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Moduli of formal A–modules under change of A

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We develop methods for computing the restriction map from the cohomology of the automorphism group of a height dn formal group law (ie the height dn Morava stabilizer group) to the cohomology of the automorphism group of an A–height n formal A–module, where A is the ring of integers in a degree d field extension of $\mathbb{Q}_p$. We then compute this map for the quadratic extensions of $\mathbb{Q}_p$ and the height 2 Morava stabilizer group at primes $p > 3$. We show that the these automorphism groups of formal modules are closed subgroups of the Morava stabilizer groups, and we use local class field theory to identify the automorphism group of an A–height 1–formal A–module with the ramified part of the abelianization of the absolute Galois group of K, yielding an action of $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ on the Lubin–Tate/Morava E–theory spectrum E_2 for each quadratic extension $K/\mathbb{Q}_p$. Finally, we run the associated descent spectral sequence to compute the $V(1)$–homotopy groups of the homotopy fixed-points of this action; one consequence is that, for each element in the $K(2)$–local homotopy groups of $V(1)$, either that element or an appropriate dual of it is detected in the Galois cohomology of the abelian closure of some quadratic extension of $\mathbb{Q}_p$.

11S31, 14L05, 55N22, 55P42, 55Q10

1 Introduction

Let K be a p–adic number field with ring of integers A, and let $\mathbb{G}^{A}_{1/n}$ denote an "A–height n formal A–module" over $\overline{\mathbb{F}}_p$, that is, $\mathbb{G}^{A}_{1/n}$ is a (one-dimensional) formal group law equipped with complex multiplication by A, and its underlying formal group law has p–height dn, where $d = [K : \mathbb{Q}_p]$. (In the base case $K = \mathbb{Q}_p$, the group $\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/n}$ is simply a p–height n formal group law over $\overline{\mathbb{F}}_p$.) The automorphism group of the underlying formal group law of $\mathbb{G}^{A}_{1/n}$ is the well-known *height dn Morava stabilizer group,* whose cohomology is the input for many spectral sequences computing stable homotopy groups of spheres and other spectra; see Devinatz and Hopkins [4; 5] for one approach, and Ravenel [19, Chapter 6] for another. Among the automorphisms of the underlying formal group law $\mathbb{G}^{A}_{1/n}$, some automorphisms commute with the complex multiplication by A, and others do not; hence the automorphism group

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of $\mathbb{G}^A_{1/n}$ is naturally a subgroup of the height dn Morava stabilizer group. (It is even a *closed* subgroup; see Proposition 4.3 and also this paper's companion and sequel paper Salch [24].)

In this paper we develop methods for computing the restriction map from the cohomology of the height dn Morava stabilizer group to the cohomology of $\mathrm{Aut}(\mathbb{G}^A_{1/n})$. In particular, in Theorem 2.9 we compute the continuous linear dual Hopf algebra of the group ring of $\mathrm{Aut}(\mathbb{G}^A_{1/n})$ (recall that, if $A = \hat{\mathbb{Z}}_p$, this linear dual Hopf algebra is called the *height n Morava stabilizer algebra*; see Ravenel [19, Section 6.3]), as well as the map from the height dn Morava stabilizer algebra to the linear dual of the group ring of $\mathrm{Aut}(\mathbb{G}^A_{1/n})$, induced by the inclusion of $\mathrm{Aut}(\mathbb{G}^A_{1/n})$ into the Morava stabilizer group.

The rest of the paper consists of applications of Theorem 2.9. We make the cohomology computations for $n = 1$ and $d = 2$ and $p > 3$; that is, for each quadratic extension $K/\mathbb{Q}_p$, we compute the restriction map in cohomology from the cohomology of the height 2 Morava stabilizer group to the cohomology of $\mathrm{Aut}(\mathbb{G}^A_1) \cong A^\times$. Here A is again the ring of integers of K. Up to isomorphism, there are only three quadratic extensions of $\mathbb{Q}_p$, one of which is unramified (and its relevant cohomological computation is Theorem 3.5), and two of which are totally ramified (and their relevant cohomological computations are Theorems 3.6 and 3.7).

By local class field theory, the norm residue symbol map is an isomorphism $A^\times \xrightarrow{\cong} \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$, where K^{nr} is the compositum of the unramified extensions of K, and K^{ab} the compositum of the abelian extensions of K; see Theorem 4.1 for a quick review of this fact. The natural isomorphism $\mathrm{Aut}(\mathbb{G}^A_1) \cong A^\times$, composed with the norm residue symbol, embeds $A^\times$ as a closed (by Proposition 4.3) subgroup of the height 2 Morava stabilizer group; hence, by the work of Goerss and Hopkins and Miller (see [7]), there exists an action of $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ on (a model for) the Lubin–Tate/Morava E–theory spectrum E_2 for each quadratic extension $K/\mathbb{Q}_p$, and by Devinatz and Hopkins [5], there exists a descent spectral sequence whose input is the (continuous) Galois cohomology of $K^{\mathrm{ab}}/K^{\mathrm{nr}}$ and whose output is the homotopy fixed point spectrum $E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})}$. In Theorem 5.2, we run this spectral sequence after smashing with the Smith–Toda complex $V(1)$ at each prime $p > 3$, and we compute the resulting map from the homotopy groups of $L_{K(2)}V(1)$ to the homotopy groups of the homotopy fixed-point spectrum $V(1) \wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})}$.

One interesting consequence is that the map

$$\pi_*(L_{K(2)}V(1)) \to \pi_*(V(1) \wedge E_2^{h\,\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^2-1})^{\mathrm{ab}}/\mathbb{Q}_p(\zeta_{p^2-1})^{\mathrm{nr}}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)}) \tag{1}$$

is injective on the sub-$\mathbb{F}_p[v_2^{\pm 1}]$–module of $\pi_*(L_{K(2)}V(1))$ generated by 1 and the element ζ_2 (from Hopkins's chromatic splitting conjecture, from Miller, Ravenel and Wilson [14], etc), and the map (1) is zero on the other May/Chevalley–Eilenberg generators of $\pi_*(L_{K(2)}V(1))$. Here ζ_{p^2-1} denotes a primitive $(p^2-1)^{\mathrm{st}}$ root of unity.

Another interesting consequence is Corollary 4.6: the product of the restriction maps

$$H_c^*(\mathrm{Aut}(\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p});\mathbb{F}_{p^2}) \to \prod_{[K:\mathbb{Q}_p]=2} H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}) \tag{2}$$

is injective in cohomological degrees ≤ 1, and for each May/Chevalley–Eilenberg basis element $x \in H_c^*(\mathrm{Aut}(\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p});\mathbb{F}_{p^2})$, either x or the Poincaré dual of x has nonzero image[1] under the map (2). (The product in the map (2) is taken over all isomorphism classes of quadratic extensions of $\mathbb{Q}_p$.) More generally, given a grading-homogeneous element $x \in \pi_*(L_{K(2)}V(1))$, either x or the (modulo $1-v_2$) Poincaré dual class of x is detected in $\pi_*(E_2^{\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})})$ for some quadratic extension $K/\mathbb{Q}$. I do not know yet if anything like this phenomenon generalizes to higher heights or to smaller primes.

This paper is a complete (and much improved) rewrite of much older material I wrote when I was in graduate school. I am grateful to T Lawson for suggesting a Galois descent argument used in the proof of Theorem 5.2; to D Ravenel for teaching me a great deal about formal modules and stable homotopy when I was a graduate student; to the anonymous referee for helpful comments and ideas which improved this paper; and to J Greenlees for his editorial help.

Conventions 1.1

- In this paper, all formal groups and formal modules are implicitly assumed to be *one-dimensional.*
- Throughout, we will use Hazewinkel's generators for BP_* (and, more generally, for the classifying ring V^A of A–typical formal A–modules, where A is a discrete valuation ring).
- By a "p–adic number field" we mean a finite field extension of the p–adic rationals $\mathbb{Q}_p$.
- When a ground field k is understood from context, we will write $\Lambda(x_1,\dots,x_n)$ for the exterior/Grassmann k–algebra with generators $x_1,\dots,x_n$, and we will write $P(x_1,\dots,x_n)$ for the polynomial k–algebra on the generators $x_1,\dots,x_n$.

[1] An appealingly coordinate-free way to describe this situation was suggested by the anonymous referee: the kernel of the map (2) is a Lagrangian, in the sense of symplectic linear algebra, for the duality pairing on the domain.

- We make use of standard conventions when dealing with Hopf algebroids, as in Ravenel [19, Appendix 1]: we write $\eta_L, \eta_R\colon A \to \Gamma$ for the left and right unit maps of a Hopf algebroid (A, Γ), and if $a \in A$, we sometimes also write a as shorthand for $\eta_L(a) \in \Gamma$.
- When $\mathbb{G}$ is an affine group scheme over a field k, we write $k[\mathbb{G}]^*$ for the Hopf algebra corepresenting $\mathbb{G}$, and given a $k[\mathbb{G}]^*$–comodule M, we write $H^*(\mathbb{G}; M)$ for the group scheme cohomology $\mathrm{Ext}^*_{k[\mathbb{G}]^*\text{–comod}}(k, M)$.
- When G is a profinite group and M a discrete G–module, we write $H^*_c(G; M)$ for the usual continuous cohomology of G, ie
$$H^*_c(G; M) = \mathrm{colim}_N\, H^*(G/N; M^N),$$
where the colimit is taken over all finite-index normal subgroups N of G.

2 Moduli of formal A–modules under change of A

The basic definition is:

Definition 2.1 Let A be a commutative ring, and let R be a commutative A–algebra. A *formal A–module over R* is a formal group law $G(X, Y) \in R[\![X, Y]\!]$ together with a ring homomorphism $\rho\colon A \to \mathrm{End}(G)$ such that $\rho(a)(X) \equiv aX \mod X^2$.

The addition in $\mathrm{End}(G)$ is the formal addition given by G, and the multiplication is composition. Chapter 21 of [9] is a good reference for formal A–modules; the paper [18] is a faster (but more abbreviated) introduction. Another reference which gives at least an attempt at an introductory account is [23].

