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Relative 2–Segal spaces

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We introduce a relative version of the 2–Segal simplicial spaces defined by Dyckerhoff and Kapranov, and Gálvez-Carrillo, Kock and Tonks. Examples of relative 2–Segal spaces include the categorified unoriented cyclic nerve, real pseudoholomorphic polygons in almost complex manifolds and the $\mathcal{R}_\bullet$–construction from Grothendieck–Witt theory. We show that a relative 2–Segal space defines a categorical representation of the Hall algebra associated to the base 2–Segal space. In this way, after decategorification we recover a number of known constructions of Hall algebra representations. We also describe some higher categorical interpretations of relative 2–Segal spaces.

18G30; 18G55, 16G20, 19G38

Introduction

Motivated by Segal's notion [38] of a Γ–space, Rezk [32] introduced Segal spaces in his study of the homotopy theory of $(\infty, 1)$–categories. Generalizing these ideas, for each integer $k \geq 1$, Dyckerhoff and Kapranov [7] introduced k–Segal spaces. Very roughly, a simplicial topological space $X_\bullet$ is called k–Segal if it satisfies a collection of locality conditions governed by polyhedral subdivisions of k–dimensional cyclic polytopes. When $k = 1$, so that the locality conditions are governed by subdivisions of the interval, the 1–Segal conditions state that, for each $n \geq 2$, the canonical map to the homotopy fibre product

$$X_n \to \overbrace{X_1 \times^R_{X_0} \cdots \times^R_{X_0} X_1}^{n \text{ factors}}$$

is a weak homotopy equivalence. Hence 1–Segal spaces reduce to Rezk's Segal spaces. The 2–Segal spaces, which were introduced independently by Gálvez-Carrillo, Kock and Tonks [12] under the name decomposition spaces, obey locality conditions governed by subdivisions of convex plane polygons. The first nontrivial conditions derive from

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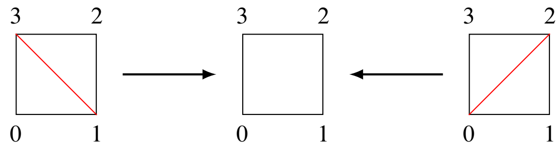
the two triangulations of the square



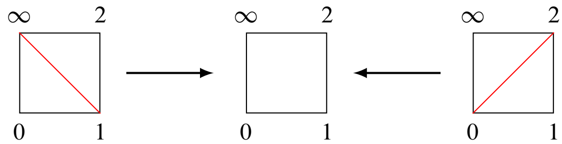
and state that the induced morphisms

$$X_{\{0,1,3\}} \times^R_{X_{\{1,3\}}} X_{\{1,2,3\}} \longleftarrow X_{\{0,1,2,3\}} \longrightarrow X_{\{0,1,2\}} \times^R_{X_{\{0,2\}}} X_{\{0,2,3\}} \tag{1}$$

are weak homotopy equivalences. A large number of examples of 2–Segal spaces from a diverse range of subjects are described in [7] and [12].

One motivation to study 2–Segal spaces is the theory of Hall algebras. Indeed, as exploited by both Dyckerhoff and Kapranov [7] and Gálvez-Carrillo, Kock and Tonks [12], the 2–Segal conditions admit a natural interpretation as higher coherence conditions on a multiplication defined on the 1–simplices X_1, the weak equivalences (1) imposing weak associativity. From the Hall algebra point of view, the most important example of a 2–Segal space is the Waldhausen $\mathcal{S}_\bullet$–construction [46] applied to an exact category, or more generally an exact ∞–category [7]. Applying suitable realization functors to this 2–Segal space recovers various familiar incarnations of the Hall algebra; see Ringel [33], Lusztig [26], Joyce [20], and Kontsevich and Soibelman [23]. However, these realizations use only the lowest 2–Segal conditions, namely (1). Taking into account the remaining conditions leads to higher categorical structures and thus to categorical Hall algebras. Applications of 2–Segal spaces to other areas, including combinatorics, topological field theories and Fukaya categories, were studied by Dyckerhoff and Kapranov [8; 9], and Gálvez-Carrillo, Kock and Tonks [12].

Partially motivated by the representation theory of Hall algebras, in this paper we introduce relative higher Segal spaces. For an integer $k \geq 1$, a relative k–Segal space over a k–Segal space $X_\bullet$ is a $(k-1)$–Segal space $Y_\bullet$ together with a morphism $Y_\bullet \to X_\bullet$ which satisfies k–dimensional locality conditions involving both $X_\bullet$ and $Y_\bullet$; by convention the 0–Segal conditions are vacuous. The simplest case is that of right relative 1–Segal spaces, the locality conditions reducing to the condition that the map $Y_\bullet \to X_\bullet$ be a right fibration of Segal spaces in the sense of Varshavskiĭ and Kazhdan [44], and de Brito [3]. More interesting is the case of relative 2–Segal spaces. The relative 2–Segal conditions on a morphism $Y_\bullet \to X_\bullet$ are governed by polyhedral subdivisions of convex plane polygons with a distinguished vertex ∞, the most basic of which are again the two triangulations of the square



and which translate into the requirement that the induced morphisms

$$Y_{\{0,1\}} \times^R_{Y_{\{1\}}} Y_{\{1,2\}} \longleftarrow Y_{\{0,1,2\}} \longrightarrow X_{\{0,1,2\}} \times^R_{X_{\{0,2\}}} Y_{\{0,2\}} \tag{2}$$

be weak equivalences. This combinatorial construction is described in Section 2.3. While the polyhedral subdivisions inducing the morphisms (1) and (2) are essentially the same, the vertex ∞ plays a crucial role in the latter, giving a rule to construct homotopy fibre products involving both $X_\bullet$ and $Y_\bullet$. Similar to the case of 2–Segal spaces, the relative 2–Segal conditions give higher coherence conditions for appropriately defined left and right actions of the algebra object X_1 on the 0–simplices Y_0. From this point of view, the weak equivalences (2) are the weak module-associativity constraints. Relative 2–Segal spaces therefore lead naturally to categorical representations of the Hall algebra of $X_\bullet$; see Theorems 4.4 and 4.5 for particular instances of this construction. In this way we obtain natural categorifications of many of the Hall algebra representations which have appeared in the literature. For example, we prove that a stable framed variant of the Waldhausen $\mathcal{S}_\bullet$–construction is relative 2–Segal over the ordinary $\mathcal{S}_\bullet$–construction, thus categorifying the Hall algebra representations studied by Soibelman [41] and Franzen [11]; see Theorem 2.10. We also prove that the $\mathcal{R}_\bullet$–construction from Grothendieck–Witt theory (ie higher algebraic KR–theory), as described by Shapiro and Yao [39] and Hornbostel and Schlichting [15], is relative 2–Segal over the $\mathcal{S}_\bullet$–construction; see Theorem 3.10. The input for the $\mathcal{R}_\bullet$–construction is a proto-exact category with duality which satisfies a reduction assumption. In the case of exact categories, the $\mathcal{R}_\bullet$–construction categorifies the Hall algebra representations of van Leeuwen [24], Enomoto [10], and Young [47; 48], while for the proto-exact category $\mathsf{Rep}_{\mathbb{F}_1}(Q)$ of representations of a quiver over $\mathbb{F}_1$, we obtain new modules over Szczesny's combinatorial Hall algebras [42]. The latter modules will be the subject of future work.

We also give examples of relative 2–Segal spaces which do not come from previously known Hall algebra representations. Starting from an almost complex manifold M with a real structure, we construct in Theorem 2.11 a relative 2–Segal semisimplicial set consisting of real pseudoholomorphic polygons in M, the base 2–Segal set being the pseudoholomorphic polygon space of Dyckerhoff and Kapranov [7]. In Theorem 3.8, we prove that the categorified unoriented twisted cyclic nerve of a category with endomorphism and compatible duality structure is relative 2–Segal over the categorified twisted cyclic nerve. This example can be viewed as a homotopical incarnation of the unoriented loop space of an orbifold.

A common theme of many of the relative 2–Segal spaces constructed in this paper is that they are in a sense unoriented. It is tempting to view these examples in the context of orientifold string theory. (The stable framed $\mathcal{S}_\bullet$–construction is different, being related to string theory with defects.) A general feature of the orientifold construction is that it imposes $\mathbb{Z}_2$–equivariance conditions on objects in the parent string theory, such as reduction of structure groups of vector bundles from general linear to orthogonal or symplectic groups, as in the $\mathcal{R}_\bullet$–construction, or replacing oriented string worldsheets with unoriented worldsheets, similar to the real pseudoholomorphic polygon and unoriented nerve constructions. In [7, Remark 3.7.8] it is speculated that there exists a sort of mirror symmetry relating the 2–Segal spaces arising from the $\mathcal{S}_\bullet$–construction with those arising from the pseudoholomorphic polygon construction. It is natural to speculate that such a mirror symmetry admits an orientifold enhancement, relating the relative 2–Segal spaces arising from the $\mathcal{R}_\bullet$– and real pseudoholomorphic polygon constructions.

Finally, we describe some applications of relative higher Segal spaces to higher category theory, thus lifting some of the results of Rezk [32], Joyal and Tierney [19], and Dyckerhoff and Kapranov [7]. It is a classical fact that 1–Segal simplicial sets can be characterized as the essential image of the fully faithful nerve functor $N_\bullet\colon \mathsf{Cat} \to \mathsf{Set}_\Delta$. In a similar vein, right relative 1–Segal simplicial sets are the essential image of the relative nerve construction applied to the category of discrete right fibrations which, via the Grothendieck construction, can be interpreted as presheaves on small categories; see Proposition 2.3. Using the work of several authors (see Joyal [17], Lurie [25], and de Brito [3]), we explain a quasicategorical generalization of these statements by considering instead right relative 1–Segal combinatorial simplicial spaces and $(\infty,1)$–presheaves. Secondly, in [7] it is proved that the category of 2–Segal simplicial sets is equivalent to both the category of multivalued categories and to the category of $\sqcup$–semisimple semibicategories. Pursuing an interpretation in terms of actions of categories as in the relative 1–Segal case, we lift these statements to the relative setting, establishing equivalences of the category of relative 2–Segal simplicial sets with both the category of modules over multivalued categories and with the category of $\mathsf{Cat}^{\sqcup}$–valued presheaves on $\sqcup$–semisimple semibicategories; see Theorems 4.8 and 4.13, respectively.

Remark After the first version of this paper was completed, a preprint by Tashi Walde [45] was posted to the arXiv which also aims at developing a theory of modules over higher Segal spaces. We comment where appropriate on the overlap.

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1 Higher Segal spaces

In this section we recall, closely following [7], some required background material from the theory of higher Segal spaces.

1.1 Simplicial objects

Let Δ be the category whose objects are the nonempty finite ordinals $[n]=\{0<\cdots<n\}$, $n\geq 0$, and whose morphisms are weakly monotone set maps. Let also Δ_{aug} be the category of all finite nonempty ordinals, of which Δ is a skeleton. Denote by $\Delta_{\mathsf{inj}}\subset\Delta$ the subcategory of injective morphisms. We sometimes consider the object $[n]\in\Delta$ as a category itself. Explicitly, the objects of $[n]$ are labelled by integers $0\leq i\leq n$ and the morphism set $\mathrm{Hom}_{[n]}(i,j)$ is empty if $i>j$ and consists of a single element if $i\leq j$.

A simplicial object of a category $\mathcal{C}$ is a functor $X_\bullet\colon\Delta^{\mathsf{op}}\to\mathcal{C}$. We write X_n for $X_{[n]}\in\mathcal{C}$ if it will not lead to confusion. The face and degeneracy maps of $X_\bullet$ are denoted by

$$\partial_i\colon X_n\to X_{n-1},\quad s_i\colon X_n\to X_{n+1}\quad\text{for } 0\leq i\leq n.$$

More generally, a functor $X_\bullet\colon\Delta_{\mathsf{inj}}^{\mathsf{op}}\to\mathcal{C}$ is a semisimplicial object of $\mathcal{C}$. A simplicial object $X_\bullet$ admits a canonical extension to a functor $\Delta_{\mathsf{aug}}^{\mathsf{op}}\to\mathcal{C}$, which we continue to denote by $X_\bullet$. For $I\in\Delta_{\mathsf{aug}}$, we write Δ^I for the simplicial set $\mathrm{Hom}_\Delta(-,I)$. In particular, $\Delta^n=\Delta^{[n]}$.

Given categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathcal{C}$ small, denote by $[\mathcal{C},\mathcal{D}]$ or $\mathcal{D}^{\mathcal{C}}$ the category of functors $\mathcal{C}\to\mathcal{D}$. Let Set, Grpd and Top be the categories of sets, small groupoids and compactly generated topological spaces, respectively. Objects of the categories $[\Delta^{\mathsf{op}},\mathsf{Top}]$ and $[\Delta^{\mathsf{op}},\mathsf{Grpd}]$ are called simplicial spaces and groupoids while objects of $\mathbb{S}=[\Delta^{\mathsf{op}},\mathsf{Set}]$ and $\mathbb{S}_\Delta=[\Delta^{\mathsf{op}},\mathbb{S}]$ are called simplicial sets and combinatorial simplicial spaces, respectively.

1.2 1–Segal spaces

Segal spaces (called 1–Segal spaces below) were introduced by Rezk [32, Section 4]; see also [38, Section 1]. The definition below is slightly different, omitting a fibrancy condition. Write $- \times^R_U -$ for the homotopy fibre product over a topological space U.

Definition 1.1 A semisimplicial space $X_\bullet\colon \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathsf{Top}$ is called 1–Segal if for every $n \geq 2$ the map

$$X_n \to X_1 \times^R_{X_0} \cdots \times^R_{X_0} X_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$, $i = 0, \dots, n-1$, is a weak homotopy equivalence.

It is straightforward to verify that a semisimplicial space $X_\bullet$ is 1–Segal if and only if one of the following two conditions hold:

(i) For every $n \geq 2$ and every $0 \leq i_1 < \cdots < i_l \leq n$ the map

$$X_n \to X_{i_1} \times^R_{X_{\{i_1\}}} \cdots \times^R_{X_{\{i_l\}}} X_{n-i_l}$$

induced by the inclusions $\{0, \dots, i_1\}, \dots, \{i_l, \dots, n\} \hookrightarrow [n]$ is a weak equivalence.

(ii) For every $n \geq 2$ and every $0 \leq i \leq n$ the map

$$X_n \to X_{\{0,\dots,i\}} \times^R_{X_{\{i\}}} X_{\{i,\dots,n\}}$$

induced by the inclusions $\{0, \dots, i\}, \{i, \dots, n\} \hookrightarrow [n]$ is a weak equivalence.

With only minor changes one can formulate the theory of 1–Segal objects (along with their higher and relative variants defined below) of a combinatorial model category, in which case $- \times^R -$ becomes a homotopy limit; see [7, Section 5]. In particular, we can speak of 1–Segal simplicial sets, groupoids or combinatorial simplicial spaces. For simplicity we will state results in terms of simplicial spaces.

Example The nerve $N_\bullet(\mathcal{C})$ of a small category $\mathcal{C}$ is the simplicial set which assigns to $[n] \in \Delta$ the set underlying the category $\mathcal{C}^{[n]}$. It is well known that $N_\bullet(\mathcal{C})$ is a 1–Segal simplicial set. In fact, the nerve functor $N_\bullet\colon \mathsf{Cat} \to \mathbb{S}$ is fully faithful with essential image the 1–Segal simplicial sets. The category $\mathcal{X}$ associated to a 1–Segal simplicial set $X_\bullet$ has objects $\mathrm{Ob}(\mathcal{X}) = X_0$ and morphisms

$$\mathrm{Hom}_{\mathcal{X}}(x_0, x_1) = \{x_0\} \times_{X_{\{0\}}} X_{\{0,1\}} \times_{X_{\{1\}}} \{x_1\}.$$

Composition of morphisms is defined using the lowest 1–Segal conditions while associativity follows from the higher 1–Segal conditions.

Example The categorified nerve $\mathcal{N}_\bullet(\mathcal{C})$ of a small category $\mathcal{C}$ is the 1–Segal simplicial groupoid which assigns to $[n] \in \Delta$ the maximal groupoid of the category $\mathcal{C}^{[n]}$ [32, Section 3.5]. Passing to classifying spaces gives a 1–Segal simplicial space $B\mathcal{N}_\bullet(\mathcal{C})$. In Rezk's framework the categorified nerve is preferred to the ordinary nerve as the former is a complete 1–Segal space.

1.3 2–Segal spaces

The 1–Segal spaces are the first in an infinite tower of higher Segal spaces introduced in [7]. In this section we focus on 2–Segal spaces, the next step in this tower. See [12] for a second approach to (unital) 2–Segal spaces.

For each integer $n \geq 2$ let $P_n \subset \mathbb{R}^2$ be a convex $(n+1)$–gon with a total order on its vertices which is consistent with the counterclockwise orientation of $\mathbb{R}^2$. The total order induces a canonical identification of the set of vertices of P_n with $[n]$. Let $\mathcal{P}$ be a polyhedral subdivision of P_n. Associating to each polygon of $\mathcal{P}$ its set of vertices defines a collection of subsets of $[n]$ and hence a simplicial subset $\Delta^{\mathcal{P}} \subset \Delta^n$. Let $X_\bullet$ be a semisimplicial space. The polyhedral subdivision $\mathcal{P}$ induces a map

$$f_{\mathcal{P}}\colon X_n \simeq (\Delta^n, X_\bullet)_R \to (\Delta^{\mathcal{P}}, X_\bullet)_R,$$

where, following [7, Section 2.2], for a semisimplicial set D the derived space of D–membranes of $X_\bullet$ is defined to be

$$(D, X_\bullet)_R = \operatorname{holim}^{\mathsf{Top}}_{\overleftarrow{\{\Delta^p \hookrightarrow D\} \in \Delta_{\mathrm{inj}}/D}} X_p.$$

Definition 1.2 A semisimplicial space $X_\bullet$ is called 2–Segal if for every $n \geq 3$ and every triangulation $\mathcal{T}$ of P_n the map $f_{\mathcal{T}}\colon X_n \to (\Delta^{\mathcal{T}}, X_\bullet)_R$ is a weak equivalence.

As in the case of 1–Segal spaces, the 2–Segal conditions can be verified using coarser subdivisions. Indeed, it is proved in [7, Proposition 2.3.2] that a semisimplicial space $X_\bullet$ is 2–Segal if and only if one of the following conditions holds:

(i) For every $n \geq 3$ and every polyhedral subdivision $\mathcal{P}$ of P_n the map $f_{\mathcal{P}}\colon X_n \to (\Delta^{\mathcal{P}}, X_\bullet)_R$ is a weak equivalence.

(ii) For every $n \geq 3$ and every $0 \leq i < j \leq n$ the map

$$f_{\{i,j\}}\colon X_n \to X_{\{i,\dots,j\}} \times^R_{X_{\{i,j\}}} X_{\{0,\dots,i,j,\dots,n\}} \tag{3}$$

induced by the inclusions $\{i,\dots,j\}, \{0,\dots,i,j,\dots,n\} \hookrightarrow [n]$ is a weak equivalence.

(iii) For every $n \geq 3$ the map (3) is a weak equivalence if $i = 0$ or $j = n$.

The following definition uses degeneracy maps and so can only be formulated in the simplicial setting.

Definition 1.3 A 2–Segal simplicial space $X_\bullet$ is called unital 2–Segal if for every $n \geq 2$ and every $0 \leq i \leq n-1$ the map

$$\partial_{\{i\}} \times s_i \colon X_{n-1} \to X_{\{i\}} \times^R_{X_{\{i,i+1\}}} X_n$$

is a weak equivalence.

One simple construction of 2–Segal spaces is the following.

Proposition 1.4 [7, Propositions 2.3.3, 2.5.3; 12, Proposition 3.5] *Let $X_\bullet$ be a 1–Segal semisimplicial space. Then $X_\bullet$ is 2–Segal. If in fact $X_\bullet$ is a simplicial space, then $X_\bullet$ is unital 2–Segal.*

1.4 The Waldhausen $\mathcal{S}_\bullet$-construction

We recall a motivating example of a unital 2–Segal space. We will work with proto-exact categories, a not necessarily additive generalization of exact categories in the sense of Quillen [31].

Definition 1.5 [7, Section 2.4] A proto-exact category is a pointed category $\mathcal{C}$, with zero object 0, together with two classes of morphisms, $\mathfrak{I}$ and $\mathfrak{D}$, called inflations and deflations and denoted by $\rightarrowtail$ and $\twoheadrightarrow$, respectively, which have the following properties:

- (i) Any morphism $0 \to U$ is in $\mathfrak{I}$ and any morphism $U \to 0$ is in $\mathfrak{D}$.
- (ii) The classes $\mathfrak{I}$ and $\mathfrak{D}$ are closed under composition and contain all isomorphisms.
- (iii) A commutative square of the form

$$\begin{array}{ccc} U & \rightarrowtail & V \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ W & \rightarrowtail & X \end{array} \tag{4}$$

  is cartesian if and only if it is cocartesian.
- (iv) Any diagram $W \rightarrowtail X \twoheadleftarrow V$ can be completed to a bicartesian diagram of the form (4).
- (v) Any diagram $W \twoheadleftarrow U \rightarrowtail V$ can be completed to a bicartesian diagram of the form (4).

Bicartesian squares of the form

$$\begin{array}{ccc} U & \rightarrowtail & V \\ \big\downarrow & & \big\downarrow \\ 0 & \rightarrowtail & X \end{array}$$

are called conflations and play the role of short exact sequences in $\mathcal{C}$. Familiar examples of proto-exact categories include abelian and, more generally, exact categories. A more exotic example is given by the category of representations of a quiver over $\mathbb{F}_1$, as described in [42].

The Waldhausen $\mathcal{S}_\bullet$–construction associates to a proto-exact category $\mathcal{C}$ a simplicial groupoid $\mathcal{S}_\bullet(\mathcal{C})$ as follows [46, Section 1.3; 7, Section 2.4]. Let $\mathsf{Ar}_n = [[1],[n]]$ be the arrow category of $[n]$. The assignment $[n] \mapsto \mathsf{Ar}_n$ defines a cosimplicial category. An object $\{(i \to j) \mapsto A_{\{i,j\}}\}_{0 \le i \le j \le n}$ of the functor category $[\mathsf{Ar}_n, \mathcal{C}]$ is a commutative diagram in $\mathcal{C}$ of the following form:

$$\begin{array}{ccccccccc}
A_{\{0,0\}} & \longrightarrow & A_{\{0,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{0,n-1\}} & \longrightarrow & A_{\{0,n\}} \\
 & & \big\downarrow & & & & \big\downarrow & & \big\downarrow \\
 & & A_{\{1,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{1,n-1\}} & \longrightarrow & A_{\{1,n\}} \\
 & & & & & & \big\downarrow & & \big\downarrow \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & & \big\downarrow & & \big\downarrow \\
 & & & & & & A_{\{n-1,n-1\}} & \longrightarrow & A_{\{n-1,n\}} \\
 & & & & & & & & \big\downarrow \\
 & & & & & & & & A_{\{n,n\}}
\end{array}$$

Let $\mathcal{W}_n(\mathcal{C}) \subset [\mathsf{Ar}_n, \mathcal{C}]$ be the full subcategory consisting of diagrams which have the following properties:

(i) For each $0 \le i \le n$ the object $A_{\{i,i\}}$ is isomorphic to $0 \in \mathcal{C}$.

(ii) All horizontal morphisms are inflations and all vertical morphisms are deflations.

(iii) Each square that can be formed in the diagram is bicartesian.

Let $\mathcal{S}_n(\mathcal{C})$ be the maximal groupoid of $\mathcal{W}_n(\mathcal{C})$. Then $\mathcal{S}_\bullet(\mathcal{C})$ is a simplicial groupoid, the degeneracy map $s_i \colon \mathcal{S}_n(\mathcal{C}) \to \mathcal{S}_{n+1}(\mathcal{C})$ inserting a row/column of identity morphisms after the i^{th} row/column and the face map $\partial_i \colon \mathcal{S}_n(\mathcal{C}) \to \mathcal{S}_{n-1}(\mathcal{C})$ deleting the i^{th} row/column and composing the obvious morphisms.

Theorem 1.6 [7, Proposition 2.4.8; 12, Theorem 10.14] *For any proto-exact category $\mathcal{C}$, the simplicial groupoid $\mathcal{S}_\bullet(\mathcal{C})$ is unital 2–Segal.*

When $\mathcal{C}$ is an exact category the simplicial space $B\mathcal{S}_\bullet(\mathcal{C})$ plays a fundamental role in the higher algebraic K–theory of $\mathcal{C}$. Indeed, we have $K_i(\mathcal{C}) = \pi_i \Omega |B\mathcal{S}_\bullet(\mathcal{C})|$, $i \geq 0$, where the basepoint of $|B\mathcal{S}_\bullet(\mathcal{C})|$ is taken to be $0 \in \mathcal{C}$. See [43; 46].

Remark A variation of the Waldhausen $\mathcal{S}_\bullet$–construction was defined in [1], giving a functor from the category of augmented stable double categories to the category of simplicial sets. It was proved that this functor is fully faithful with essential image the unital 2–Segal simplicial sets.

2 Relative higher Segal spaces

2.1 Relative 1–Segal spaces

Before introducing and studying relative 2–Segal spaces, which will be the main objects of interest in this paper, we study the more basic relative 1–Segal spaces.

Definition 2.1 Let $X_\bullet$ be a 1–Segal semisimplicial space. A morphism $F_\bullet\colon Y_\bullet \to X_\bullet$ of semisimplicial spaces is called right relative 1–Segal if for every $n \geq 1$ and every $0 \leq i \leq n$ the outside square of the diagram

(5)
$$\begin{array}{ccc} Y_n & \longrightarrow & Y_{\{i,\dots,n\}} \\ \downarrow & & \downarrow \\ Y_{\{0,\dots,i\}} & \longrightarrow & Y_{\{i\}} \\ {\scriptstyle F_{\{0,\dots,i\}}}\downarrow & & \downarrow{\scriptstyle F_{\{i\}}} \\ X_{\{0,\dots,i\}} & \longrightarrow & X_{\{i\}} \end{array}$$

is homotopy cartesian.

Similarly, a left relative 1–Segal space is a morphism $F_\bullet\colon Y_\bullet \to X_\bullet$, with $X_\bullet$ being 1–Segal, for which the outside square of all diagrams of the form

$$\begin{array}{ccccc} Y_n & \longrightarrow & Y_{\{i,\dots,n\}} & \xrightarrow{F_{\{i,\dots,n\}}} & X_{\{i,\dots,n\}} \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\{0,\dots,i\}} & \longrightarrow & Y_{\{i\}} & \xrightarrow[F_{\{i\}}]{} & X_{\{i\}} \end{array}$$

is homotopy cartesian. All results below will be formulated for right relative 1–Segal spaces; analogous results hold for left relative 1–Segal spaces.

