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# Corrigendum to the article The simplicial boundary of a CAT(0) cube complex

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We correct Theorem 3.10 of [Algebr. Geom. Topol. 13 (2013) 1299–1367] in the infinite-dimensional case. No correction is needed in the finite-dimensional case.

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In this note, we correct Theorem 3.10 of [4] and record consequent adjustments to later statements. In [4], we worked in a CAT(0) cube complex $X$ with the property that there is no infinite collection of pairwise-crossing hyperplanes. Such a complex may still contain cubes of arbitrarily large dimension. It is in this situation that Theorem 3.10 requires correction; under the stronger hypothesis that $\dim X < \infty$, the theorem and its consequences hold as written in [4]. The extant results that use the simplicial boundary (see Behrstock and Hagen [1], Chatterji, Fernós and Iozzi [2], Durham, Hagen and Sisto [3], Hagen [5] and Hagen and Susse [6]) all concern finite-dimensional CAT(0) cube complexes and are thus unaffected.

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## Corrected statement and proof of Theorem 3.10

Throughout, $X$ denotes a CAT(0) cube complex which, according to standing hypotheses in [4], has countably many cubes and hyperplanes. The following replaces [4, Theorem 3.10]:

**Theorem A** *Suppose that every collection of pairwise crossing hyperplanes in $X$ is finite. Let $v$ be an almost-equivalence class of UBSs. Then $v$ has a representative of the form $\mathcal{V} = \bigsqcup_{i\in I} \mathcal{U}_i$, where $I$ is a finite or countably infinite set, each $\mathcal{U}_i$ is minimal, and for all distinct $i, j \in I$, if $H \in \mathcal{U}_j$, then $H$ crosses all but finitely many elements of $\mathcal{U}_i$, or the same holds with the roles of $i$ and $j$ reversed. Moreover, if $\dim X < \infty$, then the following hold:*

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- *$k = |I| \leq \dim \boldsymbol{X}$;*
- *for all $1 \leq i < j \leq k$, if $H \in \mathcal{U}_j$, then $H$ crosses all but finitely many elements of $\mathcal{U}_i$;*
- *if $\mathcal{V}' = \bigsqcup_{i \in I'} \mathcal{U}'_i$ is almost-equivalent to $\mathcal{V}$, and each $\mathcal{U}'_i$ is minimal, then $|I'| = k$ and, up to reordering, $\mathcal{U}_i$ and $\mathcal{U}'_i$ are almost equivalent for all $i$.*

**Remark 1** Under the additional hypothesis that $\boldsymbol{X}$ is finite-dimensional, Theorem A can be applied in [4] everywhere that Theorem 3.10 is used; see Remark 3.

**Remark 2** Consider the following wallspace. The underlying set is $\mathbb{R}^2$, and for each integer $n \geq 0$, we have a set $\{H_i^n\}_{i \geq 0}$ of walls such that:

- $H_i^n$ is the image of an embedding $\mathbb{R} \to \mathbb{R}^2$ for all $i, n$;
- each $\{H_i^n\}_{i \geq 0}$ has $H_i^n$ separating $H_{i+1}^n$ from $H_{i-1}^n$ for all $i \geq 1$;
- whenever $n < m$, we have that $H_i^m$ crosses $H_j^n$ for $j > m$ and does not cross $H_j^n$ for $j \leq m$.

Let $\boldsymbol{X}$ be the dual cube complex, whose hyperplanes we identify with the corresponding walls. Then each $\{H_i^n\}_{i \geq 0}$ is a minimal UBS, and $\mathcal{V} = \bigsqcup_{n \geq 0} \{H_i^n\}_{i \geq 0}$ is a UBS satisfying the first conclusion. However, $\mathcal{V} = \left(\bigsqcup_{n \geq 0} \{H_i^n\}_{i > 0}\right) \sqcup \mathcal{V}'$, where $\mathcal{V}' = \{H_1^n\}_{n \geq 0}$ is a minimal UBS. So, the finite dimension hypothesis is needed to obtain the "uniqueness" clause.