The classical results on p–height and p–typicality (as in [12]) were generalized to formal A–modules, for A a discrete valuation ring (all but the first claim is proven by M Hazewinkel in Chapter 21 of [9]; the first claim is easier, and not directly used in this paper, but a proof can be found in [22]):

Theorem 2.2 *Let A be a discrete valuation ring of characteristic zero, with finite residue field. Then the classifying Hopf algebroid (L^A, L^AB) of formal A–modules admits a retract (V^A, V^AT) with the following properties:*

- *The inclusion $(V^A, V^AT) \hookrightarrow (L^A, L^AB)$ and the retraction $(L^A, L^AB) \hookrightarrow (V^A, V^AT)$ are maps of graded Hopf algebroids, and are mutually homotopy-inverse (that is, they induce an equivalence — but not an isomorphism! — on the associated stacks).*

- *If F is a formal A–module over a commutative A–algebra R and the underlying formal group law of F admits a logarithm $\log_F(X)$, then the classifying map $L^A \to R$ factors through the retraction map $L^A \to V^A$ if and only if $\log_F(X) = \sum_{n\geq 1} \alpha_n X^{q^n}$ for some $\alpha_1, \alpha_2, \ldots \in R \otimes_{\mathbb{Z}} \mathbb{Q}$, where q is the cardinality of the residue field of A.*
- *$V^A \cong A[v_1^A, v_2^A, \ldots]$ with v_n^A in grading degree $2(q^n-1)$, and $V^AT \cong V^A[t_1^A, t_2^A, \ldots]$ with t_n^A in grading degree $2(q^n-1)$.*
- *The generators $\{v_i^A\}$ for V^A, called the* Hazewinkel generators, *are defined as follows: we fix a uniformizer π for A, and let $v_0^A = \pi$. The universal A–typical formal A–module has logarithm of the form*

$$\log(x) = \sum_{i\geq 0} \lambda_i^A x^{q^i}, \tag{3}$$

and the equation

$$\pi\lambda_h^A = \sum_{i=0}^{h-1} \lambda_i^A (v_{h-i}^A)^{q^i}, \tag{4}$$

can be solved recursively for elements $v_1^A, v_2^A, \ldots \in V^A$; these are the Hazewinkel generators.

- *We have a formula*

$$\lambda_h^A = \sum_{i_1+\cdots+i_r=h} \pi^{-r} v_{i_1}^A (v_{i_2}^A)^{q^{i_1}} \cdots (v_{i_r}^A)^{q^{i_1+\cdots+i_{r-1}}}, \tag{5}$$

where π is the uniformizer and q the cardinality of the residue field of A, and all i_j are positive integers.

Definition 2.3 Let A be a discrete valuation ring of characteristic zero, with uniformizer π, and with finite residue field. Let R be a commutative A–algebra, and let $\mathbb{G}$ be a formal A–module over R.

- We say that $\mathbb{G}$ is *A–typical* if the classifying map $L^A \to R$ factors through the retraction $L^A \to V^A$.
- If $\mathbb{G}$ is A–typical, n is a nonnegative integer and R is a field, then we say that $\mathbb{G}$ has *A–height $\geq n$* if the classifying map $V^A \to R$ factors through the quotient map $V^A \to V^A/(\pi, v_1^A, \ldots, v_{n-1}^A)$. We say that $\mathbb{G}$ has *A–height n* if $\mathbb{G}$ has A–height $\geq n$ but not A–height $\geq n+1$. If $\mathbb{G}$ has A–height $\geq n$ for all n, then we say that $\mathbb{G}$ has A–height ∞.

- The inclusion $V^A \to L^A$ associates, to each formal A–module, an A–typical formal A–module $\mathrm{typ}(\mathbb{G})$ which is isomorphic to it. If $\mathbb{G}$ is an arbitrary (not necessarily A–typical) formal A–module, we say that $\mathbb{G}$ has *A–height n* if $\mathrm{typ}(\mathbb{G})$ has A–height n.

The following is proven in [9]:

Proposition 2.4 *Let p be a prime number.*

- *Every formal group law over a commutative $\hat{\mathbb{Z}}_p$–algebra admits the unique structure of a formal $\hat{\mathbb{Z}}_p$–module. Consequently, there is an equivalence of categories between formal group laws over commutative $\hat{\mathbb{Z}}_p$–algebras, and formal $\hat{\mathbb{Z}}_p$–modules. Under this correspondence, a formal $\hat{\mathbb{Z}}_p$–module is $\hat{\mathbb{Z}}_p$–typical if and only if its underlying formal group law is p–typical. If $\mathbb{G}$ is a formal $\hat{\mathbb{Z}}_p$–module of $\hat{\mathbb{Z}}_p$–height n, then its underlying formal group law has p–height n.*
- *If L and K are finite extensions of $\mathbb{Q}_p$ with rings of integers B and A, respectively, if L/K is a field extension of degree d, and if $\mathbb{G}$ is a formal B–module of B–height n, then the underlying formal A–module of $\mathbb{G}$ has A–height dn.*
- *In particular, if $K/\mathbb{Q}_p$ is a field extension of degree d, if K has ring of integers A and if $\mathbb{G}$ is a formal A–module of A–height n, then the underlying formal group law of $\mathbb{G}$ has p–height dn. Consequently,* the only formal groups which admit complex multiplication by A have underlying formal groups of p–height divisible by d.

Definition 2.5 Let $\mathbb{G}$ be an A–typical formal A–module over a commutative A–algebra R given by power series $G(X,Y) \in R[\![X,Y]\!]$ and $\rho(a)(X) \in R[\![X]\!]$ for each $a \in A$. The *strict automorphism group scheme of* $\mathbb{G}$, written $\mathrm{strictAut}(\mathbb{G})$, is the group scheme which sends a commutative A–algebra S to the group of strict automorphisms of $\mathbb{G} \otimes_R S$, ie the group (under composition) of formal power series $f(X) \in S[\![X]\!]$ such that

- $f(X) \equiv X \bmod X^2$,
- $f(G(X,Y)) = G(f(X), f(Y))$, and
- $f(\rho(a)(X)) = \rho(a)(f(X))$ for all $a \in A$.

By the usual functor-of-points argument, the strict automorphism scheme $\mathrm{strictAut}(\mathbb{G})$ of $\mathbb{G}$ is corepresented by the Hopf algebra $R \otimes_{V^A} V^A T \otimes_{V^A} R$, where R is a V^A–algebra via the ring map $V^A \to R$ classifying $\mathbb{G}$, and (as is the usual convention — see eg Chapter 6 of [19]) V^AT is a left V^A–algebra via the left unit map $\eta_L\colon V^A \to V^AT$ and V^AT is a right V^A–algebra via the right unit map $\eta_R\colon V^A \to V^AT$. (Recall that the left and right unit maps classifying the underlying A–typical formal A–module of the source and target of the universal strict isomorphism of A–typical formal A–modules.) We will write $R[\mathrm{strictAut}(\mathbb{G})]^*$ for the corepresenting Hopf algebra of $\mathrm{strictAut}(\mathbb{G})$.

It is worth being careful about notation: $\mathrm{strictAut}(\mathbb{G})$ is a profinite group scheme but often fails to be proconstant, so it is not always the case that $R[\mathrm{strictAut}(\mathbb{G})]^*$ is the continuous R–linear dual of the group ring $R[\mathrm{strictAut}(\mathbb{G})(R)]$, even when R is a field.

In Definition 2.6 we introduce a new notation which we find very convenient:

Definition 2.6 Let K be a p–adic number field with ring of integers A and residue field k, and let n be a positive integer.

- We write $\widetilde{\mathbb{G}}^A_{1/n}$ for the formal A–module over $k[v_n^A]$ classified by the map $V^A \to k[v_n^A]$ sending v_n^A to v_n^A and sending v_i^A to zero if $i \neq n$.
- Let k' be a field extension of k and $\alpha \in (k')^\times$. We write ${}_\alpha\mathbb{G}^A_{1/n}$ for the formal A–module over k' classified by the map $V^A \to k'$ sending v_n^A to α and sending v_i^A to zero if $i \neq n$.

Proposition 2.7 *Let K be a p–adic number field with ring of integers A. Let k be the residue field of A, let q be the cardinality of k and let π be a uniformizer for A. Let n be a positive integer. Then, as a quotient of $V^AT \cong A[v_1^A, v_2^A, \dots][t_1^A, t_2^A, \dots]$, the Hopf algebra corepresenting $\mathrm{strictAut}(\widetilde{\mathbb{G}}^A_{1/n})$ is*

$$k[v_n^A][t_1^A, t_2^A, \dots]/(t_i^A(v_n^A)^{q^i} - v_n^A(t_i^A)^{q^n}\ \forall i).$$

Proof Ravenel [18] proves the formula

$$\sum\nolimits^F_{i,j\geq 0}\eta_L(v_i^A)(t_j^A)^{q^i} \equiv \sum\nolimits^F_{i,j\geq 0}\eta_R(v_i^A)^{q^j}t_j^A \mod \pi, \tag{6}$$

where $\sum^F$ is the formal sum, ie the sum using the formal group law underlying the universal A–typical formal A–module (the sum is well defined because there are only finitely many terms in each grading degree). We are following the usual convention that $v_0^A = \pi$ and $t_0^A = 1$. As a consequence of (6),

$$V^A \xrightarrow{\eta_R} k[v_n^A] \otimes_{V^A} V^AT = V^AT/\eta_L(v_0^A, v_1^A, \dots, v_{n-2}^A, v_{n-1}^A, v_{n+1}^A, v_{n+2}^A, \dots)$$

is determined by

$$\sum_{i\geq 0}{}^F t_i^A \eta_R(v_n^A)^{q^i} = \sum_{j\geq 0}{}^F v_n^A (t_j^A)^{q^n}. \tag{7}$$

By induction on degree, we get

$$t_i^A \eta_R(v_n^A)^{q^i} = v_n^A (t_i^A)^{q^n}$$

in $k[v_n^A] \otimes_{V^A} V^A T \otimes_{V^A} k[v_n^A]$, since there is at most one formal summand in each grading degree on each side of (7). This gives us the relation in the statement of the theorem. □

Lemma 2.8 *Let L/K be a finite field extension of degree d and ramification degree e, with K and L p–adic number fields with rings of integers A and B, respectively. Let ℓ denote the residue field of B. Then the ring map*

$$V^A T \cong V^A[t_1^A, t_2^A, \dots] \to V^B[t_1^B, t_2^B, \dots] \cong V^B T \tag{8}$$

classifying the strict formal A–module isomorphism underlying the universal strict formal B–module isomorphism sends t_i^A to $t_{ie/d}^B$ if i is divisible by the residue degree d/e of L/K, and sends t_i^A to zero if i is not divisible by the residue degree d/e.

Furthermore, let n be a positive integer. Then the map

$$\kappa\colon V^A T \to \ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}^B_{1/n})]^* \tag{9}$$

classifying the universal strict automorphism of $\widetilde{\mathbb{G}}^B_{1/n}$ sends v_{dn}^A to

$$\frac{\pi_A}{\pi_B^e}(v_n^B)^{(q^{ne}-1)/(q^n-1)},$$

where q is the cardinality of the residue field of B, and π_A and π_B are uniformizers for A and B, respectively. Furthermore, the kernel of the map (9) contains $\eta_L(v_i^A)$ and $\eta_R(v_i^A)$ for all $i \neq dn$.