Example Let $X_\bullet$ be a 1–Segal semisimplicial space. Then the identity morphism $\mathbf{1}_{X_\bullet}\colon X_\bullet \to X_\bullet$ is both left and right relative 1–Segal.

Example Let x_* be an object of a small category $\mathcal{C}$ and let $\mathcal{C}_{/x_*}$ be the corresponding overcategory. The forgetful functor $\mathcal{C}_{/x_*} \to \mathcal{C}$ induces a simplicial morphism $N_\bullet(\mathcal{C}_{/x_*}) \to N_\bullet(\mathcal{C})$ which is right relative 1–Segal. Using instead the undercategory ${}_{x_*/}\mathcal{C}$ we obtain a left relative 1–Segal simplicial set $N_\bullet({}_{x_*/}\mathcal{C}) \to N_\bullet(\mathcal{C})$.

Example Suppose that a group G acts on a set E. Then G acts diagonally on the cartesian product E^n, $n \geq 1$. The action groupoid $\mathsf{G}\mathbin{\backslash\!\!\backslash} E^{n+1}$ is the category with objects E^{n+1} and morphisms $\operatorname{Hom}_{\mathsf{G}\backslash\!\backslash E^{n+1}}(e_\bullet, e'_\bullet) = \{g \in \mathsf{G} \mid g \cdot e_\bullet = e'_\bullet\}$. The assignment $[n] \mapsto \mathsf{G}\mathbin{\backslash\!\!\backslash} E^{n+1}$ defines a 1–Segal simplicial groupoid $\mathcal{S}_\bullet(\mathsf{G}, E)$, the face (resp. degeneracy) maps omitting (resp. repeating) the appropriate entries of $E^{\bullet+1}$. Passing to classifying spaces yields a 1–Segal space $B\mathcal{S}_\bullet(\mathsf{G}, E)$, called the Hecke–Waldhausen space of (G, E) [7, Section 2.6].

Let $\mathsf{H} \leq \mathsf{G}$ be a subgroup. The inclusion $\mathsf{H} \hookrightarrow \mathsf{G}$ defines a simplicial morphism $\mathcal{S}_\bullet(\mathsf{H}, E) \to \mathcal{S}_\bullet(\mathsf{G}, E)$. At the level of classifying spaces we obtain a morphism $B\mathcal{S}_\bullet(\mathsf{H}, E) \to B\mathcal{S}_\bullet(\mathsf{G}, E)$ which is both left and right relative 1–Segal; this can be verified in much the same way as the 1–Segal property of $B\mathcal{S}_\bullet(\mathsf{G}, E)$. At the level of geometric realizations this map is homotopy equivalent to the induced morphism $B\mathsf{H} \to B\mathsf{G}$ of classifying spaces; see [7, Proposition 2.6.7].

We have the following alternative characterization of right relative 1–Segal spaces.

Proposition 2.2 *A semisimplicial morphism $F_\bullet\colon Y_\bullet \to X_\bullet$, with $X_\bullet$ being 1–Segal, is right relative 1–Segal if and only if $Y_\bullet$ is 1–Segal and the map*

$$(F_1, \partial_0)\colon Y_1 \to X_1 \times^R_{X_0} Y_0 \tag{6}$$

is a weak equivalence.

Proof Suppose that $F_\bullet$ is right relative 1–Segal. Taking $i = n = 1$ in diagram (5) implies that the map (6) is a weak equivalence. For arbitrary i and n, both the outside square and the bottom square (which is a degenerate version of the outside square) of diagram (5) are homotopy cartesian. By the 2-out-of-3 property of weak equivalences the top square is therefore also homotopy cartesian. Hence $Y_\bullet$ is 1–Segal.

Conversely, suppose that $Y_\bullet$ is 1–Segal and that the map (6) is a weak equivalence. Then we have the following sequence of weak equivalences:

$$\begin{aligned} Y_n &\xrightarrow{\text{w.e.}} Y_{\{0,\dots,i\}} \times^R_{Y_{\{i\}}} Y_{\{i,\dots,n\}} \\ &\xrightarrow{\text{w.e.}} Y_{\{0,1\}} \times^R_{Y_{\{1\}}} Y_{\{1,\dots,i\}} \times^R_{Y_{\{i\}}} Y_{\{i,\dots,n\}} \\ &\xrightarrow{\text{w.e.}} X_{\{0,1\}} \times^R_{X_{\{1\}}} Y_{\{1\}} \times^R_{Y_{\{1\}}} Y_{\{1,\dots,i\}} \times^R_{Y_{\{i\}}} Y_{\{i,\dots,n\}} \\ &\;\vdots \\ &\xrightarrow{\text{w.e.}} X_{\{0,1\}} \times^R_{X_{\{1\}}} \cdots \times^R_{X_{\{i-1\}}} X_{\{i-1,i\}} \times^R_{X_{\{i\}}} Y_{\{i,\dots,n\}} \\ &\xleftarrow{\text{w.e.}} X_{\{0,\dots,i\}} \times^R_{X_{\{i\}}} Y_{\{i,\dots,n\}}. \end{aligned}$$

The map $Y_n \to X_{\{0,\dots,i\}} \times^R_{X_{\{i\}}} Y_{\{i,\dots,n\}}$ makes this chain of maps commute and is thus a weak equivalence. Hence $F_\bullet$ is right relative 1–Segal. □

Kazhdan and Varshavsky [44] and de Brito [3] define a right Segal fibration to be a morphism of 1–Segal spaces $F_\bullet\colon Y_\bullet \to X_\bullet$ for which the map (6) is a weak equivalence. It follows from Proposition 2.2 that right relative 1–Segal spaces and right Segal fibrations are the same objects. Using [3, Proposition 1.10], we obtain a model-category-theoretic interpretation of the relative 1–Segal conditions. Namely, right relative 1–Segal objects in $\mathbb{S}_\Delta$ are the fibrant objects of a natural left Bousfield localization of $(\mathcal{S}\mathrm{eg}_1)_{/X_\bullet}$, the overcategory model structure on $(\mathbb{S}_\Delta)_{/X_\bullet}$ induced by Rezk's 1–Segal model structure $\mathcal{S}\mathrm{eg}_1$ on $\mathbb{S}_\Delta$ [32, Theorem 7.1].

Right relative 1–Segal simplicial sets admit a simple nerve-theoretic characterization, analogous to that of 1–Segal simplicial sets; in Section 4.3 we will prove a similar result in the 2–Segal setting. To formulate this, let $\mathsf{DRFib} \subset \mathsf{Cat}^{[1]}$ be the full subcategory of discrete right fibrations and let $1\text{-}\mathsf{SegRel}\mathbb{S} \subset \mathbb{S}^{[1]}$ be the full subcategory of right relative 1–Segal simplicial sets. Let also Psh be the category whose objects are presheaves on small categories and whose morphisms are pairs

$$(\phi,\theta)\colon (\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}) \to (\mathcal{F}'\colon \mathcal{C}'^{\mathrm{op}} \to \mathsf{Set}) \tag{7}$$

consisting of a functor $\phi\colon \mathcal{C} \to \mathcal{C}'$ and a natural transformation $\theta\colon \mathcal{F} \Rightarrow \mathcal{F}' \circ \phi^{\mathrm{op}}$. The Grothendieck construction defines a functor $\int\colon \mathsf{Psh} \to \mathsf{Cat}^{[1]}$, assigning to the presheaf $\mathcal{F}$ the functor $F\colon \int_{\mathcal{C}} \mathcal{F} \to \mathcal{C}$ and assigning to a morphism (7) the following diagram:

$$\begin{array}{ccc} \int_{\mathcal{C}} \mathcal{F} & \xrightarrow{\widetilde{\theta}} & \int_{\mathcal{C}'} \mathcal{F}' \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F'} \\ \mathcal{C} & \xrightarrow[\phi]{} & \mathcal{C}' \end{array}$$

Recall that $\int_{\mathcal{C}} \mathcal{F}$ is the category whose objects are pairs $(c, \widetilde{c})$, with $c \in \mathcal{C}$ and $\widetilde{c} \in \mathcal{F}(c)$, and whose morphisms $(c_1, \widetilde{c}_1) \to (c_2, \widetilde{c}_2)$ are morphisms $c_1 \xrightarrow{x} c_2$ which satisfy $\mathcal{F}(x)(\widetilde{c}_2) = \widetilde{c}_1$. The functor $\widetilde{\theta}$ assigns to $(c, \widetilde{c}) \in \int_{\mathcal{C}} \mathcal{F}$ the object $(\phi(c), \theta_c(\widetilde{c}))$ and assigns to $x\colon (c_1, \widetilde{c}_1) \to (c_2, \widetilde{c}_2)$ the morphism $\phi(x)$.

Proposition 2.3 *The relative nerve functor*

$$N_\bullet^{[1]}\colon \mathsf{Cat}^{[1]} \to \mathbb{S}^{[1]}, \quad (\mathcal{Y} \xrightarrow{F} \mathcal{X}) \mapsto \big(N_\bullet(\mathcal{Y}) \xrightarrow{N_\bullet(F)} N_\bullet(\mathcal{X})\big)$$

is fully faithful and fits into the commutative diagram of functors

$$\begin{array}{ccccc} \mathsf{Psh} & \xrightarrow{\int} & \mathsf{Cat}^{[1]} & \xrightarrow{N_\bullet^{[1]}} & \mathbb{S}^{[1]} \\ & \searrow^{\sim} & \uparrow & & \uparrow \\ & & \mathsf{DRFib} & \xrightarrow[\sim]{} & \text{1-}\mathsf{SegRel}\mathbb{S} \end{array}$$

with indicated equivalences. In particular, there is an equivalence of categories $\text{1-}\mathsf{SegRel}\mathbb{S} \simeq \mathsf{Psh}$.

Proof That $N_\bullet^{[1]}$ is fully faithful is well known; see for example [37, Proposition 2.1]. Commutativity of the triangle in the above diagram and the fact that $\int$ induces an equivalence $\mathsf{Psh} \simeq \mathsf{DRFib}$ are both standard. To see that $N_\bullet^{[1]}$ restricts as claimed, let $(F\colon \mathcal{Y} \to \mathcal{X}) \in \mathsf{DRFib}$. Then $N_\bullet(F)\colon N_\bullet(\mathcal{Y}) \to N_\bullet(\mathcal{X})$ is a morphism of 1–Segal simplicial sets. The condition that F be a discrete right fibration is precisely the condition that the map $N_1(\mathcal{Y}) \to N_1(\mathcal{X}) \times_{N_{\{1\}}(\mathcal{X})} N_{\{1\}}(\mathcal{Y})$ be a bijection. Proposition 2.2 therefore implies that $N_\bullet(F)$ is right relative 1–Segal. The construction of a quasi-inverse $\text{1-}\mathsf{SegRel}\mathbb{S} \to \mathsf{DRFib}$ is similar. □

To end this section we explain a quasicategorical generalization of Proposition 2.3. Suppose that $X_\bullet \in \mathbb{S}_\Delta$ is a complete 1–Segal combinatorial simplicial space. The quasicategory $\mathcal{X}$ modelled by $X_\bullet$ (see [32, Section 5; 19, Section 4]) has object set the 0–simplices of X_0 and has mapping spaces

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) = \{x_0\} \times^R_{X_{\{0\}}} X_{\{0,1\}} \times^R_{X_{\{1\}}} \{x_1\}.$$

Here $\{x\}$ is regarded as the simplicial set Δ^0. The lowest 1–Segal conditions define, up to homotopy, a composition law

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) \times \mathrm{map}_{\mathcal{X}}(x_1, x_2) \to \mathrm{map}_{\mathcal{X}}(x_0, x_2)$$

which, by the remaining 1–Segal conditions, is coherently associative. Suppose now that we are given a right relative 1–Segal morphism $F_\bullet\colon Y_\bullet \to X_\bullet$. For each object $x \in \mathcal{X}$ define a Kan complex by $\mathcal{F}(x) = \{x\} \times^R_{X_0} Y_0 \in \mathbb{S}$. The diagram

$$\begin{array}{ccc}
\{x_0\} \times^R_{X_{\{0\}}} Y_{\{0,1\}} & \longrightarrow & \{x_0\} \times^R_{X_{\{0\}}} Y_{\{0\}} \\
\big\downarrow {\scriptstyle \text{w.e.}} & & \\
\{x_0\} \times^R_{X_{\{0\}}} X_{\{0,1\}} \times^R_{X_{\{1\}}} Y_{\{1\}} & & \\
\big\uparrow & & \\
\big(\{x_0\} \times^R_{X_{\{0\}}} X_{\{0,1\}} \times^R_{X_{\{1\}}} \{x_1\}\big) \times \big(\{x_1\} \times^R_{X_{\{1\}}} Y_{\{1\}}\big) & &
\end{array}$$

whose indicated arrow is a weak equivalence by the lowest right relative 1–Segal condition, defines up to homotopy an action map

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) \times \mathcal{F}(x_1) \to \mathcal{F}(x_0).$$

The remaining right relative 1–Segal conditions ensure that this action is coherently associative. In this way, we obtain an $(\infty, 1)$–presheaf on $\mathcal{X}$. Conversely, by combining [25, Proposition 5.1.1.1] (see also [17]) and [3, Theorem 1.22], we see that, up to weak equivalence, any $(\infty, 1)$–presheaf on $\mathcal{X}$ arises in this way.

2.2 Relative 2–Segal spaces

In this section we give a direct definition of relative 2–Segal spaces. In Section 2.3 we will describe a second approach using polyhedral subdivisions, in line with the definition of 2–Segal spaces.

Definition 2.4 Let $X_\bullet$ be a 2–Segal semisimplicial space. A morphism $F_\bullet\colon Y_\bullet \to X_\bullet$ of semisimplicial spaces is called relative 2–Segal if

(1) for every $n \geq 2$ and every $0 \leq i < j \leq n$ the outside square of the diagram

$$\begin{array}{ccc} Y_n & \longrightarrow & Y_{\{0,\dots,i,j,\dots,n\}} \\ \downarrow & & \downarrow \\ Y_{\{i,\dots,j\}} & \longrightarrow & Y_{\{i,j\}} \\ {\scriptstyle F_{\{i,\dots,j\}}}\downarrow & & \downarrow{\scriptstyle F_{\{i,j\}}} \\ X_{\{i,\dots,j\}} & \longrightarrow & X_{\{i,j\}} \end{array} \tag{8}$$

is homotopy cartesian, and

(2) the simplicial space $Y_\bullet$ is 1–Segal.

Remark (1) Morphisms $F_\bullet\colon Y_\bullet \to X_\bullet$ of simplicial spaces for which the first condition above holds play an important role in [12], where they are called unique lifting of factorization (ULF) functors. If, in addition, $F_\bullet$ preserves units (see below), then $F_\bullet$ is called a conservative ULF functor.

(2) The above notion of relative 2–Segal space coincides with that of Walde; see [45, Proposition 3.5.10].

(3) Rather than considering diagrams of the form (8), one could require that the outside square of all diagrams of the form

$$\begin{array}{ccccc} Y_n & \longrightarrow & Y_{\{0,\dots,i,j,\dots,n\}} & \xrightarrow{F_{\{0,\dots,i,j,\dots,n\}}} & X_{\{0,\dots,i,j,\dots,n\}} \\ \downarrow & & \downarrow & & \downarrow \\ Y_{\{i,\dots,j\}} & \longrightarrow & Y_{\{i,j\}} & \xrightarrow[F_{\{i,j\}}]{} & X_{\{i,j\}} \end{array}$$

be homotopy cartesian. However, morphisms $F_\bullet$ satisfying these conditions are less interesting from our perspective since, for example, they do not lead to categorified Hall algebra representations.

The following result will be helpful in verifying the relative 2–Segal conditions.

Proposition 2.5 *Let $F_\bullet\colon Y_\bullet \to X_\bullet$ be a morphism of semisimplicial spaces. Assume that $X_\bullet$ is 2–Segal. The following statements are equivalent:*

(i) *For every $n \geq 1$ and every $0 \leq i < j \leq n$ the outside square of diagram (8) is homotopy cartesian.*

(ii) $Y_\bullet$ *is* 2–*Segal, and for every* $n \geq 1$ *the outside square of diagram* (8) *is homotopy cartesian if* $i = 0$ or $j = n$.

(iii) $Y_\bullet$ *is* 2–*Segal, and for every* $n \geq 1$ *the square*

$$\begin{array}{ccc} Y_{\{0,\dots,n\}} & \longrightarrow & Y_{\{0,n\}} \\ {\scriptstyle F_{\{0,\dots,n\}}}\big\downarrow & & \big\downarrow{\scriptstyle F_{\{0,n\}}} \\ X_{\{0,\dots,n\}} & \longrightarrow & X_{\{0,n\}} \end{array} \tag{9}$$

is homotopy cartesian.

Proof Assume that the first condition holds. Since the bottom square of (8) is a degenerate version of the outer square, the 2-out-of-3 property of weak equivalences implies that $Y_\bullet$ is 2–Segal. Hence the second condition holds. It is clear that the second condition implies the third. Assume that the third condition holds. Then the bottom square of diagram (8) is homotopy cartesian by assumption while the top square is homotopy cartesian since $Y_\bullet$ is 2–Segal. It follows that the outside square is also homotopy cartesian, showing that the first condition holds. □

We will often show that a morphism $F_\bullet\colon Y_\bullet \to X_\bullet$ is relative 2–Segal by first proving that $Y_\bullet$ is 1–Segal, applying Proposition 1.4 and then verifying the third condition of Proposition 2.5.

We briefly describe a model-theoretic interpretation of relative 2–Segal spaces. Denote by $\mathcal{R}$ the Reedy model structure on $\mathbb{S}_\Delta$. Following [7, Section 5.2], let $\mathcal{S}\mathrm{eg}_2$ be the left Bousfield localization of $\mathcal{R}$ along the maps

$$\mathfrak{S}eg_2 = \{\Delta^{\mathcal{T}} \hookrightarrow \Delta^n \mid n \geq 3,\ \mathcal{T} \text{ is a triangulation of } P_n\}.$$

Fibrant objects of $(\mathbb{S}_\Delta, \mathcal{S}\mathrm{eg}_2)$ are then (Reedy fibrant) 2–Segal combinatorial simplicial spaces. For $X_\bullet \in \mathbb{S}_\Delta$ write $(\mathcal{S}\mathrm{eg}_2)_{/X_\bullet}$ for the induced model structure on the overcategory $(\mathbb{S}_\Delta)_{/X_\bullet}$ and let $(\mathcal{S}\mathrm{eg}_2)^{\mathsf{fib}}_{/X_\bullet}$ be its left Bousfield localization along

$$\mathfrak{S}eg_2^{\mathsf{fib}} = \{\Delta^{\{0,n\}} \hookrightarrow \Delta^n \xrightarrow{x_n} X_\bullet \mid x_n \in X_n,\ n \geq 2\}.$$

Assuming that $X_\bullet$ is 2–Segal, Proposition 2.5 implies that the fibrant objects of $((\mathbb{S}_\Delta)_{/X_\bullet}, (\mathcal{S}\mathrm{eg}_2)^{\mathsf{fib}}_{/X_\bullet})$ are the morphisms $F_\bullet\colon Y_\bullet \to X_\bullet$ for which the outside square of the diagrams (8) is homotopy cartesian, that is, the ULF functors. From this point of view, a relative 2–Segal combinatorial simplicial space is a fibrant object of $((\mathbb{S}_\Delta)_{/X_\bullet}, (\mathcal{S}\mathrm{eg}_2)^{\mathsf{fib}}_{/X_\bullet})$ whose total space is in addition 1–Segal. It is important to note that the main results of this paper do not hold for arbitrary fibrant objects of $((\mathbb{S}_\Delta)_{/X_\bullet}, (\mathcal{S}\mathrm{eg}_2)^{\mathsf{fib}}_{/X_\bullet})$.

We now formulate the relative analogue of the unital conditions.

Definition 2.6 A relative 2–Segal simplicial space $F_\bullet\colon Y_\bullet \to X_\bullet$ is called unital relative 2–Segal if for every $n \geq 2$ and every $0 \leq i \leq n-1$ the map

$$(F_{\{i\}} \circ \partial_{\{i\}}) \times s_i\colon Y_{n-1} \to X_{\{i\}} \times^R_{X_{\{i,i+1\}}} Y_n$$

is a weak equivalence.

We have the following relative analogue of Proposition 1.4.

Proposition 2.7 *Let $F_\bullet\colon Y_\bullet \to X_\bullet$ be a right relative 1–Segal semisimplicial space. Then $F_\bullet$ is relative 2–Segal. If, moreover, $F_\bullet$ is a morphism of simplicial spaces, then $F_\bullet$ is unital relative 2–Segal.*

Proof Proposition 2.2 implies that $Y_\bullet$ is 1–Segal. The relative 2–Segal morphism $Y_{\{0,\dots,n\}} \to X_{\{0,\dots,n\}} \times^R_{X_{\{0,n\}}} Y_{\{0,n\}}$ factors as the composition

$$\begin{aligned} Y_{\{0,\dots,n\}} &\to X_{\{0,\dots,n\}} \times^R_{X_{\{n\}}} Y_{\{n\}} \\ &\to X_{\{0,\dots,n\}} \times^R_{X_{\{0,n\}}} X_{\{0,n\}} \times^R_{X_{\{n\}}} Y_{\{n\}} \\ &\to X_{\{0,\dots,n\}} \times^R_{X_{\{0,n\}}} Y_{\{0,n\}}. \end{aligned}$$

The first and third morphisms are weak equivalences by the right relative 1–Segal conditions on $F_\bullet$ while the second morphism is a weak equivalence for trivial reasons. Hence the composition is a weak equivalence. The unital condition is verified in a similar way. □

Example For any 1–Segal semisimplicial space $X_\bullet$, the identity morphism $\mathbf{1}_{X_\bullet}$ is relative 2–Segal. If $X_\bullet$ is 2–Segal but not 1–Segal, then $\mathbf{1}_{X_\bullet}$ is not relative 2–Segal.

We end this section with a construction of relative 2–Segal spaces which can be seen as the 2–Segal analogue of the fact that the morphisms $N_\bullet(\mathcal{C}_{/x_*}) \to N_\bullet(\mathcal{C})$ and $N_\bullet({}_{x_*/}\mathcal{C}) \to N_\bullet(\mathcal{C})$, defined in Section 2.1, are right and left relative 1–Segal, respectively. Recall the left join functor

$$l\colon \Delta \to \Delta, \quad \{0,\dots,n\} \mapsto \{0', 0, \dots, n\}.$$

The left path space of a simplicial space $X_\bullet$ is the composition

$$P^{\lhd} X_\bullet \colon \Delta^{\mathrm{op}} \xrightarrow{l^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X_\bullet} \mathsf{Top}.$$

After identifying $P^{\lhd} X_m$ with X_{m+1} for each $m \geq 0$, the face map $\partial_i^{\lhd} \colon P^{\lhd} X_n \to P^{\lhd} X_{n-1}$ is identified with ∂_{i+1}. The remaining face maps $\partial_{0'}$ assemble to a simplicial morphism $F_\bullet^{\lhd} \colon P^{\lhd} X_\bullet \to X_\bullet$. Using instead the right join functor

$$r \colon \Delta \to \Delta, \quad \{0, \dots, n\} \mapsto \{0, \dots, n, n'\},$$

we obtain the right path space $F_\bullet^{\rhd} \colon P^{\rhd} X_\bullet \to X_\bullet$.

Proposition 2.8 *Assume that $X_\bullet$ is (unital) 2–Segal. Then the left and right path spaces*

$$F_\bullet^{\lhd} \colon P^{\lhd} X_\bullet \to X_\bullet, \quad F_\bullet^{\rhd} \colon P^{\rhd} X_\bullet \to X_\bullet$$

are (unital) relative 2–Segal.

Proof Since $X_\bullet$ is 2–Segal, the path space criterion implies that $P^{\lhd} X_\bullet$ and $P^{\rhd} X_\bullet$ are 1–Segal [7, Theorem 6.3.2; 12 Theorem 4.11]. That the remaining relative 2–Segal conditions hold is also proved in [12, Theorem 4.11]. Explicitly, the relative 2–Segal map

$$P^{\lhd} X_n = X_{\{0',0,\dots,n\}} \to X_n \times^R_{X_{\{0,n\}}} X_{\{0',0,n\}} = X_n \times^R_{X_{\{0,n\}}} P^{\lhd} X_{\{0,n\}}$$

and the relative unit map

$$P^{\lhd} X_{n-1} = X_{\{0',0,\dots,n-1\}} \to X_{\{i\}} \times^R_{X_{\{i,i+1\}}} X_{\{0',0,\dots,n\}} = X_{\{i\}} \times^R_{X_{\{i,i+1\}}} P^{\lhd} X_n$$

are 2–Segal and unit maps for $X_\bullet$, and so are weak equivalences by assumption. □

2.3 Relative polyhedral subdivisions

In this section we use combinatorial geometry to give a uniform treatment of the two conditions which define relative 2–Segal spaces. The construction below was suggested by the referee (see also [45, Section 3.2]) and replaces an earlier construction of the author in terms of polyhedral subdivisions of reflection symmetric cyclic polygons.

We require some minor modifications of the definitions from Section 1.3. For each $n \geq 1$, let $P_n^\infty \subset \mathbb{R}^2$ be a convex $(n+2)$–gon with a total order on its vertices which is consistent with the counterclockwise orientation of $\mathbb{R}^2$. There is then a canonical identification of the vertices of P_n^∞ with $[n]_\infty$, the totally ordered set $\{0 < 1 < \dots < n < \infty\}$. Note that there is a unique isomorphism $[n]_\infty \simeq [n+1]$ in Δ.

Let $F_\bullet\colon Y_\bullet \to X_\bullet$ be a morphism of semisimplicial spaces. Associated to a polyhedral subdivision $\mathcal{P}$ of P_n^∞ is the overcategory $\Delta_{\mathsf{inj}}/\Delta^{\mathcal{P}}$. Define a functor

$$F_{\mathcal{P}}\colon (\Delta_{\mathsf{inj}}/\Delta^{\mathcal{P}})^{\mathsf{op}} \to \mathsf{Top}$$

as follows. For each injective morphism $\sigma\colon \Delta^k \hookrightarrow \Delta^{\mathcal{P}}$, whose image we identify with a $(k+1)$–element subset of $[n]_\infty$, put

$$F_{\mathcal{P}}(\sigma) = \begin{cases} X_k & \text{if } \infty \notin \sigma(\Delta^k), \\ Y_{k-1} & \text{if } \infty \in \sigma(\Delta^k). \end{cases}$$

By convention we take Y_{-1} to be a point, although Y_{-1} will not play any significant role in what follows. Given a morphism

$$\begin{array}{ccc} \Delta^k & \xrightarrow{\phi} & \Delta^{k'} \\ & {\scriptstyle\sigma}\searrow \quad \swarrow{\scriptstyle\sigma'} & \\ & \Delta^{\mathcal{P}} & \end{array}$$

in $\Delta_{\mathsf{inj}}/\Delta^{\mathcal{P}}$ define a morphism $F_{\mathcal{P}}(\phi)\colon F_{\mathcal{P}}(\sigma') \to F_{\mathcal{P}}(\sigma)$ of topological spaces as follows. If the images σ and σ' both contain ∞ (resp. both do not contain ∞), then $F_{\mathcal{P}}(\phi)$ is defined by the simplicial structure of $Y_\bullet$ (resp. $X_\bullet$), while if only the image of σ' contains ∞, then $F_{\mathcal{P}}(\phi)$ is defined by the morphism $F_\bullet$.

