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A motivic Grothendieck–Teichmüller group

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We prove the Beilinson–Soulé vanishing conjecture for motives attached to the moduli spaces $\mathcal{M}_{0,n}$ of curves of genus 0 with n marked points. As part of the proof, we also show that these motives are mixed Tate. As a consequence of Levine's work, we thus obtain a well-defined category of mixed Tate motives over the moduli space of curves $\mathcal{M}_{0,n}$. We furthermore show that the morphisms between the moduli spaces $\mathcal{M}_{0,n}$ obtained by forgetting marked points and by embedding boundary components induce functors between the associated categories of mixed Tate motives. Finally, we explain how tangential base points fit into these functorialities.

The categories we construct are Tannakian, and therefore have attached Tannakian fundamental groups, connected by morphisms induced by those between the categories. This system of groups and morphisms leads to the definition of a motivic Grothendieck–Teichmüller group.

The proofs of the above results rely on the geometry of the tower of the moduli spaces $\mathcal{M}_{0,n}$. This allows us to treat the general case of motives over $\mathrm{Spec}(\mathbb{Z})$ with coefficients in $\mathbb{Z}$, working in Spitzweck's category of motives. From there, passing to $\mathbb{Q}$ coefficients, we deal with the classical Tannakian formalism and explain how working over $\mathrm{Spec}(\mathbb{Q})$ yields a more concrete description of the Tannakian groups.

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1 Introduction

M Levine [38] considers a smooth quasiprojective variety X over a number field $\mathbb{F}$. He shows that when the motive of X in $\mathrm{DM}_{/\mathbb{F},\mathbb{Q}}(\mathrm{Spec}(\mathbb{F}))$ is mixed Tate and satisfies

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the Beilinson–Soulé vanishing property, there is a well-defined Tannakian category of mixed Tate motives $\mathrm{MTM}_{/\mathbb{F},\mathbb{Q}}(X)$ whose Tannakian Hopf algebra H_X is built from a complex of algebraic cycles that compute the higher Chow groups. Moreover, Levine proves that the Tannakian group $G_X = \mathrm{Spec}(H_X)$ fits into a short exact sequence

$$1 \to G_{X,\mathrm{geom}} \to G_X \to G_{\mathrm{Spec}(\mathbb{F})} \to 1,$$

where $G_{X,\mathrm{geom}}$ can be identified with Deligne–Goncharov's motivic fundamental group $\pi_1^{\mathrm{mot}}(X,x)$ after a choice of (tangential) base point $x \in X(\mathbb{F})$.

The above exact sequence admits a Lie coalgebra counterpart

$$0 \to L^c_{\mathrm{Spec}(\mathbb{F})} \to L^c_X \to L^c_{X,\mathrm{geom}} \to 0$$

by considering the set of indecomposable elements of H_X. In [45], the author shows how, for

$$X = \mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\},$$

the explicit algebraic cycles constructed in Soudères [44] describe the coaction of $L^c_{\mathrm{Spec}(\mathbb{F})}$ on $L^c_{X,\mathrm{geom}}$.

In order to generalize this work to the moduli space $\mathcal{M}_{0,n}$ of curves in genus 0 with n marked points for any $n \geq 3$, the first step is to show that the moduli spaces $\mathcal{M}_{0,n}$ satisfy the Beilinson–Soulé vanishing conjecture.

Working over $\mathrm{Spec}(\mathbb{F})$ allows Levine to relate $H_{\mathcal{M}_{0,n}}$ to a cycle complex computing motivic cohomology. However, the moduli spaces of curves are well-defined over $\mathrm{Spec}(\mathbb{Z})$, so there is no reason to restrict ourselves to $\mathrm{Spec}(\mathbb{F})$ when considering the Beilinson–Soulé vanishing property; rather, we work in the framework of Cisinski and Déglise (see [14]). Even more generally, the Beilinson–Soulé vanishing property and the mixed Tate property hold in Spitzweck's framework (see [50]) of motives over $\mathrm{Spec}(\mathbb{Z})$ with $\mathbb{Z}$ coefficients, as proved in Theorem 3.8 below.

From Theorem 3.8, establishing the Beilinson–Soulé vanishing property for the moduli space of curves $\mathcal{M}_{0,n}$, we deduce from Spitzweck's work [50; 49] that there exists a well-defined triangulated category, $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n})$, of mixed Tate motives over $\mathcal{M}_{0,n}$ (Theorem 4.9).

We consider two families of natural morphisms between the moduli spaces $\mathcal{M}_{0,n}$, the first given by forgetting some marked points, and the second by embedding $\overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2}$ as a codimension-1 boundary component of $\overline{\mathcal{M}}_{0,n_1+n_2-2}$ on the

Deligne–Mumford compactification. Following Grothendieck's terminology in the *Esquisse d'un programme*, we call the collection of moduli spaces $\mathcal{M}_{0,n}$ equipped with these morphisms the "tower" of genus zero moduli spaces.

These morphisms induce functors between the categories $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n})$, and thence morphisms between the corresponding Tannakian groups when working with $\mathbb{Q}$ coefficients. This leads to the definition of a motivic Grothendieck–Teichmüller group, which is given in Definition 5.1.

The structure of the paper is as follows:

In Section 2 we review the framework of motivic $\mathbb{P}^1$ spectra and the stable motivic homotopy category $\mathrm{SH}(S)$. We briefly present Spitzweck's triangulated category of mixed motives over S, and review some of its properties: the Gysin/localization triangle, the projective bundle formula and the blow-up formula. We use the versions derived from the work of Déglise [15], because Spitzweck's construction relies on an oriented E_∞–ring spectrum.

In Section 3 we review the geometry of the spaces $\mathcal{M}_{0,n}$ and their Deligne–Mumford compactifications $\overline{\mathcal{M}}_{0,n}$. We first prove the triviality of the normal bundle of D_0 in $\mathcal{M}_{0,n} \cup D_0$ for any open codimension-1 stratum D_0 of $\overline{\mathcal{M}}_{0,n}$. Then we prove that the motives associated to the $\overline{\mathcal{M}}_{0,n}$ are mixed Tate over $\mathrm{Spec}(\mathbb{Z})$ and satisfy the Beilinson–Soulé vanishing property. Section 3 ends with the proof that this result also holds for the open moduli spaces $\mathcal{M}_{0,n}$.

We begin Section 4 by reviewing the construction of limit motives as developed by Spitzweck [46; 47] and Ayoub [3]. We also treat the case of motivic tangential base points. Then we show how limit motives, applied to the moduli space of curves $\mathcal{M}_{0,n}$ and to an open codimension-1 stratum D_0, lead to a natural functor between $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n})$ and $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})$. The use of tangential base points then leads to functors

$$\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathrm{Spec}(\mathbb{Z})).$$

Functoriality with respect to forgetful morphisms is a consequence of Spitzweck's construction. Working over $\mathrm{Spec}(\mathbb{Z})$ with integral coefficients, these categories are equivalent to categories of perfect representations of affine derived group schemes. The above functorialities lead to the definition of a motivic Grothendieck–Teichmüller space in this setting, which concludes Section 4.

In Section 5 we derive some consequences of the above constructions in more classical settings. In particular, working with $\mathbb{Q}$ coefficients, we obtain a Tannakian group associated to the Tannakian category given by the heart of the t–structure of $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathcal{M}_{0,n})$. This leads to a motivic Grothendieck–Teichmüller group defined in terms of automorphisms of group schemes (and not derived group schemes). Working over $\mathrm{Spec}(\mathbb{F})$, the spectrum of a number field, we show how the Deligne–Goncharov category of mixed Tate motives over the ring of its integers agrees with Spitzweck's construction of mixed Tate motives. We also show how our construction passes to this context. At the end of the section, we explain the relation between our construction and Levine's approach to mixed Tate motives and algebraic cycles.

The final section is devoted to some conjectures on the "geometric" (derived) group schemes defining the motivic Grothendieck–Teichmüller group and their relation to Betti and de Rham realizations.

2 A short review of Spitzweck's category of mixed motives

Let S be a Noetherian separated scheme of finite Krull dimension. In [50], Spitzweck constructed a E_∞–ring object $\mathrm{M}\mathbb{Z}_S$ in the category $\mathrm{Spt}^{\Sigma}_{\mathbb{P}^1}(S)$ of motivic symmetric $\mathbb{P}^1$–spectra (see [29; 19; 26]). The $\mathbb{P}^1$–spectrum $\mathrm{M}\mathbb{Z}_S$ serves in particular as the motivic Eilenberg–Mac Lane spectrum; it is also an oriented ring spectrum. This means that in $\mathrm{SH}(S)$, it is an algebra over the algebraic cobordism spectrum $\mathbf{MGL}$. Considering the category $\mathrm{Mod}_{\mathrm{M}\mathbb{Z}_S}$ of modules over $\mathrm{M}\mathbb{Z}_S$, Spitzweck uses a model structure on $\mathrm{Mod}_{\mathrm{M}\mathbb{Z}_S}$ compatible with that on $\mathrm{Spt}^{\Sigma}_{\mathbb{P}^1}(S)$ to define a triangulated category $\mathrm{DM}_{\mathbb{Z}}(S)$ of motives over S with integral coefficients, together with the adjoint functors

$$\begin{tikzcd} \mathrm{Mod}_{\mathrm{M}\mathbb{Z}_S} \arrow[r] & \mathrm{Spt}^{\Sigma}_{\mathbb{P}^1}(S) \arrow[l, bend left, "\otimes \mathrm{M}\mathbb{Z}_S"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{DM}_{\mathbb{Z}}(S) \arrow[r] & \mathrm{SH}(S) \arrow[l, bend left, "\otimes \mathrm{M}\mathbb{Z}_S"] \end{tikzcd}$$

The left to right functors $\to$ are forgetful functors, and the tensor products are those given by the symmetric monoidal structure of $\mathrm{Spt}^{\Sigma}_{\mathbb{P}^1}(S)$ (corresponding to the smash product $\wedge$ in [29]).

Below we recall some definitions and properties needed for our construction of a motivic Grothendieck–Teichmüller group. Our construction is geometric and is based on the main distinguished triangles in $\mathrm{DM}_{\mathbb{Z}}(S)$ and on the functoriality of its construction. In particular, we recall the definition of the Gysin distinguished triangles and the blow-up formula in Spitzweck's category below. Thanks to the existence of a first Chern class in Spitzweck's category and its relation with the stable motivic homotopy category, these are in fact direct consequences of Déglise's work [15]. Working over a number field and with $\mathbb{Q}$ coefficients would remove the need for the following subsections, because in that context these properties are proved in [54].

2.1 Symmetric spectrum, SH(X) and mixed motives

Let Sm_S denote the category of smooth schemes of finite type over S, and $\mathrm{Sm}_S|_{\mathrm{Nis}}$ the smooth Nisnevich site over S. We recall below some facts about Spitzweck's construction [50]. We are mostly interested in the case where $S = \mathrm{Spec}(\mathbb{Z})$.

Let $\mathrm{Spc}(S)$ be the category of (motivic) spaces over S, that is, of Nisnevich sheaves over S with values in simplicial sets. Spitzweck's construction actually uses complexes of sheaves of abelian groups. Classical comparison functors and the transfer of structures ensure that his construction passes to motivic spectra. By the Yoneda embedding, any scheme in Sm_S is a motivic space (constant in the simplicial direction); any simplicial set is also a motivic space as a constant sheaf. The terminal object is represented by S itself.

A pointed (motivic) space is a motivic space X together with a map

$$x\colon S \to X.$$

The category of pointed spaces is denoted by $\mathrm{Spc}_\bullet(S)$. To any space X, one associates a canonical pointed space $X_+ = X \sqcup *$. The category $\mathrm{Spc}_\bullet(S)$ admits a monoidal structure $\otimes$ induced by the smash product on pointed simplicial sets.

Recall that the simplicial circle is the coequalizer of

$$\Delta[0] \rightrightarrows \Delta[1].$$

Let S^1_s be the corresponding pointed space. Moreover, let S^1_T, the Tate circle, be the pointed space represented by $(\mathbb{P}^1, \{\infty\})$.

Briefly, a symmetric $\mathbb{P}^1$–spectrum E is a collection of pointed spaces $E = (E_0, E_1, \dots)$ together with structure maps $S^1_T \otimes E_n \to E_{n+1}$ and the extra data of a symmetric group action $\Sigma_n \times E_n \to E_n$ such that the composition maps

$$(S^1_T)^{\otimes p} \otimes E_n \to E_{n+p}$$

are $\Sigma_p \times \Sigma_n$–equivariant.

The iterated products of S_s^1 (resp. S_T^1) are denoted by S_s^n (resp. S_T^m). Tensoring with the simplicial circle (resp. the Tate circle) induces a simplicial (resp. Tate) suspension functor, denoted by Σ_s^1 (resp. Σ_T^1). In this manner, any motivic space X induces a symmetric $\mathbb{P}^1$–spectrum

$$\Sigma_T^\infty X_+ = (X_+, S_T^1 \otimes X_+, S_T^2 \otimes X_+, \dots).$$

We denote by $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$ the category of symmetric $\mathbb{P}^1$–spectra. The (*motivic*) *stable homotopy category* $\mathrm{SH}(S)$ is obtained from $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$ by inverting stable weak equivalences [26]. In particular, the suspension functors Σ_s and Σ_T are invertible, as are the $\mathbb{A}^1$ weak equivalences.

The category $\mathrm{SH}(S)$ is a triangulated category with shift induced by Σ_s^1. In $\mathrm{SH}(S)$, the suspension functor Σ_s will be denoted by the shift notation $[1]$. Note that S_T^1 is isomorphic to $S_s^1 \otimes (\mathbb{G}_m, \{1\})$ in $\mathrm{SH}(S)$.

In [50, Definition 4.27], Spitzweck defines a $\mathbb{P}^1$–spectrum $\mathrm{M}\mathbb{Z}_S$, or simply $\mathrm{M}\mathbb{Z}$ when there is no ambiguity about S. The $\mathbb{P}^1$–spectrum $\mathrm{M}\mathbb{Z}_S$ is an E_∞–ring object in $\mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S)$. It induces a ring object in $\mathrm{SH}(S)$, again denoted by $\mathrm{M}\mathbb{Z}_S$.

The category of motives $\mathrm{DM}_{\mathbb{Z}}(S)$ is defined as the homotopy category of modules (in $\mathbb{P}^1$–spectra) over $\mathrm{M}\mathbb{Z}_S$. For any $X \in \mathrm{Sm}_S$, the category $\mathrm{DM}_{\mathbb{Z}}(X)$ is defined similarly as the homotopy category of modules over

$$\mathrm{M}\mathbb{Z}_X = f^*\mathrm{M}\mathbb{Z}_S,$$

where $f\colon X \to S$ is the structural morphism and $f^*\colon \mathrm{Spt}_{\mathbb{P}^1}^\Sigma(S) \to \mathrm{Spt}_{\mathbb{P}^1}^\Sigma(X)$ is the pullback functor between categories of spectra. We will usually keep track of the ground scheme S and write $\mathrm{DM}_{/S,\mathbb{Z}}(X)$.

For $X \xrightarrow{f} S$ in Sm_S, we have a functor

$$\mathrm{Sm}_X \xrightarrow{M_X} \mathrm{DM}_{/S,\mathbb{Z}}(X), \quad Y \mapsto M_X(Y) = \Sigma_T^\infty(Y_+) \otimes f^*\mathrm{M}\mathbb{Z}_S = \Sigma_T^\infty(Y_+) \otimes \mathrm{M}\mathbb{Z}_X.$$

In $\mathrm{DM}_{/S,\mathbb{Z}}(X)$, the tensor unit $\mathrm{M}\mathbb{Z}_X = f^*\mathrm{M}\mathbb{Z}_S$ may also be denoted by $\mathbb{Z}_X(0)$ when we want to emphasize the structural property of its actually being the unit. We may use $\mathrm{M}\mathbb{Z}_X$ when we want to insist that properties of its construction play an important role.

The Tate object $\mathrm{M}\mathbb{Z}_X(1) = \mathbb{Z}_X(1)$ is defined by

$$\mathrm{M}\mathbb{Z}_X(1)[2] = \mathbb{Z}_X(1)[2] = \Sigma_T^1 \mathbb{Z}_X(0) = (\mathbb{P}^1, \infty) \otimes \mathrm{M}\mathbb{Z}_X,$$

and corresponds as usual to the cone of the morphism

$$M_X(\{\infty\}) \to M_X(\mathbb{P}^1)$$

shifted by -2.

The suspension $\Sigma_s^{n-2p} \circ \Sigma_T^p$ is denoted by $\Sigma^{n,p}$.

Remark 2.1 In [50, Section 10], Spitzweck showed that the functor $X \to \mathrm{DM}_{\mathbb{Z}}(X)$ satisfies the six-functors formalism.

2.2 An oriented cohomology theory

For a smooth scheme over S, $f\colon X \to S$, Spitzweck shows [50, Proposition 11.1] that the ring objects $\mathrm{M}\mathbb{Z}_S$ and $f^*\mathrm{M}\mathbb{Z}_S$ are oriented in the sense of Morel and Vezzosi [53]. In more detail, if $\mathbb{P}^\infty$ denotes the colimit of the $\mathbb{P}^n$, there is a distinguished element

$$\nu \in \mathrm{Hom}_{\mathrm{SH}(X)}(\Sigma^\infty(\mathbb{P}^\infty{}_+), \Sigma^{2,1} f^*\mathrm{M}\mathbb{Z}_S)$$

such that the element ν restricts to the canonical element induced by the unit of $f^*\mathrm{M}\mathbb{Z}_S$ in $\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(\mathbb{P}^1{}_+), \Sigma^{2,1} f^*\mathrm{M}\mathbb{Z}_S)$.

If S is regular, then for any smooth $Y \to X$, the morphism

$$\mathrm{Pic}(Y) \to \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(Y_+), \Sigma^\infty(\mathbb{P}^\infty{}_+))$$

is an isomorphism and endows $\mathrm{DM}_{\mathbb{Z}}(X)$ with an orientation as described by Déglise in [15, 2.1(Orient) axiom] (see [15, 2.3.2] or [39, Proposition 4.3.8]). Hence, for any smooth $Y \to X$, there is a morphism called the *first Chern class*,

$$c_1\colon \mathrm{Pic}(Y) \to \mathrm{Hom}_{\mathrm{DM}_{\mathbb{Z}}(X)}(M_X(Y), \mathbb{Z}_X(1)[2]),$$

which is functorial in Y and has the property that the image of the canonical bundle on $\mathbb{P}^1_X$ is the canonical projection.

Note that the formal group law attached to the first Chern class is the additive law [50, Theorem 7.10].

Thanks to Déglise's work [15], one then obtains the classical properties described in the following subsection.

