Algebraic & Geometric Topology 18 (2018) 1675–1731

Generating families and augmentations for Legendrian surfaces

DAN RUTHERFORD
MICHAEL G SULLIVAN

We study augmentations of a Legendrian surface L in the 1–jet space, J^1M, of a surface M. We introduce two types of algebraic/combinatorial structures related to the front projection of L that we call chain homotopy diagrams (CHDs) and Morse complex 2–families (MC2Fs), and show that the existence of a ρ–graded CHD or a ρ–graded MC2F is equivalent to the existence of a ρ–graded augmentation of the Legendrian contact homology DGA to $\mathbb{Z}/2$. A CHD is an assignment of chain complexes, chain maps, and homotopy operators to the 0–, 1–, and 2–cells of a compatible polygonal decomposition of the base projection of L with restrictions arising from the front projection of L. An MC2F consists of a collection of formal handleslide sets and chain complexes, subject to axioms based on the behavior of Morse complexes in 2–parameter families. We prove that if a Legendrian surface has a tame-at-infinity generating family, then it has a 0–graded MC2F and hence a 0–graded augmentation. In addition, continuation maps and a monodromy representation of $\pi_1(M)$ are associated to augmentations, and then used to provide more refined obstructions to the existence of generating families that (i) are linear at infinity or (ii) have trivial bundle domain. We apply our methods in several examples.

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1 Introduction

Pseudoholomorphic-curve-based techniques have been used to prove many results in contact and symplectic geometry over the last three decades. One such method, which has enjoyed recent success in proving rigidity results for Legendrian submanifolds and their exact Lagrangian cobordisms, is to package an appropriate class of pseudoholomorphic curves into an invariant called Legendrian contact homology (LCH), which is the homology of a differential graded algebra (DGA). One way to extract information about a Legendrian using its LCH DGA is by considering augmentations which are DGA homomorphisms into a ground ring, which we take to be $\mathbb{Z}/2$ in this article. As observed in Chekanov [5], an augmentation allows one to form a linearization of LCH which is more manageable than the full DGA. Augmentations can

Published: 9 April 2018 DOI: 10.2140/agt.2018.18.1675

arise geometrically from exact Lagrangian fillings (null-cobordisms) of a Legendrian, in which case their linearized homologies reflect the usual (relative) homology of the fillings; see Dimitroglou Rizell [7] and Ekholm [9]. In addition, augmentations of particular Legendrian surfaces have been used to provide powerful topological knot invariants through knot contact homology, with ties to string theory; see Aganagic, Ekholm, Ng, and Vafa [1], Ekholm, Etnyre, Ng, and Sullivan [10], and Ng [22]. However, not all Legendrians have augmentations.

For a 1–dimensional Legendrian knot, L, in standard contact $\mathbb{R}^3 = J^1\mathbb{R}$, the existence problem for augmentations of the LCH DGA is well understood. Fuchs [13] found an interesting combinatorial structure for a front projection called a normal ruling whose existence is equivalent to the existence of an augmentation; see Fuchs and Ishkhanov [14] and Sabloff [29]. In addition, the existence of a 0–graded normal ruling (so also a 0–graded augmentation) is equivalent to the existence of a linear-at-infinity generating family for L; see Fuchs and Rutherford [15] and Pushkar' and Chekanov [24]. Here, a generating family is a family of functions whose critical values are determined by the front projection of L. To make this connection between generating families and augmentations more precise, Henry introduced an algebraic approximation for a generating family called a *Morse complex sequence*, and established a bijection between suitable equivalence classes of Morse complex sequences and homotopy classes of augmentations; see Henry [17] and Henry and Rutherford [18].

In this article, we take up analogous problems for Legendrian surfaces in 1–jet spaces. While a few important classes of Legendrian surfaces have had their DGAs extensively studied, eg conormal tori of braids/knots and isotopy spinnings of 1–dimensional Legendrians, little has been known about the existence problem for augmentations of general Legendrian surfaces. An obstacle to extending the methods used for 1–dimensional Legendrians to the higher-dimensional case has been the difficulty in dimensions 2 and above of giving an exact computation for the differential in the LCH DGA. Building on work of Ekholm [8], recent work of the authors [26; 27] gives explicit matrix formulas for the LCH differential of any generic Legendrian surface based on a choice of cellular decomposition for the base projection. This cellular formulation of LCH is central to the present article.

1.1 Overview of results

Let M be a surface, and let L be a closed Legendrian surface in the 1–jet space J^1M of M. Given a compatible cellular decomposition, $\mathcal{E}$, of the base projection to M of L, the *cellular DGA* $(\mathcal{A}, \partial)$ of [26] has a matrix of generators associated with each 0–, 1–, and 2–cell of $\mathcal{E}$; see Section 2 for details. An augmentation $\epsilon\colon (\mathcal{A}, \partial) \to (\mathbb{Z}/2, 0)$ then produces scalar matrices assigned to each cell that can be profitably viewed as linear maps. In Section 3, by interpreting the augmentation equation $\epsilon \circ \partial = 0$ from this point of view, we arrive in Proposition 3.7 at an equivalent characterization of an augmentation as a *chain homotopy diagram* (CHD) that associates chain complexes to 0–cells, chain maps (chain isomorphisms) to 1–cells, and chain homotopy operators to 2–cells, subject to certain conditions dictated by L.

To make contact with generating families, in Section 4 we introduce the notion of a *Morse complex* 2*–family* (MC2F) for L. An MC2F is a collection of data associated to the front projection of L that is modeled on the 2–parameter family of Morse complexes that arises when L has a generating family; MC2Fs are the 2–dimensional analog of the Morse complex sequences studied by Henry. In particular, we show in Proposition 4.10 that equipping a tame-at-infinity (see Section 2.2) generating family for L with an appropriate family of gradient-like vector fields produces an MC2F for L.

Our main results are summarized in the following:

Theorem 1.1 *Let M be a surface, $L \subset J^1M$ a closed Legendrian, and ρ a divisor of the Maslov number, $m(L)$.*

The following conditions are equivalent:

1. *The LCH DGA of L has a ρ–graded augmentation to $\mathbb{Z}/2$.*
2. *L has a ρ–graded chain homotopy diagram.*
3. *L has a ρ–graded Morse complex 2–family.*

Moreover, if L has a tame-at-infinity generating family, then the LCH DGA of L has a 0–graded augmentation to $\mathbb{Z}/2$.

A generating family $F\colon E \to \mathbb{R}$ for a Legendrian in J^1M has as its domain a fiber bundle $\pi\colon E \to M$ over M. The bundle does not need to be trivial, and this can be reflected by a monodromy representation of the fundamental group of M on the homology of a fiber $E_{x_0} = \pi^{-1}(\{x_0\})$. By carrying out a similar construction for MC2Fs (see Section 4.2) and making use of the correspondence from Theorem 1.1, in Section 6.3 we associate to an augmentation ϵ and $x_0 \in M$, a *fiber homology* $H(\epsilon_{x_0})$ equipped with a *monodromy representation* $\Phi_{\epsilon,x_0}\colon \pi_1(M, x_0) \to \mathrm{GL}(H(\epsilon_{x_0}))$ which is an antihomomorphism, ie $\Phi_{\epsilon,x_0}([\sigma]\cdot[\tau]) = \Phi_{\epsilon,x_0}([\tau])\cdot\Phi_{\epsilon,x_0}([\sigma])$. Using these representations, we provide in Proposition 6.8 obstructions to the existence of generating families that are (i) linear at infinity or (ii) defined on a trivial bundle.

In the concluding Section 7, we illustrate our general results with several examples. An interesting family of Legendrians, L_Γ, arising from 3–valent graphs $\Gamma \subset M$ was introduced by Treumann and Zaslow in [34]. Using Theorem 1.1 we show that L_Γ has an augmentation if and only if the dual graph to Γ is 3–colorable; this parallels a result from [34] about constructible sheaves. We also give examples to illustrate the obstructions from Proposition 6.8.

We mention a few interesting directions for possible future study:

(i) Currently, we do not know whether the statement about generating families in Theorem 1.1 can be strengthened to an if and only if statement. A more precise question is whether every 0–graded MC2F arises from an actual generating family via an appropriate choice of gradient vector field.

(ii) The constructible sheaf invariants of Legendrian submanifolds introduced by Shende, Treumann, and Zaslow [32] have also been shown to have close ties to generating families (see Sullivan [31]) and (in dimension 1) augmentations (see Ng, Rutherford, Shende, Sivek, and Zaslow [23]). It is possible that the equivalent characterizations of augmentations for Legendrian surfaces from Theorem 1.1 could be useful for establishing a connection with sheaf-based invariants.

1.2 Organization

The proof of Theorem 1.1 is based on the following logic:

$$\text{generating family} \quad \longrightarrow \quad \text{MC2F} \quad \longleftrightarrow \text{CHD} \quad \longleftrightarrow \quad \text{augmentation.}$$

In Section 2 we review generating families, augmentations, and the cellular formulation of the LCH DGA for Legendrian surfaces from [26; 27]. In Section 3, we define CHDs and show that they are in bijection with augmentations of the cellular DGA. In Section 4, we define MC2Fs and use the analysis of Hatcher and Wagoner [16] of 2–parameter families of functions to show how a generating family for L produces an MC2F. In addition, given an MC2F, we associate continuation maps to paths in M. The properties of these maps established in Proposition 4.7 allow us to define monodromy representations for MC2Fs and are later used for translating between CHDs and MC2Fs. The construction of a CHD from an MC2F is carried out in Section 5. After establishing some tools that are useful for the construction of MC2Fs, the converse construction of an MC2F from a CHD appears in Section 6. The monodromy representations for augmentations are constructed at the end of Section 6 with obstructions to particular types of generating families observed in Proposition 6.8. Finally, in Section 7 we apply our general results to several examples.

Acknowledgements Rutherford is partially supported by grant 429536 from the Simons Foundation. Sullivan is partially supported by grant 317469 from the Simons Foundation. He thanks the Centre de Recherches Mathématiques for hosting him while some of this work was done. We thank the referee for useful comments.

