Algebraic & Geometric Topology 18 (2018) 957–974

# A characterization of quaternionic Kleinian groups in dimension 2 with complex trace fields

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Let $G$ be a nonelementary discrete subgroup of $\mathrm{Sp}(2,1)$. We show that if the sum of diagonal entries of each element of $G$ is a complex number, then $G$ is conjugate to a subgroup of $\mathrm{U}(2,1)$.

22E40, 30F40, 57S30

## 1 Introduction

Given a Kleinian group $G$ of $\mathrm{PSL}(2,\mathbb{C})$, its trace field, denoted by $\mathbb{Q}(\operatorname{tr} G)$, is defined as the field generated by the traces of its elements. Trace fields have played an important role in studying arithmetic aspects of Kleinian groups. Neumann and Reid [10] studied the trace fields of arithmetic lattices in $\mathrm{PSL}(2,\mathbb{C})$. They showed that if $G$ is a nonuniform arithmetic lattice, it is conjugate to a subgroup of $\mathrm{PSL}(2,\mathbb{Q}(\operatorname{tr} G))$.

Even though the notion of trace field was first defined for Kleinian groups in $\mathrm{PSL}(2,\mathbb{C})$, it is possible to extend the notion to complex and quaternionic Kleinian groups. Indeed, there have been a few studies concerning the trace fields of complex and quaternionic Kleinian groups. Most of studies on the trace fields of complex Kleinian groups have focused on extending the results in the case of $\mathrm{PSL}(2,\mathbb{C})$ to $\mathrm{SU}(n,1)$. McReynolds [9] showed that the trace fields of complex Kleinian groups are commensurability invariants as for real Kleinian groups. Cunha and Gusevskii [1] and Genzmer [3] studied whether a discrete subgroup of $\mathrm{SU}(2,1)$ can be realized over its trace field.

A central theme in studying the trace fields of complex Kleinian groups is to characterize complex Kleinian groups with real trace fields. It turns out that any nonelementary complex Kleinian group with real trace field preserves a totally geodesic submanifold of constant negative sectional curvature in complex hyperbolic space. Cunha and Gusevskii [1] and Fu, Li and Wang [2] proved this for Kleinian groups in $\mathrm{SU}(2,1)$, and then J Kim and S Kim [7] extended it to $\mathrm{SU}(3,1)$. Recently Kim and Kim [8] extended

Published: 12 March 2018 DOI: 10.2140/agt.2018.18.957

this result to $\mathrm{SU}(n,1)$ in general. Furthermore, they showed that any nonelementary quaternionic Kleinian group with real trace field is also conjugate to a subgroup of either $\mathrm{SO}(n,1)$ or $\mathrm{SU}(1,1)$.

For quaternionic Kleinian groups, Kim [6] proved that if a nonelementary quaternionic Kleinian group $G$ in $\mathrm{Sp}(3,1)$ has a loxodromic element fixing $0$ and $\infty$, and the sum of diagonal entries of each element of $G$ is real, then $G$ preserves a totally geodesic submanifold of constant negative sectional curvature in the quaternionic hyperbolic space. Then the result is extended to the general $\mathrm{Sp}(n,1)$ case by Kim and Kim [8].

The studies so far have focused on characterizing nonelementary discrete groups with real trace fields. It is very natural to ask what if the "real" is replaced with "complex". In this article, we give the answer for this question in the case of $\mathrm{Sp}(2,1)$. The main theorem is the following.

**Theorem 1.1** *Let $G < \mathrm{Sp}(2,1)$ be a nonelementary quaternionic Kleinian group containing a loxodromic element fixing $0$ and $\infty$. If the sum of diagonal entries of each element of $G$ is in a maximal abelian subfield of $\mathbb{H}$, then $G$ preserves a totally geodesic submanifold of $\boldsymbol{H}^2_{\mathbb{H}}$ that is isometric to $\boldsymbol{H}^2_{\mathbb{C}}$. In other words, $G$ is conjugate to a subgroup of $\mathrm{U}(2,1)$.*

In particular, the field $\mathbb{C}$ of complex numbers is one of the maximal abelian subfields of $\mathbb{H}$. Hence Theorem 1.1 answers the motivating question.

## 2 Quaternionic hyperbolic space

The materials of this chapter are borrowed from [6]. For basic notions, we refer the reader to [6], and for more information, see [5].

Let $\mathbb{H}^{2,1}$ be a quaternionic vector space of dimension 3 with a Hermitian form of signature $(2,1)$. An element of $\mathbb{H}^{2,1}$ is a column vector $p=(p_1,p_2,p_3)^t$. Throughout the paper, we choose the second Hermitian form on $\mathbb{H}^{2,1}$ given by the matrix

$$J=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}.$$

Thus

$$\langle p,q\rangle = q^*Jp=\overline{q}^tJp=\overline{q}_1p_3+\overline{q}_2p_2+\overline{q}_3p_1,$$

where $p=(p_1,p_2,p_3)^t$, $q=(q_1,q_2,q_3)^t\in\mathbb{H}^{2,1}$.

One model of a quaternionic hyperbolic 2–space $\boldsymbol{H}^2_{\mathbb{H}}$, which matches this Hermitian form, is the *Siegel domain* $\mathfrak{S}$. It is defined by identifying points of $\mathfrak{S}$ with their horospherical coordinates $p = (\zeta, v, u) \in \mathbb{H} \times \mathrm{Im}(\mathbb{H}) \times \mathbb{R}_+$. The boundary of $\mathfrak{S}$ is given by $\mathbb{H} \times \mathrm{Im}(\mathbb{H}) \times \{0\} \cup \{\infty\}$, where $\infty$ is a distinguished point at infinity. Define a map $\psi\colon \overline{\mathfrak{S}} \to \mathbb{P}\mathbb{H}^{2,1}$ by

$$\psi\colon (\zeta, v, u) \mapsto \begin{bmatrix} -|\zeta|^2 - u + v \\ \sqrt{2}\zeta \\ 1 \end{bmatrix} \text{ for } (\zeta, v, u) \in \overline{\mathfrak{S}} - \{\infty\}, \quad \text{and} \quad \psi\colon \infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $\psi$ maps $\mathfrak{S}$ homeomorphically to the set of points $p$ in $\mathbb{P}\mathbb{H}^{2,1}$ with $\langle p, p\rangle < 0$, and maps $\partial\mathfrak{S}$ homeomorphically to the set of points $p$ in $\mathbb{P}\mathbb{H}^{2,1}$ with $\langle p, p\rangle = 0$. There is a metric on $\mathfrak{S}$ called the Bergman metric, and the isometry group of $\boldsymbol{H}^2_{\mathbb{H}}$ with respect to this metric is

$$\begin{aligned} \mathrm{Sp}(2,1) &= \{A \in \mathrm{GL}(3,\mathbb{H}) : \langle p, p'\rangle = \langle Ap, Ap'\rangle,\ p, p' \in \mathbb{H}^{2,1}\} \\ &= \{A \in \mathrm{GL}(3,\mathbb{H}) : J = A^* J A\}, \end{aligned}$$

where $A\colon \mathbb{H}^{2,1} \to \mathbb{H}^{2,1}$, $x\mathbb{H} \mapsto (Ax)\mathbb{H}$ for $x \in \mathbb{H}^{2,1}$ and $A \in \mathrm{Sp}(2,1)$. As in [4], we adopt the convention that $\mathrm{Sp}(2,1)$ acts on $\boldsymbol{H}^2_{\mathbb{H}}$ on the left and the projectivization of $\mathrm{Sp}(2,1)$ acts on the right. If we write

