\tau_h)_*"'] & H_*((P_gM \times_M P_hM)^{TM}) \arrow[r] \arrow[u, "\widetilde{\tau_{1/2*}}"'] & H_{*-\dim M}(P_gM \times_M P_hM) \arrow[r] \arrow[u, "\tau_{1/2*}"'] & H_{*-\dim M}(P_{gh}M) \arrow[u, "\tau_{gh*}"']
\end{tikzcd} \tag{38}$$

which implies the diagram (37) is commutative by Lemma 4.3. □

Proof of Theorem 4.1 By the definition of the loop product, diagram (29) is same as the following diagram:

$$\begin{tikzcd}
H_p\big(\coprod_g P_gM\big)^G \otimes H_q\big(\coprod_g P_gM\big)^G \arrow[r, "\tau_*\otimes\tau_*"] \arrow[d, hook] & H_p\big(\coprod_g LM\big)^G \otimes H_q\big(\coprod_g LM\big)^G \arrow[d, hook] \\
H_p\big(\coprod_g P_gM\big) \otimes H_q\big(\coprod_g P_gM\big) \arrow[r, "\tau_*\otimes\tau_*"] \arrow[d, "\bullet"] & H_p\big(\coprod_g LM\big) \otimes H_q\big(\coprod_g LM\big) \arrow[d, "\circ"] \\
H_{p+q-\dim M}\big(\coprod_g P_gM\big) \arrow[r, "\tau_*"] \arrow[d, "\sigma_{1*}"] & H_{p+q-\dim M}\big(\coprod_g LM\big) \arrow[d, "\sigma_{2*}"] \\
H_{p+q-\dim M}\big(\coprod_g P_gM\big)^G \arrow[r, "\tau_*"] & H_{p+q-\dim M}\big(\coprod_g LM\big)^G
\end{tikzcd} \tag{39}$$

where σ_{1*}, σ_{2*} are defined as

$$\sigma_{1*}\colon H_*\big(\textstyle\coprod_g P_gM\big) \xrightarrow{\mathrm{pr}_*} H_*\big(\big(\textstyle\coprod_g P_gM\big)/G\big) \xrightarrow[\cong]{\mu_*} H_*\big(\textstyle\coprod_g P_gM\big)^G, \tag{40}$$

$$\sigma_{2*}\colon H_*(LM \times G) \xrightarrow{\mathrm{pr}_*} H_*((LM \times G)/G) \xrightarrow[\cong]{\mu_*} H_*(LM \times G)^G, \tag{41}$$

where μ_* is as defined in the proof of Lemma 2.10. The G–equivariance of τ_* implies the commutativity of the upper square, and Proposition 4.4 implies the commutativity of the middle square. The commutativity of the lower square follows from the definition of μ_* and G–equivariance of τ_*. □

5 Relation to Hochschild cohomology

In [5], the theorem of Cohen and Jones states that if M is simply connected, then its loop homology ring is isomorphic to the Hochschild cohomology of the singular cochains, hence we have $\mathbb{H}_*(LM) \cong \mathrm{HH}^*(C^*M, C^*M)$. In [2], this result is studied for global quotient orbifolds by Ángel, Backelin and Uribe, who show that if M is a simply connected manifold with G action, then its orbifold loop homology ring with coefficients in k is isomorphic to the Hochschild cohomology of the DGA $C^*(M;k)\#_k G$, hence we have $\mathbb{H}_*(L[M/G];k) \cong \mathrm{HH}^*(C^*M\#_k G, C^*M\#_k G)$. Here the DGA structure on $C^*(M;k)\#_k G$ is defined as follows. With $C^*(M;k)\#_k G = C^*(M;k) \otimes_k k[G]$ as a module, the multiplication is defined by $(x \otimes g)(y \otimes h) = xg(y) \otimes gh$, and the differential is defined by $d = d_{C^*M} \otimes \mathrm{id}_{k[G]}$.

By using our theorem, we obtain the following as a corollary.

Corollary 5.1 *Under the situation we consider in the theorem, if Φ_* is trivial and M is simply connected, then it holds that*

$$\mathbb{H}_*(L[M/G];k) \cong \mathrm{HH}^*(C^*M, C^*M) \otimes Z(k[G]) \tag{42}$$

$$\cong \mathrm{HH}^*(C^*M\#_k G, C^*M\#_k G). \tag{43}$$

Proof The first equation follows from our theorem and the theorem of Cohen and Jones, and the second one follows from the above theorem of Ángel, Backelin and Uribe. □

6 Examples

In this section, we will compute some examples of orbifold loop homology. Before that, we will prove some propositions which are useful to apply our main theorem and to compute examples.

Proposition 6.1 *If $\mathcal{G}$ is simply connected, then the action Φ_* is trivial for any field k.*

Proof Because $\pi_0(\Omega\mathcal{G})$ is represented only by the identity loop, its action on $H_*(LM)$ is trivial. □

Proposition 6.2 *If $H_{\dim M}(LM; k) = k$ for a field k, then the action Φ_* is trivial for any pair $(\mathcal{G}, G)$.*

To prove this proposition, we use the following lemma by Hepworth.

