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Nonfillable Legendrian knots in the 3–sphere

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If Λ is a Legendrian knot in the standard contact 3–sphere that bounds an orientable exact Lagrangian surface Σ in the standard symplectic 4–ball, then the genus of Σ is equal to the slice genus of (the smooth knot underlying) Λ, the rotation number of Λ is zero as well as the sum of the Thurston–Bennequin number of Λ and the Euler characteristic of Σ, and moreover, the linearized contact homology of Λ with respect to the augmentation induced by Σ is isomorphic to the (singular) homology of Σ. It was asked by Ekholm, Honda and Kálmán (2016) whether the converse of this statement holds. We give a negative answer, providing a family of Legendrian knots with augmentations which are not induced by any exact Lagrangian filling although the associated linearized contact homology is isomorphic to the homology of the smooth surface of minimal genus in the 4–ball bounding the knot.

57R17; 16E45, 53D42

1 Introduction

Let Λ be a Legendrian knot in the standard contact 3–sphere. If Λ bounds an exact orientable Lagrangian surface Σ in the standard symplectic 4–ball, then

- the genus of Σ is equal to the slice genus of (the smooth knot underlying) Λ,
- the rotation number of Λ is zero as well as the sum of the Thurston–Bennequin number of Λ and the Euler characteristic of Σ by a theorem of Chantraine [2],
- the linearized contact homology of Λ with respect to the augmentation induced by Σ is isomorphic to the (singular) homology of Σ by a theorem of Seidel; see Dimitroglou Rizell [6] and Ekholm [7].

Ekholm, Honda and Kálmán [8, Question 8.9] ask whether every augmentation for which Seidel's isomorphism holds is induced by a Lagrangian filling. We give a negative answer to this question based on the family of Legendrian knots

$$\{\Lambda_{p,q,r,s} : p,q,r,s \geq 2,\ p \equiv q \equiv r+1 \equiv s+1 \bmod 2\}$$

given by the Lagrangian projection in Figure 1.

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Throughout the paper we will always be working under the above assumption that the parities of p and q match and they are the opposite of those of r and s. The rotation and Thurston–Bennequin numbers of $\Lambda_{p,q,r,s}$ are 0 and 5, respectively. This gives a lower bound of 3 on the slice genus. On the other hand, the Seifert surface we obtain from this projection has genus 3; hence, both the slice genus and the Seifert genus are 3.

Theorem 1 *The Chekanov–Eliashberg dg–algebra of $\Lambda_{p,p,p+1,p+1}$ admits an augmentation which is not induced by any exact orientable Lagrangian filling, although the corresponding linearized contact homology is isomorphic to the homology of a surface of genus 3 with one boundary component.*

Our examples are inspired by a deformation argument in Etgü and Lekili [10] related to the Chekanov–Eliashberg algebra, where the Legendrian link of unknots linked according to the D_4–tree is shown to be significant, specifically over a base field of characteristic 2 (see the last part of the proof of Theorem 14 in [10] and Remark 7 below). As can be seen in Figure 1, $\Lambda_{p,q,r,s}$ is constructed from the D_4–link by adding twists that turn it into a Legendrian knot. We expect that a Legendrian link with similar properties is obtained if D_4 is replaced by any tree other than A_n. A similar construction was previously used by Civan, Koprowski, Etnyre, Sabloff and Walker [4] on a different link to produce infinite families of Legendrian knots not isotopic to their Legendrian mirrors.

In the next section we describe the Chekanov–Eliashberg algebra and compute the linearized contact homology of our examples. In Section 3 we gather enough information on the strictly unital A_∞–algebras obtained by dualizing the Chekanov–Eliashberg algebras of our Legendrian knots to prove that they are not quasi-isomorphic to the A_∞–algebra of cochains on a closed surface whenever the base field has characteristic 2. This proves the nonfillability of our examples by a recent result of Ekholm and Lekili [9] (see Theorem 3 below).

2 Linearized contact homology of $\Lambda_{p,q,r,s}$

We use the combinatorial description of the Chekanov–Eliashberg algebra of a Legendrian knot in the standard contact 3–sphere given in [3], using the same sign conventions as in [10], and denote the Chekanov–Eliashberg algebra of $\Lambda = \Lambda_{p,q,r,s}$ over an arbitrary base field $\mathbb{K}$ by (L, ∂).[1] This is a dg–algebra generated by Reeb chords from Λ to

[1] As a matter of fact, considering the case $\mathbb{K} = \mathbb{Z}/2$ would suffice to obtain the main result. Hence the reader may prefer to ignore all the signs throughout the paper. We choose to keep track of them to provide a more complete picture and emphasize the significance of characteristic 2 (see Remark 7).

itself (or equivalently, the crossings in a Lagrangian projection). There is a $\mathbb{Z}$–grading on (L,∂) since the rotation number of Λ is 0.