With the above notation, the relative derived membrane space is defined to be the homotopy limit

$$(\Delta^{\mathcal{P}}, F_\bullet)_R = \operatorname{holim}^{\mathsf{Top}}_{\overleftarrow{\sigma\in\Delta_{\mathsf{inj}}/\Delta^{\mathcal{P}}}} F_{\mathcal{P}}(\sigma).$$

As in [7, Proposition 2.2.19], the definition is unchanged if instead the homotopy limit is taken over the full subcategory of $\Delta_{\mathsf{inj}}/\Delta^{\mathcal{P}}$ on objects $\Delta^k \hookrightarrow \Delta^{\mathcal{P}}$ with $k \geq 1$. A refinement $\mathcal{P}$ of $\mathcal{P}'$ defines a simplicial subset $\Delta^{\mathcal{P}} \subset \Delta^{\mathcal{P}'}$ and thus a morphism $(\Delta^{\mathcal{P}'}, F_\bullet)_R \to (\Delta^{\mathcal{P}}, F_\bullet)_R$ of topological spaces. In particular, taking $\mathcal{P}'$ to be the trivial subdivision and noting that $(\Delta^{[n]_\infty}, F_\bullet)_R \simeq Y_n$, we obtain a morphism

$$f_{\mathcal{P}}^{F_\bullet}\colon Y_n \to (\Delta^{\mathcal{P}}, F_\bullet)_R.$$

Proposition 2.9 *Let $F_\bullet\colon Y_\bullet \to X_\bullet$ be a morphism of semisimplicial spaces. Assume that $X_\bullet$ is 2–Segal. Then the following statements are equivalent:*

(1) *The morphism $F_\bullet$ is relative 2–Segal.*

(2) *For every $n \geq 2$ and every polyhedral subdivision $\mathcal{P}$ of P_n^∞ the morphism $f_{\mathcal{P}}^{F_\bullet}$ is a weak equivalence.*

(3) *For every $n \geq 2$ and every triangulation $\mathcal{T}$ of P_n^∞ the morphism $f_{\mathcal{T}}^{F_\bullet}$ is a weak equivalence.*

Proof Assume that the second statement holds. If $\mathcal{P}$ is the polyhedral subdivision of P_n^{∞} consisting only of the diagonal $\{i,\infty\}$, then

$$(\Delta^{\mathcal{P}}, F_{\bullet})_R \simeq Y_{\{0,\dots,i\}} \times^R_{Y_{\{i\}}} Y_{\{i,\dots,n\}},$$

while if $\mathcal{P}'$ consists only of the diagonal $\{0,n\}$, then

$$(\Delta^{\mathcal{P}'}, F_{\bullet})_R \simeq X_{\{0,\dots,n\}} \times^R_{X_{\{0,n\}}} Y_{\{0,n\}}.$$

Hence $f_{\mathcal{P}}^{F_\bullet}$ and $f_{\mathcal{P}'}^{F_\bullet}$ reduce to the 1–Segal maps for $Y_\bullet$ and the relative 2–Segal maps for $F_\bullet$, respectively, proving that $F_\bullet$ is relative 2–Segal.

Conversely, assume that $F_\bullet$ is relative 2–Segal and let $\mathcal{P}$ be a polyhedral subdivision of P_n^{∞}. There exists a sequence $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k = \mathcal{P}$ of polyhedral subdivisions such that $\mathcal{P}_0$ is the trivial subdivision and $\mathcal{P}_j$ is obtained from $\mathcal{P}_{j-1}$ by adding a single diagonal. Then $f_{\mathcal{P}}^{F_\bullet}$ factors as the composition

$$Y_n \simeq (\Delta^{\mathcal{P}_0}, F_\bullet)_R \to (\Delta^{\mathcal{P}_1}, F_\bullet)_R \to \cdots \to (\Delta^{\mathcal{P}_{k-1}}, F_\bullet)_R \to (\Delta^{\mathcal{P}}, F_\bullet)_R.$$

The construction of the $\mathcal{P}_j$ ensures that each morphism in this composition is a weak equivalence, being induced by a 1–Segal map for $Y_\bullet$ or a relative 2–Segal map for $F_\bullet$. It follows that $f_{\mathcal{P}}^{F_\bullet}$ is a weak equivalence and the second statement holds.

The equivalence of the second and third statements is proved similarly. □

2.4 The stable framed $\mathcal{S}_\bullet$–construction

Motivated by the Hall algebra representations of [41; 11], in this section we modify the Waldhausen $\mathcal{S}_\bullet$–construction so as to construct relative 2–Segal groupoids from an abelian category together with a choice of stability condition.

Fix a field k. Let $\mathcal{C}$ be a k–linear abelian category with Grothendieck group $K_0(\mathcal{C})$. A stability function on $\mathcal{C}$ is a group homomorphism $Z\colon K_0(\mathcal{C}) \to \mathbb{C}$ such that

$$Z(A) \in \mathbb{H}_+ = \big\{m \exp(\sqrt{-1}\pi\phi) \mid m > 0,\ \phi \in (0,1]\big\} \subset \mathbb{C}$$

for all nonzero objects $A \in \mathcal{C}$ [2, Section 2]. Let $\phi(A) \in (0,1]$ be the phase of $Z(A)$. A nonzero object $A \in \mathcal{C}$ is called Z–semistable if $\phi(A') \le \phi(A)$ for all nontrivial subobjects $A' \subset A$. For each $\phi \in (0,1]$ the full subcategory $\mathcal{C}_\phi^{Z\text{-ss}} \subset \mathcal{C}$ consisting of the zero object together with all objects which are Z–semistable of phase ϕ is again abelian. We can therefore form the Waldhausen simplicial groupoid $\mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$, which is 2–Segal by Theorem 1.6.

We formulate a notion of framing following [41, Section 4]. Fix a left exact functor $\Phi\colon \mathcal{C} \to \mathsf{Vect}_k$ with values in the category of finite-dimensional vector spaces over k. A framed object of $\mathcal{C}$ is then a pair (M, s) consisting of an object $M \in \mathcal{C}$ and a section $s \in \Phi(M)$. A morphism of framed objects $(M, s) \to (M', s')$ is a pair $(\pi, \lambda) \in \mathrm{Hom}_{\mathcal{C}}(M, M') \times k$ which satisfies $\Phi(\pi)(s) = \lambda s'$. A framed object (M, s) is called stable framed if M is Z–semistable and $\phi(A) < \phi(M)$ for all proper subobjects $A \subset M$ for which $s \in \Phi(A) \subset \Phi(M)$.

Example We recall two standard examples of framings.

(1) Let $\mathsf{Rep}_k(Q)$ be the category of finite-dimensional representations of a quiver Q. Sending a representation to its dimension vector defines a surjective group homomorphism $K_0(\mathsf{Rep}_k(Q)) \to \mathbb{Z}^{Q_0}$. A tuple $\zeta = (\zeta_i)_{i \in Q_0} \in \mathbb{H}_+^{Q_0}$ defines a function $\mathbb{Z}^{Q_0} \to \mathbb{C}$, $d \mapsto \sum_{i \in Q_0} \zeta_i d_i$ which induces a stability function Z on $\mathsf{Rep}_k(Q)$. For any $f \in \mathbb{Z}_{\geq 0}^{Q_0}$ the functor $\Phi\colon U \mapsto \bigoplus_{i \in Q_0} \mathrm{Hom}_k(k^{f_i}, U_i)$ is a framing.

(2) Let $\mathsf{Coh}(X)$ be the category of coherent sheaves on a smooth projective curve X. Then $Z = -\deg + \sqrt{-1} \cdot \mathrm{rk}$ defines a stability function. The global sections functor $\Phi = H^0(X, -)$ is a framing.

Define a stable framed modification of the $\mathcal{S}_\bullet$–construction as follows. For each $n \geq 0$ let $\mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ be the maximal groupoid of the category of diagrams of the form

$$\begin{array}{ccccccccc}
A_{\{0,0\}} & \to & A_{\{0,1\}} & \to & \cdots & \to & A_{\{0,n\}} & \to & (M_0, s_0) \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & A_{\{1,1\}} & \to & \cdots & \to & A_{\{1,n\}} & \to & (M_1, s_1) \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & A_{\{n,n\}} & \to & (M_n, s_n) \\
 & & & & & & & & \downarrow \\
 & & & & & & & & (M_{n+1}, s_{n+1})
\end{array}$$

where each (M_i, s_i) is a framed object and which have the following properties:

(i) Upon forgetting the framing data $s_0, \dots, s_{n+1}$ the resulting diagram is an object of $\mathcal{S}_{n+1}(\mathcal{C}_\phi^{Z\text{-ss}})$.

(ii) Each pair (M_i, s_i), $i = 0, \dots, n$, is a stable framed object.

In the above diagram a morphism $A_{\{i,n\}} \to (M_i, s_i)$ is simply a morphism of the underlying objects of $\mathcal{C}$. The groupoids $\mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ assemble to a simplicial groupoid. There is a canonical simplicial morphism $F_\bullet\colon \mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ which forgets the rightmost column and bottom row of a diagram.

Theorem 2.10 *Let $\mathcal{C}$ be a k–linear abelian category with stability function Z and framing Φ. For each $\phi \in (0,1)$ the map $F_\bullet\colon \mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ is unital relative 2–Segal.*

Proof The proof that $\mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is 1–Segal reduces to the 2–Segal property of $\mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$, so we omit it. To verify the second of the relative 2–Segal conditions we need to show that the functor

$$\Psi_n\colon \mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times^{(2)}_{\mathcal{S}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})} \mathcal{S}_{\{0,n\}}^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$$

is an equivalence. Here $-\times^{(2)}-$ denotes the 2–pullback of groupoids.

Let $\mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}})$ be the groupoid of n–flags in $\mathcal{C}_\phi^{Z\text{-ss}}$, that is, diagrams in $\mathcal{C}_\phi^{Z\text{-ss}}$ of the form

$$0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1} \rightarrowtail A_n.$$

The forgetful functor $\mu_n\colon \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}})$ is an equivalence of groupoids [46, Section 1.3]. Similarly, let $\mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ be the groupoid of n–flags in a stable framed object. Explicitly, an object of $\mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is a diagram of the form

$$0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_{n-1} \rightarrowtail A_n \rightarrowtail (M,s) \tag{10}$$

with $A_1, \dots, A_n, M \in \mathcal{C}_\phi^{Z\text{-ss}}$ and (M,s) stable framed. We claim that the forgetful functor $\nu_n\colon \mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is an equivalence. A quasi-inverse η_n of ν_n can be constructed as follows. Given a flag (10) set $A_{\{0,k\}} = A_k$, $1 \le k \le n$, and $(M_0, s_0) = (M,s)$. Define $A_{\{1,k\}}$, $1 \le k \le n$, and M_1 as the following pushouts:

$$\begin{array}{ccc} A_{\{0,1\}} & \rightarrowtail & A_{\{0,k\}} \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & A_{\{1,k\}} \end{array} \qquad \begin{array}{ccc} A_{\{0,1\}} & \rightarrowtail & M_0 \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & M_1 \end{array}$$

Since $\mathcal{C}_\phi^{Z\text{-ss}}$ is abelian, it is automatic that each $A_{\{1,k\}}$ and M_1 are Z–semistable of phase ϕ. Let s_1 be the image of s_0 under the morphism $\Phi(M_0) \to \Phi(M_1)$. We need to show that (M_1, s_1) is stable framed. Note that s_1 is nonzero. Indeed, if $s_1 = 0$, then $s_0 \in \Phi(A_{\{0,1\}}) \subset \Phi(M_0)$ and $\phi(A_{\{0,1\}}) = \phi(M_0)$, contradicting the assumed framed stability of (M_0, s_0). Let then $B_1 \subset M_1$ be a proper subobject with $s_1 \in \Phi(B_1) \subset \Phi(M_1)$. There exists a unique object $B_0 \in \mathcal{C}$ which fits into the following exact commutative diagram:

$$\begin{array}{ccccc}
A_{\{0,1\}} & \rightarrowtail & B_0 & \twoheadrightarrow & B_1 \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & \downarrow \\
A_{\{0,1\}} & \rightarrowtail & M_0 & \twoheadrightarrow & M_1 \\
 & & \downarrow & & \downarrow \\
 & & M_0/B_0 & \xrightarrow{\sim} & M_1/B_1
\end{array}$$

Since $s_1 \in \Phi(B_1)$, using the left exactness of Φ we see that s_1 lies in the kernel of $\Phi(M_1) \to \Phi(M_1/B_1)$. It follows that s_0 is in the kernel of $\Phi(M_0) \to \Phi(M_0/B_0)$. Hence $s_0 \in \Phi(B_0) \subset \Phi(M_0)$. Since (M_0, s_0) is stable framed we have $\phi(B_0) < \phi(M_0)$. The equalities $\phi(A_{\{0,1\}}) = \phi(M_0) = \phi(M_1)$ combined with the standard see-saw properties of stability functions then give $\phi(M_1) > \phi(B_1)$, as desired. This procedure defines the top two rows of $\eta_n(A_{\{0,\bullet\}} \rightarrowtail (M_0, s_0))$ and can be iterated to define the remaining $n-2$ rows. This defines the desired quasi-inverse.

We can now prove the theorem. Consider the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) & \xrightarrow{\Psi_n} & \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times^{(2)}_{\mathcal{S}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})} \mathcal{S}^{\text{st-fr}}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}}) \\
{\scriptstyle \nu_n}\downarrow & & \downarrow{\scriptstyle \mu_n \times \nu_{\{0,n\}}} \\
\mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) & \xrightarrow[\widetilde{\Psi}_n]{} & \mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times^{(2)}_{\mathcal{F}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})} \mathcal{F}^{\text{st-fr}}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})
\end{array}$$

Noting that the groupoids $\mathcal{S}_1(\mathcal{C}_\phi^{Z\text{-ss}})$ and $\mathcal{F}_1(\mathcal{C}_\phi^{Z\text{-ss}})$ are canonically equivalent, the discussion above shows that the vertical functors are equivalences. That Ψ_n is an equivalence therefore reduces to the statement that $\widetilde{\Psi}_n$ is an equivalence, which is obvious. We omit the verification of the relative unital condition. □

Remark The notion of quotient datum on an abelian category $\mathcal{A}$ is introduced in [45, Section 5.5], where it is shown to define a relative 2–Segal groupoid over $\mathcal{S}_\bullet(\mathcal{A})$. From this point of view, Theorem 2.10 can be rephrased as the statement that a stability function Z and framing Φ on an abelian category $\mathcal{C}$ defines a (mild generalization of a) quotient datum on the Artin mapping cylinder $\mathcal{C}_\Phi$.

2.5 Real pseudoholomorphic polygons

In this section we give a relative variant of the pseudoholomorphic polygon construction of [7, Section 3.8].

Recall that an almost complex structure on a smooth manifold M is an endomorphism $J\colon TM \to TM$ of the tangent bundle which satisfies $J \circ J = -\mathbf{1}_{TM}$. A continuous map $u\colon (\Sigma, \mathrm{j}) \to (M, J)$ of almost complex manifolds is called pseudoholomorphic if it is smooth and satisfies the equation

$$du + J \circ du \circ \mathrm{j} = 0.$$

Let $\mathbb{H}$ be the Lobachevsky plane, realized as the open unit disk in $\mathbb{C}$ with centre the origin and metric

$$ds^2 = \frac{dz \wedge d\overline{z}}{(1-|z|^2)^2}.$$

The ideal boundary $\partial\mathbb{H}$ is the unit circle in $\mathbb{C}$. Denote by $\mathrm{j}_{\mathbb{H}}$ the complex structure on $\mathbb{H}$. The group $\mathsf{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by orientation-preserving isometries.

We require some basic definitions from Teichmüller theory [29]. Given $b, b' \in \partial\mathbb{H}$, let (b, b') be the oriented geodesic with limiting points b at infinite negative time and b' at infinite positive time. For each $n \geq 0$ let $P(b_0, \dots, b_n)$ be the ideal $(n+1)$–gon in $\mathbb{H}$ with vertices $b_0, \dots, b_n \in \partial\mathbb{H}$ numbered compatibly with the canonical orientation of $\partial\mathbb{H}$. A decoration of $P(b_0, \dots, b_n)$ is the data of a horocycle $\xi_i \in \mathsf{Hor}_{b_i}$ with hyperbolic centre b_i for each $i = 0, \dots, n$. The group $\mathsf{SL}_2(\mathbb{R}) \times \mathbb{R}$ acts on the set of decorated ideal $(n+1)$–gons by the formula

$$(g, a) \cdot (P(b_0, \dots, b_n), \xi_0, \dots, \xi_n) = (P(g \cdot b_0, \dots, g \cdot b_n), g \cdot \xi_0 + a, \dots, g \cdot \xi_n + a),$$

where $g \in \mathsf{SL}_2(\mathbb{R})$ and $a \in \mathbb{R}$ and we have used the canonical identification of $\mathbb{R}$–torsors $g\colon \mathsf{Hor}_{b_i} \to \mathsf{Hor}_{g \cdot b_i}$. We will often omit the decoration from the notation if it will not cause confusion.

Given a decorated ideal $(n+1)$–gon, for each $0 \leq i < j \leq n$ let $m(\xi_i, \xi_j)$ be the midpoint of the unique geodesic connecting the points $\xi_i \cap (b_i, b_j)$ and $\xi_j \cap (b_i, b_j)$. This trivializes the $\mathbb{R}$–torsor (b_i, b_j) via

$$\mathbb{R} \to (b_i, b_j), \quad 0 \mapsto m(\xi_i, \xi_j). \tag{11}$$

This trivialization is invariant under the action of $\mathbb{R} \subset \mathsf{SL}_2(\mathbb{R}) \times \mathbb{R}$.

Following [7, Section 3.8] we construct from an almost complex manifold (M, J) a semisimplicial set $\widetilde{\mathbb{T}}_\bullet(M)$. Let $\widetilde{\mathbb{T}}_0(M) = M$ and, for each $n \geq 1$, let $\widetilde{\mathbb{T}}_n(M)$ be the set of equivalence classes of pairs consisting of a decorated ideal $(n+1)$–gon P together with a continuous map $u\colon (P, \mathrm{j}_{\mathbb{H}}) \to (M, J)$ which is pseudoholomorphic on the two-dimensional interior of P. The equivalence relation is generated by the action of $\mathsf{SL}_2(\mathbb{R}) \times \mathbb{R}$ on decorated ideal polygons. Using the trivialization (11), we can identify $\widetilde{\mathbb{T}}_1(M)$ with the set of continuous maps $[-\infty, \infty] \to M$. The face map $\partial_i\colon \widetilde{\mathbb{T}}_n(M) \to \widetilde{\mathbb{T}}_{n-1}(M)$ omits the ideal boundary point b_i and forms the resulting decorated n–gon together with the restricted morphism to M. It is proved in [7, Theorem 3.8.6] that $\widetilde{\mathbb{T}}_\bullet(M)$ is a 2–Segal semisimplicial set.

Consider now the relative setting. Recall that a real structure on (M, J) is a smooth involution $\tau\colon M \to M$ which satisfies

$$d\tau - J \circ d\tau \circ J = 0.$$

A map $u\colon (\Sigma, \mathrm{j}, \sigma) \to (M, J, \tau)$ of almost complex manifolds with real structures is called real pseudoholomorphic if it is pseudoholomorphic and satisfies $u \circ \sigma = \tau \circ u$.

The real structure $\sigma\colon z \mapsto -\overline{z}$ on $\mathbb{C}$ induces a real structure on $\mathbb{H}$, again denoted by σ. The subgroup $\mathsf{SL}_2(\mathbb{R})^\sigma \leq \mathsf{SL}_2(\mathbb{R})$ which commutes with σ is isomorphic to $\mathbb{R}^\times \rtimes \mathbb{Z}_2$, the generator of $\mathbb{Z}_2$ being rotation of $\mathbb{H}$ through an angle π. Define a real ideal boundary point of $\mathbb{H}$ to be either a point of the real locus $\partial\mathbb{H}^\sigma = \{\sqrt{-1}, -\sqrt{-1}\}$ or a pair $\{b, \sigma(b)\}$ of distinct σ–conjugate ideal boundary points.

A real ideal $(n+1)$–gon $Q = Q(b_0, \dots, b_n, \dots)$ is an ideal polygon which has exactly $n+1$ real ideal boundary points, labelled so that only b_0 and b_n may lie in $\partial\mathbb{H}^\sigma$. It follows that Q is of one of the following four types:

(i) $Q(b_0, \dots, b_n, \sigma(b_n), \sigma(b_{n-1}), \dots, \sigma(b_1), \sigma(b_0))$,

(ii) $Q(b_0, \dots, b_n, \sigma(b_n), \sigma(b_{n-1}), \dots, \sigma(b_1))$,

(iii) $Q(b_0, \dots, b_n, \sigma(b_{n-1}), \dots, \sigma(b_1), \sigma(b_0))$,

(iv) $Q(b_0, \dots, b_n, \sigma(b_{n-1}), \dots, \sigma(b_1))$.

Then Q has exactly zero, one, one and two ideal vertices in $\partial\mathbb{H}^\sigma$ in cases (i)–(iv), respectively. A decoration of Q is a decoration of its underlying ideal polygon for which the horocycle at a vertex b is equal to that at $\sigma(b)$ under the canonical identification $\mathsf{Hor}_b \simeq \mathsf{Hor}_{\sigma(b)}$. The group $\mathsf{SL}_2(\mathbb{R})^\sigma \times \mathbb{R}$ acts on the set of real decorated ideal polygons.

With the above notation in place, we define a semisimplicial set $\widetilde{\mathbb{T}}^\tau_\bullet(M)$ as follows. For each $n \geq 0$ let $\widetilde{\mathbb{T}}^\tau_n(M)$ be the set of equivalence classes of pairs (Q, v) consisting of a real decorated ideal $(n+1)$–gon Q together with a real continuous map $v\colon (Q, \mathrm{j}_{\mathbb{H}}, \sigma) \to (M, J, \tau)$ which is pseudoholomorphic on the two-dimensional interior of Q. The equivalence relation is generated by the action of $\mathsf{SL}_2(\mathbb{R})^\sigma \times \mathbb{R}$ on real decorated ideal polygons. In particular, we have

$$\widetilde{\mathbb{T}}^\tau_0(M) \simeq M^\tau \sqcup C^0([-\infty, \infty], M)^\tau, \tag{12}$$

where $C^0([-\infty, \infty], M)^\tau$ denotes the set of real continuous maps $[-\infty, \infty] \to M$ with the compactified real line $\overline{\mathbb{R}} = [-\infty, \infty]$ given the $\mathbb{Z}_2$–action $x \mapsto -x$. Define face maps $\partial_i\colon \widetilde{\mathbb{T}}^\tau_n(M) \to \widetilde{\mathbb{T}}^\tau_{n-1}(M)$ by omitting the real ideal boundary point $\{b_i, \sigma(b_i)\}$. Then $\widetilde{\mathbb{T}}^\tau_\bullet(M)$ forms a semisimplicial set.

Given $(Q, v) \in \widetilde{\mathbb{T}}^\tau_n(M)$, we write $Q = Q(b_0, \dots, b_n, \dots)$ as above, and we let $\widetilde{Q} = \widetilde{Q}(b_0, \dots, b_n)$ be the decorated ideal $(n+1)$–gon obtained from Q by omitting the indicated vertices and their decorations. This defines a semisimplicial morphism

$$F_\bullet\colon \widetilde{\mathbb{T}}^\tau_\bullet(M) \to \widetilde{\mathbb{T}}_\bullet(M), \quad (Q, v) \mapsto (\widetilde{Q}, v_{|\widetilde{Q}}).$$

We can now state the main result of this section.

Theorem 2.11 *Let (M, J, τ) be an almost complex manifold with real structure. The morphism $F_\bullet\colon \widetilde{\mathbb{T}}^\tau_\bullet(M) \to \widetilde{\mathbb{T}}_\bullet(M)$ is a relative 2–Segal semisimplicial set.*

Proof For each $n \geq 2$ and $0 < i < n$ we construct an inverse of the 1–Segal map

$$\Xi_n\colon \widetilde{\mathbb{T}}^\tau_n(M) \to \widetilde{\mathbb{T}}^\tau_{\{0,\dots,i\}}(M) \times_{\widetilde{\mathbb{T}}^\tau_{\{i\}}(M)} \widetilde{\mathbb{T}}^\tau_{\{i,\dots,n\}}(M).$$

Let $(Q', v') \in \widetilde{\mathbb{T}}^\tau_{\{0,\dots,i\}}(M)$ and $(Q'', v'') \in \widetilde{\mathbb{T}}^\tau_{\{i,\dots,n\}}(M)$ with equal image in $\widetilde{\mathbb{T}}^\tau_{\{i\}}(M)$. Since $0 < i < n$ these images lie in the component $C^0([-\infty, \infty], M)^\tau$ in the decomposition (12). There exists a unique $g \in \mathsf{SL}_2(\mathbb{R})^\sigma$ with the property that $g \cdot (b'_i, \sigma(b'_i)) = (b''_i, \sigma(b''_i))$ and the restrictions

$$g \cdot v', v''\colon [-\infty, \infty] \to M$$

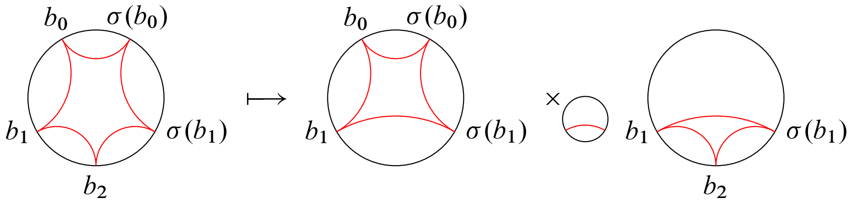


Figure 1: The map $\Xi_2\colon \widetilde{\mathbb{T}}^{\tau}_2(M) \to \widetilde{\mathbb{T}}^{\tau}_{\{0,1\}}(M) \times_{\widetilde{\mathbb{T}}^{\tau}_{\{1\}}(M)} \widetilde{\mathbb{T}}^{\tau}_{\{1,2\}}(M)$, omitting the data of the maps to M

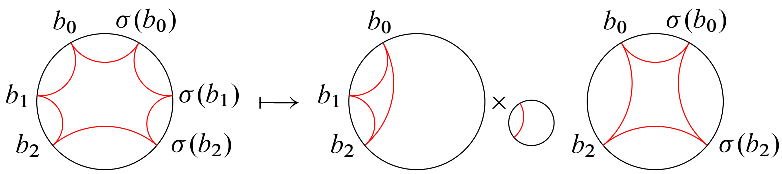


Figure 2: The map $\Psi_2\colon \widetilde{\mathbb{T}}^{\tau}_2(M) \to \widetilde{\mathbb{T}}_2(M) \times_{\widetilde{\mathbb{T}}_{\{0,2\}}(M)} \widetilde{\mathbb{T}}^{\tau}_{\{0,2\}}(M)$, omitting the data of the maps to M

are equal. Applying Morera's theorem, we see that $g \cdot (Q', v')$ and (Q'', v'') can be glued in a unique way so as to obtain a continuous map

$$g \cdot v' \cup v''\colon g \cdot Q' \cup_{(b''_i, \sigma(b''_i))} Q'' \to M$$

which is pseudoholomorphic on the two-dimensional interior. Since both Q' and Q'' (resp. v' and v'') are real, so too is $g \cdot Q' \cup_{(b''_i, \sigma(b''_i))} Q''$ (resp. $g \cdot v' \cup v''$). See Figure 1. This defines the required inverse of Ξ_n, showing that $\widetilde{\mathbb{T}}^{\tau}_{\bullet}(M)$ is 1–Segal.