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix} \in \mathrm{Sp}(2,1),$$

$A^{-1}$ is written as

$$A^{-1} = \begin{bmatrix} \bar{l} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix} \in \mathrm{Sp}(2,1).$$

Then, from $AA^{-1} = A^{-1}A = I$, we get the following identities:

(1) $a\bar{l} + b\bar{h} + c\bar{g} = 1$, (2) $a\bar{f} + b\bar{e} + c\bar{d} = 0$, (3) $a\bar{c} + |b|^2 + c\bar{a} = 0$,

(4) $d\bar{l} + e\bar{h} + f\bar{g} = 0$, (5) $d\bar{f} + |e|^2 + f\bar{d} = 1$, (6) $d\bar{c} + e\bar{b} + f\bar{a} = 0$,

(7) $g\bar{l} + |h|^2 + l\bar{g} = 0$, (8) $g\bar{f} + h\bar{e} + l\bar{d} = 0$, (9) $g\bar{c} + h\bar{b} + l\bar{a} = 1$,

(10) $\bar{l}a + \bar{f}d + \bar{c}g = 1$, (11) $\bar{l}b + \bar{f}e + \bar{c}h = 0$, (12) $\bar{l}c + |f|^2 + \bar{c}l = 0$,

(13) $\bar{h}a + \bar{e}d + \bar{b}g = 0$, (14) $\bar{h}b + |e|^2 + \bar{b}h = 1$, (15) $\bar{h}c + \bar{e}f + \bar{b}l = 0$,

(16) $\bar{g}a + |d|^2 + \bar{a}g = 0$, (17) $\bar{g}b + \bar{d}e + \bar{a}h = 0$, (18) $\bar{g}c + \bar{d}f + \bar{a}l = 1$.

**Remark 2.1** If $c=0$, then $f=0$ by (12), and hence $A$ fixes $\mathbf{0}=[0,0,1]^t$. Similarly, if $g=0$, then $d=0$ by (16), and hence $A$ fixes $\infty=[1,0,0]^t$.

Note that totally geodesic submanifolds of quaternionic hyperbolic 2–space are isometric to one of $\boldsymbol{H}^1_{\mathbb{H}}$, $\boldsymbol{H}^1_{\mathbb{C}}$, $\boldsymbol{H}^2_{\mathbb{C}}$, and $\boldsymbol{H}^2_{\mathbb{R}}$. The following proposition is essential in the proof of the main theorem.

**Proposition 2.2** *For two nonzero quaternions $a$ and $b$, if $ab$ and $ba$ are complex numbers, then $a$ and $b$ satisfy one of the following:*

(i) $a, b\in\mathbb{C}$.

(ii) $a=a_*j$ *and* $b=b_*j$ *for some* $a_*, b_*\in\mathbb{C}$.

(iii) $b=r\overline{a}$ *for some* $r\in\mathbb{R}-\{0\}$.

**Proof** First observe that $ab=a(ba)a^{-1}$. In other words, two quaternions $ab$ and $ba$ are similar. It is well known that two quaternions are similar if and only if they have the same norm and real part. By the assumption that $ab$ and $ba$ are complex numbers, it follows that

$$|\mathrm{Im}(ab)|=\sqrt{|ab|^2-\mathrm{Re}(ab)^2}=\sqrt{|ba|^2-\mathrm{Re}(ba)^2}=|\mathrm{Im}(ba)|.$$

Then one of the following holds:

(i) $\mathrm{Im}(ab)=\mathrm{Im}(ba)=0$,

(ii) $\mathrm{Im}(ab)=\mathrm{Im}(ba)\neq 0$,

(iii) $\mathrm{Im}(ab)=-\mathrm{Im}(ba)\neq 0$.

If $\mathrm{Im}(ab)=\mathrm{Im}(ba)=0$, ie $ab$ and $ba$ are real numbers, it easily follows that $b=|b/a|\overline{a}$.

If $\mathrm{Im}(ab)=\mathrm{Im}(ba)\neq 0$, then $ab=ba$, and thus

$$ab=a(ba)a^{-1}=a(ab)a^{-1}.$$

Since $a$ commutes with $ab$, which is a complex number with nonzero imaginary part, $a$ must commute with $i$. Hence $a\in\mathbb{C}$. Furthermore, $b\in\mathbb{C}$ since $b=a^{-1}(ab)$.

Lastly, if $\mathrm{Im}(ab)=-\mathrm{Im}(ba)\neq 0$, then $ab=\overline{ba}$, and thus

$$ab=a(ba)a^{-1}=a(\overline{ab})a^{-1}.$$

This implies that $a$ anticommutes with $i$. Hence $a=a_*j$ for some $a_*\in\mathbb{C}$. Applying the same argument to $ba=b(ab)b^{-1}=b(\overline{ba})b^{-1}$, it follows that $b=b_*j$ for some $b_*\in\mathbb{C}$. This completes the proof. □

The next lemma is quite elementary and the proof is easy.

**Lemma 2.3** *Let $q$ be a quaternion. If $qi\overline{q} \in \mathbb{C}$, then either $q \in \mathbb{C}$ or $q$ is of the form $q = q_* j$ for some $q_* \in \mathbb{C}$.*

**Proof** We may assume that $q \neq 0$. Note that $qi\overline{q} = q(|q|^2 i)q^{-1}$, and hence $qi\overline{q}$ is a complex number similar to $|q|^2 i \in \mathbb{C}$. By a similar argument as the proof of Proposition 2.2, $q$ either commutes or anticommutes with $i$. This leads to either $q \in \mathbb{C}$ or $q = q_* j$ for some $q_* \in \mathbb{C}$. □

## 3 Proof of Theorem 1.1

Let $F$ be a maximal abelian subfield of $\mathbb{H}$. Then $F = \mathbb{R} \oplus u\mathbb{R}$ for some nonreal quaternion $u \in \mathbb{H}$. Suppose that the sum of diagonal entries of each element of $G$ is in $F$. Since any quaternion is similar to a complex number, there exists a nonzero unit quaternion $q$ with $quq^{-1} \in \mathbb{C}$. Then the sum of the diagonal entries of each element of $D_q G D_q^{-1}$ lies in $\mathbb{C}$, where $D_q G D_q^{-1}$ is the group obtained by conjugating each element of $G$ by $D_q = \mathrm{Diag}(q, q, q)$. Hence it is sufficient to prove Theorem 1.1 in the case that $F = \mathbb{C}$.

