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Γ–structures and symmetric spaces

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Γ–structures are weak forms of multiplications on closed oriented manifolds. As was shown by Hopf the rational cohomology algebras of manifolds admitting Γ–structures are free over odd-degree generators. We prove that this condition is also sufficient for the existence of Γ–structures on manifolds which are nilpotent in the sense of homotopy theory. This includes homogeneous spaces with connected isotropy groups.

Passing to a more geometric perspective we show that on compact oriented Riemannian symmetric spaces with connected isotropy groups and free rational cohomology algebras the canonical products given by geodesic symmetries define Γ–structures. This extends work of Albers, Frauenfelder and Solomon on Γ–structures on Lagrangian Grassmannians.

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Introduction

In his seminal papers [14; 15] on the (co)homological structure of Lie groups Hopf introduced the notion of Γ–*manifolds*. By definition these are closed connected oriented manifolds M together with continuous maps

$$\psi\colon M \times M \to M$$

such that the mapping degrees of the two restrictions $\psi_x = \psi(x, -)$ and $\psi^y = \psi(-, y)$ are nonzero for some (and hence for all) $x, y \in M$. In some sense such maps ψ, which we call Γ–*structures*, capture the simplest nontrivial homological information of Lie group multiplications. Hopf proved that the rational cohomology rings of Γ–manifolds are of a surprisingly restricted type: because they admit compatible comultiplications (and are hence *Hopf algebras* in modern terminology) they are free graded commutative $\mathbb{Q}$–algebras, whose generators must be in odd degrees as M is finite-dimensional; see Hopf [14; 15] and Dieudonné [8, Chapter VI, Section 2.A] for further information. As carried out by Borel in [4, Chapitre II, Section 7], further divisibility restrictions on the mapping degrees of ψ_x and ψ^y have similar implications for the cohomology rings over finite fields.

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A closely related and much better known structure is that of an *H-space*, with both restrictions ψ_x and ψ^y being homotopic to the identity. While every compact connected oriented manifold that is an H-space is obviously a Γ–manifold, the converse fails: all odd-dimensional spheres are Γ–manifolds (see [15]), but Adams' celebrated "Hopf invariant one theorem" in [1] says that only spheres of dimension 1, 3 or 7 admit H-space structures.

In the first part of our paper, Section 1, we shall make some general remarks on the existence of Γ–structures on manifolds satisfying the above cohomological condition. As the only nontrivial requirement for a Γ–structure $\psi\colon M \times M \to M$ is the nonvanishing of the mapping degrees of ψ_x and ψ^y, a construction of such structures by obstruction theory requires the separation of rational and torsion information in the homotopy type of M. This is the underlying idea of *rational homotopy theory*, which works best for spaces whose Postnikov decompositions consist of principal fibrations and can hence be described by accessible cohomological invariants. Examples are *simple* spaces, whose fundamental groups are abelian and act trivially on higher homotopy groups, and, more generally, *nilpotent* spaces, whose fundamental groups are nilpotent and act nilpotently on higher homotopy groups.

In Section 1 we will prove the following converse of Hopf's result. From now on the notion *free algebra* stands for *free graded commutative algebra over the rationals*.

Theorem 1 *Let M be a closed connected oriented manifold which is nilpotent as a topological space. If $H^*(M;\mathbb{Q})$ is a free algebra — necessarily over odd-degree generators — then M admits a Γ–structure.*

We derive this theorem from a more general, purely topological result, Proposition 9: Let X be a connected finite nilpotent CW-complex with free rational cohomology algebra. Then there is a continuous map $X \times X \to X$ whose restrictions to each factor induce isomorphisms in rational cohomology.

Theorem 1 is in sharp contrast to the existence of H-space structures, which is a much more restrictive property.

Corollary 2 *Let $M = \mathsf{G}/\mathsf{H}$ be a compact connected homogeneous space, where G is a Lie group and $\mathsf{H} < \mathsf{G}$ is a closed connected subgroup. If $H^*(M;\mathbb{Q})$ is a free algebra, then M admits a Γ–structure.*

Theorem 1 also implies (see Corollary 13) that nilpotent manifolds with free rational cohomology algebras have virtually abelian fundamental groups.

Our abstract existence result motivates the search for explicit geometric constructions of Γ–structures. Already Hopf [15] used geodesic symmetries on odd-dimensional spheres to write down Γ–structures. It is therefore natural to consider a Riemannian symmetric space P (which we always assume to be connected) endowed with its *canonical product* (see eg Loos' book [18, Chapter II, Section 1])

$$(*) \qquad \Theta\colon P \times P \to P, \quad (x, y) \mapsto s_x(y).$$

Here s_x denotes the *geodesic symmetry* of P at the point $x \in P$, that is, the involutive isometry of P that fixes x and that reverses the direction of all geodesics emanating from x. In Section 2 we prove:

Theorem 3 *Let P be a compact symmetric space with transvection group G; that is, G is the connected closed subgroup of the isometry group of P generated by products of two geodesic symmetries of P. Assume that the isotropy subgroup $\mathsf{H} < \mathsf{G}$ of a basepoint $p \in P$ is connected. Then the following assertions are equivalent:*

- *$H^*(P;\mathbb{Q})$ is a free algebra.*
- *The canonical product Θ of P is a Γ–structure.*

We use the assumption that H is connected in Lemma 17, but we do not know whether there is any example of an oriented compact symmetric space whose isotropy groups within its transvection group are not connected and whose rational cohomology is a free algebra. Notice however that Lemma 17 and therefore Theorem 3 still hold true, if one assumes that our compact symmetric space P can be written as a quotient of a connected Lie group by a closed connected subgroup, or, in fact, that it is just a nilpotent topological space.

Theorem 3 covers Lagrangian Grassmannians of odd rank, which amounts to the main result of Albers, Frauenfelder and Solomon in [2]; see Remark 20.

