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Stable presentation length of 3–manifold groups

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We introduce the stable presentation length of a finitely presentable group. The stable presentation length of the fundamental group of a 3–manifold can be considered as an analogue of the simplicial volume. We show that, like the simplicial volume, the stable presentation length has some additive properties, and the simplicial volume of a closed 3–manifold is bounded from above and below by constant multiples of the stable presentation length of its fundamental group.

57M05, 57M27; 57M10, 57M20

1 Introduction

Mostow–Prasad rigidity (see [32] and [37]) states that a finite-volume hyperbolic 3–manifold is determined by its fundamental group. In particular, the volume of a hyperbolic 3–manifold is a topological invariant. The simplicial volume of a manifold, introduced by Gromov [15], is defined topologically, and it is proportional to the volume for a hyperbolic manifold. Furthermore, the simplicial volume of a Seifert 3–manifold is equal to zero, and the simplicial volume of a 3–manifold is additive for connected sums and decompositions along incompressible tori; see Soma [41]. Therefore, the geometrization theorem proved by Perelman [34; 35] implies that the simplicial volume of an orientable closed 3–manifold is equal to the sum of the simplicial volumes of hyperbolic pieces after the geometrization.

The simplicial volume of a closed 3–manifold is uniquely determined by its fundamental group. If the fundamental group of an orientable closed 3–manifold is freely decomposable, the 3–manifold can be decomposed into a connected sum corresponding to the free product. Hence it is sufficient to show that the claim holds for closed irreducible 3–manifolds. A closed Haken 3–manifold is determined by its fundamental group; see Waldhausen [43]. A non-Haken 3–manifold is elliptic or hyperbolic by the geometrization theorem. The simplicial volume of an elliptic manifold is equal to zero, and Mostow rigidity implies that a hyperbolic manifold is determined by its fundamental group. In order to consider a direct relation between the simplicial volume

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of a 3–manifold and its fundamental group, we will introduce the stable presentation length of a finitely presentable group.

Milnor and Thurston [31] considered some characteristic numbers of manifolds, where "characteristic" means multiplicativity for the finite-sheeted coverings; ie an invariant C of manifolds is a characteristic number if $C(N) = d \cdot C(M)$ for any d–sheeted covering $N \to M$. For example, the Euler characteristic and the simplicial volume are characteristic numbers. We say such an invariant is *volume-like*, instead of a characteristic number, in order to indicate similarity to the volume. Milnor and Thurston introduced the following volume-like invariant of a manifold, which is called the stable Δ–complexity by Francaviglia, Frigerio and Martelli [12]. The *Δ–complexity* $\sigma(M)$ of a closed n–manifold M is the minimal number of n–simplices in a triangulation of M. In this paper we use the term "triangulation" in a weak sense; ie a triangulation of a manifold M is a cellular decomposition of M such that each cell is a simplex. The Δ–complexity is not volume-like, but it is an upper volume in the sense of Reznikov [38]; ie $\sigma(N) \leq d \cdot \sigma(M)$ for any d–sheeted covering $N \to M$. Then in a natural way, we define a volume-like invariant

$$\sigma_\infty(M) = \inf_{N \to M} \frac{\sigma(N)}{\deg(N \to M)},$$

called the stable Δ–complexity of M, where the infimum is taken over the finite-sheeted coverings of M.

While the stable Δ–complexity is hard to handle, the simplicial volume following it can work similarly and has more applications. Thus the stable Δ–complexity became something obsolete, but recently, Francaviglia, Frigerio and Martelli [12] brought a further development. They introduced the stable complexity of a 3–manifold. The *complexity* $c(M)$ of 3–manifold M is the minimal number of vertices in a simple spine for M. Matveev [29, Theorem 5] showed that the complexity of M is equal to its Δ–complexity if M is irreducible and not S^3, $\mathbb{RP}^3$ or the lens space $L(3,1)$. In particular, the two complexities of M coincide if M is a hyperbolic 3–manifold. The stable complexity $c_\infty(M)$ is defined in the same way as the stable Δ–complexity. Francaviglia, Frigerio and Martelli showed that the stable complexity has same additivity as the simplicial volume of a 3–manifold, and therefore $c_\infty(M)$ is the sum of the stable complexities of the hyperbolic pieces after the geometrization [12, Corollary 5.3 and Proposition 5.10]. Moreover, the stable complexity of a 3–manifold is bounded from above and below by constant multiples of the simplicial volume. This is implied by the fact that the stable Δ–complexity of a hyperbolic 3–manifold is so.

Delzant [10] introduced a complexity $T(G)$ of a finitely presentable group G. We call it the presentation length in accordance with Agol and Liu [2]. Delzant also introduced a relative version of presentation length, and he gave an estimate of presentation length for a decomposition of group. There are some applications for the presentation length of the fundamental group of a 3–manifold. It has been used by Cooper [9] to give an upper bound for the volume of a hyperbolic 3–manifold, by White [44] to give an estimate for the diameter of a closed hyperbolic 3–manifold, and by Agol and Liu [2] to solve Simon's conjecture.

Delzant and Potyagailo [11] remarked that the volume of hyperbolic 3–manifold is not bounded from below by a constant multiple of the presentation length. They considered a relative presentation length for a thick part of a hyperbolic 3–manifold and showed that the volume is bounded from above and below by constant multiples of this relative presentation length. We will introduce the stable presentation length instead of this.

The presentation length is an upper volume. Hence we can define the stabilization of the presentation length. We will show the stable presentation length of a 3–manifold has additivity similar to the simplicial volume and the stable complexity (Theorems 5.1 and 5.3).

We conjecture that the stable presentation length for a 3–manifold is half of the stable complexity (Conjecture 4.8). This conjecture is relevant to the following problem. Francaviglia, Frigerio and Martelli asked whether the simplicial volume and the stable complexity of a 3–manifold coincide. The additivity of the simplicial volume and the stable complexity reduces this problem to the cases for the hyperbolic 3–manifolds [12, Question 6.5]. The Ehrenpreis conjecture proved by Kahn and Markovic [19] states that for any two closed hyperbolic surfaces M, N and $K > 1$, there are finite coverings of M, N which are K–quasiconformal. The simplicial volume and the stable Δ–complexity of a hyperbolic 3–manifold coincide if and only if, roughly speaking, a hyperbolic 3–manifold has a finite covering with a triangulation in which almost all the tetrahedra after straightening are nearly isometric to an ideal regular tetrahedron. Therefore, the above problem can be considered as a 3–dimensional version of the Ehrenpreis problem.

Frigerio, Löh, Pagliantini and Sauer [13] showed that the simplicial volume and the stable integral simplicial volume of a closed hyperbolic 3–manifold coincide. The integral simplicial volume of an oriented closed manifold is defined as the seminorm of the fundamental class in the integer homology. The stable integral simplicial volume is the stabilization of the integral simplicial volume in the same way as the stable Δ–complexity.

Since the integral simplicial volume is quite similar to the Δ–complexity, the above result supports the affirmative answer to that 3–dimensional version of Ehrenpreis problem, at least for the closed hyperbolic 3–manifolds. In contrast to the lower-dimensional cases, the simplicial volume and the stable integral simplicial volume of a closed hyperbolic manifold of dimension at least 4 cannot coincide [12, Theorem 2.1].

The presentation length of a group can be considered as a 2–dimensional version of the rank, which is the minimal number of generators. The relation between the presentation length and the Δ–complexity of a 3–manifold is analogous to the relation between the rank and the Heegaard genus. The Heegaard genus of a closed 3–manifold is not less than the rank of its fundamental group, and they do not coincide in general; see Boileau and Zieschang [5] and Li [25]. Lackenby [21; 22] introduced the rank gradient and the Heegaard gradient to approach the virtually Haken conjecture. The rank gradient of a finitely generated group G is defined as $\inf_H (\operatorname{rank}(H)-1)/[G:H]$, where the infimum is taken over all the finite-index subgroups H of G. Similarly, The Heegaard gradient of a finitely generated group G is defined as $\inf_N \chi_-^h(N)/\deg(N \to M)$, where $\chi_-^h(N)$ is the negative of the maximal Euler characteristic of a Heegaard surface of N, and the infimum is taken over all the finite coverings N of M. Since the virtually fibered conjecture was solved by Agol [1], we know that the rank gradient and the Heegaard gradient of a hyperbolic 3–manifold are equal to zero.

We mention a relation between the homology torsion and the stable presentation length. Let $|\operatorname{Tor}(A)|$ denote the order of the torsion part of an abelian group A. For a group G, we consider the torsion part of its abelianization $G/[G,G]$ (in other words, the first integral homology). Pervova and Petronio [36] gave the following inequality: if G is a finitely presentable group without 2–torsion,

$$T(G) \geq \log_3 |\operatorname{Tor}(G/[G,G])|.$$

As a relevant problem, Bergeron and Venkatesh [4], Lück [27] and Le [23] conjectured

$$\lim_i \frac{\log |\operatorname{Tor}(G_i/[G_i,G_i])|}{[G:G_i]} = \frac{\operatorname{vol}(M)}{6\pi}$$

for a hyperbolic 3–manifold M and an appropriate sequence $G_1 > G_2 > \cdots$ of finite-index subgroups of $G = \pi_1(M)$. Bergeron and Venkatesh gave conjectures also for lattices in more general Lie groups.

At last we give a question for amenable groups. The simplicial volume of a manifold with amenable fundamental group is equal to zero [15, Section 3.1], and the rank gradient of a finitely presentable, residually finite, infinite amenable group is also equal to zero [21, Theorem 1.2]. Similarity between the volume and the stable presentation length induce the following question.

Question 1.1 *For a finitely presentable amenable group G, is the stable presentation length $T_\infty(G)$ equal to zero?*

Organization of the paper In Section 2, we review the definition and elementary properties of the presentation length. In Section 3, we define the stable presentation length as a volume-like invariant of a finitely presentable group.

In Section 4, we consider the stable presentation length of a hyperbolic 3–manifold. For a 3–manifold with boundary, it is natural to consider its presentation length relative to the fundamental groups of the boundary components. We show that the stable presentation length of the hyperbolic 3–manifold relative to the cusp subgroups coincides with the nonrelative stable presentation length (Theorem 4.1). In fact, we show a more general result for a residually finite group and free abelian subgroups (Theorem 4.2). This result is the most technical part in this paper. The simplicial volume has a similar property; see Löh and Sauer [26, Theorem 1.5]. Namely, we can consider two versions of simplicial volume of a manifold M with boundary. One is the seminorm of the relative fundamental class, and another is for the open manifold $\operatorname{int} M$. They coincide if the fundamental groups of the boundary components are amenable. Furthermore, we show that the stable presentation length of a hyperbolic 3–manifold is bounded by constant multiples of the volume and the stable complexity.

