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Outer actions of $\mathrm{Out}(F_n)$ on small right-angled Artin groups

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We determine the precise conditions under which $\mathrm{SOut}(F_n)$, the unique index-two subgroup of $\mathrm{Out}(F_n)$, can act nontrivially via outer automorphisms on a RAAG whose defining graph has fewer than $\frac{1}{2}\binom{n}{2}$ vertices.

We also show that the outer automorphism group of a RAAG cannot act faithfully via outer automorphisms on a RAAG with a strictly smaller (in number of vertices) defining graph.

Along the way we determine the minimal dimensions of nontrivial linear representations of congruence quotients of the integral special linear groups over algebraically closed fields of characteristic zero, and provide a new lower bound on the cardinality of a set on which $\mathrm{SOut}(F_n)$ can act nontrivially.

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1 Introduction

The main purpose of this article is to study the ways in which $\mathrm{Out}(F_n)$ can act via outer automorphisms on a right-angled Artin group A_Γ with defining graph Γ. (Recall that A_Γ is given by a presentation with generators being the vertices of Γ and relators being commutators of vertices which span an edge in Γ.) Such actions have previously been studied for the extremal cases: when the graph Γ is discrete, we have $\mathrm{Out}(A_\Gamma) = \mathrm{Out}(F_m)$ for some m, and homomorphisms

$$\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$$

were investigated by Bogopol'skiĭ and Puga [1], Khramtsov [11], Bridson and Vogtmann [3], and the author [12; 13]. When the graph Γ is complete, we have $\mathrm{Out}(A_\Gamma) = \mathrm{GL}_m(\mathbb{Z})$, and homomorphisms

$$\mathrm{Out}(F_n) \to \mathrm{GL}_m(\mathbb{Z}),$$

or more general representation theory of $\mathrm{Out}(F_n)$, have been studied by Grunewald and Lubotzky [8], Potapchik and Rapinchuk [17], Turchin and Willwacher [20], and the author [12; 13].

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There are two natural ways of constructing nontrivial homomorphisms

$$\phi\colon \mathrm{Out}(F_n) \to \mathrm{Out}(A_\Gamma).$$

When Γ is a join of two graphs, Δ and Σ say, then $\mathrm{Out}(A_\Gamma)$ contains

$$\mathrm{Out}(A_\Delta) \times \mathrm{Out}(A_\Sigma)$$

as a finite-index subgroup. When additionally Δ is isomorphic to the discrete graph with n vertices, then $\mathrm{Out}(A_\Delta) = \mathrm{Out}(F_n)$, and so we have an obvious embedding ϕ.

In fact this method works also for a discrete Δ with a very large number of vertices since there are injective maps $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ constructed by Bridson and Vogtmann [3] for specific values of m growing exponentially with n.

The other way of constructing nontrivial homomorphisms ϕ becomes possible when Γ contains n vertices with identical stars. In this case it is immediate that these vertices form a clique Θ, and we have a map

$$\mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(A_\Theta) \to \mathrm{Aut}(A_\Gamma) \to \mathrm{Out}(A_\Gamma).$$

We also have the projection

$$\mathrm{Out}(F_n) \to \mathrm{Out}(H_1(F_n)) = \mathrm{GL}_n(\mathbb{Z}),$$

and combining these two maps gives us a nontrivial (though also noninjective) ϕ.

This second method does not work in other situations, by the following result of Wade.

Theorem 1.1 [21] *Let $n \geqslant 3$. Every homomorphism*

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Out}(A_\Gamma)$$

has finite image if and only if Γ does not contain n distinct vertices with equal stars.

In fact Wade proved a much more general result, in which the domain of the homomorphism is allowed to be any irreducible lattice in a real semisimple Lie group with finite centre and without compact factors, and with real rank $n-1$.

The aim of this paper is to prove:

Theorem 3.7 *Let $n \geqslant 6$. Suppose that Γ is a simplicial graph with fewer than $\frac{1}{2}\binom{n}{2}$ vertices, which does not contain n distinct vertices with equal stars and is not a join of the discrete graph with n vertices and another (possibly empty) graph. Then every homomorphism $\mathrm{SOut}(F_n) \to \mathrm{Out}(A_\Gamma)$ is trivial.*

Here $\mathrm{SOut}(F_n)$ denotes the unique index-two subgroup of $\mathrm{Out}(F_n)$.

The proof is an induction, based on an observation present in a paper of Charney, Crisp and Vogtmann [4], elaborated further in a paper of Hensel and the author [9], which states that, typically, the graph Γ contains many induced subgraphs Σ which are *invariant up to symmetry* in the sense that the subgroup of A_Γ that the vertices of Σ generate is invariant under any outer action up to an automorphism induced by a symmetry of Γ (and up to conjugacy).

To use the induction we need to show that such subgraphs are really invariant, that is, that we do not need to worry about the symmetries of Γ. To achieve this we prove

Theorem 2.28 *Every action of* $\mathrm{Out}(F_n)$ *(with* $n \geqslant 6$*) on a set of cardinality* $m \leqslant \binom{n+1}{2}$ *factors through* $\mathbb{Z}/2\mathbb{Z}$*.*

Since $\mathrm{SOut}(F_n)$ is the unique index-two subgroup of $\mathrm{Out}(F_n)$, the conclusion of this theorem is equivalent to saying that $\mathrm{SOut}(F_n)$ lies in the kernel of the action.

A crucial ingredient in the proof of this theorem is the following.

Theorem 2.27 *Let* V *be a nontrivial, irreducible* $\mathbb{K}$*–linear representation of*

$$\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}),$$

where $n \geqslant 3$*,* q *is a power of a prime* p*, and where* $\mathbb{K}$ *is an algebraically closed field of characteristic* 0*. Then*

$$\dim V \geqslant \begin{cases} 2 & \text{if } (n,p) = (3,2), \\ p^{n-1} - 1 & \text{otherwise.} \end{cases}$$

This result seems not to be present in the literature; it extends a theorem of Landazuri and Seitz [14] yielding a very similar statement for $q = p$ (see Theorem 2.26).

At the end of the paper we also offer:

Theorem 4.1 *There are no injective homomorphisms* $\mathrm{Out}(A_\Gamma) \to \mathrm{Out}(A_{\Gamma'})$ *when* Γ' *has fewer vertices than* Γ*.*

This theorem follows from looking at the $\mathbb{Z}/2\mathbb{Z}$–rank, ie the largest subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$.

2 The tools

2.1 Automorphisms of free groups

Definition 2.1 ($\mathrm{SOut}(F_n)$) Consider the composition

$$\mathrm{Aut}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z},$$

where the first map is obtained by abelianising F_n, and the second map is the determinant. We define $\mathrm{SAut}(F_n)$ to be the kernel of this map; we define $\mathrm{SOut}(F_n)$ to be the image of $\mathrm{SAut}(F_n)$ in $\mathrm{Out}(F_n)$.