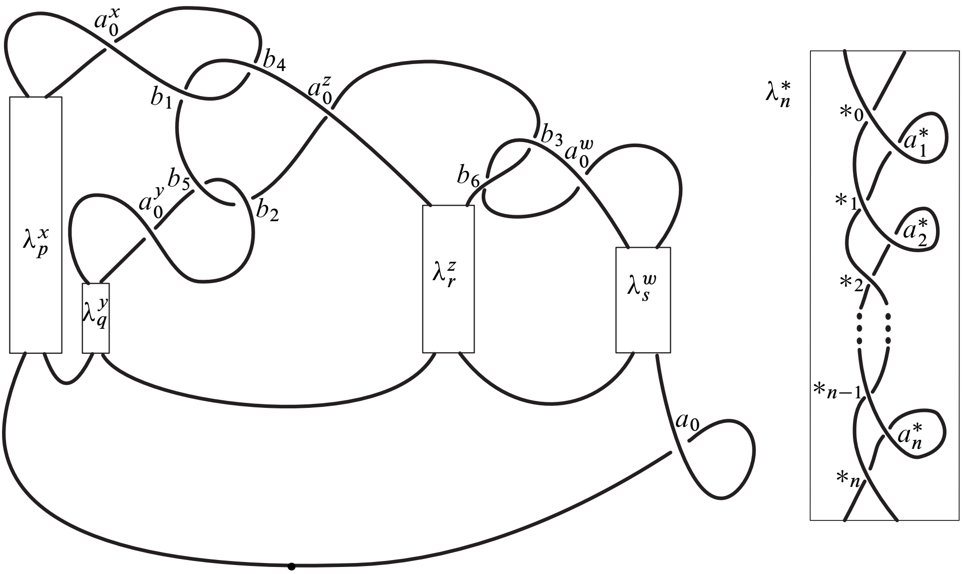

Figure 1: A Lagrangian projection of $\Lambda_{p,q,r,s}$ with $p,q,r,s \geq 2$ and $p \equiv q \equiv r+1 \equiv s+1 \bmod 2$

The generators of (L,∂) are indicated in Figure 1:

$$a_0^x,\dots,a_p^x,a_0^y,\dots,a_q^y,a_0^z,\dots,a_r^z,a_0^w,\dots,a_s^w,$$
$$x_0,\dots,x_p,y_0,\dots,y_q,z_0,\dots,z_r,w_0,\dots,w_s,$$
$$a_0,b_1,\dots,b_6,$$

with gradings

$$|*_i|=0 \quad \text{and} \quad |a_0|=|a_i^*|=1 \quad \text{for } * \in \{x,y,z,w\},$$
$$|b_1|=-|b_4|=p-r+1, \quad |b_2|=-|b_5|=q-r+1, \quad |b_3|=-|b_6|=r-s.$$

Proposition 2 *Let $\epsilon\colon L \to \mathbb{K}$ be the augmentation which maps all $*_i$ to -1 for $* \in \{x,y,z,w\}$. There is an isomorphism*

$$\mathrm{HC}_*^{\epsilon}(\Lambda_{p,p,p+1,p+1}) \cong H_{1-*}(\Sigma;\mathbb{K})$$

between the linearized contact homology of $\Lambda_{p,p,p+1,p+1}$ with respect to the augmentation ϵ and the homology of the orientable surface Σ of genus 3 with one boundary component.

Proof Counting the relevant immersed polygons with the choice of a basepoint on $\Lambda_{p,q,r,s}$ as indicated by $\bullet$ in Figure 1, the nontrivial differentials on the generators are

$$\begin{aligned}
\partial a_0 &= 1 - w_s z_r y_q x_p,\\
\partial a_0^x &= 1 + x_0 + b_1 b_4,\\
\partial a_0^y &= 1 + y_0 + b_2 b_5,\\
\partial a_0^z &= 1 + z_0 + b_4 b_1 + b_5 b_2 + z_0 b_6 b_3 + b_4 b_1 b_5 b_2,\\
\partial a_0^w &= 1 + w_0 + b_3 b_6,\\
\partial a_i^* &= 1 - *_{i-1} *_i \quad \text{for } * \in \{x, y, z, w\} \text{ and } i \geq 1.
\end{aligned}$$