To verify the second of the relative 2–Segal conditions, consider the map

$$\Psi_n\colon \widetilde{\mathbb{T}}^{\tau}_n(M) \to \widetilde{\mathbb{T}}_n(M) \times_{\widetilde{\mathbb{T}}_{\{0,n\}}(M)} \widetilde{\mathbb{T}}^{\tau}_{\{0,n\}}(M) \quad \text{for } n \geq 1.$$

Let $(P', u') \in \widetilde{\mathbb{T}}_n(M)$ and $(Q'', v'') \in \widetilde{\mathbb{T}}^{\tau}_{\{0,n\}}(M)$ with equal image in $\widetilde{\mathbb{T}}_{\{0,n\}}(M)$. Choose $g \in \mathsf{SL}_2(\mathbb{R})$ so that $g \cdot (b'_0, b'_n) = (b''_0, b''_n)$. As above, the reality condition on (Q'', v'') implies that, up to the action of $\mathsf{SL}_2(\mathbb{R})^{\sigma}$, the triple

$$\{g \cdot (P', u'),\ (Q'', v''),\ g \cdot (\sigma(P'), \tau \circ u' \circ \sigma^{-1})\}$$

can be glued in a unique way so as to obtain an element of $\widetilde{\mathbb{T}}^{\tau}_n(M)$. See Figure 2. This defines the desired inverse of Ψ_n. □

3 Relative higher Segal spaces from categories with dualities

3.1 Categories with duality

We recall some basics about categories with duality. For further details the reader is referred to [35].

Definition 3.1 A category with (strong) duality is a triple $(\mathcal{C}, P, \Theta)$ consisting of a category $\mathcal{C}$, a functor $P\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and a natural isomorphism $\Theta\colon \mathbf{1}_{\mathcal{C}} \Rightarrow P \circ P^{\mathrm{op}}$ such that $P(\Theta_U) \circ \Theta_{P(U)} = \mathbf{1}_{P(U)}$ for all objects $U \in \mathcal{C}$.

When there is no risk of confusion we will omit (P, Θ) from the notation and simply refer to $\mathcal{C}$ as a category with duality.

A form functor $(T, \varphi)\colon (\mathcal{C}, P, \Theta) \to (\mathcal{D}, Q, \Xi)$ between categories with duality consists of a functor $T\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $\varphi\colon T \circ P \Rightarrow Q \circ T^{\mathrm{op}}$ for which the diagram

$$\begin{array}{ccc} T(U) & \xrightarrow{\Xi_{T(U)}} & Q^2T(U) \\ {\scriptstyle T(\Theta_U)}\downarrow & & \downarrow{\scriptstyle Q(\varphi_U)} \\ TP^2(U) & \xrightarrow[\varphi_{P(U)}]{} & QTP(U) \end{array}$$

commutes for all objects $U \in \mathcal{C}$. In particular, let $T\colon \mathcal{C} \to \mathcal{D}$ be a duality-preserving functor, that is, a functor for which $T \circ P = Q \circ T^{\mathrm{op}}$ and $T\Theta = \Xi T$, where $T\Theta$ (resp. ΞT) is the left (resp. right) whiskering of T with Θ (resp. Ξ). Then $(T, \mathbf{1}_T)$ is a form functor. Write CatD for the category of small categories with duality with form functors as morphisms.

A (nonsingular) symmetric form in $\mathcal{C}$ is a pair (N, ψ_N), often just denoted by N, consisting of an object $N \in \mathcal{C}$ and an isomorphism $\psi_N\colon N \to P(N)$ which satisfies $P(\psi_N) \circ \Theta_N = \psi_N$. An isometry of symmetric forms $\phi\colon (M, \psi_M) \to (N, \psi_N)$ is an isomorphism $\phi\colon M \to N$ which satisfies $P(\phi) \circ \psi_N \circ \phi = \psi_M$. The groupoid of symmetric forms and their isometries is denoted by $\mathcal{C}_h$ and is called the Hermitian groupoid of $\mathcal{C}$.

For each integer $n \geq 0$ let $\boldsymbol{n}$ be the ordered set $\{0 < \cdots < n < n' < \cdots < 0'\}$. There is a unique isomorphism $\boldsymbol{n} \simeq [2n+1]$ in Δ. The morphisms $[n] \to \boldsymbol{n}$, $i \mapsto i$, define the subdivision functor $\mathsf{sd}\colon \Delta \to \Delta$. The edgewise subdivision of a simplicial object $X_\bullet$ of a category $\mathcal{C}$ is then defined to be the composition

$$X^e_\bullet\colon \Delta^{\mathrm{op}} \xrightarrow{\mathsf{sd}^{\mathrm{op}}} \Delta^{\mathrm{op}} \xrightarrow{X_\bullet} \mathcal{C}.$$

Explicitly, $X^e_n = X_{\boldsymbol{n}} = X_{2n+1}$ with face maps

$$\partial^e_i\colon X^e_n = X_{2n+1} \xrightarrow{\partial_i \circ \partial_{2n+1-i}} X_{2n-1} = X^e_{n-1},$$

the degeneracy maps admitting a similar description. The morphisms $[n] \to \boldsymbol{n}$ define a simplicial morphism $X^e_\bullet \to X_\bullet$.

In the sections below we will repeatedly apply the following result; compare [15, Section 1.5].

Lemma 3.2 *Let $X_\bullet$ be a simplicial object of Cat. For each $n \geq 0$ let (P_n, Θ_n), $n \geq 0$ be a duality structure on X_n. Suppose that*

$$P_{n-1} \circ \partial_i^{\mathrm{op}} = \partial_{n-i} \circ P_n, \quad P_{n+1} \circ s_i^{\mathrm{op}} = s_{n-i} \circ P_n,$$

and

$$\partial_i \Theta_n = \Theta_{n-1} \partial_i, \quad s_i \Theta_n = \Theta_{n+1} s_i$$

for all $n \geq 1$ and $0 \leq i \leq n$. Then $(X^e_\bullet, P^e_\bullet, \Theta^e_\bullet)$ is a simplicial object of CatD.

Proof It is clear that $(X^e_n, P^e_n, \Theta^e_n) = (X_{2n+1}, P_{2n+1}, \Theta_{2n+1})$ is a category with duality. The equality $\partial_i \Theta_n = \Theta_{n-1} \partial_i$ together with the calculation

$$\begin{aligned}
\partial^e_i \circ P^e_n &= \partial_i \circ \partial_{2n+1-i} \circ P_{2n+1} \\
&= \partial_i \circ P_{2n} \circ \partial_i^{\mathrm{op}} \\
&= P_{2n-1} \circ \partial_{2n-i}^{\mathrm{op}} \circ \partial_i^{\mathrm{op}} \\
&= P_{2n-1} \circ \partial_i^{\mathrm{op}} \circ \partial_{2n-1-i}^{\mathrm{op}} = P^e_{n-1} \circ (\partial^e_i)^{\mathrm{op}}
\end{aligned}$$

shows that the face map $(\partial^e_i, \mathbf{1}_{\partial^e_i})\colon X^e_n \to X^e_{n-1}$ is a form functor. A similar calculation shows that $(s^e_i, \mathbf{1}_{s^e_i})\colon X^e_n \to X^e_{n+1}$ is a form functor. □

3.2 Unoriented categorified nerves

As a warmup for Section 3.3 we adapt the categorified nerve construction to the relative setting.

Let $\mathcal{C}$ be a small category. Its categorified nerve $\mathcal{N}_\bullet(\mathcal{C})$ (see Section 1.2) is a 1–Segal simplicial groupoid. Suppose that (P, Θ) is a duality structure on $\mathcal{C}$. Then the groupoids $\mathcal{N}_n(\mathcal{C})$, $n \geq 0$, inherit duality structures which are compatible in the sense of Lemma 3.2. The edgewise subdivision $\mathcal{N}_\bullet^e(\mathcal{C})$ is therefore a simplicial groupoid with duality.

Definition 3.3 The Hermitian groupoid $\mathcal{N}_\bullet^e(\mathcal{C})_h$, denoted by $\mathcal{U}_\bullet(\mathcal{C})$, is called the categorified unoriented nerve of $(\mathcal{C}, P, \Theta)$.

Slightly abusively, write $(x_\bullet, \psi_\bullet) \in \mathcal{U}_n(\mathcal{C})$ for the object consisting of the diagram

$$x_0 \to \cdots \to x_n \to x_{n'} \to \cdots \to x_{0'}$$

in $\mathcal{C}$ together with isomorphisms $\psi_i \colon x_i \to P(x_{i'})$, $0 \leq i \leq 0'$, with $P(\psi_i) \circ \Theta_{x_{i'}} = \psi_{i'}$ and make the obvious squares commute. Here we use the convention that $i'' = i$. For later use, note that an object $(x_0 \xrightarrow{f} x_{0'}, \{\psi_0, \psi_{0'}\})$ of $\mathcal{U}_0(\mathcal{C})$ is determined by the pair (f, ψ_0) and that $P(f) \circ \psi_0$ defines a possibly singular symmetric form on x_0. In this way, $\mathcal{U}_0(\mathcal{C})$ can be interpreted as the groupoid of presented symmetric forms in $\mathcal{C}$, that is, possibly singular symmetric forms $\psi \colon x \to P(x)$ with the additional data of a factorization $\psi = P(f_x) \circ \psi_x$, where $\psi_x \colon x \to P(x')$ an isomorphism.

There is a canonical forgetful morphism $F_\bullet \colon \mathcal{U}_\bullet(\mathcal{C}) \to \mathcal{N}_\bullet(\mathcal{C})$ given by

$$F_n \colon \mathcal{U}_n(\mathcal{C}) \to \mathcal{N}_n(\mathcal{C}), \quad (x_\bullet, \psi_\bullet) \mapsto (x_0 \to \cdots \to x_n).$$

Proposition 3.4 *For any small category with duality $(\mathcal{C}, P, \Theta)$, the morphism*

$$F_\bullet \colon \mathcal{U}_\bullet(\mathcal{C}) \to \mathcal{N}_\bullet(\mathcal{C})$$

is unital right relative 1–Segal.

Proof Fix $0 < i < n$. An object of the 2–pullback $\mathcal{N}_{\{0,\dots,i\}}(\mathcal{C}) \times^{(2)}_{\mathcal{N}_{\{i\}}(\mathcal{C})} \mathcal{U}_{\{i,\dots,n\}}(\mathcal{C})$ is a triple

(13) $$(x_0 \to \cdots \to x_i, (x'_i \to \cdots \to x'_n \to x'_{n'} \to \cdots \to x'_{i'}, \psi_\bullet); \alpha)$$

with $\alpha \colon x_i \to x'_i$ an isomorphism. A morphism from the object (13) to a second object

$$(y_0 \to \cdots \to y_i, (y'_i \to \cdots \to y'_n \to y'_{n'} \to \cdots \to y'_{i'}, \mu_\bullet); \beta)$$

is a tuple $((p_0, \dots, p_i), (q_i, \dots, q_n, q_{n'}, \dots, q_{i'}))$ where $p_j \colon x_j \to y_j$, $1 \leq j \leq i$, and $q_j \colon x'_j \to y'_j$, $i \leq j \leq i'$, are isomorphisms which make the obvious nerve diagrams commute, the q_j respect the symmetric forms $\psi_\bullet$ and $\mu_\bullet$ and the compatibility condition

$$q_i \circ \alpha = \beta \circ p_i \tag{14}$$

holds.

We need to prove that the functor

$$\begin{aligned} \Psi_n \colon \mathcal{U}_n(\mathcal{C}) &\to \mathcal{N}_{\{0,\dots,i\}}(\mathcal{C}) \times^{(2)}_{\mathcal{N}_{\{i\}}(\mathcal{C})} \mathcal{U}_{\{i,\dots,n\}}(\mathcal{C}), \\ (x_\bullet, \psi_\bullet) &\mapsto (x_0 \to \dots \to x_i, (x_i \to \dots \to x_n \to x_{n'} \to \dots \to x_{i'}, \psi_\bullet); \mathbf{1}_{x_i}) \end{aligned}$$

is an equivalence. Consider a morphism $(p_\bullet, q_\bullet) \colon \Psi_n(x_\bullet, \psi_\bullet) \to \Psi_n(y_\bullet, \mu_\bullet)$. Since α and β are the identity maps in this case, the compatibility condition (14) reduces to $q_i = p_i$. Any morphism $\Psi_n(x_\bullet, \psi_\bullet) \to \Psi_n(y_\bullet, \mu_\bullet)$ is therefore the image of a unique isometry $(x_\bullet, \psi_\bullet) \to (y_\bullet, \mu_\bullet)$, since for an isometry the map $x_j \to y_j$ uniquely determines the map $x_{j'} \to y_{j'}$. It follows that Ψ_n is fully faithful. To show that Ψ_n is essentially surjective suppose that we are given an object of the form (13). Consider the object of $\mathcal{U}_n(\mathcal{C})$ whose underlying diagram is the top row of

$$\begin{array}{ccccccccccccccccc} x_0 & \to \cdots \to & x_{i-1} & \to & x'_i & \to \cdots \to & x'_n & \to & x'_{n'} & \to \cdots \longrightarrow & x'_{i'} & \longrightarrow & P(x_{i-1}) & \to \cdots \to & P(x_0) \\ & & \downarrow & \nearrow_{\alpha} & & & & & & & \downarrow{\scriptstyle \psi_{i'}} & \nearrow & & & \\ & & x_i & & & & & & & & P(x'_i) & & & & \end{array}$$

and whose symmetric form is $(\{\Theta_\bullet\}_{[0,i]}, \{\psi_\bullet\}_{[i,i']}, \{\mathbf{1}_\bullet\}_{[i',0']})$, the subscripts indicating the index intervals to which each map applies. In the above diagram the triangles define the corresponding horizontal morphisms and the unlabelled maps are the canonical ones. The image of this object under Ψ_n is

$$(x_0 \to \dots \to x_{i-1} \to x'_i, (x'_i \to \dots \to x'_n \to x'_{n'} \to \dots \to x'_{i'}, \psi_\bullet); \mathbf{1}_{x'_i}),$$

which is isomorphic via $((\mathbf{1}_{x_0}, \dots, \mathbf{1}_{x_{i-1}}, \alpha^{-1}), (\mathbf{1}_{x'_i}, \dots, \mathbf{1}_{x'_{i'}}))$ to the object (13). □

The constructions of this section also define a simplicial set $U_\bullet(\mathcal{C}) = N^e_\bullet(\mathcal{C})_h$ and a morphism $F_\bullet \colon U_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{C})$. While this morphism is neither left nor right relative 1–Segal, it is relevant to an interpretation of Proposition 3.4 in terms of an $(\infty, 1)$–presheaf on $\mathcal{C}$, as suggested by Section 2.1. We explain this in the remainder of the section.

First, recall the Joyal–Tierney Quillen equivalence (left adjoint on the left)

$$p_1^* \colon (\mathbb{S}, \mathcal{J}) \leftrightarrows (\mathbb{S}_\Delta, \mathcal{S}\mathrm{eg}_1) : i_1^*, \tag{15}$$

where $\mathcal{J}$ is the quasicategory model structure [19, Section 4]. The functor i_1^* is such that the n–simplices of $i_1^* X_\bullet$ are the 0–simplices of X_n. Moreover, for each complete 1–Segal space $X_\bullet \in \mathbb{S}_\Delta$ the equivalence (15) lifts to a Quillen equivalence

$$p_1^*\colon (\mathbb{S}_{/i_1^* X_\bullet}, \mathcal{J}^{\mathsf{rfib}}_{/i_1^* X_\bullet}) \leftrightarrows \big((\mathbb{S}_\Delta)_{/X_\bullet}, (\mathcal{S}\mathrm{eg}_1)^{\mathsf{rfib}}_{/X_\bullet}\big) : i_1^*,$$

where rfib denotes the induced right fibration model structure [3, Theorem 1.22].

Returning to unoriented nerves, let $\mathbb{N}_\bullet(\mathcal{C}) \in \mathbb{S}_\Delta$ be the complete 1–Segal space having $N_\bullet(\mathcal{N}_n(\mathcal{C}))$ as its n–simplices [32, Section 3.5]. In a similar way, we define $\mathbb{U}_\bullet(\mathcal{C}) \in \mathbb{S}_\Delta$. Proposition 3.4 implies that the induced morphism $\mathbb{F}_\bullet\colon \mathbb{U}_\bullet(\mathcal{C}) \to \mathbb{N}_\bullet(\mathcal{C})$ is right relative 1–Segal. Noting that $i_1^* \mathbb{N}_\bullet(\mathcal{C}) = N_\bullet(\mathcal{C})$ and $i_1^* \mathbb{U}_\bullet(\mathcal{C}) = U_\bullet(\mathcal{C})$, we obtain a right (Kan) fibration $i_1^* \mathbb{F}_\bullet\colon U_\bullet(\mathcal{C}) \to N_\bullet(\mathcal{C})$. We can therefore apply [14, Proposition A] to construct from $i_1^* \mathbb{F}_\bullet$ the desired $(\infty, 1)$–presheaf on $\mathcal{C}$. However, in our setting there is a simpler approach. As can be verified directly, the simplicial set $U_\bullet(\mathcal{C})$ is 1–Segal and hence isomorphic to the nerve of the category $\mathcal{S}$ whose objects are presented symmetric forms in $\mathcal{C}$ and whose morphisms are

$$\mathrm{Hom}_{\mathcal{S}}((x; f_x, \psi_x), (y; f_y, \psi_y)) = \{g \in \mathrm{Hom}_{\mathcal{C}}(x, y) \mid \widetilde{g} \circ f_y \circ g = f_x\},$$

where $\widetilde{g}$ is defined by $P(\widetilde{g}) = \psi_y \circ g \circ \psi_x^{-1}$. As $i_1^* \mathbb{F}_\bullet$ is a right fibration, the induced functor $F\colon \mathcal{S} \to \mathcal{C}$ is fibred in groupoids and thus determined by the presheaf

$$h^* F\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Grpd}, \quad x \mapsto \mathrm{Hom}_{\mathsf{Cat}_{/\mathcal{C}}}(\mathcal{C}_{/x}, \mathcal{S}).$$

More explicitly, the 2–Yoneda lemma defines an equivalence between $h^* F(x)$ and the fibre groupoid $\mathcal{S}_x$, the category with presented symmetric forms on x as objects and with a unique map $(x; f_x, \psi_x) \to (x; \bar{f}_x, \bar{\psi}_x)$ whenever

$$\bar{f}_x = \Theta^{-1}_{\mathsf{target}(\bar{\psi}_x)} \circ P(\psi_x \circ \bar{\psi}_x^{-1}) \circ \Theta_{\mathsf{target}(\psi_x)} \circ f_x.$$

In words, equivalences in $\mathcal{S}_x$ correspond to isomorphic changes in the target of ψ_x. After embedding Grpd into ∞–groupoids via the nerve, $h^* F$ agrees with the $(\infty, 1)$–presheaf produced by [14].

3.3 Unoriented categorified twisted cyclic nerves

Let X be a topological space with a self-homeomorphism $T\colon X \to X$. The T–twisted loop space of X is then defined to be

$$L^T X = \{\gamma \in C^0(\mathbb{R}, X) \mid \gamma(t+1) = T(\gamma(t))\}.$$

The case of the identity map $T = \mathbf{1}_X$ recovers the ordinary loop space LX. Suppose that we are also given a continuous involution $p\colon X \to X$ which satisfies

$$p = T \circ p \circ T.$$

This, together with the orientation-reversing involution $\sigma\colon \mathbb{R} \to \mathbb{R}$, $t \mapsto 1-t$, induces an involution

$$p_L\colon L^T X \to L^T X, \quad \gamma \mapsto p \circ \gamma \circ \sigma^{-1},$$

the fixed point set of which is naturally interpreted as the unoriented loop space of the stack $[X/\langle T\rangle]$. The goal of this section is to construct a relative 2–Segal simplicial space which is a categorical analogue of the unoriented loop space.

Let $\mathcal{C}$ be a small category with an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$. The categorified T–twisted cyclic nerve of $\mathcal{C}$ (see [7, Section 3.3; 5]) is the simplicial groupoid $\mathcal{NC}^T_\bullet(\mathcal{C})$ which assigns to $[n] \in \Delta$ the groupoid of all diagrams in $\mathcal{C}$ of the form

$$x_\bullet = \bigl\{x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-1} \xrightarrow{f_{n-1}} x_n \xrightarrow{f_n} T(x_0)\bigr\}. \tag{16}$$

A morphism $x_\bullet \to y_\bullet$ in $\mathcal{NC}^T_\bullet(\mathcal{C})$ is a collection of isomorphisms $x_i \to y_i$, $0 \le i \le n$, which make the obvious diagrams commute. The face map $\partial_i\colon \mathcal{NC}^T_n(\mathcal{C}) \to \mathcal{NC}^T_{n-1}(\mathcal{C})$ omits x_i and composes f_i and f_{i-1}, for $i = 1, \dots, n$. The face map ∂_0 sends (16) to

$$x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-1} \xrightarrow{f_{n-1}} x_n \xrightarrow{T(f_0)\circ f_n} T(x_1).$$

The degeneracy maps insert identity morphisms at appropriate spots.

It is proved in [7, Theorem 3.2.3] that the (noncategorified) T–twisted cyclic nerve $NC^T_\bullet(\mathcal{C})$ is a unital 2–Segal simplicial set. The analogous result holds also in the categorified setting.

Theorem 3.5 *For any small category $\mathcal{C}$ and endofunctor $T\colon \mathcal{C} \to \mathcal{C}$, the simplicial groupoid $\mathcal{NC}^T_\bullet(\mathcal{C})$ is unital 2–Segal.*

Proof Omitted. A similar result will be proved in Theorem 3.8 below. □

In addition to the pair $(\mathcal{C}, T)$, suppose now that we are given a duality structure (P, Θ) on $\mathcal{C}$ and a natural transformation

$$\lambda\colon P \Rightarrow T \circ P \circ T^{\mathrm{op}}$$

which satisfies the compatibility condition

$$TP(\lambda_x) \circ \lambda_{PT(x)} \circ \Theta_{T(x)} = T(\Theta_x) \quad \text{for } x \in \mathcal{C}. \tag{17}$$

For example, the pair $(T, \lambda) = (\mathbf{1}_{\mathcal{C}}, \mathbf{1}_P)$ satisfies this condition.

For each $n \geq 0$ define a functor $P_n\colon \mathcal{N}C_n^T(\mathcal{C})^{\mathrm{op}} \to \mathcal{N}C_n^T(\mathcal{C})$ by sending the diagram (16) to

$$P(x_n) \xrightarrow{P(f_{n-1})} P(x_{n-1}) \xrightarrow{P(f_{n-2})} \cdots \xrightarrow{P(f_1)} P(x_1) \xrightarrow{P(f_0)} P(x_0) \xrightarrow{P_*(f_n)} TP(x_n),$$

where $P_*(f_n)\colon P(x_0) \xrightarrow{\lambda_{x_0}} TPT(x_0) \xrightarrow{TP(f_n)} TP(x_n)$. The action of P_n on morphisms is the obvious one.

Lemma 3.6 *The data $\Theta_{n,x_\bullet} = (\Theta_{x_0}, \dots, \Theta_{x_n}, T(\Theta_{x_0}))$, $x_\bullet \in \mathcal{N}C_n^T(\mathcal{C})$, define a natural isomorphism $\Theta_n\colon \mathbf{1}_{\mathcal{N}C_n^T(\mathcal{C})} \Rightarrow P_n \circ P_n^{\mathrm{op}}$, giving a triple $(\mathcal{N}C_\bullet^T(\mathcal{C}), P_\bullet, \Theta_\bullet)$ which satisfies the hypotheses of Lemma 3.2.*

Proof We will prove that $\Theta_{n,x_\bullet}$ defines a morphism $x_\bullet \to P_n^2(x_\bullet)$; the remaining statements of the lemma can be verified directly. Keeping the notation (16), we need to show that the diagram

$$\begin{array}{ccc} x_n & \xrightarrow{f_n} & T(x_0) \\ \downarrow{\scriptstyle \Theta_{x_n}} & & \downarrow{\scriptstyle T(\Theta_{x_0})} \\ P^2(x_n) & \xrightarrow[P_*^2(f_n)]{} & TP^2(x_0) \end{array}$$

commutes. The definition of P_* implies that

$$P_*^2(f_n) = TP(\lambda_{x_0}) \circ TPTP(f_n) \circ \lambda_{P(x_n)}.$$

The natural transformation λ associates to $x_n \xrightarrow{f_n} T(x_0)$ the following commutative diagram:

$$\begin{array}{ccc} TPTP(x_n) & \xrightarrow{TPTP(f_n)} & TPTPT(x_0) \\ \uparrow{\scriptstyle \lambda_{P(x_n)}} & & \uparrow{\scriptstyle \lambda_{PT(x_0)}} \\ P^2(x_n) & \xrightarrow[P^2(f_n)]{} & P^2T(x_0) \end{array}$$

Combining this with the equality $P^2(f_n)\circ\Theta_{x_n}=\Theta_{T(x_0)}\circ f_n$, we compute

$$\begin{aligned}P^2_*(f_0)\circ\Theta_{x_n} &= TP(\lambda_{x_0})\circ TPTP(f_n)\circ\lambda_{P(x_n)}\circ\Theta_{x_n}\\ &= TP(\lambda_{x_0})\circ\lambda_{PT(x_0)}\circ\Theta_{T(x_0)}\circ f_n\\ &= T(\Theta_{x_0})\circ f_n,\end{aligned}$$

where in the final step the compatibility condition (17) was used. □

Lemmas 3.2 and 3.6 imply that $\mathcal{N}C^{T,e}_\bullet(\mathcal{C})$ is a simplicial groupoid with duality.