2.3 Distinguished triangles and split formulas in $\mathbf{DM}_Z(X)$

From now on, X will always denote a smooth scheme in Sm_S, where S is at worst the spectrum of a Dedekind domain of mixed characteristic. We are mostly interested in the case $S = \mathrm{Spec}(\mathbb{Z})$. Let Y be a smooth scheme in Sm_X and $p\colon P \to Y$ a projective bundle over Y of rank n. We denote by λ the canonical line bundle over P and write

$$c = c_1(\lambda)\colon M_X(P) \to \mathbb{Z}_X(1)[2]$$

for the first Chern class. The diagonal

$$\delta_i\colon P \to \underbrace{P \times_Y \cdots \times_Y P}_{i+1 \text{ times}}$$

composed with $p_* \otimes c^{\otimes i}$ gives a morphism

$$\epsilon_{P,i}\colon M_X(P) \to M_X(Y)(i)[2i].$$

Proposition 2.2 (projective bundle formula [15, Theorem 3.2]) *With the above notation, the morphism*

$$\epsilon_P\colon M_X(P) \to \bigoplus_{i=1}^{n} M_X(Y)(i)[2i]$$

given by $\epsilon_P = \sum_{i=0}^{n} \epsilon_{P,i}$ is an isomorphism.

For any $0 \leqslant r \leqslant n$, we can now define the embedding

$$\iota_r\colon M_X(Y)(r)[2r] \xrightarrow{*(-1)^r} \bigoplus_{i=1}^{n} M_X(Y)(i)[2i] \to M_X(P).$$

Let Z be a smooth closed subscheme of Y such that Z is everywhere of codimension n. As in other situations, we define the motive of Y with support in Z as

$$M_{X,\mathrm{Supp}(Z)}(Y) = M_X(Y/(Y \setminus Z)).$$

In Proposition 4.3 of [15], Déglise attaches to the pair (Z, Y) a unique isomorphism

$$\text{(purity)} \qquad \mathfrak{p}_{Y,Z}\colon M_{X,\mathrm{Supp}(Z)}(Y) \to M_X(Z)(n)[2n]$$

which is functorial with respect to Cartesian morphisms of such pairs. Furthermore, when E is a vector bundle over Y of rank n and $P = \mathbb{P}(E \oplus 1)$, $\mathfrak{p}_{P,Y}$ is the inverse of the morphism

$$M_X(Y)(n)[2n] \xrightarrow{\iota_n} M_X(P) \to M_{X,\mathrm{Supp}(Y)}(P).$$

The purity isomorphism allows us to rewrite the localization distinguished triangle as follows.

Proposition 2.3 (Gysin triangle [15, Definition 4.6]) *Let Y be a smooth scheme over X and let Z be a smooth closed subscheme of Y such that Z is everywhere of codimension n. Then there is a distinguished triangle*

$$M_X(Y \setminus Z) \xrightarrow{j_*} M_X(Y) \xrightarrow{i^*} M_X(Z)(n)[2n] \xrightarrow{\partial_{Y,Z}} M_X(Y \setminus Z)[1], \tag{1}$$

where i^* (resp. $\partial_{X,Z}$) is called the ***Gysin morphism*** *(resp.* ***residue morphism****).*

The Gysin triangle is functorial, compatible with both the projective bundle isomorphisms and the induced embeddings ι_r. Moreover, Gysin morphisms are multiplicative with respect to compositions and products [15, Corollaries 4.33 and 4.34].

After generalizing and studying Gysin morphisms f^* for projective morphisms $f\colon Y_1 \to Y_2$ in Sm_X [15, Definition 5.12], and after giving a strong dual to $M_X(Y)$ for Y smooth projective over X [15, Theorem. 5.23], Déglise goes on to prove the following blow-up formula.

Proposition 2.4 (blow-up formula [15, Theorem 5.38]) *Let Y be a smooth scheme over X and Z a smooth closed subscheme of Y purely of codimension n. Let $B_Z(Y)$ be the blow-up of Y with center Z, and let E_Z denote the exceptional divisor. Then*

$$M_X(B_Z(Y)) \simeq M_X(Y) \oplus \bigoplus_{i=1}^{n-1} M_X(Z)(i)[2i].$$

2.4 The Beilinson–Soulé vanishing property

In [38], Levine proved that if X is a smooth variety over a number field $\mathbb{F}$, with a motive of mixed Tate type, that satisfies the Beilinson–Soulé vanishing property (see (BS) below), then there exists a well-defined Tannakian (in particular, $\mathbb{Q}$–linear) category $\mathrm{MTM}_{/\mathbb{F},\mathbb{Q}}(X)$ of mixed Tate motives over X [38, Theorem 3.6.9]. Moreover, Levine proved the existence of a short exact sequence relating the Tannakian groups of $\mathrm{MTM}_{/\mathbb{F},\mathbb{Q}}(\mathbb{F})$ and of $\mathrm{MTM}_{/\mathbb{F},\mathbb{Q}}(X)$ [38, Section 6.6]. This short exact sequence is a motivic avatar of the short exact sequence for étale fundamental groups relating $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ and $\pi_1^{\mathrm{et}}(X)$.

In a similar direction, Spitzweck showed in [49] that $\mathrm{DMT}_{/\mathbb{F},\mathbb{Z}}(X)$, the triangulated category of mixed Tate motives over X (ie before applying a t–structure and obtaining $\mathrm{MTM}(-)$), is the category $\mathrm{Perf}(G_X^\bullet)$ of perfect representations of an affine derived group scheme over $\mathbb{Z}$ provided that X is a smooth connected $\mathbb{F}$–scheme of finite type (where $\mathbb{F}$ is any field) satisfying a weaker Beilinson–Soulé vanishing property (see [49, Theorem 2.2]). Corollary 8.4 in [50] extends this construction to the case where X is smooth over S and satisfies the condition of Definition 2.8. Let us give the necessary definitions and results.

Definition 2.5 For a smooth scheme X over S, let $\mathrm{DMT}_{/S,\mathbb{Z}}(X)$ be the full triangulated subcategory of compact objects in the full triangulated subcategory of $\mathrm{DM}_{/S,\mathbb{Z}}(X)$ generated by Tate objects $\mathbb{Z}_X(n)$ for $n \in \mathbb{Z}$. When working with R coefficients for any ring R, we write $\mathrm{DMT}_{/S,R}(X)$.

Spitzweck shows in Corollaries 7.19 and 7.20 of [50] that his construction recovers motivic cohomology.

Proposition 2.6 [50, 7.19 and 7.20] *Let D be a Dedekind domain of mixed characteristic. For a smooth scheme X over $S = \mathrm{Spec}(D)$, we have*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(X_+), \mathrm{M}\mathbb{Z}_S(p)[k]) &\simeq \mathrm{Hom}_{\mathrm{DM}(S)}(M_S(X), \mathbb{Z}_S(p)[k]) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(X)}(\mathbb{Z}_X(0), \mathbb{Z}_X(p)[k]) \\ &\simeq \mathrm{H}^k_{\mathrm{mot}}(X,p),\end{aligned}$$

where $\mathrm{H}^k_{\mathrm{mot}}(X,p)$ denotes the motivic cohomology in the sense of Levine [36]. This recovers the higher Chow groups of X (see [8; 9; 35])

$$\mathrm{H}^k_{\mathrm{mot}}(X,p) = \mathrm{CH}^p(X, 2p-k). \tag{2}$$

Definition 2.7 (Beilinson–Soulé vanishing property) Let X be a smooth scheme over S. One says that X satisfies the Beilinson–Soulé vanishing property (BS) if and only if

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(X_+), \mathrm{M}\mathbb{Z}_S(p)[k]) = 0 \tag{BS}$$

for all $p \geqslant 0$ when $k < 0$ and for all $p > 0$ when $k = 0$.

Spitzweck often needs only the following weaker form of this property.

Definition 2.8 (weak Beilinson–Soulé vanishing property) Let X be a smooth scheme over S. One says that X satisfies the weak Beilinson–Soulé vanishing property (wBS) if and only if

$$\forall p > 0\ \exists N \in \mathbb{Z}\ \ \forall k < N \quad \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(X_+), \mathrm{M}\mathbb{Z}_S(p)[k]) = 0. \tag{wBS}$$

Remark 2.9 Theorem 7.10 in [50] implies that, for a smooth irreducible S–scheme X, where $S=\mathrm{Spec}(D)$ is in particular regular, we have

$$\mathrm{Hom}_{\mathrm{DM}(S)}(M_S(X),\mathbb{Z}(p)[k]) = \begin{cases} 0 & \text{for } p<0, \\ 0 & \text{for } p=0 \text{ and } k\neq 0, \\ \mathbb{Z} & \text{for } p=0 \text{ and } k=0, \\ \mathscr{O}_X(X)^* & \text{for } p=1 \text{ and } k=1, \\ \mathrm{Pic}(X) & \text{for } p=1 \text{ and } k=2. \end{cases}$$

Theorem 2.10 *$S=\mathrm{Spec}(\mathbb{Z})$ satisfies the Beilinson–Soulé vanishing property (BS).*

Proof The work of Borel [10] and Beilinson [7], together with the comparison between groups of K–theory and motivic cohomology through higher Chow groups, shows that $\mathrm{Spec}(\mathbb{Q})$ satisfies the (BS) property with $\mathbb{Q}$ coefficients. The difference between $\mathrm{Spec}(\mathbb{Z})$ and $\mathrm{Spec}(\mathbb{Q})$ is concentrated in degree 1 and weight 1. For each prime, the latter has an extra generator in degree 1 and weight 1. Thus $\mathrm{Spec}(\mathbb{Z})$ satisfies the (BS) property with $\mathbb{Q}$ coefficients. As reviewed in [30, Lemma 24], the (BS) property with $\mathbb{Z}$ coefficients is a consequence of the Beilinson–Soulé vanishing property with $\mathbb{Q}$ coefficients together with the Beilinson–Lichtenbaum conjecture [30, Conjecture 17], which is equivalent to the Bloch–Kato conjecture (see [51]). Thanks to the work of V Voevodsky this last conjecture is a now a theorem, proved in [55]. This concludes the proof. □

Remark 2.11 Let $\mathbb{F}$ be a number field and $\mathcal{P}$ a set of finite places of $\mathbb{F}$. Similar arguments show that the Beilinson–Soulé vanishing property also holds when S is the spectrum of the ring $\mathcal{O}_{\mathbb{F},\mathcal{P}}$ of $\mathcal{P}$–integers of $\mathbb{F}$, ie when

$$S=\mathrm{Spec}(\mathcal{O}_{\mathbb{F},\mathcal{P}}).$$

3 Geometry of the moduli spaces $\overline{\mathcal{M}}_{0,n}$

Let n be an integer greater than or equal to 3, and let $\mathcal{M}_{0,n}$ be the moduli space of curves of genus 0 with n marked points over $\mathrm{Spec}(\mathbb{Z})$. Let $\overline{\mathcal{M}}_{0,n}$ denote its Deligne–Mumford compactification [17; 33]. This notation will not change when working over S. In particular, we will consider the case where $S=\mathrm{Spec}(\mathbb{F})$, the spectrum of a number field. The integer $l=n-3$ is the dimension of $\mathcal{M}_{0,n}$, and the boundary $\partial\overline{\mathcal{M}}_{0,n}=\overline{\mathcal{M}}_{0,n}\setminus\mathcal{M}_{0,n}$ is a strictly normal crossing divisor whose irreducible components are isomorphic to $\overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2}$ with $n_1+n_2=n+2$. If S is a finite set with n elements, we write $\mathcal{M}_{0,S}$ and $\overline{\mathcal{M}}_{0,S}$ for $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$. Note that if $(\mathbb{P}^1)^{|S|}_*$ denotes the set of all n–tuples of distinct points $z_s \in \mathbb{P}^1$ for $s \in S$, then we have

$$\mathcal{M}_{0,S} = \mathrm{PSL}_2 \backslash (\mathbb{P}^1)^{|S|}_*,$$

where PSL_2 is the algebraic group of automorphisms of $\mathbb{P}^1$ acting by Möbius transformations.

Let $S=\{1,\dots,n\}$. Recall that for any subset S' of S, there exists a natural map

$$f_{S'}\colon \mathcal{M}_{0,S} \to \mathcal{M}_{0,S'}$$

obtained by forgetting the marked points of S which do not lie in S'. This map extends to a proper morphism

$$\overline{f}_{S'} = \overline{\mathcal{M}}_{0,S} \to \overline{\mathcal{M}}_{0,S'}.$$

3.1 On the boundary of $\overline{\mathcal{M}}_{0,n}$

Let D be a codimension-1 irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$ and D_0 its open stratum,

$$D_0 = D \backslash \Big(\bigcup_{D' \neq D} D \cap D' \Big),$$

where the union is over the codimension-1 irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$ different from D. We denote the union

$$\mathcal{M}_{0,n} \cup D_0 = \overline{\mathcal{M}}_{0,n} \backslash (\partial\overline{\mathcal{M}}_{0,n} \backslash D_0)$$

by $\mathcal{M}^D_{0,n}$, and the normal bundle of D_0 in $\mathcal{M}^D_{0,n}$ by N_{D_0}. The goal of this section is to prove that N_{D_0} is trivial.

Let S denote the set $\{1,\dots,n\}$. The moduli space $\overline{\mathcal{M}}_{0,S}$ admits a stratification (see [13]) in which the codimension-0 open stratum is simply $\mathcal{M}_{0,S}$. A point in an open stratum of codimension k represents a stable curve with n marked points and k nodes. Since the genus is 0, this is a tree with $k+1$ branches, each represented by a $\mathbb{P}^1$, such that the n marked points are distributed on the $k+1$ branches in such a way that each $\mathbb{P}^1$ has at least three special points (marked points and intersection points). Moving inside a stratum makes the marked points move within their branch, but they cannot move from one branch to another. A point in a codimension-1 open stratum represents two intersecting copies of $\mathbb{P}^1$, with the marked points distributed over the two. Thus D gives a partition of S into two subsets, which completely determines the open stratum of D. In other words, D is determined by a subset T_D of S such that T_D and its complement T_D^C each contain at least 2 elements. The stratum D is isomorphic to $\overline{\mathcal{M}}_{0,T_D\cup\{e\}}\times\overline{\mathcal{M}}_{0,T_D^c\cup\{e\}}$ with e not in S.

Fix three elements i_0, i_1 and i_2 in S. The correspondence between codimension-1 irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$ and partitions $J\sqcup J^c$ of S can be made 1-to-1 by imposing the condition $|J\cap\{i_0,i_1,i_2\}|\leqslant 1$. We denote the stratum of $\partial\overline{\mathcal{M}}_{0,n}$ corresponding to such a J by D^J.

We use the following notation. When emphasizing the indexing set, we write D_S^J instead of D^J. The open stratum of D_S^J is

$$D_{0,S}^J=D_S^J\setminus\Bigl(\bigcup_{D'\neq D_S^J}D_S^J\cap D'\Bigr),$$

where the union is over the codimension-1 irreducible components D' of $\partial\overline{\mathcal{M}}_{0,S}$ that are different from D_S^J. Following the above notation, the union $\mathcal{M}_{0,S}\cup D_{0,S}^J$ is denoted by $\mathcal{M}_{0,S}^{D^J}$. Note that

$$\mathcal{M}_{0,S}^{D^J}=\overline{\mathcal{M}}_{0,S}\setminus\Bigl(\bigcup_{D'\neq D_S^J}D_S^J\cap D'\Bigr),$$

where the union is over the same components as above.

Assume that $n\geqslant 5$; thus we can assume that T_D^c contains at least 3 elements. Let $I=\{i_0,i_1,i_2,i_3\}$ be a subset of S such that i_0 is in T and i_1, i_2 and i_3 are elements of T^c. Let S_0 be $S\setminus\{i_3\}$. Consider the morphism

$$\pi_{S_0\times I}\colon\overline{\mathcal{M}}_{0,S}\xrightarrow{\bar f_{S_0}\times\bar f_I}\overline{\mathcal{M}}_{0,S_0}\times\overline{\mathcal{M}}_{0,I}.$$

Lemma 3.1 *Let $S=\{1,\dots,n\}$, $T\subset S$, $I=i_0,i_1,i_2,i_3$ and S_0 be as above. Then the image of D_0^T by $\pi_{S_0\times I}$ satisfies*

$$\pi_{S_0\times I}(D_0^T)\subset D_{0,S_0}^T\times\mathcal{M}_{0,4}.$$

Proof Let P be a point in D_0^T. As P lies in the open stratum, P represents a tree of $\mathbb{P}^1$ having only two branches, with the n marked points distributed over the branches according to the partition $T\sqcup T^c$ of S. The forgetful morphisms at worst decrease the number of branches. Hence $\bar f_{S_0}(P)$ has at most 2 branches; thus it is at worst in the open stratum of a codimension-1 irreducible component of $\overline{\mathcal{M}}_{0,S_0}$. On the one hand, since $|T| \geqslant 2$ and $i_3 \notin T$, $\overline{f}_{S_0}(P)$ is in $D^T_{S_0}$, and thus is in D^T_{0,S_0}. On the other hand, $T \cap \{i_0, i_1, i_2, i_3\} = \{i_0\}$. Therefore the tree of $\mathbb{P}^1$ corresponding to P cannot remain stable under $\overline{f}_I$. Thus $\overline{f}_I(P)$ represents a single $\mathbb{P}^1$ with n marked points, forcing it to be in $\mathcal{M}_{0,4}$. □

Proposition 3.2 *Let $n \geq 4$, and let D be a codimension-1 irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$. Then the normal bundle N_{D_0} is trivial.*

Proof The proof proceeds by induction on n. The base case $n = 4$ is clear.

Assuming that $n \geqslant 5$, write $S = \{1, \dots, n\}$ as above, and let D be D^T for some $T \subset S$. The cardinality of S being at least 5, we can assume that $|T| \geqslant 2$ and that $|T^c| \geqslant 3$. In order to have a 1-to-1 correspondence between codimension-1 irreducible components of $\partial\overline{\mathcal{M}}_{0,S}$ and partitions of S as described above, we choose i_0 in T and i_1 and i_2 in T^c. Since T^c has at least three elements, this set contains a third element i_3, different from i_1 and i_2.

A result of Keel [32, Lemma 1] shows that the morphism

$$\pi_{S_0 \times I} \colon \overline{\mathcal{M}}_{0,S} \xrightarrow{\overline{f}_{S_0} \times \overline{f}_I} \overline{\mathcal{M}}_{0,S_0} \times \overline{\mathcal{M}}_{0,I}$$

is given by a succession of blow-ups along regular smooth codimension-2 subschemes. The exceptional divisors of these blow-ups are codimension-1 irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$ of the form

$$D^{J \cup \{i_3\}} \quad \text{with} \quad J \subset S_0, \quad |J| \geqslant 2 \quad \text{and} \quad |J \cap \{i_0, i_1, i_2\}| \leqslant 1.$$

In particular, $\pi_{S_0 \times I}$ is an isomorphism outside the exceptional divisors. Hence, the image of $\mathcal{M}^{D^T}_{0,S}$ by $\pi_{S_0 \times I}$ is open in $\overline{\mathcal{M}}_{0,S_0} \times \overline{\mathcal{M}}_{0,4}$.