**Proof of Theorem A** Let $\mathcal{V}$ be a UBS representing $v$. We first establish some general facts.

**Chains** A *chain* in $\mathcal{V}$ is a set $\{U_i\}_{i=0}^{\infty} \subseteq \mathcal{V}$ of hyperplanes with the property that $U_i$ separates $U_{i+1}$ from $U_{i-1}$ for all $i \geq 1$. The chain $\{U_i\}_{i=0}^{\infty}$ is *inextensible* in $\mathcal{V}$ if there does not exist $V \in \mathcal{V}$ one of whose associated halfspaces contains $U_i$ for all $i \geq 0$. Since $\boldsymbol{X}$ contains no infinite set of pairwise crossing hyperplanes, and $\mathcal{V}$ contains no facing triple, any infinite subset of $\mathcal{V}$ contains a chain. Since $\mathcal{V}$ is unidirectional, any infinite $\mathcal{W} \subset \mathcal{V}$ contains a chain which is inextensible in $\mathcal{W}$.

**Almost-crossing** Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{V}$ be UBSs. We write $\mathcal{A} \prec \mathcal{B}$ if $B$ crosses all but finitely many $A \in \mathcal{A}$, for all $B \in \mathcal{B}$. Note that we can have $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{A}$ simultaneously (for instance, consider the hyperplanes in the standard cubulation of $\mathbb{E}^2$); in this case we say $\mathcal{A}, \mathcal{B}$ are *tied*. We have:

(i) By definition, if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C}$ is another UBS with $\mathcal{B} \prec \mathcal{C}$, then $\mathcal{A} \prec \mathcal{C}$. Similarly, if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \prec \mathcal{B}$, then $\mathcal{C} \prec \mathcal{A}$.

(ii) Suppose that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are minimal UBSs contained in the UBS $\mathcal{V}$. Suppose that $\mathcal{A} \prec \mathcal{B}$ and $\mathcal{B} \prec \mathcal{C}$. Then one of the following holds:
- $\mathcal{A} \prec \mathcal{C}$;
- $\mathcal{A}$ and $\mathcal{B}$ are tied and $\mathcal{C} \prec \mathcal{A}$.

Indeed, this follows by considering $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ as the inseparable closures of chains (see below) and using that $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \subset \mathcal{V}$ is unidirectional and contains no facing triple.

**Existence of the decomposition** The UBS $\mathcal{V}$ contains a minimal UBS $\mathcal{U}_1$. Indeed, $\mathcal{V}$ contains a chain, being infinite, and contains the inseparable closure of the chain, being inseparable. The proof of [4, Lemma 3.7] shows that this inseparable closure contains a minimal UBS.

Being infinite and unidirectional, $\mathcal{U}_1$ contains a chain $\mathcal{C}_1 = \{U_i^1\}_{i=0}^{\infty}$ which is inextensible in $\mathcal{U}_1$. By adding finitely many hyperplanes of $\mathcal{V}$ to $\mathcal{U}_1$, we can assume that $\mathcal{C}_1$ is inextensible in $\mathcal{V}$. Moreover, the inseparable closure $\overline{\mathcal{C}_1}$ of $\mathcal{C}_1$ in $\mathcal{V}$ contains a UBS (as above) and is contained in $\mathcal{U}_1$, whence, by minimality of $\mathcal{U}_1$, we have $\mathcal{U}_1 = \mathcal{F}_1 \cup \overline{\mathcal{C}_1}$ for some finite $\mathcal{F}_1$. We can remove $\mathcal{F}_1$ from $\mathcal{U}_1$ without affecting inseparability. Hence assume that $\mathcal{U}_1 = \overline{\mathcal{C}_1}$.

Let $\mathcal{V}_1 = \mathcal{V} - \mathcal{U}_1$. If $\mathcal{V}_1$ is finite, then $\mathcal{V}$ is almost-equivalent to $\mathcal{U}_1$, and we are done, with $k = 1$. Hence suppose that $\mathcal{V}_1$ is infinite. Note that $\mathcal{V}_1$ is unidirectional and has no facing triple.

Let $\mathcal{V}_1 = \mathcal{V}_1^+ \sqcup \mathcal{V}_1^-$, where $\mathcal{V}_1^+$ is the set of $V \in \mathcal{V}_1$ such that $V$ crosses all but finitely many elements of $\mathcal{U}_1$. If $V \in \mathcal{V}_1^-$, then $V$ crosses $U_0$, for otherwise $V$ would form a facing triple with $U_j, U_{j'}$ for some $j$, $j'$. Moreover, $V$ crosses only finitely many $U_j$. Indeed, otherwise, $V$ crosses all but finitely many $U_j$, and hence crosses all but finitely many hyperplanes in $\overline{\mathcal{C}_1} = \mathcal{U}_1$, and we would have $V \in \mathcal{V}_1^+$.