Definition 3.7 The Hermitian groupoid $\mathcal{N}C^{T,e}_\bullet(\mathcal{C})_h$, denoted by $\mathcal{N}U^T_\bullet(\mathcal{C})$, is called the unoriented categorified T–twisted cyclic nerve of $(\mathcal{C},P,\Theta;T,\lambda)$.

Write $(x_\bullet,\psi_\bullet)\in\mathcal{N}U^T_n(\mathcal{C})$ for the diagram

$$x_0\xrightarrow{f_0}x_1\xrightarrow{f_1}\cdots\xrightarrow{f_{n-1}}x_n\xrightarrow{f_n}x_{n'}\xrightarrow{f_{n'}}x_{(n-1)'}\xrightarrow{f_{(n-1)'}}\cdots\xrightarrow{f_{1'}}x_{0'}\xrightarrow{f_{0'}}T(x_0)$$

together with symmetric isomorphisms $\psi_\bullet$ which satisfy $\psi_{i+1}\circ f_i=P(f_{(i+1)'})\circ\psi_i$, $i=0,\dots,n-1$, and

$$\psi_{n'}\circ f_n=P(f_n)\circ\psi_n,\quad T(\psi_0)\circ f_{0'}=P_*(f_{0'})\circ\psi_{0'}.$$

The following statement gives a relative 2–Segal space which plays the role of the unoriented loop space.

Theorem 3.8 *For any small category with duality $(\mathcal{C},P,\Theta)$ and endofunctor $T\colon\mathcal{C}\to\mathcal{C}$ with compatibility data λ as above, the morphism $F_\bullet\colon\mathcal{N}U^T_\bullet(\mathcal{C})\to\mathcal{N}C^T_\bullet(\mathcal{C})$ is a unital relative 2–Segal simplicial groupoid.*

Proof We need to prove that the 1–Segal morphisms

$$\Xi_n\colon\mathcal{N}U^T_n(\mathcal{C})\to\mathcal{N}U^T_i(\mathcal{C})\times^{(2)}_{\mathcal{N}U^T_{\{i\}}(\mathcal{C})}\mathcal{N}U^T_{n-i}(\mathcal{C})$$

and the relative 2–Segal morphisms

$$\Psi_n\colon\mathcal{N}U^T_n(\mathcal{C})\to\mathcal{N}C^T_n(\mathcal{C})\times^{(2)}_{\mathcal{N}C^T_{\{0,n\}}(\mathcal{C})}\mathcal{N}U^T_{\{0,n\}}(\mathcal{C})$$

are equivalences. Since the proofs are similar we will only prove the latter. Explicitly, the functor Ψ_n sends $(x_\bullet,\psi_\bullet)\in\mathcal{N}U^T_n(\mathcal{C})$ to

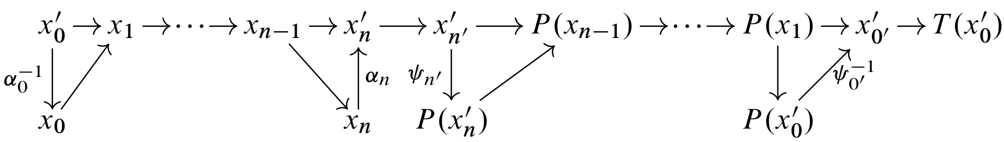
$$\big(x_0\to\cdots\to x_n\to T(x_0),\ (x_0\to x_n\to x_{n'}\to x_{0'}\to T(x_0),\psi_\bullet);(\mathbf{1}_{x_0},\mathbf{1}_{x_n})\big).$$

To see that Ψ_n is fully faithful, let $\Psi_n(x_\bullet, \psi_\bullet) \to \Psi_n(y_\bullet, \mu_\bullet)$ be a morphism determined by maps $((p_0, \dots, p_n), (q_0, q_n))$, the notation an obvious modification of that from the proof of Proposition 3.4. Since the isomorphisms α_0, α_n and β_0, β_n used to define arbitrary morphisms in $\mathcal{N}C_n^T(\mathcal{C}) \times^{(2)}_{\mathcal{N}C^T_{\{0,n\}}(\mathcal{C})} \mathcal{N}U^T_{\{0,n\}}(\mathcal{C})$ are the identities in this case, the compatibility conditions (analogous to (14)) imply $p_0 = q_0$ and $p_n = q_n$. Using the symmetry conditions imposed by the isometry condition, any morphism $\Psi_n(x_\bullet, \psi_\bullet) \to \Psi_n(y_\bullet, \mu_\bullet)$ is therefore the image of a unique morphism $(x_\bullet, \psi_\bullet) \to (y_\bullet, \mu_\bullet)$.

To prove that Ψ_n is essentially surjective, let

$$(18) \qquad \big(x_0 \to \cdots \to x_n \to T(x_0), (x'_0 \to x'_n \to x'_{n'} \to x'_{0'} \to T(x'_0), \psi_\bullet); (\alpha_0, \alpha_n)\big)$$

be an arbitrary object of $\mathcal{N}C_n^T(\mathcal{C}) \times^{(2)}_{\mathcal{N}C^T_{\{0,n\}}(\mathcal{C})} \mathcal{N}U^T_{\{0,n\}}(\mathcal{C})$. Define an object of $\mathcal{N}U_n^T(\mathcal{C})$ by the top row of the diagram



with symmetric form $(\{\Theta_\bullet\}_{[1,n-1]}, \{\psi_\bullet\}_{\{0,n,n',0'\}}, \{\mathbf{1}_\bullet\}_{[(n-1)',1']})$. The image of this object under Ψ_n is

$$\big(x'_0 \to x_1 \to \cdots \to x_{n-1} \to x'_n \to T(x'_0),\ (x'_0 \to \cdots \to x'_{0'} \to T(x'_0), \psi_\bullet); (\mathbf{1}_{x'_0}, \mathbf{1}_{x'_n})\big).$$

Then the tuple $((\alpha_0^{-1}, \mathbf{1}_{x_1}, \dots, \mathbf{1}_{x_{n-1}}, \alpha_n^{-1}), (\mathbf{1}_{x'_0}, \dots, \mathbf{1}_{x'_{0'}}))$ defines an isomorphism from the previous object to the object (18). Hence Ψ_n is an equivalence.

The proof that relative unitality holds is similar. □

Remark The homotopical counterpart of the $\mathsf{SO}(2)$–action on LX is the cyclic structure of $NC_\bullet(\mathcal{C})$ in the sense of Connes. This action extends to $\mathsf{O}(2)$ by allowing orientation-reversal of loops and corresponds to the dihedral structure of $NC_\bullet(\mathcal{C})$ induced by a (strict) duality P on $\mathcal{C}$. More generally, $NC_\bullet^T(\mathcal{C})$ has a paradihedral or, if $T^r = 1$ for some $r \geq 2$, an r–dihedral structure. Similar comments apply to categorified nerves. See [9, Section I.6] for further discussion.

3.4 The $\mathcal{R}_\bullet$–construction

The goal of this section is to prove that the $\mathcal{R}_\bullet$–construction (also called the Hermitian $\mathcal{S}_\bullet$–construction) from Grothendieck–Witt theory is relative 2–Segal over the Waldhausen $\mathcal{S}_\bullet$–construction. We work in the proto-exact setting of Section 1.4.

Definition 3.9 A proto-exact category with duality is a category with duality $(\mathcal{C}, P, \Theta)$ such that $\mathcal{C} = (\mathcal{C}, \mathfrak{I}, \mathfrak{D})$ is proto-exact and P is an exact functor. Explicitly, exactness of the functor P is the following three conditions:

(i) $P(0) \simeq 0$,

(ii) a morphism $U \xrightarrow{\phi} V$ is in $\mathfrak{I}$ if and only if $P(V) \xrightarrow{P(\phi)} P(U)$ is in $\mathfrak{D}$, and

(iii) P sends bicartesian squares to bicartesian squares.

Let N be a symmetric form in a proto-exact category with duality $\mathcal{C}$ and let $i\colon U \rightarrowtail N$ be an inflation. The orthogonal $U^{\perp}$ is defined to be the following pullback:

$$\begin{array}{ccc} U^{\perp} & \dashrightarrow & N \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & P(U) \end{array}$$

The inflation i is called isotropic if the composition $P(i)\psi_N i$ is zero and the canonical monomorphism $U \to U^{\perp}$ is an inflation.

Example (1) Let $\mathsf{Rep}_k(Q)$ be the abelian category of representations of quiver Q. Given a contravariant involution of Q together with some combinatorial data, there is an associated exact duality structure on $\mathsf{Rep}_k(Q)$ whose symmetric forms are (generalizations of) the orthogonal or symplectic quiver representations of Derksen and Weyman [4]. See also [48, Section 1.4]. A similar construction applies when $\mathsf{Rep}_k(Q)$ is replaced with $\mathsf{Rep}_{\mathbb{F}_1}(Q)$.

(2) Let $\mathsf{Vect}(X)$ be the exact category of vector bundles over a scheme X. Fix a line bundle $L \to X$ and a sign $s \in \{\pm 1\}$. Then $(P, \Theta) = (\mathrm{Hom}_{\mathcal{O}_X}(-, L), s \cdot \mathsf{can})$ defines an exact duality structure on $\mathsf{Vect}(X)$ whose symmetric forms are vector bundles over X with L–valued orthogonal ($s = 1$) or symplectic ($s = -1$) forms.

In this section we will make the following assumption on $(\mathcal{C}, P, \Theta)$.

Assumption Let N be a symmetric form in $\mathcal{C}$ and let $U \rightarrowtail N$ be isotropic with orthogonal $k\colon U^{\perp} \rightarrowtail N$. Define $M \in \mathcal{C}$ to be the following pushout:

$$\begin{array}{ccc} U & \rightarrowtail & U^{\perp} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ 0 & \dashrightarrow & M \end{array}$$

Then there is a unique symmetric form ψ_M on M such that $P(k)\psi_N k = P(\pi)\psi_M \pi$.

The symmetric form (M, ψ_M) is called the isotropic reduction of N by U and is denoted by $N /\!\!/ U$. When $\mathcal{C}$ is exact the reduction assumption is known to hold; see

[30, Lemma 5.2; 35, Lemma 2.6]. In a number of (nonexact) proto-exact examples, such as $\mathsf{Rep}_{\mathbb{F}_1}(Q)$, the reduction assumption also holds.

The category $[n]$ has a strict duality structure given at the level of objects by $i \mapsto n-i$. Then $[\mathsf{Ar}_n, \mathcal{C}]$ and its full subcategory $\mathcal{W}_n(\mathcal{C})$ inherit duality structures. Moreover, the duality structures on $\mathcal{W}_\bullet(\mathcal{C})$ satisfy the assumptions of Lemma 3.2 so that $\mathcal{W}^e_\bullet(\mathcal{C})$ is a simplicial object of CatD. Explicitly, the dual of a diagram $\{A_{\{p,q\}}\}_{0 \le p \le q \le 0'} \in \mathcal{W}^e_n(\mathcal{C})$ is the diagram $\{P(A_{\{q',p'\}})\}_{0 \le p \le q \le 0'}$ and the double dual identification at (p,q) is $\Theta_{A_{\{p,q\}}}$.

The $\mathcal{R}_\bullet$–construction of $(\mathcal{C}, P, \Theta)$, denoted by $\mathcal{R}_\bullet(\mathcal{C})$, is the Hermitian groupoid $\mathcal{W}^e_\bullet(\mathcal{C})_h$; see [39; 15].[1] Objects of $\mathcal{R}_n(\mathcal{C})$ are diagrams $\{A_{\{p,q\}}\}_{0 \le p \le q \le 0'} \in \mathcal{W}^e_n(\mathcal{C})$ together with symmetric isomorphisms $\psi_{p,q} \colon A_{\{p,q\}} \to P(A_{\{q',p'\}})$ which make all appropriate diagrams commute. In particular,

(i) for every $1 \le i \le n$ the pair $(A_{\{i,i'\}}, \psi_{i,i'})$ is a symmetric form,

(ii) for every $0 \le i \le j \le n$ the inflation $A_{\{i,j\}} \rightarrowtail A_{\{i,i'\}}$ is isotropic with orthogonal $A_{\{i,j'\}} \rightarrowtail A_{\{i,i'\}}$, and

(iii) for every $0 \le i \le j \le n$ the symmetric form $(A_{\{j,j'\}}, \psi_{j,j'})$ is isometric to the reduction $(A_{\{i,i'\}}, \psi_{i,i'}) /\!\!/ A_{\{i,j\}}$.

For example, an object of $\mathcal{R}_0(\mathcal{C})$ is a diagram

$$\begin{array}{ccc} 0 & \rightarrowtail & A_{\{0,0'\}} \\ & & \downarrow \\ & & 0 \end{array}$$

together with a symmetric form on $A_{\{0,0'\}}$. Hence $\mathcal{R}_0(\mathcal{C})$ is equivalent to the Hermitian groupoid $\mathcal{C}_h$. Similarly, an object of $\mathcal{R}_1(\mathcal{C})$ is a diagram

$$\begin{array}{ccccccc} 0 & \rightarrowtail & A_{\{0,1\}} & \rightarrowtail & A_{\{0,1'\}} & \rightarrowtail & A_{\{0,0'\}} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \rightarrowtail & A_{\{1,1'\}} & \rightarrowtail & A_{\{1,0'\}} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & \rightarrowtail & A_{\{1',0'\}} \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

[1] In [15], the notation $\mathcal{R}_\bullet(\mathcal{C})$ is used for $\mathcal{W}^e_\bullet(\mathcal{C})$, whereas what we call $\mathcal{R}_\bullet(\mathcal{C})$ is denoted by $\mathcal{R}^h_\bullet(\mathcal{C})$.

all of whose squares are bicartesian, together with the data of

(i) a symmetric form on $A_{\{0,0'\}}$ such that $A_{\{0,1\}} \rightarrowtail A_{\{0,0'\}}$ is isotropic with orthogonal $A_{\{0,1'\}} \rightarrowtail A_{\{0,0'\}}$,

(ii) a symmetric form on $A_{\{1,1'\}}$ presenting $A_{\{1,1'\}}$ as $A_{\{0,0'\}} /\!\!/ A_{\{0,1\}}$, and

(iii) isomorphisms $A_{\{0,1\}} \simeq P(A_{\{1',0'\}})$ and $A_{\{0,1'\}} \simeq P(A_{\{1,0'\}})$ such that each morphism above the diagonal agrees with the corresponding morphism below the diagonal.

In short, objects of $\mathcal{R}_n(\mathcal{C})$ are isotropic n–flags together with presentations of all subquotients and subreductions. The forgetful map $F_\bullet\colon \mathcal{R}_\bullet(\mathcal{C}) \to \mathcal{S}_\bullet(\mathcal{C})$ given by

$$F_n\colon \mathcal{R}_n(\mathcal{C}) \to \mathcal{S}_n(\mathcal{C}), \quad \{A_{\{p,q\}}, \psi_{p,q}\}_{0 \le p \le q \le 0'} \mapsto \{A_{\{p,q\}}\}_{0 \le p \le q \le n}$$

is a morphism of simplicial groupoids.

Theorem 3.10 *Let $\mathcal{C}$ be a proto-exact category with duality which satisfies the reduction assumption. Then the morphism $F_\bullet\colon \mathcal{R}_\bullet(\mathcal{C}) \to \mathcal{S}_\bullet(\mathcal{C})$ is a unital relative 2–Segal simplicial groupoid.*

Proof For each $n \ge 0$ let $\mathcal{I}_n(\mathcal{C})$ be the groupoid of isotropic n–flags in $\mathcal{C}$. An object of $\mathcal{I}_n(\mathcal{C})$ is a diagram

$$0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_n \rightarrowtail A_{n'} \rightarrowtail \cdots \rightarrowtail A_{1'} \rightarrowtail (A_{0'}, \psi_{0'}) \tag{19}$$

with $(A_{0'}, \psi_{0'})$ a symmetric form such that, for each $0 \le i \le n$, the inflation $A_i \rightarrowtail A_{0'}$ is isotropic with orthogonal $A_{i'} \rightarrowtail A_{0'}$. We claim that the forgetful functor

$$\nu_n\colon \mathcal{R}_n(\mathcal{C}) \to \mathcal{I}_n(\mathcal{C}), \quad \{A_{\{i,j\}}, \psi_{i,j}\}_{0 \le i \le j \le 0'} \mapsto \{A_{\{0,j\}}, \psi_{0,0'}\}_{0 \le j \le 0'}$$

is an equivalence. Construct a quasi-inverse η_n of ν_n as follows. Given an isotropic flag (19), put $A_{\{0,k\}} = A_k$, $1 \le k \le 1'$, and $(A_{\{0,0'\}}, \psi_{0,0'}) = (A_{0'}, \psi_{0'})$ and let $A_{\{1,k\}}$ be the following pushout:

$$\begin{array}{ccc} A_{\{0,1\}} & \rightarrowtail & A_{\{0,k\}} \\ \downarrow & & \downarrow \\ 0 & \dashrightarrow & A_{\{1,k\}} \end{array}$$

By the reduction assumption, the symmetric form $(A_{\{0,0'\}}, \psi_{0,0'})$ induces a unique compatible symmetric form on $A_{\{1,1'\}}$. We also define $A_{\{1,0'\}} = P(A_{\{0,1'\}})$, and let $A_{\{0,0'\}} \twoheadrightarrow A_{\{1',0'\}}$ be the composition

$$A_{\{0,0'\}} \xrightarrow{\psi_{0,0'}} P(A_{\{0,0'\}}) \twoheadrightarrow P(A_{\{0,1'\}}).$$

This construction defines the top two rows of $\eta_n(A_{\{0,\bullet\}})$ and can be iterated to define the remaining $n-2$ rows. It is clear that ν_n is a quasi-inverse to η_n.

We can now prove the theorem. The 1–Segal morphism for $\mathcal{R}_\bullet(\mathcal{C})$ is

$$\Xi_n\colon \mathcal{R}_n(\mathcal{C}) \to \mathcal{R}_{\{0,\dots,i\}}(\mathcal{C}) \times^{(2)}_{\mathcal{R}_{\{i\}}(\mathcal{C})} \mathcal{R}_{\{i,\dots,n\}}(\mathcal{C}).$$

Arguing as in the proof of Theorem 2.10 and using that ν_n is an equivalence, to prove that Ξ_n is an equivalence it suffices to prove that the functor

$$\widetilde{\Xi}_n\colon \mathcal{I}_n(\mathcal{C}) \to \mathcal{I}_i(\mathcal{C}) \times^{(2)}_{\mathcal{I}_{\{i\}}(\mathcal{C})} \mathcal{I}_{n-i}(\mathcal{C})$$

is an equivalence. A quasi-inverse of $\widetilde{\Xi}_n$ is defined by assigning to a pair

$$(0 \rightarrowtail A'_1 \rightarrowtail \cdots \rightarrowtail A'_i \rightarrowtail A'_{i'} \rightarrowtail \cdots \rightarrowtail A'_{1'} \rightarrowtail (A'_{0'}, \psi'_{0'})) \in \mathcal{I}_i(\mathcal{C})$$

and

$$(0 \rightarrowtail A''_{i+1} \rightarrowtail \cdots \rightarrowtail A''_n \rightarrowtail A''_{n'} \rightarrowtail \cdots \rightarrowtail A''_{(i+1)''} \rightarrowtail (A''_{i'}, \psi''_{i'})) \in \mathcal{I}_{n-i}(\mathcal{C})$$

together with an isometry $(A'_{0'}, \psi'_{0'}) /\!\!/ A'_i \simeq (A''_{i'}, \psi''_{i'})$ the object

$$(0 \rightarrowtail A'_1 \rightarrowtail \cdots \rightarrowtail A'_n \rightarrowtail A'_{n'} \rightarrowtail \cdots \rightarrowtail A'_{1'} \rightarrowtail (A'_{0'}, \psi'_{0'})) \in \mathcal{I}_n(\mathcal{C}),$$

where A'_j, $i+1 \le j \le (i+1)'$, is defined to be the following pullback:

$$\begin{array}{ccc} A'_j & \dashrightarrow & A'_{i'} \\ \downarrow & & \downarrow \\ A''_j & \rightarrowtail & A''_{i'} \end{array}$$

That this is indeed a quasi-inverse follows from the Hermitian variant of the second isomorphism theorem [30, Proposition 6.5], which generalizes to the proto-exact setting under the reduction assumption.

Arguing in the same way, to prove that the functor

$$\Psi_n\colon \mathcal{R}_n(\mathcal{C}) \to \mathcal{S}_n(\mathcal{C}) \times^{(2)}_{\mathcal{S}_{\{0,n\}}(\mathcal{C})} \mathcal{R}_{\{0,n\}}(\mathcal{C})$$

is an equivalence it suffices to prove that the functor

$$\widetilde{\Psi}_n\colon \mathcal{I}_n(\mathcal{C}) \to \mathcal{F}_n(\mathcal{C}) \times^{(2)}_{\mathcal{F}_{\{0,n\}}(\mathcal{C})} \mathcal{I}_{\{0,n\}}(\mathcal{C})$$

is an equivalence, which is obvious.

Finally, relative unitality is the condition that, for each $0 \leq i \leq n-1$, the functor

$$\Upsilon_n \colon \mathcal{R}_{n-1}(\mathcal{C}) \to \mathcal{S}_{\{i\}}(\mathcal{C}) \times^{(2)}_{\mathcal{S}_{\{i,i+1\}}(\mathcal{C})} \mathcal{R}_n(\mathcal{C})$$

is an equivalence. Note that $\mathcal{S}_0(\mathcal{C})$ is a point, $\mathcal{S}_1(\mathcal{C})$ is the maximal groupoid of $\mathcal{C}$ and the map $\mathcal{S}_0(\mathcal{C}) \xrightarrow{s_0} \mathcal{S}_1(\mathcal{C})$ sends the point to the zero object. An object of $\mathcal{S}_{\{i\}}(\mathcal{C}) \times^{(2)}_{\mathcal{S}_{\{i,i+1\}}(\mathcal{C})} \mathcal{R}_n(\mathcal{C})$ is therefore an object of $\mathcal{R}_n(\mathcal{C})$ whose maps between the i^{th} and $(i+1)^{\text{st}}$ rows/columns are the identities. It follows from this that Υ_n is an equivalence. □

When $\mathcal{C}$ is exact, the morphism $F_\bullet \colon \mathcal{R}_\bullet(\mathcal{C}) \to \mathcal{S}_\bullet(\mathcal{C})$ is closely related to the higher Grothendieck–Witt theory of $\mathcal{C}$. Indeed, $GW_i(\mathcal{C})$ is the i^{th} homotopy group of the homotopy fibre over $0 \in \mathcal{C}$ of the map $|BF_\bullet| \colon |B\mathcal{R}_\bullet(\mathcal{C})| \to |B\mathcal{S}_\bullet(\mathcal{C})|$. See [36].

4 Applications to Hall algebras

4.1 Categorical Hall algebra representations

After reviewing the relationship between 2–Segal spaces and categorical Hall algebras described in [7], in this section we construct from a relative 2–Segal space a categorical representation of the Hall algebra of the base.

Fix a combinatorial model category $\mathcal{C}$, such as $\mathbb{S}$ with its Kan model structure or Grpd with its Bousfield model structure; the latter will be the main source of examples. While the Quillen model structure on Top is not combinatorial, it is Quillen equivalent to $\mathbb{S}$ so that the results of this section, for all intents and purposes, also apply to simplicial spaces. We write $\mathrm{pt} \in \mathcal{C}$ for the final object.

Let $\mathsf{Span}(\mathcal{C})$ be the bicategory of spans in $\mathcal{C}$. Objects of $\mathsf{Span}(\mathcal{C})$ are simply objects of $\mathcal{C}$ while 1–morphisms $A \to B$ in $\mathsf{Span}(\mathcal{C})$ are spans from A to B, that is, diagrams in $\mathcal{C}$ of the form

$$A \leftarrow X \to B.$$

Composition of spans is given by homotopy pullback. A 2–morphism in $\mathsf{Span}(\mathcal{C})$ is a homotopy commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow & \downarrow & \searrow & \\ A & & & & B \\ & \nwarrow & \downarrow & \nearrow & \\ & & X' & & \end{array}$$

in $\mathcal{C}$. Give $\mathsf{Span}(\mathcal{C})$ the cartesian monoidal structure.

Associated to the bicategory $\mathsf{Span}(\mathcal{C})$ is the category $\mathsf{Span}(\mathcal{C})^{\sim}$, having the same objects as $\mathsf{Span}(\mathcal{C})$ but with morphisms being the 2–isomorphism classes spans. The constructions of this section are simplified if one uses $\mathsf{Span}(\mathcal{C})^{\sim}$ in place of $\mathsf{Span}(\mathcal{C})$, the downside being that less of the (relative) 2–Segal structure is used.

Definition 4.1 [7, Section 8.1] A transfer structure on $\mathcal{C}$ is a pair $(\mathcal{S},\mathcal{P})$ consisting of collections of morphisms $\mathcal{S}$ and $\mathcal{P}$ in $\mathcal{C}$, called smooth and proper, respectively, which satisfy the following properties:

(1) Both collections $\mathcal{S}$ and $\mathcal{P}$ are closed under composition.

(2) Given a homotopy cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X' & \xrightarrow[s']{} & Y' \end{array} \tag{20}$$

in $\mathcal{C}$ with $s' \in \mathcal{S}$ and $p' \in \mathcal{P}$, we have $s \in \mathcal{S}$ and $p \in \mathcal{P}$.

Example The pair $(\mathcal{S},\mathcal{P}) = (\mathcal{C},\mathcal{C})$ is called the trivial transfer structure on $\mathcal{C}$.

Fix a transfer structure $(\mathcal{S},\mathcal{P})$. Spans of the form $A \xleftarrow{s} X \xrightarrow{p} B$ with $s \in \mathcal{S}$ and $p \in \mathcal{P}$ are called $(\mathcal{S},\mathcal{P})$–admissible and form a subbicategory $\mathsf{Span}(\mathcal{S},\mathcal{P}) \subset \mathsf{Span}(\mathcal{C})$.