Lemma 6.3 [9] *Let Φ be the natural map defined by* (23). *Then the induced linear action $\Phi_*\colon H_0(\Omega\mathcal{G}) \otimes H_*(LM) \to H_*(LM)$ is an algebra action, in other words for any $\alpha \in H_0(\Omega\mathcal{G})$ and any $x, y \in H_*(LM)$, they satisfy*

$$\alpha \cdot (x \circ y) = (\alpha \cdot x) \circ y = x \circ (\alpha \cdot y). \tag{44}$$

Proof Let a be a loop in $\mathcal{G}$ which starts and ends at the unit. We need to show that the action $\Phi_{a*} = \Phi_*([a], -)\colon H_*(LM) \to H_*(LM)$ is trivial, where Φ_a denotes the map $\Phi(a, -)$. We also denote by Φ_a the restriction map of the map $\Phi(a, -)\colon LM \to LM$ to $M \subset LM$, where we regard M as the image of the map $c\colon M \to LM$ which assigns the constant loop.

We consider the following sequence of maps:

$$\begin{tikzcd} H_{\dim M}(M) \arrow[r, shift left, "\Phi_{a*}"] \arrow[r, shift right, "c_*"'] \arrow[rr, bend left=35, "\mathrm{id}_*"] \arrow[rr, phantom, bend left=20, "\circlearrowright"] & H_{\dim M}(LM) \arrow[r, "\mathrm{ev}_*"] & H_{\dim M}(M) \end{tikzcd} \tag{45}$$

As $H_{\dim M}(LM) = H_{\dim M}(M) = k$, we obtain $\Phi_{a*} = c_* = \mathrm{id}$. Hence for any $x \in H_*(LM)$, we have by Lemma 6.3 the equality

$$\Phi_{a*}x = \Phi_{a*}([M] \circ x) = (\Phi_{a*}[M]) \circ x = x.$$

Therefore the action of $\pi_0(\Omega\mathcal{G})$ on $H_*(LM)$ is trivial. □

Proposition 6.4 *Let k be a field. If the following conditions are satisfied, then the action Φ_* is trivial.*

(i) $\pi_1 M = 1$.

(ii) $|\pi_1\mathcal{G}| =: r < \infty$.

(iii) *The homomorphism $H_*(\Omega M; k) \to H_*(LM; k)$ induced by the inclusion map $\Omega M \to LM$ is injective; that is, the free loop fibration $\Omega M \to LM \to M$ is* ***totally noncohomologous to zero*** *(TNCZ) with respect to the field k.*

(iv) *The characteristic of k is coprime to r.*

For the proof of Proposition 6.4, we use the following lemma by Hepworth.

Lemma 6.5 [9] *Let $\mathcal{G}$ be a topological group acting continuously on a closed oriented manifold M. Then the homomorphism $\Phi'_*\colon H_*(\Omega\mathcal{G};k) \to H_*(LM;k)$ with*

$$\Phi'_*[a] = [a]\cdot[M] \tag{46}$$

commutes with the products, ie they satisfy

$$\Phi'_*[a] \circ \Phi'_*[a'] = \Phi'_*([a]\cdot[a']), \tag{47}$$

where the product on the right-hand side denotes the Pontrjagin product.

Proof of Proposition 6.4 We denote $\dim M$ by n below. Let a be a loop in $\mathcal{G}$ which starts and ends at the unit. By the arguments in the proof of Proposition 6.2, we need to show that $\Phi_{a*}[M]$ is the unit in $\mathbb{H}_0(LM)$ for the triviality of the action of $\pi_0(\Omega\mathcal{G})$. By the TNCZ assumption, we have the ring isomorphism

$$\mathbb{H}_*(LM) \cong H_*(\Omega M) \otimes \mathbb{H}_*(M), \tag{48}$$

hence we obtain the linear isomorphism

$$H_n(LM) \cong \bigoplus_{p+q=n} H_p(\Omega M) \otimes H_q(M). \tag{49}$$

By the assumption of $\pi_1 M = 1$, we have $H_0(\Omega M) \cong k$, hence $H_0(\Omega M) \otimes H_n(M)$ is a 1–dimensional vector space, and we can put

$$\Phi_{a*}[M] = x\cdot 1 + \sum_i y_i \cdot \sigma_i \otimes \delta_i, \tag{50}$$

where

$$1 = 1\otimes[M] \in H_0(\Omega M)\otimes H_n(M), \quad \sigma_i \in H_{>0}(\Omega M), \quad \delta_i \in H_{<n}(M),$$
$$\deg\sigma_i + \deg\delta_i = n, \quad x, y_i \in k.$$

We can see $x = 1$ as follows. We have a ring homomorphism induced by the evaluation map $\mathrm{ev}_*\colon \mathbb{H}_*(LM) \to \mathbb{H}_*(M)$. Hence we have

$$\mathrm{ev}_*(\sigma_i \otimes \delta_i) = \mathrm{ev}_*(\sigma_i \otimes [M]) \circ \mathrm{ev}_*(1 \otimes \delta_i). \tag{51}$$