We now suppose that $G$ is a nonelementary discrete subgroup of $\mathrm{Sp}(2,1)$ in which the sum of the diagonal entries of each element of $G$ is a complex number. Let $A$ be a loxodromic element of $G$ fixing $0$ and $\infty$, and let $B$ be an arbitrary element of $G$. In terms of matrices, we write $A$ and $B$ as

$$A = \begin{bmatrix} \lambda\mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu/\lambda \end{bmatrix}, \quad B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix}, \tag{19}$$

where $\mu, \nu \in \mathrm{Sp}(1)$ and $\lambda > 1$. For more details, see [4] or [5].

**Lemma 3.1** *The matrix $A$ of $G$ fixing $0$ and $\infty$ is an element of $\mathrm{U}(2,1)$. In other words, $\mu, \nu \in \mathrm{U}(1)$.*

**Proof** For a matrix $X$, we denote by $\mathrm{tr}(X)$ the sum of the diagonal entries of $X$. Let $\mu = \mu_0 + \mu_1 i + \mu_2 j + \mu_3 k$ and $\nu = \nu_0 + \nu_1 i + \nu_2 j + \nu_3 k$ for $\mu_t, \nu_t \in \mathbb{R}$, where $t = 0, 1, 2, 3$. Then

$$\mathrm{tr}(A) = (\lambda + 1/\lambda)\,(\mu_0 + \mu_1 i + \mu_2 j + \mu_3 k) + (\nu_0 + \nu_1 i + \nu_2 j + \nu_3 k) \in \mathbb{C},$$

and hence

$$(\lambda + 1/\lambda)\mu_t + \nu_t = 0 \quad \text{for } t = 2, 3. \tag{20}$$

Furthermore, considering

$$\begin{aligned}\operatorname{tr}(A^2) &= (\lambda^2 + 1/\lambda^2)\mu^2 + \nu^2 \\ &= (\lambda^2 + 1/\lambda^2)(\mu_0^2 - \mu_1^2 - \mu_2^2 - \mu_3^2 + 2\mu_0\mu_1 i + 2\mu_0\mu_2 j + 2\mu_0\mu_3 k) \\ &\qquad + (\nu_0^2 - \nu_1^2 - \nu_2^2 - \nu_3^2 + 2\nu_0\nu_1 i + 2\nu_0\nu_2 j + 2\nu_0\nu_3 k) \in \mathbb{C},\end{aligned}$$

we have that for $t = 2, 3$,

$$(\lambda^2 + 1/\lambda^2)\mu_0\mu_t + \nu_0\nu_t = \mu_t[(\lambda^2+1/\lambda^2)\mu_0 - (\lambda+1/\lambda)\nu_0] = 0. \tag{21}$$

If $\mu_2 = \mu_3 = 0$, (20) implies that $\nu_2 = \nu_3 = 0$. Then $\mu, \nu \in \mathbb{C}$, and so $\mu, \nu \in \mathrm{U}(1)$. From now on, we assume that $\mu_2 \neq 0$ or $\mu_3 \neq 0$. By (21),

$$(\lambda^2 + 1/\lambda^2)\mu_0 - (\lambda + 1/\lambda)\nu_0 = 0, \tag{22}$$

and we can write

$$\nu = \frac{\lambda^4+1}{\lambda(\lambda^2+1)}\mu_0 + \nu_1 i - \Big(\lambda + \frac{1}{\lambda}\Big)(\mu_2 j + \mu_3 k).$$

Now let us consider $A^4$. Then

$$\begin{aligned}\operatorname{tr}(A^4) &= \Big(\lambda^4 + \frac{1}{\lambda^4}\Big)\mu^4 + \nu^4 \\ &= \Big(\lambda^4 + \frac{1}{\lambda^4}\Big)(\mu_0 + \mu_1 i + \mu_2 j + \mu_3 k)^4 \\ &\qquad + \Big(\frac{\lambda^4+1}{\lambda(\lambda^2+1)}\mu_0 + \nu_1 i - \Big(\lambda + \frac{1}{\lambda}\Big)(\mu_2 j + \mu_3 k)\Big)^4 \in \mathbb{C}.\end{aligned}$$

Since $\mu, \nu \in \mathrm{Sp}(1)$, we get

$$\mu_0^2+\mu_1^2+\mu_2^2+\mu_3^2 = 1 \quad \text{and} \quad |\nu|^2 = \Big(\frac{\lambda^4+1}{\lambda(\lambda^2+1)}\mu_0\Big)^2 + \nu_1^2 + \Big(\lambda+\frac{1}{\lambda}\Big)^2(\mu_2^2+\mu_3^2) = 1.$$

Using these identities and calculating the $j$–part of $\operatorname{tr}(A^4)$, we have that

$$\frac{4\mu_0\mu_2(\lambda^2-1)(\lambda^6-1)(4\lambda^2\mu_0^2 - (\lambda^2+1)^2)}{\lambda^4(\lambda^2+1)^2} = 0.$$

Since $\lambda > 1$ and $0 \le \mu_0^2 < 1$, it follows that $\mu_0\mu_2 = 0$. By repeating the same argument for the $k$–part of $\operatorname{tr}(A^4)$, one gets $\mu_0\mu_3 = 0$. Since $\mu_2 \neq 0$ or $\mu_3 \neq 0$, we have $\mu_0 = 0$ and hence $\nu_0 = 0$ by (22). That is, $\mu$ and $\nu$ are purely imaginary, and so

$\overline{\mu} = -\mu$ and $\overline{\nu} = -\nu$. Since $|\mu| = |\nu| = 1$, we know that $\mu^3 = -\mu$ and $\nu^3 = -\nu$. If we write $\nu = \nu_1 i - (\lambda + 1/\lambda)(\mu_2 j + \mu_3 k)$ as before, since

$$\operatorname{tr}(A^3) = \left(\lambda^3 + \frac{1}{\lambda^3}\right)\mu^3 + \nu^3 = -\left(\lambda^3 + \frac{1}{\lambda^3}\right)\mu - \nu \in \mathbb{C},$$

the $j$–part of $\operatorname{tr}(A^3)$ is zero; ie

$$-\left(\lambda^3 + \frac{1}{\lambda^3}\right)\mu_2 + \left(\lambda + \frac{1}{\lambda}\right)\mu_2 = -\left(\lambda + \frac{1}{\lambda}\right)\left(\lambda - \frac{1}{\lambda}\right)^2 \mu_2 = 0.$$

Since $\lambda > 1$, we have $\mu_2 = 0$. Similarly, considering the $k$–part of $\operatorname{tr}(A^3)$, we also get $\mu_3 = 0$. This contradicts to the assumption that $\mu_2 \neq 0$ or $\mu_3 \neq 0$. Therefore, $\mu_2 = \mu_3 = 0$ and thus $\mu, \nu \in \mathrm{Sp}(1) \cap \mathbb{C} = \mathrm{U}(1)$. □