Conjugating ∂ by the automorphism $\mathrm{id} + \epsilon$ gives another differential ∂^ϵ on L,

$$\begin{aligned}
\partial^\epsilon a_0 = w_s + z_r + y_q + x_p - w_s z_r - w_s y_q - w_s x_p - z_r y_q - z_s x_p - y_q x_p\\
+ w_s z_r y_q + w_s z_r x_p + w_s y_q x_p + z_r y_q x_p - w_s z_r y_q x_p,
\end{aligned}$$

$$\partial^\epsilon a_0^x = x_0 + b_1 b_4, \quad \partial^\epsilon a_0^y = y_0 + b_2 b_5, \quad \partial^\epsilon a_0^w = w_0 + b_3 b_6,$$

$$\partial^\epsilon a_0^z = z_0 + b_4 b_1 + b_5 b_2 - b_6 b_3 + z_0 b_6 b_3 + b_4 b_1 b_5 b_2,$$

$$\partial^\epsilon a_i^* = *_{i-1} + *_i - *_{i-1} *_i \quad \text{for } * \in \{x, y, z, w\} \text{ and } i \geq 1.$$

Applying the elementary transformation

$$a_0 \mapsto a_0 - (-1)^p \left(\sum_{i=0}^{p} (-1)^i a_i^x + \sum_{i=0}^{q} (-1)^i a_i^y - \sum_{i=0}^{r} (-1)^i a_i^z - \sum_{i=0}^{s} (-1)^i a_i^w \right)$$

simplifies the computation of linearized contact homology $\mathrm{HC}_*^\epsilon(\Lambda)$ of Λ associated to the augmentation ϵ and, more importantly, the description of the A_∞–algebras that will be discussed in the next section.

At this point, we have the following presentation of the differential ∂_1^ϵ on the linearized complex which computes $\mathrm{HC}_*^\epsilon(\Lambda_{p,q,r,s})$:

$$\partial_1^\epsilon a_0 = \partial_1^\epsilon b_j = 0,$$

$$\partial_1^\epsilon a_0^x = x_0, \quad \partial_1^\epsilon a_0^y = y_0, \quad \partial_1^\epsilon a_0^z = z_0, \quad \partial_1^\epsilon a_0^w = w_0,$$

$$\partial_1^\epsilon a_i^* = *_{i-1} + *_i \quad \text{for } * \in \{x, y, z, w\} \text{ and } i \geq 1.$$

It is clear that $\mathrm{HC}_*^\epsilon(\Lambda_{p,q,r,s})$ is spanned by $a_0, b_1, \ldots, b_6$. Moreover, if $p = q = r - 1 = s - 1$, then $|b_i| = 0$ for all i and we get the graded isomorphism in the statement. □

3 The augmentation ϵ is not induced by a Lagrangian filling

In this section, we prove that $\Lambda_{p,q,r,s}$ has no exact Lagrangian filling associated to the augmentation ϵ defined in Proposition 2 by using the following result of Ekholm and Lekili [9, Theorem 4].

Theorem 3 (Ekholm and Lekili) *If Λ has an exact Lagrangian filling Σ, then there is an A_∞ quasi-isomorphism between $\mathrm{RHom}_{\mathrm{CE}^*}(\mathbb{K},\mathbb{K})$ and the A_∞–algebra $C^*(S;\mathbb{K})$ of (singular) cochains on the closed surface S obtained by capping the boundary of Σ, where $\mathrm{CE}^* = L_{-*}$ and $\mathbb{K}$ is equipped with a CE–module structure by the augmentation $\epsilon_\Sigma\colon \mathrm{CE} \to \mathbb{K}$ induced by the filling Σ.*

In order to describe the A_∞–algebra $\mathrm{RHom}_{\mathrm{CE}^*}(\mathbb{K},\mathbb{K})$ in the above statement, we utilize the isomorphism between $\mathrm{RHom}_{\mathrm{CE}^*}(\mathbb{K},\mathbb{K})$ and the linear dual $\mathcal{B}$ of the Legendrian A_∞–coalgebra LC_* which satisfies $\mathrm{CE}^* = \Omega\mathrm{LC}_*$, defined in the more general setting of [9]. In the current setting, the strictly unital A_∞–algebra $\mathcal{B}$ can be obtained from the nonunital A_∞–algebra on the linearized cochain complex defined in [4] — which is also the endomorphism algebra of ϵ in the Aug_- category of [1] — by adding a copy of $\mathbb{K}$ to make it unital (see [9, Remark 24]).

The description of $\mathcal{B} \cong \mathrm{RHom}_{\mathrm{CE}^*}(\mathbb{K},\mathbb{K})$ we provide is based on the presentation of (L, ∂^ϵ) obtained at the end of the proof of Proposition 2. Abusing the notation, we denote the duals of the generators of L by the generators themselves. The nontrivial A_∞–products on $\mathcal{B}$ (besides those dictated by strict unitality) are