As T is also a subset of S_0, let $D^T_{S_0}$ be the corresponding codimension-1 component of $\partial\overline{\mathcal{M}}_{0,S_0}$ and D^T_{0,S_0} its open stratum. Lemma 3.1 above shows that

$$\pi_{S_0 \times I}(D^T_0) \subset D^T_{0,S_0} \times \mathcal{M}_{0,4}.$$

Thus, the image of $\pi_{S_0 \times I}(\mathcal{M}^{D^T}_{0,S})$ is open in $\overline{\mathcal{M}}_{0,S_0} \times \overline{\mathcal{M}}_{0,4}$ and is included in $\mathcal{M}^{D^T}_{0,S_0} \times \overline{\mathcal{M}}_{0,4}$, which is also open in $\overline{\mathcal{M}}_{0,S_0} \times \overline{\mathcal{M}}_{0,4}$. As a consequence, $\pi_{S_0 \times I}(\mathcal{M}^{D^T}_{0,S})$ is open in $\mathcal{M}^{D^T}_{0,S_0} \times \overline{\mathcal{M}}_{0,4}$.

Since $\pi_{S_0 \times I}$ is an isomorphism away from the exceptional divisors, the triviality of $N_{D^T_0}$ in $\mathcal{M}^{D^T}_{0,S}$ is equivalent to the triviality of the normal bundle of $\pi_{S_0 \times I}(D^T_0)$ in

$$\pi_{S_0 \times I}(\mathcal{M}^{D^T}_{0,S}).$$

But by the above discussion, this is a consequence of the triviality of $N_{D^T_{0,S_0}}$ in

$$\mathcal{M}_{0,S_0}^{D^T_{0,S_0}}.$$

The proposition follows by induction, the base case $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1$ with $\partial\overline{\mathcal{M}}_{0,4} \simeq \{0,1,\infty\}$ being trivial. $\square$

3.2 The motive of $\overline{\mathcal{M}}_{0,n}$

In this subsection, let $S = \mathrm{Spec}(\mathbb{Z})$. The main goal here is to prove that the motive $M_S(\overline{\mathcal{M}}_{0,n})$

- is a (finite) direct sum of motives of the type $\mathbb{Z}_S(p)[2p]$ with $p \geqslant 0$;
- satisfies the (BS) property (see Theorem 3.5).

The key ingredients are the decomposition of $M_S(\overline{\mathcal{M}}_{0,n})$ into a direct sum of Tate motives using the blow-up formula, and the Beilinson–Soulé property for the base scheme $S = \mathrm{Spec}(\mathbb{Z})$.

Definition 3.3 Let X be a smooth scheme over S. We say that X is *effective of Tate type* $(p,2p)$ or simply of type ET when $M_S(X)$ is a finite direct sum of motives $\mathbb{Z}_S(p_i)[2p_i]$ with $p_i \geqslant 0$.

A direct application of the blow-up formula given in Proposition 2.4 gives the following.

Lemma 3.4 *Let X be a smooth scheme over S and Z a smooth closed subscheme of X. We assume that both X and Z are of type* ET. *Then the blow-up $\mathrm{Bl}_Z(X)$ of X with center Z is also of type* ET.

Theorem 3.5 *Let n be an integer greater or equal to 3. The motive $M_S(\overline{\mathcal{M}}_{0,n})$ is isomorphic to*

$$M_S(\overline{\mathcal{M}}_{0,n}) = \bigoplus_i \mathbb{Z}_S(p_i)[2p_i],$$

where the direct sum is finite and each $p_i \geq 0$. Moreover, $M_S(\overline{\mathcal{M}}_{0,n})$ satisfies (BS)*; that is, for any pair of integers p and k such that $p \geqslant 0$ and $k<0$ or $p>0$ and $k=0$, we have*

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty(\overline{\mathcal{M}}_{0,n+}), \mathrm{M}\mathbb{Z}_S(p)[k]) = \mathrm{Hom}_{\mathrm{DM}(S)}(M_S(\overline{\mathcal{M}}_{0,n}), \mathbb{Z}_S(p)[k]) = 0.$$

Proof Note that the second part of the theorem follows directly from the first part using Lemma 3.6 below.

We do the proof of the first part by induction on n.

Note that $\overline{\mathcal{M}}_{0,3}$ is isomorphic to $S=\mathrm{Spec}(\mathbb{Z})$ and $\overline{\mathcal{M}}_{0,4}$ is simply $\mathbb{P}^1_S$. Hence, using the (BS) property for $S=\mathrm{Spec}(\mathbb{Z})$ (Theorem 2.10) and the projective bundle formula, we see that $\overline{\mathcal{M}}_{0,3}$ and $\overline{\mathcal{M}}_{0,4}$ are of type ET.

Now fix $n\geqslant 5$. Let I_n be the set $\{1,\dots,n\}$ (denoted by S in the previous section). In [32, Theorems 1 and 2], Keel proved that the morphism

$$\overline{\mathcal{M}}_{0,I_n}\to\overline{\mathcal{M}}_{0,I_{n-1}}\times\overline{\mathcal{M}}_{0,I_4}$$

is a sequence of blow-ups

$$\overline{\mathcal{M}}_{0,I_n}\xrightarrow{\sim}B_{n-3}\to\cdots\to B_k\to\cdots\to B_1=\overline{\mathcal{M}}_{0,I_{n-1}}\times\overline{\mathcal{M}}_{0,I_4},$$

where $B_{k+1}\to B_k$ is the blow-up along disjoint centers isomorphic to some irreducible components of $\partial\overline{\mathcal{M}}_{0,I_{n-1}}$.

As $B_1\simeq\overline{\mathcal{M}}_{0,I_{n-1}}\times\overline{\mathcal{M}}_{0,I_4}$, the induction hypothesis and the Künneth formula show that B_1 is of type ET. But any irreducible component of $\partial\overline{\mathcal{M}}_{0,I_{n-1}}$ is isomorphic to $\overline{\mathcal{M}}_{0,n_1}\times\overline{\mathcal{M}}_{0,n_2}$ for some n_1 and n_2 satisfying $n_1+n_2=n-1+2=n+1$; thus the Künneth formula and the induction hypothesis show that the centers of the blow-up $B_{k+1}\to B_k$ are also of type ET.

Now an argument by induction on k together with the blow-up formula suffice to prove that B_k is of type ET for all k. Hence $\overline{\mathcal{M}}_{0,I_n}\simeq B_{n-3}$ is also of type ET. □

Note that the above proof is similar to the proof of Proposition 4.4 in [43]. The proof in [43] uses a cohomological setting, which explains the minus signs in the shifts and twists in [43]. One could also bypass part of Keel's result in [32] by observing that the map

$$\overline{\mathcal{M}}_{0,I_n}\to\prod_{i=4}^{n}\overline{\mathcal{M}}_{0,\{1,2,3,i\}}$$

collapses all irreducible components of the form D^T with $|T\cap\{1,2,3\}|\leqslant 1$ and $|T|\geqslant 3$. Normalizing the marked points z_1, z_2 and z_3 to 1, ∞ and 0, respectively, we see that $\overline{\mathcal{M}}_{0,I_n}$ is the result of blowing up $(\mathbb{P}^1)^{n-3}$ along the poset given by all the intersections of the divisors $t_i=t_j$ and $t_i=\varepsilon$ with $i\neq j$ and $\varepsilon=0,1,\infty$. This is exactly the situation of [43, Proposition 4.4]. To obtain the above theorem, that proof only needs to be modified each time it uses the blow-up formula in order to take the ET type property into account.

Lemma 3.6 *Let X be a smooth scheme over S. Assume that X is effective of Tate type $(p,2p)$. Then $M_S(X)$ satisfies the Beilinson–Soulé property (BS).*

Proof By definition, $M_S(X)$ is a direct sum of Tate motives of the form $\mathbb{Z}_S(i)[2i]$ for $i \geqslant 0$. Using Proposition 2.6, in order to show that $M_S(X)$ satisfies (BS), it is enough to show that for any $i \geqslant 0$ and pair (p,k) such that either $p \geqslant 0$ with $k<0$ or $p>0$ with $k=0$, we have

$$\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Z}_S(i)[2i], \mathbb{Z}(p)[k]) = 0.$$

However, the above Hom group is simply

$$\mathrm{Hom}_{\mathrm{DM}(S)}(\mathbb{Z}_S(0), \mathbb{Z}(p-i)[k-2i]).$$

If $p-i \leq 0$, one can use Remark 2.9. When $p-i>0$, the result follows from the (BS) property of $S=\mathrm{Spec}(\mathbb{Z})$ given in Theorem 2.10 because $k-2i<0$. □

Corollary 3.7 *Let $n \geqslant 4$ and let D be an irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$. Then D is of type* ET *and satisfies the (BS) property. Moreover, if $\mathcal{S}$ is a nonempty intersection of k irreducible codimension-1 components of $\partial\overline{\mathcal{M}}_{0,n}$, then $\mathcal{S}$ is of type* ET *and satisfies the (BS) property.*

Proof The closed stratum $\mathcal{S}$ is isomorphic to a product

$$\overline{\mathcal{M}}_{0,l_1+3} \times \overline{\mathcal{M}}_{0,l_2+3} \times \cdots \times \overline{\mathcal{M}}_{0,l_{k+1}+3}$$

with $l_1+l_2+\cdots+l_{k+1}=n-3-k$ (see [13]). The corollary follows from the Künneth formula and Theorem 3.5. □

3.3 The motive of $\mathcal{M}_{0,n}$

In this section, we prove that the motives of the open moduli spaces of curves $\mathcal{M}_{0,n}$ are mixed Tate motives satisfying the (BS) property.

Let us first recall some facts about the boundary of $\overline{\mathcal{M}}_{0,n}$ and its stratified structure. We already recalled that $\partial\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ is a normal crossing divisor (see [33, Theorem. 2.7]). Let $\mathcal{S}$ be the intersection of k irreducible codimension-1 components of $\partial\overline{\mathcal{M}}_{0,n}$. Then $\mathcal{S}$ is isomorphic to the product of $k+1$ moduli spaces of curves

$$\mathcal{S} \simeq \overline{\mathcal{M}}_{0,n_1} \times \cdots \times \overline{\mathcal{M}}_{0,n_{k+1}}$$

such that $\sum_{i=1}^{k+1}(n_i-3) = n-3-k$.

Writing $\partial\overline{\mathcal{M}}_{0,n}$ as the union of its irreducible components

$$\partial\overline{\mathcal{M}}_{0,n}=\bigcup_{i=1}^{N} D_i,$$

we may assume that $\mathcal{S}=\bigcap_{i=1}^{k} D_i$. The open stratum $\mathring{\mathcal{S}}$ is defined as

$$\mathring{\mathcal{S}}=\mathcal{S}\setminus\Bigl(S\cap\Bigl(\bigcup_{i=k+1}^{N} D_i\Bigr)\Bigr)$$

and is isomorphic (see [13]) to

$$\mathring{\mathcal{S}}\simeq\mathcal{M}_{0,n_1}\times\cdots\times\mathcal{M}_{0,n_{k+1}}.$$

Theorem 3.8 *Let n be an integer greater than or equal to 3. The motive $M_S(\mathcal{M}_{0,n})$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$, the triangulated category of mixed Tate motives. Moreover, the motive $M_S(\mathcal{M}_{0,n})$ satisfies (BS), ie we have*

$$\mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^{\infty}(\mathcal{M}_{0,n+}),\mathrm{M}\mathbb{Z}_S(p)[k])=\mathrm{Hom}_{\mathrm{DM}(S)}(M_S(\mathcal{M}_{0,n}),\mathbb{Z}_S(p)[k])=0$$

for all $p\geqslant 0$ when $k<0$ and for all $p>0$ when $k=0$.

This statement holds in a more general situation. Let X_0 be a smooth scheme over S whose motive $M_S(X_0)$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Let $D=\bigcup_{i=1}^{l} Z_i$ be a strict normal crossing divisor of X_0. Assume that any irreducible component of any intersection of the Z_i has a motive in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Let $U=X_0\setminus D$.

Theorem 3.9 *$M_S(U)$ is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS).*

Proof The proof is a double induction on the dimension n of X_0 and l.

Let $Z'=\bigcup_{i=1}^{l-1} Z_i$ and $X=X_0\setminus Z'$. The intersection $Z=Z_l\cap X$ is of codimension $d=1$. The Gysin triangle (1) ensures that $M_S(U)$ sits in the distinguished triangle

$$\to M_S(U)\to M_S(X)\to M_S(Z)(d)[2d]\to M_S(U)[1]\to\cdots.$$

Applying the $\mathrm{Hom}_{\mathrm{DM}(S)}$ functor, we obtain an exact sequence

$$\begin{aligned}\mathrm{H}^k_{\mathrm{mot}}(X,p)\to\mathrm{H}^k_{\mathrm{mot}}(U,p)&=\mathrm{Hom}_{\mathrm{DM}(S)}(M(U)[1],\mathbb{Z}_S(p)[k+1])\\&\to\mathrm{Hom}_{\mathrm{DM}(S)}(M_S(Z)(d)[2d],\mathbb{Z}_S(p)[k+1])=\mathrm{H}^{k+1-2d}_{\mathrm{mot}}(Z,p-d).\end{aligned}$$

When $n=1$ and $l=1$, we have $X=X_0$ and $Z=Z_1=D$. Hence both are in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfy (BS), which implies the theorem for U.

When $n > 1$ or $k > 1$, we see by induction on l that X is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Then $Z = Z_l \cap X$ is equal to

$$Z = Z_l \setminus \left(\bigcup_{i=1}^{l-1} Z_i \cap Z_l \right).$$

By induction on n, we see that Z is in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies (BS). Thus the above exact sequence, induced by the Gysin triangle, implies the theorem for U. □

Proof of Theorem 3.8 We apply Theorem 3.9 to the case where $X = \overline{\mathcal{M}}_{0,n}$ and $D = \partial\overline{\mathcal{M}}_{0,n}$. In this case, Theorem 3.5 and Corollary 3.7 ensure that the hypotheses are satisfied. Note that in this case, Z is an open codimension-1 stratum of the compactification, hence it is isomorphic to a product of open moduli spaces of curves. One could perform the above induction directly for the moduli spaces of curves. □

Remark 3.10 • Any strict normal crossing divisor D of X_0 induces a stratification of X_0 where the strata are given by irreducible components of the intersection of the Z_i. Let $U = X \setminus \left(\bigcup_{i\in I} \overline{\mathscr{S}}_i\right)$ be the complement of a union of closed strata defined by the divisor D. We assume that, in this description of U, I is minimal and the strata $\overline{\mathscr{S}}_i$ have maximal dimension d_i. This removes some ambiguities in the choices of the strata and some possible redundancy. Theorem 3.9 remains valid when the strata $\overline{\mathscr{S}}_i$ have a motive in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfy (BS).

In this case, the proof goes by induction on the dimension $d = \max(d_i)$ of X_0 and the number k of strata of dimension d. As above, the proof relies on the Gysin triangle and on the long exact sequence for $\mathrm{H}^k_{\mathrm{mot}}$. An important point is that closed strata of dimension 0 are disjoint and that open strata (ie closed strata minus closed strata of lower dimension) are disjoint.

• The duality and Gysin morphism given by Déglise in [15, Section 5] give an "open relative motive" $M_S(X \setminus A; B)$, where X is smooth projective and A and B are two strict normal crossing divisors sharing no common irreducible components. This is explained by Levine in Part I, Chapter IV, Section 2.3 of [36].

• F Brown, in [11, Section 2.2], uses a partial compactification $\mathcal{M}^{\delta}_{o,n}$ of $\mathcal{M}_{0,n}$ attached to a dihedral structure δ on $\{1,\dots,n\}$. Theorem 3.9 also shows that the motive $M_S(\mathcal{M}^{\delta}_{0,n})$ is also in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$ and satisfies the (BS) property.

4 A motivic Grothendieck–Teichmüller group

In this section, we define an integral motivic Grothendieck–Teichmüller space

$$\mathrm{GT}^h_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Z})$$

over $\mathbb{Z}$. For any $n \geqslant 3$, Spitzweck gives an equivalence between $\mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n})$ and the perfect representations of an affine derived group scheme $G^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$. These groups sit as middle terms in short exact sequences relating $G^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$, $G^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,3}} = G^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathrm{Spec}(\mathbb{Z})}$ and $K^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$, the last of which represents a "geometric part". These exact sequences are compatible with the natural morphisms in the tower of the $\mathcal{M}_{0,n}$ (namely the morphisms forgetting marked points and embedding codimension-1 components).

Following Grothendieck's idea developed in [24, Section 2], $\mathrm{GT}^h_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Z})$ is then defined as the automorphism space of the tower given by the $K^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$.

In the following section, we will give a nonderived version using rational coefficients, as the Tannakian formalism is available in that context; indeed, in the rational context, the Tannakian formalism associates groups $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},\mathcal{M}_{0,n}}$ to the categories of mixed Tate motives over $\mathcal{M}_{0,n}$. In this nonderived setting, the motivic Grothendieck–Teichmüller group is defined as the automorphism group of the tower of the geometric parts $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},\mathcal{M}_{0,n}}$. Furthermore, working with rational coefficients and over the spectrum of a number field, Levine [38] showed that the group $K_{/\mathrm{Spec}(\mathbb{Q}),\mathbb{Q},\mathcal{M}_{0,n}}$ is identified with Deligne–Goncharov motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathcal{M}_{0,n})$ [16], hence the description of $K^{\bullet}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$ as a "geometric part".

4.1 Tangential base points and normal bundle

In this section we describe the final requirements for developing a motivic Grothendieck–Teichmüller construction:

A natural functor $\mathrm{DMT}_{/S,\mathbb{Z}}(X \setminus Z) \to \mathrm{DMT}_{/S,\mathbb{Z}}(N_Z^0) \to \mathrm{DMT}_{/S,\mathbb{Z}}(Z)$, where N_Z^0 denotes the normal bundle of Z in X minus its zero section This functor allows us to obtain a derived group morphism

$$G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,k}\times\mathcal{M}_{0,l}} \to G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z},\mathcal{M}_{0,n}}$$

induced by the inclusion of $D \simeq \overline{\mathcal{M}}_{0,k} \times \overline{\mathcal{M}}_{0,l}$, an irreducible component of $\overline{\mathcal{M}}_{0,n}$, into $\overline{\mathcal{M}}_{0,n}$. This is a motivic version of the morphism between fundamental groups presented in [13] in the topological context, or in [40] in the étale case.

Motivic tangential base points or motivic base points at infinity In general, base points provide an augmentation to the differential graded (E_∞) algebras underlying the description of mixed Tate categories as comodule categories. In particular, this is the case in [38], where a relative bar construction makes it possible to identify $K_{/\mathrm{Spec}(\mathbb{Q}),\mathbb{Q},\mathcal{M}_{0,4}}$ with the Deligne–Goncharov fundamental group. The authors also give sections in the (derived) group setting to the morphism induced by $p^*\colon \mathrm{DMT}_{/S,\mathbb{Z}}(S) \to \mathrm{DMT}_{/S,\mathbb{Z}}(X)$ and associated to the structural morphism $p\colon X \to S$. Tangential base points are used to compensate the lack of S–points (such as in the case of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ over $S = \mathrm{Spec}(\mathbb{Z})$) and to preserve symmetries.