2 Background

2.1 Legendrian surfaces

Let M be a 2–dimensional manifold. Then the 1–jet space $J^1M = T^*M \times \mathbb{R}_z$ is a 5–dimensional contact manifold with a standard contact structure $\xi = \ker(dz - y\,dx)$, where $x = (x_1, x_2)$ are local coordinates for M (which we denote sometimes by M_x) and $y = (y_1, y_2) \in T^*_x M$ are fiber coordinates. A *Legendrian* (surface) $L \subset J^1M$ is a 2–dimensional submanifold such that $TL \subset \xi$.

Let $\Pi_F \colon J^1M \to J^0M = M \times \mathbb{R}_z$ be the so-called *front projection*. Similarly, let $\Pi_B \colon J^1M \to M$ be the *base projection*. We usually consider Legendrians that have generic front and base projections; see [26, Section 2.2] for a detailed discussion. Figure 1 illustrates the generic singularities which arise in $\Pi_F(L)$ and $\Pi_B(L)$. At a swallowtail point, a pair of cusp edges and a crossing arc all meet. We call a swallowtail point *upward* (resp. *downward*) if the sheet that connects the two cusp edges appears above (resp. below) the two crossing sheets. In the base projection, the image of the cusp edges divides a disk neighborhood of a swallowtail point into two parts. We refer to the region between the two cusp edges, above which the cusp sheets exist, as the *swallowtail region*.

A generic loop $\gamma \subset L$ is assigned an integer $m(\gamma) = D(\gamma) - U(\gamma) \in \mathbb{Z}$, where $D(\gamma)$ (resp. $U(\gamma)$) is the number of times γ crosses with a cusp edge of L in the downward (resp. upward) direction. This assignment gives a well-defined cohomology class $m \in \mathrm{Hom}(H_1(L;\mathbb{Z}),\mathbb{Z}) = H^1(L;\mathbb{Z})$, and the *Maslov number* $m(L) \in \mathbb{Z}_{\geq 0}$ of L is the nonnegative generator of the image of m. A *Maslov potential*, μ, for L is a locally constant function

$$\mu \colon L \setminus (\Sigma_{\mathrm{cusp}} \cup \Sigma_{\mathrm{st}}) \to \mathbb{Z}/m(L),$$

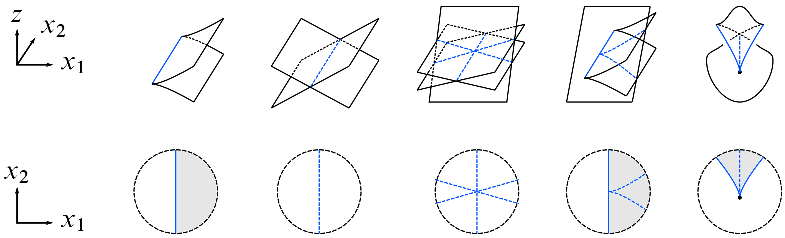

Figure 1: Generic singularities of front projections are pictured along with their base projections (left to right): cusps, crossings, triple points, cusp-sheet intersection, and swallowtail points. Additional codimension-2 singularities in the base projection arise as transverse intersections of two crossing/cusp arcs that are disjoint in $\Pi_F(L)$.

where $\Sigma_{\text{cusp}} \cup \Sigma_{\text{st}} \subset L$ is the union of all cusp and swallowtail points, such that μ increases by 1 when passing from the lower sheet to the upper sheet at any cusp edge. Maslov potentials exist and, when L is connected, are unique up to an overall additive constant.

2.2 Generating families

We review generating families in the Legendrian setting; for more details and applications, see for example [4; 33; 30]. Let $\pi\colon E \to M$ be a locally trivial fiber bundle over M with manifold fiber N. Given $F\colon E \to \mathbb{R}$ and $x \in M$, we denote its restriction to a fiber by $f_x\colon \pi^{-1}(x) \cong N \to \mathbb{R}$. We denote by $\eta = (\eta_1, \dots, \eta_n) \in N$ locally defined fiber coordinates and refer to a point in E as $e = (x, \eta)$. Suppose that $dF\colon E \to T^*E$ is transverse to the fiber normal bundle

$$N_E = \{(e, \nu) \in T^*E \mid \nu = 0 \text{ on } \ker(d\pi(e))\}.$$

In coordinates, this is equivalent to 0 being a regular value of $(x, \eta) \mapsto \partial_\eta F(x, \eta)$. This transversality condition ensures that the set of fiber critical points of F,

$$\Sigma_F = \{(x, \eta) \in E \mid (df_x)_\eta = 0\} = (dF)^{-1}(N_E),$$

is a manifold. There is then a Legendrian immersion of Σ_F into J^1M given in coordinates by

$$i_F\colon \Sigma_F \to J^1M, \quad (x, \eta) \mapsto (x, y, z) = (x, \partial_x F(x, \eta), F(x, \eta)).$$

When i_F is an embedding with $i_F(\Sigma_F) = L$, we say that F is a *generating family* for L. If F is a generating family for L, then so too is $F \circ \phi$, where ϕ: $E \to E$ is a fiber-preserving diffeomorphism. In addition, *stabilizations* of F, defined by $\bar{F}$: $E \times \mathbb{R}^m \to \mathbb{R}$, $\bar{F}(e, \mu) = F(e) + Q(\mu)$ for some nondegenerate quadratic form Q: $\mathbb{R}^m \to \mathbb{R}$, are also generating families for L.

In order to apply the tools of Morse theory to F, it is important to make some assumption about the behavior of F outside of compact sets. The following two conditions are commonly used in the generating family literature. A generating family F: $E \to \mathbb{R}$ is *linear at infinity* (resp. *quadratic at infinity*) if $E = E' \times \mathbb{R}^k$, where E' is a locally trivial fiber bundle with closed manifold fibers and, outside of a compact subset of E, the generating family F agrees with a fixed nonzero linear form (resp. a fixed nondegenerate quadratic form) on $\mathbb{R}^k$. We say F is *tame at infinity* if F is either linear or quadratic at infinity. Note that in the linear-at-infinity case, the $\mathbb{R}^k$ factor must have $k \geq 1$, while $k = 0$ is allowed in the quadratic-at-infinity case. If M is noncompact, then a quadratic-at-infinity generating family cannot produce a compact Legendrian.

Remark 2.1 It can be shown that, after a fiber-preserving diffeomorphism, a stabilization of a linear (resp. quadratic) at infinity generating family can again be made linear (resp. quadratic) at infinity. This is an important point for defining generating family homology invariants using tame-at-infinity generating families; see [30].

2.3 Augmentations

A *differential graded algebra* (DGA) in this article is an associative graded unital algebra $\mathcal{A}$, equipped with a differential; that is, a derivation ∂: $\mathcal{A} \to \mathcal{A}$ which squares to 0 and decreases the grading by 1. We consider DGAs with ground ring $\mathbb{Z}/2$ that are graded by $\mathbb{Z}/m$ for some $m \in \mathbb{Z}_{\geq 0}$ (where $\mathbb{Z}/m = \mathbb{Z}$ when $m = 0$). The DGAs we consider are freely generated by elements of homogeneous degree.

An *augmentation* ϵ: $(\mathcal{A}, \partial) \to (\mathbb{Z}/2, 0)$ is an algebra morphism ϵ: $\mathcal{A} \to \mathbb{Z}/2$ such that $\epsilon(1) = 1$ and $\epsilon \circ \partial = 0$. Given a divisor $\rho \,|\, m$, we say that ϵ is *ρ–graded* if ρ preserves grading mod m, equivalently, if $\epsilon(a_i) \neq 0$ for a generator $a_i \in \mathcal{A}$ implies $|a_i| = 0 \bmod \rho$.

In the context of Legendrian contact homology, the standard notion of equivalence used for DGAs is *stable tame isomorphism*, which also implies homotopy equivalence. The existence of a ρ–graded augmentation is invariant under stable tame isomorphism; see [5] or [26, Section 2.1.1].

2.4 The cellular DGA

We refer the reader to [11; 12], for example, for the pseudoholomorphic-based definition of the DGA underlying Legendrian contact homology (LCH). Instead, for the remainder of this section we review the stable tame isomorphic cellular DGA. The cellular DGA was introduced in [26, Section 3], and proven to be stable-tame isomorphic to the usual LCH DGA in [27].

Definition 2.2 Let $L \subset J^1M$ be a Legendrian surface with generic base projection. A *compatible polygonal decomposition* $\mathcal{E}$ for L is a polygonal cell decomposition of $\Pi_B(L) \subset M$ that contains the base projection of all cusp edges, crossing arcs, and swallowtail points of the front projection of L in its 1–skeleton and that we equip as follows:

(1) We choose an orientation for each 1–cell.

(2) We label, in the domain of each 2–cell, two of its 0–cells as "initial" and "terminal" vertices v_0 and v_1. If $v_0 = v_1$ we must also choose a direction for the path around the circle from v_0 to v_1.

(3) At each swallowtail point, we choose a labeling of the two corners that border the crossing locus. One region is labeled S and the other T.

Convention 2.3 In this article, to simplify the exposition, we will assume in addition that near swallowtail points, the 1–skeleton of $\mathcal{E}$ agrees with the projection of the singular set with the three 1–cells oriented away from the swallowtail point. The cellular DGA can be defined without this assumption. See Figure 4.

Let e^d_α be a cell from $\mathcal{E}$, where $0 \le d \le 2$ is the dimension. We let $L(e^d_\alpha)$ denote the *set of sheets* of L above e^d_α. This is defined as the set of those connected components of L above e^d_α that are *not contained in a cusp edge*; ie

$$L(e^d_\alpha) = \pi_0\big(\Pi_B^{-1}(e^d_\alpha) \cap (L \setminus \Sigma_{\mathrm{cusp}})\big).$$

Note that we do consider a swallowtail point above a 0–cell to be a sheet. Each set $L(e^d_\alpha) = \{S^\alpha_p\}$ has a partial order by (pointwise) descending z–coordinate,

$$S^\alpha_p \prec S^\alpha_q \quad \Longleftrightarrow \quad z(S^\alpha_p) > z(S^\alpha_q);$$

two sheets are incomparable if and only if they meet at a crossing arc above e^d_α in $\Pi_F(L)$. When the sheets of $L(e^d_\alpha)$ are totally ordered by z–coordinates, we use $\{1, 2, 3, \dots, n\}$ for the indexing set, so that $S^\alpha_i \prec S^\alpha_{i+1}$.