$$\mu^1_{\mathcal{B}}(*_i) = a^*_i + a^*_{i+1} \quad \text{for } * \in \{x,y,z,w\} \text{ and } i \geq 0,$$
$$\mu^1_{\mathcal{B}}(x_p) = a^x_p, \quad \mu^1_{\mathcal{B}}(y_q) = a^y_q, \quad \mu^1_{\mathcal{B}}(z_r) = a^z_r, \quad \mu^1_{\mathcal{B}}(w_s) = a^w_s,$$
$$\mu^2_{\mathcal{B}}(b_1,b_4) = (-1)^{p+1}a_0 + a^x_0, \quad \mu^2_{\mathcal{B}}(b_2,b_5) = (-1)^{p+1}a_0 + a^y_0,$$
$$\mu^2_{\mathcal{B}}(b_3,b_6) = (-1)^{p}a_0 + a^w_0,$$
$$\mu^2_{\mathcal{B}}(b_4,b_1) = \mu^2_{\mathcal{B}}(b_5,b_2) = (-1)^{p}a_0 + a^z_0, \quad \mu^2_{\mathcal{B}}(b_6,b_3) = (-1)^{p+1}a_0 - a^z_0,$$
$$\mu^2_{\mathcal{B}}(x_{i-1},x_i) = (-1)^{p+i}a_0 - a^x_i, \quad \mu^2_{\mathcal{B}}(y_{i-1},y_i) = (-1)^{p+i}a_0 - a^y_i,$$
$$\mu^2_{\mathcal{B}}(z_{i-1},z_i) = (-1)^{p+i+1}a_0 - a^z_i, \quad \mu^2_{\mathcal{B}}(w_{i-1},w_i) = (-1)^{p+i+1}a_0 - a^w_i,$$
$$\mu^2_{\mathcal{B}}(w_s,z_r) = \mu^2_{\mathcal{B}}(w_s,y_q) = \mu^2_{\mathcal{B}}(w_s,x_p) = \mu^2_{\mathcal{B}}(z_r,y_q) = \mu^2_{\mathcal{B}}(z_r,x_p) = \mu^2_{\mathcal{B}}(y_q,x_p) = -a_0,$$
$$\mu^3_{\mathcal{B}}(z_0,b_6,b_3) = a_0 + a^z_0,$$

$$\mu^3_{\mathcal{B}}(w_s, z_r, y_q) = \mu^3_{\mathcal{B}}(w_s, z_r, x_p) = \mu^3_{\mathcal{B}}(w_s, y_q, x_p) = \mu^3_{\mathcal{B}}(z_r, y_q, x_p) = a_0,$$
$$\mu^4_{\mathcal{B}}(b_4, b_1, b_5, b_2) = a_0 + a^z_0, \quad \mu^4_{\mathcal{B}}(w_s, z_r, y_q, x_p) = -a_0,$$

and the gradings in $\mathcal{B}$ are

$$\begin{aligned} |a_0| = |a^*_i| = 2 \quad &\text{and} \quad |*_i| = 1 \quad \text{for } * \in \{x, y, z, w\} \text{ and } i \geq 0,\\ |b_1| = p - r + 2, \quad &|b_2| = q - r + 2, \quad |b_3| = r - s + 1,\\ |b_4| = -p + r, \quad &|b_5| = -q + r, \quad |b_6| = -r + s, \end{aligned}$$

since the gradings of the generators in CE^*, LC_* and $\mathcal{B}$ which correspond to a generator of grading gr in L are $-\mathrm{gr}$, $-\mathrm{gr}-1$ and $\mathrm{gr}+1$, respectively.

A homological perturbation argument, as suggested by [11, Proposition 1.12 and Remark 1.13], provides a minimal model $(\mathcal{A}, \mu^\bullet_{\mathcal{A}})$ quasi-isomorphic to $(\mathcal{B}, \mu^\bullet_{\mathcal{B}})$. From the above description of the A_∞–products we see the decomposition $\mathcal{B} = \mathcal{A} \oplus \mathcal{C}$, where $\mathcal{A}$ is generated by

$$\{1, a_0, b_i : i = 1, \ldots, 6\}$$

and gives a minimal model for $\mathcal{B}$, whereas $\mathcal{C}$ is the subalgebra generated by the rest of the generators of $\mathcal{B}$ and acyclic with respect to the differential μ^1. As in [11], let the inclusion F^1 and the projection G^1 be the first-order pieces of recursively defined A_∞–algebra homomorphisms $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$. Also let $T^1\colon \mathcal{C} \to \mathcal{C}$ be the contracting homotopy with $\mu^1_{\mathcal{B}} T^1 + T^1 \mu^1_{\mathcal{B}} = F^1 G^1 - \mathrm{id}$ defined by

$$\begin{aligned} T^1(a^x_i) &= -x_i + x_{i+1} + \cdots + (-1)^{p-i+1} x_p,\\ T^1(a^y_i) &= -y_i + y_{i+1} + \cdots + (-1)^{q-i+1} y_q,\\ T^1(a^z_i) &= -z_i + z_{i+1} + \cdots + (-1)^{r-i+1} z_r,\\ T^1(a^w_i) &= -w_i + w_{i+1} + \cdots + (-1)^{s-i+1} w_s, \end{aligned}$$