Unless stated otherwise, in Section 4 we use the ground scheme $S = \mathrm{Spec}(\mathbb{Z})$ with $\mathbb{Z}$ coefficients. Let n be an integer greater than or equal to 4. Let D be an irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$, and D_0 its open stratum (see Section 3.1). The open stratum is isomorphic to

$$D_0 \sim \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$$

with $n_1 + n_2 = n + 2$.

Proposition 4.1 *There is a natural functor*

$$\mathcal{L}^{\mathrm{DM}}_{D,\mathcal{M}_{0,n}}\colon \mathrm{DM}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DM}_{/S,\mathbb{Z}}(D_0)$$

sending Tate objects to Tate objects and hence inducing a natural functor

$$\mathcal{L}_{D,\mathcal{M}_{0,n}}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DMT}_{/S,\mathbb{Z}}(D_0).$$

Moreover, its composition with the "structural functor"

$$p^*\colon \mathrm{DMT}_{/S,\mathbb{Z}}(S) \to \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n})$$

is isomorphic to

$$p^*_{D_0}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(S) \to \mathrm{DMT}_{/S,\mathbb{Z}}(D_0),$$

where $p^*_{D_0}\colon D_0 \to S$ is the structural morphism of the open stratum D_0.

This proposition is a consequence of Proposition 15.19 in [46], which we discuss below. Let X a be a smooth scheme over S and $Z \xrightarrow{i} X$ a regular closed embedding such that Z is smooth over S. In our application, we can also assume that Z is a divisor of X. Let X^0 be the open complement and N^0_Z the normal bundle of Z in X with zero section removed. In [46], Spitzweck defines, as a consequence of his Proposition 15.19, a natural functor

$$\mathcal{L}^{\mathrm{DM}}_{X,Z}\colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DM}_{/S,\mathbb{Z}}(N^0_Z).$$

We apply this functor to the situation $X = \mathcal{M}_{0,n} \cup D_0$ and compose it with the pullback functor induced by an everywhere-nonzero S–section $\sigma\colon D_0 \to N^0_{D_0}$ given by Proposition 3.2. In order to show that it sends Tate objects to Tate objects, we need to review the construction of the functor $\mathcal{L}^{\mathrm{DM}}_{X,Z}$. The geometric part of this construction relies on the (affine) deformation to the normal cone.

The deformation to the normal cone is a key geometric construction needed in the studies of Gysin maps. It was explicitly used and formalized by W Fulton [22]. It plays an important role in defining *specialization maps*, for example in the microlocal theory of sheaves [31], and it was developed and generalized in [41; 28] to higher deformations in order to study the A_∞ structure of cycle modules.

We recall the construction of the deformation to the normal cone. Let X, Z and X^0 be smooth schemes over S:

$$\begin{tikzcd} Z \arrow[r, "i_Z"] \arrow[dr, "p_Z"'] & X \arrow[d, "p_X"] & X^0 = X \setminus Z \arrow[l, "j_{X^0}"'] \arrow[dl, "p_{X^0}"] \\ & S & \end{tikzcd} \tag{3}$$

where i_Z is a regular closed embedding. We consider the blow-up $\mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1_S$ of $X \times \mathbb{A}^1_S$ along $Z \times \{0\}$,

$$\pi\colon \mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1_S \to X \times \mathbb{A}^1_S.$$

Since everything is defined over the base scheme S, we drop the subscript S (as in $\mathbb{A}^1_S$) whenever this does not lead to confusion. The preimage of $X \times \mathbb{G}_m$ under π is, by definition, isomorphic to $X \times \mathbb{G}_m$. The preimage $X \times \{0\}$ has two components, one being $\mathrm{Bl}_Z X$ while the other is $\mathbb{P}(N_Z \oplus \mathcal{O}_Z)$, where N_Z denotes the normal bundle of Z in X. These two branches intersect each other at $\mathbb{P}(N_Z)$, which is the exceptional divisor of $\mathrm{Bl}_Z X$. The deformation of Z to the normal cone is defined as

$$D(X,Z) = (\mathrm{Bl}_{Z\times\{0\}} X \times \mathbb{A}^1) \setminus \mathrm{Bl}_Z X.$$

In terms of spectra, if $\mathcal{J}_Z$ denotes the sheaf of ideals defining Z, the deformation $D(X,Z)$ is given by $\mathrm{Spec}(\mathcal{A}_{X,Z})$, where

$$\mathcal{A}_{X,Z} = \bigoplus_{n\in\mathbb{Z}} \mathcal{J}_Z^n t^{-n} \subset \mathcal{O}_X[t,t^{-1}] \tag{4}$$

with the convention that $\mathcal{J}_Z^n = \mathcal{O}_X$ for all $n \leqslant 0$. The geometric situation is described by the following diagram:

$$
\begin{tikzcd}
Z \arrow[d, "\sim"'] \arrow[r, bend left, dotted, "s_0"] & N_Z \arrow[l, "p_{N_Z}"] \arrow[r, "i_{N_Z}"] \arrow[dl, "p_{N_Z}"] \arrow[dd, "f|_{\{0\}}"] & D(X,Z) \arrow[dd, "f"] \arrow[dr, "\pi|_{D(X,Z)}"] & & X \times \mathbb{G}_m \arrow[lll, "j_{X \times \mathbb{G}_m}"'] \arrow[r, "p_{\to X}"] \arrow[dd, "f|_{\mathbb{G}_m}"] \arrow[dr, "="] & X \\
Z \times \{0\} \arrow[dr] & & & X \times \mathbb{A}^1 \arrow[dl] & & X \times \mathbb{G}_m \arrow[u, "p_{\to X}"'] \arrow[dl, "p_{\to \mathbb{G}_m}"] \\
& \{0\} \arrow[r, "i_0"] & \mathbb{A}^1 & & \mathbb{G}_m \arrow[ll, "j_{\mathbb{G}_m}"'] &
\end{tikzcd}
$$

In the above diagram, the two "big rectangles" are Cartesian. The map f is smooth because Z itself is smooth over S; hence the maps $f|_{\{0\}}$ and $f|_{\mathbb{G}_m}$ are also smooth. This was observed by J Ayoub in [2] at the beginning of Section 1.6.1, after diagram (1.37).

The open deformation $D^0(X,Z)$ is obtained by removing the strict transform of $Z \times \mathbb{A}^1$ from $D(X,Z)$. This strict transform is the closure of $Z \times \mathbb{G}_m$ in $D(X,Z)$. The properties of $D^0(X,Z)$ are summarized in the Cartesian diagram

$$
\begin{tikzcd}
Z & N_Z^0 \arrow[l, "p^0_{N_Z}"] \arrow[r, "i_{N_Z^0}"] \arrow[d, "f|_{\{0\}}"] & D^0(X,Z) \arrow[d, "f"] & X^0 \times \mathbb{G}_m \arrow[l, "j_{X^0 \times \mathbb{G}_m}"] \arrow[r, "p_{\to X^0}"] \arrow[d, "p_{\to \mathbb{G}_m}"] & X^0 \\
& \{0\} \arrow[r, "i_{\{0\}}"] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l, "j_{\mathbb{G}_m}"] &
\end{tikzcd}
\tag{5}
$$

which is "an open immersion" of the previous one with closed complement given over $\{0\}$ and $\mathbb{G}_m$ by $s_0(Z)$ and Z, respectively.

From this geometric situation, Spitzweck obtained in [46, Proposition 15.19] an isomorphism

$$
i_Z^* j_{X^0 *} \mathrm{M}\mathbb{Z}_{X_0} \simeq p^0_{N_Z *} \mathrm{M}\mathbb{Z}_{N_Z^0} \tag{6}
$$

by comparing the inclusions of $X \xrightarrow{\sim} X \times \{0\}$ and $X \xrightarrow{\sim} \times X \times \{1\}$ in the strict transform of $X \times \mathbb{A}^1$ in $D(X,Y)$, and similarly for Z.

Spitzweck next used one of his main results [46, Corollary 15.14] to identify the homotopy category of modules over ${p^0_{N_Z}}_*\mathrm{M}\mathbb{Z}_{N^0_Z}$ with the full triangulated subcategory of $\mathrm{DM}_{/S,\mathbb{Z}}(N^0_Z)$ generated by homotopy colimits of the pullbacks by $p^0_{N_Z}$ of objects from $\mathrm{DM}_{/S,\mathbb{Z}}(Z)$. The composition of $\mathrm{DM}_{/S,\mathbb{Z}}(X^0) \to \mathcal{H}(i_Z^* {j_{X^0}}_*\text{–Mod})$ with the previous identification and the one given by (6) gives a functor

$$\mathcal{L}^{\mathrm{DM}}_{X,Z}\colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DM}_{/S,\mathbb{Z}}(N^0_Z).$$

Let $p_{X^0}\colon X^0 \to S$ be the structural morphism. The composition of

$$p^*_{X^0}\colon \mathrm{DM}_{/S,\mathbb{Z}}(S) \to \mathrm{DM}_{/S,\mathbb{Z}}(X^0)$$

with $\mathcal{L}_{X,Z}$ is isomorphic to the map $\mathrm{DM}_{/S,\mathbb{Z}}(S) \to \mathrm{DM}_{/S,\mathbb{Z}}(N^0_Z)$ induced by the structural morphism of N^0_Z, for the following reasons:

- The compatibility with objects lifted from the base is given at the end of [46, Corollary 15.14].
- The condition of Corollary 15.14, requiring that $M \otimes ({p^0_{N_Z}}_*\mathrm{M}\mathbb{Z}_{N^0_Z})$ is isomorphic to ${p^0_{N_Z}}_* \circ p^{0*}_{N_Z}(M)$ for any M in $\mathrm{DM}_{\mathbb{Z}}(Z)$, is satisfied.
- These same two properties ensure that $\mathcal{L}_{X,Z}$ sends Tate objects to Tate objects.

This material was also developed in [47], with some further details.

Proof of Proposition 4.1 We apply the above discussion to the case $X = \mathcal{M}_{0,n} \cup D_0$, $X^0 = \mathcal{M}_{0,n}$ and $Z = D_0$. Then we compose this functor by the nonzero section of $p^0_{N_Z}\colon N^0_Z \to Z$ given by Proposition 3.2, to obtain

$$\mathcal{L}_{D,\mathcal{M}_{0,n}}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DMT}_{/S,\mathbb{Z}}(D_0). \qquad \square$$

Remark 4.2 In this remark, we develop another approach to limit motives: the nearby cycle functor [3]. This method was used by Ayoub in [4; 5] in the case of a curve over a field. The following construction agrees with Spitzweck's; see [47; 46]. We explain below how deforming to the normal cone allows us to obtain a limit motive functor from Ayoub's nearby cycle functor. For the remainder of this remark, we assume that the hypotheses of Ayoub's formalism are satisfied; that is, we assume that the functor $\mathrm{DM}_{/S,R}(-)$ associating to $X \in \mathrm{Sm}_S$ the triangulated category $\mathrm{DM}_{/S,R}(X)$ (using R coefficients) comes from a monoidal stable homotopic algebraic derivator on diagrams of quasiprojective schemes over S. This assumption applies directly to our situation when working with rational coefficients ($R = \mathbb{Q}$) and Beilinson's E_∞–ring spectrum $\mathrm{M}\mathbb{Q}_X$ as in Section 5.1.

Let us again give diagram (5):

$$\begin{tikzcd}
N_Z^0 \arrow[r, "i_{N_Z^0}"] \arrow[d, "f|_{\{0\}}"] & D^0(X,Z) \arrow[d, "f"] & X^0 \times \mathbb{G}_m \arrow[l, "j_{X^0\times\mathbb{G}_m}"'] \arrow[r, "p_{\to X^0}"] \arrow[d, "p_{\to\mathbb{G}_m}"] & X^0 \\
\{0\} \arrow[r, "i_{\{0\}}"] \arrow[dr, "="'] & \mathbb{A}^1 \arrow[d] & \mathbb{G}_m \arrow[l, "j_{\mathbb{G}_m}"'] \arrow[dl, "q_{\mathbb{G}_m}"] & \\
& S & &
\end{tikzcd}$$

which corresponds to the situation of a "specialization functor" over the base $\mathbb{A}^1$ as described by Ayoub [3, Sections 3.1, 3.4 and 3.5]. The nearby cycles functor from Ayoub gives us

$$\Psi_f \colon \mathrm{DM}_{/S,R}(X^0 \times \mathbb{G}_m) \to \mathrm{DM}_{/S,R}(N_Z^0).$$

The limit motives functor is then obtained by composing with $p^*_{\to X^0}$:

$$\mathcal{L}^{\mathrm{DM}\,\Psi}_{X,Z} \colon \mathrm{DM}_{/S,R}(X^0) \xrightarrow{p^*_{\to X^0}} \mathrm{DM}_{/S,R}(X^0 \times \mathbb{G}_m) \xrightarrow{\Psi_f} \mathrm{DM}_{/S,R}(N_Z^0).$$

When $X^0 = \mathcal{M}_{0,n}$ and $Z = D_0$, we compose, as previously, this functor with the one induced by the nonzero section of $p^0_{N_Z} \colon N_Z^0 \to Z$ given in Proposition 3.2. Thus we obtain

$$\mathcal{L}^{\mathrm{DM}\,\Psi}_{D,\mathcal{M}_{0,n}} \colon \mathrm{DM}_{/S,R}(\mathcal{M}_{0,n}) \to \mathrm{DM}_{/S,R}(D_0).$$

The compatibility with mixed Tate categories follows from two facts:

(1) Tate objects in $\mathrm{DM}_{/S,R}(X^0)$, denoted by $R_{X^0}(i)$ below, are lifted from the ones in $\mathrm{DM}_{/S,R}(S)$. Hence, their pullback in $\mathrm{DM}_{/S,R}(X^0 \times \mathbb{G}_m)$ can be seen either as lifted from $\mathbb{G}_m$ or as lifted from S.

(2) In the first case, the compatibility of the specialization functor with smooth morphisms [3, Definition 3.1.1] ensures that

$$\Psi_f(p^*_{\to\mathbb{G}_m} R_{\mathbb{G}_m}(i)) \simeq f|^*_{\{0\}} \circ \Psi_{\mathrm{id}_{\mathbb{A}^1}}(R_{\mathbb{G}_m}(i)).$$

Now, viewing the Tate objects as lifted from S by $q^*_{\mathbb{G}_m}$, we can apply Proposition 3.5.10 in [3], which ensures that

$$\Psi_{\mathrm{id}_{\mathbb{A}^1}} \circ q^*_{\mathbb{G}_m} \sim \mathrm{id}.$$

Hence, the functor

$$\mathcal{L}_{X,Z}^{\mathrm{DM}\,\Psi}\colon \mathrm{DM}_R(X^0) \to \mathrm{DM}_R(N_Z^0)$$

sends Tate objects to Tate objects (eventually composing with the natural isomorphic transformation). It also ensures that its composition with

$$p_{X^0}\colon \mathrm{DM}_{/S,R}(S) \to \mathrm{DM}_{/S,R}(X^0)$$

is the map $\mathrm{DM}_{/S,R}(S) \to \mathrm{DM}_{/S,R}(N_Z^0)$ induced by the structural morphism of N_Z^0.

This gives us the desired functor on mixed Tate categories

$$\mathcal{L}_{D,\mathcal{M}_{0,n}}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DMT}_{/S,\mathbb{Z}}(D_0).$$

After this long remark, we come to the more delicate aspect of tangential base points in general situations. The general situation is the following: $X \xrightarrow{p_X} S$ is a smooth scheme with a strict normal crossing divisor $Z = \bigcup_{i\in I} Z_i$. We denote by Z_J the intersection $\bigcap_{i\in J} Z_j$ for $J \subset I$. The Z_J are also smooth over S. In our applications, where $X = \overline{\mathcal{M}}_{0,n}$ and $Z = \partial\overline{\mathcal{M}}_{0,n}$, the Z_J are irreducible. We assume this continues to hold for the following description. If not, the description below still works if extra care is given to keep track of the various irreducible components of the Z_J.

As before let X^0 denote $X \setminus Z$. Let J be a subset of I and let Z_J^0 be the "open stratum"

$$Z_J \setminus \Bigl(\bigcup_{i\in I\setminus J} Z_i \cap Z_J\Bigr).$$

Let N_i (resp. N_i^0) denote the normal bundle of Z_i in X (resp. with zero section removed); N_J (resp. N_J^0) is defined as the fiber product of the $N_j|_{Z_J}$ (resp. $N_j^0|_{Z_J}$) over Z_J and N_{J^0} (resp. $N_{J^0}^0$) its restriction to Z_J^0.

We are interested in generalizing the previous situation by constructing a functor

$$\mathcal{L}_{X,J}\colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DM}_{/S,\mathbb{Z}}(N_{J^0}^0)$$

compatible with mixed Tate categories and structural pullback functors. Then we want to apply this functor to the case where Z_J is an S–point, that is, of maximal codimension.

First, we observe that by setting

$$X' = X \setminus \Big(\bigcup_{i \in I \setminus J} Z_i \Big),$$

we can assume that $I = J$. In this case, Z_J^0 (resp. N_{J^0}, $N_{J^0}^0$) is simply Z_J (resp. N_J, N_J^0). We treat only this situation below.

In the strict normal crossing divisor situation, N_J equals N_{Z_J}, the normal bundle of Z_J in X. Moreover, locally with affine coordinates, or when N_{Z_J} is trivial, there is an isomorphism between N_J^0 and $Z_J \times_S (\mathbb{G}_m)^{|J|}$.

Lemma 4.3 (consequence of [46, Proposition 15.22]) *There is a natural functor*

$$\mathcal{L}^{\mathrm{DM}}_{X,J} \colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DM}_{/S,\mathbb{Z}}(N_J^0)$$

preserving Tate objects. The functor $\mathcal{L}^{\mathrm{DM}}_{X,J}$ is compatible with structural pullback morphisms. Hence we obtain a functor between mixed Tate categories

$$\mathcal{L}_{X,J} \colon \mathrm{DMT}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DMT}_{/S,\mathbb{Z}}(N_J^0)$$

such that its composition with $\mathrm{DMT}_{/S,\mathbb{Z}}(S) \to \mathrm{DMT}_{/S,\mathbb{Z}}(X^0)$ equals the functor

$$\mathrm{DMT}_{/S,\mathbb{Z}}(S) \to \mathrm{DMT}_{/S,\mathbb{Z}}(N_J^0)$$

induced by the structural morphism of N_J^0.

Review of the proof in [46] The proof consists of a generalization of the isomorphism (6),

$$i_J^* j_{X^0 *} \mathrm{M}\mathbb{Z}_{X^0} \simeq p^0_{N_J *} \mathrm{M}\mathbb{Z}_{N_J^0},$$

where i_J denotes the regular embedding $Z_J \to X$.

This isomorphism is obtained by taking the fiber product over $X \times \mathbb{A}^1$ of the deformation $D(X, Z_j)$ (resp. $D^0(X, Z_j)$) for all Z_j. Then Spitzweck's construction goes essentially as in the case where there is only one Z_j, by observing that $\mathrm{M}\mathbb{Z}_{X^0}$ is the pullback from $\mathrm{M}\mathbb{Z}_S$ by $(p_X \circ j_{X^0})^*$.