The algebra $\mathcal{A}$ is freely generated as follows. For each cell e^d_α we associate one generator for each pair of sheets S^α_p, $S^\alpha_q \in L(e^d_\alpha)$ satisfying $S^\alpha_p \prec S^\alpha_q$. We denote these generators as $a^\alpha_{p,q}$, $b^\alpha_{p,q}$, or $c^\alpha_{p,q}$ in the case where e^d_α is a 0–cell, 1–cell, or 2–cell respectively. Sometimes we suppress the superscript α from notation. The grading of $\mathcal{A}$ requires a choice of Maslov potential, μ, and is defined on generators by

$$|c_{p,q}| = \mu(S_p) - \mu(S_q) + 1, \quad |b_{p,q}| = \mu(S_p) - \mu(S_q), \quad |a_{p,q}| = \mu(S_p) - \mu(S_q) - 1.$$

2.4.1 The differential without swallowtail points In reviewing the differential, we start with the case that L *does not have swallowtail points.* We choose for each cell a bijection ι between $\{1, \dots, n_\alpha\}$ and the indexing set for $L(e^d_\alpha)$ that is compatible with the partial ordering of $L(e^d_\alpha)$ in the sense that

$$S_p \prec S_q \quad \Longrightarrow \quad \iota(p) < \iota(q).$$

Using the bijection, we arrange the generators corresponding to e^d_α into a strictly upper triangular $n_\alpha \times n_\alpha$ matrix, which we label A, B, or C accordingly. Note that entries in the upper triangular part of A or B that correspond to pairs of sheets that cross are 0.

Next, suppose that a cell $e^{d'}_\beta$ appears along the boundary of e^d_α with $d' < d$. We then place the generators associated to $e^{d'}_\beta$ into a corresponding $n_\alpha \times n_\alpha$ *boundary matrix* X in the following manner: Each sheet in $L(e^{d'}_\beta)$ belongs to the closure of a unique sheet in $L(e^d_\alpha)$. This identifies the indexing set of $L(e^{d'}_\beta)$ with a subset of $\{1, \dots, n_\alpha\}$, and we place the generators associated to $e^{d'}_\beta$ into the corresponding rows and columns of X. The remaining rows and columns correspond to sheets of $L(e^d_\alpha)$ that meet a cusp edge above $e^{d'}_\beta$, and such sheets come in pairs. When $d' = 0$ (resp. $d' = 1$), we insert the 2×2 block $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ $\left(\text{resp. } \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right)$ along the diagonal in the columns and rows that represent each cusping pair of sheets.

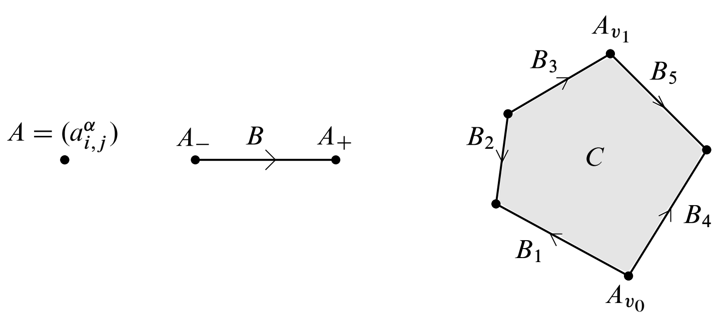

Figure 2: Formulas for the differentials for the pictured cells are given in (2-1); note $\partial C = A_{v_1} C + C A_{v_0} + (I + B_3)(I + B_2)^{-1}(I + B_1) + (I + B_5)^{-1}(I + B_4)$.

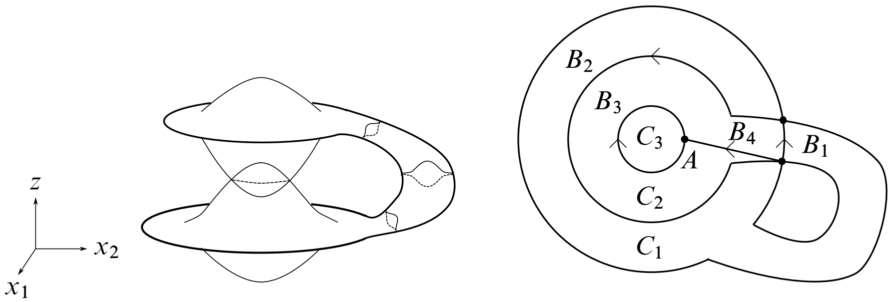

Figure 3: The front projection (left) of a Legendrian sphere pictured with a compatible polygonal decomposition (right). Arrows indicate the orientation of 1–cells.

For a 1–cell, let A_+ (resp. A_-) be the boundary matrices for the terminal (resp. initial) vertex. For a 2–cell, let A_{v_0} and A_{v_1} be the boundary matrices associated to the chosen initial and terminal vertices, v_0 and v_1. In addition, let $B_1, \dots, B_j$ and $B_{j+1}, \dots, B_m$ denote the boundary matrices associated to the successive boundary edges that appear in the domain of the characteristic map for the 2–cell, as we travel the two paths along the boundary of D^2 from v_0 to v_1. (If $v_0 = v_1$, then one of these paths is constant as specified in the definition of $\mathcal{E}$.) The differential $\partial\colon \mathcal{A} \to \mathcal{A}$ is then determined by the following matrix formulas, where ∂ is applied entry-by-entry:

$$
\begin{aligned}
&\partial A = A^2,\\
&\partial B = A_+(I+B) + (I+B)A_-,\\
&\partial C = A_{v_1}C + CA_{v_0} + (I+B_j)^{\eta_j}\cdots(I+B_1)^{\eta_1}\\
&\qquad\qquad + (I+B_m)^{\eta_m}\cdots(I+B_{j+1})^{\eta_{j+1}},
\end{aligned}
\tag{2-1}
$$

where $\eta_i \in \{-1,+1\}$ compares the orientation of the 1–cell with the orientation of the path from v_0 to v_1 on which it lies. See Figure 2.

Example 2.4 In [26, Example 5.1], we compute the cellular DGA of the Legendrian sphere whose front diagram appears in the left diagram of Figure 3. We use the polygonal decomposition in the right diagram. We do not repeat here the full computations of the differential (see [26, Equation 23]) other than to quote some specifics: both the matrices A and B_4 are 4×4–matrices and the $(2,3)$–entry for A is 0; if we let $\widehat{N}_{3,4}$ be the matrix with all 0's except for a 1 in entries $(1,2)$ and $(3,4)$ (this notation is explained in [26, Example 5.1]), then

$$\partial A = A^2 \quad \text{and} \quad \partial B_4 = A(I+B_4) + (I+B_4)\widehat{N}_{3,4}.$$

We revisit this Legendrian sphere in Examples 3.8 and 4.5 when we discuss CHDs and CM2Fs.

An augmentation ϵ for this differential is given by

$$1 = \epsilon(a_{1,2}) = \epsilon(a_{1,4}) = \epsilon(a_{3,4}) = \epsilon(b^4_{2,4}) \tag{2-2}$$

and $\epsilon(x) = 0$ for all other generators x. Checking that $\epsilon \circ \partial(B_4) = 0$ amounts to the matrix calculation $(E_{1,2} + E_{3,4} + E_{1,4})(I + E_{2,4}) + (I + E_{2,4})\widehat{N}_{3,4} = 0$, and that $\epsilon \circ \partial(x) = 0$ holds on other generators is verified by similar routine matrix computations.

2.4.2 Adjustments for swallowtail points In this article, we focus our arguments on the case of upward swallowtail points as pictured in Figure 1. The downward swallowtail is similar; for details see [26, Sections 3.6–3.12]. Suppose near a swallowtail point e^0_{st} that L has n sheets (resp. $(n-2)$ sheets) inside (resp. outside) the swallowtail region, and the sheets in position k, $k+1$, $k+2$ (with respect to descending z–coordinate) above the swallowtail region meet at the swallowtail point. Recall that the two 2–cell corners within the swallowtail region that border the crossing locus at the swallowtail point have been labeled with S and T.

Let

$$\begin{aligned} A_S &= [I + E_{k+2,k+1}]\widehat{A}_{k,k+2}[I + E_{k+2,k+1}], \\ A_T &= [I + E_{k+1,k+2}]\widehat{A}_{k,k+1}[I + E_{k+1,k+2}], \\ S &= I + \widehat{A}_{k,k+1}E_{k+2,k} + E_{k+1,k+2} = I + \sum_{i<k} a_{i,k}E_{i,k} + E_{k+1,k+2}, \\ T &= I + E_{k+1,k+2}, \end{aligned} \tag{2-3}$$

where $E_{i,j}$ is the matrix with all 0's except for a 1 in the $(i,j)^{\text{th}}$ entry, and $\widehat{A}_{i,j}$ is the $(n-2)\times(n-2)$ matrix A over the swallowtail point enlarged by the 2×2 block $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ in columns (and rows) i and j.

Let B_{cr} denote the matrix over the 1–cell associated to the crossing locus with endpoint at e^0_{st}. If the ordering of the sheets used to form B_{cr} agrees with that of the 2–cell marked by S (resp. T) then in the differential ∂B_{cr} set the boundary matrix $A_\pm$ associated to e^0_{st} equal to A_S (resp. A_T). By assumption on $\mathcal{E}$ in Convention 2.3, all other 1–cells with endpoints at e^0_{st} have $n-2$ sheets, and we take the boundary matrix to just be A.

For the 2–cell that includes the region marked by S (resp. T), in (2-1) we replace the $I + B_i$ factor associated to the cusp edge that begins at the swallowtail point with the product $(I + B_i)S$ (resp. $(I + B_i)T$).

3 Augmentations are CHDs

In this section, we examine augmentations of the cellular DGA. By viewing the image of the matrices A, B, and C under ϵ as linear maps we establish in Proposition 3.7 an equivalent characterization of augmentations as *chain homotopy diagrams* which assign chain complexes, chain isomorphisms, and chain homotopies to the cells of $\mathcal{E}$.