and $T^1(*_i) = 0$ for $* \in \{x, y, z, w\}$ and $i \geq 0$. As we demonstrate below, the homotopy T^1 suffices to compute the A_∞–products $\mu^\bullet_{\mathcal{A}}$ on $\mathcal{A}$ recursively, using in [11, Equation (1.18)]

$$(1)\quad F^d(\alpha_d, \ldots, \alpha_1) = \sum_r \sum_{s_1, \ldots, s_r} T^1\big(\mu^r_{\mathcal{B}}(F^{s_r}(\alpha_d, \ldots, \alpha_{d-s_r+1}), \ldots, F^{s_1}(\alpha_{s_1}, \ldots, \alpha_1))\big),$$

$$(2)\quad \mu^d_{\mathcal{A}}(\alpha_d, \ldots, \alpha_1) = \sum_r \sum_{s_1, \ldots, s_r} G^1\big(\mu^r_{\mathcal{B}}(F^{s_r}(\alpha_d, \ldots, \alpha_{d-s_r+1}), \ldots, F^{s_1}(\alpha_{s_1}, \ldots, \alpha_1))\big),$$

where both sums are over partitions $s_1 + \cdots + s_r = d$ with $r \geq 2$.

To begin with, the only nontrivial $\mu^2_{\mathcal{A}}$ products (besides those dictated by strict unitality) are

$$\mu^2_{\mathcal{A}}(b_1, b_4) = \mu^2_{\mathcal{A}}(b_2, b_5) = \mu^2_{\mathcal{A}}(b_6, b_3) = (-1)^{p+1} a_0,$$
$$\mu^2_{\mathcal{A}}(b_4, b_1) = \mu^2_{\mathcal{A}}(b_5, b_2) = \mu^2_{\mathcal{A}}(b_3, b_6) = (-1)^{p} a_0,$$

as a consequence of (2) for $d = 2$.

Moreover, $\mu^3_{\mathcal{A}}$ vanishes since

- $\mu^3_{\mathcal{B}}$ is trivial on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$, and
- $\mu^2_{\mathcal{B}}$ vanishes on $\operatorname{im}(F^2) \otimes \mathcal{A} \subset \operatorname{im}(T^1) \otimes \mathcal{A} \subset \mathcal{C} \otimes \mathcal{A}$ and on $\mathcal{A} \otimes \operatorname{im}(F^2) \subset \mathcal{A} \otimes \mathcal{C}$,

where $F^2 = T^1 \circ \mu^2_{\mathcal{B}} \circ (F^1 \otimes F^1) = T^1 \circ \mu^2_{\mathcal{B}}$ by (1) and since F^1 is the inclusion.

Proceeding further, (2) for $d = 4$ simplifies as

$$\mu^4_{\mathcal{A}}(\alpha_4, \dots, \alpha_1) = \\ G^1\big(\mu^4_{\mathcal{B}}(\alpha_4, \dots, \alpha_1) + \mu^3_{\mathcal{B}}(F^2(\alpha_4, \alpha_3), \alpha_2, \alpha_1) + \mu^2_{\mathcal{B}}(F^2(\alpha_4, a_3), F^2(\alpha_2, \alpha_1))\big).$$

This is because the other four terms in (2) vanish on $\mathcal{A}^{\otimes 4}$. Among these terms,

$$G^1\big(\mu^2_{\mathcal{B}}(F^3(\alpha_4, \alpha_3, \alpha_2), \alpha_1)\big) \quad \text{and} \quad G^1\big(\mu^2_{\mathcal{B}}(\alpha_4, F^3(\alpha_3, \alpha_2, \alpha_1))\big)$$

vanish because $\mu^2_{\mathcal{B}}$ vanishes on $\mathcal{C} \otimes \mathcal{A}$ and $\mathcal{A} \otimes \mathcal{C}$, whereas $G^1\big(\mu^3_{\mathcal{B}}(\alpha_4, \alpha_3, F^2(\alpha_2, \alpha_1))\big)$ and $G^1\big(\mu^3_{\mathcal{B}}(\alpha_4, F^2(\alpha_3, \alpha_2), \alpha_1)\big)$ vanish since $\mu^3_{\mathcal{B}}$ vanishes on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{C}$ and $\mathcal{A} \otimes \mathcal{C} \otimes \mathcal{A}$.

At this point, we also observe that all three summands on the right-hand side of the above formula for $\mu^4_{\mathcal{A}}$ vanish on quadruples which are not of the form $(b_i, b_{i\pm 3}, b_j, b_{j\pm 3})$.