According to Lemma 3.1, $A$ is written as

$$A = \begin{bmatrix} \lambda e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & (1/\lambda)e^{i\theta} \end{bmatrix}, \quad \text{where } \lambda > 1 \text{ and } \theta, \phi \in [0, 2\pi).$$

**Lemma 3.2** *For any element $B \in G$, every diagonal entry of $B$ is a complex number.*

**Proof** Let $B$ be the matrix

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & l \end{bmatrix}.$$

Since the sum of the diagonal entries of every element of $G$ is in $\mathbb{C}$, we have that

$$\begin{aligned} \operatorname{tr}(B) &= a + e + l \in \mathbb{C}, \\ \operatorname{tr}(AB) &= \lambda e^{i\theta} a + e^{i\phi} e + \frac{e^{i\theta}}{\lambda} l \in \mathbb{C}, \\ \operatorname{tr}(A^{-1}B) &= \frac{e^{-i\theta}}{\lambda} a + e^{-i\phi} e + \lambda e^{-i\theta} l \in \mathbb{C}. \end{aligned}$$

Solving for $a, e$, and $l$, we conclude that $a, e, l \in \mathbb{C}$. This shows that every element of $G$ has complex diagonal entries. □

**Lemma 3.3** *Let $A$, $B_1$ and $B_2$ be elements of $G$ of the form*

$$A = \begin{bmatrix} \lambda e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & (1/\lambda)e^{i\theta} \end{bmatrix}, \quad B_1 = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & l_1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & l_2 \end{bmatrix}$$

*for some $\lambda > 1$ and $\theta, \phi \in [0, 2\pi)$. Then $b_1 d_2$, $c_1 g_2$, $d_1 b_2$, $f_1 h_2$, $g_1 c_2$ and $h_1 f_2$ are all complex numbers. Furthermore, $b_1 i d_2$ and $h_1 i f_2$ are complex numbers provided $\phi$ is not an integer multiple of $\pi$, and $c_1 i g_2$, $d_1 i b_2$, $f_1 i h_2$ and $g_1 i c_2$ are complex numbers provided $\theta$ is not an integer multiple of $\pi$.*

**Proof** Since $B_1$ and $B_2$ are in $G$, we know that $a_1$, $e_1$, $l_1$, $a_2$, $e_2$ and $l_2$ are all complex numbers by Lemma 3.2. Consider the elements $B_1 A B_2$, $B_1 A^2 B_2$, $B_1 A^3 B_2$, $B_1 A^4 B_2$ in $G$ and also their respective $(1,1)$–entries:

$$\lambda e^{i\theta} a_1 a_2 + b_1 e^{i\phi} d_2 + \lambda^{-1} c_1 e^{i\theta} g_2,$$
$$\lambda^2 e^{2i\theta} a_1 a_2 + b_1 e^{2i\phi} d_2 + \lambda^{-2} c_1 e^{2i\theta} g_2,$$
$$\lambda^3 e^{3i\theta} a_1 a_2 + b_1 e^{3i\phi} d_2 + \lambda^{-3} c_1 e^{3i\theta} g_2,$$
$$\lambda^4 e^{4i\theta} a_1 a_2 + b_1 e^{4i\phi} d_2 + \lambda^{-4} c_1 e^{4i\theta} g_2.$$

These are also all complex numbers by Lemma 3.2. Since $a_1, a_2 \in \mathbb{C}$, the following are all complex numbers as well:

$$\lambda \cos\phi(b_1 d_2) + \lambda \sin\phi(b_1 i d_2) + \cos\theta(c_1 g_2) + \sin\theta(c_1 i g_2) = z_1,$$
$$\lambda^2 \cos 2\phi(b_1 d_2) + \lambda^2 \sin 2\phi(b_1 i d_2) + \cos 2\theta(c_1 g_2) + \sin 2\theta(c_1 i g_2) = z_2,$$
$$\lambda^3 \cos 3\phi(b_1 d_2) + \lambda^3 \sin 3\phi(b_1 i d_2) + \cos 3\theta(c_1 g_2) + \sin 3\theta(c_1 i g_2) = z_3,$$
$$\lambda^4 \cos 4\phi(b_1 d_2) + \lambda^4 \sin 4\phi(b_1 i d_2) + \cos 4\theta(c_1 g_2) + \sin 4\theta(c_1 i g_2) = z_4.$$

One can easily show that the determinant of the associated $4 \times 4$ matrix to the system of linear equations above is

$$\lambda^3 \sin\phi \sin\theta(\lambda^2 - 2\cos(\phi - \theta)\lambda + 1)(\lambda^2 - 2\cos(\phi + \theta)\lambda + 1).$$

Hence we have that $b_1 d_2$, $b_1 i d_2$, $c_1 g_2$ and $c_1 i g_2$ are all complex numbers if both $\phi$ and $\theta$ are not integer multiples of $\pi$. It can be easily checked that $b_1 d_2$ and $c_1 g_2$ are still complex numbers even if either $\phi$ or $\theta$ is an integer multiple of $\pi$. Therefore, $b_1 d_2$ and $c_1 g_2$ are complex numbers. Furthermore, if $\phi$ is not an integer multiple of $\pi$, then $b_1 i d_2$ is a complex number. If $\theta$ is not an integer multiple of $\pi$, then $c_1 i g_2$ is a complex number. Similarly, from the other diagonal entries of $B_1 A B_2$, $B_1 A^2 B_2$, $B_1 A^3 B_2$, $B_1 A^4 B_2$ in $G$, the lemma is proved. □

Applying Lemma 3.3 to $B_1 = B_2 = B$ and $B_1 = B$, $B_2 = B^{-1}$ (or $B_2 = B$, $B_1 = B^{-1}$), we immediately have:

**Corollary 3.4** *Let $B$ be the matrices written in (19). Then*

(i) $bd$, $db$, $fh$, $hf$, $cg$, $gc$, $b\overline{h}$, $f\overline{d}$, $c\overline{g}$, $\overline{h}b$, $\overline{d}f$, $\overline{g}c \in \mathbb{C}$.

(ii) $bid$, $hif$, $bi\overline{h}$, $\overline{d}if \in \mathbb{C}$ *unless* $\phi \equiv 0 \pmod{\pi}$.

(iii) $cig$, $dib$, $fih$, $gic$, $ci\overline{g}$, $di\overline{f}$, $\overline{g}ic$, $\overline{h}ib \in \mathbb{C}$ *unless* $\theta \equiv 0 \pmod{\pi}$.

Let $B$ be an arbitrary element of $G$ (written as in (19)) that does not fix both $0$ and $\infty$. Suppose that $cg = 0$. Then $B$ fixes either $0$ or $\infty$; see Remark 2.1. This means that $A$ and $B$ have a common fixed point. However this is impossible since $G$ is discrete. Hence $cg \neq 0$.

## 3.1 The $bd \neq 0$ case

We will first deal with the case that there exists an element $B$ of $G$ with $bd \neq 0$, which we assume throughout this section. As seen in Corollary 3.4, both $bd$ and $db$ are complex numbers. Applying Proposition 2.2 for $b$ and $d$, one of the following holds:

(1) $b, d \in \mathbb{C}$.