In Section 5, we show additivity of the stable presentation length. We give a proof similar to the proof for the stable complexity by using Delzant's result (Theorem 2.7) and Theorem 4.2. We also show that the stable presentation length of a Seifert 3–manifold vanishes (Theorem 5.2). These results imply that the stable presentation length of a closed 3–manifold is equal to the sum of the stable presentation lengths of hyperbolic pieces after the geometrization.

In Section 6, we give some examples of stable presentation length. The stable presentation lengths of the surface groups are the only examples of nonzero stable presentation length we compute explicitly in this paper. We also give examples for fundamental groups of some hyperbolic 3–manifolds. Those examples support Conjecture 4.8.

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2 Preliminaries for presentation length

We review the definition of presentation length (also known as Delzant's T–invariant) and some elementary facts. See Delzant [10] for details.

Definition 2.1 Let G be a finitely presentable group. We define the *presentation length* $T(G)$ of G by

$$T(G) = \min_{\mathcal{P}} \sum_{i=1}^{m} \max\{0, |r_i| - 2\},$$

where we take the minimum over presentations $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of G, and let $|r_i|$ denote the word length of r_i.

To a presentation $\mathcal{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of G, we associate the *presentation complex* P, the 2–dimensional cellular complex consisting of a single 0–cell and of 1–cells and 2–cells corresponding to the generators and relators. Then $\pi_1(P)$ is isomorphic to G. By dividing a k–gon of a presentation complex into $k-2$ triangles, $T(G)$ can be realized by a *triangular presentation* of G, ie a presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ in which each word length $|r_i|$ is equal to 2 or 3. If G has no 2–torsion, we can assume $|r_i| = 3$. From now on, a presentation complex is always assumed to be triangular; ie each of its 2–cells is a triangle or a bigon. $T(G)$ is the minimal number of triangles in a presentation complex for G.

Delzant [10] also introduced a relative version of the presentation length. We need this in order to estimate the presentation length under a decomposition of group. Before defining the relative presentation length, we prepare the notion of an orbihedron, due to Haefliger [16], which is an analogue of an orbifold. An *orbihedron* is a cellular complex with isotropy groups on cells such that after an appropriate subdivision of cells, the star neighborhood of each cell c is isomorphic to the quotient of a cellular complex by a certain cellular action of the isotropy group which fixes the preimage of c. This local structure gives the notion of a covering space of an orbihedron, analogously to an orbifold. The universal covering of an orbihedron is a covering without further nontrivial connected coverings. The fundamental group of an orbihedron is the deck transformation of its universal covering. Consequently, an orbihedron is isomorphic to the quotient of its universal covering by its fundamental group. If the isotropy group on every cell in the universal covering of an orbihedron is trivial, its isotropy groups are identified with subgroups of the fundamental group up to conjugacy. Note that isotropy groups of an orbihedron are possibly infinite, unlike an orbifold.

Definition 2.2 Let G be a finitely presentable group. Suppose that $C_1, \dots, C_l$ are subgroups of G. A *(relative) presentation complex* P for $(G; C_1, \dots, C_l)$ is a 2–dimensional orbihedron satisfying the following conditions:

- Any 2–cell of P is a triangle or a bigon.
- The 0–cells of P consist of l vertices with isotropy groups $C_1, \dots, C_l$. The isotropy groups of the 1–cells and 2–cells are trivial.
- The isotropy groups of the universal covering of P are trivial, and the fundamental group $\pi_1^{\mathrm{orb}}(P)$ of P as an orbihedron is isomorphic to G. This isomorphism makes the isotropy groups $C_1, \dots, C_l$ be subgroups of G up to conjugacy.

We define the *relative presentation length* $T(G; C_1, \dots, C_l)$ as the minimal number of triangles in a relative presentation complex for $(G; C_1, \dots, C_l)$. We say that a presentation complex P is *minimal* if P realizes the presentation length.

Our definition requires that the isotropy is only on the vertices, but this is not essential. Indeed, if an isotropy of a 2–complex is on edges or 2–cells, we can construct a presentation complex by replacing edges with bigons. We can consider only the conjugacy classes of $C_1, \dots, C_l < G$. By definition, we have $T(G; \{1\}) = T(G)$. We can allow a presentation complex for G to have more than one vertex, namely, $T(G; \{1\}, \dots, \{1\}) = T(G; \{1\})$. This follows by contracting vertices of a presentation complex along edges without changing the fundamental group. More generally, the following holds.

Proposition 2.3 [10, Lemma I.1.3] *For a finitely presentable group G and its subgroups $C, C', C_1, \dots, C_l$, suppose that C' is contained in a conjugate of C. Then*

$$T(G; C, C', C_1, \dots, C_l) = T(G; C, C_1, \dots, C_l).$$

The relative presentation length is usually finite, though the definition does not require this. The construction in the proof will be used for the proof of Theorem 4.2.

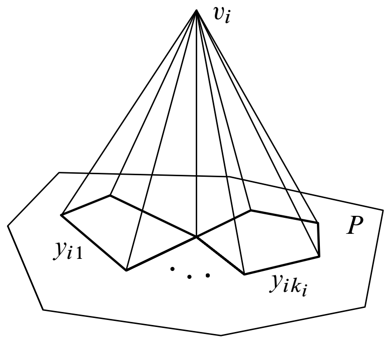

Figure 1: Construction of a relative presentation complex

Proposition 2.4 *Let G be a finitely presentable group. Suppose that $C_1, \dots, C_l$ are finitely generated subgroups of G. Then there is a finite presentation complex for $(G; C_1, \dots, C_l)$; in other words, we have $T(G; C_1, \dots, C_l) < \infty$.*

Proof Take a presentation complex P for G. Let $y_{i1}, \dots, y_{ik_i}$ be generators of C_i for $1 \le i \le l$. There exist simplicial paths $a_{i1}, \dots, a_{ik_i}$ in P corresponding to $y_{i1}, \dots, y_{ik_i}$. We construct a complex P' by attaching cones of $a_{i1}, \dots, a_{ik_i}$ to P; see Figure 1. Put isotropy C_i on the vertex of the i^{th} cone. Then P' is a finite presentation complex for $(G; C_1, \dots, C_l, \{1\})$. □

Delzant [10] showed how the presentation length behaves under a decomposition into a graph of groups. A *graph of groups* $\mathcal{G}$, in the sense of Serre [40], is a collection of the following data:

- an underlying connected graph Γ, consisting a vertex set V, an edge set E and maps $o_\pm\colon E \to V$ from edges to their endpoints;
- vertex groups $\{G_v\}$ and edge groups $\{C_e\}$ for $v \in V$ and $e \in E$;
- injections $\{\iota_\pm\colon C_e \hookrightarrow G_{o_\pm(e)}\}$ for $e \in E$.

The fundamental group $\pi_1(\mathcal{G})$ can be characterized as follows. A graph of spaces $\mathcal{X}$ corresponding to $\mathcal{G}$ is a collection of CW-complexes $\{X_v\}$, $\{X_e\}$ and π_1–injective maps $\{i_\pm\colon X_e \to X_{o_\pm(e)}\}$, where $\pi_1(X_v) = G_v$, $\pi_1(X_e) = C_e$ and $i_\pm$ induces $\iota_\pm$. We construct a space

$$X_{\mathcal{X}} = \Big(\bigsqcup_{v \in V} X_v \sqcup \bigsqcup_{e \in E} (X_e \times [-1, 1]) \Big) \Big/ \sim,$$

where the gluing relation is that $(x, \pm 1) \sim i_{\pm}(x)$ for $x \in X_e$. Then $\pi_1(\mathcal{G}) = \pi_1(X_{\mathcal{X}})$. For a given group G, we say that $\mathcal{G}$ is a decomposition of G if $G \cong \pi_1(\mathcal{G})$.

Let $\mathcal{G}$ be a decomposition of a group G. Suppose that $G_1, \dots, G_n$ are the vertex groups of $\mathcal{G}$ and $C_1, \dots, C_l$ are the edge groups of $\mathcal{G}$. We construct presentation complexes P_i for $(G_i; C_{i1}, \dots, C_{il_i})$, where C_{ij} for $1 \le j \le l_i$ are the edge groups corresponding to the edges that have the i^{th} vertex as an endpoint. We can construct a presentation complex P for $(G; C_1, \dots, C_l)$ by gluing $P_1, \dots, P_n$ along their vertices. Then the number of triangles of P is the sum of those of the P_i. Therefore, we have the following proposition.

Proposition 2.5 [10, Lemma I.1.4] *Let G, C_i and C_{ij} be as above. Then*

$$T(G; C_1, \dots, C_l) \le \sum_{i=1}^{n} T(G_i; \{C_{ij}\}_{1 \le j \le l_i}).$$

We need to consider a "good" decomposition in order to estimate the presentation length from below.

Definition 2.6 Let $\mathcal{G}$ be a decomposition of G, and let $C_1, \dots, C_l$ be the edge subgroups of $\mathcal{G}$. A subgroup C of G is *rigid* if it satisfies the following condition: If G acts on a tree T without inversion and C contains a nontrivial stabilizer of an edge of T, then C fixes a vertex of T. We say $\mathcal{G}$ is *rigid* if every edge group of $\mathcal{G}$ is rigid.

Let C_{ij} be as Proposition 2.5. $\mathcal{G}$ is *reduced* if there is no decomposition $\mathcal{G}'$ of G_i such that C_{ij} is a vertex group of $\mathcal{G}'$ for any G_i and C_{ij}.

With this preparation, we can state the following highly nontrivial fact.

Theorem 2.7 [10, Theorem II] *Let $\mathcal{G}$, G_i and C_{ij} be as in Proposition 2.5. Suppose that $\mathcal{G}$ is rigid and reduced. Then*

$$T(G) \ge \sum_{i=1}^{n} T(G_i; \{C_{ij}\}_{1 \le j \le l_i}).$$

Since a free product decomposition of a group is rigid and reduced, we have the following theorem.

Corollary 2.8 [10, Corollary I] *Let $G = A * B$ be a free product of finitely presentable groups. Then $T(G) = T(A) + T(B)$.*

We will mainly consider the fundamental group of an orientable 3–manifold. A connected sum decomposition of a 3–manifold induces a free product decomposition of the fundamental group, which concerns Corollary 2.8. A (possibly disconnected) orientable surface S embedded in an irreducible orientable 3–manifold M is an *essential surface* if S does not contain a sphere, each component S_i of S induces an injection $\pi_1(S_i) \hookrightarrow \pi_1(M)$, and no pair of components of S are parallel. A decomposition of an irreducible orientable 3–manifold along an essential surface induces a decomposition of its fundamental group into a graph of groups. Then a component of the decomposed manifold corresponds to a vertex group, and a component of the essential surface corresponds to an edge group. We can apply Theorem 2.7 in this case.

Proposition 2.9 [10, Proposition I.6.1] *Let $\mathcal{G}$ be a decomposition of the fundamental group of an irreducible orientable 3–manifold M. Suppose that $\mathcal{G}$ corresponds to a decomposition of M along an essential surface. Then $\mathcal{G}$ is rigid and reduced.*

3 Definition of stable presentation length

The (relative) presentation length is an upper volume; ie it has the following submultiplicative property.