In order to prove Proposition 5, it suffices to compute

$$\mu^4_{\mathcal{A}}(b_i, b_j, b_k, b_l),$$

where (b_i, b_j, b_k, b_l) is a cyclic permutation of (b_1, b_2, b_5, b_4), (b_1, b_5, b_2, b_4) or (b_2, b_5, b_2, b_5). To this end, a straightforward, if tedious, computation reveals that the nontrivial $\mu^4_{\mathcal{A}}$ products among these ten are

$$\mu^4_{\mathcal{A}}(b_4, b_1, b_2, b_5) = a_0, \quad \mu^4_{\mathcal{A}}(b_5, b_2, b_4, b_1) = \mu^4_{\mathcal{A}}(b_5, b_2, b_5, b_2) = -a_0$$

if p is even, and

$$\mu^4_{\mathcal{A}}(b_4, b_1, b_2, b_5) = \mu^4_{\mathcal{A}}(b_4, b_1, b_5, b_2) = a_0, \quad \mu^4_{\mathcal{A}}(b_2, b_5, b_2, b_5) = -a_0$$

if p is odd.

Another important ingredient for the proof of Proposition 5 is the following formality statement:

Lemma 4 *Over a field $\mathbb{K}$ of characteristic 2, the algebra of cochains $C^*(S;\mathbb{K})$ on a closed orientable surface S is a formal dg–algebra.*

Over the base field $\mathbb{K}=\mathbb{R}$, there is the classical (and much more general) formality result in [5]. Since we were not able to locate an extension of this result to nonzero characteristic cases in the literature, we provide a proof of Lemma 4 at the end of this section. Note that the characteristic condition in the statement of Lemma 4 can be removed by a straightforward modification of the proof.

Proposition 5 *If the characteristic of the base field $\mathbb{K}$ is 2, then the A_∞–algebra $\mathcal{A}$ is not A_∞ quasi-isomorphic to the algebra of cochains on a closed orientable surface S.*

Proof By Lemma 4, it suffices to prove that there is no A_∞ quasi-isomorphism between $\mathcal{A}$ and $H^*(S;\mathbb{K})$.

Suppose that $\mathcal{F}$ is an A_∞–algebra homomorphism from $\mathcal{A}$ to $H^*(S;\mathbb{K})$. Since $\mathcal{A}$ is a minimal A_∞–algebra, $\mu^1_{\mathcal{A}}$ is trivial. We have established that $\mu^3_{\mathcal{A}}$ vanishes as well. As a consequence, a particular set of A_∞–functor equations, satisfied by the family of graded multilinear maps $\mathcal{F}^d\colon \mathcal{A}^{\otimes d}\to H^*(S;\mathbb{K})[1-d]$, simplifies to

$$
\begin{aligned}
&\mathcal{F}^1(\mu^4_{\mathcal{A}}(b_i,b_j,b_k,b_l))\\
&\quad=\mathcal{F}^3(b_i,b_j,b_k)\cup\mathcal{F}^1(b_l)+\mathcal{F}^1(b_i)\cup\mathcal{F}^3(b_j,b_k,b_l)+\mathcal{F}^2(b_i,b_j)\cup\mathcal{F}^2(b_k,b_l)\\
&\qquad\qquad+\mathcal{F}^3(\mu^2_{\mathcal{A}}(b_i,b_j),b_k,b_l)+\mathcal{F}^3(b_i,\mu^2_{\mathcal{A}}(b_j,b_k),b_l)+\mathcal{F}^3(b_i,b_j,\mu^2_{\mathcal{A}}(b_k,b_l)).
\end{aligned}
$$

In the rest of the proof we refer to the above equation as $\mathrm{Eq}_{(i,j,k,l)}$ and consider the sum of all the equations $\mathrm{Eq}_{(i,j,k,l)}$ where (i,j,k,l) is a cyclic permutation of $(1,2,5,4)$, $(1,5,2,4)$ or $(2,5,2,5)$.

First of all, the computation preceding this proposition implies that the sum of the left-hand side of these ten equations is equal to $(-1)^{p+1}\mathcal{F}^1(a_0)$ regardless of the characteristic. In contrast, if $\mathrm{char}(\mathbb{K})=2$, the right-hand side of the sum of these ten equations is 0; hence, $\mathcal{F}^1(a_0)=0$ and therefore $\mathcal{F}$ is not a quasi-isomorphism.