Proof The claim about the behavior of the map (8) on the generators $t_1^A, t_2^A, \dots$ is a generalization of Lemma 3.11(a) of [18]. Proving this claim requires some explanation of how the generators $t_1^A, t_2^A, \dots$ in $V^A T$ work; see eg the proof of [19, Theorem A.2.1.27(d)]. Write $\mathbb{G}^B_{\mathrm{univ}}$ for the universal B–typical formal B–module, and write $\log_{\mathbb{G}^B_{\mathrm{univ}}}(X) = \sum_{n\geq 0} \lambda_n^B X^{q^n}$ for its logarithm. Then $\mathbb{G}^B_{\mathrm{univ}}$ is the source of the universal strict isomorphism of B–typical formal B–modules; write $\mathbb{G}^B_{\mathrm{univ},0}$ for its target. Then $\mathbb{G}^B_{\mathrm{univ},0}$ has logarithm $\log_{\mathbb{G}^B_{\mathrm{univ},0}}(X) = \sum_{n\geq 0} \eta_R(\lambda_n^B) X^{q^n}$, where $\eta_R\colon V^A \to V^A T$ is the right unit map. The universal strict isomorphism $f\colon \mathbb{G}^B_{\mathrm{univ}} \to \mathbb{G}^B_{\mathrm{univ},0}$ has inverse given by the formal sum

$$f^{-1}(X) = \sum_{n\geq 0}^{\mathbb{G}^B_{\mathrm{univ}}} t^B_n X^{q^n}. \tag{10}$$

The map (8) is determined as follows: for any p–adic number ring C, if we let F_C denote the underlying formal group law of the universal C–typical formal C–module, then the log coefficients $\lambda^C_0, \lambda^C_1, \dots$ are the coefficients in the unique power series

$$\log_{F_C}(X) = \sum_{n\geq 0} \lambda^C_n X^{q^n_C}$$

satisfying the conditions

$$\lambda^C_0 = 1 \quad \text{and} \quad F_C(X,Y) = \log^{-1}_{F_C}(\log_{F_C}(X) + \log_{F_C}(Y)),$$

where q_C is the cardinality of the residue field of C. Now, if we write

- K_{nr} for the maximal unramified extension of K contained in L,
- A_{nr} for the ring of integers of K_{nr}, and
- $\gamma'\colon V^{A_{\mathrm{nr}}} \to V^B$ for the map classifying the underlying A_{nr}–typical formal A_{nr}–module of the universal B–typical formal B–module,

then, by the definition of γ', applying γ' to the coefficients of $F_{A_{\mathrm{nr}}}$ yields F_B. In particular, γ' applied to $\log_{F_{A_{\mathrm{nr}}}}(X)$ yields $\log_{F_B}(X)$. Since L/K_{nr} is totally ramified, $q_B = q_{A_{\mathrm{nr}}} = q$, and consequently we have equalities of power series

$$\sum_{n\geq 0} \gamma'(\lambda^{A_{\mathrm{nr}}}_n) X^{q^n} = \gamma'(\log_{F_{A_{\mathrm{nr}}}}(X)) = \log_{F_B}(X) = \sum_{n\geq 0} \lambda^B_n X^{q^n},$$

ie γ' sends $\lambda^{A_{\mathrm{nr}}}_n$ to λ^B_n for all n. Hence the map of Hopf algebroids

$$\gamma\colon (V^{A_{\mathrm{nr}}}, V^{A_{\mathrm{nr}}}T) \to (V^B, V^B T)$$

sends $\eta_R(\lambda^{A_{\mathrm{nr}}}_n)$ to $\eta_R(\lambda^B_n)$ for all n, ie solving (10) yields that $\gamma(t^{A_{\mathrm{nr}}}_n) = t^B_n$.

The unramified case is similar: the map $V^A \to V^{A_{\mathrm{nr}}}$ sends λ^A_n to $\lambda^{A_{\mathrm{nr}}}_{ne/d}$ if n is divisible by the residue degree $d/e = [K_{\mathrm{nr}} : K]$, and sends λ^A_n to zero if n is not divisible by the residue degree, and solving (10) yields that the map

$$\begin{aligned} k(\alpha)[\mathrm{strictAut}({}_\alpha\mathbb{G}^A_{1/dn})]^* &\cong k(\alpha)[t^A_1, t^A_2, \dots]/(t^A_i \alpha^{q^{ei}-1} - (t^A_i)^{q^{en}}\ \forall i) \\ &\to \ell[t^{A_{\mathrm{nr}}}_1, t^{A_{\mathrm{nr}}}_2, \dots]/(t^{A_{\mathrm{nr}}}_i \alpha^{q^{ei}-1} - (t^{A_{\mathrm{nr}}}_i)^{q^{en}}\ \forall i) \cong \ell[\mathrm{strictAut}({}_\alpha\mathbb{G}^{A_{\mathrm{nr}}}_{1/en})]^* \end{aligned}$$

sends t_n^A to $t_{ne/d}^{A_{\mathrm{nr}}}$ if n is divisible by the residue degree $d/e = [K_{\mathrm{nr}} : K]$, and sends t_n^A to zero if n is not divisible by the residue degree.

Now for the claim about the kernel of the map (9): we break the problem into two parts, an unramified part and a totally ramified part. From Lemma 3.11(b) of [18], we have that the map $\gamma\colon V^A \to V^{A_{\mathrm{nr}}}$ classifying the underlying A–typical formal A–module of the universal A_{nr}–typical formal A_{nr}–module sends v_n^A to $v_{ne/d}^{A_{\mathrm{nr}}}$ if the residue degree d/e divides n, and $\gamma(v_n^A) = 0$ if d/e does not divide n.

Meanwhile, the map $\gamma'\colon V^{A_{\mathrm{nr}}} \to V^B$ can be computed using (4):

$$\gamma'\biggl(\sum_{i=0}^{h-1} \lambda_i^{A_{\mathrm{nr}}} (v_{h-i}^{A_{\mathrm{nr}}})^{q^i}\biggr) = \frac{\pi_A}{\pi_B} \sum_{i=0}^{h-1} \lambda_i^B (v_{h-i}^B)^{q^i}, \tag{11}$$

and the fact that $\gamma'(\lambda_i^{A_{\mathrm{nr}}}) = \lambda_i^B$. Modulo $v_1^B, v_2^B, \dots, v_{n-2}^B, v_{n-1}^B, v_{n+1}^B, v_{n+2}^B, \dots,$ (5) reads

$$\lambda_h^B = \begin{cases} \pi_B^{-h/n} (v_n^B)^{(q^h-1)/(q^n-1)} & \text{if } n \mid h, \\ 0 & \text{if } n \nmid h, \end{cases}$$

and consequently (11) reads

$$\begin{aligned} \sum_{i=0}^{h-1} \lambda_i^B \gamma'(v_{h-i}^{A_{\mathrm{nr}}})^{q^i} &= \frac{\pi_A}{\pi_B} \pi_B^{-(h-n)/n} (v_n^B)^{(q^{h-n}-1)/(q^n-1)} (v_n^B)^{q^{h-n}} \\ &= \frac{\pi_A}{\pi_B^{h/n}} (v_n^B)^{(q^h-1)/(q^n-1)}. \end{aligned}$$

Now an easy induction gives us that $\gamma'(v_h^A)$ has positive π_B–adic valuation, and hence is zero in $\ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}_{1/n}^B)]^*$, as long as $h < en$, and when $h = en$ we get the formula $\gamma'(v_{en}^{A_{\mathrm{nr}}}) = (\pi_A/\pi_B^e)(v_n^B)^{(q^{en}-1)/(q^n-1)}$, which, combined with the unramified computation above, immediately yields that the map (9) sends v_{dn}^A to $(\pi_A/\pi_B^e)(v_n^B)^{(q^{ne}-1)/(q^n-1)}$, as claimed.

Now for a slightly more involved induction: suppose we have shown that $\kappa(v_{n(e+a)}^{A_{\mathrm{nr}}}) = 0$ for $a = 1, \dots, j-1$. Then equations (5) and (11) yield the equation

$$\kappa(v_{n(e+j)}^{A_{\mathrm{nr}}}) + \lambda_{jn}^B \kappa(v_{ne}^{A_{\mathrm{nr}}})^{q^{nj}} = \frac{\pi_A}{\pi_B^{e+j}} (v_n^B)^{(q^{n(e+j)}-1)/(q^n-1)},$$

ie

$$\kappa(v_{n(e+j)}^{A_{\mathrm{nr}}}) = \left(\frac{\pi_A}{\pi_B^{e+j}} - \left(\frac{\pi_A}{\pi_B^e}\right)^{q^{nj}} \frac{1}{\pi_B^j} \right) (v_n^B)^{(q^{n(e+j)}-1)/(q^n-1)}, \tag{12}$$

and the right-hand side of (12) is zero, since the scalars π_A and π_B live in the residue field of B, which is the finite field with q elements, hence the q^{th} power map is the identity. So $\kappa(v^{A_{\mathrm{nr}}}_{n(e+j)}) = 0$ for all $j > 0$; an easy computation using equations (5) and (11) also yields that $\kappa(v^{A_{\mathrm{nr}}}_j) = 0$ for $j > en$ not divisible by n.

An easy consequence of equation (6) is that $\eta_L(v_i)$ is congruent to $\eta_R(v_i)$ modulo $(\eta_L(v^A_0), \eta_L(v^A_1), \ldots, \eta_L(v^A_{i-1}))$. Hence $\eta_R(v_i)$ is in the kernel of κ for all $i < dn$. Now we carry out an induction to show that $\kappa(\eta_R(v^A_i)) = 0$ for all $i > dn$ as well: suppose that we have already shown that $\kappa(\eta_R(v^A_{dn+j})) = 0$ for $j = 1, \ldots, i-1$. Then, reducing formula (6) modulo the kernel of κ, we have

$$\sum\nolimits^F_{j\geq 0}\eta_L(v^A_{dn})(t^A_j)^{q^{en}} = \sum\nolimits^F_{j\geq 0, i\leq dn}\eta_R(v^A_i)^{q^{je/d}} t^A_j$$

(using the fact that the cardinality of the residue field of A is $q^{e/d}$), and in grading degree $2(q^{dn+i}-1)$, this equation reads

$$\eta_L(v^A_{dn})(t^A_i)^{q^{en}} = \eta_R(v^A_{dn})^{q^{ie/d}} t^A_i +^F \eta_R(v^A_{dn+i}). \tag{13}$$