We observe that whenever the canonical product Θ defines a Γ–structure on P, then the mapping degrees of Θ_x and Θ^y are (up to sign) powers of 2. In view of Borel's work [4], this means that properties of the rational cohomology algebras imply additional restrictions on the cohomology algebras over finite fields of characteristic different from two. This generalizes the results [3, Corollaries 3.3 and 4.10] due to Araki.

Unfortunately Theorem 3 does not cover all the manifolds from Corollary 2. For example it remains an open problem to provide a geometric construction of Γ–structures on complex and quaternionic Stiefel manifolds, which are not symmetric spaces. The cohomology of these Stiefel manifolds can be found for instance in the book by Mimura and Toda [22, Theorem 3.10, page 119].

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1 Postnikov decompositions and Γ–structures

In this section we present a homotopy-theoretic construction of Γ–structures. Recall that the homotopy type of a path-connected CW-complex X can be analysed by means of its *Postnikov decomposition*; see, for example, [12, pages 410ff]: Choose a basepoint in X and, for $n \geq 0$, let X_n be obtained by killing all homotopy groups of X above degree n by attaching cells of dimension at least $n+2$. Up to homotopy equivalence we can assume that each inclusion $p_{n+1}\colon X_{n+1} \to X_n$ is a fibration. The long exact homotopy sequence shows that the fibre is an Eilenberg–Mac Lane space $K(\pi_{n+1}, n+1)$, where $\pi_{n+1} := \pi_{n+1}(X)$. In particular, X_1 is the classifying space $B\pi_1(X)$.

Recall that X is called *simple* if $\pi_1 = \pi_1(X)$ is abelian and acts trivially on higher homotopy groups. For simple X each fibration $p_{n+1}\colon X_{n+1} \to X_n$ is *principal* — see [12, Theorem 4.69] — that is, it is the pullback of the path-loop fibration

$$K(\pi_{n+1}, n+1) \to PK(\pi_{n+1}, n+2) \to K(\pi_{n+1}, n+2)$$

along a map $X_n \to K(\pi_{n+1}, n+2)$. By definition this map determines the n^{th} *k–invariant* $k_n \in H^{n+2}(X_n; \pi_{n+1})$. This class is equal to the image of the fundamental class in $H^{n+1}(K(\pi_{n+1}, n+1); \pi_{n+1})$ under the transgressive differential d_{n+2} in the Leray–Serre spectral sequence for the fibration p_{n+1}. Furthermore, the k–invariant k_n is equal to zero if and only if the fibration $p_{n+1}\colon X_{n+1} \to X_n$ is fibre homotopy equivalent to the trivial fibration. We denote by $(k_n)_{\mathbb{Q}} \in H^{n+2}(X_n; \pi_{n+1} \otimes \mathbb{Q})$ the image of k_n under the coefficient homomorphism $\pi_{n+1} \to \pi_{n+1} \otimes \mathbb{Q}$.

In the following we collect some well-known facts on the cohomology of Eilenberg–Mac Lane spaces.

Lemma 4 *Let C be a (finite or infinite) cyclic group, and let $n > 0$ be a positive integer. Then:*

- $H^*(K(C,n);\mathbb{Z})$ *is a finitely generated group in each degree.*
- *For $C = \mathbb{Z}$ the cohomology algebra $H^*(K(\mathbb{Z},n);\mathbb{Q})$ is free with one generator in degree n.*
- *For $|C| < \infty$ the reduced cohomology $\widetilde{H}^*(K(C,n);\mathbb{Q})$ is equal to 0.*

Let $m > 0$ be a positive integer and let $\mu_m\colon C \to C$ be multiplication by m. Then:

- *For $C = \mathbb{Z}$ the induced map*
$$\mu_m^*\colon H^*(K(\mathbb{Z},n);\mathbb{Q}) \to H^*(K(\mathbb{Z},n);\mathbb{Q})$$
is given by multiplication with m^k on $H^{kn}(K(\mathbb{Z},n);\mathbb{Q})$ for $k \geq 0$.
- *For all C the map*
$$\mu_m^*\colon \widetilde{H}^*(K(C,n);\mathbb{Z}/m) \to \widetilde{H}^*(K(C,n);\mathbb{Z}/m)$$
is equal to 0.

Proof We first prove all but the last statement by induction on n. For $n = 1$ we have $K(C,1) = BC$, the classifying space of C, and thus the assertions are clear for $C = \mathbb{Z}$ (recall $B\mathbb{Z} = S^1$). For $|C| < \infty$ the classifying space BC is an infinite-dimensional lens space, the cohomology $\widetilde{H}^*(BC;\mathbb{Z})$ is equal to $\mathbb{Z}/|C|$ in even degrees and 0 in odd degrees, and $\mu_m^*\colon H^*(BC;\mathbb{Z}) \to H^*(BC;\mathbb{Z})$ is given by multiplication with m^k on $H^{2k}(BC;\mathbb{Z})$. Together with the universal coefficient theorem this completes the case $n = 1$.

For the inductive step we recall that the cohomology (with coefficients in some commutative ring R) of the base and the fibre of the path-loop fibration
$$K(C,n) \to PK(C,n+1) \to K(C,n+1)$$
each appear on one of the two coordinate axes of the E_2–term of the Leray–Serre spectral sequence and that this spectral sequence is natural with respect to homomorphisms $C \to C$. The spectral sequence converges to $H^*(PK(C,n+1);R)$, which vanishes in positive degrees, because $PK(C,n+1)$ is contractible.

By induction this shows that $H^*(K(C,n+1);\mathbb{Z})$ is finitely generated in each degree, that $\widetilde{H}^*(K(C,n+1);\mathbb{Q}) = 0$ for $|C| < \infty$, and that $H^*(K(\mathbb{Z},n+1);\mathbb{Q})$ is a free algebra in one generator of degree $n+1$. The last implication is based on the multiplicative structure of the spectral sequence.