As previously, the compatibility with mixed Tate objects and pullback by structural morphism relies on [46, Corollary 15.14]. □

Remark 4.4 (on higher deformations to the normal cone and the nearby cycle functor) As previously, especially when working with rational coefficients and the Beilinson spectrum, one might prefer to use Ayoub's nearby cycles functor. We now introduce a higher deformation to the normal cone as presented in [28, Section 3.1.3] following M Rost [41, Section (10.6)]. Let $\mathcal{J}_j$ denote the sheaf of ideals defining the Z_j, and let k denote the cardinality of J. We consider only the case $J = I$. We assume that $J = \{1, \dots, k\}$ as this induces an easier notation. Then the subalgebra of $\mathcal{O}_X[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}]$,

$$\mathcal{A}_{X,J} = \bigoplus_{(a_1,\dots,a_k)\in\mathbb{Z}^k} \mathcal{J}_1^{a_1} \cdots J_k^{a_k} t_1^{a_1} \cdots t_k^{a_k},$$

is quasicoherent over $\mathcal{O}_X[t_1, \dots, t_n]$. In the above definition, as previously, we set $\mathcal{J}_j^{a_j} = \mathcal{O}_X$ when $a_j \leqslant 0$. The simultaneous deformation of the Z_j is defined as

$$D(X, J) = D(X; Z_1, \dots, Z_k) = \operatorname{Spec}(\mathcal{A}_{X,J}). \tag{7}$$

Inverting the t_j, one obtains

$$\mathcal{A}_{X,J}[t_1^{-1}, \dots, t_k^{-1}] = \mathcal{O}_X[t_1, t_1^{-1}, \dots, t_k, t_k^{-1}].$$

Hence there is a canonical isomorphism between $X \times \mathbb{G}_m^k$ and the restriction of $D(X, J)$ over $\mathbb{G}_m^k$. The following commutative diagram holds:

$$\begin{tikzcd}
D(X,J) \arrow[dr] \arrow[dd, "f_J"'] & & D(X,J)|_{\mathbb{G}_m^k} \arrow[ll, "j_{D|_{\mathbb{G}_m^k}}"'] \arrow[dr, "="] \arrow[dd, "f_J|_{\mathbb{G}_m^k}"'] & X \\
& X \times \mathbb{A}^k \arrow[dl] & & X \times \mathbb{G}_m^k \arrow[u, "p_{\to X}"'] \arrow[ll] \arrow[dl, "p_{\to \mathbb{G}_m^k}"] \\
\mathbb{A}^k & & \mathbb{G}_m^k \arrow[ll, "j_{\mathbb{G}_m^k}"] &
\end{tikzcd} \tag{8}$$

where the square and the parallelograms are Cartesian.

The construction is compatible with permutation of the coordinates on $\mathbb{A}^k$ and permutation of the Z_j. Inverting only $t_1, \dots, t_l$ (with $l < k$), we obtain

$$D(X, J)|_{\mathbb{G}_m^l \times \mathbb{A}^{k-l}} = \mathbb{G}_m^l \times D(X; Z_{l+1}, \dots, Z_k).$$

In [28], F Ivorra described the fiber of $D(X,J)$ over $t_1=\cdots=t_l=0$. The description goes by induction on l. For $l=1$, $Z_{[l]}$ is simply Z_1 and $N_{[l]}$ is N_1, the normal bundle of Z_1 in X. For $l\geqslant 2$, let $Z_{[l]}$ be the intersection

$$Z_1\cap\cdots\cap Z_l$$

and let $N_{[l]}$ be the normal bundle of $N_{[l-1]}|_{Z[l]}$ in $N_{[l-1]}$, which can be written as

$$N_{[l]}=N(N_{[l-1]},N_{[l-1]}|_{Z_{[l]}}).$$

Now let $D^{[l]}$ be the deformation

$$D(N_{[l]};N_{[l]}|_{Z_{l+1}\cap Z_{[l]}},\dots,N_{[l]}|_{Z_k\cap Z_{[l]}}).$$

Then we have

$$D(X,J)|_{t_1=\cdots=t_l=0}=D^{[l]}.$$

The fiber over $(0,\dots,0)$, ie when all the t_j are zero, is isomorphic to N_J, the normal bundle of Z_J in X. As a last remark, the description of $\mathcal{A}_{X,J}$ shows that the restriction of $D(X;J)$ to the diagonal is isomorphic to $D(X;Z_J)$.

Now, as in the case of a simple deformation, one can remove the strict transform of $Z\times\mathbb{A}^k$ in $D(X,J)$ and obtain an "open deformation" $D^0(X,J)$ whose restriction to $\mathbb{G}_m^k$ is simply $X^0\times\mathbb{G}_m^k$. Its fiber over $(0,\dots,0)$ is isomorphic to N_J^0 (as described in Spitzweck's construction above) and its restriction to the diagonal Δ_k gives

$$\begin{tikzcd}
N_J^0 \arrow[r, "i_{N_Z^0}"] \arrow[d, "f_J|_{\{0\}}"] & D^0(X,J)|_{\Delta_k} \arrow[d, "f_J"] & X^0\times\mathbb{G}_m \arrow[l, "j_{X^0\times\mathbb{G}_m}"'] \arrow[r, "p_{\to X^0}"] \arrow[d, "p_{\to\mathbb{G}_m}"] & X^0 \\
\{0\} \arrow[r, "i_{\{0\}}"] \arrow[dr, "="'] & \mathbb{A}^1 \arrow[d] & \mathbb{G}_m \arrow[l, "j_{\mathbb{G}_m}"'] \arrow[dl, "q_{\mathbb{G}_m}"] & \\
& S & &
\end{tikzcd}$$

We now proceed as in Remark 4.2 in order to obtain a functor

$$\mathcal{L}_{X,J}^{\mathrm{DM},\Psi}\colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0)\xrightarrow{p^*_{\to X^0}}\mathrm{DM}_{/S,\mathbb{Z}}(X^0\times\mathbb{G}_m)\xrightarrow{\Psi_{f_J}}\mathrm{DM}_{/S,\mathbb{Z}}(N_J^0).$$

Compatibilities with mixed Tate categories and pullback by structural morphisms are as in Remark 4.2.

Conjecture 1 The geometric construction of the higher deformation to the normal cone $D(X,J)$ makes it possible to use a succession of specialization functors, each corresponding to an $\mathbb{A}^1$ factor. This procedure does not depend on choices when considering only the mixed Tate motives categories. In this case, it agrees with our construction using the diagonal.

We explain now how the functor $\mathcal{L}_{X,J}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(X^0) \to \mathrm{DMT}_{/S,\mathbb{Z}}(N_J^0)$ from Lemma 4.3 leads to a *tangential base point* or *base point at infinity* for the moduli space of curves $\mathcal{M}_{0,n}$. Let $n \geqslant 4$. Let v be a point in $\overline{\mathcal{M}}_{0,n}$ given by a closed stratum of $\partial\overline{\mathcal{M}}_{0,n}$ of maximal codimension. The stratum v is the nonempty intersection of exactly $n-3 = \dim_S(\overline{\mathcal{M}}_{0,n})$ irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$:

$$v = \bigcap_{\substack{D \text{ closed stratum of } \partial\overline{\mathcal{M}}_{0,n}\\ \mathrm{codim}(D)=1\\ v\in D}} D = \bigcap_{j\in J} D_j,$$

where $J=\{1,\dots,n-3\}$ corresponds to a numbering of the closed codimension-1 strata D with $v\in D$. The normal bundle N_v of v in $\overline{\mathcal{M}}_{0,n}$ is trivial because v is a point.

Definition 4.5 A *tangential base point* x_v of $\mathcal{M}_{0,n}$ is the choice of a closed stratum v of maximal codimension in $\partial\overline{\mathcal{M}}_{0,n}$ and of a nonzero S–point in $N_v^0 = N_J^0$ with the notation of Lemma 4.3. Here $N_{J^0}^0 = N_J^0$ because $v = Z_J$ cannot have a nonempty intersection with any other components of $\partial\overline{\mathcal{M}}_{0,n}$.

Proposition 4.6 *For any tangential base point x_v of $\mathcal{M}_{0,n}$, there is a natural functor*

$$\tilde{x}_v^{\mathrm{DM},*}\colon \mathrm{DM}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DM}_{/S,\mathbb{Z}}(S)$$

sending Tate objects to Tate objects and hence inducing a natural functor

$$\tilde{x}_v^{*}\colon \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DMT}_{/S,\mathbb{Z}}(S).$$

Both functors are compatible, in the sense of Lemma 4.3, with the pullback by the structural functor from $\mathrm{DM}_{/S,\mathbb{Z}}(S)$.

Proof The boundary $\partial\overline{\mathcal{M}}_{0,n}$ can be written as the union of its codimension-1 irreducible components,

$$\partial\overline{\mathcal{M}}_{0,n} = \bigcup_{i\in I} D_i.$$

The closed stratum v defines a subset J of I by

$$v = \bigcap_{j \in J} D_j.$$

With

$$X = \overline{\mathcal{M}}_{0,n} \setminus \Big(\bigcup_{i \in I \setminus J} D_i \Big) \quad \text{and} \quad Z_j = D_j \setminus \Big(\bigcup_{i \in I \setminus J} D_i \cap D_j \Big),$$

Lemma 4.3 gives a functor

$$\mathcal{L}^{\mathrm{DM}}_{X,J} \colon \mathrm{DM}_{/S,\mathbb{Z}}(X^0) = \mathrm{DM}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \to \mathrm{DM}_{/S,\mathbb{Z}}(N^0_J).$$

The functor $\widetilde{x}_v^{\mathrm{DM},*}$ is obtained by composing $\mathcal{L}^{\mathrm{DM}}_{X,J}$ with the pullback of the S–point in N^0_J. In this application, we have simply $Z_J = v$ and $Z^0_J = Z_J$. □

There is a canonical system of tangential base points over Spec($\mathbb{Z}$) on $\mathcal{M}_{0,n}$. A point $v = \bigcap_{j \in J} D_j$ of this system is given by a closed stratum of maximal codimension of $\partial \overline{\mathcal{M}}_{0,n}$. In order to choose an S–point in its normal bundle, we choose a dihedral structure δ on the marked points, and vertex coordinates x_j corresponding to the point v (see [11, Definition 2.18]). The chosen vertex coordinates might differ only by the choice of their numbering. These vertex coordinates induce a basis on N_J. The sum of the vectors of this basis depends only on the dihedral structure δ; it is the S–point in N^0_J attached to v and δ. Changing the dihedral structure δ amounts to introducing signs rather than taking the sum of the basis vectors (see [11, Section 2.7]).

Definition 4.7 Let $P_{n,\infty}$ denote the set of canonical tangential base points described in the previous paragraph.

Brown develops his notion of base points at infinity for $\mathcal{M}_{0,n}$ in relation to the question of unipotent closures and periods of the moduli space of curves in genus 0; see [11, Definition 3.16 and Example 3.17 and before Section 6.3].

Remark 4.8 The results of the above subsection and of Section 3 hold by the same arguments in a "more classical" motivic category, for instance the one developed by Cisinski and Déglise [14]: rational coefficients over a general base and the Beilinson E_∞–ring spectrum. In Section 5, we will explore the case where the base is either a number field or the ring of integers of a number field with some primes inverted and rational coefficients.

4.2 The motivic short exact sequence

Derived group schemes were studied in particular by B Toën in [52] and M Spitzweck in [49]. They can be considered as spectra of E_∞ algebras (with a specific cosimplicial structure). Now, using [49], we define a derived group scheme associated to the category $\mathrm{DM}_{\mathbb{Z}}(\mathcal{M}_{0,n})$. Recall that $S=\operatorname{Spec}(\mathbb{Z})$. Using Theorems 2.10 and 3.8, we can directly apply Theorem 8.4 of [50].

Theorem 4.9 [50] *Let $S=\operatorname{Spec}(\mathbb{Z})$ and $n\geqslant 3$. There is an affine derived group scheme $G^\bullet_{/S,\mathbb{Z},n}=\operatorname{Spec}(A_n)$ over $\mathbb{Z}$ such that*

$$\operatorname{Perf}(G^\bullet_{/S,\mathbb{Z},n})\simeq \mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}),$$

where Perf denotes the category of perfect representations and $\mathrm{DMT}_{/S,\mathbb{Z}}(X)$ the full triangulated subcategory of compact objects in the full triangulated subcategory of $\mathrm{DM}_{/S,\mathbb{Z}}(X)$ generated by Tate objects $\mathrm{M}\mathbb{Z}_X(p)=\mathbb{Z}_X(p)$ for $p\in\mathbb{Z}$ (see Definition 2.5)*. We shall write simply $G^\bullet_{/S,\mathbb{Z}}$ for $G^\bullet_{/S,\mathbb{Z},3}$.*

We similarly define $G^\bullet_{/S,\mathbb{Z},n_1,n_2}$ associated to $\mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n_1}\times\mathcal{M}_{0,n_2})$. The natural functors between categories $\mathrm{DMT}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n})$ induce morphisms between affine derived group schemes.

Proposition 4.10 *The structural morphism $\phi_n\colon\mathcal{M}_{0,n}\to S=\operatorname{Spec}(\mathbb{Z})$ induces a surjective morphism*

$$G^\bullet_{/S,\mathbb{Z},n}\xrightarrow{\phi_n}G^\bullet_{/S,\mathbb{Z},3}=G^\bullet_{/S,\mathbb{Z}}$$

induced by the natural pullback p_n^* at the category level. Any choice of S–point $x\in\mathcal{M}_{0,n}(S)$ provides a morphism $x\colon G^\bullet_{/S,\mathbb{Z}}\to G^\bullet_{/S,\mathbb{Z},n}$ satisfying $\phi_n\circ x\sim\mathrm{id}$. The equivalent statement holds for $G^\bullet_{/S,\mathbb{Z},n_1,n_2}$, where, by an abuse of notation, the morphisms between derived groups are denoted the same as the morphisms between schemes.

Proof In order to pass from functors between categories to morphisms between the corresponding affine derived group schemes $G^\bullet_{/S,\mathbb{Z},n}$ and $G^\bullet_{/S}(\mathbb{Z})$, it is enough to obtain a morphism between the corresponding E_∞ algebras A_n and A_3 with $G^\bullet_{/S,\mathbb{Z},n}=\operatorname{Spec}(A_n)$ and $G^\bullet_{/S,\mathbb{Z},3}=G^\bullet_{/S,\mathbb{Z}}=\operatorname{Spec}(A_{n-1})$. Spitzweck [49] describes the algebra A_n as essentially the first degree of the Čech resolution of $B_n\to\mathbb{Z}$, where B_n is an E_∞ algebra in the category of graded complex of abelian groups $\operatorname{Cpx}(\mathrm{Ab})^{\mathbb{Z}}$.

The cosimplicial structure of the Čech resolution induces the group structure of $G^{\bullet}_{/S,\mathbb{Z},n}$. The above construction is functorial in B_n. Hence we only need to obtain the morphism $B_3 \to B_n$ (the functor Spec is contravariant). B_n is itself obtained from $P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}$, the periodization of $\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}$ (see [48]), as

$$P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}} = \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i,i}\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}.$$

Let r_n be the right adjoint functor to

$$\mathrm{Cpx}(\mathrm{Ab}) \to \mathrm{Mod}_{\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}},$$

and let $r_n^{\mathbb{Z}}$ be the graded version of r_n and right adjoint to

$$\mathrm{Cpx}(\mathrm{Ab})^{\mathbb{Z}} \to (\mathrm{Mod}_{\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}})^{\mathbb{Z}}.$$

B_n is then defined as $r_n^{\mathbb{Z}}(P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}})$. By an abuse of notation, we also write r_n and $r_n^{\mathbb{Z}}$ for the corresponding induced functors on the homotopy categories. The following triangle is commutative:

$$\begin{tikzcd}
\mathrm{Cpx}(\mathrm{Ab}) \arrow[rr] \arrow[dr] & & \mathrm{Mod}_{\mathrm{M}\mathbb{Z}_S} \arrow[dl, "\phi_n^*"] \\
& \mathrm{Mod}_{\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}} &
\end{tikzcd}$$

and it induces a similar commutative triangle between the homotopy categories. Thus, the diagram between right adjoints (and their graded versions) is also commutative:

$$r_n = r_{n-1} \circ (\phi_n)_*.$$

Hence we only need to show how the functor ϕ_n^* induces a natural morphism

$$P\mathrm{M}\mathbb{Z}_S = P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,3}} \to (\phi_n)_*(P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}).$$

The unit of the adjunction for the functor ϕ_n^* gives a morphism

$$\mathrm{M}\mathbb{Z}_S \to (\phi_n)_*(\phi_n)^*\mathrm{M}\mathbb{Z}_S = (\phi_n)_*\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}.$$

Applying periodization to both sides then concludes the proof.

By the same argument, an S–point $x\colon S \to \mathcal{M}_{0,n}$ gives a morphism $x\colon G^{\bullet}_{/S,\mathbb{Z}} \to G^{\bullet}_{/S,\mathbb{Z},n}$. The composition $\phi_n \circ x$ is homotopic to the identity, because the relation $\phi_n \circ x = \mathrm{id}$ at the scheme level passes through at each stage of the argument by functoriality. □

Remark 4.11 The above argument holds for any morphism $f\colon X \to Y$ in Sm_S.

Let $K^\bullet_{/S,\mathbb{Z},n}$ be the kernel of ϕ_n. For any $n \geqslant 4$, we obtain a short exact sequence

$$1 \to K^\bullet_{/S,\mathbb{Z},n} \to G^\bullet_{/S,\mathbb{Z},n} \to G^\bullet_{/S,\mathbb{Z}} \to 1. \tag{SES_n}$$

Proposition 4.12 *The morphism $\psi_{n,i}\colon \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ forgetting the i^{th} marked point induces a commutative diagram*

$$\begin{tikzcd}
1 \arrow[r] & K^\bullet_{/S,\mathbb{Z},n} \arrow[r] \arrow[d, "\tilde{\psi}_{n,i}"'] & G^\bullet_{/S,\mathbb{Z},n} \arrow[r, "\phi_n"'] \arrow[d, "\psi_{n,i}"'] & G^\bullet_{/S,\mathbb{Z}} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & K^\bullet_{/S,\mathbb{Z},n-1} \arrow[r] & G^\bullet_{/S,\mathbb{Z},n-1} \arrow[r, "\phi_{n-1}"'] & G^\bullet_{/S,\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

Permutations of the marked points on $\mathcal{M}_{0,n}$ induce an action of the symmetric group on $G^\bullet_{/S,\mathbb{Z},n}$ and $K^\bullet_{/S,\mathbb{Z},n}$. A similar statement holds for $G^\bullet_{/S,\mathbb{Z},n_1,n_2}$ and $K^\bullet_{/S,\mathbb{Z},n_1,n_2}$.

Proof The functoriality of the pullback functors means that the equivalent of the right-hand square for the categories $\mathrm{DMT}_{/S,\mathbb{Z}}$ also commutes. Remark 4.11 above then ensures that the desired morphisms exist, by arguments similar to those developed in Proposition 4.10.