Proposition 3.1 *For a finitely presentable group G, let H be a finite-index subgroup of G. Let $d = [G : H]$ denote the index of H in G. Suppose that $C_1, \dots, C_l$ are subgroups of G. Then*

$$T(H; \{gC_ig^{-1} \cap H\}_{1\le i\le l, g\in G}) \le d \cdot T(G; C_1, \dots, C_l).$$

In particular, $T(H) \le d \cdot T(G)$.

We remark that $\{gC_ig^{-1} \cap H\}_{1\le i\le l, g\in G}$ is a finite family of subgroups up to conjugate in H, since H is a finite-index subgroup of G.

Proof Let P be a minimal presentation complex for $(G; C_1, \dots, C_l)$. There exists a d–sheeted covering $\widetilde{P}$ of P as an orbihedron which corresponds to $H \le G$. Then the isotropies on the vertices of $\widetilde{P}$ are $\{gC_ig^{-1} \cap H\}_{1\le i\le l, g\in G}$. Therefore, $\widetilde{P}$ is a presentation complex for $(H; \{gC_ig^{-1} \cap H\}_{1\le i\le l, g\in G})$ with $d \cdot T(G; C_1, \dots, C_l)$ triangles. □

Proposition 3.1 leads to the definition of stable presentation length as an analogue of the stable Δ–complexity by Milnor and Thurston [31]. Stable presentation length is a volume-like invariant; ie it is multiplicative for finite-index subgroups.

Definition 3.2 We define the *stable presentation length* $T_\infty(G)$ of a finitely presentable group G by

$$T_\infty(G) = \inf_{H \leq G} \frac{T(H)}{[G:H]},$$

where the infimum is taken over all the finite-index subgroups H. Furthermore, suppose that $C_1, \dots, C_l$ are subgroups of G. Define the (*relative*) *stable presentation length* as

$$T_\infty(G; C_1, \dots, C_l) = \inf_{H \leq G} \frac{T(H; \{gC_i g^{-1} \cap H\}_{1 \leq i \leq l, g \in G})}{[G:H]}.$$

Proposition 3.3 *Let G, H, d and $C_1, \dots, C_l$ be as in Proposition 3.1. Then*

$$T_\infty(H; \{gC_i g^{-1} \cap H\}_{1 \leq i \leq l, g \in G}) = d \cdot T_\infty(G; C_1, \dots, C_l).$$

In particular, $T_\infty(H) = d \cdot T_\infty(G)$.

Proof Take a finite-index subgroup G' of G. Then $H' = G' \cap H$ is also a finite-index subgroup of G. We have

$$T(H'; \{gC_i g^{-1} \cap H'\}_{1 \leq i \leq l, g \in G}) \leq [G' \colon H'] \cdot T(G'; \{gC_i g^{-1} \cap G'\}_{1 \leq i \leq l, g \in G})$$

by Proposition 3.1. Hence we can calculate $T_\infty(G; C_1, \dots, C_l)$ by taking the infimum for only the subgroups of H. Therefore,

$$\begin{aligned} T_\infty(H; \{gC_i g^{-1} \cap H\}_{1 \leq i \leq l, g \in G}) &= \inf_{H' \leq H} \frac{T(H'; \{gC_i g^{-1} \cap H'\}_{1 \leq i \leq l, g \in G})}{[H:H']} \\ &= d \cdot \inf_{H' \leq H} \frac{T(H'; \{gC_i g^{-1} \cap H'\}_{1 \leq i \leq l, g \in G})}{[G:H']} \\ &= d \cdot T_\infty(G; C_1, \dots, C_l). \end{aligned}$$ □

4 Stable presentation length for hyperbolic 3–manifolds

We consider the stable presentation length of the fundamental group of a compact 3–manifold M. We write

$$\begin{aligned} T(M) &= T(\pi_1(M)), & T(M; \partial M) &= T(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l)), \\ T_\infty(M) &= T_\infty(\pi_1(M)), & T_\infty(M; \partial M) &= T_\infty(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l)), \end{aligned}$$

where $S_1, \dots, S_l$ are the components of ∂M. We call those in the right column and bottom row the relative and stable presentation lengths of M, respectively.

If M is a 3–manifold with boundary, we can also consider the relative presentation length $T(M; \partial M)$. For instance, let M be a finite-volume cusped hyperbolic 3–manifold. We consider M as a compact 3–manifold with boundary. The interior of M admits a hyperbolic metric. Let $S_1, \dots, S_l$ be the components of ∂M. The 2–skeleton of an ideal triangulation of M (ie a cellular decomposition of the space obtained by smashing each boundary component of M to a point such that every 3–cell is tetrahedron and its vertices are the points from boundary components of M) can be regarded as a relative presentation complex of $(\pi_1(M); \pi_1(S_1), \dots, \pi_1(S_l))$. We show that this relative stable presentation length coincides with the absolute stable presentation length.

Theorem 4.1 *If M is a finite-volume hyperbolic 3–manifold, $T_\infty(M; \partial M) = T_\infty(M)$.*

More generally, we show the following theorem. Since $\pi_1(M)$ is linear for a hyperbolic 3–manifold M, it is residually finite [18].

Theorem 4.2 *Let G be a finitely presentable group, and let $C_1, \dots, C_l$ be free abelian subgroups of G whose ranks are at least 2. Suppose G is residually finite. Then it holds that $T_\infty(G; C_1, \dots, C_l) = T_\infty(G)$.*

We remark that it is necessary to suppose the rank of C_i is at least 2. The inequality does not hold for the case of Theorem 6.2 since $T_\infty(\pi_1(\Sigma_{g,b})) = 0$.

For an integer $p > 1$, the *p–characteristic covering* of the torus T^2 is the covering which corresponds to the subgroup $p\mathbb{Z} \times p\mathbb{Z} < \mathbb{Z} \times \mathbb{Z} \cong \pi_1(T^2)$. A *p–characteristic covering* of M is a finite covering whose restriction on each cusp is a union of p–characteristic coverings of the torus. A hyperbolic 3–manifold M admits p–characteristic coverings for arbitrarily large p [18, Lemma 4.1]. We can use them for a proof of Theorem 4.1. In general, however, a residually finite group G with $C_1, \dots, C_l$ may not have such subgroups. Nonetheless, we can take a nearly orthogonal basis of a subgroup of C_i with respect to a basis of C_i $(1 \le i \le l)$.

A *lattice* in $\mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ which spans $\mathbb{R}^n$. A lattice in $\mathbb{R}^n$ has a nearly orthogonal basis as in Lemma 4.3, called a *reduced basis*. We refer to Cassels [7, Section VIII.5.2] for a proof. Lenstra, Lenstra and Lovász [24] gave a polynomial-time algorithm to find a reduce basis. We will use the following lemma with a 1–norm on $\mathbb{R}^n$.

Lemma 4.3 *Given a norm $\|\cdot\|$ in $\mathbb{R}^n$, there is a constant ϵ_n such that the following holds. If Λ is a lattice in $\mathbb{R}^n$, then there is a basis $(v_1,\dots,v_n)$ of Λ such that*

$$d(\Lambda)\geq \epsilon_n\|v_1\|\cdots\|v_n\|,$$

where $d(\Lambda)$ is the covolume of Λ, which is the determinant of the matrix whose columns are the v_i.

Proof of Theorem 4.2 We first show that $T_\infty(G;C_1,\dots,C_l)\leq T_\infty(G)$.

Assume that $T_\infty(G;C_1,\dots,C_l)\leq T(G)$ for any groups G and $C_1,\dots,C_l$ satisfying the condition. Then $T_\infty(H;\{gC_ig^{-1}\cap H\}_{1\leq i\leq l,g\in G})\leq T(H)$ for a finite-index subgroup H of G. Proposition 3.3 implies that $T_\infty(G;C_1,\dots,C_l)\leq T(H)/[G:H]$. By taking the infimum over H, we obtain $T_\infty(G;C_1,\dots,C_l)\leq T_\infty(G)$. Hence it is sufficient to show that $T_\infty(G;C_1,\dots,C_l)\leq T(G)$. For simplicity, we assume $l=1$ and write $C=C_1$ and $r=\mathrm{rank}(C)\geq 2$.

Take a minimal presentation complex P for G. Let $\alpha_1,\dots,\alpha_r$ be simplicial paths in P representing generators $x_1,\dots,x_r$ of C. Let a_i denote the length of α_i for $1\leq i\leq r$. Since a finite-index subgroup H of G contains a finite-index normal subgroup $\bigcap_{g\in G} gHg^{-1}$ of G, it is sufficient to consider the finite-index normal subgroups of G. Suppose that H is a finite-index normal subgroup of G. Let d denote the index of $H<G$. Let $\widetilde{P}$ be the covering of P corresponding to H. Let $\{C_1',\dots,C_m'\}$ be subgroups of H representing the conjugacy classes of $\{gCg^{-1}\cap H\}_{g\in G}$. We can regard C_i' as a finite-index subgroup of C by the natural inclusion $\iota_i\colon C_i'\hookrightarrow C$. Since H is normal in G, all the images of the ι_i coincide and have index d/m in C. We regard $C\cong\mathbb{Z}^r$ as a lattice in $\mathbb{R}^r$ and put the 1–norm $\|\cdot\|$ in $\mathbb{R}^r$ with respect to the basis $(x_1/a_1,\dots,x_r/a_r)$.

We construct a presentation complex $\widetilde{P}'$ for $(H;\{C_i'\}_{1\leq i\leq m})$ by attaching 2–cells to $\widetilde{P}$. We take a reduced basis $(y_1,\dots,y_r)$ of $\iota_i(C_i')$ as in Lemma 4.3. Let $\beta_{i1},\dots,\beta_{ir}$ be paths in $\widetilde{P}$ representing $y_1,\dots,y_r\in\iota_i(C_i')$ such that the length of β_{ij} is $\|y_j\|$. We obtain a presentation complex $\widetilde{P}'$ by attaching cones of the β_{ij} as in the proof of Proposition 2.4. The number of the triangles of $\widetilde{P}'$ is

$$d\cdot T(G)+m(\|y_1\|+\cdots+\|y_r\|).$$

It holds that $d/m\geq\epsilon_r\|y_1\|\cdots\|y_r\|$ by Lemma 4.3. Hence

$$T_\infty(G;C)\leq\frac{T(H;\{C_i'\}_{1\leq i\leq m})}{d}\leq T(G)+\frac{\|y_1\|+\cdots+\|y_r\|}{\epsilon_r\|y_1\|\cdots\|y_r\|}.$$

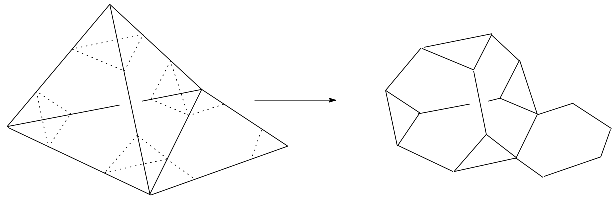

Figure 2: Truncation of the presentation complex Q

Since G is residually finite, there is a normal subgroup H of G such that every $\|y_j\|$ for $1 \le j \le r$ is arbitrarily large. We have supposed that $r \ge 2$. Therefore, we obtain $T_\infty(G;C) \le T(G)$.