(2) $b$ and $d$ are of the form $b = b_* j$ and $d = d_* j$, where $b_*, d_* \in \mathbb{C}$.

(3) $d = r\overline{b}$ for some $r \in \mathbb{R} - \{0\}$.

We will consider these cases separately as follows.

**Case 1** Suppose that $b, d \in \mathbb{C}$. Since $b\overline{h}, \overline{h}b \in \mathbb{C}$ and $b$ is nonzero, $h \in \mathbb{C}$. Similarly, since $f\overline{d}, \overline{d}f \in \mathbb{C}$ and $d$ is nonzero, $f \in \mathbb{C}$. Then, from (2) and (13), it can be seen that $c$ and $g$ are also complex numbers. Thus every entry of $B$ is a complex number, and hence $B \in \mathrm{U}(2,1)$.

Let $B'$ be any other element of $G$ that does not fix both $0$ and $\infty$, and write

$$B' = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & l' \end{bmatrix}.$$

Then by Lemma 3.2, we know that $a', e', l' \in \mathbb{C}$. Furthermore, by Lemma 3.3, we have

$$bd', db', b\overline{h'}, f'\overline{d}, cg', gc' \in \mathbb{C}.$$

Since $b$ and $d$ are nonzero complex numbers, it follows that $b', d', h', f' \in \mathbb{C}$. Moreover, from (2) and (13), it follows that $c', g' \in \mathbb{C}$. Thus every entry of $B'$ is a complex number; ie $B' \in \mathrm{U}(2,1)$. Therefore, $G$ is a subgroup of $\mathrm{U}(2,1)$, which preserves a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ in $\boldsymbol{H}^2_{\mathbb{H}}$.

**Case 2** Now we suppose that $b = b_* j$ and $d = d_* j$ for some $b_*, d_* \in \mathbb{C}$. Since $b\bar{h}, \bar{h}b \in \mathbb{C}$ by Corollary 3.4 and $b$ is nonzero, $h = h_* j$ for some $h_* \in \mathbb{C}$. In the same way, since $f\bar{d}, \bar{d}f \in \mathbb{C}$ and $d$ is nonzero, $f = f_* j$ for some $f_* \in \mathbb{C}$. Furthermore, by (2), we have that

$$a\bar{f} + b\bar{e} + c\bar{d} = -af_* j + b_* j\bar{e} - cd_* j.$$

Since $e$ is a complex number, $j\bar{e} = ej$. Hence

$$-af_* j + b_* j\bar{e} - cd_* j = (-af_* + b_* e - cd_*)j = 0.$$

Since $d_* \neq 0$, we conclude that $c$ is a complex number. Similarly, from (13), it can be derived that $g \in \mathbb{C}$. To summarize, $a, e, l, c, g \in \mathbb{C}$, and $b$, $d$, $f$ and $h$ are of the form $q_* j$ where $q_* \in \mathbb{C}$. Then for $z_1, z_2 \in \mathbb{C}$,

$$A \begin{bmatrix} z_1 \\ z_2 j \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda e^{i\theta} z_1 \\ e^{i\phi} z_2 j \\ (1/\lambda)e^{i\theta} \end{bmatrix} \sim \begin{bmatrix} z_1' \\ z_2' j \\ 1 \end{bmatrix}$$

for some $z_1', z_2' \in \mathbb{C}$ and

$$B \begin{bmatrix} z_1 \\ z_2 j \\ 1 \end{bmatrix} = \begin{bmatrix} a & b_* j & c \\ d_* j & e & f_* j \\ g & h_* j & l \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 j \\ 1 \end{bmatrix} = \begin{bmatrix} az_1 + b_* j z_2 j + c \\ d_* j z_1 + e z_2 j + f_* j \\ g z_1 + h_* j z_2 j + l \end{bmatrix}$$
$$= \begin{bmatrix} az_1 - b_* \bar{z}_2 + c \\ (d_* \bar{z}_1 + e z_2 + f_*) j \\ g z_1 - h_* \bar{z}_2 + l \end{bmatrix} \sim \begin{bmatrix} z_1'' \\ z_2'' j \\ 1 \end{bmatrix}$$

for some $z_1'', z_2'' \in \mathbb{C}$. Note that $\begin{bmatrix} z_1 & z_2 j & 1 \end{bmatrix} = \begin{bmatrix} z_1 & j\bar{z}_2 & 1 \end{bmatrix}$ for $z_1, z_2 \in \mathbb{C}$. Hence $A$ and $B$ leave invariant a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ of polar vectors $\begin{bmatrix} z_1 & j z_2 & 1 \end{bmatrix}^t$, where $z_1, z_2 \in \mathbb{C}$.

Let $B'$ be any other element of $G$ that does not fix both $0$ and $\infty$. Let

$$B' = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & l' \end{bmatrix}.$$

Then $a', e', l' \in \mathbb{C}$ by Lemma 3.2. Applying Lemma 3.3 to $B$ and $B'$, one can conclude that $bd', d'b, b'd, db' \in \mathbb{C}$. Then, by Proposition 2.2, $b'$ and $d'$ are of the form $q_* j$ where $q_* \in \mathbb{C}$ since $b$ and $d$ have the same form. By a similar argument, one can show that $f'$ and $h'$ are of the same form $q_* j$. Moreover, $c', g' \in \mathbb{C}$ because $cg', gc' \in \mathbb{C}$ by Lemma 3.3 and $cg \neq 0$. Therefore, $B'$ is of the same form as $B$, and we conclude that every element of $G$ preserves a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ consisting of $\begin{bmatrix} z_1 & jz_2 & 1 \end{bmatrix}^t$, where $z_1, z_2 \in \mathbb{C}$.

**Case 3** Lastly, suppose that $d = r_1\overline{b}$ for some $r_1 \in \mathbb{R} - \{0\}$. Then consider $AB$ written as

$$AB = \begin{bmatrix} \lambda e^{i\theta}a & \lambda e^{i\theta}b & \lambda e^{i\theta}c \\ e^{i\phi}d & e^{i\phi}e & e^{i\phi}f \\ \lambda^{-1}e^{i\theta}g & \lambda^{-1}e^{i\theta}h & \lambda^{-1}e^{i\theta}l \end{bmatrix}.$$

The element $AB$ falls into the $bd \neq 0$ case. In addition, if $\phi + \theta \not\equiv 0 \pmod{\pi}$, then $AB$ can never fall into Case 3 and thus it must fall into either Case 1 or 2, and we are done. Hence, from now on, we assume that $\phi + \theta \equiv 0 \pmod{\pi}$. Moreover, in order to avoid repetition, assume that $b$ and $d$ are neither complex numbers nor of the form $b = b_* j$ and $d = d_* j$ for $b_*, d_* \in \mathbb{C}$. Under these hypotheses, we claim that $\phi \equiv 0 \pmod{\pi}$. If not, $bid = r_1 bi\overline{b} \in \mathbb{C}$ holds by Corollary 3.4. Lemma 2.3 implies either $b \in \mathbb{C}$ or $b = b_* j$ for $b_* \in \mathbb{C}$, which contradicts our hypothesis. Therefore, $\phi \equiv 0 \pmod{\pi}$, and thus $\theta \equiv 0 \pmod{\pi}$ since $\phi + \theta \equiv 0 \pmod{\pi}$.