It is easy to see that both $\mathrm{SAut}(F_n)$ and $\mathrm{SOut}(F_n)$ are index-two subgroups of, respectively, $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$.

The group $\mathrm{SAut}(F_n)$ has a finite presentation given by Gersten [7], and from this presentation one can immediately obtain the following result.

Proposition 2.2 (Gersten [7]) *The abelianisation of* $\mathrm{SAut}(F_n)$, *and therefore of* $\mathrm{SOut}(F_n)$, *is trivial for all* $n \geqslant 3$.

It follows that $\mathrm{SOut}(F_n)$ is the unique subgroup of $\mathrm{Out}(F_n)$ of index two.

We will now look at symmetric and alternating subgroups of $\mathrm{Out}(F_n)$ and list some corollaries of their existence.

Proposition 2.3 [2, Proposition 1] *Let* $n \geqslant 3$. *There exists a symmetric subgroup of rank* n,

$$\mathrm{Sym}_n < \mathrm{Out}(F_n),$$

such that any homomorphism $\phi\colon \mathrm{Out}(F_n) \to G$ *that is not injective on* Sym_n *has image of cardinality at most* 2.

The symmetric group is precisely the symmetric group operating on some fixed basis of F_n. It is easy to see that it intersects $\mathrm{SOut}(F_n)$ in an alternating group Alt_n. Whenever we talk about the alternating subgroup Alt_n of $\mathrm{SOut}(F_n)$, we mean this subgroup. Note that $\mathrm{SOut}(F_n)$ actually contains an alternating subgroup of rank $n+1$, which is a supergroup of our Alt_n; we will denote it by Alt_{n+1}. There is also a symmetric supergroup Sym_{n+1} of Alt_{n+1} contained in $\mathrm{Out}(F_n)$.

The proof of [2, Proposition 1] actually allows one to prove the following proposition.

Proposition 2.4 *Let $n \geqslant 3$. Then* $\mathrm{SOut}(F_n)$ *is the normal closure of any nontrivial element of* Alt_n.

Following the proof of [2, Theorem A], we can now conclude:

Corollary 2.5 *Let*

$$\phi\colon \mathrm{SOut}(F_n) \to \mathrm{GL}_k(\mathbb{Z})$$

be a homomorphism with $n \geqslant 6$ and $k < n$. Then ϕ is trivial.

Proof For $n \geqslant 6$, the alternating group Alt_{n+1} does not have nontrivial complex representations below dimension n. Thus $\phi|_{\mathrm{Alt}_{n+1}}$ is not injective, and is therefore trivial since Alt_{n+1} is simple. Now we apply Proposition 2.4. □

More can be said about linear representations of $\mathrm{Out}(F_n)$ in somewhat larger dimensions; see [12; 13; 20].

Another related result that we will use is the following.

Theorem 2.6 [12] *Let $n \geqslant 6$ and $m < \binom{n}{2}$. Then every homomorphism* $\mathrm{Out}(F_n) \to \mathrm{Out}(F_m)$ *has image of cardinality at most 2, provided that $m \neq n$.*

In fact, in the next section we will go back to the proof of the above theorem and show:

Theorem 2.7 *Let $n \geqslant 6$ and $m < \frac{1}{2}\binom{n}{2}$. Then every homomorphism*

$$\mathrm{SOut}(F_n) \to \mathrm{Out}(F_m)$$

is trivial, provided that $m \neq n$.

2.2 Homomorphisms $\mathrm{SOut}(F_n) \to \mathrm{Out}(F_m)$

To study such homomorphisms we need to introduce finite subgroups B_n and B of $\mathrm{SOut}(F_n)$ that will be of particular use. Let F_n be freely generated by $\{a_1, \dots, a_n\}$.

Definition 2.8 Let us define $\delta \in \mathrm{Out}(F_n)$ by $\delta(a_i) = {a_i}^{-1}$ for each i. (Formally speaking, this defines an element in $\mathrm{Aut}(F_n)$; we take δ to be the image of this element in $\mathrm{Out}(F_n)$.) Define $\sigma_{12} \in \mathrm{Sym}_n < \mathrm{Out}(F_n)$ to be the transposition swapping a_1 with a_2. Define $\xi \in \mathrm{SOut}(F_n)$ by

$$\xi = \begin{cases} \delta & \text{if } n \text{ is even,} \\ \delta\sigma_{12} & \text{if } n \text{ is odd,} \end{cases}$$

and set $B_n = \langle \mathrm{Alt}_{n+1}, \xi \rangle \leqslant \mathrm{SOut}(F_n)$.

We also set A to be either Alt_{n-1}, ie the pointwise stabiliser of $\{1,2\}$ when Alt_{n+1} acts on $\{1,2,\ldots,n+1\}$ in the natural way (in the case of odd n), or Alt_{n+1} (in the case of even n). Furthermore, we set $B=\langle A,\xi\rangle$.

It is easy to see that B_n is a finite group: it is a subgroup of the automorphism group of the (suitably marked) $(n{+}1)$–*cage graph*, that is, a graph with 2 vertices and $n+1$ edges connecting them.

To prove Theorem 2.7 we need to introduce some more notation from [12]. Throughout, when we talk about modules or representations, we work over the complex numbers.

Definition 2.9 A B–module V admits a *convenient split* if and only if V splits as a B–module into

$$V=U\oplus U',$$

where U is a sum of trivial A–modules and ξ acts by negation on U'.

Definition 2.10 A graph X with a G–action is called *G–admissible* if and only if it is connected, has no vertices of valence 2, and any G–invariant forest in X contains no edges. Here by "invariant" we mean setwise invariant.

Proposition 2.11 [12] *Let $n\geqslant 6$. Suppose that X is a B_n–admissible graph of rank smaller than $\binom{n+1}{2}$ such that the following hold:*

(1) *the B–module $H_1(X;\mathbb{C})$ admits a convenient split;*
(2) *any vector in $H_1(X;\mathbb{C})$ which is fixed by Alt_{n+1} is also fixed by ξ;*
(3) *the action of B_n on X restricted to A is nontrivial.*

Then X is the $(n{+}1)$–cage.

The above proposition does not (unfortunately) appear in this form in [12]; it does however follow from the proof of [12, Proposition 6.7].

Proof of Theorem 2.7 Let $\phi\colon \mathrm{SOut}(F_n)\to\mathrm{Out}(F_m)$ be a homomorphism. Using Nielsen realisation for free groups (due to, independently, Culler [6], Khramtsov [10] and Zimmermann [23]) we construct a finite connected graph X with fundamental group F_m, on which B_n acts in a way realising the outer action $\phi|_{B_n}$. We easily arrange for X to be B_n–admissible by collapsing invariant forests. Note that $V=H_1(F_m;\mathbb{C})$ is naturally isomorphic to $H_1(X;\mathbb{C})$ as a B_n–module.