If i is not divisible by the residue degree d/e of L/K, then we have already shown that $\kappa(t^A_i) = 0$, and consequently (13) implies that $\kappa(\eta_R(v^A_{dn+i})) = 0$, as desired. So suppose instead that i is divisible by the residue degree d/e. We already know that

$$\kappa(\eta_R(v^A_{dn})) = \kappa(\eta_L(v^A_{dn})) = \frac{\pi_A}{\pi^e_B}(v^B_n)^{(q^{ne}-1)/(q^n-1)},$$

and that

$$t^B_{ie/d}(v^B_n)^{q^{ie/d}} = v^B_n (t^B_{ie/d})^{q^n} \tag{14}$$

in $\ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}^B_{1/n})]^*$. Hence,

$$\begin{aligned}
&\frac{\pi_A}{\pi^e_B}(v^B_n)^{(q^{ne}-1)/(q^n-1)}(t^B_{ie/d})^{q^{en}} \\
&\qquad = \kappa(\eta_L(v^A_{dn})(t^A_i)^{q^{en}}) \\
&\qquad = \kappa(\eta_R(v^A_{dn})^{q^{ie/d}} t^A_i +^F \eta_R(v^A_{dn+i})) \\
&\qquad = \kappa(\eta_R(v^A_{dn})^{q^{ie/d}} t^A_i) +^F \kappa(\eta_R(v^A_{dn+i})) \\
&\qquad = \left(\frac{\pi_A}{\pi^e_B}(v^B_n)^{(q^{ne}-1)/(q^n-1)}\right)^{q^{ie/d}} t^B_{ie/d} +^F \kappa(\eta_R(v^A_{dn+i})),
\end{aligned} \tag{15}$$

with the third equality due to the formal group law on $\ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}^B_{1/n})]^*$ being precisely the one classified by κ. We have $(\pi_A/\pi^e_B)^q = \pi_A/\pi^e_B$, since π_A/π^e_B is an element of $\ell \cong \mathbb{F}_q$, and repeated use of (14) then implies that

$$(16) \qquad \frac{\pi_A}{\pi_B^e}(v_n^B)^{(q^{ne}-1)/(q^n-1)}(t_{ie/d}^B)^{q^{en}} = \left(\frac{\pi_A}{\pi_B^e}(v_n^B)^{(q^{ne}-1)/(q^n-1)}\right)^{q^{ie/d}} t_{ie/d}^B,$$

and consequently (15) implies that $\kappa(\eta_R(v_{dn+i}^A)) = 0$, completing the inductive step. □

Lemma 2.8 shows that the map

$$\kappa\colon V^AT \to \ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}_{1/n}^B)]^*$$

factors through the projection $V^AT \to k \otimes_{V^A} V^AT \otimes_{V^A} k = k[\mathrm{strictAut}(\widetilde{\mathbb{G}}_{1/dn}^A)]^*$, where k is the residue field of A. This gives us a well-defined map of Hopf algebras

$$k[\mathrm{strictAut}(\widetilde{\mathbb{G}}_{1/dn}^A)]^* \to \ell[\mathrm{strictAut}(\widetilde{\mathbb{G}}_{1/n}^B)]^*,$$

which we compute in Theorem 2.9:

Theorem 2.9 *Let L/K be a finite field extension of degree d, with K and L p–adic number fields with rings of integers A and B, respectively. Let k and ℓ be the residue fields of A and B, let e be the ramification degree of L/K, let q be the cardinality of ℓ, and let π_A and π_B be uniformizers for A and B, respectively. Let n be a positive integer.*

Then the underlying formal A–module of $\widetilde{\mathbb{G}}_{1/n}^B$ is $\widetilde{\mathbb{G}}_{1/dn}^A$. Furthermore, if ℓ' is a field extension of ℓ and $\beta \in (\ell')^\times$, then the underlying formal A–module of ${}_\beta\mathbb{G}_{1/n}^B$ is ${}_\alpha\mathbb{G}_{1/dn}^A$, where

$$\alpha = \frac{\pi_A}{\pi_B^e}\beta^{(q^{en}-1)/(q^n-1)}.$$

Furthermore, the ring map

$$(17) \quad k(\alpha)[\mathrm{strictAut}({}_\alpha\mathbb{G}_{1/dn}^A)]^* = k(\alpha)[t_1^A, t_2^A, \dots]/(t_i^A\alpha^{q^{ei}-1} - (t_i^A)^{q^{en}} \ \forall i)$$
$$\to \ell[t_1^B, t_2^B, \dots]/(t_i^B\beta^{q^i-1} - (t_i^B)^{q^n} \ \forall i) = \ell[\mathrm{strictAut}({}_\beta\mathbb{G}_{1/n}^B)]^*$$

classifying the strict formal A–module automorphism of ${}_\alpha\mathbb{G}_{1/dn}^A$ underlying the universal strict formal B–automorphism of ${}_\beta\mathbb{G}_{1/n}^B$ sends t_i^A to $t_{ie/d}^A$ if i is divisible by the residue degree d/e of L/K, and sends t_i^A to zero if i is not divisible by the residue degree d/e.

Proof These claims all follow from Proposition 2.7 and Lemma 2.8. □

3 The $n = 1, d = 2$ case

Recall the following computation from Theorem 6.3.22 of [19] (this computation is also carried out again, in detail, in the present paper's companion paper [24]):

Theorem 3.1 (cohomology of the height 2 Morava stabilizer group at large primes) *Let k be a finite field of characteristic $p > 3$. Then*

$$H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p}); k) \cong \Lambda(\zeta_2) \otimes_k k\{1, h_{10}, h_{11}, \eta_2 h_{10}, \eta_2 h_{11}, \eta_2 h_{10} h_{11}\},$$

with bidegrees as follows:

(18)

cohomology class	cohomology degree	internal degree
1	0	0
h_{10}	1	$2(p-1)$
h_{11}	1	$2p(p-1)$
ζ_2	1	0
$h_{10}\eta_2$	2	$2(p-1)$
$h_{11}\eta_2$	2	$2p(p-1)$
$h_{10}\zeta_2$	2	$2(p-1)$
$h_{11}\zeta_2$	2	$2p(p-1)$
$h_{10}h_{11}\eta_2$	3	0
$h_{10}\eta_2\zeta_2$	3	$2(p-1)$
$h_{11}\eta_2\zeta_2$	3	$2p(p-1)$
$h_{10}h_{11}\eta_2\zeta_2$	4	0

where the cup products in $\mathbb{F}_p\{1, h_{10}, h_{11}, h_{10}\eta_2, h_{11}\eta_2, h_{10}h_{11}\eta_2\}$ are all zero aside from the Poincaré duality cup products, ie each class has the obvious dual class such that the cup product of the two is $h_{10}h_{11}\eta_2$, and the remaining cup products are all zero. The internal/topological degrees are defined modulo $|v_2| = 2(p^2-1)$.

In the cobar complex for the Hopf algebra $\mathbb{F}_p[{}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p}]^*$, we have cocycle representatives

$$h_{10} = [t_1], \tag{19}$$

$$h_{11} = [t_1^p], \tag{20}$$

$$\zeta_2 = [t_2 + t_2^p - t_1^{p+1}], \tag{21}$$

$$h_{10}\eta_2 = [t_1 \otimes t_2 - t_1 \otimes t_2^p + t_1 \otimes t_1^{p+1} + t_1^2 \otimes t_1^p], \tag{22}$$

$$h_{11}\eta_2 = [t_1^p \otimes t_2^p - t_1^p \otimes t_2 + t_1^p \otimes t_1^{p+1} + t_1^{2p} \otimes t_1]. \tag{23}$$

Proposition 3.2 *Let K be a p–adic number field with ring of integers A. Let k be the residue field of A, let π be a uniformizer for A, let k' be a field extension of k, and let $\alpha \in (k')^{\times}$. Then the profinite group scheme ${}_{\alpha}\mathbb{G}_1^A$ is the proconstant group scheme taking the value $1+\pi A$, the group (under multiplication) of 1–units in A. That is, the Hopf algebra $k'[{}_{\alpha}\mathbb{G}_1^A]^*$ is the continuous k'–linear dual of the topological group ring $k'[1+\pi A]$.*

Proof It follows from the Barsotti–Tate module generalization of the well-known Dieudonné–Manin classification of p–divisible groups over algebraically closed fields (see [13]; also see [17] for a nice treatment of the theory of Barsotti–Tate modules) that ${}_{\alpha}\mathbb{G}_1^A \otimes_{k'} \bar{k}' \cong {}_{\beta}\mathbb{G}_1^A \otimes_k \bar{k}'$ for all $\alpha, \beta \in k'$, where $\bar{k}'$ is the algebraic closure of k', and that $\operatorname{strictAut}({}_{\alpha}\mathbb{G}_1^A \otimes_{k'} \bar{k}')$ is the proconstant group scheme with value $1+\pi A$. So we just need to show that $\operatorname{strictAut}({}_{\alpha}\mathbb{G}_1^A)$ is already proconstant.

Theorem 2.9 gives us that

$$k'[\operatorname{strictAut}({}_{\alpha}\mathbb{G}_1^A)]^* \cong k'[t_1, t_2, \dots]/(t_i\alpha^{q^i-1} - t_i^q \ \forall i),$$

where q is the cardinality of k. Our argument is essentially the same as that of the proof of Theorem 6.2.3 in Ravenel's book [19]: an affine profinite group scheme $\cdots \to G_2 \to G_1 \to G_0$ over a field k is proconstant if and only if the corepresenting Hopf algebroid $k[G_n]^*$ has a k–linear basis $\{y_i\}_{i\in I}$ of idempotent elements such that $y_i y_j = 0$ for all $i \neq j$ (see eg [26]). In the case of $\operatorname{strictAut}({}_{\alpha}\mathbb{G}_1^A)$, it is profinite by virtue of being the limit (over n) of the strict automorphism group scheme of a formal A–module n–bud, and the strict automorphism group scheme of a formal A–module n–bud is corepresented by the Hopf algebra

$$k'[t_1, t_2, \dots, t_m]/(t_i\alpha^{q^i-1} - t_i^q : i = 1, \dots, m),$$

where m is the integer floor of $\log_q n$. This Hopf algebra splits, as a k'–algebra, as the tensor product of copies of $k'[t_i]/(t_i\alpha^{q^i-1} - t_i^q)$ for various i. The map of k'–algebras $k'[s]/(s-s^q) \to k'[t_i]/(t_i\alpha^{q^i-1} - t_i^q)$ sending s to $\alpha^{(1-q^i)/(q-1)}t_i$ is an isomorphism of k'–algebras, and $k'[s]/(s-s^q)$ admits a k'–linear basis of idempotents whose pairwise products are all zero, namely, $1-s^{q-1}$ and $-\sum_{j=1}^{q-1}(a^i s)^j$ for $i = 1, \dots, q-1$ (this basis is taken from Theorem 6.2.3 of [19]), where a is any generator for $\mathbb{F}_q^{\times} \cong k^{\times} \subseteq (k')^{\times}$. So $\operatorname{strictAut}({}_{\alpha}\mathbb{G}_1^A)$ is indeed "already" (ie without any need to change base to an algebraic closure) proconstant over k'. □

Theorem 3.3 is easy and classical, essentially a part of local class field theory:

Theorem 3.3 *Let $p > 3$, and let $K/\mathbb{Q}_p$ be a quadratic extension with ring of integers A. Let π be a uniformizer for A and let k be the residue field of A. Then the continuous group cohomology of the profinite group $1+\pi A$ of 1–units in A is*

$$H_c^*(1+\pi A; k) \cong \Lambda(h_1, h_2),$$

with h_1 and h_2 in cohomological degree 1.