3.1 Ordered complexes

Let V be a vector space over $\mathbb{Z}/2$ with specified basis $\mathcal{B} = \{v_p \mid p \in I\}$. We use the inner product notation to denote the bilinear form $\langle v_p, v_q \rangle = \delta_{p,q}$, so that for $w = \sum \alpha_i v_i$ the i^{th} coefficient is $\alpha_i = \langle w, v_i \rangle \in \mathbb{Z}/2$.

Definition 3.1 Suppose that the basis $\mathcal{B}$ is equipped with a partial order $\prec$. A linear transformation $T\colon V \to V$ is *strictly upper triangular* if

$$\langle T(v_q), v_p \rangle \neq 0 \quad \Longrightarrow \quad v_p \prec v_q.$$

An *ordered complex* is a triple $(V, \mathcal{B}, d)$ such that $d\colon V \to V$ is a differential (ie $d^2 = 0$) that is strictly upper triangular. An ordered complex is *m–graded* if basis vectors $v_p \in \mathcal{B}$ are assigned degrees $|v_p| \in \mathbb{Z}/m$, and d has degree $+1$ (mod m) with respect to the resulting grading on V.

Remark 3.2 Our ordered complexes are *cohomologically graded* (at least mod m) since $\deg(d) = +1$. This is in contrast with the boundary operator on the cellular DGA, which has $\deg(\partial) = -1$.

3.2 Handleslide maps

Definition 3.3 Let V be a $\mathbb{Z}/2$–vector space with basis $\mathcal{B} = \{v_p \mid p \in I\}$. Given $u, l \in I$ such that $u \neq l$, the *handleslide map* $h_{u,l}$ is the linear map satisfying

$$h_{u,l}(v_k) = v_k + \delta_{k,l} v_u. \tag{3-1}$$

Note that since this article works with $\mathbb{Z}/2$–coefficients, $h_{u,l}^{-1} = h_{u,l}$. When the indexing set I is $\{1, \ldots, n\}$, the matrix for $h_{u,l}$ is $I + E_{u,l}$.

3.3 Vector spaces associated to cells

Let $L \subset J^1 M$ be a Legendrian equipped with a Maslov potential μ and a compatible polygonal decomposition $\mathcal{E}$. To each d–cell $e^d_\alpha \in \mathcal{E}$ we associate the vector space spanned by the (noncusping) sheets of L above e^d_α,

$$V(e^d_\alpha) = \mathrm{Span}_{\mathbb{Z}/2}\, L(e^d_\alpha).$$

Recall that $L(e^d_\alpha)$ is partially ordered by descending z–coordinate. In addition, each $V(e^d_\alpha)$ has a $\mathbb{Z}/m(L)$–grading arising from

$$|S^\alpha_p| = \mu(S^\alpha_p).$$

3.4 Boundary differentials and maps

In the following definitions we initially assume that L has no swallowtail points, and then give modifications for the general case.

Suppose that a 0–cell e^0_β appears along the boundary of e^d_α with $d = 1$ or 2, and write $e^0_\beta \xrightarrow{j} e^d_\alpha$ for a corresponding inclusion[1] of e^0_β into the boundary of D^d, viewed as the domain of the characteristic map $D^d \to \overline{e^d_\alpha} \subset M$. Assuming that $V(e^0_\beta)$ has been given a differential d_β such that $(V(e^0_\beta), L(e^0_\beta), d_\beta)$ is an ordered complex, we define a *boundary differential*

$$\hat{d}_\beta = \hat{d}(e^0_\beta \xrightarrow{j} e^d_\alpha)\colon V(e^d_\alpha) \to V(e^d_\alpha)$$

as follows. The natural inclusion $i\colon L(e^0_\beta) \hookrightarrow L(e^d_\alpha)$ (where $i(S^\beta_p) = S^\alpha_q$ when $S^\beta_p \subset \overline{S^\alpha_q}$ in L) extends to an injection $i\colon V(e^0_\beta) \hookrightarrow V(e^d_\alpha)$. We have

$$V(e^d_\alpha) = i(V(e^0_\beta)) \oplus V_{\mathrm{cusp}},$$

where V_{cusp} is spanned by the (possibly zero) sheets that meet a cusp edge above e^0_β. We define $\hat{d}_\beta$ to satisfy

$$\hat{d}_\beta = d_\beta \oplus d_{\mathrm{cusp}},$$

where $d_{\mathrm{cusp}}(S^\alpha_b) = S^\alpha_a$ when sheets S^α_b and S^α_a meet at a cusp edge above e^0_β with S^α_a (resp. S^α_b) the upper (resp. lower) sheet.

Next, suppose that for a 1–cell, e^1_β, we are given a chain isomorphism

$$f\colon (V(e^1_\beta), \hat{d}_-) \to (V(e^1_\beta), \hat{d}_+),$$

where $\hat{d}_-$ and $\hat{d}_+$ are the boundary differentials associated to the initial and terminal vertices of e^1_β. Additionally, let $e^1_\beta \xrightarrow{j} e^2_\alpha$ be an appearance of e^1_β along the boundary of e^2_α. (Technically, a lift of e^1_β to the domain of the characteristic map of e^2_α.) We

[1] There may be more than one since e^0_β may appear more than once along the boundary of e^d_α.

extend f to a *boundary morphism*

$$\widehat{f} = \widehat{f}(e^1_\beta \xrightarrow{j} e^2_\alpha)\colon (V(e^2_\alpha), \widehat{d}_-) \to (V(e^2_\alpha), \widehat{d}_+)$$

using the direct sum decomposition $V(e^2_\alpha) = i(V(e^1_\beta)) \oplus V_{\mathrm{cusp}}$ as

$$\widehat{f} = f \oplus \mathrm{id}\,.$$

3.4.1 Adjustments for swallowtail points Suppose now that e^0_{st} is an upward swallowtail point. (The downward case is similar.) Label adjacent cells as e^1_S, e^1_T, e^1_{cr}, e^2_S, and e^2_T, so that e^2_S and e^2_T contain the corners labeled S and T; e^1_{cr} contains the crossing locus; and e^1_S and e^1_T sit below the cusp edges that border the S and T corners. See Figure 4.

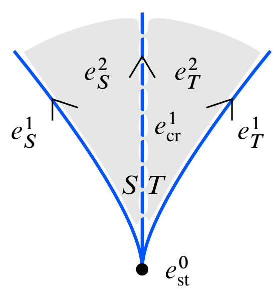

Figure 4: Notation for cells near a swallowtail point

We make the following adjustments:

(1) Given $(V(e^0_{\mathrm{st}}), d)$, the boundary differentials for $V(e^2_T)$, $V(e^1_{\mathrm{cr}})$, and $V(e^2_S)$ are defined as follows. Above e^2_T, the sheets of L are totally ordered, so we write

$$L(e^2_T) = \{S_1, \dots, S_n\} \quad \text{with } S_i \prec S_{i+1}.$$

Let $S_k, S_{k+1}, S_{k+2} \in L(e^2_T)$ be the sheets whose closures contain the swallowtail point, so that S_{k+1} and S_{k+2} meet at the crossing arc. First, we define $d_{k,k+1}\colon V(e^2_T) \to V(e^2_T)$ as if sheets S_k and S_{k+1} meet at a cusp above e^0_{st}, ie identify $V(e^0_{\mathrm{st}})$ with the subspace spanned by $\{S_1, \dots, \widehat{S}_k, \widehat{S}_{k+1}, S_{k+2}, S_n\} \subset L(e^2_T)$, and extend d to $d_{k,k+1}$ via $d_{k,k+1}S_{k+1} = S_k$. Then, define

$$\widehat{d}_T = h_{k+1,k+2}\, d_{k,k+1}\, h_{k+1,k+2}, \tag{3-2}$$

where $h_{k+1,k+2}$ is the handleslide map, given by $h_{k+1,k+2}(S_l) = S_l + \delta_{l,k+2} S_{k+1}$.

The boundary differentials on $V(e^1_{\mathrm{cr}})$ and $V(e^2_S)$ are defined so that the bijections $L(e^2_S) \cong L(e^1_{\mathrm{cr}}) \cong L(e^2_T)$ (from identifying sheets whose closures intersect above e^1_{cr})

extend to isomorphisms of complexes. Note that if sheets above e^2_S are also labeled with descending z–coordinate, then the isomorphism $Q\colon V(e^2_T)\to V(e^2_S)$ interchanges S_{k+1} and S_{k+2}. Because of this, the boundary differential $\widehat{d}_S\colon V(e^2_S)\to V(e^2_S)$ would be

$$\widehat{d}_S = Q\widehat{d}_T Q^{-1} = h_{k+2,k+1}\, d_{k,k+2}\, h_{k+2,k+1}, \tag{3-3}$$

where $d_{k,k+2}$ is formed as if S_k and S_{k+2} meet at a cusp above e^0_{st}.

Boundary differentials for $V(e^1_T)$ and $V(e^1_S)$ (and neighboring cells outside the swallowtail region) are defined using the bijections $L(e^1_S)\cong L(e^0_{\mathrm{st}})\cong L(e^1_T)$.

(2) Suppose we have a chain isomorphism $f\colon (V(e^1_X),\widehat{d}_-)\to (V(e^1_X),\widehat{d}_+)$, for $X=S$ or T. (Here, $\widehat{d}_-$ is the differential for the swallowtail point, since we have assumed in Convention 2.3 all 1–cells are oriented away from the swallowtail point.) We define the boundary morphism $\widehat{f}\colon (V(e^2_X),\widehat{d}_-)\to (V(e^2_X),\widehat{d}_+)$ via

$$\widehat{f} = (f\oplus \mathrm{id})\circ H_X,$$

where we decompose $V(e^2_X)$ in the usual way into $i(V(e^1_X))\oplus V_{\mathrm{cusp}}$, and H_X is defined by

$$H_S = \Bigl(\prod_{\{i \mid \langle d_0 S^0_k, S^0_i\rangle \neq 0\}} h_{i,k}\Bigr) h_{k+1,k+2} \quad\text{and}\quad H_T = h_{k+1,k+2}, \tag{3-4}$$

where d_0 denotes the differential on $V(e^0_{\mathrm{st}})$. (Note that sheets above e^0_{st} are totally ordered, and the handleslide maps in the product all commute.)