Now let $m > 0$ and $\mu_m\colon \mathbb{Z} \to \mathbb{Z}$ be multiplication by m. The naturality of the spectral sequence shows inductively that $\mu_m^*\colon H^*(K(\mathbb{Z}, n+1); \mathbb{Q}) \to H^*(K(\mathbb{Z}, n+1); \mathbb{Q})$ is multiplication by m^k in degree $k(n+1)$. This finishes the inductive step.

It remains to show the last statement of Lemma 4: for all cyclic groups C and all $m, n > 0$ the map

$$\mu_m^*\colon \widetilde{H}^*(K(C,n); \mathbb{Z}/m) \to \widetilde{H}^*(K(C,n); \mathbb{Z}/m)$$

is equal to 0.

First, let us assume that $m = p^r$, where p is a prime number and $r > 0$. If C is finite, then the inclusion of the (unique) Sylow p–subgroup $P \to C$ induces an isomorphism

$$\widetilde{H}^*(K(C,n); \mathbb{Z}/p^r) \cong \widetilde{H}^*(K(P,n); \mathbb{Z}/p^r)$$

by a spectral sequence argument. It is therefore enough to concentrate on the case $C = \mathbb{Z}/p^{\ell}$, where $\ell > 0$, if C is finite and the case $C = \mathbb{Z}$ if C is infinite.

We work by induction on r. Let $r = 1$; hence $m = p$. It is well known — see [6; 7; 25] or [21, Theorem 6.19] — that $H^*(K(C,n); \mathbb{Z}/p)$ is a polynomial algebra with free generators of the form $\mathcal{P}(\iota_n)$, where $\iota_n \in H^n(K(C,n); \mathbb{Z}/p)$ is the fundamental class and $\mathcal{P}$ is some mod p cohomology operation. Because the map μ_p^* is multiplication by p and hence zero on $H^n(K(C,n); \mathbb{Z}/p)$ the assertion for $r = 1$ is implied by the naturality of the operations $\mathcal{P}$.

Next, assuming the assertion for $r-1$, the assertion for r follows by use of the exact Bockstein sequence

$$\cdots \to \widetilde{H}^*(K(C,n); \mathbb{Z}/p^{r-1}) \to \widetilde{H}^*(K(C,n); \mathbb{Z}/p^r) \to \widetilde{H}^*(K(C,n); \mathbb{Z}/p) \to \cdots,$$

which is associated to the short exact coefficient sequence

$$0 \to \mathbb{Z}/p^{r-1} \to \mathbb{Z}/p^r \to \mathbb{Z}/p \to 0$$

and which is natural with respect to homomorphisms $C \to C$. A simple diagram chase together with the equality $\mu_{p^r} = \mu_{p^{r-1}} \circ \mu_p$ then shows the assertion for r.

After having finished the proof for $m = p^r$ we will now deal with the general case $m = p_1^{r_1} \cdots p_k^{r_k}$ with pairwise different primes p_i and $r_i > 0$. According to the decomposition

$$\mathbb{Z}/m \cong \mathbb{Z}/p_1^{r_1} \times \cdots \times \mathbb{Z}/p_k^{r_k}$$

we obtain a splitting

$$\widetilde{H}^*(K(C,n);\mathbb{Z}/m) \cong \widetilde{H}^*(K(C,n);\mathbb{Z}/p_1^{r_1}) \oplus \cdots \oplus \widetilde{H}^*(K(C,n);\mathbb{Z}/p_k^{r_k})$$

which is natural with respect to self-maps of $K(C,n)$. On $\widetilde{H}^*(K(C,n);\mathbb{Z}/p_i^{r_i})$ the multiplication $\mu_m\colon C \to C$ induces the zero map, because μ_m factors through $\mu_{p_i^{r_i}}$. This implies the last assertion of Lemma 4 for general $m > 0$. □

Let X be a connected finite simple CW-complex whose rational cohomology algebra is free. It must be finitely generated with all generators in odd degrees, because X is assumed to be finite. Serre's finiteness theorem (or a direct inspection of the Postnikov decomposition in connection with Lemma 4) implies that the homotopy groups $\pi_*(X)$ are finitely generated in each degree. Hence they are finite products of cyclic groups.

Lemma 5 *For each $n \geq 0$ the following hold:*

- *$H^*(X_n;\mathbb{Q})$ is a free algebra with generators in degrees $\leq n$ corresponding to the duals of the generators of $\pi_i(X) \otimes \mathbb{Q}$, where $i \leq n$.*
- *The canonical map $X \to X_n$ induces an injective map in rational cohomology.*
- *The rationalized k–invariant $(k_n)_{\mathbb{Q}} \in H^{n+2}(X_n;\pi_{n+1} \otimes \mathbb{Q})$ vanishes.*

Proof The first assertion implies the second one, because the canonical map $X \to X_n$ induces an isomorphism in rational cohomology up to degree n and the cohomology algebra $H^*(X;\mathbb{Q})$ is free. The second assertion implies the third one by the following argument. The rationalized k–invariant $(k_n)_{\mathbb{Q}}$ is the image of the fundamental class in $H^{n+1}(K(\pi_{n+1},n+1);\mathbb{Q})$ under the differential d_{n+2} in the spectral sequence for the fibration $X_{n+1} \to X_n$. If this differential were nonzero, then the induced map $H^*(X_n;\mathbb{Q}) \to H^*(X_{n+1};\mathbb{Q})$ would not be injective. However then the map $H^*(X_n;\mathbb{Q}) \to H^*(X;\mathbb{Z})$ would not be injective, either, by use of the factorization $X \to X_{n+1} \to X_n$. This contradicts the second assertion.