Proof By Proposition II.5.5 in [15], the p–adic exponential and logarithm maps yield an isomorphism of profinite groups between $1+\pi A$ and the group πA under addition. As a profinite abelian group, $\pi A \cong A \cong \widehat{\mathbb{Z}}_p \times \widehat{\mathbb{Z}}_p$, and it is classical that the continuous cohomology $H_c^*(\widehat{\mathbb{Z}}_p; k) = \operatorname{colim}_{n\to\infty} H^*(\mathbb{Z}/p^n\mathbb{Z}; k)$ is the colimit of the sequence of graded abelian groups

$$\begin{array}{ccccccc}
H^*(\mathbb{Z}/p\mathbb{Z};k) & \longrightarrow & H^*(\mathbb{Z}/p^2\mathbb{Z};k) & \longrightarrow & H^*(\mathbb{Z}/p^3\mathbb{Z};k) & \longrightarrow & \cdots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\Lambda(h)\otimes_k k[b] & \longrightarrow & \Lambda(h)\otimes_k k[b] & \longrightarrow & \Lambda(h)\otimes_k k[b] & \longrightarrow & \cdots
\end{array}$$

where the horizontal maps send h to h and b to 0, ie $H_c^*(\widehat{\mathbb{Z}}_p; k)$ is an exterior algebra on one generator. □

Proposition 3.4 is well known, and not difficult; see eg Section I.6.6 of [21].

Proposition 3.4 *Let $p>2$. Then there are, up to isomorphism, exactly three quadratic extensions of $\mathbb{Q}_p$: the unramified extension $\mathbb{Q}_p(\zeta_{p^2-1})$, where ζ_{p^2-1} is a primitive $(p^2-1)^{st}$ root of unity; and two totally ramified extensions $\mathbb{Q}_p(\sqrt{p})$ and $\mathbb{Q}_p(\sqrt{ap})$, where a is any integer satisfying $1 \le a < p$ which does not have a square root in $\mathbb{F}_p$.*

Theorem 3.5 *Let $p > 3$. Then*

$$H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]}); \mathbb{F}_p) \cong \Lambda(h_{20}, h_{21}),$$

with h_{20} and h_{21} each in cohomological degree 1 and internal degree 0, and the restriction map

$$\begin{aligned}\Lambda(\zeta_2)\otimes_k k\{1, h_{10}, h_{11}, \eta_2 h_{10}, \eta_2 h_{11}, \eta_2 h_{10} h_{11}\} &\cong H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}); k)\\ &\xrightarrow{\mathrm{res}} H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]}); \mathbb{F}_p) \cong \Lambda(h_{20}, h_{21}),\end{aligned}$$

induced by the inclusion of the profinite subgroup

$$\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]}) \subset \mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}),$$

is the map of graded $\mathbb{F}_p$–algebras determined by

$$\begin{aligned} \mathrm{res}(\zeta_2) &= h_{20} + h_{21}, & \mathrm{res}(h_{10}\eta_2) &= 0, \\ \mathrm{res}(h_{10}) &= 0, & \mathrm{res}(h_{11}\eta_2) &= 0, \\ \mathrm{res}(h_{11}) &= 0, & \mathrm{res}(h_{10}h_{11}\eta_2) &= 0. \end{aligned}$$

Proof We will use cocycle representatives (19)–(23) for cohomology classes in $H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}); k)$. In the cobar complex for $\mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]})]^*$, one easily computes that t_2 and t_2^p are 1–cocycles (since Theorem 2.9 describes $\mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]})]^*$ as a quotient Hopf algebra of $\mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p})]^*$, we can use Ravenel's formulas for the comultiplication on $S(2) \cong \mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p})]^*$ from Section 6.3 of [19], and simply reduce them modulo $t_1, t_3, t_5, \dots$ to get the comultiplication on $\mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]})]^*$) which are, modulo coboundaries, $\mathbb{F}_p$–linearly independent; hence t_2 and t_2^p represent two linearly independent classes in $H^1(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]}); \mathbb{F}_p)$, so, by Theorem 3.3, the cohomology classes of t_2 and t_2^p are a minimal set of $\mathbb{F}_p$–algebra generators for $H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\zeta_{p^2-1}]}); \mathbb{F}_p)$. We write h_{20} and h_{21} for the cohomology classes of t_2 and t_2^p, respectively. Applying Theorem 2.9, the map res is simply reduction modulo $t_1, t_3, t_5, \dots$ on cocycle representatives; hence, $\mathrm{res}(\zeta_2) = h_{20} + h_{21}$ and res vanishes on all other generators for the ring $H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}); k)$. □

Theorem 3.6 *Let $p > 3$. Then*

$$H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p) \cong \Lambda(h_{10}, h_{20}),$$

with h_{10} and h_{20} each in cohomological degree 1, h_{10} in internal degree $2(p-1)$ and h_{20} in internal degree 0, and the restriction map

$$\Lambda(\zeta_2) \otimes_k k\{1, h_{10}, h_{11}, \eta_2 h_{10}, \eta_2 h_{11}, \eta_2 h_{10} h_{11}\} \cong H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}); k) \\ \xrightarrow{\mathrm{res}} H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p) \cong \Lambda(h_{10}, h_{20}),$$

induced by the inclusion of the profinite subgroup

$$\mathrm{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]}) \subset \mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}),$$

is the map of graded $\mathbb{F}_p$–algebras determined by

$$\begin{aligned} \operatorname{res}(\zeta_2) &= 2h_{20}, & \operatorname{res}(h_{10}\eta_2) &= 0, \\ \operatorname{res}(h_{10}) &= h_{10}, & \operatorname{res}(h_{11}\eta_2) &= 0, \\ \operatorname{res}(h_{11}) &= h_{10}, & \operatorname{res}(h_{10}h_{11}\eta_2) &= 0. \end{aligned}$$

Proof This is a very similar computation to Theorem 3.5. We use cocycle representatives (19)–(23) for the $\mathbb{F}_p$–algebra generators of $H_c^{*,*}(\operatorname{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}); k)$. Theorem 2.9 tells us that, as a quotient of the Hopf algebra

$$\mathbb{F}_p[\operatorname{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p})]^* \cong \mathbb{F}_p[t_1, t_2, \dots]/(t_i^{p^2} - t_i),$$

the Hopf algebra $\mathbb{F}_p[\operatorname{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]})]^*$ is $\mathbb{F}_p[t_1, t_2, \dots]/(t_i^p - t_i)$. The 1–cocycles t_1 and $t_2 - \frac{1}{2}t_1^2$ in the cobar complex for $\mathbb{F}_p[\operatorname{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]})]^*$ are easily seen to be $\mathbb{F}_p$–linearly independent modulo coboundaries, so Theorem 3.3 again implies that the cohomology classes of t_1 and $t_2 - \frac{1}{2}t_1^2$ are a minimal set of $\mathbb{F}_p$–algebra generators for $H_c^{*,*}(\operatorname{strictAut}({}_1\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]}); \mathbb{F}_p)$.

Reducing cocycles (19)–(21) modulo $(t_1^p - t_1, t_2^p - t_2)$ immediately yields the given formulas for $\operatorname{res}(\zeta_2)$, $\operatorname{res}(h_{10})$ and $\operatorname{res}(h_{11})$. For $\operatorname{res}(h_{10}\eta_2)$, we see that reducing (22) modulo $(t_1^p - t_1, t_2^p - t_2)$ yields the 2–cocycle $t_1 \otimes t_1^2 + t_1^2 \otimes t_1$, which is the coboundary of $\frac{1}{3}t_1^3$, so $\operatorname{res}(h_{10}\eta_2) = 0$, and a similar computation yields $\operatorname{res}(h_{11}\eta_2) = 0$. □

Theorem 3.7 *Let $p > 3$, and choose an integer a such that $0 < a < p$ and such that a does not have a square root in $\mathbb{F}_p$. Then $\mathbb{F}_{p^2}$ has a $(p+1)^{st}$ root ω of a, and the underlying formal $\widehat{\mathbb{Z}}_p$–module of ${}_\omega\mathbb{G}_1^{\widehat{\mathbb{Z}}_p(\sqrt{ap})}$ is ${}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p}$, and*

$$H_c^{*,*}(\operatorname{strictAut}({}_\omega\mathbb{G}_1^{\widehat{\mathbb{Z}}_p(\sqrt{ap})}); \mathbb{F}_p) \cong \Lambda(h_{10}, h_{20}),$$

with h_{10} and h_{20} each in cohomological degree 1, h_{10} in internal degree $2(p-1)$ and h_{20} in internal degree 0, and the restriction map

$$\begin{aligned} &\Lambda(\zeta_2) \otimes_{\mathbb{F}_{p^2}} \mathbb{F}_{p^2}\{1, h_{10}, h_{11}, \eta_2 h_{10}, \eta_2 h_{11}, \eta_2 h_{10} h_{11}\} \\ &\quad \cong H_c^{*,*}(\operatorname{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}); \mathbb{F}_{p^2}) \\ &\qquad\qquad \xrightarrow{\operatorname{res}} H_c^{*,*}(\operatorname{strictAut}({}_\omega\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{ap}]}); \mathbb{F}_{p^2}) \cong \Lambda(h_{10}, h_{20}), \end{aligned}$$

induced by the inclusion of the profinite subgroup

$$\operatorname{strictAut}({}_\omega\mathbb{G}_1^{\widehat{\mathbb{Z}}_p[\sqrt{p}]}) \subset \operatorname{strictAut}({}_1\mathbb{G}_{1/2}^{\widehat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}),$$

is the map of graded $\mathbb{F}_p$–algebras determined by

$$\begin{aligned} \mathrm{res}(\zeta_2) &= 2h_{20}, & \mathrm{res}(h_{10}\eta_2) &= 0,\\ \mathrm{res}(h_{10}) &= h_{10}, & \mathrm{res}(h_{11}\eta_2) &= 0,\\ \mathrm{res}(h_{11}) &= \omega^{p-1}h_{10}, & \mathrm{res}(h_{10}h_{11}\eta_2) &= 0. \end{aligned}$$

Proof This is a very similar computation to Theorem 3.6. The existence of ω in $\mathbb{F}_{p^2}$ is very easy: a is a $(p-1)^{\mathrm{st}}$ root of unity since $a \in \mathbb{F}_p$; hence, ω is a $(p+1)(p-1)^{\mathrm{st}}$ root of unity; hence, ω is fixed by the square of the Frobenius on the algebraic closure $\overline{\mathbb{F}}_p$. So $\omega \in \mathbb{F}_{p^2}$.