Remark 3.4 See Figure 5 and the proof of Proposition 4.10 for a Morse-theoretic explanation of the maps H_S and H_T.

Lemma 3.5 *For $X=S$ or T, the boundary morphisms*

$$\widehat{f}\colon (V(e^2_X),\widehat{d}_-)\to (V(e^2_X),\widehat{d}_+)$$

are chain isomorphisms.

Proof Note that $f\oplus\mathrm{id}$ is a chain isomorphism from $(V(e^2_X), d_{k,k+1})\to (V(e^2_X),\widehat{d}_+)$ (since the differentials respect the direct sum $i(V(e^1_x))\oplus V_{\mathrm{cusp}}$, and f was a chain isomorphism). Thus, it suffices to check that H_S and H_T are chain isomorphisms from $(V(e^2_X),\widehat{d}_-)$ to $(V(e^2_X), d_{k,k+1})$.

In the notation from (2-3), with respect to the basis $S_1, \dots, S_n$ for $V(e^2_X)$ (sheets ordered with descending z–coordinate above e^2_X), the relevant linear maps have the matrices given in the table

X	linear map	matrix
S or T	$d_{k,k+1}$	$\widehat{A}_{k,k+1}$
S	$\widehat{d}_-$	A_S
	H_S	$S = I + \widehat{A}_{k,k+1} E_{k+2,k} + E_{k+1,k+2}$
T	$\widehat{d}_-$	A_T
	H_T	$T = I + E_{k+1,k+2}$

where the entries of the underlying $(n-2)\times(n-2)$ matrix A are specialized as

$$a_{i,j} \mapsto \langle d_0 S^0_j, S^0_i \rangle. \tag{3-5}$$

(For the matrix for H_S, start with the definition of H_S to compute

$$\Bigg(\prod_{\{i \mid \langle d_0 S^0_k, S^0_i\rangle \neq 0\}} (I+E_{i,k})\Bigg)(I+E_{k+1,k+2}) = \Bigg(\prod_{i<k}(I+a_{i,k}E_{i,k})\Bigg)(I+E_{k+1,k+2})$$
$$= I + \sum_{i<k} a_{i,k}E_{i,k} + E_{k+1,k+2},$$

which is equal to the matrix S.)

Thus, we need to verify the matrix identities

$$\widehat{A}_{k,k+1} S = S A_S \quad \text{and} \quad \widehat{A}_{k,k+1} T = T A_T. \tag{3-6}$$

In [26, Lemma 3.4], the equations

$$\partial S = \widehat{A}_{k,k+1} S + S A_S \quad \text{and} \quad \partial T = \widehat{A}_{k,k+1} T + T A_T$$

are established in the cellular DGA. Since $\partial T = 0$, and $\partial S = (\widehat{A}_{k,k+1})^2 E_{k+2,k}$, the left-hand sides vanish once $a_{i,j}$ is specialized as in (3-5) (since then $\widehat{A}_{k,k+1}$ is the matrix of a differential). □

3.5 Augmentations as chain homotopy diagrams

Definition 3.6 A *chain homotopy diagram* for $(L, \mathcal{E})$ is a triple

$$\mathcal{D} = (\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\})$$

consisting of:

(1) For each 0–cell, e^0_α, a differential, d_α, making $(V(e^0_\alpha), L(e^0_\alpha), d_\alpha)$ into an ordered complex.

(2) For each 1–cell, e^1_β, a chain map

$$f_\beta\colon (V(e^1_\beta), \widehat{d}_-) \to (V(e^1_\beta), \widehat{d}_+)$$

such that $f_\beta - \mathrm{id}$ is strictly upper triangular. Here, $\widehat{d}_-$ and $\widehat{d}_+$ denote the boundary differentials associated to the 0–cells at the initial and terminal endpoint of e^1_β. Note that the condition on $f_\beta - \mathrm{id}$ implies that f_β is an isomorphism.

(3) For each 2–cell, e^2_γ, a strictly upper triangular chain homotopy

$$K_\gamma\colon (V(e^2_\gamma), \widehat{d}_{v_0}) \to (V(e^2_\gamma), \widehat{d}_{v_1})$$

between the chain isomorphisms $\widehat{f}_j^{\eta_j} \circ \cdots \circ \widehat{f}_1^{\eta_1}$ and $\widehat{f}_m^{\eta_m} \circ \cdots \circ \widehat{f}_{j+1}^{\eta_{j+1}}$. Here, $\widehat{d}_{v_0}$ and $\widehat{d}_{v_1}$ denote the boundary differentials for the vertices v_0 and v_1; the $\widehat{f}_i$ with $1 \le i \le j$ (resp. with $j+1 \le i \le m$) are the boundary morphisms associated to the edges of e^2_γ as they appear in the counterclockwise (resp. clockwise) path from v_0 to v_1 in the domain of a characteristic map for e^2_γ; and the exponents are $+1$ (resp. -1) when the orientation of the 1–cell agrees (resp. disagrees) with the orientation of this path.

Suppose L is equipped with a Maslov potential μ, so that the vector spaces $V(e^d_\alpha)$ are all graded by $\mathbb{Z}/m(L)$. Given a divisor $\rho \mid m(L)$, we say that a CHD $\mathcal{D}$ is *ρ–graded* if the maps d_α, f_β, and K_γ all have respective degrees $+1$, 0, and -1 mod ρ.

Proposition 3.7 *For any $\rho \mid m(L)$, there is a bijection between ρ–graded augmentations of the cellular DGA of $(L, \mathcal{E})$ and ρ–graded chain homotopy diagrams for $(L, \mathcal{E})$.*

Proof First, consider triples of linear maps $(\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\})$ with the only restriction being that d_α, $f_\beta - \mathrm{id}$, and K_γ each are strictly upper triangular. There is a bijection between such triples and the set of all algebra homomorphisms from the cellular DGA $\mathcal{A}$ to $\mathbb{Z}/2$ that arises from replacing a linear map with its matrix with respect to $L(e^d_\alpha)$:

$$(\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\}) \mapsto (\epsilon\colon \mathcal{A} \to \mathbb{Z}/2),$$

$$\epsilon(a^\alpha_{p,q}) = \langle d_\alpha S^\alpha_q, S^\alpha_p \rangle, \quad \epsilon(b^\beta_{p,q}) = \langle f_\beta S^\beta_q, S^\beta_p \rangle, \quad \epsilon(c^\gamma_{p,q}) = \langle K_\gamma S^\gamma_q, S^\gamma_p \rangle.$$

(This is a bijection because all matrix coefficients of the $(\{d_\alpha\}, \{f_\beta - \mathrm{id}\}, \{K_\gamma\})$ corresponding to pairs S_p and S_q for which there is no corresponding generator of $\mathcal{A}$ are forced to be 0 by the strictly upper triangular condition, eg the generator $a^\alpha_{p,q}$ exists if and only if $S^\alpha_p \prec S^\alpha_q$.)

The above correspondence restricts to a bijection between CHDs and augmentations since the requirements on the maps d_α, f_β, and K_γ from the definition of CHD are equivalent to the matrix equations arising from applying $\epsilon \circ \partial = 0$ to the corresponding A, B, and C matrices. In more detail, we have:

(1) For $A = (a^\alpha_{p,q})$,

$$\epsilon \circ \partial(A) = 0 \iff [\epsilon(A)]^2 = 0 \iff (d_\alpha)^2 = 0;$$

ie $\epsilon \circ \partial(A) = 0$ if and only if $(V(e^0_\alpha), d_\alpha)$ is a chain complex.

(2) For $B = (b^\beta_{p,q})$,

$$\epsilon \circ \partial(B) = 0 \iff \epsilon(A_+)(I + \epsilon(B)) = (I + \epsilon(B))\epsilon(A_-) \iff \widehat{d}_+ \circ f_\beta = f_\beta \circ \widehat{d}_-;$$

ie $\epsilon \circ \partial(B) = 0$ if and only if $f_\beta\colon (V(e^1_\beta), \widehat{d}_-) \to (V(e^1_\beta), \widehat{d}_+)$ is a chain isomorphism. (Note that $(I + \epsilon(B))$ is the matrix of f_β. It is also important to observe that the $\epsilon(A_\pm)$ are the matrices for the boundary differentials $\widehat{d}_\pm$. This is readily verified from comparing the *boundary matrices* used in defining ∂B with the *boundary differentials* associated to $V(e^1_\beta)$. In particular, (i) the 2×2 blocks $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ inserted when forming $A_\pm$ reflect the definition of $\widehat{d}_\pm$ on the subspace $V_{\mathrm{cusp}} \subset V(e^1_\beta)$, and (ii) as already observed in Lemma 3.5 when e^1_β is the crossing 1–cell at a swallowtail point, $A_- = A_S$ (or A_T depending on whether the chosen total ordering of $L(e^1_\beta)$ used to form B agrees with the ordering above the S or T 2–cell) is the matrix of the boundary differential $\widehat{d}_-$.)

(3) For $C = (c^\gamma_{p,q})$, considering first the case that e^2_γ does not border swallowtail points, we have

$$\begin{aligned}
&\epsilon \circ \partial(C) = 0 \\
&\iff \epsilon(A_{v_1})\epsilon(C) + \epsilon(C)\epsilon(A_{v_0}) \\
&\qquad\qquad = (I + \epsilon(B_j))^{\eta_j} \cdots (I + \epsilon(B_1))^{\eta_1} + (I + \epsilon(B_m))^{\eta_m} \cdots (I + \epsilon(B_{j+1}))^{\eta_{j+1}} \\
&\iff \widehat{d}_{v_1} K_\gamma + K_\gamma \widehat{d}_{v_0} = \widehat{f}_j^{\eta_j} \circ \cdots \circ \widehat{f}_1^{\eta_1} - \widehat{f}_m^{\eta_m} \circ \cdots \circ \widehat{f}_{j+1}^{\eta_{j+1}};
\end{aligned}$$

that is, $\epsilon \circ \partial(C) = 0$ if and only if $K_\gamma\colon (V(e^2_\gamma), \widehat{d}_{v_0}) \to (V(e^2_\gamma), \widehat{d}_{v_1})$ is a chain homotopy between the chain isomorphisms $\widehat{f}_j^{\eta_j} \circ \cdots \circ \widehat{f}_1^{\eta_1}$ and $\widehat{f}_m^{\eta_m} \circ \cdots \circ \widehat{f}_{j+1}^{\eta_{j+1}}$. (We used that since the B_i are nilpotent,

$$\epsilon(I + B_i)^{-1} = \epsilon(I + B_i + B_i^2 + \cdots) = I + \epsilon(B_i) + \epsilon(B_i)^2 + \cdots = (I + \epsilon(B_i))^{-1}.$$

Again, it is important to verify that $\epsilon(A_{v_i})$ (resp. $I + \epsilon(B_i)$) is the matrix of the boundary differential associated to v_i (resp. boundary morphism for the corresponding f_i). The case of boundary differentials is as before, while the $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ inserted into B_i is consistent with $\widehat{f_i}$ acting as the identity on the component $V_{\mathrm{cusp}} \subset V(e^2_\gamma)$.)