It is hence enough to prove the first assertion by induction on n. This assertion is clear for $n = 0$, because X_0 is homotopy equivalent to a point. In the inductive step the assumption $(k_n)_{\mathbb{Q}} = 0$ implies

$$H^*(X_{n+1};\mathbb{Q}) \cong H^*(X_n;\mathbb{Q}) \otimes H^*(K(\pi_{n+1},n+1);\mathbb{Q})$$

by Lemma 4, the Künneth theorem applied to the splitting of π_{n+1} into cyclic groups, and the multiplicative properties of the Leray–Serre spectral sequence. From this the first assertion follows for $n+1$. □

Proposition 6 *For all $n \geq 0$ and $m > 0$ there is a self-map $f_{m,n}\colon X_n \to X_n$ with the following properties:*

- *The induced map $f_{m,n}^*\colon H^*(X_n;\mathbb{Q}) \to H^*(X_n;\mathbb{Q})$ is an isomorphism.*
- *The induced map $f_{m,n}^*\colon \widetilde{H}^*(X_n;\mathbb{Z}/m) \to \widetilde{H}^*(X_n;\mathbb{Z}/m)$ is equal to 0.*

Proof We apply induction on n. Again the case $n = 0$ is clear.

For the inductive step we assume the assertion holds for a fixed n and all $m > 0$. By Lemma 5 the k–invariant $k_n \in H^{n+2}(X_n;\pi_{n+1})$ is a torsion class. Let τ denote the order of the torsion subgroup of $H^{n+2}(X_n;\mathbb{Z})$. Then the restriction of the canonical map

$$H^{n+2}(X_n;\mathbb{Z}) \to H^{n+2}(X_n;\mathbb{Z}) \otimes \mathbb{Z}/\tau \to H^{n+2}(X_n;\mathbb{Z}/\tau)$$

to the torsion subgroup of $H^{n+2}(X_n;\mathbb{Z})$ is injective. This is clear for the first map, and for the second map it follows from the universal coefficient theorem. In particular, the self-map $f_{\tau,n}\colon X_n \to X_n$ provided by the induction hypothesis induces the zero map on the torsion subgroup of $H^{n+2}(X_n;\mathbb{Z})$. We now consider a splitting

$$\pi_{n+1} \cong C_1 \times \cdots \times C_k$$

into cyclic groups and obtain a corresponding splitting

$$H^{n+2}(X_n;\pi_{n+1}) \cong H^{n+2}(X_n;C_1) \oplus \cdots \oplus H^{n+2}(X_n;C_k),$$

which is natural in X_n. Now, for each $1 \leq i \leq k$ there is a self-map $f_i\colon X_n \to X_n$ inducing an isomorphism in rational cohomology and the zero map on the torsion subgroup of $H^{n+2}(X_n;C_i)$: if $C_i = \mathbb{Z}$ we take $f_{\tau,n}\colon X_n \to X_n$ as explained before, and if $C_i = \mathbb{Z}/m$ we take the self-map $f_{m,n}\colon X_n \to X_n$ provided by the induction hypothesis. Let $f := f_k \circ \cdots \circ f_1\colon X_n \to X_n$. By construction we have:

- The map f induces an isomorphism $H^*(X_n;\mathbb{Q}) \to H^*(X_n;\mathbb{Q})$.
- $f^*(k_n) = 0$.

In the pullback square of fibrations

$$\begin{CD}
K(\pi_{n+1}, n+1) @>{=}>> K(\pi_{n+1}, n+1) \\
@V{\text{incl.}}VV @VVV \\
f^*(X_{n+1}) @>{F}>> X_{n+1} \\
@VVV @VV{p_{n+1}}V \\
X_n @>{f}>> X_n
\end{CD}$$

the map F induces an isomorphism in rational cohomology, by a spectral sequence argument, and because f induces an isomorphism in rational cohomology. Because $f^*(k_n) = 0$ the induced fibration $f^*(X_{n+1}) \to X_n$ is fibre homotopy trivial and hence we get a homotopy equivalence $X_n \times K(\pi_{n+1}, n+1) \simeq f^*(X_{n+1})$. Let

$$\alpha_n\colon X_n \times K(\pi_{n+1}, n+1) \simeq f^*(X_{n+1}) \xrightarrow{F} X_{n+1}$$

be the composition of the resulting maps. The map α_n induces an isomorphism in rational cohomology by construction.

Next, let $\mu := \mu_{|k_n|}\colon \pi_{n+1} \to \pi_{n+1}$ be the multiplication by the order of the torsion class $k_n \in H^{n+2}(X_n; \pi_{n+1})$. We wish to define a map β_n fitting into a commutative diagram:

$$\begin{array}{ccc}
K(\pi_{n+1}, n+1) & \xrightarrow{K(\mu, n+1)} & K(\pi_{n+1}, n+1) \\
\downarrow & & \downarrow{\scriptstyle \text{incl.}} \\
X_{n+1} & \xrightarrow{\beta_n} & X_n \times K(\pi_{n+1}, n+1) \\
\downarrow{\scriptstyle p_{n+1}} & & \downarrow{\scriptstyle \text{proj.}} \\
X_n & \xrightarrow{=} & X_n
\end{array}$$

The only nontrivial task is the construction of the map $X_{n+1} \to K(\pi_{n+1}, n+1)$ appearing in the middle horizontal line. For this we consider the diagram:

$$\begin{array}{ccccc}
K(\pi_{n+1}, n+1) & \xrightarrow{=} & K(\pi_{n+1}, n+1) & \xrightarrow{K(\mu, n+1)} & K(\pi_{n+1}, n+1) \\
\downarrow & & \downarrow & & \downarrow \\
X_{n+1} & \longrightarrow & PK(\pi_{n+1}, n+2) & \xrightarrow{PK(\mu, n+1)} & PK(\pi_{n+1}, n+2) \\
\downarrow & & \downarrow & & \downarrow \\
X_n & \xrightarrow{k_n} & K(\pi_{n+1}, n+2) & \xrightarrow{K(\mu, n+2)} & K(\pi_{n+1}, n+2)
\end{array}$$

By assumption the composition in the lower row is homotopic to a constant map. We use the homotopy lifting property to homotope the composition $X_{n+1} \to PK(\pi_{n+1}, n+2)$ in the second row to a map $X_{n+1} \to PK(\pi_{n+1}, n+2)$ which factors through the fibre inclusion in the right-hand column.