We have a linear representation

$$\mathrm{SOut}(F_n) \xrightarrow{\phi} \mathrm{Out}(F_m) \to \mathrm{GL}(V).$$

The corresponding induced linear representation

$$\mathrm{Out}(F_n) \to \mathrm{GL}(W)$$

has dimension $\dim W = 2 \dim V = 2m$. Since we are assuming that

$$m < \frac{1}{2}\binom{n}{2},$$

the combination of [12, Lemma 3.8 and Proposition 3.11] tells us that W splits as an $\mathrm{Out}(F_n)$–module as

$$W = W_0 \oplus W_1 \oplus W_{n-1} \oplus W_n,$$

where the action of $\mathrm{Out}(F_n)$ is trivial on W_0 but not on W_n, and the action of the subgroup $\mathrm{SOut}(F_n)$ is trivial on both. Moreover, as Sym_{n+1} modules, W_1 is the sum of standard and W_{n-1} of signed standard representations. We also know that δ acts on W_i as multiplication by $(-1)^i$.

When n is even this immediately tells us that, as a $B = B_n$–module, we have

$$W = U \oplus U',$$

where $U = W_0 \oplus W_n$ is sum of trivial $A = \mathrm{Alt}_{n+1}$–modules, and $\xi = \delta$ acts by negation on

$$U' = W_1 \oplus W_{n-1}.$$

When n is odd we can still write

$$W = U \oplus U'$$

as a B–module, with A acting trivially on U and ξ acting by negation on U'. Here we have $W_0 \oplus W_n < U$, but U also contains the trivial A–modules contained in $W_1 \oplus W_{n-1}$. The module U' is the sum of the standard A–modules. Thus W admits a convenient split.

Now we claim that V also admits a convenient split as a B–module. To define the induced $\mathrm{Out}(F_n)$–module W we need to pick en element $\mathrm{Out}(F_n) \smallsetminus \mathrm{SOut}(F_n)$; we have already defined such an element, namely σ_{12}. The involution σ_{12} commutes with ξ and conjugates A to itself. Thus, as an A module, V could only consist of the trivial and standard representations, since these are the only A–modules present in W.

Moreover, any trivial A–module in V is still a trivial A–module in W, and so ξ acts on it by negation. Therefore V also admits a convenient split as a B–module. Thus we have verified assumption (1) of Proposition 2.11.

Observe that the $\mathrm{SOut}(F_n)$–module V embeds into W. In W every Alt_{n+1}–fixed vector lies in $W_0 \oplus W_n$, and here ξ acts as the identity. Thus assumption (2) of Proposition 2.11 is satisfied in W and therefore also in V.

We have verified the assumptions (1) and (2) of Proposition 2.11; we also know that the conclusion of Proposition 2.11 fails, since the $(n+1)$–cage has rank n, which would force $m = n$, contradicting the hypothesis of the theorem. Hence we know that assumption (3) of Proposition 2.11 fails, and so A acts trivially on X. But this implies that $A \leqslant \ker \phi$.

Note that A is a subgroup of the simple group Alt_{n+1}, and so we have

$$\mathrm{Alt}_{n+1} \leqslant \ker \phi.$$

But then Proposition 2.4 tells us that ϕ is trivial. □

2.3 Automorphisms of RAAGs

Throughout the paper, Γ will be a simplicial graph, and A_Γ will be the associated RAAG, that is the group generated by the vertices of Γ with a relation of two vertices commuting if and only if they are joined by an edge in Γ.

We will often look at subgraphs of Γ, and we always take them to be induced subgraphs. Thus we will make no distinction between a subgraph of Γ and a subset of the vertex set of Γ.

Given an induced subgraph $\Sigma \subseteq \Gamma$ we define A_Σ to be the subgroup of A_Γ generated by (the vertices of) Σ. Abstractly, A_Σ is isomorphic to the RAAG associated to Σ (since Σ is an induced subgraph).

Definition 2.12 (links, stars, and extended stars) Given a subgraph $\Sigma \subseteq \Gamma$ we define

- $\mathrm{lk}(\Sigma) = \{w \in \Gamma \mid w \text{ is adjacent to } v \text{ for all } v \in \Sigma\}$;
- $\mathrm{st}(\Sigma) = \Sigma \cup \mathrm{lk}(\Sigma)$;
- $\widehat{\mathrm{st}}(\Sigma) = \mathrm{lk}(\Sigma) \cup \mathrm{lk}(\mathrm{lk}(\Sigma))$.

Definition 2.13 (joins and cones) We say that two subgraphs $\Sigma, \Delta \subseteq \Gamma$ form a *join* $\Sigma * \Delta \subseteq \Gamma$ if and only if $\Sigma \subseteq \mathrm{lk}(\Delta)$ and $\Delta \subseteq \mathrm{lk}(\Sigma)$.

A subgraph $\Sigma \subseteq \Gamma$ is a *cone* if and only if there exists a vertex $v \in \Sigma$ such that $\Sigma = v * (\Sigma \smallsetminus \{v\})$. In particular, a singleton is a cone.

Definition 2.14 (join decomposition) Given a graph Σ we say that

$$\Sigma = \Sigma_1 * \cdots * \Sigma_k$$

is the *join decomposition* of Σ when each Σ_i is nonempty and is not a join of two nonempty subgraphs.

Each of the graphs Σ_i is called a *factor*, and the join of all the factors which are singletons is called the *clique factor*.

We will often focus on a specific finite-index subgroup $\mathrm{Out}^0(A_\Gamma)$ of $\mathrm{Out}(A_\Gamma)$, called the *group of pure outer automorphisms of* A_Γ. To define it we need to discuss a generating set of $\mathrm{Out}(A_\Gamma)$ due to Laurence [15] (it was earlier conjectured to be a generating set by Servatius [18]).

$\mathrm{Aut}(A_\Gamma)$ is generated by the following classes of automorphisms:

(1) inversions;

(2) partial conjugations;

(3) transvections;

(4) graph symmetries.

Here, an *inversion* maps one generator of A_Γ to its inverse, fixing all other generators.

A *partial conjugation* needs a vertex v; it conjugates all generators in one connected component of $\Gamma \smallsetminus \mathrm{st}(v)$ by v and fixes all other generators.

A *transvection* requires vertices v, w with $\mathrm{st}(v) \supseteq \mathrm{lk}(w)$. For such v and w, a transvection is the automorphism which maps w to wv and fixes all other generators.

A *graph symmetry* is an automorphism of A_Γ which permutes the generators according to a combinatorial automorphism of Γ.

The group $\mathrm{Aut}^0(A_\Gamma)$ of pure automorphisms is defined to be the subgroup generated by generators of the first three types, ie without graph symmetries. The group $\mathrm{Out}^0(A_\Gamma)$ of pure outer automorphisms is the quotient of $\mathrm{Aut}^0(A_\Gamma)$ by the inner automorphisms.