Theorem 4.2 [7, Proposition 8.1.7] *Let $X_\bullet$ be a 2–Segal object of $\mathcal{C}$. Assume that the span*

$$m_X = \left\{ X_{\{0,1\}} \times X_{\{1,2\}} \xleftarrow{(\partial_2,\partial_0)} X_{\{0,1,2\}} \xrightarrow{\partial_1} X_{\{0,2\}} \right\} \in \mathrm{Hom}_{\mathsf{Span}(\mathcal{C})}(X_1 \otimes X_1, X_1) \tag{21}$$

is $(\mathcal{S},\mathcal{P})$–admissible. Then (X_1, m_X) is a semigroup in $\mathsf{Span}(\mathcal{S},\mathcal{P})$. Moreover, if $X_\bullet$ is unital and the span

$$I_X = \left\{ \mathrm{pt} \xleftarrow{\mathsf{can}} X_0 \xrightarrow{s_0} X_1 \right\} \in \mathrm{Hom}_{\mathsf{Span}(\mathcal{C})}(\mathrm{pt}, X_1)$$

is $(\mathcal{S},\mathcal{P})$–admissible, then (X_1, m_X, I_X) is a monoid in $\mathsf{Span}(\mathcal{S},\mathcal{P})$.

The pair $\mathcal{H}(X_\bullet) = (X_1, m_X)$ is called the $(\mathcal{S},\mathcal{P})$–universal Hall algebra of $X_\bullet$. Slightly abusively, the associativity isomorphism a_X for m_X is omitted from the notation. Note that at the universal level the transfer structure $(\mathcal{S},\mathcal{P})$ simply determines the subbicategory $\mathsf{Span}(\mathcal{S},\mathcal{P}) \subset \mathsf{Span}(\mathcal{C})$ to which $\mathcal{H}(X_\bullet)$ belongs. In particular, we can always take the trivial transfer structure. A more important role is played by the transfer structure when passing from the universal Hall algebra to its concrete realizations. To explain this procedure, fix a monoidal category $(\mathcal{V}, \otimes, \mathbf{1}_{\mathcal{V}})$.

Definition 4.3 [7, Section 8.1] A $\mathcal{V}$–valued theory with transfer on $\mathcal{C}$ is the data of

(1) a transfer structure $(\mathcal{S}, \mathcal{P})$ on $\mathcal{C}$,

(2) a contravariant functor $(-)^*\colon \mathcal{S} \to \mathcal{V}$ and a covariant functor $(-)_*\colon \mathcal{P} \to \mathcal{V}$ with common values on objects, denoted by $\mathfrak{h}$, and which map weak equivalences to isomorphisms, and

(3) an isomorphism $\mathfrak{h}(\mathrm{pt}) \simeq \mathbf{1}_{\mathcal{V}}$ and multiplicativity data for $\mathfrak{h}$, that is, natural maps
$$\mathfrak{h}(X) \otimes \mathfrak{h}(Y) \to \mathfrak{h}(X \otimes Y)$$
which satisfy associativity and unitality conditions

such that for a homotopy cartesian diagram of the form (20) with $s, s' \in \mathcal{S}$ and $p, p' \in \mathcal{P}$, we have $p'_* \circ s^* = s'^* \circ p_*$.

Applying a $\mathcal{V}$–valued theory with transfer $\mathfrak{h}$ to the $(\mathcal{S}, \mathcal{P})$–universal Hall algebra $\mathcal{H}(X_\bullet)$ gives a semigroup in $\mathcal{V}$, denoted by
$$\mathcal{H}(X_\bullet; \mathfrak{h}) = (\mathfrak{h}(X_1), \partial_{1*} \circ (\partial_2 \times \partial_0)^*).$$
Dually, if the opposite span $m_X^{\mathrm{op}} \in \mathrm{Hom}_{\mathsf{Span}(\mathcal{C})}(X_1, X_1 \otimes X_1)$ is $(\mathcal{S}, \mathcal{P})$–admissible, then (X_1, m_X^{op}) is a cosemigroup in $\mathsf{Span}(\mathcal{S}, \mathcal{P})$, called the $(\mathcal{S}, \mathcal{P})$–universal Hall coalgebra, and passing to theories with transfer gives cosemigroups in monoidal categories. These statements can be proved in the same way as Theorem 4.2.

Example Let $\mathcal{S}_\bullet(\mathcal{C})$ be the Waldhausen groupoid of an essentially small exact category $\mathcal{C}$. Suppose that $\mathcal{C}$ is finitary in the sense that $\mathrm{Ext}^n_{\mathsf{env}(\mathcal{C})}(U, V)$, $n = 0, 1$, is finite for all $U, V \in \mathcal{C}$. Here $\mathsf{env}(\mathcal{C})$ denotes the abelian envelope of $\mathcal{C}$ [31]. Then $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}))$ categorifies the Hall algebra of $\mathcal{C}$, as defined in various contexts in [33; 16; 34]. To see this, let k be a field of characteristic zero. Consider the transfer structure on Grpd in which $\mathcal{S}$ and $\mathcal{P}$ are the collections of weakly proper and locally proper morphisms, respectively (see [7, Section 8.2]). A Vect_k–valued theory with transfer $\mathfrak{F}_0$ on Grpd is then be defined by taking finitely supported k–valued functions which are constant on isomorphism classes. The resulting k–algebra $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ is the standard Hall algebra of $\mathcal{C}$. Explicitly, $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C});\mathfrak{F}_0)$ is the k–vector space with basis the characteristic functions $\{1_U\}_{U\in\pi_0(\mathcal{S}_1(\mathcal{C}))}$ and multiplication

$$1_U\cdot 1_V=\sum_{W\in\pi_0(\mathcal{S}_1(\mathcal{C}))}F^W_{U,V}1_W.$$

Here $F^W_{U,V}$ is the number of admissible subobjects of W which are isomorphic to U and have quotient isomorphic to V.

For certain categories $\mathcal{C}$, such as $\mathsf{Coh}(X)$ for a smooth projective variety X or $\mathsf{mod}(A)$ for a finitely generated algebra A, the $\mathcal{S}_\bullet$–construction defines a 2–Segal simplicial Artin stack. This allows to recover the perverse sheaf-theoretic [26], motivic [20; 22] and cohomological [23] Hall algebras of $\mathcal{C}$. See [7, Section 8.5].

By using relative 2–Segal spaces we can easily modify the above results to construct module objects.

Theorem 4.4 *Let $X_\bullet$ be as in Theorem 4.2 and let $F_\bullet\colon Y_\bullet\to X_\bullet$ be a relative 2–Segal object of $\mathcal{C}$. Assume that the left action span*

$$\mu^l_Y=\bigl\{X_{\{0,1\}}\times Y_{\{1\}}\xleftarrow{(F_1,\partial_0)}Y_{\{0,1\}}\xrightarrow{\partial_1}Y_{\{1\}}\bigr\}\in\mathrm{Hom}_{\mathsf{Span}(\mathcal{C})}(X_1\otimes Y_0,Y_0)\tag{22}$$

is $(\mathcal{S},\mathcal{P})$–admissible. Then (Y_0,μ^l_Y) is a left (X_1,m_X)–module in $\mathsf{Span}(\mathcal{S},\mathcal{P})$. Moreover, if $F_\bullet$ is unital, then (Y_0,μ^l_Y) is a left (X_1,m_X,I_X)–module. Analogous statements hold for the right action span

$$\mu^r_Y=\bigl\{Y_{\{0\}}\times X_{\{0,1\}}\xleftarrow{\partial_1\times F_1}Y_{\{0,1\}}\xrightarrow{\partial_0}Y_{\{1\}}\bigr\}\in\mathrm{Hom}_{\mathsf{Span}(\mathcal{C})}(Y_0\otimes X_1,Y_0).$$

Proof We will prove the theorem for the left action span. It follows directly from the definitions that we have

$$\mu^l_Y\circ(m_X\otimes\mathbf{1}_{Y_0})=\left\{\begin{array}{ccc}(X_{\{0,1,2\}}\times Y_{\{2\}})\times^R_{X_{\{0,2\}}\times Y_{\{2\}}}Y_{\{0,2\}} & \longrightarrow Y_{\{0,2\}} & \longrightarrow Y_{\{0\}}\\ \downarrow & \downarrow & \\ X_{\{0,1,2\}}\times Y_{\{2\}} & \longrightarrow X_{\{0,2\}}\times Y_{\{2\}} & \\ \downarrow & & \\ X_{\{0,1\}}\times X_{\{1,2\}}\times Y_{\{2\}} & & \end{array}\right\}$$

and

$$\mu^l_Y\circ(\mathbf{1}_{X_1}\otimes\mu^l_Y)=\left\{\begin{array}{ccc}(X_{\{0,1\}}\times Y_{\{1,2\}})\times^R_{X_{\{0,1\}}\times Y_{\{1\}}}Y_{\{0,1\}} & \longrightarrow Y_{\{0,1\}} & \longrightarrow Y_{\{0\}}\\ \downarrow & \downarrow & \\ X_{\{0,1\}}\times Y_{\{1,2\}} & \longrightarrow X_{\{0,1\}}\times Y_{\{1\}} & \\ \downarrow & & \\ X_{\{0,1\}}\times X_{\{1,2\}}\times Y_{\{2\}} & & \end{array}\right\}$$

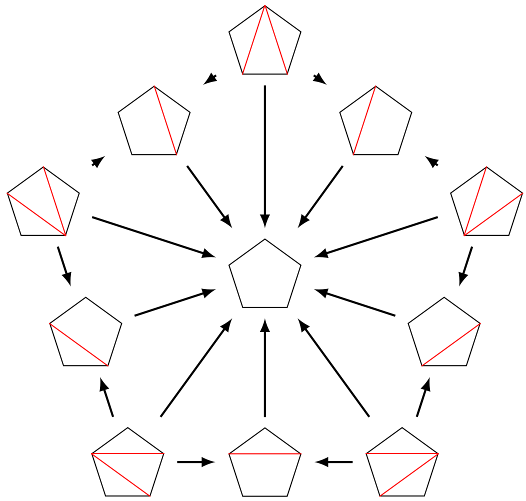


Figure 3: The poset of polyhedral subdivisions of P_3^∞

as spans $X_1 \otimes X_1 \otimes Y_0 \to Y_0$. We claim that both $\mu_Y^l \circ (m_X \otimes \mathbf{1}_{Y_0})$ and $\mu_Y^l \circ (\mathbf{1}_{X_1} \otimes \mu_Y^l)$ are 2–isomorphic to the span

$$\sigma^l = \left\{ X_{\{0,1\}} \times X_{\{1,2\}} \times Y_{\{2\}} \xleftarrow{(F_{\{0,1\}} \circ \partial_2, F_{\{1,2\}} \circ \partial_0, \partial_0 \circ \partial_1)} Y_{\{0,1,2\}} \xrightarrow{\partial_1 \circ \partial_2} Y_{\{0\}} \right\}.$$

Indeed, the composition

$$Y_{\{0,1,2\}} \xrightarrow{\alpha_1} (X_{\{0,1,2\}} \times Y_{\{2\}}) \times^R_{X_{\{0,2\}} \times Y_{\{2\}}} Y_{\{0,2\}} \longrightarrow X_{\{0,1,2\}} \times^R_{X_{\{0,2\}}} Y_{\{0,2\}}$$

is a weak equivalence by the relative 2–Segal condition on $F_\bullet$. Since the second functor is a weak equivalence for trivial reasons, it follows that α_1 is a weak equivalence and so defines a 2–isomorphism $\sigma^l \to \mu_Y^l \circ (m_X \otimes \mathbf{1}_{Y_0})$. Similarly, the composition

$$Y_{\{0,1,2\}} \xrightarrow{\alpha_2} (X_{\{0,1\}} \times Y_{\{1,2\}}) \times^R_{X_{\{0,1\}} \times Y_{\{1\}}} Y_{\{0,1\}} \longrightarrow Y_{\{1,2\}} \times^R_{Y_{\{1\}}} Y_{\{0,1\}}$$

is a weak equivalence by the 1–Segal condition on $Y_\bullet$ and α_2 defines a 2–isomorphism $\sigma^l \to \mu_Y^l \circ (\mathbf{1}_{X_1} \otimes \mu_Y^l)$. We claim that the composition

$$\alpha_Y^l \colon \mu_Y^l \circ (m_X \otimes \mathbf{1}_{Y_0}) \xrightarrow{\alpha_1^{-1}} \sigma^l \xrightarrow{\alpha_2} \mu_Y^l \circ (\mathbf{1}_{X_1} \otimes \mu_Y^l)$$

is a module associator for the left action of (X_1, m_X) on Y_0 determined by μ_Y^l. Without the unital assumption, this amounts to verifying that α_Y^l satisfies module-theoretic Mac Lane coherence; see [28, Section 2.3] or diagram (23) below. We will verify this using the setting of Section 2.3. The poset of the eleven polyhedral subdivisions of the relative pentagon P_3^∞, ordered by refinement, is illustrated in Figure 3. At the level of the map $F_\bullet\colon Y_\bullet \to X_\bullet$, each node $\mathcal{P}$ of Figure 3 defines a span

$$X_1 \times X_1 \times X_1 \times Y_0 \leftarrow (\Delta^{\mathcal{P}}, F_\bullet)_R \to Y_0.$$

Similarly, each arrow of Figure 3 defines a 2–isomorphism of spans. The spans associated to the five vertices of the pentagon are precisely the spans appearing in the diagram expressing module-theoretic Mac Lane coherence while the composed 2–isomorphisms along the edges of the pentagon are precisely the arrows in the coherence diagram. It follows that module-theoretic Mac Lane coherence holds. Hence (Y_0, μ_Y^l) is a left (X_1, m_X)–module.

Finally, in the unital setting the action of I_X on Y_0 is given by the span

$$\begin{array}{ccccc}
(X_{\{0\}} \times Y_{\{1\}}) \times^R_{X_{\{0,1\}} \times Y_{\{1\}}} Y_{\{0,1\}} & \longrightarrow & Y_{\{0,1\}} & \longrightarrow & Y_{\{0\}} \\
\downarrow & & \downarrow & & \\
X_{\{0\}} \times Y_{\{1\}} & \longrightarrow & X_{\{0,1\}} \times Y_{\{1\}} & & \\
\downarrow & & & & \\
\mathrm{pt} \times Y_{\{1\}} & & & &
\end{array}$$

which is 2–isomorphic to the identity span $Y_0 \xleftarrow{\mathbf{1}_{Y_0}} Y_0 \xrightarrow{\mathbf{1}_{Y_0}} Y_0$. Indeed, the composition of functors

$$Y_{\{0\}} \xrightarrow{I_Y} (X_{\{0\}} \times Y_{\{1\}}) \times^R_{X_{\{0,1\}} \times Y_{\{1\}}} Y_{\{0,1\}} \longrightarrow X_{\{0\}} \times^R_{X_{\{0,1\}}} Y_{\{0,1\}}$$

is an equivalence by the unital relative 2–Segal condition while the second functor is trivially an equivalence. To complete the proof we need to verify that I_Y is compatible with the module associator in the sense that diagram (24) below commutes. This is a straightforward exercise which can be completed in much the same way as the above verification of Mac Lane coherence. □

The left $\mathcal{H}(X_\bullet)$–module $\mathcal{M}(Y_\bullet) = (Y_0, \mu_Y^l)$ is called the $(\mathcal{S}, \mathcal{P})$–universal left Hall module of $F_\bullet$. From a $\mathcal{V}$–valued theory with transfer $\mathfrak{h}$ we obtain a left $\mathcal{H}(X_\bullet; \mathfrak{h})$–module $\mathcal{M}(Y_\bullet; \mathfrak{h})$ in $\mathcal{V}$. In the same way we get right modules over $\mathcal{H}(X_\bullet)$ and $\mathcal{H}(X_\bullet; \mathfrak{h})$. Note that the left and right module structures do not define a bimodule.

Example Let $X_\bullet$ be 2–Segal object of $\mathcal{C}$. By Proposition 2.8 the right path space $F_\bullet^{\triangleright}\colon P^{\triangleright} X_\bullet \to X_\bullet$ is relative 2–Segal. Since $F_n^{\triangleright} = \partial_{n'}$, the spans $\mu^l_{P^{\triangleright} X}$ and m_X are equal. Hence $\mathcal{M}(X_\bullet^{\triangleright}) = \mathcal{H}(X_\bullet)$ as left $\mathcal{H}(X_\bullet)$–modules. On the other hand, the right $\mathcal{H}(X_\bullet)$–module structure on $\mathcal{M}(X_\bullet^{\triangleright})$ is closely related to the coproduct Δ on $\mathcal{H}(X_\bullet)$. Consider for example $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C});\mathfrak{F}_0)$ for a finitary exact category $\mathcal{C}$. Recall that the Green bilinear form on $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C});\mathfrak{F}_0)$, defined by

$$(1_U, 1_V) = \frac{\delta_{U,V}}{|\mathrm{Aut}(U)|},$$

satisfies the Hopf property $(1_U \otimes 1_V, \Delta 1_W) = (1_U \cdot 1_V, 1_W)$. Then the right and left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C});\mathfrak{F}_0)$–actions are adjoint with respect to $(-,-)$.

Remark (1) Generalizing the previous observation, it is proved in Proposition 5.6.10 of [45] that for a relative 2–Segal simplicial groupoid $Y_\bullet \to X_\bullet$, the right and left $\mathcal{H}(X_\bullet)$–module structures of $\mathcal{M}(Y_\bullet)$ are related via a categorical Green bilinear form.

(2) The universal Hall algebra $\mathcal{H}(X_\bullet)$ is itself a $\mathcal{H}(X_\bullet)$–bimodule which, however, does not arise from a relative 2–Segal space over $X_\bullet$.

Example Let $\mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \to \mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ be the relative 2–Segal groupoid associated to a stability function Z and a framing Φ on an abelian category $\mathcal{C}$. The universal left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}}))$–module $\mathcal{M}(\mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}))$ categorifies the stable framed Hall algebra representations of [41; 11] which appear in framed Donaldson–Thomas theory.

Example Let $\mathcal{R}_\bullet(\mathcal{C}) \to \mathcal{S}_\bullet(\mathcal{C})$ be the relative 2–Segal groupoid associated to a proto-exact category with duality via the $\mathcal{R}_\bullet$–construction. For finitary exact $\mathcal{C}$, the universal left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}))$–module $\mathcal{M}(\mathcal{R}_\bullet(\mathcal{C}))$ categorifies the Hall algebra representations of [47]. Explicitly, $\mathcal{M}(\mathcal{R}_\bullet(\mathcal{C});\mathfrak{F}_0)$ is the k–vector space with basis $\{1_{(M,\psi_M)}\}_{(M,\psi_M)\in\pi_0(\mathcal{R}_0(\mathcal{C}))}$ and left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C});\mathfrak{F}_0)$–module structure

$$1_U \star 1_{(M,\psi_M)} = \sum_{(N,\psi_N)\in\pi_0(\mathcal{R}_0(\mathcal{C}))} G_{U,M}^N 1_{(N,\psi_N)}.$$

The structure constant $G_{U,M}^N$ is the number of isotropic subobjects of N which are isomorphic to U and have reduction isometric to M.

Similarly, the simplicial stack version of $\mathcal{R}_\bullet(\mathcal{C})$ is relative 2–Segal over $\mathcal{S}_\bullet(\mathcal{C})$ and recovers the perverse sheaf-theoretic [10], motivic and cohomological [48] Hall algebra representations which appear in the representation theory of quantum enveloping algebras and orientifold Donaldson–Thomas theory.

4.2 Hall monoidal module categories

We briefly describe a variant of the constructions of Section 4.1. Let k be a field of characteristic zero. Let $X_\bullet$ be a unital 2–Segal groupoid[2] which is admissible with respect to weakly proper and locally proper maps. Write $\mathsf{Fun}_0(X_1)$ for the (abelian) category of finitely supported functors $X_1 \to \mathsf{Vect}_k$. The span m_X from Theorem 4.2 induces a bifunctor

$$\otimes^X = \partial_{1*} \circ (\partial_2 \times \partial_0)^*\colon \mathsf{Fun}_0(X_1) \otimes \mathsf{Fun}_0(X_1) \to \mathsf{Fun}_0(X_1).$$

It is proved in [6, Theorem 2.49] that the triple $\mathcal{H}^{\otimes}(X_\bullet) = (\mathsf{Fun}_0(X_1), \otimes^X, I_X)$ is a monoidal category. More generally, each componentwise cocontinuous monoidal left derivator of groupoids $\mathbb{D}$ determines a monoidal category structure on $\mathbb{D}(X_1)$; see [45, Theorem 5.0.1(1)]. The example $\mathcal{H}^{\otimes}(X_\bullet)$ arises for $\mathbb{D} = \mathsf{Fun}_0(-, \mathsf{Vect}_k)$.

In a similar way, we have the following relative construction. The same statement is proved in [45, Proposition 5.2.6(2)], as a special case of [45, Theorem 5.0.1(2)].

Theorem 4.5 *Assume that $F_\bullet\colon Y_\bullet \to X_\bullet$ is an admissible unital relative 2–Segal groupoid. Then the left action span μ_Y^l defines a bifunctor*

$$\otimes^Y = \partial_{1*} \circ (F_1 \times \partial_0)^*\colon \mathsf{Fun}_0(X_1) \otimes \mathsf{Fun}_0(Y_0) \to \mathsf{Fun}_0(Y_0)$$

which gives $\mathcal{M}^{\otimes}(Y_\bullet) = (\mathsf{Fun}_0(Y_0), \otimes^Y)$ the structure of a left $\mathcal{H}^{\otimes}(X_\bullet)$–module category in the sense of [28].

Proof The proof is very similar to that Theorem 4.4 and is therefore omitted. □

We give two related instances of Theorem 4.5.

Example Let $\mathsf{Vect}_{\mathbb{F}_1}$ be the proto-exact category of finite-dimensional vector spaces over $\mathbb{F}_1$. Objects of $\mathsf{Vect}_{\mathbb{F}_1}$ are finite pointed sets and morphisms are partial bijections. Let $\mathsf{Vect}_{\mathbb{F}_1}^{\mathsf{sk}} \subset \mathsf{Vect}_{\mathbb{F}_1}$ be the skeleton of standard ordinals. Writing $\mathfrak{S}_n$ for the symmetric group on n letters, we have an equivalence of groupoids

$$\mathcal{S}_1(\mathsf{Vect}_{\mathbb{F}_1}^{\mathsf{sk}}) \simeq \bigsqcup_{n \geq 0} B\mathfrak{S}_n.$$

Objects of $\mathcal{H}^{\otimes}(\mathcal{S}_\bullet(\mathsf{Vect}_{\mathbb{F}_1}^{\mathsf{sk}}))$ (henceforth denoted by $\mathcal{H}^{\otimes}(\mathsf{Vect}_{\mathbb{F}_1}^{\mathsf{sk}})$) are thus sequences of finite-dimensional representations of symmetric groups over k, only finitely many of

[2] With minor modifications $X_\bullet$ could be a simplicial object of a combinatorial model category.

which are nontrivial. The monoidal product is induction of representations. Using the results of [27], it follows that $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})$ is equivalent to the category $\mathbf{P}$ of polynomial functors $\mathsf{Vect}_k \to \mathsf{Vect}_k$ [6, Section 2.5.1].

The functor $P = \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{F}_1}}(-, \{*, 1\})$ defines a strict exact duality structure on $\mathsf{Vect}_{\mathbb{F}_1}$. Note that each object of $\mathsf{Vect}_{\mathbb{F}_1}$ is canonically isomorphic to its dual. In particular, P preserves $\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}$. A symmetric form on $\mathbb{F}_1^n$ is an element $\pi \in \mathfrak{S}_n$ which squares to the identity; conjugate such elements determine isometric symmetric forms. It follows that symmetric forms on $\mathbb{F}_1^n$ are determined uniquely by their Witt index $0 \le w \le \lfloor \frac{1}{2} n \rfloor$, the number of 2–cycles of any representative π. The isometry group of π, which is its centralizer in $\mathfrak{S}_n$, is isomorphic to $(\mathbb{Z}_2 \wr \mathfrak{S}_w) \times \mathfrak{S}_{n-2w}$. We therefore obtain an equivalence of groupoids

$$\mathcal{R}_0(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}) \simeq \bigsqcup_{w \ge 0} \bigsqcup_{d \ge 0} B((\mathbb{Z}_2 \wr \mathfrak{S}_w) \times \mathfrak{S}_d),$$

and we see that objects of $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})$ are finite sequences of representations of groups of the form $(\mathbb{Z}_2 \wr \mathfrak{S}_w) \times \mathfrak{S}_d$. The left $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})$–action on $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})$ is induction of representations along subgroups of the form

$$\mathfrak{S}_n \times ((\mathbb{Z}_2 \wr \mathfrak{S}_w) \times \mathfrak{S}_d) \le (\mathbb{Z}_2 \wr \mathfrak{S}_{n+w}) \times \mathfrak{S}_d.$$

From this description we see that

$$\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}) = \bigoplus_{d=0}^{\infty} \mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; d)$$

as $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})$–modules, the index d labelling the fixed difference between the dimension and twice the Witt index. Moreover, we have

$$\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; d) \simeq \mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; 0) \times \mathbf{P}^d,$$

where $\mathbf{P}^d \subset \mathbf{P}$ is the full subcategory of degree d homogeneous polynomial functors. Using again results of [27], we find that $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; 0)$ is equivalent to $\mathbf{P}_{\mathbb{Z}_2}$, the category of polynomial functors $\mathsf{Vect}_k^{\mathbb{Z}_2\text{-gr}} \to \mathsf{Vect}_k$, where $\mathsf{Vect}_k^{\mathbb{Z}_2\text{-gr}}$ denotes the category of $\mathbb{Z}_2$–graded finite-dimensional vector spaces over k and we view $\mathbf{P}_{\mathbb{Z}_2}$ as a $\mathbf{P}$–module category via the forgetful functor $\mathsf{Vect}_k^{\mathbb{Z}_2\text{-gr}} \to \mathsf{Vect}_k$.

Upon passing to Grothendieck groups we obtain an isomorphism

$$K_0(\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; d)) \simeq \biggl(\bigoplus_{w=0}^{\infty} R_k(\mathbb{Z}_2 \wr \mathfrak{S}_w) \biggr) \otimes_{\mathbb{Z}} R_k(\mathfrak{S}_d)$$

as modules over the algebra

$$K_0(\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1})) \simeq \bigoplus_{n=0}^{\infty} R_k(\mathfrak{S}_n),$$

where $R_k(-)$ denotes the representation ring over k. In the case $k = \mathbb{C}$ these modules have been studied in [40] and are closely related to the work of Zelevinsky [49], who studied the algebra structure on $K_0(\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; 0))$ arising from induction of representations of wreath symmetric products.

We now give a sort of quantization of the previous example.

Example Let q be a prime power and consider the exact category $\mathsf{Vect}_{\mathbb{F}_q}$. Let $\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q} \subset \mathsf{Vect}_{\mathbb{F}_q}$ be the skeleton consisting of the vector spaces $\mathbb{F}_q^n$, $n \geq 0$. We have an equivalence of groupoids

$$S_1(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}) \simeq \bigsqcup_{n \geq 0} B\mathsf{GL}_n(\mathbb{F}_q).$$

The category $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$, whose monoidal product is parabolic induction of finite-dimensional representations of GL, has appeared in the work of Joyal and Street [18]. When $k = \mathbb{C}$ the associated algebra $K_0(\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}))$, which is the complex representation ring of the tower of general linear groups over $\mathbb{F}_q$, has been studied by Green [13] and Zelevinsky [49].