The map β_n induces an isomorphism in rational cohomology, again by a spectral sequence argument combined with Lemma 4.

With the maps α_n and β_n in hand we are ready to conclude the inductive step. Let $m > 0$ be arbitrary. Using the induction hypothesis, Lemma 4 and the Künneth formula it is easy to construct a self-map

$$f'_{m,n}\colon X_n \times K(\pi_{n+1}, n+1) \to X_n \times K(\pi_{n+1}, n+1)$$

which induces an isomorphism in rational cohomology and the zero map in reduced $\mathbb{Z}/m$–cohomology. But then the composition

$$f_{m,n+1}\colon X_{n+1} \xrightarrow{\beta_n} X_n \times K(\pi_{n+1}, n+1) \xrightarrow{f'_{m,n}} X_n \times K(\pi_{n+1}, n+1) \xrightarrow{\alpha_n} X_{n+1}$$

is as required. □

For further use we isolate the following information from the proof of Proposition 6.

Corollary 7 *For each n there are maps $\alpha_n\colon X_n \times K(\pi_{n+1}, n+1) \to X_{n+1}$ and $\beta_n\colon X_{n+1} \to X_n \times K(\pi_{n+1}, n+1)$ inducing isomorphisms in rational cohomology.*

Theorem 1 for simple spaces now follows from the next proposition.

Proposition 8 *Let X be a connected finite CW-complex which is a simple topological space and whose rational cohomology algebra is free. Then there is a continuous map*

$$\psi\colon X \times X \to X$$

such that, for all $x, y \in X$, the maps $\psi(x, -)\colon X \to X$ and $\psi(-, y)\colon X \to X$ induce isomorphisms in rational cohomology.

Proof It is clear that such a map exists on X_0. Assume that we have already constructed a map $\psi_n\colon X_n \times X_n \to X_n$ with the required properties.

Together with the product on $K(\pi_{n+1}, n+1)$ induced by the addition map on π_{n+1} we obtain a multiplication ψ'_{n+1} on $X'_{n+1} := X_n \times K(\pi_{n+1}, n+1)$ with the required properties. Now we define ψ_{n+1} as the composition

$$X_{n+1} \times X_{n+1} \xrightarrow{\beta_n \times \beta_n} X'_{n+1} \times X'_{n+1} \xrightarrow{\psi'_{n+1}} X'_{n+1} \xrightarrow{\alpha_n} X_{n+1},$$

where α_n and β_n are taken from Corollary 7.

Once we have constructed ψ_n with $n > 2 \dim X$, the construction of ψ is complete by the cellular approximation theorem. □

For the next notions and results compare [20, Sections 3.1. and 3.2]. A path-connected CW-complex X is called *nilpotent* if $\pi_1(M)$ is a nilpotent group and acts nilpotently on the higher homotopy groups. Note that simple complexes are automatically nilpotent. For a nilpotent complex the fibrations $p_{n+1}\colon X_{n+1} \to X_n$ in the Postnikov decomposition are in general not principal, but they admit *finite principal refinements*

$$X_{n+1} =: Y_{r_n} \xrightarrow{q_{r_n}} Y_{r_n-1} \xrightarrow{q_{r_n-1}} \cdots \xrightarrow{q_2} Y_1 \xrightarrow{q_1} Y_0 := X_n,$$

where each $q_i\colon Y_i \to Y_{i-1}$ is a principal fibration with fibre $K(G_i, n+1)$ for some abelian group G_i, which is classified by a cohomology class in $H^{n+2}(Y_{i-1}; G_i)$. The same argument as in the proof of Proposition 8 then shows:

Proposition 9 *Let X be a connected finite CW-complex which is a nilpotent topological space. Assume that the rational cohomology algebra of X is free. Then there is a continuous map $\psi\colon X \times X \to X$ such that, for all $x, y \in X$, the maps $\psi(x,-)\colon X \to X$ and $\psi(-,y)\colon X \to X$ induce isomorphisms in rational cohomology.*

This proposition immediately implies Theorem 1.

Lemma 10 *Let $M = \mathsf{G}/\mathsf{H}$ be a homogeneous space, where G is a connected Lie group and $\mathsf{H} < \mathsf{G}$ is a closed connected subgroup. Then M is a simple topological space.*

Proof This fact is well known and we include a proof for the readers' convenience. Because H is connected we get an exact sequence

$$\cdots \to \pi_2(\mathsf{G}/\mathsf{H}) \to \pi_1(\mathsf{H}) \to \pi_1(\mathsf{G}) \to \pi_1(\mathsf{G}/\mathsf{H}) \to 1.$$

The topological group G has abelian fundamental group, and hence the same holds for G/H.

Next, let $f\colon S^n \to \mathsf{G}/\mathsf{H}$ and $\mu\colon S^1 \to \mathsf{G}/\mathsf{H}$ be based maps, where we take the south pole of any sphere as basepoint. Recall that $[\mu]_*([f]) \in \pi_n(\mathsf{G}/\mathsf{H})$, the result of the action of $[\mu] \in \pi_1(\mathsf{G}/\mathsf{H})$ on $[f] \in \pi_n(\mathsf{G}/\mathsf{H})$, is represented by the map $\mu * f\colon S^n \to \mathsf{G}/\mathsf{H}$, defined as follows. Consider the one-point union $S^n \vee S^n$, where the second sphere is piled above the first one, identifying the north pole of the first with the south pole of the second. We take the south pole of the first sphere as basepoint of $S^n \vee S^n$. Now consider the composition $S^n \xrightarrow{\text{height}} [0,1] \xrightarrow{\mu} \mathsf{G}/\mathsf{H}$ on the lower sphere, and the map $f\colon S^n \to \mathsf{G}/\mathsf{H}$ on the upper sphere, and compose the resulting map $S^n \vee S^n \to \mathsf{G}/\mathsf{H}$ with the basepoint-preserving coproduct $S^n \to S^n \vee S^n$. This defines $\mu * f$. Note that for $n = 1$ this results in the usual conjugation action of $\pi_1(\mathsf{G}/\mathsf{H})$ on itself.