By Corollary 3.4, we have that $f\overline{d}$, $\overline{d}f$, $b\overline{h}$, $\overline{h}b \in \mathbb{C}$. Applying Proposition 2.2 to $f, \overline{d}$ and $b, \overline{h}$ respectively, it can be easily seen that $f = r_2' d = r_2' r_1 \overline{b} = r_2 \overline{b}$ and $h = r_3 b$ for some $r_2, r_3 \in \mathbb{R} - \{0\}$. From (5) and (14),

$$2r_1 r_2 |b|^2 = 2r_3 |b|^2 = 1 - |e|^2,$$

and thus,

$$r_3 = r_1 r_2 = \frac{1 - |e|^2}{2|b|^2} \quad \text{and} \quad h = r_1 r_2 b.$$

Moreover, using (2), (4), (11), (13), we have the following equations:

$$r_2 ab + r_1 cb + b\overline{e} = 0, \quad r_1 lb + r_2 gb + r_1 r_2 b\overline{e} = 0,$$
$$r_1 r_2 \overline{c} b + \overline{l} b + r_2 be = 0, \qquad r_1 r_2 \overline{a} b + \overline{g} b + r_1 be = 0.$$

These equations are written as

$$-r_1 r_2 b\overline{e} = r_1 r_2^2 ab + r_1^2 r_2 cb = r_1 lb + r_2 gb,$$
$$-r_1 r_2 be = r_1^2 r_2 \overline{c} b + r_1 \overline{l} b = r_1 r_2^2 \overline{a} b + r_2 \overline{g} b.$$

Since $b \neq 0$, these equations simplify to

$$r_1 r_2^2 a + r_1^2 r_2 c = r_1 l + r_2 g,$$
$$r_1 r_2^2 a - r_1^2 r_2 c = r_1 l - r_2 g.$$

Hence we finally get that $r_1 r_2^2 a = r_1 l$ and $r_1^2 r_2 c = r_2 g$; ie $l = r_2^2 a$ and $g = r_1^2 c$ since $r_1, r_2 \neq 0$. Now $B$ is written as

$$B = \begin{bmatrix} a & b & c \\ r_1\overline{b} & e & r_2\overline{b} \\ r_1^2 c & r_1 r_2 b & r_2^2 a \end{bmatrix}, \quad \text{where } a, e \in \mathbb{C} \text{ and } r_1, r_2 \in \mathbb{R} - \{0\}. \tag{23}$$

Since $cg = r_1^2 c^2 \in \mathbb{C}$ by Corollary 3.4, either $c \in \mathbb{C}$ or $c$ is purely imaginary.

**Case 3.1** $\boldsymbol{c \in \mathbb{C}}$ From (11), $(r_2\overline{a} + r_1\overline{c})b + be = 0$, and thus we have that

$$beb^{-1} = -r_2\overline{a} - r_1\overline{c}.$$

By hypothesis, $a, c \in \mathbb{C}$ and $r_1, r_2 \in \mathbb{R} - \{0\}$. Hence the above identity implies that $beb^{-1}$ is also a complex number. We have two complex numbers $e$ and $beb^{-1}$ which are similar. Therefore, we have one of the following:

(i) $\mathrm{Im}(e) = \mathrm{Im}(beb^{-1}) = 0$.

(ii) $\mathrm{Im}(e) = \mathrm{Im}(beb^{-1}) \neq 0$ and $b \in \mathbb{C}$

(iii) $\mathrm{Im}(e) = -\mathrm{Im}(beb^{-1}) \neq 0$ and $b = b_* j$ with $b_* \in \mathbb{C}$.

Since we are now assuming that neither of the last two cases holds, the first case holds; ie $e \in \mathbb{R}$.

Since $\phi, \theta \equiv 0 \pmod{\pi}$ as mentioned before, $e^{i\phi}, e^{i\theta} \in \mathbb{R}$. Therefore, for any $z_1, z_2, z_3 \in \mathbb{C}$,

$$A \begin{bmatrix} z_1 \\ \overline{b} z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \lambda e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & (1/\lambda)e^{i\theta} \end{bmatrix} \begin{bmatrix} z_1 \\ \overline{b} z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} z_1' \\ \overline{b} z_2' \\ z_3' \end{bmatrix}$$

for some $z_1', z_2', z_3' \in \mathbb{C}$. In addition,

$$B \begin{bmatrix} z_1 \\ \overline{b} z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a & b & c \\ r_1\overline{b} & e & r_2\overline{b} \\ r_1^2 c & r_1 r_2 b & r_2^2 a \end{bmatrix} \begin{bmatrix} z_1 \\ \overline{b} z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a z_1 + |b|^2 z_2 + c z_3 \\ \overline{b}(r_1 z_1 + e z_2 + r_2 z_3) \\ r_1^2 c z_1 + r_1 r_2 |b|^2 z_2 + r_2^2 a z_3 \end{bmatrix} = \begin{bmatrix} z_1'' \\ \overline{b} z_2'' \\ z_3'' \end{bmatrix}$$

for some $z_1'', z_2'', z_3'' \in \mathbb{C}$.

Let $B'$ be any other element of $G$ that does not fix both $0$ and $\infty$. By applying Lemma 3.3 to $B$ and $B'$ as in the previous case, one can check that $B'$ has the same form as $B$; ie

$$B' = \begin{bmatrix} a' & b' & c' \\ r_3\overline{b'} & e' & r_4\overline{b'} \\ r_3^2 c' & r_3 r_4 b' & r_4^2 a' \end{bmatrix}, \quad \text{where } a', c' \in \mathbb{C},\ e' \in \mathbb{R} \text{ and } r_3, r_4 \in \mathbb{R} - \{0\}.$$

Then, considering the diagonal entries of $B'B$, it follows that

$$a'a + r_1 b'\overline{b} + r_1^2 c'c \in \mathbb{C}, \quad r_1 \overline{b} b' + ee' + r_2 r_3 r_4 \overline{b} b' \in \mathbb{C}.$$

Since $a, a', c, c' \in \mathbb{C}$, $e, e' \in \mathbb{R}$ and $r_1, r_2, r_3, r_4 \neq 0$, we have that $b'\overline{b} \in \mathbb{C}$ and $\overline{b} b' \in \mathbb{C}$. Applying Proposition 2.2 for $\overline{b}$ and $b'$, we have $b' = rb$ for some $r \in \mathbb{R}$ since $b$ is neither a complex number nor of the form $b = b_* j$ for some $b_* \in \mathbb{C}$. Hence

$$B' = \begin{bmatrix} a' & rb & c' \\ r_3 r\overline{b} & e' & r_4 r\overline{b} \\ r_3^2 c' & r_3 r_4 rb & r_4^2 a' \end{bmatrix}.$$

Then it is easy to see that $B'$ also preserves a copy of $\boldsymbol{H}_{\mathbb{C}}^2$ of polar vectors $\begin{bmatrix} z_1 & \overline{b} z_2 & z_3 \end{bmatrix}^t$. Therefore, we conclude that $G$ preserves a copy of $\boldsymbol{H}_{\mathbb{C}}^2$ of polar vectors $\begin{bmatrix} z_1 & \overline{b} z_2 & z_3 \end{bmatrix}^t$, where $z_1, z_2, z_3 \in \mathbb{C}$.