Let us quote the following result of Charney, Crisp and Vogtmann:

Proposition 2.15 [4, Corollary 3.3] *There exists a finite subgroup*

$$Q < \mathrm{Out}(A_\Gamma)$$

consisting solely of graph symmetries such that

$$\mathrm{Out}(A_\Gamma) = \mathrm{Out}^0(A_\Gamma) \rtimes Q.$$

Corollary 2.16 *Suppose that any action of G on a set of cardinality at most k is trivial, and assume that Γ has k vertices. Then any homomorphism*

$$\phi \colon G \to \mathrm{Out}(A_\Gamma)$$

has image contained in $\mathrm{Out}^0(A_\Gamma)$.

Proof Proposition 2.15 tells us that

$$\mathrm{Out}(A_\Gamma) = \mathrm{Out}^0(A_\Gamma) \rtimes Q$$

for some group Q acting faithfully on Γ. Hence we can postcompose ϕ with the quotient map

$$\mathrm{Out}^0(A_\Gamma) \rtimes Q \to Q$$

and obtain an action of G on the set of vertices of Γ. By assumption this action has to be trivial, and thus $\phi(G)$ lies in the kernel of this quotient map, which is $\mathrm{Out}^0(A_\Gamma)$. □

Definition 2.17 (G–invariant subgraphs) Given a homomorphism $G \to \mathrm{Out}(A_\Gamma)$ we say that a subgraph $\Sigma \subseteq \Gamma$ is G*-invariant* if and only if the conjugacy class of A_Σ is preserved (setwise) by G.

Definition 2.18 Having an invariant subgraph $\Sigma \subseteq \Gamma$ allows us to discuss two additional actions:

- Since, for any subgraph Σ, the normaliser of A_Σ in A_Γ is equal to $A_\Sigma C(A_\Sigma)$, where $C(A_\Sigma)$ is the centraliser of A_Σ (see eg [5, Proposition 2.2]), any invariant subgraph Σ gives us an *induced (outer) action* $G \to \mathrm{Out}(A_\Sigma)$.
- When Σ is invariant, we also have the *induced quotient action*

$$G \to \mathrm{Out}(A_\Gamma / \langle\langle A_\Sigma \rangle\rangle) \simeq \mathrm{Out}(A_{\Gamma \smallsetminus \Sigma}).$$

Let us quote the following.

Lemma 2.19 [9, Lemmata 4.2, 4.3] *For any homomorphism* $G \to \mathrm{Out}^0(A_\Gamma)$ *we have:*

(1) *for every subgraph* $\Sigma \subseteq \Gamma$ *which is not a cone,* $\mathrm{lk}(\Sigma)$ *is* G*–invariant;*

(2) *connected components of* Γ *which are not singletons are* G*–invariant;*

(3) $\widehat{\mathrm{st}}(\Sigma)$ *is* G*–invariant for every subgraph* Σ*;*

(4) *if* Σ *and* Δ *are* G*–invariant, then so is* $\Sigma \cap \Delta$*;*

(5) *if* Σ *is* G*–invariant, then so is* $\mathrm{st}(\Sigma)$*.*

Definition 2.20 (trivialised subgraphs) Let $\phi\colon G \to \mathrm{Out}(A_\Gamma)$ be given. We say that a subgraph $\Sigma \subseteq \Gamma$ is *trivialised* if and only if Σ is G–invariant and the induced action is trivial.

Lemma 2.21 *Let* $\phi\colon G \to \mathrm{Out}(A_\Gamma)$ *be a homomorphism. Suppose that* Σ *is a connected component of* Γ *which is trivialised by* G*. Consider the graph*

$$\Gamma' = (\Gamma \smallsetminus \Sigma) \sqcup \{s\},$$

where s *denotes a new vertex not present in* Γ*. There exists an action*

$$\psi\colon G \to \mathrm{Out}(A_{\Gamma'})$$

for which $\{s\}$ *is invariant and such that the quotient actions*

$$G \to \mathrm{Out}(A_{\Gamma \smallsetminus \Sigma})$$

induced by ϕ *and* ψ *by removing, respectively,* Σ *and* s*, coincide.*

Proof Consider an epimorphism $f\colon A_\Gamma \to A_{\Gamma'}$ defined on vertices of Γ by

$$f(v) = \begin{cases} v & \text{if } v \notin \Sigma, \\ s & \text{if } v \in \Sigma. \end{cases}$$

The kernel of f is normally generated by elements vu^{-1}, where $v, u \in \Sigma$ are vertices. Since the induced action of G on A_Σ is trivialised, the action preserves each element vu^{-1} up to conjugacy. But this in particular means that G preserves the (conjugacy class of) the kernel of f, and hence ϕ induces an action

$$G \to \mathrm{Out}(A_{\Gamma'})$$

which we call ψ. It is now immediate that ψ is as required. $\square$

2.4 Finite groups acting on RAAGs

Definition 2.22 Suppose that Γ has k vertices. Then the abelianisation of A_Γ is isomorphic to $\mathbb{Z}^k$, and we have the natural map

$$\mathrm{Out}(A_\Gamma) \to \mathrm{Out}(H_1(A_\Gamma)) = \mathrm{GL}_k(\mathbb{Z}).$$

We will refer to the kernel of this map as the *Torelli subgroup*.

We will need the following consequence of independent (and more general) results of Toinet and Wade.

Theorem 2.23 (Toinet [19]; Wade [21]) *The Torelli group is torsion free.*

Lemma 2.24 *Let $\phi\colon H \to \mathrm{Out}(A_\Gamma)$ be a homomorphism with a finite domain. Suppose that $\Gamma = \Sigma_1 \cup \cdots \cup \Sigma_m$ and each Σ_i is trivialised by H. Then so is Γ.*

Proof Consider the action

$$\psi\colon H \to \mathrm{Out}(H_1(A_\Gamma)) = \mathrm{GL}_k(\mathbb{Z})$$

obtained by abelianising A_Γ, where k is the number of vertices of Γ. This $\mathbb{Z}$–linear representation ψ preserves the images of the subgroups A_{Σ_i} and is trivial on each of them. Thus the representation is trivial, and so $\phi(H)$ lies in the Torelli group. But the Torelli subgroup is torsion free. Hence ϕ is trivial. □

Lemma 2.25 *Let $\phi\colon G \to \mathrm{Out}(A_\Gamma)$ be a homomorphism. Let*

$$\Gamma = (\Gamma_1 \cup \cdots \cup \Gamma_n) \sqcup \Theta,$$

where $n \geqslant 1$, each Γ_i is trivialised by G, and Θ is a discrete graph with m vertices. Suppose that for some $l \in \{m, m+1\}$ any homomorphism

$$G \to \mathrm{Out}(F_l)$$

is trivial. Then Γ is trivialised, provided that G is the normal closure of a finite subgroup H and that G contains a perfect subgroup P, which in turn contains H.