By the above exact sequence the map μ lifts to a based map $\overline{\mu}\colon S^1 \to \mathsf{G}$. Using the left multiplication of G on G/H and the above explicit description of $\mu * f$ it is easy to show that $\mu * f$ is based homotopic to f. □

Together with our previous results this implies Corollary 2.

Remark 11 Of course the above argument is modelled along the lines of rational homotopy theory. In particular our Proposition 6 is implied by [27, Theorem (12.2)] (note that by Lemma 5 our X is a formal space in the sense of rational homotopy theory). However we found it somewhat difficult to trace a complete proof of this theorem in the literature. We feel that this fact and the special focus of our paper justifies the ad hoc, but self-contained discussion above instead of an in-depth exploration of rational homotopy theory.

We are grateful to Dieter Kotschick for pointing out the following lemma and corollary.

Lemma 12 *Each Γ–manifold has virtually abelian fundamental group.*

Proof Let $\psi\colon M \times M \to M$ be a Γ–structure and $x, y \in M$. Without loss of generality we can assume that $\psi_x, \psi^y\colon M \to M$ preserve a basepoint in M. Let $H_1, H_2 < \pi_1(M)$ be defined as the images of

$$(\psi_x)_*\colon \pi_1(M) \to \pi_1(M) \quad \text{and} \quad (\psi^y)_*\colon \pi_1(M) \to \pi_1(M).$$

The maps $\psi_x, \psi^y\colon M \to M$ factor through the connected coverings of M defined by H_1 and H_2, respectively. Because ψ_x and ψ^y have nonzero mapping degrees, these coverings are finite and hence H_1 and H_2 are of finite index in $\pi_1(M)$. This implies that $H_1 \cap H_2 < \pi_1(M)$ is also of finite index.

Since the map $\psi_*\colon \pi_1(M) \times \pi_1(M) \to \pi_1(M)$ is a group homomorphism, elements in H_1 commute with elements in H_2. This implies that the finite-index subgroup $H_1 \cap H_2 < \pi_1(M)$ is abelian. □

Together with Theorem 1 this implies:

Corollary 13 *Let M be a closed connected oriented manifold which is nilpotent as a topological space. If $H^*(M;\mathbb{Q})$ is a free algebra — necessarily over odd-degree generators — then $\pi_1(M)$ is virtually abelian.*

It remains an interesting open problem whether this conclusion can be drawn without the use of Theorem 1.

2 Canonical products on symmetric spaces

In this section we prove Theorem 3. One implication is a special case of Hopf's theorem [14; 15]. Let P be a compact symmetric space whose isotropy groups within its transvection group are connected. This assumption implies that P is orientable. We are left to show that if the rational cohomology $H^*(P;\mathbb{Q})$ of P is a free algebra, then the canonical product Θ on P defined in Equation $(*)$ on page 879 is a Γ–structure.

We first observe that the degree of the map $\Theta_x\colon P \to P$ given by $y \mapsto \Theta(x,y) = s_x(y)$ with $x \in P$ fixed is

$$\deg(\Theta_x) = (-1)^{\dim(P)}.$$

For fixed $y \in P$ we will examine the map

$$\theta := \Theta^y\colon P \to P, \quad x \mapsto \Theta(x,y) = s_x(y).$$

If the degree of θ does not vanish, then Θ is a Γ–structure. We will reduce our considerations to irreducible simply connected symmetric spaces of compact type using well-known features of compact symmetric spaces, which can be found in the classical literature such as [13; 18; 19] or [29, Chapter 8].

We start with three preliminary lemmata. Just like compact Lie groups (see eg [16, Theorem 4.29, page 198]) compact symmetric spaces admit finite covers that split off flat factors:

Lemma 14 *Every compact symmetric space P is finitely covered by a product $T \times \widetilde{Q}$ of a flat torus T and a simply connected compact symmetric space $\widetilde{Q}$.*

Proof The deck transformation group Δ of the universal cover $\widetilde{P}$ of P is a finitely generated discrete subgroup of the abelian centralizer $C_{\mathsf{Iso}(\widetilde{P})}(\mathsf{Trans}(\widetilde{P}))$ of the transvection group $\mathsf{Trans}(\widetilde{P})$ of $\widetilde{P}$ in its isometry group (see [24, Lemma 1.2, page 194] and [29, Theorem 8.3.11, page 244]). Since $\widetilde{P} \cong \mathbb{R}^k \times \widetilde{Q}$, where $\widetilde{Q}$ is a simply connected compact symmetric space, the isometry group of $\widetilde{P}$ splits as $\mathsf{Iso}(\widetilde{P}) = \mathsf{Iso}(\mathbb{R}^k) \times \mathsf{Iso}(\widetilde{Q})$. Thus any element of Δ has the form $f \times g$ for some $f \in C_{\mathsf{Iso}(\mathbb{R}^k)}(\mathsf{Trans}(\mathbb{R}^k)) \cong \mathbb{R}^k$ and some $g \in C_{\mathsf{Iso}(\widetilde{Q})}(\mathsf{Trans}(\widetilde{Q}))$. Since $\widetilde{Q}$ is a symmetric space of compact type $C_{\mathsf{Iso}(\widetilde{Q})}(\mathsf{Trans}(\widetilde{Q}))$ is finite. Let N be a common multiple of the orders of elements of $C_{\mathsf{Iso}(\widetilde{Q})}(\mathsf{Trans}(\widetilde{Q}))$; then $g^N = e$ for all $g \in C_{\mathsf{Iso}(\widetilde{Q})}(\mathsf{Trans}(\widetilde{Q}))$. Since Δ is abelian, the N^{th} power is an endomorphism of Δ whose image Δ^N acts trivially on the $\widetilde{Q}$ factor of $\widetilde{P}$. As Δ^N has finite index in Δ, the space $\widetilde{P}/\Delta^N \cong T \times \widetilde{Q}$, which is the desired cover of P, is compact. □