By the same arguments, permutations of the marked points act on $G^\bullet_{/S,\mathbb{Z},n}$. □

Proposition 4.13 *Let D be an irreducible component of $\partial\overline{\mathcal{M}}_{0,n}$ and let D_0 be its open stratum* (see Section 3.1)*. Then D_0 is isomorphic to*

$$D_0 \sim \mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$$

with $n_1 + n_2 = n + 2$.

The inclusion $i_D\colon \overline{\mathcal{M}}_{0,n_1} \times \overline{\mathcal{M}}_{0,n_2} \to \overline{\mathcal{M}}_{0,n}$ induces a gluing morphism

$$i_{n_1,n_2,D}\colon G^\bullet_{/S,\mathbb{Z},n_1,n_1} \to G^\bullet_{/S,\mathbb{Z},n}$$

and a morphism

$$\tilde{\imath}_{n_1,n_2,D}\colon K^\bullet_{/S,\mathbb{Z},n_1,n_2} \to K^\bullet_{/S,\mathbb{Z},n}.$$

Moreover, the projections $\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \to \mathcal{M}_{0,n_i}$ induce morphisms

$$p_{n_1,n_2}\colon G^\bullet_{/S,\mathbb{Z},n_1,n_2} \to G^\bullet_{/S,\mathbb{Z},n_1} \times G^\bullet_{/S,\mathbb{Z},n_2}$$

and

$$\tilde{p}_{n_1,n_2}\colon K^\bullet_{/S,\mathbb{Z},n_1,n_2} \to K^\bullet_{/S,\mathbb{Z},n_1} \times K^\bullet_{/S,\mathbb{Z},n_2}.$$

The above morphisms make the following diagram commutative:

$$\begin{tikzcd}
1 \arrow[r] & K^{\bullet}_{/S,\mathbb{Z},n} \arrow[rr] & & G^{\bullet}_{/S,\mathbb{Z},n} \arrow[r, "\phi_n"'] & G^{\bullet}_{/S,\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & K^{\bullet}_{/S,\mathbb{Z},n_1,n_2} \arrow[rr] \arrow[u, "\tilde{i}_{n_1,n_2,D}"] \arrow[d, "\tilde{p}_{n_1,n_2}"] & & G^{\bullet}_{/S,\mathbb{Z},n_1,n_2} \arrow[r, "\phi_{n_1,n_2}"'] \arrow[u, "i_{n_1,n_2,D}"'] \arrow[d, "p_{n_1,n_2}"] & G^{\bullet}_{/S,\mathbb{Z}} \arrow[r] \arrow[u, "="] & 1 \\
& K^{\bullet}_{/S,\mathbb{Z},n_1} \times K^{\bullet}_{/S,\mathbb{Z},n_2} \arrow[rr] & & G^{\bullet}_{/S,\mathbb{Z},n_1} \times G^{\bullet}_{/S,\mathbb{Z},n_2} & &
\end{tikzcd}$$

As previously, by a slight abuse we use the same notation for the morphisms between derived groups and the corresponding morphisms between the associated schemes.

Proof The morphisms p_{n_1,n_2} are obtained as in Proposition 4.10 and Remark 4.11 using the morphisms

$$\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2} \to \mathcal{M}_{0,n_i}$$

for $i = 1, 2$.

In order to obtain a morphism

$$G^{\bullet}_{/S,\mathbb{Z},n_1,n_2} \to G^{\bullet}_{/S,\mathbb{Z},n},$$

we can proceed as in Proposition 4.12, working directly in terms of periodizations of $\mathrm{M}\mathbb{Z}_{D_0}$ and $\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}$.

Proposition 4.1 gives a functor

$$\begin{aligned}\mathrm{DM}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n}) \xrightarrow{\mathcal{L}^{\mathrm{DM}}_{\mathcal{M}_{0,n},D_0}} \mathrm{DM}_{/S,\mathbb{Z}}(N^{\mathbf{0}}_{D_0})& \\ \xrightarrow{\sigma^*} \mathrm{DM}_{/S,\mathbb{Z}}(D_0) &= \mathrm{DM}_{/S,\mathbb{Z}}(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}),\end{aligned}$$

where $N^{\mathbf{0}}_{D_0}$ denotes the normal bundle of D_0 in $\mathcal{M}_{0,n} \cup D_0$ with the zero section removed and σ is an everywhere-nonzero section of $N^{\mathbf{0}}_{D_0}$ as in Proposition 3.2. The arguments of Proposition 4.12 apply to the morphism σ. Hence, it is enough to obtain a morphism

$$L_{\mathcal{M}_{0,n},D_0}\colon r^{\mathbb{Z}}_n(P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}) \to r^{\mathbb{Z}}_{N^0_{D_0}}(P\mathrm{M}\mathbb{Z}_{N^0_{D_0}})$$

induced by $\mathcal{L}^{\mathrm{DM}}_{\mathcal{M}_{0,n},D_0}$. In the above formula, $r^{\mathbb{Z}}_{N^0_{D_0}}$ denotes the graded version of the right adjoint to

$$\mathrm{Cpx}(\mathrm{Ab}) \to \mathrm{Mod}_{\mathrm{M}\mathbb{Z}_{N^0_{D_0}}}.$$

The heart of the construction of $\mathcal{L}^{\mathrm{DM}}_{\mathcal{M}_{0,n},D_0}$ is the equivalence (6),

$$i^*_{D_0} j_{\mathcal{M}_{0,n}\,*}(\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}) \simeq p^0_{N_{D_0}\,*}(\mathrm{M}\mathbb{Z}_{N^0_{D_0}}).$$

Recall that $p^0_{N_{D_0}}\colon N^0_{D_0} \to D_0$ is induced by the defining morphism of the normal bundle $N_{D_0} \to D_0$. Pushing forward the above equivalence along $i_{D_0}\colon D_0 \to \mathcal{M}_{0,n} \cup D_0$ gives the morphism in $\mathrm{DM}_{/S,\mathbb{Z}}(\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}\cup D_0})$

$$(9)\quad (j_{\mathcal{M}_{0,n}})_*(\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}) \to (i_{D_0})_* \circ i^*_{D_0} \circ (j_{\mathcal{M}_{0,n}})_*(\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}) \simeq (i_{D_0})_* \circ (p^0_{N_{D_0}})_*(\mathrm{M}\mathbb{Z}_{N^0_{D_0}}),$$

where the first map is given by the unit of the adjunction $((i_{D_0})_*, (i_{D_0})^*)$.

Following the arguments and notations of the previous Proposition 4.12, we now pass to periodizations and apply $r^{\mathbb{Z}}_{\mathcal{M}_{0,n}\cup D_0}$. This concludes the proof, since we have

$$r^{\mathbb{Z}}_{\mathcal{M}_{0,n}\cup D_0}(j^{\mathbb{Z}}_{\mathcal{M}_{0,n}\,*}(P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}})) = r^{\mathbb{Z}}_n(P\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}})$$

and

$$r^{\mathbb{Z}}_{\mathcal{M}_{0,n}\cup D_0}((i_{D_0})_* \circ (p^0_{N_{D_0}})_*(P\mathrm{M}\mathbb{Z}_{N^0_{D_0}})) = r^{\mathbb{Z}}_{N^0_{D_0}}(P\mathrm{M}\mathbb{Z}_{N^0_{D_0}}). \qquad \square$$

The compatibility property with structural morphisms in Proposition 4.6 shows that the short exact sequence

$$1 \to K^\bullet_{/S,\mathbb{Z},n} \to G^\bullet_{/S,\mathbb{Z},n} \to G^\bullet_{/S,\mathbb{Z}} \to 1$$

is split by any choice of a tangential S–point of $\mathcal{M}_{0,n}$. We restrict ourselves to the family of tangential base points in $P_{n,\infty}$.

Proposition 4.14 *Let x_v be a tangential base point of $\mathcal{M}_{0,n}$ (with $n \geqslant 4$) in $P_{n,\infty}$. Then x_v induces a splitting*

$$\begin{tikzcd} 1 \arrow[r] & K^\bullet_{/S,\mathbb{Z},n} \arrow[r] & G^\bullet_{/S,\mathbb{Z},n} \arrow[r, "\phi_n"'] & G^\bullet_{/S,\mathbb{Z}} \arrow[r] \arrow[l, bend right, "\widetilde{x}_v"'] & 1. \end{tikzcd}$$

Proof The morphism $\widetilde{x}_v$ is defined in terms of the E_∞ algebras A_n and A_3 using the periodization of $\mathrm{M}\mathbb{Z}_{\mathcal{M}_{0,n}}$ and $\mathrm{M}\mathbb{Z}_S$ as in the proof of Proposition 4.13 above. We see that it splits the above exact sequence due to the compatibility between tangential base point functors and structural morphism functors given in Proposition 4.6. $\square$

4.3 A motivic Grothendieck–Teichmüller group

Thanks to the above subsection, the motivic short exact sequence (SES$_n$) is compatible with the geometry of the tower of $\mathcal{M}_{0,n}$, namely the gluing morphisms $\tilde{\imath}_{n_1,n_2,D}$ and the forgetful morphisms $\widetilde{\psi}_{n,i}$. Following Grothendieck's ideas [24, Section 2], we would like to define the motivic Grothendieck–Teichmüller group as the group of automorphisms of the tower of the $K^\bullet_{/S,\mathbb{Z},n}$ (the geometric part) compatible with the natural morphisms. However the "derived nature" of these objects requires some extra care in this definition. In this subsection, we define the motivic Grothendieck–Teichmüller group space over $\mathbb{Z}$ (with $\mathbb{Z}$ coefficients).

The affine derived group schemes $G^\bullet_{/S,\mathbb{Z},n}$ and $K^\bullet_{/S,\mathbb{Z},n}$ (as well as those corresponding to the products $\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}$) naturally give simplicial objects in affine derived schemes using the group structure. This induced simplicial structure is, in fact, given by the cosimplicial structure of the Čech resolution of $B_n \to \mathbb{Z}$. The category of simplicial objects in derived schemes is a simplicial model category, and we can use the notion of homotopy automorphism space as defined by B Fresse [21, Part II, Section 2.2]. Very briefly, the homotopy automorphism space $\mathrm{Aut}^h(O^\bullet)$ of an object $O^\bullet$ is a simplicial monoid whose connected component (its π_0) gives invertible homotopy classes of endomorphisms of $O^\bullet$:

$$\pi_0(\mathrm{Aut}^h(O^\bullet)) = [O^\bullet, O^\bullet]^\times = \mathrm{Aut}_{\mathrm{Ho(simple\ derived\ affine\ schemes)}}(O^\bullet).$$

We use Fresse's definition of the homotopy automorphism space [21, pages 57–58] to force the equivariance with the action of the permutation groups.

Definition 4.15 Let Aut^h_n be the disjoint union of the connected components of $\mathrm{Aut}^h(K^\bullet_{/S,\mathbb{Z},n})$ having a ϕ: $K^\bullet_{/S,\mathbb{Z},n} \to K^\bullet_{/S,\mathbb{Z},n}$ as a vertex satisfying

$$\forall \sigma \in \Sigma_n \quad [\phi][\sigma] = [\sigma][\phi] \quad \text{in } \pi_0(\mathrm{Aut}^h(K^\bullet_{/S,\mathbb{Z},n})),$$

where Σ_n denotes the group of permutation on n elements. The spaces $\mathrm{Aut}^h_{n_1,n_2}$ are defined similarly with respect to $K^\bullet_{/S,\mathbb{Z},n_1,n_2}$ and the induced $\Sigma_{n_1} \times \Sigma_{n_2}$–action.

First, let $\mathrm{GT}^{h,\leqslant 4}_{/S,\mathbb{Z}}$ be simply Aut^h_4.

The homotopy automorphisms of the tower up to level n are then defined by induction. Let $\mathrm{Map}_{n\to n-1}$ (resp. $\mathrm{Map}_{n_1\times n_2\to k}$ for $k \geqslant 4$) denote the homotopy mapping space $\mathrm{Map}(K^\bullet_{/S,\mathbb{Z},n}, K^\bullet_{/S,\mathbb{Z},n-1})$ (resp. $\mathrm{Map}(K^\bullet_{/S,\mathbb{Z},n_1,n_2}, K^\bullet_{/S,\mathbb{Z},k})$) as defined in [21, II.2.1, page 51]. For a fixed $n \geqslant 5$, $\mathrm{GT}^{h,\leqslant n}_{/S,\mathbb{Z}}$ is defined in three steps by the homotopy Cartesian squares which we define below.

We define $\widetilde{\mathrm{GT}}^{h,\leqslant n}_{/S,\mathbb{Z}}$ as the "product" of $\mathrm{GT}^{h,\leqslant n-1}_{/S,\mathbb{Z}}$ and Aut^h_n; the compatibility with the morphisms forgetting marked points is forced by using the following homotopy Cartesian squares:

$$\begin{tikzcd}
\widetilde{\mathrm{GT}}^{h,\leqslant n}_{/S,\mathbb{Z}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\underline{h}\lrcorner"] & \mathrm{Aut}^h_n \arrow[d] \\
\mathrm{GT}^{h,\leqslant n-1}_{/S,\mathbb{Z}} \arrow[r] & \prod_{i=1}^n \mathrm{Map}_{n\to n-1}
\end{tikzcd}$$

where the horizontal (resp. vertical) map is given on the i^{th} factor by composing on the right (resp. left) by $\widetilde{\psi}_{n,i}\colon K^\bullet_{/S,\mathbb{Z},n} \to K^\bullet_{/S,\mathbb{Z},n-1}$.

For n_1 and n_2 with $n_1+n_2=n+2$, the homotopy automorphisms in $\mathrm{Aut}^h_{n_1,n_2}$ must be compatible with projections on $\mathcal{M}_{0,n_1}$ and $\mathcal{M}_{0,n_2}$. We define $\widetilde{\mathrm{Aut}}^h_{n_1,n_2}$ by the homotopy Cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathrm{Aut}}^h_{n_1,n_2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\underline{h}\lrcorner"] & \mathrm{Aut}^h_{n_1,n_2} \arrow[d] \\
\mathrm{Aut}^h_{n_1} \times \mathrm{Aut}^h_{n_2} \arrow[r] & \mathrm{Map}_{n_1\times n_2\to n_1} \times \mathrm{Map}_{n_1\times n_2\to n_2}
\end{tikzcd}$$

where the horizontal (resp. vertical) map is given on the i^{th} factor by composing on the right (resp. left) by $\widetilde{p}_{n_i}\colon K^\bullet_{/S,\mathbb{Z},n_1,n_2} \to K^\bullet_{/S,\mathbb{Z},n_i}$ (with $i=1,2$).

Now we force the compatibility with the gluing morphisms $\tilde{\imath}_{n_1,n_2,D}\colon K^\bullet_{/S,\mathbb{Z},n_1,n_2} \to K^\bullet_{/S,\mathbb{Z},n}$. The space $\mathrm{GT}^{h,\leqslant n}_{/S,\mathbb{Z}}$ is defined by the homotopy Cartesian diagram

$$\begin{tikzcd}
\mathrm{GT}^{h,\leqslant n}_{/S,\mathbb{Z}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\underline{h}\lrcorner"] & \widetilde{\mathrm{GT}}^{h,\leqslant n}_{/S,\mathbb{Z}} \arrow[d] \\
\prod_{n_1+n_2=n+2} \widetilde{\mathrm{Aut}}^h_{n_1,n_2} \arrow[r] & \prod_{D\subset\partial\overline{\mathcal{M}}_{0,n}} \mathrm{Map}_{n_1\times n_2\to n} \times \prod_{n_1+n_2=n+2} \mathrm{Aut}^h_{n_1} \times \mathrm{Aut}^h_{n_2}
\end{tikzcd}$$

where the products $\prod_{D\subset\partial\overline{\mathcal{M}}_{0,n}}$ run through the set of irreducible components of $\partial\overline{\mathcal{M}}_{0,n}$ and the maps $\mathrm{Map}_{n_1\times n_2\to n}$ in the above diagram are given by composition with gluing morphisms $\tilde{\imath}_{n_1,n_2,D}$. The maps into $\mathrm{Aut}^h_{n_i}$ are the projections given by the construction.

Definition 4.16 (motivic Grothendieck–Teichmüller space over $S=\mathrm{Spec}(\mathbb{Z})$ with $\mathbb{Z}$ coefficients) Let $\mathrm{GT}^h_{/S}(\mathbb{Z})$ be the inverse limit of the $\mathrm{GT}^{h,\leqslant n}_{/S,\mathbb{Z}}$:

$$\mathrm{GT}^h_{/S}(\mathbb{Z})=\varprojlim(\mathrm{GT}^{h,\leqslant n}_{/S,\mathbb{Z}}).$$

Remark 4.17 In the above definition, $\mathrm{GT}^h_{/S}(\mathbb{Z})$ inherits a simplicial structure from each simplicial set Aut^h_n. As in the étale fundamental group setting (see [27]), the motivic short exact sequence (SES$_n$) gives an action of $G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z})$ (the integral points of $G^\bullet_{/S,\mathbb{Z}}$) on each $K^\bullet_{/S,\mathbb{Z},n}$ and $K^\bullet_{/S,\mathbb{Z},n_1,n_2}$. This action is compatible with permutations of marked points, gluing morphisms and morphisms forgetting marked points. Considering homotopy automorphism spaces (and not only their π_0) makes its possible to hope for a monomorphism

$$G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z})\hookrightarrow \mathrm{GT}^h_{/S}(\mathbb{Z}),$$

where the simplicial structure of $G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z})$ comes from its group structure.

Considering only the set of homotopy classes of morphisms would force us to consider $\pi_0(G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z}))$ and not $G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z})$.

In order to prove the above statement, which is really a motivic homotopy theory result, it might be better to begin by investigating more deeply the structure of the affine derived group scheme $K^\bullet_{/S,\mathbb{Z},n}$ as proposed in Section 6.

Remark 4.18 The approach in [25] only requires that the automorphisms of braid groups preserve the "inertia subgroups". While the approach in [25] is more workable, it is not as precise in its geometric implications as that described here. Remark 6.1 outlines the relation between "inertia subgroups" and morphisms $\tilde{\imath}_{n_1,n_2,D}$.

5 Comparison with classical motivic constructions

In this section we explore how the above situation evolves when working in the more classical setting of rational coefficients, and also when working over a number field.

5.1 Rational coefficients

In this subsection we describe the situation with rational coefficients. The main advantage is that the t–structure is available, which allows us to use the Tannakian formalism on the *nonderived* category of mixed Tate motives.

In order to work with rational coefficients, one can consider the Beilinson spectrum $H_{B,S}$ (see [14, Definition 13.1.2]) and work with the homotopy category of modules over it as our derived category of motives with $\mathbb{Q}$ coefficients $\mathrm{DM}_{/S,\mathbb{Q}}(S)$ (resp. $\mathrm{DM}_{\mathbb{Q}}(\mathcal{M}_{0,n})$ for $n \geqslant 4$). The proofs of the corresponding statements in the previous sections go through with identical arguments. Another way to work with rational coefficients is to work with the rationalization $\mathrm{M}\mathbb{Q}_S$ of Spitzweck's spectrum $\mathrm{M}\mathbb{Z}_S$. These two approaches are equivalent thanks to Theorems 7.14 and 7.18 in [50], which give an isomorphism

$$\mathrm{M}\mathbb{Q}_S \simeq H_{B,S}.$$

More generally, using pullbacks by structural morphisms, Spitzweck's work ensures that $\mathrm{M}\mathbb{Q}_X \simeq H_{B,X}$.