**Case 3.2** ***c* is purely imaginary** Now we suppose that the previous case does not happen for any element of $G$. Hence assume that $c$ is not a complex number.

**Claim** $$r_2 = -1.$$

**Proof of claim** From (3) and $\overline{c} = -c$, we have

$$\operatorname{Re}(a\overline{c} + |b|^2 + c\overline{a}) = 2\operatorname{Re}(a\overline{c}) + |b|^2 = -2\operatorname{Re}(ac) + |b|^2 = 0.$$

Also since $ac$ and $ca$ are similar, we have $2\operatorname{Re}(ca) = 2\operatorname{Re}(ac) = |b|^2$. In addition, once we prove that the $(1,3)$–entry of $B^2$, namely $ac + r_2|b|^2 + r_2^2 ca$, is purely imaginary, then we have that

$$0 = 2\operatorname{Re}(ac + r_2|b|^2 + r_2^2 ca) = |b|^2 + 2r_2|b|^2 + r_2^2|b|^2 = (r_2+1)^2|b|^2.$$

Since $b \neq 0$, it follows that $r_2 = -1$. For this reason, we only need to show that the $(1,3)$–entry of $B^2$ is purely imaginary. This follows if both the $(1,2)$–entry and the $(2,1)$–entry of $B^2$ are nonzero since it is assumed that the previous case does not happen for any element of $G$.

Putting $a=a_0+a_1i$ and $c=c_1i+c_2j+c_3k$, the identity $a\overline{c}+|b|^2+c\overline{a}=0$ of (3) implies that

$$|b|^2+2a_1c_1=0.$$

By a straight computation, the $(1,2)$–entry of $B^2$ is

$$ab+be+r_1r_2cb.$$

From (11), we have that $be=-r_2\overline{a}b-r_1\overline{c}b=-r_2\overline{a}b+r_1cb$. The last equation follows from the assumption that $c$ is purely imaginary. Then the $(1,2)$–entry of $B^2$ is written as

$$ab+be+r_1r_2cb=(a-r_2\overline{a}+r_1(r_2+1)c)b.$$

Note that $a$ is a complex number, $c$ is purely imaginary and not a complex number, and $b\neq 0$. Hence, if $r_2\neq -1$, the $(1,2)$–entry of $B^2$ can never be zero. In a similar way, the $(2,1)$–entry of $B^2$ is also nonzero if $r_2\neq -1$. Hence if $r_2\neq -1$, the $(1,3)$–entry of $B^2$ must be purely imaginary, and then $r_2=-1$ as mentioned above. This makes a contradiction. Therefore, $r_2=-1$. □

Now $B$ is written as

$$B=\begin{bmatrix} a & b & c\\ r_1\overline{b} & e & -\overline{b}\\ r_1^2c & -r_1b & a\end{bmatrix},\quad \text{where } a,e\in\mathbb{C} \text{ and } r_1\in\mathbb{R}-\{0\}. \tag{24}$$

We look at the matrix $BA$:

$$BA=\begin{bmatrix} a & b & c\\ r_1\overline{b} & e & -\overline{b}\\ r_1^2c & -r_1b & a\end{bmatrix}\begin{bmatrix} \lambda e^{i\theta} & 0 & 0\\ 0 & e^{i\phi} & 0\\ 0 & 0 & (1/\lambda)e^{i\theta}\end{bmatrix}=\begin{bmatrix} \lambda a e^{i\theta} & be^{i\phi} & ce^{i\theta}/\lambda\\ \lambda r_1\overline{b}e^{i\theta} & ee^{i\phi} & -\overline{b}e^{i\theta}/\lambda\\ \lambda r_1^2ce^{i\theta} & -r_1be^{i\phi} & ae^{i\theta}/\lambda\end{bmatrix}.$$

Since $e^{i\phi},e^{i\theta}\in\mathbb{R}$, the $(1,2)$–entry of $BA$ is neither a complex number nor of the form $q_*j$ for $q_*\in\mathbb{C}$. Hence $BA$ is of the same form as $B$ in (24). Then the modulus of the $(1,2)$–entry of $BA$ should equal to the modulus of the $(2,3)$–entry of $BA$. Hence, we have that

$$|be^{-2i\theta}|=\left|\frac{-\overline{b}e^{i\theta}}{\lambda}\right| \quad\text{and}\quad |b|=\frac{|b|}{\lambda},\quad \text{and so } \lambda=1.$$

However, this contradicts the assumption that $\lambda>1$. Therefore, the case that $c$ is purely imaginary and not a complex number can never happen.

## 3.2 The $bd = 0$ case

So far, we have looked at the case that there exists an element $B$ of $G$ with $bd \neq 0$. From now on, we consider the remaining case that every element of $G$ satisfies $bd = 0$. If $bd = 0$, by considering $B^{-1}$, we also have $fh = 0$. Then, using the identities (1)–(18), it can be easily checked that $b = d = f = h = 0$. For example, if $b = f = 0$, then by (6), $d = 0$ because $c \neq 0$. Then by (15), $h = 0$. Therefore, every element of $G$ is of the form

$$\begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & l \end{bmatrix}, \quad \text{where } a, e, l \in \mathbb{C}.$$

Applying Proposition 2.2 for $c$ and $g$, since $c, g \neq 0$, one of the following holds:

(i) $c,\ g \in \mathbb{C}$.

(ii) $c$ and $g$ are of the form $c = c_* j$ and $g = g_* j$ where $c_*, g_* \in \mathbb{C}$.

(iii) $g = r\bar{c}$ for some $r \in \mathbb{R} - \{0\}$.

First, if $c,\ g \in \mathbb{C}$, then $B \in \mathrm{U}(2,1)$. For any other element

$$B' = \begin{bmatrix} a' & 0 & c' \\ 0 & e' & 0 \\ g' & 0 & l' \end{bmatrix} \in G, \quad \text{where } a', e', l' \in \mathbb{C}, \tag{25}$$

we have $c', g' \in \mathbb{C}$ since $cg', gc' \in \mathbb{C}$ by Lemma 3.3. Hence $B' \in \mathrm{U}(2,1)$. This implies that $G$ is a subgroup of $\mathrm{U}(2,1)$.