Conversely, we show that $T_\infty(G) \le T_\infty(G;C)$. Similar to the above argument, it is sufficient to show that $T_\infty(G) \le T(G;C)$. Take a minimal presentation complex Q for $(G;C)$. We construct a presentation complex for G by truncating a neighborhood of the vertex of Q (see Figure 2) and attaching 2–cells. Let Q' be the truncated complex. Let Γ be the sectional graph of the truncation in Q'. Attaching edges to Γ if necessary, we may assume that Γ is connected and the natural map from $\pi_1(\Gamma)$ to C is surjective. We contract vertices of Q' along edges of Γ to obtain a 2–complex Q''. We obtain a bouquet Γ' in Q'' from Γ. Then we have the natural surjection $p\colon \pi_1(\Gamma') \to C$. Attaching more edges to Γ' if necessary, we may assume that there are edges $\gamma_1, \dots, \gamma_r$ such that the images of the elements $[\gamma_1], \dots, [\gamma_r] \in \pi_1(\Gamma')$ forms a basis of C. Let $\gamma'_1, \dots, \gamma'_s$ be the other edges of Γ'. Write $z_j = p[\gamma_j]$ and $z'_k = p[\gamma'_k]$ for $1 \le j \le r$ and $1 \le k \le s$. We can present z'_k as a product of the z_j, and let b_k denote its word length. We obtain a presentation complex Q''' for G by attaching triangles to Q'' along Γ', where $r(r-1)$ attached triangles correspond to the commutators $[z_i, z_j] = z_i z_j z_i^{-1} z_j^{-1}$ $(1 \le i, j \le r)$ and at most $b_1 + \dots + b_s - s$ attached triangles correspond to the presentation of z'_k by the z_j. Let K denote the union of Γ' and the attached triangles.

Suppose that H is a finite-index normal subgroup of G, and that d and $\{C'_1, \dots, C'_m\}$ are as above. Let $\widetilde{Q}$ be the covering of Q'' corresponding to H. Let $\widetilde{K}_1, \dots, \widetilde{K}_m$ be the components of the covering of K in $\widetilde{Q}$ corresponding to $\{C'_1, \dots, C'_m\}$. Each covering $\widetilde{K}_i \to K$ has degree d/m. In order to construct a presentation complex $\widetilde{Q}'$ for H, we contract simplices of $\widetilde{K}_i \subset \widetilde{Q}$ in the following manner.

We describe the way of contraction on the universal covering of K. Regard $\pi_1(K) = C$ as a lattice in $\mathbb{R}^r$. Take a reduced basis of $\pi_1(\widetilde{K}_i)(< \pi_1(K))$. Let F be the fundamental

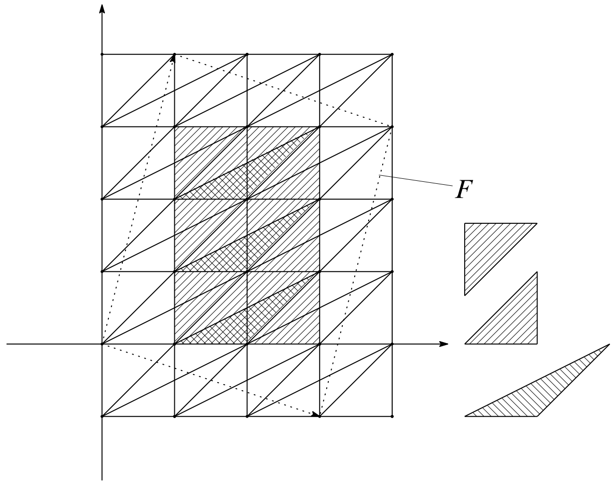

Figure 3: Contraction of simplices in F

domain of $\pi_1(\widetilde{K}_i)$ defined by this reduced basis. We contract simplices in the interior of F into a point.

We give an example in Figure 3. Suppose $z_1 = (1,0)$, $z_2 = (0,1)$ and $z_1' = (2,1)$. The 2–complex K consists of three triangles corresponding to the commutator $[z_1, z_2]$ and $z_1' = z_1^2 z_2$. Now let $((3,-1),(1,4))$ be taken as a basis of a lattice $\pi_1(\widetilde{K}_i)$. Then we contract 15 triangles whose projection is in the interior of F.

This construction does not change the fundamental group of $\widetilde{Q}$. (If $r \geq 3$, this construction may change the homotopy type of $\widetilde{Q}$.) Thus we obtain a presentation complex $\widetilde{Q}'$ for H.

The number of triangles of $\widetilde{Q}'$ is at most

$$d \cdot T(G;C) + m(e+f),$$

where

$$\begin{aligned} e &= e_{11} + \cdots + e_{1r} + e_{21} + \cdots + e_{2s}, \\ f &= f_1 + f_{21} + \cdots + f_{2s}, \end{aligned}$$

and e_{1j} and e_{2k} are the numbers of edges of $\widetilde{Q}'$ which respectively derive from γ_j and γ_k', f_1 is the number of the triangles of $\widetilde{Q}'$ which derive from ones corresponding to the commutators $[x_i, x_j]$, and f_{2k} is the number of the triangles of $\widetilde{Q}'$ which derive from ones corresponding to the presentation of z_k' by the z_j. We have that $d \cdot T(G;C) + me$ triangles of $\widetilde{Q}'$ derive from the hexagons of Q' and mf triangles of $\widetilde{Q}'$ derive from the triangles of K.

If the edges and triangles are not contracted by the above construction, they are near the boundary of F in the above picture. Hence there exists a constant $\delta_r > 0$ such that

$$\begin{aligned} e_{1j} &\le \delta_r \operatorname{vol}(\partial F), & e_{2k} &\le b_k \delta_r \operatorname{vol}(\partial F), \\ f_1 &\le r(r-1)\delta_r \operatorname{vol}(\partial F), & f_{2k} &\le (b_k - 1) e_{2k}, \end{aligned}$$

where $\operatorname{vol}(\partial F)$ is the surface area of F with respect to the standard Euclidean metric of $\mathbb{R}^r$. Therefore,

$$\begin{aligned} T_\infty(G) \le \frac{T(H)}{d} &\le T(G;C) + \frac{m}{d}(e+f) \\ &\le T(G;C) + \left(r^2 + \sum_{k=1}^{s} b_k^2\right)\delta_r \cdot \frac{\operatorname{vol}(\partial F)}{\operatorname{vol}(F)}. \end{aligned}$$

Since G is residually finite and F is defined by a reduced basis, there is a normal subgroup H of G such that $\operatorname{vol}(\partial F)/\operatorname{vol}(F)$ is arbitrarily small. $\square$

Cooper [9] showed that $\operatorname{vol}(M) < \pi \cdot T(M)$ for a closed hyperbolic 3–manifold M. The isoperimetric inequality by Agol and Liu [2, Lemma 4.4] implies that this inequality also holds for a cusped hyperbolic 3–manifold. Delzant and Potyagailo [11] remarked that a converse inequality does not hold; namely, the infimum of $\operatorname{vol}(M)/T(M)$ for the hyperbolic 3–manifolds is zero. Indeed, hyperbolic Dehn surgery [42, Chapters 4 and 6] gives infinitely many hyperbolic manifolds whose presentation lengths are divergent while their volumes are bounded. Delzant and Potyagailo used a relative presentation length $T(\pi_1(M);\mathcal{E})$ to bound the volume from below, where $\mathcal{E}$ consists of the elementary subgroups of $\pi_1(M)$ whose translation lengths are less than a Margulis number. They also showed that $\operatorname{vol}(M) \le \pi \cdot T(\pi_1(M);\mathcal{E})$ [11, Theorem B]. In particular, $\operatorname{vol}(M) \le \pi \cdot T(M;\partial M)$. We use the stable presentation length to bound the volume instead of $T(\pi_1(M);\mathcal{E})$. Cooper's inequality immediately implies that $\operatorname{vol}(M) \le \pi \cdot T_\infty(M)$. A converse estimate holds for the stable presentation length.

Proposition 4.4 *The infimum of* $\operatorname{vol}(M)/T_\infty(M)$ *for hyperbolic* 3*–manifolds is positive.*

In order to show this, we mention a connection between the presentation length and the complexity of a 3–manifold. For a closed 3–manifold M, the Δ–complexity (or Kneser complexity) $\sigma(M)$ is defined as the minimal number of tetrahedra in a triangulation of M. We also define $\sigma(M)$ for a cusped finite-volume hyperbolic 3–manifold M by ideal triangulations. The complexity $c(M)$ of Matveev [29] is the minimal number of vertices in a simple spine of M. It holds that $\sigma(M) = c(M)$ if M is irreducible and not $S^3, \mathbb{RP}^3$ or the lens space $L(3,1)$, in particular, if M is a hyperbolic 3–manifold [29, Theorem 5].

We respectively define the stable Δ–complexity $\sigma_\infty(M)$ and the stable complexity $c_\infty(M)$ of a 3–manifold M as $\inf \sigma(N)/d$ and $\inf c(N)/d$ by taking the infimum over all the finite coverings N of M, where d is the degree of the covering. It holds that $\sigma_\infty(M) = c_\infty(M)$ if M is a hyperbolic 3–manifold. We have that $c_\infty(M)$ vanishes for a Seifert 3–manifold M, and c_∞ has additivity for the prime decomposition and the JSJ decomposition.

Proposition 4.5 *For a closed 3–manifold M, it holds that $T(M) \leq \sigma(M) + 1$.*

Proof We take a minimal triangulation of M. Consider the 2–skeleton P_0 of this triangulation. P_0 has $2\sigma(M)$ triangles. Since a 2–complex P in M has a fundamental group isomorphic to $\pi_1(M)$ as long as $M \setminus P$ consists of 3–balls, we can remove $\sigma(M) - 1$ triangles from P_0 without changing the fundamental group. Therefore, we obtain a presentation complex for $\pi_1(M)$ with $\sigma(M) + 1$ triangles. □

Proposition 4.6 *For a cusped finite-volume hyperbolic 3–manifold M, it holds that $T(M) \leq \sigma(M) + 3$.*

Proof We take a minimal ideal triangulation of M. Consider the dual spine P_0 of this triangulation. P_0 has $\sigma(M)$ 2–cells, $2\sigma(M)$ edges and $\sigma(M)$ vertices. The $\sigma(M)$ 2–cells can be decomposed into $4\sigma(M)$ triangles. We contract $\sigma(M) - 1$ vertices along edges. Since every edge of P_0 is incident on three triangles, we obtain a presentation complex of $\pi_1(M)$ with $\sigma(M) + 3$ triangles. □

Since the fundamental group of a 3–manifold is residually finite [18], M admits arbitrarily large finite covering if $\pi_1(M)$ is infinite. This implies the following corollary.