In the case that e^2_γ contains the S or T corner at a swallowtail point, the definition of the $\widehat{f_i}$ for the cusp edge bordering the corner acquires a factor of H_S or H_T, while an S or T matrix is inserted at the corresponding part of the product in the definition of ∂C. As observed in Lemma 3.5, S and T are the respective matrices of H_S and H_T, so it follows that $\epsilon \circ \partial(C) = 0$ is still equivalent to K_γ being a chain homotopy of the required form. □

Example 3.8 The CHD associated to the augmentation from Example 2.4 is the following. Consider the polygonal decomposition in Figure 3. For $\beta = 1, 2, 3, 4$, let e^1_β be the 1–cell associated with the matrix B_β. For $\gamma = 1, 2, 3$, let e^2_γ be the 2–cell associated with the matrix C_γ. Let e^0_1 be the 0–cell associated with the matrix A, let e^0_2 be the 0–cell at the initial endpoint of e^1_4, and let e^0_3 be the third 0–cell. Counting the number of (noncusping) sheets over each cell, we get

$$\begin{aligned} (\mathbb{Z}/2)^4 &= V(e^0_1) = V(e^1_3) = V(e^1_4) = V(e^2_2) = V(e^2_3), \\ (\mathbb{Z}/2)^2 &= V(e^1_1) = V(e^1_2) = V(e^2_1), \\ (\mathbb{Z}/2)^0 &= V(e^0_2) = V(e^0_3). \end{aligned}$$

The differential assigned to e^0_1 is defined by

$$d_1 S_2 = S_1, \quad d_1 S_4 = S_1 + S_3, \quad d_1 S_1 = d_1 S_3 = 0,$$

while the differentials of the other 0–cells are both trivial. The chain isomorphism assigned to e^1_4 is defined by

$$f_4(S_i) = S_i + \delta_{i,4} S_2,$$

while the chain isomorphisms of the other 1–cells are all the identity. Here we use that $\epsilon(b^4_{2,4}) = 1$ and $\epsilon(b^k_{i,j}) = 0$ for all other b–generators. Let e^0_1 (resp. e^0_2) be the terminal (resp. initial) vertex for all 2–cells. The chain homotopies assigned to the 2–cells are all trivial.

4 Morse complex 2–families

In this section, we introduce Morse complex 2–families (MC2Fs), which are detailed combinatorial approximations of generating families. In Section 4.2, using an MC2F we produce combinatorial continuation maps associated to paths in the base surface, again in analogy with Morse theory. Finally, in Proposition 4.10 we show that pairing a generating family F with an appropriate family of gradient-like vector fields produces an MC2F, and we observe how properties of F are reflected in the associated continuation maps.

4.1 Definition of MC2Fs

Let $L \subset J^1M$ with Maslov potential μ have generic front and base projections. Write

$$\Sigma = \Pi_B(\Sigma_{\mathrm{cusp}} \cup \Sigma_{\mathrm{st}} \cup \Sigma_{\mathrm{cr}})$$

for the base projection of the singular set of L (cusps, swallowtail points, and crossing arcs). Let $R_\nu \subset M \setminus \Sigma$ be a region, ie an open connected subset. Following earlier definitions, we let $L(R_\nu)$ denote the set of sheets of L above R_ν, ie components of $\Pi_B^{-1}(R_\nu) \cap L$. Sheets in $L(R_\nu)$ are totally ordered by descending z–coordinate, so we always index sheets as $L(R_\nu) = \{S_1^\nu, S_2^\nu, \dots, S_n^\nu\}$ with $z(S_i) > z(S_{i+1})$ pointwise. The $\mathbb{Z}/2$–vector space spanned by $L(R_\nu)$ is denoted $V(R_\nu)$, and is assigned a $\mathbb{Z}/m$–grading via the Maslov potential.

Definition 4.1 Let $\rho \mid m(L)$. A ρ–graded *Morse complex 2–family* (*MC2F*), $\mathcal{C}$, for L is a triple $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ which consists of the following data, subject to Axioms 4.2 and 4.3:

(1) A *superhandleslide set*, H_{-1}, which is a finite set of points in $M \setminus \Sigma$. Each point $x \in H_{-1}$ is assigned upper and lower lifts $u_x, l_x \in L$ satisfying

$$z(u_x) > z(l_x) \qquad \text{and} \qquad \mu(u_x) - \mu(l_x) = -1 \pmod{\rho}.$$

(2) A *handleslide set*, which is an immersed compact 1–manifold $H\colon X \to M$, where $X = \bigsqcup_i X_i$ with each X_i equal to S^1 or $[0,1]$. When restricted to the interior of X, the manifold H is transverse to (the strata of) Σ; it is disjoint from H_{-1}; and its only self-intersections are transverse double points in $M \setminus \Sigma$. Moreover, H is equipped with continuous upper and lower endpoint lifts $u, l\colon X \to L$ satisfying

$$z(u) > z(l) \qquad \text{and} \qquad \mu(u) - \mu(l) = 0 \pmod{\rho}.$$

(3) Set

$$\Sigma_{\mathcal{C}} = \Sigma \cup H(X) \cup H_{-1}.$$

For each connected component $R_\nu \subset M \setminus \Sigma_{\mathcal{C}}$, the vector space $V(R_\nu)$ is assigned a differential d_ν making $(V(R_\nu), L(R_\nu), d_\nu)$ into a ρ–graded ordered complex, ie d_ν is strictly upper triangular and

$$\deg(d_\nu) = +1 \pmod{\rho}.$$

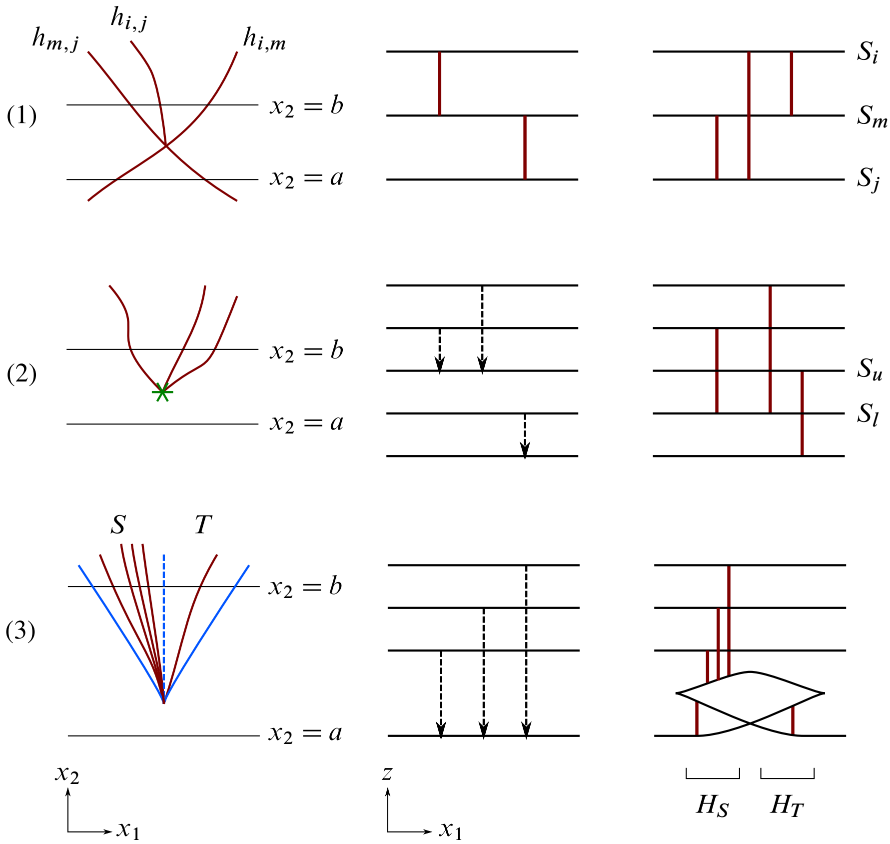

Figure 5: The three types of endpoints for handleslide arcs in H allowed by Axiom 4.2. The left column depicts the base projection (to M) of H (in red), H_{-1} (a green star), and the singular set, Σ (in blue). The center and right column depict the slices of the front projection at $x_2 = a$ and $x_2 = b$ respectively; a dotted black arrow from S_i to S_j indicates that $\langle dS_j, S_i \rangle = 1$. The three types of endpoints allowed are (1) at double points of H, (2) at superhandleslide points, and (3) at swallowtail points.

Before stating Axioms 4.2 and 4.3 we introduce some terminology. When considering the handleslide set of $\mathcal{C}$ locally in $M \setminus \Sigma$, a handleslide arc whose upper (resp. lower) lift is S_i (resp. S_j) is called an (i, j)*–handleslide arc*. Note that the indices i and j are not globally well defined for a given component of H, since they may change when the image of H crosses Σ. The phrase (i, j)*–superhandleslide point* has a similar meaning.

Axiom 4.2 Endpoints of handleslide arcs (for components of H with $X_i = [0, 1]$) are as follows (see also Figure 5):

(1) Let $x \in M \setminus \Sigma$ be a double point of H where for some $i < m < j$ an (i,m)–handleslide arc intersects an (m,j)–handleslide arc. Then a unique (i,j)–handleslide arc has a unique endpoint at x.