We use cocycle representatives (19)–(23) for the $\mathbb{F}_{p^2}$–algebra generators of

$$H_c^{*,*}(\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}); \mathbb{F}_{p^2}).$$

Theorem 2.9 tells us that, as a quotient of the Hopf algebra

$$\mathbb{F}_p[\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})]^* \cong \mathbb{F}_{p^2}[t_1, t_2, \dots]/(t_i^{p^2} - t_i),$$

the Hopf algebra $\mathbb{F}_{p^2}[\mathrm{strictAut}({}_\omega\mathbb{G}_1^{\hat{\mathbb{Z}}_p[\sqrt{ap}]})]^*$ is $\mathbb{F}_{p^2}[t_1, t_2, \dots]/(t_i^p - \omega^{p^i-1}t_i)$, ie

$$\mathbb{F}_{p^2}[t_1, t_2, \dots]/(t_i^p - \omega^{p-1}t_i \text{ for } i \text{ odd},\ t_i^p - t_i \text{ for } i \text{ even}).$$

The 1–cocycles t_1 and $t_2 - \frac{1}{2}\omega^{p-1}t_1^2$ in the cobar complex for

$$\mathbb{F}_{p^2}[\mathrm{strictAut}({}_\omega\mathbb{G}_1^{\hat{\mathbb{Z}}_p[\sqrt{ap}]})]^*$$

are again easily seen to be $\mathbb{F}_p$–linearly independent modulo coboundaries, so Theorem 3.3 again implies that the cohomology classes of t_1 and $t_2 - \frac{1}{2}\omega^{p-1}t_1^2$ are a minimal set of $\mathbb{F}_p$–algebra generators for $H_c^{*,*}(\mathrm{strictAut}({}_\omega\mathbb{G}_1^{\hat{\mathbb{Z}}_p[\sqrt{ap}]}); \mathbb{F}_{p^2})$.

Reducing cocycles (19)–(21) modulo $(t_1^p - \omega^{p-1}t_1, t_2^p - t_2)$ immediately yields the given formulas for $\mathrm{res}(h_{10})$, $\mathrm{res}(h_{11})$ and $\mathrm{res}(\zeta_2)$. For $\mathrm{res}(h_{10}\eta_2)$, we see that reducing (22) modulo $(t_1^p - \omega^{p-1}t_1, t_2^p - t_2)$ yields the 2–cocycle $\omega^{p-1}(t_1 \otimes t_1^2 + t_1^2 \otimes t_1)$, which is the coboundary of $\frac{1}{3}\omega^{p-1}t_1^3$, so $\mathrm{res}(h_{10}\eta_2) = 0$, and a similar computation yields $\mathrm{res}(h_{11}\eta_2) = 0$. □

4 Relations with local class field theory

The profinite group scheme $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\widehat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is isomorphic to the constant profinite group scheme on the group of units $\mathcal{O}_{D_{1/n,\mathbb{Q}_p}}^{\times}$ in the maximal order in the central division algebra over $\mathbb{Q}_p$ of Hasse invariant $1/n$; see Remark 4.2. This group of units $\mathcal{O}_{D_{1/n,\mathbb{Q}_p}}^{\times}$ plays an important role in attempts to generalize classical local class field theory to a "nonabelian local class field theory" capable of describing the representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of degree > 1 (the degree 1 representations are already described by classical local class field theory; see Theorem 4.1 for the reason why). The book [8] is a standard reference for the nonabelian generalizations, and there one can read about the successes that have been had in producing certain nonabelian generalizations of local class field theory which describe ℓ–adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $\ell \neq p$.

It is a much more difficult problem, however, to produce a nonabelian local class field theory which describes p–adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and much less is known about this case; this is one of the major goals of research in the theory of p–adic Galois representations. One issue that makes p–adic representations more difficult is the dramatic failure of semisimplicity for integral p–adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and related categories of representations: the group $\mathcal{O}_{D_{1/n,\mathbb{Q}_p}}^{\times}$ — whose ℓ–adic representations are closely related to ℓ–adic degree n representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $\ell \neq p$ by the local Langlands correspondences; see [8] — has a great deal of nonvanishing mod p cohomology, ie the cohomology of the height n Morava stabilizer group.

In this section, motivated by a remark of E Artin,[2] we compute the restriction map induced in cohomology by the inclusion of $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ as a subgroup of $\mathcal{O}_{D_{1/2,\mathbb{Q}_p}}^{\times}$ for all quadratic extensions $K/\mathbb{Q}_p$, with $p > 3$; this inclusion is constructed using local class field theory in Proposition 4.3.

Recall (from eg Serre's chapter on local class field theory in [2]) Artin reciprocity for p–adic number fields:

Theorem 4.1 *Let K be a p–adic number field with ring of integers A. Let π denote a uniformizer for A, let k denote the residue field of A, and let K^{ab}, K^{tr} and K^{nr}*

[2] "My own belief is that we know it already, though no one will believe me — that whatever can be said about non-Abelian class field theory follows from what we know now, since it depends on the behavior of the broad field over the intermediate fields — and there are sufficiently many Abelian cases," Artin, 1946; the quotation appears in [3], which refers the reader to page 312 of [6].

denote the compositum of all the finite abelian, tamely ramified abelian and unramified Galois extensions of K, respectively, in some fixed algebraic closure for K. Then there exist commutative diagrams of profinite groups with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & A_{\cong}^{\times} & \longrightarrow & K_{\theta}^{\times} & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow \\
1 & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{nr}}/K) & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & 1+\pi A_{\cong} & \longrightarrow & A_{\cong}^{\times} & \longrightarrow & k_{\cong}^{\times} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{tr}}) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}) & \longrightarrow & \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}}) & \longrightarrow & 1
\end{array}$$

where $1+\pi$ is the group (under multiplication) of 1–units in A, where θ is the limit, over all finite abelian extensions L/K, of the norm residue symbol maps $K^{\times}/N_{L/K}L^{\times} \xrightarrow{\cong} \mathrm{Gal}(L/K)$, and where η agrees with the embedding of $\mathbb{Z}$ into its profinite completion under the composite

$$\mathbb{Z} \to \mathrm{Gal}(K^{\mathrm{nr}}/K) \xrightarrow{\cong} \mathrm{Gal}(\bar{k}/k) \xrightarrow{\cong} \hat{\mathbb{Z}}.$$

Remark 4.2 By the computation of the Brauer group $\mathrm{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$ of a local field K (again, see Serre's chapter on local class field theory in [2]), every finite-rank central division algebra over K is classified, up to isomorphism, by some rational number $r/s \in \mathbb{Q}/\mathbb{Z}$, called the *Hasse invariant* of the division algebra; if r/s is a reduced fraction, then s^2 is the rank of the division algebra. It is well known (again, see Serre's chapter on local class field theory in [2], or [13]) that the endomorphism ring of an A–height n formal A–module over the algebraic closure of k is isomorphic to the maximal order (ie maximal compact subring) in the central division algebra with Hasse invariant $1/n$; and, furthermore, every degree n field extension of K embeds, by a ring homomorphism, into that division algebra. Here A is the ring of integers of K and k is the residue field of A.

Since the profinite group scheme $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})$ is already proconstant after base change to $\mathbb{F}_{p^n}$ (this is Theorem 6.2.3 of [19]), it is unnecessary to base change all the way to $\overline{\mathbb{F}}_p$; the above results from local class field theory provide an embedding of $A^{\times} \cong \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ into $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ for each field extension $K/\mathbb{Q}_p$ of degree n. Here A again denotes the ring of integers of K.

Proposition 4.3 *Let $K/\mathbb{Q}_p$ be a field extension of degree n. Let A denote the ring of integers of K, and let π denote a uniformizer for A and k the residue field of A. Let q be the cardinality of k, and let ω denote a $((q^{en}-1)/(q^n-1))^{th}$ root of π^e/p in $\mathbb{F}_{p^n}$. Let*

$$i_K\colon \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}) \hookrightarrow \mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$$

be the homomorphism of profinite groups defined as the composite of the norm residue symbol $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}) \xrightarrow{\cong} A^\times \cong \mathrm{Aut}({}_\omega\mathbb{G}_1^A)$ *with the natural embedding* $\mathrm{Aut}({}_\omega\mathbb{G}_1^A) \hookrightarrow \mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p})$ *of* $\mathrm{Aut}({}_\omega\mathbb{G}_1^A)$ *as the automorphisms of the underlying formal $\hat{\mathbb{Z}}_p$–module of* ${}_\omega\mathbb{G}_1^A$ (which is ${}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p}$, by Theorem 2.9) *which preserve the complex multiplication by A.*

Then the image of i_K is a closed *subgroup of* $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$.

Proof Let G_a denote the automorphism group of the underlying formal $\hat{\mathbb{Z}}_p$–module a–bud of

$${}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n},$$

so that $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is, as a profinite group, the limit of the sequence of finite groups $\cdots \to G_3 \to G_2 \to G_1$. Let H_a denote the subgroup of $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ consisting of those automorphisms whose underlying formal $\hat{\mathbb{Z}}_p$–module a–bud automorphism commutes with the complex multiplication by A, ie those whose underlying formal $\hat{\mathbb{Z}}_p$–module a–bud automorphism is an automorphism of the underlying formal A–module a–bud of ${}_\omega\mathbb{G}_1^A$. The index of H_a in $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is at most the cardinality of G_a, hence is finite. Now we use the theorem of Nikolov and Segal from [16]: every finite-index subgroup of a topologically finitely generated profinite group is an open subgroup. The group $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is topologically finitely generated since

- its pro-p–subgroup $\mathrm{strictAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is a p–adic analytic Lie group, hence topologically finitely generated (see [11], or Theorem 5.11 of [10] for an English-language summary of the relevant result), and
- $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is a split extension of the finite group $\mathbb{F}_{p^n}^\times$ by the topologically finitely generated group $\mathrm{strictAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p}; f \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$.

So H_a is an open subgroup of $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$. Every open subgroup of a profinite group is also closed; consequently, each H_a is a closed subgroup of

$\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$, and consequently so is the intersection $\bigcap_a H_a$. But $\bigcap_a H_a$ is the group of all formal power series which are automorphisms of ${}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}$ and whose polynomial truncations, of any length, commute with the complex multiplication by A. Consequently $\bigcap_a H_a = \mathrm{Aut}({}_\omega\mathbb{G}_1^A)$ is a closed subgroup of $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$. □

Definition 4.4 Let p be a prime number, n a positive integer and i an integer. The profinite group scheme $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is proconstant (see Remark 4.2); in this definition, and in the rest of the paper, we will write $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ to mean the profinite group given by evaluating that group scheme on $\mathbb{F}_{p^n}$ (ie $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ is the usual Morava stabilizer group, as in Chapter 6 of [19]).