Proof We can quotient out all of the groups A_{Γ_i} and obtain an induced quotient action

$$(*) \qquad G \to \mathrm{Out}(A_\Theta).$$

We claim that this map is trivial. To prove the claim we have to consider two cases: the first case occurs when $l = m$ in the hypothesis of our lemma, that is, every homomorphism

$$G \to \mathrm{Out}(F_m)$$

is trivial. Since Θ is a discrete graph with m vertices, we have $\mathrm{Out}(A_\Theta) = \mathrm{Out}(F_m)$, and so the homomorphism $(*)$ is trivial.

The second case occurs when $l = m + 1$ in the hypothesis of our lemma. In this situation we quotient A_Γ by each subgroup A_{Γ_i} for $i > 1$, but instead of quotienting out A_{Γ_1}, we use Lemma 2.21. This way we obtain an outer action on a free group with $m + 1$ generators, and such an action has to be trivial by assumption. Thus we can take a further quotient and conclude again that the induced quotient action $(*)$ on A_Θ is trivial. This proves the claim.

Now consider the action of G on the abelianisation of A_Γ. We obtain a map

$$\psi\colon G \to \mathrm{GL}_k(\mathbb{Z}),$$

where k is the number of vertices of Γ. Since each Γ_i is trivialised, and the induced quotient action on A_Θ is trivial, we see that $\psi(G)$ lies in the abelian subgroup of $\mathrm{GL}_n(\mathbb{Z})$ formed by block upper-triangular matrices with identity blocks on the diagonal, and a single nontrivial block of fixed size above the diagonal. But P is perfect, and so $\psi(P)$ must lie in the Torelli subgroup of $\mathrm{Out}(A_\Gamma)$. This is however torsion free by Theorem 2.23, and so H must in fact lie in the kernel of ϕ. We conclude that the action of G on Γ is also trivial, since G is the normal closure of H. □

2.5 Some representation theory

Let us mention a result about representations of $\mathrm{PSL}_n(\mathbb{Z}/p\mathbb{Z})$, for prime p, due to Landazuri and Seitz:

Theorem 2.26 [14] *Suppose that we have a nontrivial, irreducible projective representation* $\mathrm{PSL}_n(\mathbb{Z}/p\mathbb{Z}) \to \mathrm{PGL}(V)$, *where* $n \geqslant 3$, p *is prime, and* V *is a vector space over a field* $\mathbb{K}$ *of characteristic other than* p. *Then*

$$\dim V \geqslant \begin{cases} 2 & \text{if } (n,p) = (3,2), \\ p^{n-1} - 1 & \text{otherwise.} \end{cases}$$

We offer an extension of their theorem for algebraically closed fields of characteristic 0, which we will need to discuss actions of $\mathrm{Out}(F_n)$ and $\mathrm{SOut}(F_n)$ on finite sets.

Theorem 2.27 *Let V be a $\mathbb{K}$–linear representation of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ that is nontrivial and irreducible, where $n \geqslant 3$, q is a power of a prime p, and where $\mathbb{K}$ is an algebraically closed field of characteristic 0. Then*

$$\dim V \geqslant \begin{cases} 2 & \text{if } (n,p) = (3,2), \\ p^{n-1} - 1 & \text{otherwise.} \end{cases}$$

Proof Let $\phi\colon \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}) \to \mathrm{GL}(V)$ denote our representation. Consider Z, the subgroup of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ generated by diagonal matrices with all nonzero entries equal. Note that Z is the centre of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$. Hence V splits as an $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$–module into intersections of eigenspaces of all elements of Z. Since V is irreducible, we conclude that $\phi(Z)$ lies in the centre of $\mathrm{GL}(V)$.

First suppose that $q = p$. Consider the composition

$$\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}) \to \mathrm{GL}(V) \to \mathrm{PGL}(V).$$

We have just showed that Z lies in the kernel of this composition, and so our representation descends to a representation of $\mathrm{PSL}_n(\mathbb{Z}/p\mathbb{Z}) \cong \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})/Z$. This new, projective representation is still irreducible. It is also nontrivial since otherwise V would have to be a 1–dimensional nontrivial $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$–representation. There are no such representations since $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ is perfect when $p = q$. Now Theorem 2.26 yields the result.

Suppose now that $q = p^\alpha$, where $\alpha > 1$. Let $N \trianglelefteq \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ be the kernel of the natural map $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$. As an N–module, by Maschke's theorem, V splits as

$$V = \bigoplus_{i=1}^{k} U_i,$$

where each $U_i \neq \{0\}$ is a direct sum of irreducible N–modules, and irreducible submodules $W \leqslant U_i$ and $W' \leqslant U_j$ are isomorphic if and only if $i = j$.

Observe that we get an induced action of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})/N \cong \mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ on the set $\{U_i, U_2, \dots, U_k\}$. As V is an irreducible $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$–module, the action is transitive.

Note that an action of a group on a finite set S induces a representation on the vector space with basis S. If $k > 1$ then this representation is not the sum of trivial ones because of the transitivity just described, and so

$$k \geqslant \begin{cases} 2 & \text{if } (n,p) = (3,2), \\ p^{n-1} - 1 & \text{otherwise} \end{cases}$$

since our theorem holds for $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$. Since $\dim U_i \geqslant 1$ for all i, we get $\dim V \geqslant k$ and our result follows.

Let us henceforth assume that $k = 1$. We have

$$V = U_1 = \bigoplus_{j=1}^{l} W,$$

where W is an irreducible N–module.

Note that we have an alternating group $\mathrm{Alt}_n < \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ satisfying

$$\mathrm{Alt}_n \cap N = \{1\}.$$

Let $\sigma \in \mathrm{Alt}_n$ be an element of order $o(\sigma)$ equal to 2 or 3.

Consider the group $M = \langle N, \sigma \rangle < \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$. Note that $M \cong N \rtimes \mathbb{Z}_{o(\sigma)}$. The module V splits as a direct sum of irreducible M–modules by Maschke's theorem. Let X be such an irreducible M–module.

Note that X, as an N–module, is a direct sum of, say, m copies of the N–module W (with $m \geqslant 1$). Frobenius reciprocity (see eg [22, Corollary 4.1.17]) tells us that the multiplicity m of W (as an N–module) in X is equal to the multiplicity of the M–module X in the M–module induced from the N–module W. Hence the multiplicity of W in the M–module induced from the N–module W is at least m^2. But it is bounded above by $o(\sigma)$, and $o(\sigma) \leqslant 3$, which forces $m = 1$ since $m \geqslant 1$.

This shows in particular that X as an N–module is isomorphic to W. It also shows that the M–module induced from W contains a submodule isomorphic to X. Since

$$M \cong N \rtimes \mathbb{Z}_{o(\sigma)},$$

an easy calculation shows that σ acts on this copy of X as a scalar multiple of the identity matrix, ie via a central matrix. This is true for every irreducible M–submodule X of V, and hence σ commutes with N when acting on V. Since the above statement is true for each $\sigma \in \mathrm{Alt}_n$ of order 2 or 3, we conclude that ϕ factors through $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})/[N, \mathrm{Alt}_n]$. Note that we need to consider elements σ of order 3 when we are dealing with the case $n = 4$.