Let us denote the Tate objects $\mathrm{M}\mathbb{Q}_X(n)$ by $\mathbb{Q}_X(n)$ when there is no need to insist on the spectrum they are coming from, and simply by $\mathbb{Q}(n)$ when X is sufficiently clear. As previously, the derived category of mixed Tate motives $\mathrm{DMT}_{/S,\mathbb{Q}}(X)$ is defined as the full triangulated subcategory of compact objects in the full triangulated subcategory of $\mathrm{DM}_{/S,\mathbb{Q}}(X)$ generated by Tate objects $\mathbb{Q}_X(n)$ for $n \in \mathbb{Z}$.

Hence, from the previous section, we have a family of diagrams between derived categories of mixed Tate motives with rational coefficients ($S = \mathrm{Spec}(\mathbb{Z})$)

$$\begin{tikzcd}
\mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n-1}) \arrow[d, "\psi^*_{n,i}"'] & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \arrow[l, "\phi^*_{n-1}"'] \arrow[d, "="] \\
\mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n}) \arrow[d, "\mathcal{L}_{D,\mathcal{M}_{0,n}}"'] \arrow[u, phantom] & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \arrow[l, "\phi^*_n"'] \arrow[d, "="] \\
\mathrm{DMT}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2}) & \mathrm{DMT}_{/S,\mathbb{Q}}(S) \arrow[l]
\end{tikzcd} \tag{10}$$

for any $n \geqslant 4$, $n_1 + n_2 = n + 2$ and D a closed codimension-1 stratum of $\overline{\mathcal{M}}_{0,n}$, and where, in the above diagram, the tangential base points x_v and $x_{v'}$ lie in $P_{n,\infty}$ and in $P_{n-1,\infty}$, respectively, and are moreover compatible in the sense that $\psi_{n,i}(v) = v'$ and $\mathrm{d}\,\psi_{n,i}(x_v) = x_{v'}$ at the scheme level.

As remarked in [38], the arguments of [34] go through provided that the Beilinson–Soulé vanishing property (BS) holds (see also [37]). Thus, when X over S (and thus S) satisfies the (BS) property, we obtain a Tannakian category $\mathrm{MTM}_{/S}(X)$ of mixed Tate motives over X as the heart of $\mathrm{DMT}_{/S,\mathbb{Q}}$ by the t–structure (with duality and tensor structure inherited from those in $\mathrm{DMT}_{/S,\mathbb{Q}}$). The fiber functor is induced by the weight-graded piece Gr_W^*:

$$\omega\colon M \mapsto \bigoplus_n \mathrm{Hom}(\mathbb{Q}_X(n), \mathrm{Gr}_n^W(M)).$$

The diagram (10) is compatible with the t–structure and induces a similar diagram between Tannakian categories of mixed Tate motives $\mathrm{MTM}_{/S,\mathbb{Q}}(-)$. The Tannakian formalism and the weight filtration allow us to identify the categories $\mathrm{MTM}_{/S,\mathbb{Q}}(-)$ with the categories of graded representations of graded pro-unipotent affine algebraic groups $G_{/S,\mathbb{Q},n}$ and $G_{/S,\mathbb{Q},n_1,n_2}$. The grading of these groups encodes the $\mathbb{G}_m$ action induced by the weight grading.

We may drop the subscript $/S$ when the base scheme is sufficiently clear, and simply write $G_{\mathbb{Q},n}$ and $G_{\mathbb{Q},n_1,n_2}$. As in the above section, $G_{/S,\mathbb{Q},3}$ is denoted by $G_{/S,\mathbb{Q}}$ or simply $G_{\mathbb{Q}}$; it is the Tannakian group scheme associated to $\mathrm{MTM}_{/S,\mathbb{Q}}(S)$. The groups $G_{/S,\mathbb{Q},n}$ are sometime referred to as motivic fundamental groups of $\mathcal{M}_{0,n}$. However, we prefer to use the expression *Tannakian groups* of $\mathcal{M}_{0,n}$, as these groups are obtained from the categories $\mathrm{MTM}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n})$ by the Tannakian formalism. Hence the diagram (10) (over $S=\mathrm{Spec}(\mathbb{Z})$) leads to a diagram of group schemes

$$\begin{tikzcd}
1 \arrow[r] & K_{/S,\mathbb{Q},n-1} \arrow[r] & G_{/S,\mathbb{Q},n-1} \arrow[r, "\phi_{n-1}"] & G_{/S,\mathbb{Q}} \arrow[r] \arrow[l, dotted, bend right, "\widetilde{x}_{v'}"'] & 1 \\
1 \arrow[r] & K_{/S,\mathbb{Q},n} \arrow[r] \arrow[u, "\widetilde{\psi}_{n,i}"] & G_{/S,\mathbb{Q},n} \arrow[r, "\phi_n"'] \arrow[u, "\psi_{n,i}"] & G_{/S,\mathbb{Q}} \arrow[r] \arrow[u, dashed, "="] \arrow[l, dotted, bend right, "\widetilde{x}_v"'] & 1 \\
1 \arrow[r] & K_{/S,\mathbb{Q},n_1,n_2} \arrow[r] \arrow[u, "\widetilde{i}_{n_1,n_2,D}"] \arrow[d, "\widetilde{p}_{n_1,n_2}"'] & G_{/S,\mathbb{Q},n_1,n_2} \arrow[r, "\phi_{n_1,n_2}"'] \arrow[u, "i_{n_1,n_2,D}"] \arrow[d, "p_{n_1,n_2}"'] & G_{/S,\mathbb{Q}} \arrow[r] \arrow[u, dashed, "="] & 1 \\
 & K_{/S,\mathbb{Q},n_1}\times K_{/S,\mathbb{Q},n_2} \arrow[r] & G_{/S,\mathbb{Q},n_1}\times G_{/S,\mathbb{Q},n_2} & &
\end{tikzcd} \tag{11}$$

where the three first lines are exact.

Definition 5.1 (motivic Grothendieck–Teichmüller group over $S=\mathrm{Spec}(\mathbb{Z})$ with $\mathbb{Q}$ coefficients) Let $\mathrm{GT}^{\mathrm{mot}}_{/S}(\mathbb{Q})$ be the group of automorphisms g of the tower of groups

$$(K_{/S,\mathbb{Q},n})_{n\geq 4} \cup (K_{/S,\mathbb{Q},n_1,n_2})_{n_1,n_2\geq 4}.$$

Each element g is given by two collections of morphisms $(g_n)_{n\geq 4}$ and $(g_{n_1,n_2})_{n_1,n_2\geq 4}$ such that each g_n (resp. g_{n_1,n_2}) is an automorphism of $K_{/S,\mathbb{Q},n}$ (resp. $K_{/S,\mathbb{Q},n_1,n_2}$) and the g_n and the g_{n_1,n_2} commute with the action of the symmetric group on $K_{/S,\mathbb{Q},n}$ and also with the morphisms $\tilde{\imath}_{n_1,n_2,D}$, $\widetilde{p}_{n_1,n_2}$ and $\widetilde{\psi}_{n,i}$.

5.2 Working over a number field and its ring of integers

Working over the integers gives a very general description of the categories

$$\mathrm{MTM}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n}),$$

which can be applied in various contexts. But working over a number field allows us to have a more concrete description of the above groups and their Hopf algebraic avatars (in the Tannakian formalism) in terms of algebraic cycles, as described in [38].

Before explaining this in more detail, let us take the opportunity to compare the above category $\mathrm{MTM}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{Z}))$ of mixed Tate motives over $\mathbb{Z}$ with the one defined by Goncharov and Deligne in [16], which, by construction, is a subcategory of $\mathrm{MTM}_{/\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{Q}))$.

The structural morphism $p_{\mathbb{Q}}\colon \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$ induces a functor of the derived motivic categories

$$p_{\mathbb{Q}}^*\colon \mathrm{DM}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{Z})) \to \mathrm{DM}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{Q}))$$

sending Tate objects to Tate objects. This functor is compatible with the t–structure, and hence induces a functor between mixed Tate categories

$$\begin{aligned} p_{\mathbb{Q}}^*\colon \mathrm{MTM}_{/\mathrm{Spec}\mathbb{Z},\mathbb{Q}}(\mathrm{Spec}(\mathbb{Z})) \to \mathrm{MTM}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}&(\mathrm{Spec}(\mathbb{Q})) \\ &= \mathrm{MTM}_{/\mathrm{Spec}(\mathbb{Q}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{Q})). \end{aligned}$$

Using Remark 2.11, the same result holds when $\mathbb{Z}$ is replaced by $\mathcal{O}_{\mathbb{F},\mathcal{P}}$, the ring of $\mathcal{P}$–integers of a number field $\mathbb{F}$ (here $\mathcal{P}$ denotes a set of finite places of $\mathbb{F}$); see [16; 23]:

$$p_{\mathbb{F},\mathcal{P}}^*\colon \mathrm{MTM}_{/S,\mathbb{Q}}(S) \to \mathrm{MTM}_{/S,\mathbb{Q}}(\mathrm{Spec}(\mathbb{F})) = \mathrm{MTM}_{/\mathrm{Spec}(\mathbb{F}),\mathbb{Q}}(\mathrm{Spec}(\mathbb{F})),$$

where $S = \mathrm{Spec}(\mathcal{O}_{\mathbb{F},\mathcal{P}})$.

The functor $p_{\mathbb{F},\mathcal{P}}^*$ sends the Tate object $\mathbb{Q}_S(i)$ to $\mathbb{Q}_{\mathbb{F}}(i)$ and induces the inclusion

$$\begin{aligned} \mathcal{O}_{\mathbb{F},\mathcal{P}}^* \otimes \mathbb{Q} = \mathrm{Ext}_{\mathrm{MTM}_{/S}(S)}&(\mathbb{Q}_S(0), \mathbb{Q}_S(1)) \\ &\to \mathbb{F}^* \otimes \mathbb{Q} = \mathrm{Ext}_{\mathrm{MTM}_{/\mathrm{Spec}(\mathbb{F})}(\mathrm{Spec}(\mathbb{F}))}(\mathbb{Q}_{\mathbb{F}}(0), \mathbb{Q}_{\mathbb{F}}(1)) \end{aligned}$$

of the extension groups. Hence it induces an equivalence between the category $\mathrm{MTM}_{/S}(S)$ and the category $\mathrm{MTM}^{\mathrm{DG}}(\mathcal{O}_{\mathbb{F},\mathcal{P}})$ previously defined by Deligne and Goncharov (see [16, Sections 1.4 and 1.7]). Recall that, by definition, the category $\mathrm{MTM}^{\mathrm{DG}}(\mathcal{O}_{\mathbb{F},\mathcal{P}})$ is the Tannakian subcategory of $\mathrm{MTM}_{/\mathrm{Spec}(\mathbb{F})}(\mathrm{Spec}(\mathbb{F}))$ such that the coaction of $\mathrm{Ext}(\mathbb{Q}_{\mathbb{F}}(0), \mathbb{Q}_{\mathbb{F}}(1))$ on the canonical fiber functor factors through $\mathcal{O}_{\mathbb{F},\mathcal{P}}$. We summarize the above discussion as the following result.

Proposition 5.2 *There is an equivalence of categories*

$$\mathrm{MTM}_{/S}(S) \simeq \mathrm{MTM}^{\mathrm{DG}}(\mathcal{O}_{\mathbb{F},\mathcal{P}}).$$

Now let us work over a number field $\mathbb{F}$, ie with $S = \mathrm{Spec}(\mathbb{F})$. Theorem 3.8 continues to hold, and the moduli spaces of curves $\mathcal{M}_{0,n}$ $(n \geqslant 3)$ have a motive in $\mathrm{DMT}_{/S,\mathbb{Q}}(S)$ and satisfy the (BS) property. Hence Levine's results show that the Tannakian group $G_{/S,\mathbb{Q},n}$ associated to $\mathrm{MTM}_{/S}(\mathcal{M}_{0,n})$ is the spectrum of a Hopf $\mathbb{Q}$ algebra $H_{/S,\mathbb{Q},n}$ built from algebraic cycles (see [38]). More precisely, let $V^k_{/S,\mathbb{Q},n}(p)$ be the $\mathbb{Q}$–vector space freely generated by the closed irreducible subvarieties

$$Z \subset \mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k} \times \mathbb{A}^p$$

such that the projection

$$\mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k} \times \mathbb{A}^p \to \mathcal{M}_{0,n} \times (\mathbb{P}^1 \setminus \{1\})^{2p-k}$$

restricted to Z is dominant, flat and equidimensional of dimension 0 (ie quasifinite).

The group $\Sigma_{2p-k} \rtimes (\mathbb{Z}/2\mathbb{Z})^{2p-k}$ acts on $V^k_{/S,\mathbb{Q},n}(p)$ by permutation of the $\mathbb{P}^1 \setminus \{1\}$ factors and by the inversion $t_i \mapsto 1/t_i$ on the same factors. Let Alt_{2p-k} be the corresponding alternating projection. The symmetric group Σ_p acts on $V^k_{/S,\mathbb{Q},n}(p)$ by permutation of the $\mathbb{A}^1$ factors; we let Sym_p denote the corresponding symmetric projection.

The vector space $\mathcal{N}^k_{/S,\mathbb{Q},n}(p)$ is defined as $\mathrm{Sym}_p \circ \mathrm{Alt}_{2p-k}(V^k_{/S,\mathbb{Q},n}(p))$. For fixed p (and n), they form a complex with differential induced by the intersection with the faces of $(\mathbb{P}^1 \setminus \{1\})^{2p-k}$ given by $t_i = 0$ and $t_i = \infty$. Concatenation of factors and pullback by the diagonal $\Delta_n\colon \mathcal{M}_{0,n} \to \mathcal{M}_{0,n} \times \mathcal{M}_{0,n}$ induce a product structure on

$$\mathcal{N}_{/S,\mathbb{Q},n} = \bigoplus_{p \geqslant 0} \Big(\bigoplus_{k \geqslant 0} \mathcal{N}^k_{/S,\mathbb{Q},n}(p) \Big).$$

This endows $\mathcal{N}_{/S,\mathbb{Q},n}$ with the structure of a differential graded commutative algebra for the cohomological degree k. The bar construction of $\mathcal{N}_{/S,\mathbb{Q},n}$ as a motivic Hopf algebra is given in [9; 38]; a short review can be found in [45]. As a consequence of Levine's results in [38] and of Theorem 3.8, we obtain the following statement.

Corollary 5.3 [38] *Let $\mathbb{F}$ be a number field and $S=\mathrm{Spec}(\mathbb{F})$. Let $H_{/S,\mathbb{Q},n}$ be Hopf algebra given by the H^0 of the bar construction of $\mathcal{N}_{/S,\mathbb{Q},n}$. Then there is an isomorphism*

$$G_{/S,\mathbb{Q},n} \simeq \mathrm{Spec}(H_{/S,\mathbb{Q},n}),$$

where $G_{/S,\mathbb{Q},n}$ is the Tannakian (pro-unipotent graded) group associated to the category $\mathrm{MTM}_{/S,\mathbb{Q}}(\mathcal{M}_{0,n})$. Moreover, in the exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K_{/S,\mathbb{Q},n} \arrow[r] & G_{/S,\mathbb{Q},n} \arrow[r, "\phi_n"'] & G_{/S,\mathbb{Q}} \arrow[r] \arrow[l, bend right, dotted, "\tilde{x}_v"'] & 1, \end{tikzcd}$$

any choice of a (tangential) base point x_v in $P_{n,\infty}$ defines an action of the rational points of $G_{/S,\mathbb{Q}}$ on $K_{/S,\mathbb{Q},n}$ coming from the action over $\mathrm{Spec}(\mathbb{Z})$. Furthermore, any (possibly different) choice of a tangential base point $x_{v'}$ in $P_{n,\infty}$ identifies $K_{/S,\mathbb{Q},n}$ with Deligne–Goncharov motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathcal{M}_{0,n}, x_{v'})$.

The same holds for G_{n_1,n_2}. Families of base points x_v can be chosen in a compatible way.

The equivalence presented in Proposition 5.2 between the mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$ and the Deligne–Goncharov subcategory of mixed Tate motives over $\mathrm{Spec}(\mathbb{Q})$ allow us to obtain the following result.

Corollary 5.4 *There is an injective map from the group of rational points*

$$G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},3}(\mathbb{Q}) = G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathbb{Q})$$

to $\mathrm{GT}^{\mathrm{mot}}_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Q})$.

Proof As in the étale setting, the short motivic exact sequence

$$\begin{tikzcd} 1 \arrow[r] & K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n} \arrow[r] & G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n} \arrow[r, "\phi_n"'] & G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}} \arrow[r] \arrow[l, bend right, dotted, "\tilde{x}_v"'] & 1 \end{tikzcd}$$

gives an action of the rational points of $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}$ on $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n}$, and hence a morphism

$$G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathbb{Q}) \to \mathrm{Aut}(K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n}) \quad \text{for any } n \geqslant 4.$$

Similarly, there is an action of $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathbb{Q})$ on the $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n_1,n_2}$. These actions are compatible with the diagram (11) used in the definition of $\mathrm{GT}^{\mathrm{mot}}_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Q})$, and give a morphism

$$G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathbb{Q}) \to \mathrm{GT}^{\mathrm{mot}}_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Q}). \tag{12}$$

Let $\widetilde{x}_v = \vec{01}$ denote the standard tangential base point of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ at 0 associated to the standard affine coordinate of $\mathbb{P}^1 \setminus \infty$. The work of Brown in [12] shows that the algebra $\pi_1^{\mathrm{mot}}(\mathcal{M}_{0,4}, \vec{01})$ of functions over the Deligne–Goncharov motivic fundamental group is isomorphic to the algebra of functions over $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}$. Brown's result is purely motivic, and implies that the action of $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}$ on $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},4}$ is faithful because of Levine's isomorphism

$$K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},4} \simeq \pi_1^{\mathrm{mot}}(\mathcal{M}_{0,4}, \vec{01}),$$

recalled in Corollary 5.3. This implies the injectivity of the above morphism (12). Brown's result uses Deligne and Goncharov's category of mixed Tate motives over $\mathrm{Spec}(\mathbb{Z})$, which is a subcategory of the category of motives over $\mathrm{Spec}(\mathbb{Q})$. Levine's isomorphism also holds in the category of mixed Tate motives over $\mathrm{Spec}(\mathbb{Q})$. Proposition 5.2 ensures that both results pass to the setting developed here. □

The relation between the Betti realization of $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}$ and the classical pro-unipotent Grothendieck–Teichmüller group is reviewed in [20, Part I, Outlook, pages 423–424]. The Betti realization is used in particular to identify the pro-unipotent completion of the free group on two generators with the Betti realization of $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},4}$. A similar approach could perhaps be used to prove Conjecture 4 in Section 6 below. A more detailed review of the relations between the pro-unipotent and pro-l completions of the fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, the motivic Tannakian group $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}$ and the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ can be found in [1, Section 25].