Assume now that q is odd and fix a sign $s \in \{\pm 1\}$. Take the exact duality structure with $P = \mathrm{Hom}_{\mathsf{Vect}_{\mathbb{F}_q}}(-, \mathbb{F}_q)$ with $\Theta = s \cdot \mathsf{can}$. Symmetric forms in $\mathsf{Vect}_{\mathbb{F}_q}$ are orthogonal or symplectic vector spaces. Identify the dual of $\mathbb{F}_q^n$ with itself via the dual basis. It follows that we have an equivalence of groupoids

$$R_0(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}) \simeq \bigsqcup_{n \geq 0} \bigsqcup_{\varepsilon \in \mathsf{W}_n} B\mathsf{G}^{\varepsilon}_n,$$

where W_n is the Witt group of $\mathbb{F}_q^n$ ($\mathbb{Z}_2$ if $s = 1$ and trivial if $s = -1$) and $\mathsf{G}^{\varepsilon}_n = \mathsf{O}^{\varepsilon}_n(\mathbb{F}_q)$ if $s = +1$ and $\mathsf{G}_n = \mathsf{Sp}_{2n}(\mathbb{F}_q)$ if $s = -1$. The left $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$–action on $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$ is given by parabolic induction between GL and G representations. When $k = \mathbb{C}$ the $K_0(\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}))$–modules $K_0(\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}))$ were studied by van Leeuwen [24], who showed that they are generated by cuspidal elements with respect to the natural Hall comodule structure.

It is natural to regard the module $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$ as a sort of q–analogue of the modules $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; d)$, $d = 0, 1$. This is consistent with the philosophy that the $q \to 1$

limit of $\mathsf{G}(\mathbb{F}_q)$ is its Weyl group, namely $\mathbb{Z}_2 \wr \mathfrak{S}_n$ for the symplectic Sp_{2n} and even orthogonal O_{2n} groups and $(\mathbb{Z}_2 \wr \mathfrak{S}_n) \times \mathbb{Z}_2$ for the odd orthogonal group O_{2n+1}. The extra factor of $\mathbb{Z}_2$ for O_{2n+1} reflects the fact that the subcategories of $\mathcal{M}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$ with fixed Witt type ε are isomorphic as left $\mathcal{H}^{\otimes}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q})$–modules, the same statement holding also at $q = 1$. Working instead with Ringel-style Hall algebras and modules allows for the following more precise statement, which can be verified directly: the $q = 1$ specialization of the $\mathcal{H}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}; \mathfrak{F}_0)$–module $\mathcal{M}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_q}; \mathfrak{F}_0)$ is isomorphic to the direct sum of the $\mathcal{H}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; \mathfrak{F}_0)$–modules $\mathcal{M}(\mathsf{Vect}^{\mathsf{sk}}_{\mathbb{F}_1}; \mathfrak{F}_0)$, $d = 0, 1$.

4.3 Modules over multivalued categories

In this section and the next we describe two higher categorical interpretations of relative 2–Segal simplicial sets; the first is in terms of modules over multivalued categories while the second is of a Hall algebraic nature.

Let $2\text{-}\mathsf{Seg}\mathbb{S} \subset \mathbb{S}$ be the full subcategory of unital 2–Segal simplicial sets. Let also $\mu\mathsf{Cat}$ denote the category of small multivalued categories. Objects of $\mu\mathsf{Cat}$ are tuples $\mathfrak{X} = (\mathfrak{X}_0, \mathfrak{X}_1, m_{\mathfrak{X}}, a_{\mathfrak{X}}, e_{\mathfrak{X}}, i^l_{\mathfrak{X}}, i^r_{\mathfrak{X}})$ consisting of sets $\mathfrak{X}_0$ and $\mathfrak{X}_1$ with source and target maps $\partial_1, \partial_0\colon \mathfrak{X}_1 \to \mathfrak{X}_0$, a composition law $m_{\mathfrak{X}} \in \mathrm{Hom}_{\mathsf{Span}(\mathsf{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1, \mathfrak{X}_1)$, an associator isomorphism

$$(m_{\mathfrak{X}} \circ (m_{\mathfrak{X}} \times \mathbf{1}_{\mathfrak{X}_1}) \xrightarrow{a_{\mathfrak{X}}} m_{\mathfrak{X}} \circ (\mathbf{1}_{\mathfrak{X}_1} \times m_{\mathfrak{X}})) \in \mathrm{Hom}_{\mathsf{Span}(\mathsf{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1, \mathfrak{X}_1)$$

which satisfies Mac Lane coherence, a unit map $e_{\mathfrak{X}}\colon \mathfrak{X}_0 \to \mathfrak{X}_1$ and compatible left and right unit isomorphisms $i^l_{\mathfrak{X}}, i^r_{\mathfrak{X}} \in \mathrm{Hom}_{\mathsf{Span}(\mathsf{Set})}(\mathfrak{X}_1, \mathfrak{X}_1)$. A morphism of multivalued categories $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$ is the data of maps $\varphi_i\colon \mathfrak{X}_i \to \mathfrak{X}'_i$, $i = 0, 1$, compatible with the source, target and unit maps and a morphism of spans $\widetilde{\varphi}_2\colon \varphi_1 \circ m_{\mathfrak{X}} \to m_{\mathfrak{X}'} \circ (\varphi_1 \times_{\varphi_0} \varphi_1)$.

Theorem 4.6 [7, Theorem 3.3.6] *The generalized nerve construction defines an equivalence of categories* $\mathsf{N}_\bullet\colon \mu\mathsf{Cat} \xrightarrow{\sim} 2\text{-}\mathsf{Seg}\mathbb{S}$.

In the relative setting we will require the notion of a module over a multivalued category.

Definition 4.7 Let $\mathfrak{X}$ be a multivalued category. A unital left $\mathfrak{X}$–module is a tuple $\mathfrak{Y} = (\mathfrak{Y}_0, \mathfrak{F}_0, \mu_{\mathfrak{Y}}, \alpha_{\mathfrak{Y}}, \iota_{\mathfrak{Y}})$ consisting of

(i) a set $\mathfrak{Y}_0$ together with a map $\mathfrak{F}_0\colon \mathfrak{Y}_0 \to \mathfrak{X}_0$,

(ii) a left action span $\mu_{\mathfrak{Y}} \in \mathrm{Hom}_{\mathsf{Span}(\mathsf{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$,

(iii) a module associator isomorphism

$$(\mu_{\mathfrak{Y}} \circ (m_{\mathfrak{X}} \times \mathbf{1}_{\mathfrak{Y}_0}) \xrightarrow{\alpha_{\mathfrak{Y}}} \mu_{\mathfrak{Y}} \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu_{\mathfrak{Y}})) \in \mathrm{Hom}_{\mathsf{Span(Set)}}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$$

which satisfies module-theoretic Mac Lane coherence: the diagram

$$\begin{array}{ccc} & \mu \circ (m \circ (m \times \mathbf{1}_{\mathfrak{X}_1}) \times \mathbf{1}_{\mathfrak{Y}_0}) & \\ \swarrow{\scriptstyle \alpha \circ (\mu \times \mathbf{1}_{\mathfrak{X}_1} \times \mathbf{1}_{\mathfrak{Y}_0})} & & \searrow{\scriptstyle \mu \circ (a \times \mathbf{1}_{\mathfrak{Y}_0})} \\ \mu \circ (m \times \mu) & & \mu \circ (m \circ (\mathbf{1}_{\mathfrak{X}_1} \times m) \times \mathbf{1}_{\mathfrak{Y}_0}) \\ \Big\downarrow{\scriptstyle \alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mathbf{1}_{\mathfrak{X}_1} \times \mu)} & & \Big\downarrow{\scriptstyle \alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times m \times \mathbf{1}_{\mathfrak{Y}_0})} \\ \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu)) & \xleftarrow[\mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \alpha)]{} & \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times (\mu \circ (m \circ \mathbf{1}_{\mathfrak{Y}_0}))) \end{array} \tag{23}$$

commutes in $\mathrm{Hom}_{\mathsf{Span(Set)}}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$, and

(iv) a unit isomorphism $\iota_{\mathfrak{Y}}\colon \mu_{\mathfrak{Y}} \circ (e_{\mathfrak{X}} \times \mathbf{1}_{\mathfrak{Y}_0}) \to \mathbf{1}_{\mathfrak{Y}_0}$ in $\mathrm{Hom}_{\mathsf{Span(Set)}}(\mathfrak{Y}_0, \mathfrak{Y}_0)$ for which the diagram

$$\begin{array}{ccc} \mu \circ (m \circ (\mathbf{1}_{\mathfrak{X}_1} \times e) \times \mathbf{1}_{\mathfrak{Y}_0}) & \xrightarrow{\alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times e \times \mathbf{1}_{\mathfrak{Y}_0})} & \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu \circ (e \times \mathbf{1}_{\mathfrak{Y}_0})) \\ & {\scriptstyle \mu \circ (i^l_{\mathfrak{X}} \times \mathbf{1}_{\mathfrak{Y}_0})} \searrow \quad \swarrow {\scriptstyle \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \iota_{\mathfrak{Y}})} & \\ & \mu & \end{array} \tag{24}$$

commutes in $\mathrm{Hom}_{\mathsf{Span(Set)}}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$.

Define a category $\mu\mathsf{Cat}\text{-}\mathsf{mod}$ as follows. Objects are unital left modules over multivalued categories. A morphism $(\varphi, \vartheta)\colon (\mathfrak{Y} \to \mathfrak{X}) \to (\mathfrak{Y}' \to \mathfrak{X}')$ consists of a morphism $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$, a map $\vartheta_0\colon \mathfrak{Y}_0 \to \mathfrak{Y}'_0$ which satisfies $\mathfrak{F}'_0 \circ \vartheta_0 = \varphi_0 \circ \mathfrak{F}_0$ and a morphism of spans

$$\widetilde{\vartheta}_1\colon \vartheta_0 \circ \mu_{\mathfrak{Y}} \to \mu_{\mathfrak{Y}'} \circ (\varphi_1 \times_{\varphi_0} \vartheta_0).$$

Informally, ϑ defines a morphism of $\mathfrak{X}$–modules $\mathfrak{Y} \to \varphi^* \mathfrak{Y}'$.

Let $2\text{-}\mathsf{SegRel}\mathbb{S} \subset \mathbb{S}^{[1]}$ be the full subcategory of unital relative 2–Segal simplicial sets.

Theorem 4.8 *The equivalence* $\mathsf{N}_\bullet\colon \mu\mathsf{Cat} \xrightarrow{\sim} 2\text{-}\mathsf{Seg}\mathbb{S}$ *lifts to an equivalence*

$$\mathsf{N}^{\mathsf{rel}}_\bullet\colon \mu\mathsf{Cat}\text{-}\mathsf{mod} \xrightarrow{\sim} 2\text{-}\mathsf{SegRel}\mathbb{S}.$$

Proof We begin by modifying the proof of [7, Theorem 3.3.6] so as to define a functor $2\text{-}\mathsf{SegRel}\mathbb{S} \to \mu\mathsf{Cat}\text{-}\mathsf{mod}$. Let $F_\bullet\colon Y_\bullet \to X_\bullet$ be a unital relative 2–Segal simplicial set.

The multivalued category $\mathfrak{X}$ associated to $X_\bullet$ via Theorem 4.6 has $\mathfrak{X}_0 = X_0$ and $\mathfrak{X}_1 = X_1$ with the canonical maps $\partial_1, \partial_0\colon \mathfrak{X}_1 \to \mathfrak{X}_0$ and composition span

$$m_{\mathfrak{X}} = \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \xleftarrow{(\partial_2,\partial_0)} X_2 \xrightarrow{\partial_1} \mathfrak{X}_1\}.$$

Turning to the relative data, let $\mathfrak{Y}_0 = Y_0$ and $\mathfrak{F}_0 = F_0$. Define an action span by

$$\mu_{\mathfrak{Y}} = \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \xleftarrow{(F_1,\partial_0)} Y_1 \xrightarrow{\partial_1} \mathfrak{Y}_0\}.$$

The morphisms of spans

$$\mu_{\mathfrak{Y}} \circ (m_{\mathfrak{X}} \times \mathbf{1}_{\mathfrak{Y}_0}) \leftarrow \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \leftarrow Y_2 \to \mathfrak{Y}_0\} \to \mu_{\mathfrak{Y}} \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu_{\mathfrak{Y}}),$$

each of which is constructed as in the proof of Theorem 4.4 and is an isomorphism by the relative 2–Segal conditions, combine to define the associator $\alpha_{\mathfrak{Y}}$. The unit isomorphism $\iota_{\mathfrak{Y}}$ is defined to be the inverse of the relative unit bijection $Y_{\{0\}} \to X_{\{0\}} \times_{X_{\{0,1\}}} Y_{\{0,1\}}$. Mac Lane coherence and unit compatibility are verified as in the proof of Theorem 4.4. Hence $\mathfrak{Y}$ is a unital left $\mathfrak{X}$–module. At the level of morphisms the functor $2\text{-}\mathsf{SegRel}\mathbb{S} \to \mu\mathsf{Cat}\text{-}\mathsf{mod}$ is defined in the obvious way.

To describe a quasi-inverse $\mathsf{N}^{\mathsf{rel}}_\bullet\colon \mu\mathsf{Cat}\text{-}\mathsf{mod} \to 2\text{-}\mathsf{SegRel}\mathbb{S}$, let $\mathfrak{X} \in \mu\mathsf{Cat}$ with associated unital 2–Segal simplicial set $X_\bullet = \mathsf{N}_\bullet(\mathfrak{X})$. Given a unital left $\mathfrak{X}$–module $\mathfrak{Y}$, let $Y_0 = \mathfrak{Y}_0$ and let Y_1 be the middle set of the span $\mu_{\mathfrak{Y}}$. We have canonical maps $\partial_i\colon Y_1 \to Y_0$ and $F_i\colon Y_i \to X_i$, $i = 0, 1$, and, by inverting the map which defines $\iota_{\mathfrak{Y}}$, a map $s_0\colon Y_0 \to Y_1$. Moreover, these maps obey the 1–truncated simplicial identities. For each $n \geq 2$ define

$$\widetilde{Y}_n = \varprojlim^{\mathsf{Set}}_{\{\sigma\colon \Delta^k \hookrightarrow \Delta^{P_n^\infty}\}_{k \leq 2}} F_{P_n^\infty}(\sigma),$$

where we use the notation introduced in Section 2.3. Given $0 \leq i < j < k \leq n$ and an element $\widetilde{y} \in \widetilde{Y}_n$, write y_{ij} and y_{ijk} for the corresponding elements of $Y_{\{i,j\}}$ and $X_{\{i,j,k\}}$, respectively. The associator $\alpha_{\mathfrak{Y}}$ defines a collection of bijections

$$\alpha_{ijk}\colon X_{\{i,j,k\}} \times_{X_{\{i,k\}}} Y_{\{i,k\}} \to Y_{\{i,j\}} \times_{Y_{\{j\}}} Y_{\{j,k\}}$$

which we use to define

$$Y_n = \bigl\{\widetilde{y} \in \widetilde{Y}_n \mid \alpha_{ijk}(y_{ijk}, y_{ik}) = (y_{ij}, y_{jk}) \text{ for all } 0 \leq i < j < k \leq n\bigr\}.$$

Module-theoretic Mac Lane coherence implies that the Y_n assemble to a 1–Segal simplicial set $Y_\bullet$ and that the canonical maps $F_n\colon Y_n \to X_n$ assemble to a simplicial morphism $F_\bullet$. To see that $F_\bullet$ satisfies the remaining relative 2–Segal conditions, consider the following commutative diagram:

$$\begin{array}{ccc} Y_n & \longrightarrow & X_n \times_{X_{\{0,n\}}} Y_{\{0,n\}} \\ \downarrow & & \downarrow \\ Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 & \longrightarrow & X_{\{0,1,2\}} \times_{X_{\{0,2\}}} \cdots \times_{X_{\{0,n-1\}}} X_{\{0,n-1,n\}} \times_{X_{\{0,n\}}} Y_{\{0,n\}} \end{array}$$

The vertical morphisms are bijections by the 1– and 2–Segal conditions on $Y_\bullet$ and $X_\bullet$, respectively. The bottom arrow is the iterated application of the module associator, relating the bracketings in which n elements of $\mathfrak{X}$ act on $\mathfrak{Y}$ from right to left and from left to right, and is therefore a bijection. Hence $F_\bullet$ is relative 2–Segal. The relative unit bijection $Y_{\{0\}} \to X_{\{0\}} \times_{X_{\{0,1\}}} Y_{\{0,1\}}$ is the inverse of $\iota_{\mathfrak{Y}}$. The compatibility of $\iota_{\mathfrak{Y}}$, $i^l_{\mathfrak{X}}$ and $\alpha_{\mathfrak{Y}}$ implies that the higher relative unit bijections hold. The functor $\mathsf{N}^{\mathsf{rel}}_\bullet$ assigns to a morphism (φ, ϑ) the pair $(\phi_\bullet, \theta_\bullet)$, where $\phi_\bullet = \mathsf{N}_\bullet(\varphi)$, θ_0 is equal to $\widetilde{\vartheta}_0$ and θ_n, $n \geq 1$, are the canonically induced maps, which are well defined by the definitions of Y_n and Y'_n, $n \geq 2$. □

Remark Theorem 4.8 could also have been formulated in terms of right modules.

There is a semisimplicial variant of Theorem 4.8 where $\mu\mathsf{Cat}\text{-}\mathsf{mod}$ is replaced with $\frac{1}{2}\mu\mathsf{Cat}\text{-}\mathsf{mod}$, the category of left modules over multivalued semicategories, the prefix "semi" indicating that all data related to unit morphisms is omitted. As a simple special case, let $X_\bullet$ be a 2–Segal semisimplicial set with $X_0 = X_1 = \mathrm{pt}$. Then $X_\bullet$ defines a distributive monoidal endofunctor on Set by $F \otimes F' = X_2 \times F \times F'$. As explained in [7, Section 3.7], the associator reduces to a bijection $a: X_2 \times X_2 \to X_2 \times X_2$ which satisfies the pentagon equation

$$a_{23} \circ a_{13} \circ a_{12} = a_{12} \circ a_{23}.$$

Conversely, a bijective solution a of the pentagon equation on a set X_2 extends to a 2–Segal semisimplicial set with $\mathsf{N}_0(X_2, a) = \mathsf{N}_1(X_2, a) = \mathrm{pt}$ and $\mathsf{N}_n(X_2, a)$, $n \geq 2$, the set of tuples $\{x_{ijk} \in X_2 \mid 0 \leq i < j < k \leq n\}$ which satisfy

$$a(x_{ijk}, x_{ikl}) = (x_{ijl}, x_{jkl}), \quad 0 \leq i < j < k < l \leq n.$$

Modifying this construction, a relative 2–Segal semisimplicial set $Y_\bullet \to X_\bullet$ with $Y_0 = \mathrm{pt}$ defines a monoidal endofunctor $F \boxtimes G = Y_1 \times F \times G$ which is a left $\otimes$–module. The module associator is a bijection $\alpha: X_2 \times Y_1 \to Y_1 \times Y_1$ which satisfies the a–pentagon equation

$$\alpha_{23} \circ \alpha_{13} \circ a_{12} = \alpha_{12} \circ \alpha_{23}.$$

Moreover, from a bijective solution to the a–pentagon equation we can construct a relative 2–Segal semisimplicial set with $\mathsf{N}_0(Y_1,\alpha)=\mathrm{pt}$, $\mathsf{N}_1(Y_1,\alpha)=Y_1$ and $\mathsf{N}_n(Y_1,\alpha)$, $n\geq 2$, the subset of $\mathsf{N}_n(X_2,a)\times\{y_{ij}\in Y_1\mid 0\leq i<j\leq n\}$ consisting of tuples which satisfy

$$\alpha(x_{ijk},y_{ik})=(y_{ij},y_{jk}),\quad 0\leq i<j<k\leq n.$$

The structure map $\mathsf{N}_\bullet(Y_1,\alpha)\to\mathsf{N}_\bullet(X_2,a)$ is the canonical projection.

Example For any group G, the map

$$a\colon \mathsf{G}\times\mathsf{G}\to\mathsf{G}\times\mathsf{G},\quad (x,y)\mapsto(xy,y)$$

is a bijective solution to the pentagon equation and so defines a 2–Segal semisimplicial set $\mathsf{N}_\bullet(\mathsf{G})$; see [21; 7, Example 3.7.7].

Let now $\rho\colon \mathsf{G}\to\mathrm{Aut}(M)$ be a left G–action on a set M. Then

$$\alpha\colon \mathsf{G}\times M\to M\times M,\quad (x,m)\mapsto(\rho(x)m,m)$$

solves the a–pentagon equation. Moreover, α is a bijection if and only if ρ gives M the structure of a G–torsor. Hence, associated to each G–torsor M is a relative 2–Segal simplicial set $\mathsf{N}_\bullet(\mathsf{G},M)\to\mathsf{N}_\bullet(\mathsf{G})$. When $M=\mathsf{G}$ with G acting by left multiplication the above construction recovers the left path $P^{\lhd}\mathsf{N}_\bullet(\mathsf{G})\to\mathsf{N}_\bullet(\mathsf{G})$.

4.4 Presheaves on Hall 2–categories

We begin by constructing a module over a Hall-type algebra from a relative 2–Segal semisimplicial set. Fix a field k. Let $X_\bullet$ be a 2–Segal simplicial set for which the map (∂_2,∂_0) from the span (21) has finite fibres. The Hall category $H(X_\bullet)$ [7, Section 3.4] is the k–linear category with $\mathrm{Ob}(H(X_\bullet))=X_0$ and $\mathrm{Hom}_{H(X_\bullet)}(a,b)=\mathfrak{F}_0(X_{a\to b})$, the finitely supported k–valued functions on the set

$$X_{a\to b}=\{a\}\times_{X_0}X_1\times_{X_0}\{b\}.$$

Composition of morphisms is defined by push-pull along the span

$$X_{b\to c}\times X_{a\to b}\leftarrow\{p\in X_2\mid \partial_{\{0\}}p=a,\ \partial_{\{1\}}p=b,\ \partial_{\{2\}}p=c\}\to X_{a\to c}. \tag{25}$$

Writing 1_x fo the characteristic function of $x\in X_{a\to b}$, composition in $H(X_\bullet)$ becomes $1_x\cdot 1_{x'}=\sum_{x''}f^{x''}_{x,x'}1_{x''}$ where

$$f^{x''}_{x,x'}=\bigl|\{p\in X_2\mid \partial_2 p=x,\ \partial_1 p=x',\ \partial_0 p=x''\}\bigr|.$$

Similarly, a relative 2–Segal simplicial set $F_\bullet\colon Y_\bullet \to X_\bullet$ for which the map (F_1, ∂_0) from the span (22) has finite fibres defines a presheaf $\mathcal{F}\colon H(X_\bullet)^{\mathrm{op}} \to \mathsf{Set}$ by setting $\mathcal{F}(a) = \mathfrak{F}_0(F_0^{-1}(a))$, $a \in X_0$, and using push-pull along the span

$$X_{a\to b} \times F_0^{-1}(b) \leftarrow \{q \in Y_1 \mid \partial_1 q \in F_0^{-1}(a),\ \partial_0 q \in F_0^{-1}(b)\} \to F_0^{-1}(a) \tag{26}$$

to define the action map $\mathrm{Hom}_{H(X_\bullet)}(a,b) \times \mathcal{F}(b) \to \mathcal{F}(a)$. Writing $1_\xi \in \mathcal{F}(a)$ for the characteristic function of $\xi \in F_0^{-1}(a)$, we have $1_x \star 1_{\xi'} = \sum_{\xi''} g_{x,\xi'}^{\xi''} 1_{\xi''}$, where

$$g_{x,\xi'}^{\xi''} = \big|\{q \in Y_1 \mid \partial_1 q = \xi'',\ \partial_0 q = \xi',\ F_1(q) = x\}\big|.$$

Before categorifying the above construction, we recall some preliminary definitions from [7, Section 3.5.B]. A category $\mathcal{C}$ is called $\sqcup$–semisimple if it is equivalent to an overcategory $\mathsf{Set}_{/B}$, in which case a simple object of $\mathcal{C}$ is an object which is isomorphic to a point $\{b\} \in \mathsf{Set}_{/B}$. Denote by $\|\mathcal{C}\|$ the set of isomorphism classes of simple objects of $\mathcal{C}$. A functor between $\sqcup$–semisimple categories is called additive (resp. simple) if it preserves coproducts (resp. simple objects). The $\sqcup$–semisimple categories, their additive functors and their natural transformations form a bicategory $\mathsf{Cat}^{\sqcup}$. Let $\mathsf{Cat}^{\sqcup !} \subset \mathsf{Cat}^{\sqcup}$ be the subbicategory whose morphisms are instead simple additive functors.

Define a 2–functor by $A\colon \mathsf{Span}(\mathsf{Set}) \to \mathsf{Cat}^{\sqcup}$ by

$$Z \mapsto \mathsf{Set}_{/Z}, \quad (Z \xleftarrow{s} W \xrightarrow{p} Z') \mapsto (\mathsf{Set}_{/Z} \xrightarrow{p_* \circ s^*} \mathsf{Set}_{/Z'})$$

with the obvious action on 2–morphisms. Define also $B\colon \mathsf{Cat}^{\sqcup} \to \mathsf{Span}(\mathsf{Set})$ by

$$\mathcal{C} \mapsto \|\mathcal{C}\|, \quad (\mathcal{C} \xrightarrow{\phi} \mathcal{C}') \mapsto \Big(\|\mathcal{C}\| \leftarrow \bigsqcup_{a \in \|\mathcal{C}\|} F'(\phi(a)) \to \|\mathcal{C}'\|\Big),$$

where $F'\colon \mathcal{C}' \to \mathsf{Set}_{/Z'}$ is some equivalence.

Proposition 4.9 [7, Proposition 3.5.4] *The 2–functors $A\colon \mathsf{Span}(\mathsf{Set}) \leftrightarrow \mathsf{Cat}^{\sqcup} : B$ are mutually inverse 2–equivalences which restrict to 2–equivalences $\mathsf{Set} \leftrightarrow \mathsf{Cat}^{\sqcup !}$.*

For later purposes, fix natural equivalences

$$\kappa\colon \mathbf{1}_{\mathsf{Cat}^{\sqcup}} \Rightarrow A \circ B, \quad \lambda\colon B \circ A \Rightarrow \mathbf{1}_{\mathsf{Span}(\mathsf{Set})}$$

realizing the 2–equivalences of Proposition 4.9.