(2) Suppose $p \in H_{-1}$ is a (u,l)–superhandleslide point, and let d_ν be the differential associated to any region of $M \setminus \Sigma_{\mathcal{C}}$ adjacent to p. Then, for any $i < u < l < j$, at p there are $\langle d_\nu S_u, S_i \rangle$ endpoints of (i,l)–handleslide arcs; and $\langle d_\nu S_j, S_l \rangle$ endpoints of (u,j)–handleslide arcs.

(3) Suppose $p \in M$ is an upward swallowtail point such that outside (resp. inside) the swallowtail region L has $n-2$ (resp. n) sheets, and such that sheet S_k (resp. sheets S_k, S_{k+1}, and S_{k+2}) contains the swallowtail point in their closure.

Denote by d_0 the differential associated to the $(n-2)$–sheeted region of $M \setminus \Sigma_{\mathcal{C}}$ near p. Then at p there are $\langle d_0 S_k, S_i \rangle$ endpoints of (i,k)–handleslide arcs locally contained within the swallowtail region as well as two additional $(k+1,k+2)$–handleslide arcs, one on each side of the crossing locus near p.

The downward swallowtail case is similar, but vertically reflected.

Axiom 4.3 When two regions R_0 and R_1 share a border along an arc, $A \subset \Sigma_{\mathcal{C}}$, the complexes $(V(R_0), d_0)$ and $(V(R_1), d_1)$ are related as follows:

(1) Suppose A belongs to an (i,j)–handleslide arc. We require that the handleslide map

$$h_{i,j}\colon (V(R_0), d_0) \xrightarrow{\cong} (V(R_1), d_1)$$

be a chain isomorphism.

(2) Suppose A belongs to the crossing locus. We have a bijection $L(R_0) \cong L(R_1)$ by identifying sheets whose closures (in L) intersect above A. We require that the induced isomorphism $V(R_0) \cong V(R_1)$ be an isomorphism of complexes.

Equivalently, label sheets above R_0 and R_1 with descending z–coordinate as, respectively, $S^0_1, \dots, S^0_n$ and $S^1_1, \dots, S^1_n$. If sheets S^i_k and S^i_{k+1} meet at the crossing arc above A, we require that the map

$$Q\colon (V(R_0), d_0) \xrightarrow{\cong} (V(R_1), d_1), \quad Q(S^0_i) = S^1_{\tau(i)}$$

be an isomorphism, where $\tau = (k\ \ k+1)$ denotes the transposition.

(3) Suppose A belongs to the cusp locus. We require that the complexes be related as in the boundary differential construction of Section 3.4.

In more detail, suppose that above A the sheets S^1_k and S^1_{k+1} meet at a cusp edge. Include $V(R_0)$ into $V(R_1)$ via

$$S^0_i \mapsto \begin{cases} S^1_i & \text{if } i < k, \\ S^1_{i+2} & \text{if } i \geq k, \end{cases}$$

and write $V_{\text{cusp}} = \text{Span}_{\mathbb{Z}/2}\{S^1_k, S^1_{k+1}\}$. We require that, with respect to the direct sum decomposition $V(R_1) = V(R_0) \oplus V_{\text{cusp}}$, the differential be given by $d_1 = d_0 \oplus d_{\text{cusp}}$, where $d_{\text{cusp}} S_{k+1} = S_k$.

We record some observations about the definition.

Observation 4.4 (1) For Axiom 4.2(2) about the appearance of H near a superhandleslide $p \in H_{-1}$ it suffices to check the condition for a single choice of adjacent region at p. It then follows from Axiom 4.3(1) that the condition will hold for all adjacent regions, since the differentials associated to different regions bordering p are related by a sequence of handleslide maps that do not change the matrix coefficients $\langle dS_u, S_i \rangle$ and $\langle dS_j, S_l \rangle$ with $i < u < l < j$.

(2) If sheets S_k and S_{k+1} cross along at least one boundary arc of a region R_ν then $\langle d_\nu S_{k+1}, S_k \rangle = 0$. (This follows from Axiom 4.3(2). Otherwise, the differential in the neighboring region would not be upper triangular.)

(3) If sheets S_k and S_{k+1} meet at a cusp along at least one boundary arc of a region R_ν then $\langle dS_{k+1}, S_k \rangle = 1$. (Use Axiom 4.3(3).)

(4) An (i, j)–handleslide arc cannot intersect a crossing locus involving sheets S_i and S_j, and cannot cross a cusp edge involving S_i or S_j. (This is because the lifts satisfy the inequality $z(u) > z(l)$, and cannot be continuously extended past a cusp point.)

(5) Given a swallowtail point p and a differential d_0 for the outside of the swallowtail region, once handleslide arcs are placed near p as required in Axiom 4.2(3), at least locally, there is always a unique way to assign differentials $\{d_\nu\}$ to the regions within the swallowtail region so that Axiom 4.3 holds. See Proposition 6.2.

Example 4.5 In Figure 6, we construct an MC2F for our running example from Figure 3. A comparison with Example 3.8 gives an example of the correspondence between CHDs and CM2Fs that will be established in general in Sections 5 and 6. Recall the labeling of the cells from Example 3.8. Suppose the 0–cell e^0_1 sits inside the region bounded by the union of the two red handleslide arcs and the 1–cell e^1_4

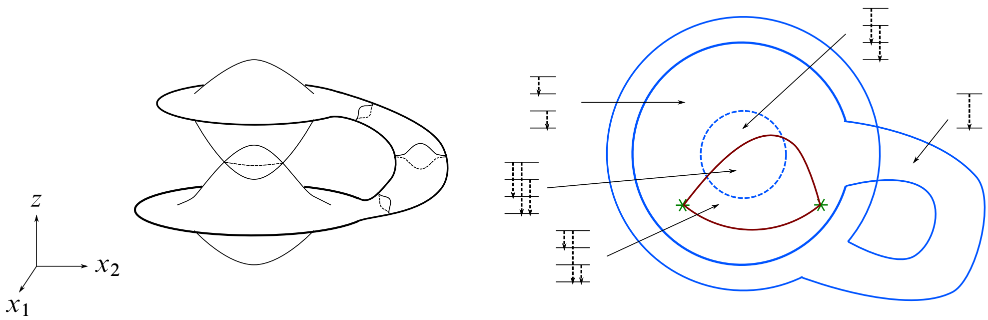

Figure 6: An MC2F for the Legendrian sphere from Figure 3. The two green stars are $(2,3)$–superhandleslide points. The lower red arc is a $(1,3)$–handleslide arc. The upper red arc is a $(2,4)$–handleslide arc (resp. $(3,4)$–handleslide arc) when it is outside (resp. inside) the crossing circle. The differentials d_ν are indicated by the dotted arrows.

intersects the upper red arc once. The fact that e^1_4 is the unique 1–cell to intersect some handleslide arc an odd number of times results in it being the unique 1–cell with a nontrivial chain isomorphism in Example 3.8.

Remark 4.6 (1) The definition of an MC2F is based on the generic bifurcations of Morse complexes in 2–parameter families of functions; see Proposition 4.10. That the differentials d_ν have degree $+1$ (mod ρ) corresponds to working with Morse cohomology complexes rather than homology. Here, the grading is given by the Morse index, but the differential counts positive gradient trajectories rather than negative trajectories.

(2) The reader familiar with the Gromov compactness/gluing proof of $d^2 = 0$ in Morse or Floer theory can interpret Axiom 4.2(1)–(2) as gluing various configurations of broken trajectories to produce boundaries of the moduli space of handleslide trajectories.

4.2 Combinatorial continuation maps

Suppose that $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ is an MC2F for $L \subset J^1M$. In the following, using $\mathcal{C}$ we associate continuation maps to paths in M. For paths that are disjoint from the singular set of L the continuation maps have properties at the chain level that will be important for constructing a CHD from an MC2F.

Let $\sigma\colon [0,1] \to M$ be a smooth path that is transverse to the strata of $\Sigma_{\mathcal{C}}$. Suppose $\sigma(i)$ lies in the component $R_i \subset M \setminus \Sigma_{\mathcal{C}}$ for $i = 0, 1$. We define the *continuation map*

$$f(\sigma)\colon (V(R_0), d_0) \to (V(R_1), d_1) \tag{4-1}$$

to be the composition

$$f(\sigma) = f_m \circ \cdots \circ f_1$$

with the maps $f_1, \dots, f_m$ associated to those $0 < s_1 < \cdots < s_m < 1$ where $\sigma(s_l)$ intersects $\Sigma_{\mathcal{C}}$ as follows:

(1) When $\sigma(s_l)$ intersects an (i, j)–*handleslide*,

$$f_l = h_{i,j}.$$

(2) When $\sigma(s_l)$ intersects a *crossing*, f_l is the map Q from Axiom 4.3(2).

(3) When $\sigma(s_l)$ intersects a *cusp*, notate the regions bordering the cusp edge as R' and R'' so that the two cusp sheets exist above R'' and not above R'. Write $V(R'') = V(R') \oplus V_{\text{cusp}}$. If σ passes from R' to R'' as s increases, then $f_l\colon V(R') \to V(R'')$ is the inclusion. If σ passes from R'' to R', then $f_l\colon V(R'') \to V(R')$ is the projection.

Proposition 4.7 *Let $\sigma, \tau\colon [0,1] \to H$ be paths transverse to $\Sigma_{\mathcal{C}}$. Then:*

(1) *The continuation map $f(\sigma)$ is a quasi-isomorphism.*

(2) *If $\sigma(1) = \tau(0)$, then*

$$f(\sigma * \tau) = f(\tau) \circ f(\sigma).$$

(3) *If σ and τ are path homotopic (ie homotopic relative endpoints) in M, then $f(\sigma), f(\tau)\colon (V(R_0), d_0) \to (V(R_1), d_1)$ are chain homotopic.*

If σ and τ are disjoint from the singular set of L, ie disjoint from crossing and cusp arcs, then:

(4) *The matrix of $f(\sigma) - \mathrm{id}$ is strictly upper triangular.*

(5) *The inverse path $\sigma^{-1}(s) = \sigma(1-s)$ has*

$$f(\sigma^{-1}) = (f(\sigma))^{-1}.$$

(6) *If σ and τ are path homotopic via a homotopy whose image is also disjoint from crossings and cusps, then there is a strictly upper triangular homotopy operator $K\colon V(R_0) \to V(R_1)$ between $f(\tau)$ and $f(\sigma)$:*

$$f(\sigma) - f(\tau) = d_1 K + K d_0.$$

If the image of the homotopy is also disjoint from superhandleslide points, then $f(\sigma) = f(\tau)$.