If $V, V' \in \mathcal{V}_1^-$, and $W$ separates $V, V'$, then $W$ crosses only finitely many elements of $\mathcal{U}_1$, and hence either $W \in \mathcal{V}_1^-$ or $W \in \mathcal{U}_1$. Moreover, $W$ crosses $U_0$. Now, since $\mathcal{U}_1 = \overline{\mathcal{C}_1}$, no element of $\mathcal{U}_1$ crosses $U_0$, so $W \in \mathcal{V}_1^-$. Hence $\mathcal{V}_1^-$ is inseparable.

On the other hand, suppose $V, V' \in \mathcal{V}_1^+$ and $W$ separates $V, V'$. Then $W$ must cross all but finitely many elements of $\mathcal{U}_1$, so $W \in \mathcal{V}_1^+$ (the possibility that $W \in \mathcal{U}_1$ is ruled out since $\mathcal{U}_1$ is the inseparable closure of $\mathcal{C}_1$). Hence $\mathcal{V}_1^+$ is inseparable.

Thus each of $\mathcal{V}_1^+$ or $\mathcal{V}_1^-$ is either finite or a UBS. By definition, $\mathcal{U}_1 \prec \mathcal{V}_1^+$. We now check that, if $\mathcal{V}_1^-$ is infinite, then $\mathcal{V}_1^- \prec \mathcal{U}_1$. Define a map $f : \mathcal{V}_1^- \to \mathbb{N}$ by declaring $f(V)$ to be the largest $j$ for which $V$ crosses $U_j$. If $f^{-1}(j)$ is infinite for some $j$, then $f^{-1}(j)$ contains a chain $\mathcal{D}$, all of whose hyperplanes are separated by $U_j$ from $U_{j'}$ whenever $j' > j+1$, contradicting unidirectionality of $\mathcal{V}$. Hence $f^{-1}(j)$ is finite for all $j$. Together with the facts that each $V \in \mathcal{V}_1^-$ crosses $U_0$ and $\mathcal{U}_1 = \overline{\mathcal{C}_1}$, this implies that $\mathcal{V}_1^- \prec \mathcal{U}_1$. In summary, one of the following holds:

(1) $\mathcal{V} = \mathcal{U}_1 \sqcup \mathcal{V}_1^+ \sqcup \mathcal{V}_1^-$, where $\mathcal{U}_1, \mathcal{V}_1^+$ are UBSs and $|\mathcal{V}_1^-| < \infty$. In this case, $\mathcal{U}_1 \cup \mathcal{V}_1^-$ is inseparable, so, by enlarging $\mathcal{U}_1$, we can write $\mathcal{V} = \mathcal{U}_1 \sqcup \mathcal{V}_1^+$ where $\mathcal{U}_1 \prec \mathcal{V}_1^+$.

(2) $\mathcal{V} = \mathcal{U}_1 \sqcup \mathcal{V}_1^+ \sqcup \mathcal{V}_1^-$, where $\mathcal{U}_1, \mathcal{V}_1^-$ are UBSs and $|\mathcal{V}_1^+| < \infty$. As above, we can, by enlarging $\mathcal{U}_1$ leaving $\mathcal{V}$ unchanged, write $\mathcal{V} = \mathcal{U}_1 \sqcup \mathcal{V}_1^-$, with $\mathcal{V}_1^- \prec \mathcal{U}_1$.

(3) $\mathcal{V} = \mathcal{U}_1 \sqcup \mathcal{V}_1^+ \sqcup \mathcal{V}_1^-$, and $\mathcal{V}_1^- \prec \mathcal{U}_1 \prec \mathcal{V}_1^+$.

We now apply the above construction of minimal sub-UBSs to $\mathcal{V}_1^+$ (in case (1)), to $\mathcal{V}_1^-$ (in case (3)), or to both (in case (2)). Continuing in this way, using the above facts about $\prec$, we find a countable set $I$ and a subset $\bigsqcup_{i\in I} \mathcal{U}_i \subseteq \mathcal{V}$ such that each $\mathcal{U}_i$ is a minimal UBS, and $\mathcal{U}_i \prec \mathcal{U}_j$ or $\mathcal{U}_j \prec \mathcal{U}_i$ for all $i, j \in I$, and every chain in $\mathcal{V}$ is contained in $\bigsqcup_{i\in I} \mathcal{U}_i$. Hence $\mathcal{V} - \mathcal{U}_i$ consists of those hyperplanes of $\mathcal{V}$ which do not lie in any chain. Since any infinite set of hyperplanes in $\mathcal{V}$ contains a chain, there are finitely many such hyperplanes, so $\mathcal{V}$ is almost-equivalent to $\bigsqcup_{i\in I} \mathcal{U}_i$.