Mennicke's proof of the congruence subgroup property [16] tells us that N is normally generated (as a subgroup of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$) by the p^{th} powers of the elementary matrices.

Now $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ itself is generated by elementary matrices; let us denote such a matrix by E_{ij} with the usual convention. Observe that for all $\sigma \in \mathrm{Alt}_n$ we have

$$\phi(E_{\alpha\beta}^{-1} E_{ij}^p E_{\alpha\beta}) = \phi(\sigma^{-1} E_{\alpha\beta}^{-1} E_{ij}^p E_{\alpha\beta}\sigma) = \phi(E_{\sigma(\alpha)\sigma(\beta)}^{-1} E_{ij}^p E_{\sigma(\alpha)\sigma(\beta)}).$$

Choose $\sigma \in A_n$ such that $\sigma(\alpha) = i$ and $\sigma(\beta) = j$. We conclude that $\phi(N)$ lies in the centre of $\phi(\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}))$. In particular, $\phi(N)$ is abelian, and hence (as $\mathbb{K}$ is algebraically closed) $\dim W = 1$ as W is an irreducible N–module. Since V is a direct sum of N–modules isomorphic to W, the group N acts via matrices in the centre of $\mathrm{GL}(V)$. Hence N lies in the kernel of the composition

$$\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}) \xrightarrow{\phi} \mathrm{GL}(V) \to \mathrm{PGL}(V).$$

We have already shown that Z lies in this kernel, and so our representation descends to a projective representation of $\mathrm{PSL}_n(\mathbb{Z}/p\mathbb{Z})$. If we can show that this representation is nontrivial, we can then apply Theorem 2.26 and our proof will be finished.

Suppose that this projective representation is trivial. This means that V is a 1–dimensional, nontrivial $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$–representation. This is, however, impossible since the abelianisation of $\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ is trivial when $n \geqslant 3$. □

2.6 Actions of $\mathrm{Out}(F_n)$ on finite sets

Theorem 2.28 *Every action of* $\mathrm{Out}(F_n)$ *(with* $n \geqslant 6$*) on a set of cardinality* $m \leqslant \binom{n+1}{2}$ *factors through* $\mathbb{Z}/2\mathbb{Z}$.

Proof Suppose that we are given such an action. It gives us

$$\mathrm{Out}(F_n) \to \mathrm{Sym}_m \hookrightarrow \mathrm{GL}_{m-1}(\mathbb{C}),$$

where Sym_m denotes the symmetric group of rank m, and the second map is the standard irreducible representation of Sym_m. Since

$$m - 1 < \binom{n+1}{2},$$

the composition factors through the natural map $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$ induced by abelianising F_n by [12, Theorem 3.13]. Thus we have

$$\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_{m-1}(\mathbb{C})$$

with finite image. The congruence subgroup property [16] tells us that the map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_{m-1}(\mathbb{C})$ factors through a congruence map

$$\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/p^\alpha\mathbb{Z})$$

for some positive integer α and some prime p. Now

$$m-1 < 2^{n-1}-1 \leqslant p^{n-1}-1,$$

and so the restricted map $\mathrm{SL}_n(\mathbb{Z}/p^\alpha\mathbb{Z}) \to \mathrm{GL}_{m-1}(\mathbb{C})$ must be trivial by Theorem 2.27. Thus the given action factors through $\mathrm{GL}_n(\mathbb{Z}/p^\alpha\mathbb{Z})/\,\mathrm{SL}_n(\mathbb{Z}/p^\alpha\mathbb{Z})$, an abelian group. Therefore $\mathrm{SOut}(F_n)$ lies in the kernel of ϕ since it is perfect (Proposition 2.2), and we are finished. □

Corollary 2.29 *Every action of* $\mathrm{SOut}(F_n)$ (*with* $n \geqslant 6$) *on a set of cardinality* $m \leqslant \frac{1}{2}\binom{n+1}{2}$ *is trivial.*

Proof Every action of an index-k subgroup of a group G on a set of cardinality m can be induced to an action of G on a set of cardinality km. □

3 The main result

Definition 3.1 Let D_n denote the discrete graph with n vertices.

Definition 3.2 Let $\phi\colon G \to \mathrm{Out}(A_\Gamma)$ be a homomorphism, and let n be fixed. We define two properties of the action (with respect to n):

- $\mathfrak{C}$ (clique) For every G–invariant clique Σ in Γ with at least n vertices there exists a G–invariant subgraph Θ of Γ such that $\Theta \cap \Sigma$ is a proper nonempty subgraph of Σ.
- $\mathfrak{D}$ (discrete) For every G–invariant subgraph Δ of Γ isomorphic to D_n, there exists a G–invariant subgraph Θ of Γ such that $\Theta \cap \Delta$ is a proper nonempty subgraph of Δ.

Lemma 3.3 *Let* $\phi\colon G \to \mathrm{Out}(A_\Gamma)$ *be an action satisfying* $\mathfrak{C}$ *and* $\mathfrak{D}$. *Let* Ω *be a* G*–invariant subgraph of* Γ. *Then both the induced action and the induced quotient action satisfy* $\mathfrak{C}$ *and* $\mathfrak{D}$.

Proof Starting with a subgraph Σ or Δ in either Ω or $\Gamma \smallsetminus \Omega$, we observe that the subgraph is a subgraph of Γ, and so using the relevant property we obtain a G–invariant subgraph Θ. We now only need to observe that $\Theta \cap \Omega$ is G–invariant by Lemma 2.19(4), and the image of Θ in $\Gamma \smallsetminus \Omega$ is invariant under the induced quotient action

$$G \to \mathrm{Out}(A_{\Gamma \smallsetminus \Omega}). \qquad \square$$

Theorem 3.4 *Let us fix positive integers n and $m \geqslant n$. Suppose that a group G satisfies all of the following:*

(1) *G is the normal closure of a finite subgroup H.*

(2) *All homomorphisms*
$$G \to \mathrm{Out}(F_k)$$
are trivial when $k \neq n$ and $k < m$.

(3) *All homomorphisms*
$$G \to \mathrm{GL}_k(\mathbb{Z})$$
are trivial when $k < n$.

(4) *Any action of G on a set of cardinality smaller than m is trivial.*

Let
$$\phi\colon G \to \mathrm{Out}(A_\Gamma)$$
be a homomorphism, where Γ has fewer than m vertices. Then ϕ is trivial, provided that the action satisfies properties $\mathfrak{C}$ and $\mathfrak{D}$ (with respect to n).

Proof Formally, the proof is an induction on the number of vertices of Γ and splits into two cases.

Before we proceed, let us observe that assumption (4) allows us to apply Corollary 2.16, and hence to use Lemma 2.19 whenever we need to.

Case 1 Suppose that Γ does not admit proper nonempty G–invariant subgraphs.