We write $\mathbb{F}_{p^n}(i)$ for $\mathbb{F}_{p^n}$ with the action of $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ given by the i^{th} power of the cyclotomic character, ie a given element $z \in \mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ acts on $\mathbb{F}_{p^n}(i)$ by multiplication by $\overline{z}^i$, where $\overline{z}$ is the image of z under the reduction map $\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}) \to \mathbb{F}_{p^n}^\times$, ie $\overline{z}$ is the leading term of z as a power series in $\mathbb{F}_{p^n}[\![X]\!]$.

In particular, $\mathrm{strictAut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$ acts trivially on $\mathbb{F}_{p^n}(i)$ for all i,

$$\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$$

acts trivially on $\mathbb{F}_{p^n}(0)$ and $\mathbb{F}_{p^n}(i) = \mathbb{F}_{p^n}(i + p^n - 1)$.

Now we use the computations in Theorems 3.5, 3.6 and 3.7 to make the same cohomological computations, but with coefficients twisted by powers of the cyclotomic character, as in Definition 4.4. This cohomology with twisted coefficients is exactly what we need in order to compute the E_2–terms of the descent spectral sequences of Theorem 5.2.

Theorem 4.5 *Let $p > 3$. Then, for each of the three isomorphism classes of quadratic extensions $K/\mathbb{Q}_p$* (see Proposition 3.4), *the cohomology algebra*

$$H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_{p^2}(0))$$

is an exterior algebra on two generators in cohomological degree 1, and:

- *If $K/\mathbb{Q}_p$ is unramified, then $H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_{p^2}(i))$ is isomorphic as a bigraded $\mathbb{F}_{p^2}$–vector space to $H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_{p^2}(0))$ if i is divisible by $p^2 - 1$, and $H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_{p^2}(i)) \cong 0$ if i is not divisible by $p^2 - 1$.*

- *If $K/\mathbb{Q}_p$ is totally ramified, then $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(i))$ is isomorphic as a bigraded $\mathbb{F}_{p^2}$–vector space to $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(0))$ if i is divisible by $p-1$, and $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(i))\cong 0$ if i is not divisible by $p-1$.*

Meanwhile, $H^*_c(\mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2});\mathbb{F}_{p^2}(i))$ is isomorphic as a bigraded $\mathbb{F}_{p^2}$–vector space to $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(0))$ if i is divisible by p^2-1, and $H^*_c(\mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2});\mathbb{F}_{p^2}(i))\cong 0$ if i is not divisible by p^2-1.

Finally, the inclusion

$$i_K\colon \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\hookrightarrow \mathcal{O}^{\times}_{D_{1/2,\mathbb{Q}_p}}\cong \mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2}\otimes_{\mathbb{F}_p}\mathbb{F}_{p^2})$$

constructed in Proposition 4.3 induces the following restriction map in cohomology in the three cases:

$K/\mathbb{Q}_p$ unramified *The map*

$$\begin{aligned}\Lambda(\zeta_2)\otimes_{\mathbb{F}_{p^2}}&\mathbb{F}_{p^2}\{1,h_{10},h_{11},\eta_2h_{10},\eta_2h_{11},\eta_2h_{10}h_{11}\}\\ &\cong H^*_c\big(\mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2});\mathbb{F}_{p^2}(i(p^2-1))\big)\\ &\qquad\xrightarrow{\mathrm{res}} H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(i(p^2-1)))\cong\Lambda(h_{20},h_{21})\end{aligned}$$

sends ζ_2 to $h_{20}+h_{21}$ and is zero on all other generators.

$K=\mathbb{Q}_p(\sqrt{p})$ *The map*

$$\begin{aligned}\Lambda(\zeta_2)\otimes_{\mathbb{F}_{p^2}}&\mathbb{F}_{p^2}\{1,h_{10},h_{11},\eta_2h_{10},\eta_2h_{11},\eta_2h_{10}h_{11}\}\\ &\cong H^*_c\big(\mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2});\mathbb{F}_{p^2}(i(p^2-1))\big)\\ &\qquad\xrightarrow{\mathrm{res}} H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(i(p^2-1)))\cong\Lambda(h_{10},h_{20})\end{aligned}$$

sends ζ_2 to $2h_{20}$, sends h_{10} and h_{11} to h_{10}, and is zero on all other generators.

$K=\mathbb{Q}_p(\sqrt{ap})$ for a a nonsquare in $\widehat{\mathbb{Z}}_p^{\times}$ *The map*

$$\begin{aligned}\Lambda(\zeta_2)\otimes_{\mathbb{F}_{p^2}}&\mathbb{F}_{p^2}\{1,h_{10},h_{11},\eta_2h_{10},\eta_2h_{11},\eta_2h_{10}h_{11}\}\\ &\cong H^*_c\big(\mathrm{Aut}({}_1\mathbb{G}^{\widehat{\mathbb{Z}}_p}_{1/2});\mathbb{F}_{p^2}(i(p^2-1))\big)\\ &\qquad\xrightarrow{\mathrm{res}} H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}});\mathbb{F}_{p^2}(i(p^2-1)))\cong\Lambda(h_{10},h_{20})\end{aligned}$$

sends ζ_2 to $2h_{20}$, sends h_{10} to h_{10} and h_{11} to $\omega^{p-1}h_{10}$, and is zero on all other generators. (Here $\omega\in\mathbb{F}_{p^2}$ is a $(p+1)^{\mathrm{st}}$ root of a.)

(See Theorem 3.1 for the multiplicative structure of $H^*_c\big(\mathrm{strictAut}({}_1\mathbb{G}^{\hat{\mathbb{Z}}_p}_{1/2}; \mathbb{F}_{p^2}(0))\big)$ and the cohomological degrees of its generators.)

Proof If A has residue field k and $\mathbb{G}$ is a formal A–module over a field extension of k, then we have the short exact sequence of profinite groups

$$1 \to \mathrm{strictAut}(\mathbb{G}) \to \mathrm{Aut}(\mathbb{G}) \to k^\times \to 1, \tag{24}$$

since the strict automorphisms are simply the automorphisms whose leading coefficient (in $k^\times$) is 1. So the quotient map

$$\mathrm{Aut}({}_a\mathbb{G}^A_1)/\,\mathrm{strictAut}({}_a\mathbb{G}^A_1) \to \mathrm{Aut}({}_1\mathbb{G}^{\hat{\mathbb{Z}}_p}_{1/2})/\,\mathrm{strictAut}({}_1\mathbb{G}^{\hat{\mathbb{Z}}_p}_{1/2})$$

is simply the monomorphism $k^\times \hookrightarrow \mathbb{F}^\times_{p^2}$, where k is the residue field of k. Consequently:

- if $K/\mathbb{Q}_p$ is unramified, the action of $\mathrm{Aut}({}_a\mathbb{G}^A_1)/\,\mathrm{strictAut}({}_a\mathbb{G}^A_1)$ on $\mathbb{F}_{p^2}(i)$ is trivial if and only if i is divisible by p^2-1, and
- if $K/\mathbb{Q}_p$ is totally ramified, the action of $\mathrm{Aut}({}_a\mathbb{G}^A_1)/\,\mathrm{strictAut}({}_a\mathbb{G}^A_1)$ on $\mathbb{F}_{p^2}(i)$ is trivial if and only if i is divisible by $p-1$.

The rest follows easily from Theorems 3.5, 3.6 and 3.7, and the (immediately collapsing) Lyndon–Hochschild–Serre spectral sequence for the extension of profinite groups (24). □

Corollary 4.6 (how much of the cohomology of the height 2 Morava stabilizer group is visible in the cohomology of Galois groups?) *The product*

$$H^*_c(\mathrm{Aut}({}_1\mathbb{G}^{\hat{\mathbb{Z}}_p}_{1/2}); \mathbb{F}_{p^2}(0)) \to \prod_{[K:\mathbb{Q}_p]=2} H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_{p^2}(0)) \tag{25}$$

of the restriction maps from Theorem 4.5 is injective in cohomological degrees ≤ 1. *Furthermore, for each May/Chevalley–Eilenberg basis element*

$$x \in H^*_c(\mathrm{Aut}({}_1\mathbb{G}^{\hat{\mathbb{Z}}_p}_{1/2}); \mathbb{F}_{p^2}(0)),$$

either x or the Poincaré dual of x has nonzero image under the map (25).

5 Topological consequences

Remark 5.1 As a consequence of Proposition 4.3, if $K/\mathbb{Q}_p$ is a degree n extension, then $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ is a *closed* subgroup of the Morava stabilizer group

$\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\widehat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n})$. Consequently we can use the machinery of [5] to construct and compute the homotopy fixed-point spectrum $E_n^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})}$ using a homotopy fixed-point spectral sequence

$$H_c^*(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \pi_*(E_n)) \Rightarrow \pi_*(E_n^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})}),$$

ie

$$H_c^*(\mathrm{Aut}(\mathbb{G}); \pi_*(E_n)) \Rightarrow \pi_*(E_n^{h\,\mathrm{Aut}(\mathbb{G})}) \tag{26}$$

where $\mathbb{G}$ is a formal A–module, with A the ring of integers of K.

We carry this out in the quadratic case for $p > 3$, in Theorem 5.2, after smashing with $V(1)$.

It is worth explaining the relationship of spectral sequence (26) to the ideas suggested by Ravenel [18], who asked there whether there exist certain "algebraic extensions of the sphere spectrum" which would imply the existence of a spectral sequence whose input is the cohomology of the classifying Hopf algebroid of formal A–modules, ie the cohomology of the moduli stack of formal A–modules, and whose output would be the stable homotopy groups of a spectrum which resembles the sphere spectrum (indeed, it would be an "algebraic extension of the sphere spectrum"). The paper [25] shows that Ravenel's algebraic extensions of the sphere spectrum do not exist, except in trivial cases. However, the input for spectral sequence (26) is the cohomology of a *formal neighborhood of a point* in the moduli stack of formal A–modules, while the output is the stable homotopy groups of a spectrum which resembles the $K(n)$*–local* sphere spectrum. So we regard spectral sequence (26) as a kind of "local substitute" for the (nonexistent) spectral sequence Ravenel asked about in [18].

Theorem 5.2 *Let $p > 3$. For each of the three isomorphism classes of quadratic extensions of $\mathbb{Q}_p$* (see Proposition 3.4), *we compute the $V(1)$–homotopy groups of the homotopy fixed-point spectrum $E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\rtimes\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)}$:*

$K/\mathbb{Q}_p$ unramified *We have*

$$\pi_*(V(1) \wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\rtimes\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)}) \cong \Lambda(h_{20}, h_{21}) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2^{\pm 1}],$$

with homotopy degrees $|h_{20}| = |h_{21}| = -1$ and $|v_2| = 2(p^2-1)$. The natural map from the homotopy groups of the $K(2)$–local Smith–Toda $V(1)$ is the ring map

$$\begin{aligned}\pi_*(L_{K(2)}V(1)) &\cong \Lambda(\zeta_2) \otimes_{\mathbb{F}_p} \mathbb{F}_p\{1, h_{10}, h_{11}, \eta_2 h_{10}, \eta_2 h_{11}, \eta_2 h_{10} h_{11}\} \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2^{\pm 1}] \\ &\to \Lambda(h_{20}, h_{21}) \otimes_{\mathbb{F}_p} \mathbb{F}_p[v_2^{\pm 1}] \cong \pi_*(V(1) \wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\rtimes\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)})\end{aligned}$$

sending v_2 to v_2, sending ζ_2 to $h_{20}+h_{21}$ and sending h_{10}, h_{11}, $\eta_2 h_{10}$, $\eta_2 h_{11}$ and $\eta_2 h_{10}h_{11}$ to zero.