**Uniqueness and dimension bound when $\dim X < \infty$** Observe that for any finite subset of $I$, $X$ contains a set of pairwise crossing hyperplanes of the same cardinality, so $|I| = k \leq \dim \mathcal{X} < \infty$. The uniqueness statement follows exactly as in the proof of [4, Theorem 3.10]; finite dimension is needed precisely because that argument uses that $k < \infty$.

**Ordering $I$ when $\dim X < \infty$** To complete the proof, it suffices to consider the UBS $\bigsqcup_{i=1}^k \mathcal{U}_i$, where each $\mathcal{U}_i$ is a minimal UBS and, for all $i, j$, either $\mathcal{U}_i \prec \mathcal{U}_j$ or $\mathcal{U}_j \prec \mathcal{U}_i$. Let $\Gamma$ be the graph with a vertex for each $\mathcal{U}_i$, with a directed edge from $\mathcal{U}_i$ to $\mathcal{U}_j$ if $\mathcal{U}_i \prec \mathcal{U}_j$ but $\mathcal{U}_j \not\prec \mathcal{U}_i$. By induction and the properties of $\prec$ established above, $\Gamma$ cannot contain a directed cycle, ie $\Gamma$ is a finite directed acyclic graph, whose vertices thus admit a linear order respecting the direction of edges. Hence we can order (and relabel) the $\mathcal{U}_i$ so that $\mathcal{U}_i \prec \mathcal{U}_j$ when $i < j$. □

**Remark 3** The correction of Theorem 3.10 affects the rest of [4] as follows:

- Since UBSs are not used in Sections 1 or 2, none of the statements there is affected.
- In Section 3, Theorem 3.10 should be adjusted as above. Following Theorem 3.10, one should then add the standing hypothesis that $X$ is finite-dimensional. The same standing hypothesis should be added in Section 4.
- In Section 5, all of the statements involving the simplicial boundary already hypothesize finite dimension, so no statement in that section is affected.
- In Sections 6.1 and 6.2, the hypothesis that $X$ is strongly locally finite must be replaced everywhere by the hypothesis that $X$ is locally finite and finite dimensional, because of the dependence on Theorem 3.14.

**Remark 4** (the infinite-dimensional case) In the infinite-dimensional, strongly locally finite case, many results in [4] are still available with sufficient care. For example, one can still define the simplicial boundary as the simplicial complex associated to the almost-containment partial ordering on the set of almost-equivalence classes of UBSs, but Remark 2 shows that in this case, simplices of $\partial_{\triangle} X$ may contain more $0$–simplices than expected. Many statements, when rephrased in terms of UBSs rather than boundary simplices, still hold in this generality. For example, the proof of Theorem 3.19 still shows that any UBS whose equivalence class is maximal in the above partial ordering determines a geodesic ray, and Theorems 3.23 and 3.30 hold as written.

## References

[1] **J Behrstock**, **M F Hagen**, *Cubulated groups: thickness, relative hyperbolicity, and simplicial boundaries*, Groups Geom. Dyn. 10 (2016) 649–707 MR

[2] **I Chatterji**, **T Fernós**, **A Iozzi**, *The median class and superrigidity of actions on* CAT(0) *cube complexes*, J. Topol. 9 (2016) 349–400 MR

[3] **M G Durham**, **M F Hagen**, **A Sisto**, *Boundaries and automorphisms of hierarchically hyperbolic spaces*, preprint (2016) arXiv

[4] **M F Hagen**, *The simplicial boundary of a CAT(0) cube complex*, Algebr. Geom. Topol. 13 (2013) 1299–1367 MR

[5] **M F Hagen**, *Cocompactly cubulated crystallographic groups*, J. Lond. Math. Soc. 90 (2014) 140–166 MR

[6] **M F Hagen**, **T Susse**, *Hierarchical hyperbolicity of all cubical groups*, preprint (2016) arXiv

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