Because the degree of σ_i is positive, the degree of $\sigma_i \otimes [M]$ is larger than n, hence $\mathrm{ev}_*(\sigma_* \otimes [M])$ is equal to 0. Therefore, we obtain

$$\mathrm{ev}_*\,\Phi_{a*}[M] = x\cdot \mathrm{ev}_*\,1 + \sum_i y_i \cdot \mathrm{ev}_*(\sigma_i \otimes \delta_i) = x \cdot \mathrm{ev}_*\,1.$$

Moreover, we have the diagram

$$\begin{tikzcd} H_n(M) \arrow[r, "\Phi_{a*}"] \arrow[rr, bend left, "\mathrm{id}_*", "\circlearrowright"'] & H_n(LM) \arrow[r, "\mathrm{ev}_*"] & H_n(M) \end{tikzcd} \tag{52}$$

Thus we have

$$x \cdot [M] = x \cdot \mathrm{ev}_* 1 = \mathrm{ev}_* \circ \Phi_{a*}[M] = [M], \tag{53}$$

hence we obtain $x = 1$. Furthermore, we can show that the y_i are all 0 as follows. Because the δ_i are all nilpotent, we have

$$(\Phi_{a*}[M] - 1)^N = 0 \tag{54}$$

for sufficiently large N. By the assumption $|\pi_1\mathcal{G}| = r < \infty$ and Lemma 6.5, we have

$$(\Phi_{a*}[M])^r = (\Phi'_*[a])^r = \Phi'_*[a]^r = \Phi'_*[1] = 1. \tag{55}$$

Since the characteristic of k is coprime to r, the common divisor of $(\Phi_{a*}[M] - 1)^N$ and $(\Phi_{a*}[M])^r - 1$ is $\Phi_{a*}[M] - 1$. Hence we conclude that the y_i are all 0, which implies $\Phi_{a*}[M] = 1$. □

Example 6.6 We consider the case $(M, \mathcal{G}, G, k) = (S^3, \mathrm{Spin}(3), \pi, k)$, where

$$\pi = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^3 = (ca)^5 \rangle$$

and k is an algebraic closed field whose characteristic is not 2, 3 or 5. The finite group π acts on S^3 and the quotient manifold S^3/π is called the *Poincaré homology sphere*. Then we have an algebra isomorphism

$$\begin{aligned} \mathbb{H}_*(L(S^3/\pi); k) &\cong \mathbb{H}_*(S^3; k) \otimes Z(k[\pi]) \\ &\cong \mathbb{H}_*(S^3; k) \otimes k^9. \end{aligned}$$

Here we use the fact that if k is an algebraic closed field and G is a finite group, then $Z(k[G]) \cong k^{c(G)}$, where $c(G)$ denotes the number of conjugacy classes of G.

Remark If $k = \mathbb{Q}$, Vigué [21] shows that the following are equivalent.

(i) The homomorphism $H_*(\Omega M; k) \to H_*(LM; k)$ induced by the inclusion $\Omega M \to LM$ is injective.

(ii) $H^*(M; k)$ is a free graded commutative algebra.

Remark For $M = S^n$ or $\mathbb{C}\mathrm{P}^n$, Menichi [16] shows that the following are equivalent.

(i) The homomorphism induced by the inclusion $H_*(\Omega M; k) \to H_*(LM; k)$ is injective.

(ii) The Euler number $\chi(M)$ is 0 in k.

Remark The necessary condition for the free loop fibration

$$\Omega M \to LM \to M \tag{56}$$

to be the TNCZ fibration with respect to a finite field $\mathbb{F}_p$ has been studied for many homogeneous spaces by Kuribayashi [13]. For example:

- For $M = \mathbb{C}\mathrm{P}^n$ or $\mathbb{H}\mathrm{P}^n$, the fibration (56) is TNCZ if and only if the Euler number $\chi(M)$ is 0 in $\mathbb{F}_p$.
- For $M = \mathrm{SU}(m+n)/\mathrm{SU}(n)$ or $\mathrm{Sp}(m+n)/\mathrm{Sp}(n)$, the fibration (56) is TNCZ for any $m, n \geq 0,\ p > 0$.
- For $\mathrm{SO}(m+n)/\mathrm{SO}(n)$ and $p > 2$, the fibration (56) is TNCZ if and only if n is odd.
- For $\mathrm{SO}(m+n)/\mathrm{SO}(n)$ and $p = 2$, the fibration (56) is TNCZ if $m \leq 4$ or $1 \leq m \leq 8$ and $n \geq 43$.
- For $U(m+n)/U(m) \times U(n)$ and $\mathrm{Sp}(m+n)/\mathrm{Sp}(m) \times \mathrm{Sp}(n)$, the fibration (56) is *not* TNCZ for any $m, n \geq 0,\ p > 0$.

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Graduate School of Mathematical Sciences, The University of Tokyo
Tokyo, Japan

`asao@ms.u-tokyo.ac.jp`

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