Corollary 4.7 *If M is a closed 3–manifold or a finite-volume hyperbolic 3–manifold, it holds that $T_\infty(M) \leq \sigma_\infty(M)$.*

The stable complexity of a hyperbolic 3–manifold is bounded from above and below by constant multiples of its volume. For a finite-volume hyperbolic 3–manifold M, it holds that $\mathrm{vol}(M) \leq V_3 \sigma(M)$, where V_3 is the volume of an ideal regular tetrahedron, which is the maximum of the volumes of geodesic tetrahedra in hyperbolic 3–space.

This implies that $\mathrm{vol}(M) \leq V_3\sigma_\infty(M)$. Conversely, there exists a constant $C > 0$ such that $\sigma_\infty(M) \leq C\,\mathrm{vol}(M)$ holds for any hyperbolic manifold M. This follows from the fact, by Jørgensen and Thurston, that a thick part of a hyperbolic 3–manifold can be decomposed by uniformly thick tetrahedra. Proofs of this fact are given by Francaviglia, Frigerio and Martelli [12, Proposition 2.5] in the case where M is closed, and by Breslin [6] and Kobayashi and Rieck [20] otherwise. Proposition 4.4 follows from this inequality and Corollary 4.7.

We conjecture an equality between stable presentation length and stable complexity.

Conjecture 4.8 *For a finite-volume hyperbolic 3–manifold M, it holds that*

$$T_\infty(M) = \tfrac{1}{2}\sigma_\infty(M).$$

We will give some examples where $T_\infty(M) \leq \frac{1}{2}\sigma_\infty(M)$ in Section 6.2. It holds that $T(M) \geq \frac{1}{2}\sigma(M)$ if a minimal (relative) presentation complex for $\pi_1(M)$ injects into M. This is because M can be decomposed into $2T(M)$ tetrahedra.

If Conjecture 4.8 holds, then $T_\infty(M) = (1/(2V_3)) \cdot \mathrm{vol}(M)$ for a hyperbolic 3–manifold M which is commensurable with the figure-eight knot complement M_1. Indeed, $\sigma_\infty(M_1) = 2$ since M_1 can be decomposed into two ideal regular tetrahedra. Conjecture 4.8 implies a best possible refinement of Cooper's inequality $\mathrm{vol}(M) < 2V_3 \cdot T(M)$.

5 Additivity of stable presentation length

We will show additivity of the stable presentation length of 3–manifold groups in the same manner as the simplicial volume. The proofs of Theorem 5.1 and 5.3 are similar. Let G be a finitely presentable group and let $\{G_i\}$ be decomposed groups of G. We will construct a presentation complex for a finite-index subgroup of G by gluing finite coverings of presentation complexes for the G_i. This implies an inequality between $T_\infty(G)$ and $\sum_i T_\infty(G_i)$. In order to show the converse inequality, we will obtain presentation complexes for finite-index subgroups of the G_i by decomposing a finite covering of a presentation complex for G.

We first show additivity for a free product. This holds for any finitely presentable group.

Theorem 5.1 *For finitely presentable groups G_1 and G_2, it holds that*

$$T_\infty(G_1 * G_2) = T_\infty(G_1) + T_\infty(G_2).$$

Proof From Corollary 2.8, we will use additivity of presentation length for a free product. Write $G = G_1 * G_2$. We first show that $T_\infty(G) \le T_\infty(G_1) + T_\infty(G_2)$. For $i = 1, 2$, let P_i be a presentation complex for G_i, take a d_i–index subgroup H_i of G_i, and let $\widetilde{P}_i$ denote the covering of P_i corresponding to H_i. Since each $\widetilde{P}_i$ has d_i vertices, we can glue d_2 copies of $\widetilde{P}_1$ and d_1 copies of $\widetilde{P}_2$ along the vertices to obtain a $d_1 d_2$–sheeted covering $\widetilde{P}$ of $P_1 \vee P_2$. The wedge sum $P_1 \vee P_2$ is a presentation complex for G. Then $\pi_1(\widetilde{P})$ is isomorphic to a free product $H_1^{*d_2} * H_2^{*d_1} * F_k$, where F_k is a free group. Corollary 2.8 implies that $T(\pi_1(\widetilde{P})) = d_2 \cdot T(H_1) + d_1 \cdot T(H_2)$. Therefore,

$$T_\infty(G) \le \frac{T(\pi_1(\widetilde{P}))}{d_1 d_2} = \frac{T(H_1)}{d_1} + \frac{T(H_2)}{d_2}.$$

Since we took H_1 and H_2 arbitrarily, we obtain $T_\infty(G) \le T_\infty(G_1) + T_\infty(G_2)$.

Conversely, we show that $T_\infty(G_1) + T_\infty(G_2) \le T_\infty(G)$. Let P_i be as above. Then $P = P_1 \vee P_2$ is a presentation complex for G. Take a d–index subgroup H of G. Let $\widetilde{P}$ denote the covering of P corresponding to H. Then $\widetilde{P}$ is homotopic to

$$P_{11} \vee \cdots \vee P_{1m} \vee P_{21} \vee \cdots \vee P_{2n} \vee S^1 \vee \cdots \vee S^1,$$

where P_{ij} is a covering of P_i. Let d_{ij} be the degree of the covering $P_{ij} \to P_i$. Then $\sum_{j=1}^m d_{1j} = \sum_{j=1}^n d_{2j} = d$. Since $H = \pi_1(\widetilde{P})$ is isomorphic to

$$\pi_1(P_{11}) * \cdots * \pi_1(P_{1m}) * \pi_1(P_{21}) * \cdots * \pi_1(P_{2n}) * F_k,$$

Corollary 2.8 and Proposition 3.3 imply that

$$\begin{aligned} T(H) &= T(\pi_1(P_{11})) + \cdots + T(\pi_1(P_{1m})) + T(\pi_1(P_{21})) + \cdots + T(\pi_1(P_{2n})) \\ &\ge d_{11} \cdot T_\infty(\pi_1(P_1)) + \cdots + d_{1m} \cdot T_\infty(\pi_1(P_1)) \\ &\qquad\qquad + d_{21} \cdot T_\infty(\pi_1(P_2)) + \cdots + d_{2n} \cdot T_\infty(\pi_1(P_2)) \\ &= d \cdot T_\infty(G_1) + d \cdot T_\infty(G_2). \end{aligned}$$

Therefore, $T_\infty(G_1) + T_\infty(G_2) \le T(H)/d$. Since we took H arbitrarily, we obtain $T_\infty(G_1) + T_\infty(G_2) \le T_\infty(G)$. □

Before we show additivity for the JSJ decomposition, we show that the stable presentation length for a Seifert 3–manifold vanishes.

Theorem 5.2 *For a compact Seifert 3–manifold M,*

$$T_\infty(M) = T_\infty(M; \partial M) = 0.$$

Proof Since a Seifert 3–manifold can be regarded as an S^1–bundle over an 2–orbifold, M is covered by an S^1–bundle over a surface. Hence we can assume M is an S^1–bundle over a compact surface.

If M has boundary, M is a product of S^1 and a surface. Then M admits a d–sheeted covering homeomorphic to M for any $d \leq 1$. Thus $T_\infty(M) = T_\infty(M; \partial M) = 0$ by Proposition 3.3.

We consider an S^1–bundle over a closed surface Σ_g of genus g. The homeomorphic class of an S^1–bundle over Σ_g is determined by the Euler number e. Let $M(\Sigma_g, e)$ denote the S^1–bundle over Σ_g of the Euler number e. Since $\pi_1(M(S^2, e))$ is finite or isomorphic to $\mathbb{Z}$, we have $T_\infty(M(S^2, e)) = 0$. Suppose $g \geq 1$. Then $\pi_1(M(\Sigma_g, e))$ has a presentation

$$\langle x_1, y_1, \dots, x_g, y_g, z \mid [x_1, y_1] \cdots [x_g, y_g] z^e,\ [x_i, z],\ [y_i, z] \text{ for } 1 \leq i \leq g \rangle,$$

where the x_i and y_i correspond to generators of the fundamental group of the base surface, z is a generator of the fundamental group of the ordinary fiber, and $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$. Therefore,

$$T(\pi_1(M(\Sigma_g, e))) \leq 8g + |e| - 2.$$

For any integer $d \geq 1$, the bundle $M(\Sigma_g, e)$ admits $M(\Sigma_{g'}, de)$ as a d–sheeted covering along the base space, where $g' = d(g-1) + 1$. Furthermore, $M(\Sigma_{g'}, de)$ admits $M(\Sigma_{g'}, e)$ as a d–sheeted covering along the fiber direction. Thus we obtain a d^2–sheeted covering $M(\Sigma_{g'}, e) \to M(\Sigma_g, e)$. Hence

$$T_\infty(\pi_1(M(\Sigma_g, e))) \leq \frac{T(\pi_1(M(\Sigma_{g'}, e)))}{d^2} \leq \frac{8(d(g-1)+1) + |e| - 2}{d^2}.$$

The right-hand side converges to zero when d increases. □

Finally we show additivity for the JSJ decomposition.

Theorem 5.3 *Let M be a compact irreducible 3–manifold with empty or incompressible tori boundary. Suppose $M = M_1 \cup \cdots \cup M_h$ is the JSJ decomposition, where $M_1, \dots, M_h$ are Seifert or hyperbolic 3–manifolds with incompressible tori boundary. Then*

$$T_\infty(M) = T_\infty(M_1) + \cdots + T_\infty(M_h).$$

Proof We remark that the fundamental group of a compact 3–manifold is residually finite by Hempel [18] and the geometrization theorem.