Lemma 15 *Let P_1 and P_2 be two compact oriented symmetric spaces. Then the canonical product on $P_1 \times P_2$ is a Γ–structure if and only if the canonical products on P_1 and on P_2 are both Γ–structures.*

Proof The Riemannian product $P_1 \times P_2$ is again a compact oriented symmetric space and the geodesic symmetries of $P_1 \times P_2$ are products of geodesic symmetries of P_1 and of P_2. The claim now follows easily by the multiplicativity of mapping degrees. □

Lemma 16 *Let $p\colon \widehat{P} \to P$ be an orientation-preserving Riemannian covering between two compact oriented symmetric spaces. Then the canonical product on $\widehat{P}$ is a Γ–structure if and only if the canonical product on P is a Γ–structure.*

Proof The canonical products $\widehat{\Theta}$ on $\widehat{P}$ and Θ on P are related by

$$\Theta \circ (p \times p) = p \circ \widehat{\Theta}.$$

Let $\hat{y} \in \widehat{P}$ be a chosen origin and $y := p(\hat{y})$. Then the maps $\hat{\theta} = \widehat{\Theta}^{\hat{y}}\colon \hat{x} \mapsto \hat{s}_{\hat{x}}(\hat{y})$ and $\theta = \Theta^{y}\colon x \mapsto s_x(y)$ satisfy $p \circ \hat{\theta} = \theta \circ p$. Since p is a covering, we get

$$\deg(p)\deg(\hat{\theta}) = \deg(\theta)\deg(p),$$

where $\deg(p)$ coincides with the number of sheets of p. Division by $\deg(p)$ yields $\deg(\hat{\theta}) = \deg(\theta)$. □

Using these lemmata, we can proceed with our proof of Theorem 3. Let P be a symmetric space as in Theorem 3 such that $H^*(P;\mathbb{Q})$ is an exterior algebra generated by homogeneous elements in odd degrees. By Lemma 14 and the de Rham decomposition for symmetric spaces there is a Riemannian product

$$\widehat{P} := T \times \widetilde{P}_1 \times \cdots \times \widetilde{P}_m$$

of a flat torus T (a single point and a circle are considered zero- and one-dimensional tori respectively) and irreducible simply connected compact symmetric spaces $\widetilde{P}_1, \dots, \widetilde{P}_m$, where $m \geq 0$, together with a finite Riemannian covering map $p\colon \widehat{P} \to P$.

Lemma 17 *The covering $p\colon \widehat{P} \to P$ induces an isomorphism between the graded $\mathbb{Q}$–algebras $H^*(\widehat{P};\mathbb{Q})$ and $H^*(P;\mathbb{Q})$.*

Proof We prove this claim in two different ways. Firstly, let $\hat{y} \in \widehat{P}$ and $y := p(\hat{y})$. Let $\widehat{\mathsf{G}}$ and G denote the transvection groups of $\widehat{P}$ and P. The isotropy group $\widehat{\mathsf{H}} \subset \widehat{\mathsf{G}}$ of $\hat{y}$ is connected, because $\widehat{P}$ is a product of a torus and simply connected compact symmetric spaces, and the isotropy group $\mathsf{H} \subset \mathsf{G}$ of y is connected by assumption. Their linear isotropy actions are identified by p. By [29, Theorem 8.5.8] the real cohomology algebra of any compact symmetric space S is isomorphic to the algebra of those elements in $\bigwedge^* T_x S$, where $x \in S$, which are invariant under the action of the isotropy subgroup of x within the transvection group of S. Thus p induces an isomorphism $H^*(\widehat{P};\mathbb{R}) \cong H^*(P;\mathbb{R})$ and also an isomorphism of the rational cohomologies.

Secondly, the fundamental group $\pi_1(\widehat{P}) < \pi_1(P)$ is abelian and acts trivially on $\pi_n(\widehat{P}) = \pi_n(P)$ for $n > 1$, because $P = \mathsf{G}/\mathsf{H}$ with connected H and by Lemma 10. Hence $\widehat{P}$ is a simple space. Furthermore, because p is a finite covering, the induced map $\pi_*(\widehat{P}) \otimes \mathbb{Q} \to \pi_*(P) \otimes \mathbb{Q}$ is an isomorphism. By the Whitehead–Serre theorem for simple spaces (or inspecting the induced map between Postnikov decompositions of $\widehat{P}$ and P) the induced map in rational cohomology is an isomorphism as well. □

Lemma 17 and Künneth's formula imply

$$H^*(P;\mathbb{Q}) \cong H^*(T;\mathbb{Q}) \otimes H^*(\widetilde{P}_1;\mathbb{Q}) \otimes \cdots \otimes H^*(\widetilde{P}_m;\mathbb{Q}).$$

Note that $H^*(P;\mathbb{Q})$ is generated by homogeneous elements in odd degrees if and only if the same holds for $H^*(\widetilde{P}_j;\mathbb{Q})$ for all $j \in \{1,\dots,m\}$. Since the mapping degree of θ on an r–dimensional flat torus is 2^r, we are left to verify Theorem 3 only for irreducible simply connected compact symmetric spaces $P = \mathsf{G}/\mathsf{H}$ whose rational cohomology algebra is generated by homogeneous elements in odd degrees.