6 Some open questions

The main goal of this article is the proof of Theorem 3.8, which, thanks to Levine's results in [38], yields a Tannakian category of mixed Tate motives over $\mathcal{M}_{0,n}$ whose Tannakian group is given by the spectrum of $H_{/S,\mathbb{Q},n}$. This now makes it possible to describe $H_{/S,\mathbb{Q},n}$ by explicit algebraic cycles, hence generalizing the construction of [44]. This will be the topic of another article.

Studying the functoriality of these Tannakian groups with respect to the geometric morphisms between the $\mathcal{M}_{0,n}$ and working over $\mathbb{Z}$ with $\mathbb{Z}$ coefficients is a natural approach, because the constructions presented in this paper are purely geometric in nature. However, these results raise a new series of questions concerning a deeper understanding of $K^{\bullet}_{/S,\mathbb{Z},n}$, questions which are more motivic homotopy-theoretic. Below, we propose some open problems in this direction.

Working with $\mathbb{Z}$ coefficients over $S=\mathrm{Spec}(\mathbb{Z})$ forces us to consider the triangulated categories $\mathrm{DMT}_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Z}}(\mathcal{M}_{0,n})$ and affine derived group schemes $G^{\bullet}_{/S,\mathbb{Z},n}$. The geometric part $K^{\bullet}_{/S,\mathbb{Z},n}$ of $G^{\bullet}_{/S,\mathbb{Z},n}$ was defined in Section 4.2 as the kernel of the morphism induced by the structure map

$$1 \to K^{\bullet}_{/S,\mathbb{Z},n} \to G^{\bullet}_{/S,\mathbb{Z},n} \xrightarrow[\phi_n]{} G^{\bullet}_{/S,\mathbb{Z}} \to 1.$$

A similar definition was given for $K^{\bullet}_{/S,\mathbb{Z},n_1,n_2}$.

Conjecture 2 The induced morphism

$$K^{\bullet}_{/S,\mathbb{Z},n_1,n_2} \xrightarrow[p_{n_1,n_2}]{} K^{\bullet}_{/S,\mathbb{Z},n_1} \times K^{\bullet}_{/S,\mathbb{Z},n_2}$$

is an isomorphism. More generally, for X and Y smooth over S satisfying the (BS) property and having a motive in $\mathrm{DMT}_{/S,\mathbb{Z}}(S)$, we conjecture that there is an isomorphism

$$K^{\bullet}_{/S,\mathbb{Z},X\times Y} \simeq K^{\bullet}_{/S,\mathbb{Z},X} \times K^{\bullet}_{/S,\mathbb{Z},Y},$$

where we use a clear "extension" of the notation $K^{\bullet}_{/S,\mathbb{Z},n} = K^{\bullet}_{/S,\mathbb{Z},\mathcal{M}_{0,n}}$.

This conjecture would endow the family of affine derived group schemes $K^{\bullet}_{/S,\mathbb{Z},n}$ with an operadic structure given by the gluing morphisms $\tilde{\imath}_{n_1,n_2,D}$ (along the line of [20, Part I, Section 4.3.5, page 155]). In order to take the action of the full symmetric group into account, the $K^{\bullet}_{/S,\mathbb{Z},n}$ should also have a cyclic operad structure. This is due to a shift between the arity of the operad and the number n of marked points; it is also caused by fixing one of the marked points (for example the first one) as a gluing point.

Remark 6.1 Let D_0 be an open codimension-1 stratum of $\overline{\mathcal{M}}_{0,n}$, and let $N^0_{D_0}$ be its normal bundle with the zero section removed. Due to the triviality of N_{D_0} (Proposition 3.2), the more general statement in Conjecture 2 implies the identification

$$K^{\bullet}_{/S,\mathbb{Z},N^0_{D_0}} \simeq K^{\bullet}_{/S,\mathbb{Z},D_0} \times K^{\bullet}_{/S,\mathbb{Z},\mathbb{G}_m} \simeq K^{\bullet}_{/S,\mathbb{Z},n_1} \times K^{\bullet}_{/S,\mathbb{Z},n_2} \times K^{\bullet}_{/S,\mathbb{Z},\mathbb{G}_m}.$$

Hence, the unit of $K^\bullet_{/S,\mathbb{Z},D_0}$ or a (tangential) base point of D_0 leads, by means of the composition with $\tilde{\imath}_{n_1,n_2,D}$, to a morphism

$$K^\bullet_{/S,\mathbb{Z},\mathbb{G}_m} \to K^\bullet_{/S,\mathbb{Z},N^0_{D_0}} \xrightarrow{\tilde{\imath}_{n_1,n_2,D}} K^\bullet_{/S,\mathbb{Z},n}.$$

This gives a reasonable way to also consider automorphisms of $K^\bullet_{/S,\mathbb{Z},\mathbb{G}_m}$ in the definition of $\mathrm{GT}^h_{/S}(\mathbb{Z})$ (resp. $\mathrm{GT}^{\mathrm{mot}}_{/S}(\mathbb{Q})$). These automorphisms must be compatible with the above composition and the rest of the tower. This approach provides a very concrete and geometric foundation for the classical hypothesis that automorphisms "preserve inertia subgroups" as, for example, in [42; 25].

Conjecture 2 and Remark 6.1 would allow us to consider only homotopy automorphisms of $K^\bullet_{/S,\mathbb{Z},n}$ and $K^\bullet_{/S,\mathbb{Z},\mathbb{G}_m}$ in the definition of $\mathrm{GT}^h_{/S}(\mathbb{Z})$. It would be even better to define $\mathrm{GT}^h_{/S}(\mathbb{Z})$ as the homotopy automorphisms of an "enriched" operad. As above, the operadic structure takes into account the action of the permutation groups and the gluing morphisms $\tilde{\imath}_{n_1,n_2,D}$. The "enriched part" would take into account the morphisms forgetting marked points and $K^\bullet_{/S,\mathbb{Z},\mathbb{G}_m}$.

Because the $\mathcal{M}_{0,n}$ are rational $K(\pi,1)$ spaces, the structure of $K^\bullet_{/S,\mathbb{Z},n}$ could be described more precisely as follows.

Conjecture 3 Let P be a finite set of points in $\mathbb{P}^1$, with $|P| \geqslant 2$. Then the affine derived group scheme $K^\bullet_{/S,\mathbb{Z},\mathbb{P}^1\setminus P}$ is an affine group scheme. Furthermore, the affine derived group scheme $K^\bullet_{/S,\mathbb{Z},n}$ is an affine group scheme

$$K^\bullet_{/S,\mathbb{Z},n} = K_{/S,\mathbb{Z},n} = \mathrm{Spec}(R_n),$$

where R_n is a commutative (but not cocommutative) Hopf algebra.

The first step in proving this conjecture would consist in dealing with the cases of $\mathbb{G}_m$, $\mathbb{P}^1 \setminus \{0,1,\infty\} = \mathcal{M}_{0,4}$ and $\mathbb{P}^1 \setminus P$. From there, the conjecture for $K^\bullet_{/S,\mathbb{Z},n}$ should follow by induction because the forgetful morphism $\mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ has fiber $\mathbb{P}^1 \setminus P$ with $|P| = n-1$. Conjecture 3 would make it possible to consider classical automorphisms in the definition of the Grothendieck–Teichmüller space. However, even if Conjecture 3 holds, the definition of $\mathrm{GT}^h_{/S}(\mathbb{Z})$ in terms of homotopy automorphisms of an "enriched" operad should be preserved, in order to investigate the properties of the morphism $G^\bullet_{/S,\mathbb{Z}}(\mathbb{Z}) \to \mathrm{GT}^h_{/S}(\mathbb{Z})$.

As we already saw in the situation of rational coefficients, a choice of tangential base points x_v in $P_{n,\infty}$ endows the $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n}$ with an action of $G_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q}}(\mathbb{Q})$, giving it a motivic structure. As $\mathrm{GT}^{\mathrm{mot}}(\mathbb{Q})$ acts on the tower of the $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n}$, it acts on the tower of their (Betti/de Rham) realizations. The corresponding towers are respectively denoted by $\mathcal{R}eal_{\mathrm{Betti}}(K_*)$ and $\mathcal{R}eal_{\mathrm{de\ Rham}}(K_*)$. We expect the following result.

Conjecture 4 We have the isomorphism

$$\mathrm{GT}^{\mathrm{mot}}(\mathbb{Q}) \simeq \mathrm{Aut}(\mathcal{R}eal_{\mathrm{Betti}}(K_*)) \simeq \mathrm{GT}(\mathbb{Q}).$$

The weight filtration on the $K_{/\mathrm{Spec}(\mathbb{Z}),\mathbb{Q},n}$ induces a weight filtration on $\mathrm{GT}^{\mathrm{mot}}(\mathbb{Q})$. Denote by $\mathrm{GRT}^{\mathrm{mot}}_{/\mathrm{Spec}(\mathbb{Z})}(\mathbb{Q})$ the induced sum of graded pieces. With this notation, we have

$$\mathrm{GRT}^{\mathrm{mot}}(\mathbb{Q}) \simeq \mathrm{Aut}(\mathcal{R}eal_{\mathrm{de\ Rham}}(K_*)) \simeq \mathrm{GRT}(\mathbb{Q}).$$

Note that in the above formulas, the second isomorphism is a consequence of the work of Bar-Natan [6] following Drinfel'd's work in [18].

Usually, the pro-unipotent version of the Grothendieck–Teichmüller space is a group scheme GT. In Conjecture 4 above, we only consider its $\mathbb{Q}$–points, because our definition of $\mathrm{GT}^{\mathrm{mot}}(\mathbb{Q})$ and $\mathrm{GT}^{h}_{/S}(\mathbb{Z})$ arises in the context of fixed coefficients (namely $\mathbb{Q}$ and $\mathbb{Z}$ coefficients). When working with a general coefficient ring R, however, the arguments given in this article should yield a more general group scheme definition for the motivic Grothendieck–Teichmüller group defined here.

References

[1] **Y André**, *Une introduction aux motifs: motifs purs, motifs mixtes, périodes*, Panoramas et Synthèses 17, Soc. Math. France, Paris (2004) MR

[2] **J Ayoub**, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, I*, Astérisque 314, Soc. Math. France, Paris (2007) MR

[3] **J Ayoub**, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique, II*, Astérisque 315, Soc. Math. France, Paris (2007) MR

[4] **J Ayoub**, *L'algèbre de Hopf et le groupe de Galois motiviques d'un corps de caractéristique nulle, II*, J. Reine Angew. Math. 693 (2014) 151–226 MR

[5] **J Ayoub**, *Une version relative de la conjecture des périodes de Kontsevich–Zagier*, Ann. of Math. 181 (2015) 905–992 MR

[6] **D Bar-Natan**, *On associators and the Grothendieck–Teichmuller group, I*, Selecta Math. 4 (1998) 183–212 MR

[7] **A A Beĭlinson**, *Higher regulators and values of L–functions*, from "Current problems in mathematics", Itogi Nauki i Tekhniki 24, Akad. Nauk SSSR, Moscow (1984) 181–238 MR In Russian; translated in J. Soviet Math. 30 (1985) 2036–2070

[8] **S Bloch**, *Algebraic cycles and higher K–theory*, Adv. in Math. 61 (1986) 267–304 MR

[9] **S Bloch**, **I Kříž**, *Mixed Tate motives*, Ann. of Math. 140 (1994) 557–605 MR

[10] **A Borel**, *Stable real cohomology of arithmetic groups*, Ann. Sci. École Norm. Sup. 7 (1974) 235–272 MR

[11] **F C S Brown**, *Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$*, Ann. Sci. Éc. Norm. Supér. 42 (2009) 371–489 MR

[12] **F Brown**, *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. 175 (2012) 949–976 MR

[13] **X Buff**, **J Fehrenbach**, **P Lochak**, **L Schneps**, **P Vogel**, *Espaces de modules des courbes, groupes modulaires et théorie des champs*, Panoramas et Synthèses 7, Soc. Math. France, Paris (1999) MR

[14] **D-C Cisinski**, **F Déglise**, *Triangulated categories of mixed motives*, preprint (2009) arXiv

[15] **F Déglise**, *Around the Gysin triangle, II*, Doc. Math. 13 (2008) 613–675 MR

[16] **P Deligne**, **A B Goncharov**, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. 38 (2005) 1–56 MR

[17] **P Deligne**, **D Mumford**, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. 36 (1969) 75–109 MR

[18] **V G Drinfel'd**, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Algebra i Analiz 2 (1990) 149–181 MR In Russian; translated in Leningrad Math. J. 2 (1991) 829–860

[19] **B I Dundas**, **M Levine**, **P A Østvær**, **O Röndigs**, **V Voevodsky**, *Motivic homotopy theory*, Springer (2007) MR

[20] **B Fresse**, *Homotopy of operads and Grothendieck–Teichmüller groups, I: The algebraic theory and its topological background*, Mathematical Surveys and Monographs 217, Amer. Math. Soc., Providence, RI (2017) MR

[21] **B Fresse**, *Homotopy of operads and Grothendieck–Teichmüller groups, II: The applications of (rational) homotopy theory methods*, Mathematical Surveys and Monographs 217, Amer. Math. Soc., Providence, RI (2017) MR

[22] **W Fulton**, *Intersection theory*, 2nd edition, Ergeb. Math. Grenzgeb. 2, Springer (1998) MR

[23] **A B Goncharov**, *Multiple polylogarithms and mixed Tate motives*, preprint (2000) arXiv

[24] **A Grothendieck**, *Esquisse d'un programme*, from "Geometric Galois actions, I" (L Schneps, P Lochak, editors), London Math. Soc. Lecture Note Ser. 242, Cambridge Univ. Press (1997) 5–48 MR

[25] **D Harbater**, **L Schneps**, *Fundamental groups of moduli and the Grothendieck–Teichmüller group*, Trans. Amer. Math. Soc. 352 (2000) 3117–3148 MR

[26] **M Hovey**, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra 165 (2001) 63–127 MR

[27] **Y Ihara**, *On the embedding of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *into* $\widehat{\mathrm{GT}}$, from "The Grothendieck theory of dessins d'enfants" (L Schneps, editor), London Math. Soc. Lecture Note Ser. 200, Cambridge Univ. Press (1994) 289–321 MR

[28] **F Ivorra**, *Cycle modules and the intersection* A_∞*–algebra*, Manuscripta Math. 144 (2014) 165–197 MR

[29] **J F Jardine**, *Motivic symmetric spectra*, Doc. Math. 5 (2000) 445–552 MR

[30] **B Kahn**, *Algebraic* K*–theory, algebraic cycles and arithmetic geometry*, from "Handbook of K–theory" (E M Friedlander, D R Grayson, editors), volume I, Springer (2005) 351–428 MR

[31] **M Kashiwara**, **P Schapira**, *Sheaves on manifolds*, Grundl. Math. Wissen. 292, Springer (1994) MR

[32] **S Keel**, *Intersection theory of moduli space of stable* n*–pointed curves of genus zero*, Trans. Amer. Math. Soc. 330 (1992) 545–574 MR

[33] **F F Knudsen**, *The projectivity of the moduli space of stable curves, II: The stacks* $M_{g,n}$, Math. Scand. 52 (1983) 161–199 MR

[34] **M Levine**, *Tate motives and the vanishing conjectures for algebraic* K*–theory*, from "Algebraic K–theory and algebraic topology" (P G Goerss, J F Jardine, editors), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. 407, Kluwer, Dordrecht (1993) 167–188 MR

[35] **M Levine**, *Bloch's higher Chow groups revisited*, from "K–theory", Astérisque 226, Soc. Math. France, Paris (1994) 235–320 MR

[36] **M Levine**, *Mixed motives*, Mathematical Surveys and Monographs 57, Amer. Math. Soc., Providence, RI (1998) MR

[37] **M Levine**, *Mixed motives*, from "Handbook of K–theory" (E M Friedlander, D R Grayson, editors), volume I, Springer (2005) 429–521 MR

[38] **M Levine**, *Tate motives and the fundamental group*, from "Cycles, motives and Shimura varieties" (V Srinivas, editor), Tata Inst. Fund. Res. Stud. Math. 21, Tata Inst. Fund. Res., Mumbai (2010) 265–392 MR

[39] **F Morel**, **V Voevodsky**, $\mathbf{A}^1$*–homotopy theory of schemes*, Inst. Hautes Études Sci. Publ. Math. 90 (1999) 45–143 MR

[40] **H Nakamura**, *Coupling of universal monodromy representations of Galois–Teichmüller modular groups*, Math. Ann. 304 (1996) 99–119 MR

[41] **M Rost**, *Chow groups with coefficients*, Doc. Math. 1 (1996) 319–393 MR

[42] **L Schneps**, *Special loci in moduli spaces of curves*, from "Galois groups and fundamental groups" (L Schneps, editor), Math. Sci. Res. Inst. Publ. 41, Cambridge Univ. Press (2003) 217–275 MR

[43] **I Soudères**, *Motivic double shuffle*, Int. J. Number Theory 6 (2010) 339–370 MR

[44] **I Soudères**, *Cycle complex over the projective line minus three points: toward multiple zeta values cycles*, preprint (2012) arXiv

[45] **I Soudères**, *A relative basis for mixed Tate motives over the projective line minus three points*, preprint (2013) arXiv

[46] **M Spitzweck**, *Operads, algebras and modules in model categories and motives*, PhD thesis, Bonn Universität (2001) Available at `http://hss.ulb.uni-bonn.de/2001/0241/0241.htm`

[47] **M Spitzweck**, *Motivic approach to limit sheaves*, from "Geometric methods in algebra and number theory" (F Bogomolov, Y Tschinkel, editors), Progr. Math. 235, Birkhäuser, Boston (2005) 283–302 MR

[48] **M Spitzweck**, *Periodizable motivic ring spectra*, preprint (2009) arXiv

[49] **M Spitzweck**, *Derived fundamental groups for Tate motives*, preprint (2010) arXiv

[50] **M Spitzweck**, *A commutative $\mathbb{P}^1$–spectrum representing motivic cohomology over Dedekind domains*, preprint (2013) arXiv

[51] **A Suslin**, **V Voevodsky**, *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, from "The arithmetic and geometry of algebraic cycles" (B B Gordon, J D Lewis, S Müller-Stach, S Saito, N Yui, editors), NATO Sci. Ser. C Math. Phys. Sci. 548, Kluwer, Dordrecht (2000) 117–189 MR

[52] **B Toën**, *Homotopical and higher categorical structures in algebraic geometry*, preprint (2003) arXiv

[53] **G Vezzosi**, *Brown–Peterson spectra in stable $\mathbb{A}^1$–homotopy theory*, Rend. Sem. Mat. Univ. Padova 106 (2001) 47–64 MR

[54] **V Voevodsky**, *Triangulated categories of motives over a field*, from "Cycles, transfers, and motivic homology theories", Ann. of Math. Stud. 143, Princeton Univ. Press (2000) 188–238 MR

[55] **V Voevodsky**, *On motivic cohomology with $\mathbb{Z}/l$–coefficients*, Ann. of Math. 174 (2011) 401–438 MR

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