Note that this is in particular the case when Γ is a single vertex, which is the base case of our induction.

We claim that Γ is either discrete, or a clique. To prove the claim, let us suppose that Γ is not discrete.

Let v be a vertex of Γ with a nonempty link. Lemma 2.19(3) tells us that $\widehat{\mathrm{st}}(v)$ is G–invariant, and thus it must be equal to Γ. Hence Γ is a join, and therefore admits a join decomposition.

If each factor of the decomposition is a singleton, then Γ is a clique as claimed. Otherwise, the decomposition contains a factor Σ which is not a singleton and not a join, and so in particular not a cone. Thus Lemma 2.19(1) informs us that $\mathrm{lk}(\Sigma)$ is G–invariant. This is a contradiction, since this link is a proper nonempty subgraph. We have thus shown the claim.

Suppose that Γ is a clique with say, k vertices. Property $\mathfrak{C}$ immediately tells us that $k < n$, and so we are dealing with a homomorphism

$$\phi\colon G \to \mathrm{Out}(A_\Gamma) = \mathrm{GL}_k(\mathbb{Z}),$$

where $k < n$. Such a homomorphism is trivial by assumption (3).

Suppose that Γ is a discrete graph with, say, k vertices. Property $\mathfrak{D}$ immediately tells us that $k \neq n$, and so we are dealing with a homomorphism

$$\phi\colon G \to \mathrm{Out}(A_\Gamma) = \mathrm{Out}(F_k),$$

where $k \neq n$ and $k < m$. Such a homomorphism is trivial by assumption (2).

Case 2 Suppose that Γ admits a proper nonempty G–invariant subgraph Σ.

Lemma 3.3 guarantees that the induced action

$$G \to \mathrm{Out}(A_\Sigma)$$

satisfies the assumptions of our theorem, and thus, using the inductive hypothesis, we conclude that this induced action is trivial.

We argue in an identical manner for the induced quotient action

$$G \to \mathrm{Out}(A_{\Gamma \smallsetminus \Sigma})$$

and conclude that it is also trivial.

These two observations imply that in particular the restriction of these two actions to the finite group H from assumption (1) is trivial. Now Lemma 2.24 tells us that H lies in the kernel of ϕ, and hence so does G as it is a normal closure of H by assumption (1). $\square$

Lemma 3.5 *Suppose that* Γ *does not contain* n *distinct vertices with identical stars. Then property* $\mathfrak{C}$ *holds for any action* $G \to \mathrm{Out}^0(A_\Gamma)$.

Proof Let Σ be a G–invariant clique in Γ with at least n vertices. Since we know that no n vertices of Γ have identical stars, we need to have distinct vertices of Σ, say v and w, with $\mathrm{st}(v) \neq \mathrm{st}(w)$. Without loss of generality we may assume that there exists $u \in \mathrm{st}(v) \smallsetminus \mathrm{st}(w)$. In particular this implies that u and w are not adjacent.

Consider $\Lambda = \mathrm{lk}(\{u, w\})$: it is invariant by Lemma 2.19(1) since $\{u, w\}$ is not a cone, it intersects Σ nontrivially since the intersection contains v, and the intersection is also proper since $w \notin \Lambda$. Thus property $\mathfrak{C}$ is satisfied. □

Proposition 3.6 *In Theorem 3.4, we can replace the assumption on the action satisfying $\mathfrak{D}$ by the assumption that Γ is not a join of D_n and another (possibly empty) graph, provided that G satisfies additionally*

(5) *G contains a perfect subgroup P, which in turn contains H.*

Proof We are going to proceed by induction on the number of vertices of Γ as before. Assuming the inductive hypothesis, we will either show the conclusion of the theorem directly, or we will show that in fact property $\mathfrak{D}$ holds.

Note that the base case of the induction (Σ being a singleton) always satisfies $\mathfrak{D}$.

Let Δ be as in property $\mathfrak{D}$, and suppose that the property fails for this subgraph.

Case 1 Suppose that there exists a vertex u of Δ with a nonempty link.

Let v be a vertex of $\Gamma \smallsetminus \Delta$ joined to some vertex of Δ. Consider $\widehat{\mathrm{st}}(v)$; this subgraph is G–invariant by Lemma 2.19(3). If $\widehat{\mathrm{st}}(v)$ intersects Δ and does not contain it, then Δ does satisfy property $\mathfrak{D}$. We may thus assume that $\Delta \subseteq \widehat{\mathrm{st}}(v)$.

We would like to apply induction to $\widehat{\mathrm{st}}(v)$, and conclude that this subgraph, and hence Δ, are trivialised. This would force Δ to satisfy property $\mathfrak{D}$.

There are two cases in which we cannot apply the inductive hypothesis to $\widehat{\mathrm{st}}(v)$: this subgraph might be equal to Γ, or it might be a join of a subgraph isomorphic to D_n and another subgraph.

In the former case, Γ is a join of two nonempty graphs. If there exists a factor Θ of the join decomposition of Γ which is not a singleton and which does not contain Δ, then let us look at $\mathrm{lk}(\Theta)$. This is a proper subgraph of Γ, it is G–invariant by Lemma 2.19(1), and it is not a join of D_n and another graph since Γ is not. Thus we may apply the inductive hypothesis to $\mathrm{lk}(\Theta)$ and conclude that it is trivialised. But $\Delta \subseteq \mathrm{lk}(\Theta)$, and so Δ is also trivialised and thus satisfies $\mathfrak{D}$.

If Γ has no such factor Θ in its join decomposition, then $\Gamma = \mathrm{st}(\Sigma)$, where Σ is a nonempty clique. The clique Σ is a proper subgraph since it does not contain Δ. It is G–invariant by Lemma 2.19(1), and so the inductive hypothesis tells us that it is trivialised.

The induced quotient action $G \to \mathrm{Out}(A_{\Gamma \smallsetminus \Sigma})$ is also trivialised by induction as $\Gamma \smallsetminus \Sigma$ cannot be a join of D_n and another graph as before. We now apply Lemma 2.24 for the subgroup H and conclude that H, and hence its normal closure G, act trivially.

Now we need to look at the situation in which $\widehat{\mathrm{st}}(v)$ is a proper subgraph of Γ, but it is a join of D_n and another graph.

Let us look at Λ, the intersection of $\widehat{\mathrm{st}}(v)$ with the link of all factors of the join decomposition of $\widehat{\mathrm{st}}(v)$ isomorphic to D_n. The subgraph Λ is G–invariant by Lemma 2.19(1) and (4). It is a proper subgraph of Γ, and so the inductive hypothesis tells us that Λ is trivialised. If Λ contains Δ then we are done.

The graph Λ does not contain Δ if and only if Δ is a factor of the join decomposition of $\widehat{\mathrm{st}}(v)$. Observe that we can actually use another vertex of $\Gamma \smallsetminus \Delta$ in place of v, provided that this other vertex is joined by an edge to some vertex of Δ. Thus we may assume that Δ is a factor of the join decomposition of every $\widehat{\mathrm{st}}(v)$ where v is as described. This is however only possible when $\mathrm{st}(\Delta)$ is a connected component of Γ. There must be at least one more component since Γ is not a join of Δ and another graph.