Since the rational cohomology of an *inner* compact symmetric space, which is a compact symmetric space all of whose geodesic symmetries belong to its transvection group G, has only contributions in even degrees (see eg [29, proof of Theorem 8.6.7] and [11, Theorem VII, page 467]), we may assume that $P = \mathsf{G}/\mathsf{H}$ is an *outer* symmetric space. Using the classification of simply connected irreducible compact symmetric spaces one could at this point determine all outer symmetric spaces that satisfy the assumptions of Theorem 3 by checking case-by-case the rational cohomology of such spaces given in [28] (see also [22; 26; 11]). But we prefer a more conceptual approach that we essentially learned from Oliver Goertsches.

Since the Lie algebras of G and of H form a *Cartan pair* $(\mathfrak{g},\mathfrak{h})$ (see [11, pages 448 and 465]) with $\rho := \operatorname{rank}(\mathfrak{g}) - \operatorname{rank}(\mathfrak{h}) > 0$, we have that $H^*(\mathsf{G}/\mathsf{H};\mathbb{Q})$ is isomorphic to a tensor product of a 2^ρ–dimensional exterior algebra and a quotient of a symmetric algebra (see [11, Theorem IV, page 463] and [17, Theorem 3]). Therefore G/H satisfies the hypotheses of Theorem 3 if and only if $\dim(H^*(\mathsf{G}/\mathsf{H};\mathbb{Q})) = 2^\rho$.

Remark 18 In [10] Goertsches gave a Lie-theoretic description of these spaces. They are precisely those symmetric spaces where the number of Weyl chambers of $\mathfrak{g}$ that intersect a given Weyl chamber of $\mathfrak{h}$ is equal to one. Nontrivial intersections of Weyl chambers of $\mathfrak{g}$ with Weyl chambers of $\mathfrak{h}$ are called *compartments* in [9].

From [23, Section 3] one sees that these spaces P are those spaces where the involution of $\mathfrak{g}$ associated with P is the canonical extension of an order-two automorphism of the Dynkin diagram of $\mathfrak{g}$ (see also [5, pages 33ff] and [9, pages 1128 and 1129]). The simply connected irreducible compact outer symmetric spaces $P = \mathsf{G}/\mathsf{H}$ of this kind are of one of two types (see [23, page 305]):

- Those of splitting rank, which are symmetric spaces where the rank of G is the sum of the rank of H and the rank of P. The irreducible simply connected compact symmetric spaces of splitting rank are the simply connected compact simple Lie groups, the odd-dimensional round spheres, $\mathsf{SU}_{2n}/\mathsf{Sp}_n$ with $n \geq 3$, and the exceptional space $\mathsf{E}_6/\mathsf{F}_4$.
- The spaces $\mathsf{SU}_{2n+1}/\mathsf{SO}_{2n+1}$ with $n \in \mathbb{N}$.

The referee made us aware that the mapping degree of θ has already been calculated by Araki in these cases:

- $\deg(\theta) = 2^{\mathrm{rank}(P)}$ if P is of splitting rank; see [3, Theorem 3.1].
- $\deg(\theta) = 2^n$ for $P = \mathsf{SU}_{2n+1}/\mathsf{SO}_{2n+1}$; see [3, Theorem 4.9].

This concludes the proof of Theorem 3.

Remark 19 In [3] Araki actually considered the map

$$\mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{H}, \quad g\mathsf{H} \mapsto g\sigma(g^{-1})\mathsf{H},$$

where σ is the involutive automorphism of G such that H is the identity component of its fixed-point set. In our terms we have $\sigma(g) = s_{e\mathsf{H}} \circ g \circ s_{e\mathsf{H}}$. Using $s_{e\mathsf{H}}(g\mathsf{H}) = \sigma(g)\mathsf{H}$ and $s_{g\mathsf{H}} = g \circ s_{e\mathsf{H}} \circ g^{-1}$ one sees that Araki's map coincides with our map θ if one chooses eH as basepoint.

Remark 20 (Lagrangian Grassmannians of odd rank) The Lagrangian Grassmannian $\mathcal{L} := \mathsf{U}_{2n+1}/\mathsf{O}_{2n+1}$ can be identified with the set of all Lagrangian subspaces of $\mathbb{C}^{2n+1}$. Since $\mathcal{L} = (\mathsf{U}_{2n+1}/\{\pm I\})/\mathsf{SO}_{2n+1}$, it meets the assumptions of Theorem 3. The reflection at a Lagrangian subspace is an orthogonal antisymplectic involution of $\mathbb{C}^{2n+1}$ and vice-versa. Identifying $\mathcal{L}$ with the space of all orthogonal antisymplectic involutions of $\mathbb{C}^{2n+1}$ it is shown in [2] that the conjugation

$$\mathcal{L} \times \mathcal{L} \to \mathcal{L}, \quad (R_1, R_2) \mapsto R_1 R_2 R_1$$

is a Γ–structure on $\mathcal{L}$. If one identifies $\mathcal{L}$ with the fixed-point set $\mathrm{Fix}(\tau)$ of the transposition map τ of U_{2n+1} by mapping the Lagrangian subspace $A(\mathbb{R}^{2n+1})$, where $A \in \mathsf{U}_{2n+1}$, to the matrix $AA^{\mathsf{T}} \in \mathrm{Fix}(\tau)$, the Γ–structure defined above can be written as

$$\mathrm{Fix}(\tau) \times \mathrm{Fix}(\tau) \to \mathrm{Fix}(\tau), \quad (A, B) \mapsto AB^{-1}A$$

(see [2, page 930]). Since $\mathrm{Fix}(\tau)$ is a totally geodesic submanifold of U_{2n+1}, the Γ–structure considered in [2] is just the canonical product on $\mathcal{L}$ induced by the geodesic symmetries of the Lie group U_{2n+1}.

Remark 21 (compact Lie groups) A compact Lie group with a bi-invariant metric is a symmetric space of splitting rank. The canonical product on a compact Lie group, considered as a symmetric space, discussed here is a Γ–structure different from the Lie-theoretic product. If one chooses the identity as basepoint, then θ is actually the squaring map.

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