We first show that $T_\infty(M) \le T_\infty(M_1) + \cdots + T_\infty(M_h)$. Take d_i–sheeted coverings $f_i\colon \widetilde{M}_i \to M_i$ for $1 \le i \le h$. Then there exist an integer p independent of i and coverings $g_i\colon N_i \to \widetilde{M}_i$ such that each $f_i \circ g_i\colon N_i \to M_i$ is a p–characteristic covering, ie the restriction of the covering on each component of ∂M_i is the covering corresponding to $p\mathbb{Z} \times p\mathbb{Z} < \mathbb{Z} \times \mathbb{Z}$ [12, Proposition 5.7]. We can glue copies N_{ij} of N_i along the boundary to obtain a d–sheeted covering $f\colon N \to M$. Then $f^{-1}(M_i) = N_{11} \cup \cdots \cup N_{il_i}$. Each copy $g_{ij}\colon N_{ij} \to \widetilde{M}_i$ of g_i is a $d/l_i d_i$–sheeted covering. $N = \bigcup_{i,j} N_{ij}$ is the JSJ decomposition. Therefore, we obtain

$$\begin{aligned} T(\pi_1(N); \{\pi_1(\partial N_{ij})\}) &\le \sum_{i,j} T(N_{ij}; \partial N_{ij}) \\ &\le \sum_{i,j} \frac{d}{l_i d_i} T(\widetilde{M}_i; \partial \widetilde{M}_i) = \sum_i \frac{d}{d_i} T(\widetilde{M}_i; \partial \widetilde{M}_i) \end{aligned}$$

by Proposition 2.5. Hence

$$T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\}) \le \frac{T(\pi_1(N); \{\pi_1(\partial N_{ij})\})}{d} \le \sum_i \frac{T(\widetilde{M}_i; \partial \widetilde{M}_i)}{d_i}.$$

Since we took $\widetilde{M}_i$ arbitrarily, we obtain

$$T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\}) \le \sum_i T_\infty(\widetilde{M}_i; \partial \widetilde{M}_i).$$

Furthermore, $T_\infty(M) = T_\infty(\pi_1(M); \{\pi_1(\partial M_i)\})$ and $T_\infty(\widetilde{M}_i) = T_\infty(\widetilde{M}_i; \partial \widetilde{M}_i)$ by Theorem 4.2.

Conversely, we show that $T_\infty(M_1) + \cdots + T_\infty(M_h) \le T_\infty(M)$. Take a d–sheeted covering $p\colon \widetilde{M} \to M$. Then the components M_{ij} of $p^{-1}(M_i)$ are the components of the JSJ decomposition of $\widetilde{M}$. Let d_{ij} denote the degree of the covering $M_{ij} \to M_i$. Then $\sum_j d_{ij} = d$. We have

$$\sum_j T(M_{ij}; \partial M_{ij}) \ge \sum_j d_{ij} \cdot T_\infty(M_i; \partial M_i) = d \cdot T_\infty(M_i; \partial M_i)$$

by definition. Theorem 2.7 implies that

$$\sum_{i,j} T(M_{ij}; \partial M_{ij}) \le T(\widetilde{M}).$$

Therefore, it holds that

$$\sum_i T_\infty(M_i; \partial M_i) \le \frac{T(\widetilde{M})}{d}.$$

Since we took $\widetilde{M}$ arbitrarily, we obtain

$$\sum_i T_\infty(M_i; \partial M_i) \le T_\infty(M).$$

Furthermore, $T_\infty(M_i) = T_\infty(M_i; \partial M_i)$ by Theorem 4.2. □

Corollary 5.4 *There exists a constant $C > 0$ such that the following holds. If M is a closed 3–manifold, then*

$$C \cdot T_\infty(M) \le \|M\| \le \frac{\pi}{V_3} T_\infty(M),$$

where $\|M\|$ is the simplicial volume of M and V_3 is the volume of an ideal regular tetrahedron.

Proof We can assume that M is orientable by taking the double covering. Let $M = M_1 \# \cdots \# M_n$ be the prime decomposition. Each connected summand M_i is irreducible or homeomorphic to $S^1 \times S^2$. Let $M_i = M_{i1} \cup \cdots \cup M_{ih_i}$ be the JSJ decomposition if M_i is irreducible. The geometrization theorem implies that each JSJ component M_{ij} is Seifert fibered or hyperbolic. Let $N_1, \dots, N_m$ denote the hyperbolic components among M_{ij}. Then

$$\|M\| = \frac{1}{V_3}(\mathrm{vol}(N_1) + \cdots + \mathrm{vol}(N_m))$$

by additivity and proportionality of simplicial volume [15]. Now we have

$$T_\infty(M) = T_\infty(N_1) + \cdots + T_\infty(N_m)$$

by Theorems 5.1, 5.2 and 5.3. Therefore, we are reduced to proving the result for hyperbolic 3–manifolds. A hyperbolic 3–manifold M satisfies the above inequalities by Cooper's inequality and Proposition 4.4. □

6 Examples of stable presentation length

6.1 Surface groups

We calculate the explicit value of the stable presentation length of a surface group, which coincides with the simplicial volume of the surface.

Theorem 6.1 *Let Σ_g be a closed orientable surface of genus $g \geq 1$. Then*

$$T_\infty(\pi_1(\Sigma_g)) = 4g - 4 = -2\chi(\Sigma_g).$$

Proof If $g = 1$, then $\pi_1(\Sigma_g) \cong \mathbb{Z} \times \mathbb{Z}$ has a finite-index proper subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Hence $T_\infty(\pi_1(\Sigma_g)) = 0$ by Proposition 3.3.

Suppose that $g \geq 2$. Since there is a presentation

$$\pi_1(\Sigma_g) = \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle,$$

we have $T(\pi_1(\Sigma_g)) \leq 4g - 2$. In order to estimate from below, take a minimal presentation complex P for $\pi_1(\Sigma_g)$. We put a hyperbolic metric on Σ_g. There exists a map $f \colon P \to \Sigma_g$ inducing an isomorphism between their fundamental groups. We can choose an f that maps every 2–cell of P to a geodesic triangle in Σ_g.

We claim that f is surjective. If not, there is a point p in $\Sigma_g - f(P)$. Then f induces an injection from $\pi_1(\Sigma_g)$ to $\pi_1(\Sigma_g - \{p\})$. Since $\pi_1(\Sigma_g - \{p\})$ is a free group and $\pi_1(\Sigma_g)$ is not a free group, we have a contradiction. Now $\operatorname{area}(\Sigma_g) = (4g-4)\pi$ and the area of a geodesic triangle in Σ_g is smaller than π. Hence we obtain $(4g-4)\pi < \pi \cdot T(\pi_1(\Sigma_g))$.

We finally compute $T_\infty(\pi_1(\Sigma_g))$. Since $\Sigma_{d(g-1)+1}$ covers Σ_g with degree d,

$$T_\infty(\pi_1(\Sigma_g)) \leq \frac{1}{d} T(\pi_1(\Sigma_{d(g-1)+1})) \leq \frac{1}{d}(4(d(g-1)+1)-2).$$

Hence we obtain $T_\infty(\pi_1(\Sigma_g)) \leq 4g-4$ by sending $d \to \infty$. Conversely, the fact that $4g-4 < (1/d)T(\pi_1(\Sigma_{d(g-1)+1}))$ for any $d \geq 1$ implies $4g-4 \leq T_\infty(\pi_1(\Sigma_g))$. $\square$

Theorem 6.2 *Let $\Sigma_{g,b}$ denote the compact orientable surface of genus g whose boundary components are $S_1, \dots, S_b$. Suppose that $b > 0$ and $2g - 2 + b > 0$. Then*

$$\begin{aligned} T_\infty(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b)) &= T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b)) \\ &= 4g - 4 + 2b = -2\chi(\Sigma_{g,b}). \end{aligned}$$

Proof We have that the surface $\Sigma_{g,b}$ admits a hyperbolic metric with cusps $S_1, \dots, S_b$. An ideal triangulation of this hyperbolic surface gives a presentation complex for $(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b))$, which consists of $4g-4+2b$ triangles. Therefore, $T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b)) \leq 4g - 4 + 2b$.

For the converse inequality, we put a hyperbolic metric with geodesic boundary on $\Sigma_{g,b}$. Take a minimal presentation complex P for $(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b))$. Let P'

be the complex obtained by truncating P. There is a continuous map $f\colon P' \to \Sigma_{g,b}$ that sends the truncated section $\partial P'$ of P' to the corresponding boundary components and that induces an isomorphism between their fundamental groups. Then f induces a map $Df\colon DP' \to D\Sigma_{g,b}$ between their doubles. Since Df induces an isomorphism between the fundamental groups, Df is surjective by the proof of Theorem 6.1. Therefore, f is also surjective. After straightening f relative to the boundary, the 2–cells of P' map to right-angled hexagons, whose areas are equal to π. Then

$$(4g-4+2b)\pi = \mathrm{area}(\Sigma_{g,b}) \leq \pi \cdot T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b)).$$

Now we have $T(\pi_1(\Sigma_{g,b}); \pi_1(S_1), \dots, \pi_1(S_b)) = 4g-4+2b$. Since these values are already volume-like, their stable presentation lengths coincide with their presentation lengths. □

6.2 Bianchi groups

We consider the stable presentation lengths of Bianchi groups $\mathrm{PSL}(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$; namely,

$$\mathcal{O}_d = \begin{cases} \mathbb{Z}\left[\frac{1}{2}(1+\sqrt{-d})\right] & \text{if } -d \equiv 1 \bmod 4, \\ \mathbb{Z}[\sqrt{-d}] & \text{if } -d \equiv 2, 3 \bmod 4. \end{cases}$$

It is known that the fundamental group of every finite-volume cusped arithmetic hyperbolic 3–manifold is commensurable with a Bianchi group [33, Proposition 4.1]. Hatcher [17] showed that some Bianchi groups preserve tessellations of $\mathbb{H}^3$ by ideal uniform polyhedra. Consequently, hyperbolic 3–manifolds obtained from certain ideal uniform polyhedra are arithmetic.

We will give upper bounds of stable presentation lengths of some arithmetic link components by constructing explicit presentations of their fundamental groups. We consider the complements of links in $T^2 \times [0,1]$. Since the complement of the Hopf link is homeomorphic to $T^2 \times [0,1]$, the complement of a link in $T^2 \times [0,1]$ is homeomorphic to the complement of a link in S^3. As we will see later, there are infinitely many links in $T^2 \times [0,1]$ whose complements admit hyperbolic structures.

For a hyperbolic link K in $T^2 \times [0,1]$, the two boundary components of $T^2 \times [0,1]$ and the components of K correspond to the cusps. We will take finite coverings of $T^2 \times [0,1] \setminus K$ induced by those of $T^2 \times [0,1]$. These coverings are the complements of links $K_{m,n}$ in $T^2 \times [0,1]$. Analogously to the ones for links in S^3, we can obtain

Wirtinger presentations of $\pi_1(T^2 \times [0,1] \setminus K_{m,n})$ from their diagrams. We will need additional generators to obtain presentations of shorter lengths.

From an ideal triangulation of $T^2 \times [0,1] \setminus K$, we can obtain an explicit presentation complex. For instance, the 2–skeleton of an ideal triangulation is a presentation complex for $\pi_1(T^2 \times [0,1] \setminus K)$ relative to the fundamental groups of the cusps. If there is an alternating diagram of K, We can systematically obtain an ideal polyhedral decomposition of $T^2 \times [0,1] \setminus K$ from the diagram analogously to the ideal decomposition of alternating links due to Menasco [30]. This argument will be applied in Sections 6.2.1 and 6.2.2. The ideal decomposition in Section 6.2.2 was explained in detail by Champanerkar, Kofman and Purcell [8, Section 3].