Second, if $c$ and $g$ are of the form $c = c_* j$ and $g = g_* j$, where $c_*, g_* \in \mathbb{C}$, then for $z_1, z_2, z_3 \in \mathbb{C}$, we have

$$A \begin{bmatrix} z_1 j \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \lambda e^{i\theta} & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & (1/\lambda) e^{i\theta} \end{bmatrix} \begin{bmatrix} z_1 j \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \lambda e^{i\theta} z_1 j \\ e^{i\phi} z_2 \\ (1/\lambda) e^{i\theta} z_3 \end{bmatrix} = \begin{bmatrix} z'_1 j \\ z'_2 \\ z'_3 \end{bmatrix},$$

$$B \begin{bmatrix} z_1 j \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ g & 0 & l \end{bmatrix} \begin{bmatrix} z_1 j \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a z_1 j + c z_3 \\ e z_2 \\ g z_1 j + l z_3 \end{bmatrix} = \begin{bmatrix} z''_1 j \\ z''_2 \\ z''_3 \end{bmatrix},$$

for some $z'_1, z'_2, z'_3, z''_1, z''_2, z''_3 \in \mathbb{C}$. Hence, $A$ and $B$ leave invariant a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ of polar vectors $\begin{bmatrix} z_1 j & z_2 & z_3 \end{bmatrix}^t$, where $z_1, z_2, z_3 \in \mathbb{C}$.

For any other element $B' \in G$ of the form (25), since $cg', gc' \in \mathbb{C}$ by Lemma 3.3, $c'$ and $g'$ are also of the form $c' = c'_* j$, $g' = g'_* j$ for $c'_*,\ g'_* \in \mathbb{C}$. Therefore, every element of $G$ preserves a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ of polar vectors $\begin{bmatrix} z_1 j & z_2 & z_3 \end{bmatrix}^t$, where $z_1, z_2, z_3 \in \mathbb{C}$.

Lastly, in the case that $g = r\overline{c}$ for some $r \in \mathbb{R} - \{0\}$, we assume that $c \notin \mathbb{C}$ and that $c$ is not of the form $c = c_* j$ for $c_* \in \mathbb{C}$ to avoid repetition. From (1), we have that $a\overline{l} + rc^2 = 1$, and so $c^2 \in \mathbb{C}$. Then $c$ should be purely imaginary because $c \notin \mathbb{C}$. By (3), we have $\mathrm{Re}(ca) = 0$, so $a \in \mathbb{R}$. Then for $z_1, z_2, z_3 \in \mathbb{C}$,

$$B \begin{bmatrix} cz_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a & 0 & c \\ 0 & e & 0 \\ r\overline{c} & 0 & l \end{bmatrix} \begin{bmatrix} cz_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} c(az_1 + z_3) \\ ez_2 \\ r|c|^2 z_1 + lz_3 \end{bmatrix} = \begin{bmatrix} cz_1' \\ z_2' \\ z_3' \end{bmatrix}$$

for some $z_1', z_2', z_3' \in \mathbb{C}$.

**Claim** $$\theta \equiv 0 \pmod{\pi}.$$

**Proof of claim** The $(1,1)$–entry of $BAB$ is a complex number; ie

$$\lambda e^{i\theta} a^2 + \frac{rce^{i\theta}\overline{c}}{\lambda} \in \mathbb{C}.$$

Since $\lambda e^{i\theta} a^2 \in \mathbb{C}$, we have $ce^{i\theta}\overline{c} = |c|^2 \cos\theta - (cic)\sin\theta \in \mathbb{C}$. So if $\theta \not\equiv 0 \pmod{\pi}$, $cic \in \mathbb{C}$. Then by Lemma 2.3, either $c \in \mathbb{C}$ or $c = c_* j$ for $c_* \in \mathbb{C}$. This contradicts our assumption. Thus $\theta \equiv 0 \pmod{\pi}$. □

Due to the claim above, $A$ is written as

$$A = \begin{bmatrix} \pm\lambda & 0 & 0 \\ 0 & e^{i\phi} & 0 \\ 0 & 0 & \pm 1/\lambda \end{bmatrix}.$$

Then, for $z_1, z_2, z_3 \in \mathbb{C}$,

$$A \begin{bmatrix} cz_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} \pm c\lambda z_1 \\ e^{i\phi} z_2 \\ \pm(1/\lambda) z_3 \end{bmatrix} = \begin{bmatrix} cz_1' \\ z_2' \\ z_3' \end{bmatrix}$$

for some $z_1', z_2', z_3' \in \mathbb{C}$.

Thus $A$ and $B$ leave invariant a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ of polar vectors $\begin{bmatrix} cz_1 & z_2 & z_3 \end{bmatrix}^t$, where $z_1, z_2, z_3 \in \mathbb{C}$. For any other element $B' \in G$ of the form (25), by Lemma 3.3, $cg' \in \mathbb{C}$ and $g'c \in \mathbb{C}$. Since $c$ is purely imaginary, Proposition 2.2 implies that $g' = r'c$ for some $r' \in \mathbb{R} - \{0\}$ and $g'$ is purely imaginary. Since $c'g', g'c' \in \mathbb{C}$ by Corollary 3.4,

$c' = r''g'$ for some $r'' \in \mathbb{R} - \{0\}$ by Proposition 2.2. Also, by a similar argument as above, we have $a' \in \mathbb{R}$ using (3). Therefore, $B'$ is written as

$$B' = \begin{bmatrix} a' & 0 & r'r''c \\ 0 & e' & 0 \\ r'c & 0 & l' \end{bmatrix},$$

where $e', l' \in \mathbb{C}$, $a' \in \mathbb{R}$, $r', r'' \in \mathbb{R} - \{0\}$, and $c$ is purely imaginary. Then, for $z_1, z_2, z_3 \in \mathbb{C}$,

$$B' \begin{bmatrix} cz_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} a' & 0 & r'r''c \\ 0 & e' & 0 \\ r'c & 0 & l' \end{bmatrix} \begin{bmatrix} cz_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} c(a'z_1 + r'r''z_3) \\ e'z_2 \\ -r'|c|^2 z_1 + l'z_3 \end{bmatrix} = \begin{bmatrix} cz'_1 \\ z'_2 \\ z'_3 \end{bmatrix}$$

for some $z'_1, z'_2, z'_3 \in \mathbb{C}$. Therefore, every element of $G$ preserves a copy of $\boldsymbol{H}^2_{\mathbb{C}}$ of polar vectors $\begin{bmatrix} cz_1 & z_2 & z_3 \end{bmatrix}^t$, where $z_1, z_2, z_3 \in \mathbb{C}$. □

**Acknowledgements** The authors are indebted to the referee for some valuable comments and especially for suggesting a mild generalization of the main theorem in the first draft of the manuscript. J Kim thanks John Parker for useful discussions while staying in Hunan University and Yueping Jiang for the hospitality while staying in Changsha. This work was supported by the Project "PoINT" of Jeju National University in 2015.

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Received: 19 January 2017 Revised: 30 August 2017