To prove the vanishing of the right-hand side for $\mathrm{char}(\mathbb{K})=2$, we consider the terms on this side in three separate groups and argue that each group adds up to 0. First observe that, since the cup product $\cup$ is (graded-)commutative, each of the first two terms on the right-hand side of the equation $\mathrm{Eq}_{(i,j,k,l)}$ appears in exactly one other equation, namely $\mathrm{Eq}_{(l,i,j,k)}$ or $\mathrm{Eq}_{(j,k,l,i)}$. For the same reason, the third term on the right-hand side of $\mathrm{Eq}_{(i,j,k,l)}$ is canceled by that of $\mathrm{Eq}_{(k,l,i,j)}$, unless of course $(i,j)=(k,l)$. This leaves us with the sum of the third terms of $\mathrm{Eq}_{(2,5,2,5)}$ and $\mathrm{Eq}_{(5,2,5,2)}$,

$$\mathcal{F}^2(b_2,b_5)\cup\mathcal{F}^2(b_2,b_5)+\mathcal{F}^2(b_5,b_2)\cup\mathcal{F}^2(b_5,b_2),$$

where each summand vanishes by the behavior of the cup product on surfaces. Finally, remember that we have $\mu^2_{\mathcal{A}}(b_i,b_j)=0$ for $|i-j|\neq 3$ and, when $\mathrm{char}(\mathbb{K})=2$,

$$\mu^2_{\mathcal{A}}(b_4,b_1)=\mu^2_{\mathcal{A}}(b_2,b_5)=\mu^2_{\mathcal{A}}(b_5,b_2)=a_0.$$

This suffices to conclude that each of the last three terms on the right-hand side of any one of the equations is either 0 or it appears in exactly two of our equations, eg the fifth term in $\mathrm{Eq}_{(2,4,1,5)}$ is equal to the fifth term in $\mathrm{Eq}_{(2,5,2,5)}$. □

Corollary 6 *The Legendrian knot $\Lambda_{p,q,r,s}$ admits an augmentation which is not induced by an exact orientable Lagrangian filling.*

Remark 7 When $\mathrm{char}(\mathbb{K})\neq 2$, our proof of Proposition 5 breaks down because the right-hand side of the sum of the ten equations we consider in the last step of the proof is equal to

$$2(-1)^p(\mathcal{F}^3(a_0,b_2,b_5)+\mathcal{F}^3(b_2,a_0,b_5)+\mathcal{F}^3(b_2,b_5,a_0)),$$

which is not necessarily 0 in general.

Proof of Lemma 4 We prove the formality of the dg–algebra $C=C^*(S,\mathbb{K})$ of (simplicial) cochains with the cup product on the closed surface S associated to the triangulation given in Figure 2 by providing a zig-zag of explicit dg–algebra quasi-isomorphisms connecting C and the cohomology algebra $H=H^*(S,\mathbb{K})$ of S.

We denote the generators of C by

$$e_i,\alpha_j,\beta_j,\theta_k,\gamma_k\quad\text{for } 1\leq j\leq g \text{ and } 1\leq k\leq 4g,$$

which represent the duals of the simplices

$$e_i^0,\;\; a_j,\;\; b_j,\;\; e_k^1,\;\; e_k^2,$$

as indicated in Figure 2.

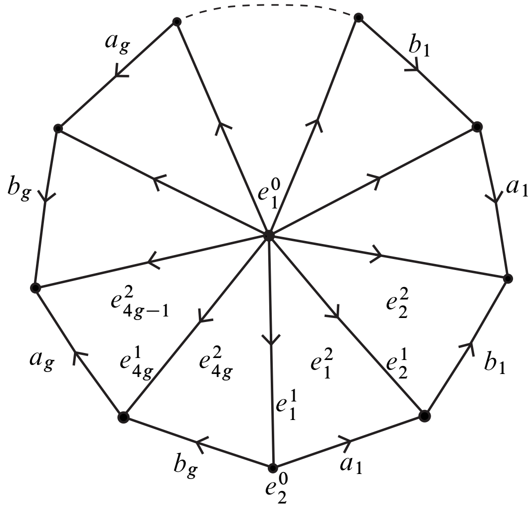

Figure 2: A triangulation of a closed, orientable surface S of genus g

The nontrivial differentials and products can be read from the triangulation as

$$\partial e_1 = \partial e_2 = \theta_1 + \cdots + \theta_{4g},$$
$$\partial\alpha_j = \gamma_{4j-3} + \gamma_{4j-1}, \quad \partial\beta_j = \gamma_{4j-2} + \gamma_{4j}, \quad \partial\theta_k = \gamma_{k-1} + \gamma_k,$$
$$e_i e_i = e_i, \quad e_1\theta_k = \theta_k, \quad e_1\gamma_k = \gamma_k, \quad \theta_k e_2 = \theta_k, \quad \gamma_k e_2 = \gamma_k,$$
$$e_2\alpha_j = \alpha_j e_2 = \alpha_j, \quad e_2\beta_j = \beta_j e_2 = \beta_j,$$
$$\theta_{4j-3}\alpha_j = \gamma_{4j-3}, \quad \theta_{4j-2}\beta_j = \gamma_{4j-2},$$
$$\theta_{4j}\alpha_j = \gamma_{4j-1}, \quad \theta_{4j+1}\beta_j = \gamma_{4j}.$$

(In the above equations and the rest of the proof, indices should always be interpreted modulo $4g$.)