We can also obtain a small presentation complex from an ideal even triangulation. An (ideal) triangulation $\mathcal{T}$ of a 3–manifold M determines the projection p from the tetrahedra to the end-compactification of M. Following Rubinstein and Tillmann [39], $\mathcal{T}$ is an *even* triangulation if the preimage $p^{-1}(\tau)$ of each edge τ in $\mathcal{T}$ consists of an even number of edges. A *vertex coloring* of $\mathcal{T}$ is a map from the vertices in $\mathcal{T}$ to $\{0,1,2,3\}$ such that its restriction to the vertices of each tetrahedron is bijective. Although the universal covering of an even triangulation admits a vertex coloring, the deck transformations may not preserve the coloring. This difference determines a monodromy homomorphism from $\pi_1(M)$ to the symmetric group $\mathrm{Sym}(4)$ on $\{0,1,2,3\}$, which is called a *symmetric representation* for an even triangulation in [39].

Lemma 6.3 *Suppose that a finite-volume hyperbolic 3–manifold M admits an ideal even triangulation $\mathcal{T}$ with n tetrahedra. Then $T_\infty(M) \le \frac{1}{2}n$.*

Proof Let $\rho\colon \pi_1(M) \to \mathrm{Sym}(4)$ be a symmetric representation. The manifold M has a finite covering M' which corresponds to $\ker(\rho)$. Then the lifted triangulation $\mathcal{T}'$ of $\mathcal{T}$ for M' admits a vertex coloring. We take the $\deg(M' \to M) \cdot \frac{1}{2}n$ triangles in $\mathcal{T}'$ which do not contain a vertex of color 0. The union of these triangles is a presentation complex for $\pi_1(M')$ relative to the fundamental groups of the cusps of colors $\{1,2,3\}$. Therefore, Theorem 4.2 implies that $T_\infty(M) = T_\infty(M')/\deg(M' \to M) \le \frac{1}{2}n$. □

We will give explicit examples for the above construction in Sections 6.2.1 and 6.2.2.

6.2.1 $\boldsymbol{d=3}$ **(figure-eight knot complement)** The figure-eight knot complement M_1 is obtained from two ideal regular tetrahedra, thus $\mathrm{vol}(M_1) = 2V_3 = 2.0298\ldots$ and $\sigma(M_1) = \sigma_\infty(M_1) = 2$. It is known that $\pi_1(M_1)$ is isomorphic to an index-12 subgroup

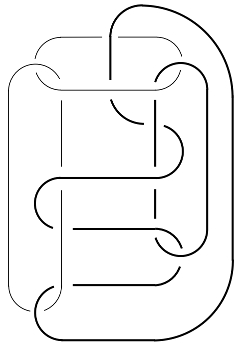

Figure 4: A link whose complement is $M_{1,1}$

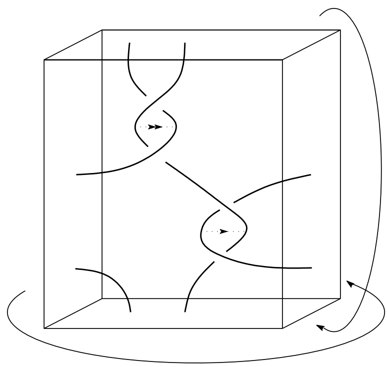

Figure 5: $M_{1,1}$ as the complement of a link in $T^2 \times [0,1]$

of $\mathrm{PSL}(2,\mathcal{O}_3)$. The index follows from Humbert's formula [42, Theorem 7.4.1] for $\mathrm{vol}(\mathbb{H}^3/\mathrm{PSL}(2,\mathcal{O}_d))$.

For a general case, suppose that a hyperbolic 3–manifold M is obtained from ideal regular tetrahedra. Then the action of $\pi_1(M)$ on $\mathbb{H}^3$ preserves the tessellation by ideal regular tetrahedra of $\mathbb{H}^3$. Since the symmetry group of this tessellation is commensurable with $\mathrm{PSL}(2,\mathcal{O}_3)$, the groups $\pi_1(M)$, $\pi_1(M_1)$ and $\mathrm{PSL}(2,\mathcal{O}_3)$ are commensurable as shown in [17, Section 3, Example 2].

Proposition 6.4 $$T_\infty(M_1) \leq 1.$$

Proof Let $M_{1,1}$ denote the complement of the link in Figure 4. By taking the Hopf sublink consisting of the two components shown using thin lines, we regard $M_{1,1}$ as the complement of a link in $T^2 \times [0,1]$, which is constructed by gluing the piece in Figure 5 along the faces of the top and bottom, left and right.

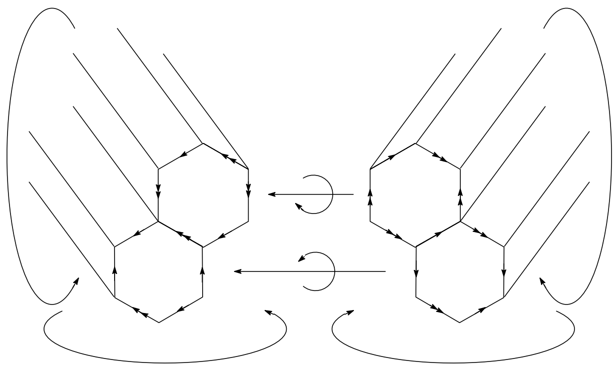

Figure 6: A decomposition of $M_{1,1}$

The manifold $M_{1,1}$ can be decomposed into four ideal regular hexagonal pyramids as shown in Figure 6. Since the union of two ideal regular hexagonal pyramids can be decomposed into six ideal regular tetrahedra, the manifold $M_{1,1}$ is obtained from 12 ideal regular tetrahedra. Since the manifolds $M_{1,1}$ and M_1 are commensurable, we have $T_\infty(M_{1,1})/T_\infty(M_1) = \mathrm{vol}(M_{1,1})/\mathrm{vol}(M_1) = 6$.

Let $M_{m,n}$ denote the mn–sheeted covering of $M_{1,1}$ which is the m–sheeted covering along s and the n–sheeted covering along t as in Figure 7. The diagram gives a Wirtinger presentation of $\pi_1(M_{m,n})$. We put a base point of $\pi_1(M_{m,n})$ at the upper left front point. The generators are

$$x_{ij},\ y_{ij},\ z_{ij},\ w_{ij},\ x_{m+1,j},\ y_{m+1,j},\ x_{i,n+1},\ z_{i,n+1},\ x_{m+1,n+1},\ s,\ t,$$

and the relators are

$$\begin{aligned}
z_{ij} &= y_{ij}x_{ij}y_{ij}^{-1}, & w_{ij} &= z_{ij}y_{ij}z_{ij}^{-1}, & st &= ts,\\
x_{i+1,j+1} &= w_{ij}^{-1}z_{i,j+1}w_{ij}, & y_{i+1,j} &= x_{i+1,j+1}^{-1}w_{ij}x_{i+1,j+1}, & &\\
x_{m+1,j} &= sx_{1,j}s^{-1}, & x_{m+1,n+1} &= sx_{1,n+1}s^{-1}, & y_{m+1,j} &= sy_{1,j}s^{-1},\\
x_{i,n+1} &= tx_{i,1}t^{-1}, & x_{m+1,n+1} &= tx_{m+1,1}t^{-1}, & z_{i,n+1} &= tz_{i,1}t^{-1},
\end{aligned}$$

for $1 \le i \le m$, $1 \le j \le n$. The generators x_{ij}, y_{ij}, z_{ij}, w_{ij} correspond to the arcs in the diagram, Some relators correspond to the crossings of the link, and the others come from the actions of s and t.

We add generators a_{ij} and b_{ij} for smaller presentation length. Thus we obtain an explicit presentation of $\pi_1(M_{m,n})$: the generators are

$$x_{ij},\ y_{ij},\ z_{ij},\ w_{ij},\ a_{ij},\ b_{ij},\ x_{m+1,j},\ y_{m+1,j},\ x_{i,n+1},\ z_{i,n+1},\ x_{m+1,n+1},\ s,\ t,$$

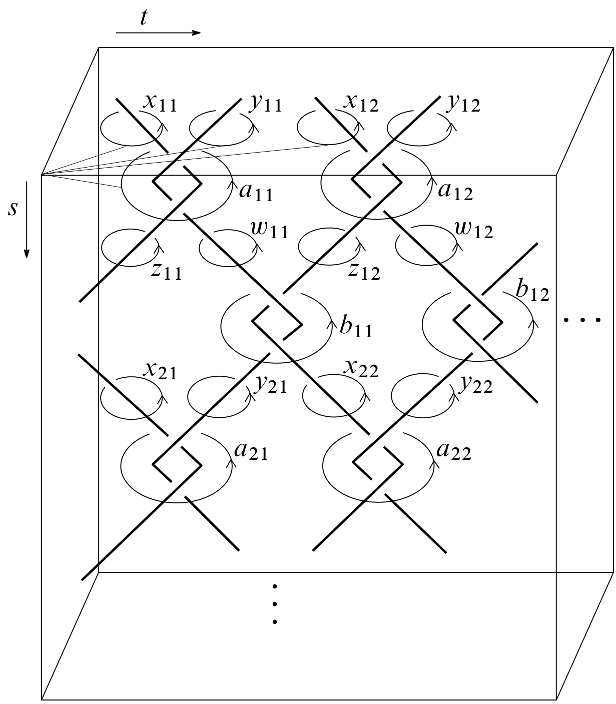

Figure 7: Generators of $\pi_1(M_{m,n})$

and the relators are

$$
\begin{array}{llll}
a_{ij} = y_{ij}x_{ij}, & a_{ij} = z_{ij}y_{ij}, & a_{ij} = w_{ij}z_{ij}, & \\
b_{ij} = z_{i,j+1}w_{ij}, & b_{ij} = w_{ij}x_{i+1,j+1}, & b_{ij} = x_{i+1,j+1}y_{i+1,j}, & \\
x_{m+1,j} = sx_{1,j}s^{-1}, & x_{m+1,n+1} = sx_{1,n+1}s^{-1}, & y_{m+1,j} = sy_{1,j}s^{-1}, & \\
x_{i,n+1} = tx_{i,1}t^{-1}, & x_{m+1,n+1} = tx_{m+1,1}t^{-1}, & z_{i,n+1} = tz_{i,1}t^{-1}, & st = ts,
\end{array}
$$

for $1 \le i \le m, 1 \le j \le n$. Therefore,

$$T_\infty(M_{1,1}) \le \inf_{m,n} \frac{T(M_{m,n})}{mn} \le \inf_{m,n} \frac{6mn+4m+4n+6}{mn} = 6. \qquad \square$$

Remark 6.5 In fact, Proposition 6.4 follows from Lemma 6.3 since a triangulation made of ideal regular tetrahedra is even. For future use, however, we gave an explicit presentation of $\pi_1(M_{m,n})$.