$K=\mathbb{Q}_p(\sqrt{p})$ $\pi_*(V(1)\wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\rtimes\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)})\cong\Lambda(h_{10},h_{20})\otimes_{\mathbb{F}_p}\mathbb{F}_p[b^{\pm1}]$, *with homotopy degrees $|h_{10}|=2p-3$ and $|h_{20}|=-1$ and $|b|=2(p-1)$. The natural map from the homotopy groups of the $K(2)$–local Smith–Toda $V(1)$ is the ring map*

$$\begin{aligned}\pi_*(L_{K(2)}V(1))&\cong\Lambda(\zeta_2)\otimes_{\mathbb{F}_p}\mathbb{F}_p\{1,h_{10},h_{11},\eta_2h_{10},\eta_2h_{11},\eta_2h_{10}h_{11}\}\otimes_{\mathbb{F}_p}\mathbb{F}_p[v_2^{\pm1}]\\&\to\Lambda(h_{10},h_{20})\otimes_{\mathbb{F}_p}\mathbb{F}_p[v_2^{\pm1}]\cong\pi_*(V(1)\wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})\rtimes\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)})\end{aligned}$$

sending v_2 to b^{p+1}, sending ζ_2 to $2h_{20}$, sending h_{10} to h_{10}, sending h_{11} to $h_{10}b^{p-1}$ and sending η_2h_{10}, η_2h_{11} and $\eta_2h_{10}h_{11}$ to zero.

$K=\mathbb{Q}_p(\sqrt{ap})$, with a a nonsquare $\pi_*(V(1)\wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})})$ *is isomorphic to $\Lambda(h_{10},h_{20})\otimes_{\mathbb{F}_{p^2}}\mathbb{F}_{p^2}[b^{\pm1}]$, with homotopy degrees $|h_{10}|=2p-3$ and $|h_{20}|=-1$ and $|b|=2(p-1)$. The natural map from the homotopy groups of the $K(2)$–local Smith–Toda $V(1)$ base-changed to $\mathbb{F}_{p^2}$ is the ring map*

$$\begin{aligned}&\pi_*(V(1)\wedge E_2^{h\,\mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p}\otimes_{\mathbb{F}_p}\mathbb{F}_{p^2})})\\&\quad\cong\Lambda(\zeta_2)\otimes_{\mathbb{F}_{p^2}}\mathbb{F}_{p^2}\{1,h_{10},h_{11},\eta_2h_{10},\eta_2h_{11},\eta_2h_{10}h_{11}\}\otimes_{\mathbb{F}_{p^2}}\mathbb{F}_{p^2}[v_2^{\pm1}]\\&\qquad\to\Lambda(h_{10},h_{20})\otimes_{\mathbb{F}_{p^2}}\mathbb{F}_{p^2}[v_2^{\pm1}]\cong\pi_*(V(1)\wedge E_2^{h\,\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})})\end{aligned}$$

sending v_2 to b^{p+1}, sending ζ_2 to $2h_{20}$, sending h_{10} to h_{10}, sending h_{11} to $\omega^{p-1}h_{10}b^{p-1}$ with ω a $(p-1)^{\mathrm{st}}$ root of a and sending η_2h_{10}, η_2h_{11} and $\eta_2h_{10}h_{11}$ to zero.

Proof See [4; 5] for the equivalence

$$L_{K(n)}S\simeq E_n^{h\,\mathrm{Aut}({}_1\mathbb{G}_{1/n}^{\hat{\mathbb{Z}}_p}\otimes_{\mathbb{F}_p}\mathbb{F}_{p^n})\rtimes\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)}.$$

Since $V(1)$ is $E(1)$–acyclic, $L_{K(2)}V(1)$ is weakly equivalent to $L_{E(2)}V(1)$, so we get weak equivalences $L_{K(2)}V(1)\simeq L_{E(2)}V(1)\simeq V(1)\wedge L_{E(2)}S$ since $E(2)$–localization is smashing; see [20] for the proof of Ravenel's smashing conjecture. Since $V(1)$ is finite, $(E_n\wedge V(1))^{hG}\simeq E_n^{hG}\wedge V(1)$, and now we use the $X=V(1)$ case of the conditionally convergent descent spectral sequence (see eg 4.6 of [1], or [5])

$$E_2^{s,t} \cong H_c^s(G;(E_n)_t(X)) \Rightarrow \pi_{t-s}((E_n \wedge X)^{hG}),$$
$$d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

In the case $n=2$ and $X=V(1)$, we have $(E_2)_* \cong W(\mathbb{F}_{p^2})[\![u_1]\!][u^{\pm 1}]$ with v_1 acting by $u_1 u^{1-p}$, and consequently $(E_2)_*(V(1)) \cong \mathbb{F}_{p^2}[u^{\pm 1}]$. One needs to know the action of $\mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})$ or $\mathrm{Aut}({}_\omega\mathbb{G}_1^A) \cong \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ on $\mathbb{F}_{p^2}[u^{\pm 1}]$ to compute the E_2–term of the spectral sequence; but $\mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})$ has the finite-index pro-p–subgroup $\mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})$, and similarly, $\mathrm{Aut}({}_\omega\mathbb{G}_1^A) \cong \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ has the finite-index pro-p–subgroup $\mathrm{strictAut}({}_\omega\mathbb{G}_1^A) \cong \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{tr}})$. As a pro-$p$–group admits no nontrivial continuous action on a one-dimensional vector space over a field of characteristic p, we only need to know the actions of

$$\mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) / \mathrm{strictAut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \cong \mathbb{F}_{p^2}^\times,$$
$$\mathrm{Aut}({}_\omega\mathbb{G}_1^A) / \mathrm{strictAut}({}_\omega\mathbb{G}_1^A) \cong \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{nr}}) \cong k^\times,$$

on $\mathbb{F}_{p^2}[u^{\pm 1}]$; ie for each j, the $\mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})$–module $\mathbb{F}_{p^2}\{u^j\}$ is $\mathbb{F}_{p^2}(i)$ for some i, as in Definition 4.4; specifically it is $\mathbb{F}_{p^2}(j)$ (see Section 1 of [4]).

Hence Theorem 4.5 provides the E_2–term of the descent spectral sequence for $n=2$ and $X=V(1)$ in each of the four cases

$$\begin{aligned}
G &= \mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}),\\
G &= \mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p),\\
G &= \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}),\\
G &= \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p),
\end{aligned}$$

and in each case, there is a horizontal vanishing line of finite height already at the E_2–page of the spectral sequence (this is computed in Theorem 4.5), hence the spectral sequence converges strongly.

The E_2–term of the descent spectral sequence, along with the map of E_2–terms induced by the inclusion of the closed subgroup

$$\mathrm{Gal}(K^{\mathrm{nr}}/K^{\mathrm{ab}}) \subseteq \mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}),$$

is computed in Theorem 4.5. In the case of $G = \mathrm{Aut}({}_1\mathbb{G}_{1/2}^{\hat{\mathbb{Z}}_p} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2})$ and $G = \mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ for $K = \mathbb{Q}_p(\zeta_{p^2-1})$ and $K = \mathbb{Q}_p(\sqrt{p})$, we computed the cohomology of a $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$–form of the Hopf algebra $\mathbb{F}_{p^2}[G]^*$ in Theorems 3.1, 3.5 and 3.6; since the nonabelian Galois cohomology group $H^1(\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p); GL_n(\mathbb{F}_{p^2}))$ classifying $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$–forms of n–dimensional $\mathbb{F}_{p^2}$–vector spaces vanishes (this is a well-known generalization of Hilbert's Theorem 90), the invariants of the $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$–action on $H^*_c(G)$ agree, up to isomorphism of graded $\mathbb{F}_p$–vector spaces, with the results of Theorems 3.1, 3.5 and 3.6 (this Galois descent argument was suggested to me by T Lawson). There is no room for differentials in the descent spectral sequences, so $E_2 \cong E_\infty$ in each spectral sequence. □

In Theorem 5.2, we have indexed $\pi_*(L_{K(2)}V(1))$ with the homotopy degrees:

homotopy class	degree	homotopy class	degree
1	0	$h_{10}\zeta_2$	$2p-4$
h_{10}	$2p-3$	$h_{11}\zeta_2$	$2p^2-2p-2$
h_{11}	$2p^2-2p-1$	$h_{10}h_{11}\eta_2$	-3
ζ_2	-1	$h_{10}\eta_2\zeta_2$	$2p-5$
$h_{10}\eta_2$	$2p-4$	$h_{11}\eta_2\zeta_2$	$2p^2-2p-3$
$h_{11}\eta_2$	$2p^2-2p-2$	$h_{10}h_{11}\eta_2\zeta_2$	-4

It is possible that classes with these names (eg ζ_2, from [14]) differ by some power of v_2 from the classes with these names used by others in the field; the necessary power of v_2 is easily found by comparing grading degrees.

As an amusing way to restate some of the conclusions of Theorem 5.2, we can write some Galois cohomology rings as natural quotients of the homotopy groups of the $K(2)$–local Smith–Toda complex $V(1)$:

Corollary 5.3 *Let $p > 3$ and let $K/\mathbb{Q}_p$ be a quadratic extension. Then the mod p continuous cohomology $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_p)$ of the Galois group $\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}})$ satisfies the isomorphisms of graded algebras*

- $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_p) \cong \pi_*(L_{K(2)}V(1))/(1-v_2, h_{10}, h_{11}) \otimes_{\mathbb{F}_p} \Lambda(h_2)$ *with h_2 in degree 1 if $K/\mathbb{Q}_p$ is unramified, and*
- $H^*_c(\mathrm{Gal}(K^{\mathrm{ab}}/K^{\mathrm{nr}}); \mathbb{F}_p) \cong \pi_*(L_{K(2)}V(1))/(1-v_2, h_{10}-h_{11}, \eta_2 h_{10})$ *if $K/\mathbb{Q}_p$ is totally ramified and π^2/p is a square modulo p.*

Throughout, π denotes a uniformizer for the ring of integers of K, and we have regraded $\pi_*(L_{K(2)}V(1))$ so that its grading is by $K(2)$–Adams filtration, not by homotopy degree.

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