Note that the component $\mathrm{st}(\Delta)$ is invariant by Lemma 2.19(5).

Suppose that the clique factor Σ of $\mathrm{lk}(\Delta)$ is nontrivial. As before, Σ is trivialised. Observing that $\Gamma \smallsetminus \Sigma$ is disconnected, and if it is discrete then it is has more than n vertices, allows us to apply the inductive hypothesis to the quotient action induced by Σ, and so, arguing as before, we see that Γ is trivialised.

Now suppose that $\mathrm{lk}(\Delta)$ has a trivial clique component. The join decomposition of the component $\mathrm{st}(\Delta)$ consists of at least two factors, each of which is invariant by Lemma 2.19(1). Let Θ be such a factor. Removing Θ leaves us with a disconnected graph smaller than Γ. Thus, we may apply the inductive hypothesis, provided that $\Gamma \smallsetminus \Theta$ is not D_n. This might however occur: in this situation $\mathrm{st}(\Delta) \smallsetminus \Theta$ fulfils the role of the graph Θ from the definition of $\mathfrak{D}$, and so we can use the inductive hypothesis nevertheless.

We now apply Lemma 2.24 to the subgroup H and the induced quotient actions determined by removing two distinct factors of $\mathrm{st}(\Delta)$, and we conclude that H, and hence its normal closure G, act trivially on A_Γ.

Case 2 Suppose that $\mathrm{lk}(u) = \varnothing$ for every vertex u of Δ.

We write $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k \sqcup \Theta$, where the subgraphs Γ_i are nondiscrete connected components of Γ, and Θ is discrete. By assumption $\Delta \subseteq \Theta$.

If $k \geqslant 2$, then removing any component Γ_i leaves us with a smaller graph, to which we can apply the inductive hypothesis. Then we use Lemma 2.25.

If $k = 0$, then Θ is not isomorphic to D_n by assumption. Then we know that the action ϕ is trivial by assumption (2).

If $k = 1$, then we need to look more closely at Γ_1. If Γ_1 does not have factors isomorphic to D_n in its join decomposition, then by induction we know that Γ_1 is trivialised. Now we use Lemma 2.25.

Suppose that Γ_1 contains a subgraph Ω isomorphic to D_n in its join decomposition. If Γ_1 has a nontrivial clique factor, then this factor is invariant, induction tells us that it is trivialised, and the induced quotient action is also trivial. Thus the entire action of H is trivial thanks to Lemma 2.24, and thus the action of G is trivial as G is the normal closure of H.

If the clique factor is trivial, then taking links of different factors of the join decomposition of Γ_1 allows us to repeat the argument we just used and conclude that H, and thus G, act trivially. $\square$

Theorem 3.7 *Let $n \geqslant 6$. Suppose that Γ is a simplicial graph with fewer than $\frac{1}{2}\binom{n}{2}$ vertices. Let $\phi\colon \mathrm{SOut}(F_n) \to \mathrm{Out}(A_\Gamma)$ be a homomorphism. Then ϕ is trivial, provided that there are no n vertices in Γ with identical stars and that Γ is not a join of the discrete graph with n vertices and another (possibly empty) graph.*

Proof We start by showing that $G = \mathrm{SOut}(F_n)$ satisfies the assumptions (1)–(4) of Theorem 3.4 and (5) of Proposition 3.6 with $m = \frac{1}{2}\binom{n}{2}$.

(1) Let $H = \mathrm{Alt}_n$. The group G is the normal closure of H by Proposition 2.4.

(2) All homomorphisms
$$G \to \mathrm{Out}(F_k)$$
are trivial when $k \neq n$ and $k < m$ by Theorem 2.7.

(3) All homomorphisms

$$G \to \mathrm{GL}_k(\mathbb{Z})$$

are trivial when $k < n$ by Corollary 2.5.

(4) Any action of G on a set of cardinality smaller than m is trivial by Corollary 2.29.

(5) G is perfect by Proposition 2.2.

To verify property $\mathfrak{C}$ we use Lemma 3.5, and using Proposition 3.6 we replace property $\mathfrak{D}$. Now we apply Theorem 3.4. $\square$

4 From larger to smaller RAAGs

In this section we will look at homomorphisms $\mathrm{Out}(A_\Gamma) \to \mathrm{Out}(A_{\Gamma'})$, where Γ' has fewer vertices than Γ.

Theorem 4.1 *There are no injective homomorphisms* $\mathrm{Out}(A_\Gamma) \to \mathrm{Out}(A_{\Gamma'})$ *when* Γ' *has fewer vertices than* Γ.

Proof For a group G we define its $\mathbb{Z}_2$*–rank* to be the largest n such that $(\mathbb{Z}_2)^n$ embeds into G.

We claim that the $\mathbb{Z}_2$–rank of $\mathrm{Out}(A_\Gamma)$ is equal to $|\Gamma|$, the number of vertices of Γ.

Firstly, note that for every vertex of Γ we have the corresponding inversion in $\mathrm{Out}(A_\Gamma)$, and these inversions commute; hence the $\mathbb{Z}_2$–rank of $\mathrm{Out}(A_\Gamma)$ is at least $|\Gamma|$.

For the upper bound, observe that the $\mathbb{Z}_2$–rank of $\mathrm{GL}_n(\mathbb{R})$ is equal to n, since we can simultaneously diagonalise commuting involutions in $\mathrm{GL}_n(\mathbb{R})$. Thus, the $\mathbb{Z}_2$–rank of $\mathrm{GL}_n(\mathbb{Z})$ is equal to n as well (since it is easy to produce a subgroup of this rank).

Finally, note that the kernel of the natural map $\mathrm{Out}(A_\Gamma) \to \mathrm{GL}_n(\mathbb{Z})$ with $n = |\Gamma|$ is torsion free by Theorem 2.23, and so the $\mathbb{Z}_2$–rank of $\mathrm{GL}_n(\mathbb{Z})$ is bounded below by the $\mathbb{Z}_2$–rank of $\mathrm{Out}(A_\Gamma)$. $\square$

Remark 4.2 The proof of the above theorem works for many subgroups of $\mathrm{Out}(A_\Gamma)$ as well; specifically it applies to $\mathrm{Out}^0(A_\Gamma)$, the group of *untwisted* outer automorphisms $\mathrm{U}(A_\Gamma)$, and the intersection $\mathrm{U}^0(A_\Gamma) = \mathrm{U}(A_\Gamma) \cap \mathrm{Out}^0(A_\Gamma)$.

It also works when the domain of the homomorphisms is $\mathrm{Aut}(A_\Gamma)$, or more generally any group with $\mathbb{Z}_2$–rank larger than the number of vertices of Γ'.

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