A semibicategory is the data of a bicategory with all mention of unit 1–morphisms omitted. A semibicategory is called $\sqcup$–semisimple if it is small, its morphism categories are $\sqcup$–semisimple and composition of morphisms is additive in each variable. A lax 2–functor between $\sqcup$–semisimple semibicategories is called admissible if the associated functors on morphism categories are simple additive.

Definition 4.10 Two admissible lax 2–functors $\phi, \phi' \colon \mathbb{X} \to \mathbb{X}'$ of $\sqcup$–semisimple semibicategories are called equivalent if they agree on objects and there exists a collection of natural isomorphisms

$$\mathrm{Hom}_{\mathbb{X}}(a,b) \underset{\phi'_{a,b}}{\overset{\phi_{a,b}}{\rightrightarrows}} \mathrm{Hom}_{\mathbb{X}'}(\phi(a),\phi(b)), \quad U_{a,b} \colon \phi_{a,b} \Downarrow \phi'_{a,b}, \qquad a,b \in \mathrm{Ob}(\mathbb{X})$$

which commute with the coherence maps for ϕ, ϕ' and the associativity isomorphisms.

If $\mathbb{X}$, $\mathbb{X}'$ are bicategories, then $\phi, \phi' \colon \mathbb{X} \to \mathbb{X}'$ are equivalent if there exists an invertible icon $U \colon \phi \Rightarrow \phi'$. Slightly abusively, we will refer to $\{U_{a,b}\}_{a,b \in \mathrm{Ob}(\mathbb{X})}$ as an icon even if $\mathbb{X}$, $\mathbb{X}'$ are not bicategories. The $\sqcup$–semisimple semibicategories and their equivalence classes of admissible lax 2–functors form a category $\frac{1}{2}\mathsf{biCat}^{\sqcup}$.

Let $X_\bullet$ be a 2–Segal semisimplicial set. Its Hall 2–category $\mathbb{H}(X_\bullet)$ is the $\sqcup$–semisimple semibicategory with

$$\mathrm{Ob}(\mathbb{H}(X_\bullet)) = X_0, \quad \mathrm{Hom}_{\mathbb{H}(X_\bullet)}(a,b) = \mathsf{Set}_{/X_{a \to b}}.$$

Push-pull along the span (25) defines composition of 1–morphisms. Associated to a morphism $\phi_\bullet \colon X_\bullet \to X'_\bullet$ is the admissible lax 2–functor $\phi \colon \mathbb{H}(X_\bullet) \to \mathbb{H}(X'_\bullet)$ given by ϕ_0 on objects and by pushforward along ϕ_1 on morphisms.

Theorem 4.11 [7, Theorem 3.5.8] *The assignment $X_\bullet \mapsto \mathbb{H}(X_\bullet)$ extends to an equivalence between the category of 2–Segal semisimplicial sets and $\frac{1}{2}\mathsf{biCat}^{\sqcup}$.*

Proof A proof is outlined in [7]. We fill in some details which will be needed below. We first construct a functor $\Xi \colon \frac{1}{2}\mathsf{biCat}^{\sqcup} \to \frac{1}{2}\mu\mathsf{Cat}$. Define $\mathfrak{X} = \Xi(\mathbb{X}) \in \frac{1}{2}\mu\mathsf{Cat}$ by

$$\mathfrak{X}_0 = \mathrm{Ob}(\mathbb{X}), \quad \mathfrak{X}_1 = \bigsqcup_{a,b \in \mathrm{Ob}(\mathbb{X})} \|\mathrm{Hom}_{\mathbb{X}}(a,b)\|$$

with span $m_{\mathfrak{X}}$ obtained by applying Proposition 4.9 to the functors which define composition of 1–morphisms in $\mathbb{X}$. Given $\phi \colon \mathbb{X} \to \mathbb{X}'$, define $\varphi = \Xi(\phi)$ so that φ_0 is equal to ϕ on objects and φ_1 is given by the functors $\phi_{a,b}$. If $\phi, \phi' \colon \mathbb{X} \to \mathbb{X}'$

are equivalent, say via an icon $U\colon \phi \Rightarrow \phi'$, then $\varphi_0 = \varphi_0'$ and $\varphi_1 = \varphi_1'$ due to the isomorphisms $U_{a,b}(f)\colon \phi_{a,b}(f) \xrightarrow{\sim} \phi'_{a,b}(f)$, $f \in \mathrm{Hom}_{\mathbb{X}}(a,b)$.

Define also a functor $\Omega\colon \frac{1}{2}\mu\mathsf{Cat} \to \frac{1}{2}\mathsf{biCat}^{\sqcup}$ by setting $\mathbb{X} = \Omega(\mathfrak{X})$, where $\mathrm{Ob}(\mathbb{X}) = \mathfrak{X}_0$ and $\mathrm{Hom}_{\mathbb{X}}(a,b) = \mathsf{Set}_{/\mathfrak{X}_{a\to b}}$. Given $\varphi\colon \mathfrak{X} \to \mathfrak{X}'$, let $\phi = \Omega(\varphi)$ be the morphism which is equal to φ_0 on objects and whose component functor $\phi_{a,b}$ is pushforward along $\varphi_1\colon \mathfrak{X}_{a\to b} \to \mathfrak{X}'_{\varphi_0(a)\to\varphi_0(b)}$.

We claim that Ξ and Ω are mutually inverse equivalences. Let $\epsilon\colon \Xi \circ \Omega \Rightarrow \mathbf{1}_{\frac{1}{2}\mu\mathsf{Cat}}$ be the natural isomorphism whose component $\epsilon_{\mathfrak{X}}\colon \Xi(\Omega(\mathfrak{X})) \to \mathfrak{X}$ has $(\epsilon_{\mathfrak{X}})_0$ equal to the identity and has $(\epsilon_{\mathfrak{X}})_1$ equal to the map

$$\bigsqcup_{a,b\in\mathfrak{X}_0} \lambda_{\mathfrak{X}_{a\to b}}\colon \bigsqcup_{a,b\in\mathfrak{X}_0} \|\mathsf{Set}_{/\mathfrak{X}_{a\to b}}\| \to \mathfrak{X}_1.$$

Similarly, let $\eta\colon \mathbf{1}_{\frac{1}{2}\mathsf{biCat}^{\sqcup}} \Rightarrow \Omega \circ \Xi$ be the natural transformation whose component $\eta_{\mathbb{X}}\colon \mathbb{X} \to \Omega(\Xi(\mathbb{X}))$ is the identity on objects and is equal to the functor $\kappa_{\mathrm{Hom}_{\mathbb{X}}(a,b)}$ on morphism categories. Given $\phi\colon \mathbb{X} \to \mathbb{X}'$, we need to check that the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\eta_{\mathbb{X}}} & \Omega(\Xi(\mathbb{X})) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\Omega(\Xi(\phi))} \\ \mathbb{X}' & \xrightarrow[\eta_{\mathbb{X}'}]{} & \Omega(\Xi(\mathbb{X}')) \end{array}$$

commutes in $\frac{1}{2}\mathsf{biCat}^{\sqcup}$, which amounts to giving an invertible icon $U\colon \eta_{\mathbb{X}'} \circ \phi \Rightarrow \Omega(\Xi(\phi)) \circ \eta_{\mathbb{X}}$. The required natural transformation $U_{a,b}$ is defined as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{X}}(a,b) & \xrightarrow{\kappa_{\mathrm{Hom}_{\mathbb{X}}(a,b)}} & \mathsf{Set}_{/\|\mathrm{Hom}_{\mathbb{X}}(a,b)\|} \\ {\scriptstyle\phi_{a,b}}\downarrow & \overset{\kappa_{\phi_{a,b}}}{\Nearrow} & \downarrow{\scriptstyle\|\phi_{a,b}\|_*} \\ \mathrm{Hom}_{\mathbb{X}'}(\phi(a),\phi(b)) & \xrightarrow[\kappa_{\mathrm{Hom}_{\mathbb{X}'}(\phi(a),\phi(b))}]{} & \mathsf{Set}_{/\|\mathrm{Hom}_{\mathbb{X}'}(\phi(a),\phi(b))\|} \end{array}$$

This establishes an equivalence $\frac{1}{2}\mathsf{biCat}^{\sqcup} \simeq \frac{1}{2}\mu\mathsf{Cat}$. Using the semisimplicial variant of Theorem 4.6 we get an equivalence from $\frac{1}{2}\mathsf{biCat}^{\sqcup}$ to the category of 2–Segal semisimplicial sets which is a quasi-inverse of the functor $X_\bullet \mapsto \mathbb{H}(X_\bullet)$. □

Motivated by Proposition 2.3, the goal of the remainder of this section is to interpret relative 2–Segal semisimplicial sets in terms of a certain class of presheaves. To this end, define a $\mathsf{Cat}^{\sqcup}$–valued presheaf on a $\sqcup$–semisimple semibicategory to be a 2–functor $\mathbb{F}\colon \mathbb{X}^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}$ with $\mathbb{X} \in \frac{1}{2}\mathsf{biCat}^{\sqcup}$. Here $\mathbb{X}^{\mathsf{op}}$ is the semibicategory obtained from $\mathbb{X}$

by reversing only its 1–cells. A morphism

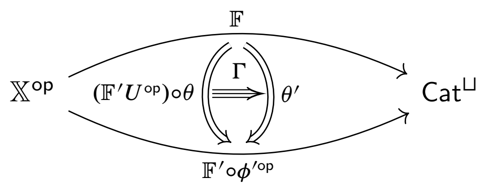
$$(\phi, \theta)\colon (\mathbb{F}\colon \mathbb{X}^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}) \to (\mathbb{F}'\colon \mathbb{X}'^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup})$$

consists of a lax 2–functor $\phi\colon \mathbb{X} \to \mathbb{X}'$ and an oplax natural transformation $\theta\colon \mathbb{F} \Rightarrow \mathbb{F}' \circ \phi^{\mathsf{op}}$. The pair (ϕ, θ) is called admissible if ϕ is admissible and the functors $\theta_a\colon \mathbb{F}(a) \to \mathbb{F}'(\phi(a))$ are simple additive. The composition $\mathbb{F} \xrightarrow{(\phi,\theta)} \mathbb{F}' \xrightarrow{(\xi,\zeta)} \mathbb{F}''$ is defined by the right whiskering of ζ and ϕ^{op}: namely $(\xi, \zeta) \circ (\phi, \theta) = (\xi \circ \phi, (\zeta\phi^{\mathsf{op}}) \circ \theta)$.

Definition 4.12 Two admissible morphisms

$$(\phi, \theta), (\phi', \theta')\colon (\mathbb{F}\colon \mathbb{X}^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}) \to (\mathbb{F}'\colon \mathbb{X}'^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup})$$

are called equivalent if there exists an invertible icon $U\colon \phi \Rightarrow \phi'$ and an invertible modification of the following form:



Denote by $\mathsf{Psh}^{\sqcup}$ the category of $\mathsf{Cat}^{\sqcup}$–valued presheaves on $\sqcup$–semisimple semibicategories and their equivalence classes of admissible morphisms.

We now construct a functor from the category of relative 2–Segal semisimplicial sets to $\mathsf{Psh}^{\sqcup}$. Given $F_\bullet\colon Y_\bullet \to X_\bullet$, define $\mathbb{F}\colon \mathbb{H}(X_\bullet)^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}$ as follows. Put $\mathbb{F}(a) = \mathsf{Set}_{/F_0^{-1}(a)}$, $a \in X_0$, and use the span (26) to define a functor

$$\mathbb{F}_{a,b}\colon \mathsf{Set}_{/X_{a\to b}} \to [\mathsf{Set}_{/F_0^{-1}(b)}, \mathsf{Set}_{/F_0^{-1}(a)}],$$

the image of which consists of additive functors by Proposition 4.9. Here and in what follows we use the equivalence $[\mathcal{C} \times \mathcal{D}, \mathcal{E}] \simeq [\mathcal{C}, [\mathcal{D}, \mathcal{E}]]$ for categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, the first two of which are small. The relative 2–Segal bijections induce the coherence isomorphisms for $\mathbb{F}$. Given a morphism

$$\begin{array}{ccc} Y_\bullet & \xrightarrow{\theta_\bullet} & Y'_\bullet \\ {\scriptstyle F_\bullet}\big\downarrow & & \big\downarrow{\scriptstyle F'_\bullet} \\ X_\bullet & \xrightarrow[\phi_\bullet]{} & X'_\bullet \end{array}$$

of relative 2–Segal semisimplicial sets, we need to define a morphism

$$(\phi, \theta)\colon (\mathbb{F}\colon \mathbb{H}(X_\bullet)^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}) \to (\mathbb{F}'\colon \mathbb{H}(X'_\bullet)^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}).$$

Let ϕ be the lax 2–functor associated to $\phi_\bullet$. Let $\theta_a \colon \mathbb{F}(a) \to \mathbb{F}'(\phi(a))$ be the pushforward along $\theta_0|_{F_0^{-1}(a)} \colon F_0^{-1}(a) \to F_0'^{-1}(\phi(a))$. The natural transformation

$$\begin{array}{ccc} \mathsf{Set}_{/X_{a\to b}} & \xrightarrow{(\mathbb{F}'\circ\phi^{\mathrm{op}})_{a,b}} & [\mathsf{Set}_{/F_0'^{-1}(\phi(b))}, \mathsf{Set}_{/F_0'^{-1}(\phi(a))}] \\ {\scriptstyle \mathbb{F}_{a,b}}\big\downarrow & \overset{\theta_{a,b}}{\nearrow\!\!\!\!\nearrow} & \big\downarrow{\scriptstyle -\circ\theta_b} \\ [\mathsf{Set}_{/F_0^{-1}(b)}, \mathsf{Set}_{/F_0^{-1}(a)}] & \xrightarrow[\theta_a\circ -]{} & [\mathsf{Set}_{/F_0^{-1}(b)}, \mathsf{Set}_{/F_0'^{-1}(\phi(a))}] \end{array}$$

is induced by the morphism of spans determined by the restriction of the map $\theta_1 \colon Y_1 \to Y_1'$ to the middle set of the span (26).

Theorem 4.13 *The assignment $F_\bullet \mapsto \mathbb{F}$ extends to an equivalence between the category of relative 2–Segal semisimplicial sets and $\mathsf{Psh}^{\sqcup}$.*

Proof Lift $\Xi \colon \frac{1}{2}\mathsf{biCat}^{\sqcup} \to \frac{1}{2}\mu\mathsf{Cat}$ to a functor $\Xi^{\mathrm{rel}} \colon \mathsf{Psh}^{\sqcup} \to \frac{1}{2}\mu\mathsf{Cat}\text{-}\mathsf{mod}$ as follows. Define $\Xi^{\mathrm{rel}}(\mathbb{F} \colon \mathbb{X}^{\mathrm{op}} \to \mathsf{Cat}^{\sqcup}) = (\mathfrak{F} \colon \mathfrak{Y} \to \mathfrak{X})$ by $\mathfrak{X} = \Xi(\mathbb{X})$ and $\mathfrak{Y}_0 = \bigsqcup_{a\in\mathrm{Ob}(\mathbb{X})} \|\mathbb{F}(a)\|$ with the canonical map $\mathfrak{F}_0 \colon \mathfrak{Y}_0 \to \mathfrak{X}_0$. The span $\mu_{\mathfrak{Y}}$ is obtained by applying Proposition 4.9 to the functors $\mathbb{F}_{a,b}$ while the associator $\alpha_{\mathfrak{Y}}$ is defined using the coherence isomorphisms for $\mathbb{F}$. Given a morphism $(\phi,\theta) \colon \mathbb{F} \to \mathbb{F}'$, set $\varphi = \Xi(\phi)$, let $\vartheta_0 \colon \mathfrak{Y}_0 \to \mathfrak{Y}_0'$ be the map determined by the functors θ_a and let $\widetilde{\vartheta}_1 \colon \vartheta_0 \circ \mu_{\mathfrak{Y}} \to \mu_{\mathfrak{Y}'} \circ (\varphi_1 \times_{\varphi_0} \vartheta_0)$ be the morphism obtained by applying Proposition 4.9 to the following coherence natural transformations:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{X}}(a,b) \times \mathbb{F}(b) & \xrightarrow{\phi_{a,b}\times\theta_b} & \mathrm{Hom}_{\mathbb{X}'}(\phi(a),\phi(b)) \times \mathbb{F}'(\phi(b)) \\ {\scriptstyle \mathbb{F}_{a,b}}\big\downarrow & \overset{\theta_{a,b}}{\nearrow\!\!\!\!\nearrow} & \big\downarrow{\scriptstyle \mathbb{F}'_{a,b}} \\ \mathbb{F}(a) & \xrightarrow[\theta_a]{} & \mathbb{F}'(\phi(a)) \end{array}$$

Suppose that (ϕ,θ) and (ϕ',θ') are equivalent morphisms, say via an icon U and a modification Γ. Then $\varphi = \varphi'$. Since $U_a \colon \phi(a) \to \phi'(a)$ is the identity we have $((\mathbb{F}' U^{\mathrm{op}}) \circ \theta)_a = \theta_a$ and hence a natural isomorphism $\Gamma_a \colon \theta_a \Rightarrow \theta_a'$, showing that $\vartheta_0 = \vartheta_0'$. The equality $\widetilde{\vartheta}_1 = \widetilde{\vartheta}_1'$ follows from the fact that the components $\{\Gamma_a\}_{a\in\mathrm{Ob}(\mathbb{X})}$ commute with morphisms in $\mathbb{X}$.

Similarly, construct a lift $\Omega^{\mathrm{rel}} \colon \frac{1}{2}\mu\mathsf{Cat}\text{-}\mathsf{mod} \to \mathsf{Psh}^{\sqcup}$ of $\Omega \colon \frac{1}{2}\mu\mathsf{Cat} \to \frac{1}{2}\mathsf{biCat}^{\sqcup}$. Define $\Omega^{\mathrm{rel}}(\mathfrak{F} \colon \mathfrak{Y} \to \mathfrak{X}) = (\mathbb{F} \colon \mathbb{X}^{\mathrm{op}} \to \mathsf{Cat}^{\sqcup})$ by $\mathbb{X} = \Omega(\mathfrak{X})$ with $\mathbb{F}(a) = \mathsf{Set}_{/\mathfrak{F}_0^{-1}(a)}$ and functors $\mathbb{F}_{a,b}$ determined by $\mu_{\mathfrak{Y}}$. Given a morphism $(\varphi,\vartheta)\colon \mathfrak{F} \to \mathfrak{F}'$, put $\phi = \Omega(\varphi)$ and let θ be the oplax natural transformation whose component functor $\theta_a\colon \mathbb{F}(a) \to \mathbb{F}'(\phi(a))$ is pushforward along the map $\vartheta_{0|\mathfrak{F}_0^{-1}(a)}\colon \mathfrak{F}_0^{-1}(a) \to \mathfrak{F}_0'^{-1}(\phi(a))$ and whose coherence natural transformations $\theta_{a,b}$ are induced by the morphism $\widetilde{\vartheta}_1$.

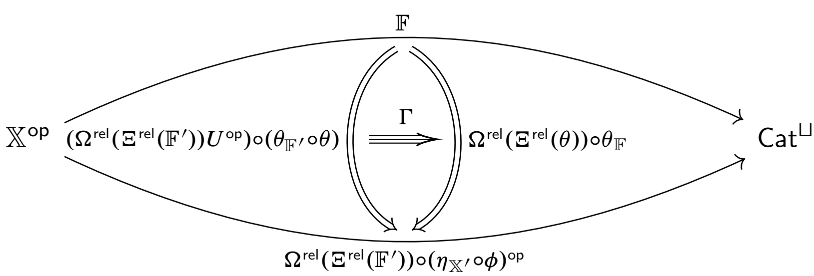
To prove that Ξ^{rel} and Ω^{rel} are inverse equivalences, we first lift $\epsilon\colon \Xi \circ \Omega \Rightarrow \mathbf{1}_{\frac{1}{2}\mu\mathsf{Cat}}$ to $\epsilon^{\mathsf{rel}}\colon \Xi^{\mathsf{rel}} \circ \Omega^{\mathsf{rel}} \Rightarrow \mathbf{1}_{\frac{1}{2}\mu\mathsf{Cat}\text{-}\mathsf{mod}}$. On the total space $\epsilon^{\mathsf{rel}}_{\mathfrak{F}}\colon \Xi^{\mathsf{rel}}(\Omega^{\mathsf{rel}}(\mathfrak{F})) \to \mathfrak{F}$ is given by $\bigsqcup_{a\in\mathfrak{X}_0} \lambda_{\mathfrak{F}_0^{-1}(a)}$ on objects while the morphism $\vartheta_0 \circ \mu_{\Xi^{\mathsf{rel}}(\Omega^{\mathsf{rel}}(\mathfrak{Y}))} \to \mu_{\mathfrak{Y}} \circ (\varphi_1 \times_{\varphi_0} \vartheta_0)$ is $\lambda_{\mu_{\mathfrak{Y}}}$. Similarly, lift $\eta\colon \mathbf{1}_{\frac{1}{2}\mathsf{biCat}^{\sqcup}} \Rightarrow \Omega \circ \Xi$ to $\eta^{\mathsf{rel}}\colon \mathbf{1}_{\mathsf{Psh}^{\sqcup}} \Rightarrow \Omega^{\mathsf{rel}} \circ \Xi^{\mathsf{rel}}$ as follows. Writing $\eta^{\mathsf{rel}}_{\mathbb{F}} = (\phi_{\mathbb{F}}, \theta_{\mathbb{F}})$, put $\phi_{\mathbb{F}} = \eta_{\mathbb{X}}$. Since $\eta_{\mathbb{X}}$ is the identity on objects, we can define $\theta_{\mathbb{F}}\colon \mathbb{F} \to \Omega^{\mathsf{rel}}(\Xi^{\mathsf{rel}}(\mathbb{F})) \circ \eta_{\mathbb{X}}^{\mathsf{op}}$ so that its component functor $(\theta_{\mathbb{F}})_a$ is $\lambda_{\mathbb{F}(a)}$. Associated to the functor $\mathbb{F}_{a,b}$ is the natural transformation

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{X}}(a,b)\times\mathbb{F}(b) & \xrightarrow{\lambda_{\operatorname{Hom}_{\mathbb{X}}(a,b)\times\mathbb{F}(b)}} & \mathsf{Set}_{/\|\operatorname{Hom}_{\mathbb{X}}(a,b)\times\mathbb{F}(b)\|} \\ {\scriptstyle \mathbb{F}_{a,b}}\big\downarrow & \overset{\lambda_{\mathbb{F}_{a,b}}}{\nearrow\!\!\!\!\nearrow} & \big\downarrow{\scriptstyle \|\mathbb{F}_{a,b}\|_*} \\ \mathbb{F}(a) & \xrightarrow[\lambda_{\mathbb{F}(a)}]{} & \mathsf{Set}_{/\|\mathbb{F}(a)\|} \end{array}$$

which we take to be the coherence data $(\theta_{\mathbb{F}})_{a,b}$. It remains to check that, for an admissible morphism $(\phi,\theta)\colon \mathbb{F} \to \mathbb{F}'$, the diagram

$$\begin{array}{ccc} \mathbb{F} & \xrightarrow{\eta^{\mathsf{rel}}_{\mathbb{F}}} & \Omega^{\mathsf{rel}}(\Xi^{\mathsf{rel}}(\mathbb{F})) \\ {\scriptstyle (\phi,\theta)}\big\downarrow & & \big\downarrow{\scriptstyle \Omega^{\mathsf{rel}}(\Xi^{\mathsf{rel}}(\phi,\theta))} \\ \mathbb{F}' & \xrightarrow[\eta^{\mathsf{rel}}_{\mathbb{F}'}]{} & \Omega^{\mathsf{rel}}(\Xi^{\mathsf{rel}}(\mathbb{F}')) \end{array}$$

commutes in $\mathsf{Psh}^{\sqcup}$. That is, we need an invertible icon $U\colon \phi_{\mathbb{F}'} \circ \phi \Rightarrow \eta_{\mathbb{X}}(\phi) \circ \phi_{\mathbb{F}}$, which exists by Theorem 4.11, and the following invertible modification:



The required natural transformations Γ_a are defined as follows:

$$\begin{array}{ccc} \mathbb{F}(a) & \xrightarrow{\lambda_{\mathbb{F}(a)}} & \mathsf{Set}_{/\|\mathbb{F}(a)\|} \\ {\scriptstyle\theta_a}\big\downarrow & \overset{\lambda_{\theta_a}}{\nearrow\!\!\!\nearrow} & \big\downarrow{\scriptstyle\|\theta_a\|_*} \\ \mathbb{F}'(\phi(a)) & \xrightarrow[\lambda_{\mathbb{F}'(\phi(a))}]{} & \mathsf{Set}_{/\|\mathbb{F}'(\phi(a))\|} \end{array}$$

This establishes the equivalence $\mathsf{Psh}^{\sqcup} \simeq \frac{1}{2}\mu\mathsf{Cat}\text{-}\mathsf{mod}$. Now apply Theorem 4.8. $\square$

The simplicial variant of Theorem 4.11 is an equivalence between 2-Seg$\mathbb{S}$ and the category of $\sqcup$–semisimple bicategories with simple units. The modification of Theorem 4.13 is a compatible equivalence between 2-SegRel$\mathbb{S}$ and the category of $\mathsf{Cat}^{\sqcup}$–valued presheaves over $\sqcup$–semisimple bicategories with simple units.

Example In the above language, the semisimplicial set $\widetilde{\mathbb{T}}_{\bullet}(M)$ of Section 2.5 defines a $\sqcup$–semisimple semibicategory $\mathbb{X}(M)$ with $\mathrm{Ob}(\mathbb{X}(M)) = M$ and $\mathrm{Hom}_{\mathbb{X}(M)}(a,b) = \mathsf{Set}_{/M_{a\to b}}$, where $M_{a\to b} \subset C^0([-\infty,\infty], M)$ is the subset of paths from a to b, with composition given by counting (suitably interpreted) pseudoholomorphic polygons in M with prescribed boundary conditions. The relative 2–Segal semisimplicial morphism $\widetilde{\mathbb{T}}^{\tau}_{\bullet}(M) \to \widetilde{\mathbb{T}}_{\bullet}(M)$ of Theorem 2.11 becomes the presheaf $\mathbb{F}^{\tau}\colon \mathbb{X}(M)^{\mathsf{op}} \to \mathsf{Cat}^{\sqcup}$ given by $\mathbb{F}^{\tau}(a) = \mathsf{Set}_{/a^{\tau}\sqcup M^{\tau}_{a\to\tau(a)}}$, where $a^{\tau} = a$ if $a \in M^{\tau}$ and $a^{\tau} = \varnothing$ otherwise and $M^{\tau}_{a\to\tau(a)} \subset M_{a\to\tau(a)}$ is the subset of real paths. Counts of real pseudoholomorphic n–gons, with $n \leq 4$, obeying one ordinary and two real boundary conditions determine the required functors $\mathbb{F}^{\tau}(b) \to \mathbb{F}^{\tau}(a)$.

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