It is also possible to construct an explicit relative presentation complex as in the proof of Lemma 6.3. The manifold $M_{1,1}$ has four cusps S_0, S_1, S_2 and S_3, where S_0 and S_1 are the boundary component of $T^2 \times [0,1]$. We construct a fundamental

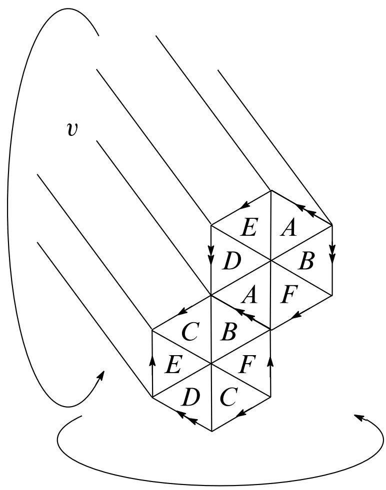

Figure 8: A fundamental domain X of $M_{1,1}$

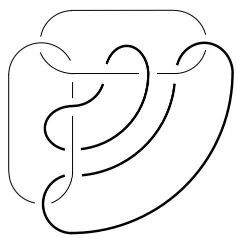

Figure 9: A link whose complement is M_2'

domain X of $M_{1,1}$ as a union of 12 ideal regular tetrahedra such that S_0 corresponds to a single vertex v of X (Figure 8). Then we obtain a presentation complex for $(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3))$ from the triangles in ∂X which do not contain v. Hence $T(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3)) \leq 6$. Theorem 4.2 implies that

$$T_\infty(M_{1,1}) = T_\infty(\pi_1(M_{1,1}); \pi_1(S_1), \pi_1(S_2), \pi_1(S_3)) \leq 6.$$

6.2.2 $\boldsymbol{d = 1}$ **(Whitehead link complement)** The Whitehead link complement M_2 is obtained from one ideal regular octahedron. Since $\mathrm{vol}(M_2) = 3.6638\ldots$, we have $\sigma(M_2) = 4$ and $3.6 < \sigma_\infty(M_2) \leq 4$. It is unknown whether $\sigma_\infty(M_2) = 4$ or not. It is known that $\pi_1(M_2)$ is an index-12 subgroup of $\mathrm{PSL}(2, \mathcal{O}_1)$. If a hyperbolic manifold M is obtained from ideal regular octahedra, the group $\pi_1(M)$ is commensurable with $\mathrm{PSL}(2, \mathcal{O}_1)$ as shown in [17, Section 3, Example 3].

Proposition 6.6 $T_\infty(M_2) \leq 2$.

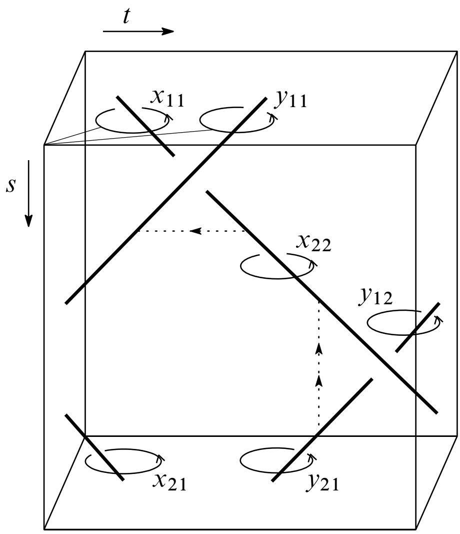

Figure 10: Generators of $\pi_1(M_2')$

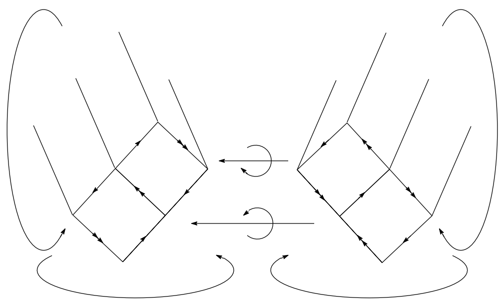

Figure 11: A decomposition of M_2'

Proof Since there is an even triangulation of M_2 with four ideal tetrahedra [39, Example 4], Lemma 6.3 implies the assertion. As with the above proposition, however, we consider links in $T^2 \times [0,1]$. Let M_2' denote the complement of the link in Figure 9. We regard M_2' as the complement of a link in $T^2 \times [0,1]$ as shown in Figure 10.

The manifold M_2' can be decomposed into four ideal regular square pyramids as shown in Figure 11. Since a union of two ideal regular square pyramids is an ideal regular octahedron, the manifold M_2' is obtained from two ideal regular octahedra. Since M_2' and M_2 are commensurable, we have $T_\infty(M_2')/T_\infty(M_2) = \mathrm{vol}(M_2')/\mathrm{vol}(M_2) = 2$.

We obtain a Wirtinger presentation of $\pi_1(M_2')$: the generators are

$$x_{11},\ x_{21},\ x_{22},\ y_{11},\ y_{12},\ y_{21},\ s,\ t,$$

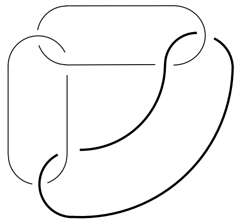

Figure 12: The alternating 3–chain link

and the relators are

$$x_{22} = y_{11}x_{11}y_{11}^{-1}, \quad y_{21} = x_{22}^{-1}y_{12}x_{22},$$
$$x_{21} = sx_{11}s^{-1}, \qquad y_{21} = sy_{11}s^{-1},$$
$$y_{12} = ty_{11}t^{-1}, \qquad x_{22} = tx_{21}t^{-1}, \qquad st = ts.$$

After we take large coverings along $T^2 \times [0,1]$ as with Proposition 6.4, the relators which do not contain s or t contribute an estimate of the stable presentation length. Therefore, $T_\infty(M_2') \leq 4$. □

We remark that it is possible to prove that $T_\infty(M_2') \leq 4$ by constructing a relative presentation complex as with Proposition 6.4.

6.2.3 $d = 7$ (magic manifold) Let M_3 denote the complement of the alternating 3–chain link in Figure 12. Gordon and Wu [14] called M_3 the magic manifold for the reason that it gives various interesting examples of Dehn fillings. Martelli and Petronio [28] classified the nonhyperbolic Dehn fillings of M_3. The manifold M_3 is obtained from two ideal uniform triangular prisms. Since $\mathrm{vol}(M_3) = 5.3334\ldots$, we have $\sigma(M_3) = 6$ and $5.2 < \sigma_\infty(M_3) \leq 6$. The group $\pi_1(M_3)$ is an index-6 subgroup of $\mathrm{PSL}(2, \mathcal{O}_7)$ as shown in [42, Chapter 6, Example 6.8.2].

Proposition 6.7 $$T_\infty(M_3) \leq 3.$$

Proof The manifold M_3 is homeomorphic to the complement of a link in $T^2 \times [0,1]$ as shown in Figure 13. We obtain an explicit presentation of $\pi_1(M_3)$: the generators are

$$x_{11},\ x_{21},\ y_{11},\ y_{12},\ a,\ s,\ t,$$

and the relators are

$$a = y_{12}x_{11}, \qquad a = x_{11}x_{21}, \qquad a = x_{21}y_{11},$$
$$x_{21} = sx_{11}s^{-1}, \quad y_{12} = ty_{11}t^{-1}, \quad st = ts.$$

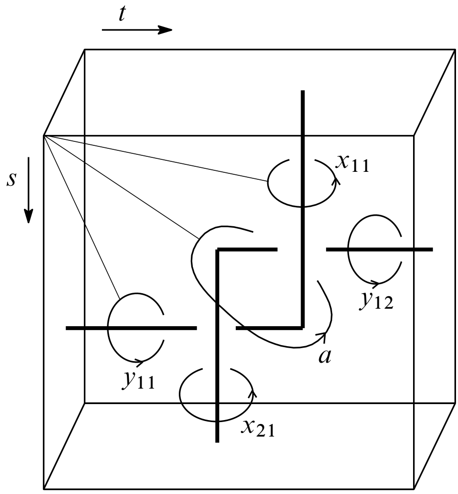

Figure 13: Generators of $\pi_1(M_3)$

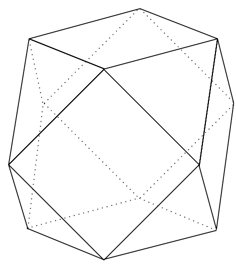

Figure 14: A cuboctahedron

After we take large coverings along $T^2 \times [0,1]$ as with the above propositions, the relators which do not contain s or t contribute an estimate of the stable presentation length. Therefore, $T_\infty(M_3) \leq 3$. □

6.2.4 $\boldsymbol{d = 2}$ If a hyperbolic manifold M is obtained from ideal uniform cuboctahedra (Figure 14), the group $\pi_1(M)$ is commensurable with $\mathrm{PSL}(2, \mathcal{O}_2)$ as shown in [17]. Let M_4 denote the complement of the link in Figure 15. This link was shown in [3, Figure 1b]. The manifold M_4 is obtained from four ideal uniform cuboctahedra. Let N_4 denote the double of an ideal uniform cuboctahedron along the ideal squares. Then M_4 is the double of N_4 along the 3–punctured spheres. We regard five components of the link in Figure 15 as horizontal and the seven others as vertical. If we cut M_4 along six horizontal 4–punctured spheres and eight vertical 3–punctured spheres, then we obtain four ideal uniform cuboctahedra. Since $\mathrm{vol}(M_4) = 48.1843\ldots$ and a cuboctahedron

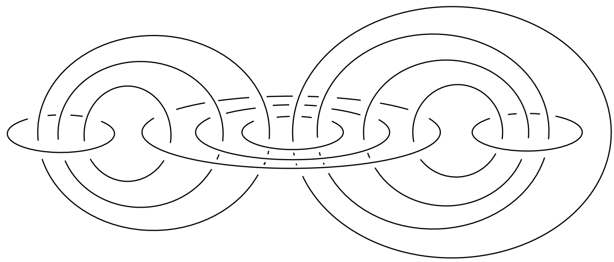

Figure 15: A link whose complement is M_4

can be decomposed into 14 tetrahedra compatible with the decomposition of M_4, we have $48 \le \sigma(M_4) \le 56$ and $47.4 < \sigma_\infty(M_4) \le 56$. Although this ideal triangulation of M_4 is not even, the argument in the proof of Lemma 6.3 can be applied.

Proposition 6.8 $T_\infty(M_4) \le 28$.

Proof Let P denote the 2–skeleton of the decomposition of M_4 into four ideal uniform cuboctahedra. Let Q denote the subcomplex of P consisting of the cells which do not contain a fixed vertex v. Then the 2–cells of Q consist of twelve triangles and eight squares. By decomposing each square of Q into two triangles, we obtain a presentation complex Q' for $\pi_1(M_4)$ relative to the fundamental groups of the other cusps than v. Since the 2–cells of Q consist of 28 triangles, we have $T_\infty(M_4) \le 28$ by Theorem 4.2. □

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