We now define another dg–algebra, quasi-isomorphic to C and with a simplified differential, so that the rest of the proof is more transparent. This new dg–algebra C' is generated by

$$e, \varphi_j, \psi_j, \nu, \epsilon_1, \zeta_1, \xi_l, \nu_l \quad \text{for } 1 \le j \le g,\ 1 \le l \le 4g-1,$$

so that the map $\Phi\colon C' \to C$, defined by

$$e \mapsto e_1 + e_2, \quad \epsilon_1 \mapsto e_1, \quad \zeta_1 \mapsto \theta_1 + \cdots \theta_{4g},$$
$$\varphi_j \mapsto \alpha_j + \theta_{4j-2} + \theta_{4j-1}, \quad \psi_j \mapsto \beta_j + \theta_{4j-1} + \theta_{4j},$$
$$\xi_l \mapsto \theta_l, \quad \nu_l \mapsto \gamma_{l-1} + \gamma_l, \quad \nu \mapsto \gamma_{4g},$$

is a dg–algebra quasi-isomorphism. More precisely, on C', the nontrivial differentials are

$$\partial'\epsilon_1 = \zeta_1, \quad \partial'\xi_l = \nu_l,$$

e is the identity element, and the remaining products are

$$\begin{gathered}
\varphi_j\psi_j = \nu + \nu_1 + \cdots + \nu_{4j-2}, \qquad \psi_j\varphi_j = \nu + \nu_1 + \cdots + \nu_{4j-1},\\
\xi_{4j-3}\varphi_j = \nu + \nu_1 + \cdots + \nu_{4j-3}, \quad \xi_{4j-2}\psi_j = \nu + \nu_1 + \cdots + \nu_{4j-2},\\
\xi_{4j}\varphi_j = \nu + \nu_1 + \cdots + \nu_{4j-1}, \quad \xi_{4j+1}\psi_j = \nu + \nu_1 + \cdots + \nu_{4j} \quad \text{for } j < g,\\
\xi_1\psi_g = \nu,\\
\epsilon_1\epsilon_1 = \epsilon_1, \quad \epsilon_1\xi_l = \xi_l, \quad \epsilon_1\varphi_j = \xi_{4j-2} + \xi_{4j-1},\\
\epsilon_1\psi_j = \xi_{4j-1} + \xi_{4j}, \quad \epsilon_1\psi_g = \zeta_1 + \xi_1 + \cdots \xi_{4g-2} \quad \text{for } j < g,\\
\epsilon_1\nu = \nu,\\
\epsilon_1\zeta_1 = \zeta_1, \quad \epsilon_1\nu_l = \nu_l, \quad \zeta_1\varphi_j = \nu_{4j-2} + \nu_{4j-1},\\
\zeta_1\psi_j = \nu_{4j-1} + \nu_{4j} \quad \text{for } j < g,\\
\zeta_1\psi_g = \nu_1 + \cdots + \nu_{4g-2}.
\end{gathered}$$

In the next step, we define yet another dg–algebra $\widehat{C}$ by stabilizing C', ie $\widehat{C}$ contains C' as a subalgebra and the inclusion map is a dg–algebra quasi-isomorphism. Namely, we add the generators

$$\epsilon_k, \zeta_k \quad \text{for } 2 \le k \le 2g+1$$

with

$$|\epsilon_k| = 0, \quad |\zeta_k| = 1 \quad \text{and} \quad \widehat{\partial}\epsilon_k = \zeta_k, \quad \widehat{\partial}\zeta_k = 0$$

to those of C', and extend the algebra structure to $\widehat{C}$ by adding the nontrivial products

$$\begin{gathered}
\epsilon_{2j}\psi_j = \xi_1 + \cdots + \xi_{4j-2}, \quad \epsilon_{2j+1}\varphi_j = \xi_1 + \cdots + \xi_{4j-1},\\
\zeta_{2j}\psi_j = \nu_1 + \cdots + \nu_{4j-2}, \quad \zeta_{2j+1}\varphi_j = \nu_1 + \cdots + \nu_{4j-1}
\end{gathered}$$

for $j = 1, \dots, g$.

Finally, it is clear that the map $\widehat{\Phi}$: $H \to \widehat{C}$ defined on the cohomology algebra $H = H^*(S;\mathbb{K})$ by

$$e \mapsto e, \quad \overline{\varphi}_j \mapsto \varphi_j + \zeta_{2j}, \quad \overline{\psi}_j \mapsto \psi_j + \zeta_{2j+1}, \quad \nu \mapsto \nu$$

is a dg–algebra quasi-isomorphism, proving the formality of $C = C^*(S;\mathbb{K})$. □

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