When $\mathcal{C}$ is ρ–graded, all of the above continuation maps (resp. homotopy operators) have degree 0 (resp. -1) mod ρ.

Proof This is based on one standard approach to continuation maps in Morse theory, as in [19].

Item (1) follows from Axiom 4.3, which shows that each individual factor

$$f_l\colon (V(R_{l-1}), d_{l-1}) \to (V(R_l), d_l)$$

is a quasi-isomorphism, where R_{l-1} (resp. R_l) are the regions containing $\sigma(s)$ as $s \to s_l^-$ (resp. as $s \to s_l^+$).

Item (2) is obvious from the definition.

(4) and (5) follow from the definition since $h_{i,j}^{-1} = h_{i,j}$, and the matrix of each $h_{i,j}$ is upper triangular with 1's on the diagonal.

To prove (6), we consider a homotopy $I\colon [0,1] \times [0,1] \to M$ from σ to τ given by $I(s,t) = \sigma_t(s)$, with $\sigma_t(i) = \sigma(i) = \tau(i)$ for $i = 0, 1$, such that the image of I is disjoint from all crossing and cusp arcs. By taking I sufficiently generic, we can assume $I^{-1}(H)$ is an immersed 1–manifold whose nonembedded points are as in the definition of MC2F, ie the interior of $I^{-1}(H)$ has at worst transverse double points, and all endpoints of $I^{-1}(H)$ in the interior of $[0,1] \times [0,1]$ are as in Axiom 4.2(1)–(2). Moreover, we can assume the projection to the t direction is a Morse function $\pi_t\colon I^{-1}(H) \to \mathbb{R}$, and all critical points of π_t, double points of $I^{-1}(H)$, and superhandleslide points occur at different values of t. We subdivide $0 = t_0 < t_1 < \cdots < t_N = 1$ so that each interval $[t_i, t_{i+1}]$ contains only one such t–value that is located in the interior of the interval. See Figure 7. To complete the proof, we check that $f(\sigma_{t_i}) \sim f(\sigma_{t_{i+1}})$.

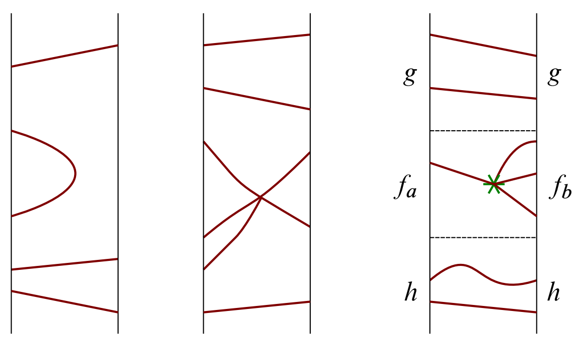

Figure 7: The handleslide set $I^{-1}(H)$ for $t \in [t_i, t_{i+1}]$, as considered in Cases 1, 2, and 3 from the proof of Proposition 4.7

Case 1 (t_i, t_{i+1}) contains a *critical point* of $\pi_t|_{I^{-1}(H)}$. Then the products that define $f(\sigma_{t_i})$ and $f(\sigma_{t_{i+1}})$ agree except for a consecutive pair of handleslides maps $h_{i,j}h_{i,j}$ that appears in only one of the two. Since $h_{i,j}^{-1} = h_{i,j}$ we get $f(\sigma_{t_i}) = f(\sigma_{t_{i+1}})$.

Case 2 (t_i, t_{i+1}) contains an interior *double point* of $I^{-1}(H)$.

For any $u_1 > l_1$ and $u_2 > l_2$, a straightforward computation gives the relations for handleslide maps,

$$h_{u_1,l_1}h_{u_2,l_2} = \begin{cases} h_{u_2,l_2}h_{u_1,l_1} & \text{when } l_1 \neq u_2 \text{ and } l_2 \neq u_1, \\ h_{u_2,l_2}h_{u_1,l_1}h_{u_1,l_2} & \text{when } l_1 = u_2. \end{cases} \tag{4-2}$$

Let $u_1 > l_1$ and $u_2 > l_2$ denote the indices of the upper and lower lifts of the two interior points of $I^{-1}(H)$ that intersect. If $l_1 \neq u_2$ and $l_2 \neq u_1$, then $f(\sigma_{t_i})$ and $f(\sigma_{t_{i+1}})$ differ by the transposition of a pair of consecutive factors: that is, $h_{u_1,l_1}h_{u_2,l_2}$ is interchanged with $h_{u_2,l_2}h_{u_1,l_1}$. The first formula from (4-2) shows that $f(\sigma_{t_i}) = f(\sigma_{t_{i+1}})$. Supposing that $l_1 = u_2$, Axiom 4.2(1) applies to show that the products defining $f(\sigma_{t_i})$ and $f(\sigma_{t_{i+1}})$ are related as in the second equation of (4-2) with the caveat that the h_{u_1,l_2} may appear in some other location, including on the left-hand side. Since h_{u_1,l_2} is self-inverse and commutes with h_{u_2,l_2} and h_{u_1,l_1}, the equality $f(\sigma_{t_i}) = f(\sigma_{t_{i+1}})$ follows.

Case 3 (t_i, t_{i+1}) contains a (u,l)–*superhandleslide point*, p.

We can factor

$$f(\sigma_{t_i}) = gf_a h \quad \text{and} \quad f(\sigma_{t_{i+1}}) = gf_b h,$$

where

$$\begin{aligned} h&\colon (V(R_0), d_0) \to (V(R'), d'), \\ f_a, f_b&\colon (V(R'), d') \to (V(R''), d''), \\ g&\colon (V(R''), d'') \to (V(R_1), d_1), \end{aligned}$$

with f_a and f_b corresponding to the segments of σ_{t_i} and $\sigma_{t_{i+1}}$ that contain the intersections of these paths with the collection of handleslides with endpoints at p, as in Axiom 4.2(2). See Figure 7. Since any two of the handleslides with endpoints at p give handleslide maps h_{i_1,j_1} and h_{i_2,j_2} with $j_1 \neq i_2$ (because $i_2 \leq u < l \leq j_1$), the matrix of $f_a - f_b$ is

$$\sum_{i<u} \langle d'' S_u, S_i \rangle E_{i,l} + \sum_{l<j} \langle d' S_j, S_l \rangle E_{u,j}.$$

(As in Observation 4.4(1), the coefficients $\langle dS_u, S_i \rangle$ and $\langle dS_j, S_l \rangle$ are the same when d is the differential from any of the regions that border p, including d' and d''.)

Taking K to have matrix $E_{u,l}$ it follows that

$$f_a - f_b = d'' K + K d',$$

so that

$$f(\sigma_{t_i}) - f(\sigma_{t_{i+1}}) = g f_a h - g f_b h = d_1(gKh) + (gKh) d_0.$$

Note that since g and h (resp. K) are upper triangular (resp. strictly upper triangular), it follows that the homotopy operator gKh is strictly upper triangular.

With Cases 1–3 established, we note that a homotopy operator $\widetilde{K}$ between $f(\sigma)$ and $f(\tau)$ is the sum of the homotopy operators K_i between each $f(\sigma_{t_i})$ and $f(\sigma_{t_{i+1}})$. Thus, it follows that $\widetilde{K}$ is indeed upper triangular, and is 0 if the image of I is disjoint from superhandleslide points.

Finally, to establish (3), the previous argument is extended to allow the possibility that the image of the homotopy I intersects crossings and cusps. Assuming I generic, this leads to several new codimension-2 strata of $I^{-1}(\Sigma_{\mathcal{C}})$ to be considered in producing the chain homotopy $f(\sigma_{t_i}) \sim f(\sigma_{t_{i+1}})$. The list includes:

(a) Local maxima and minima of π_t restricted to a crossing or cusp arc.

(b) Transverse crossings of two crossing, cusp, or handleslide arcs. In the case of the intersection of two crossing and/or cusp arcs, we may assume that two disjoint pairs of sheets are involved.

(c) The generic codimension-2 singularities of front projections as in Figure 1: triple points, cusp-sheet intersections, and swallowtail points.

We leave this straightforward, but somewhat lengthy case-by-case check mostly to the reader, commenting here on a few interesting points.

Note that in fact $f(\sigma_{t_i}) = f(\sigma_{t_{i+1}})$ in all cases except some local maxima/minima of cusp arcs. In the case of a local minimum, an identity map factor in $f(\sigma_{t_i})$ is replaced with either $j \circ p$ or $p \circ j$, where

$$V(R') \xrightarrow{j} V(R'') = V(R') \oplus V_{\mathrm{cusp}} \quad \text{and} \quad V(R'') = V(R') \oplus V_{\mathrm{cusp}} \xrightarrow{p} V(R')$$

are the inclusion and projection. One has

$$p \circ j = \mathrm{id} \quad \text{and} \quad j \circ p - \mathrm{id} = d_{R''} K + K d_{R''},$$

where $K(S_a) = S_b$ for the cusp sheets S_a and S_b (with S_a above S_b) and $K(S_i) = 0$ for $i \neq a$.

We examine also the case of an (upward) swallowtail point. The tangency to the cusp edge at the swallowtail can be assumed to be nonvertical, and we consider the case where the swallowtail sheets exist above $\sigma(t_{i+1})$ but not $\sigma(t_i)$. Assuming the swallowtail sheets are S_k, S_{k+1}, S_{k+2}, so that the sheets meeting at cusp edges are labeled S_k and S_{k+1}, the continuation map $f(\sigma_{t_{i+1}})$ is obtained from $f(\sigma_{t_i})$ via inserting the product

$$pH_S QH_T j,$$

where H_S, Q, and H_T have matrices

$$H_S = I + E_{k+1,k+2} + \sum_{i<k} a_{i,k} E_{i,k}, \quad Q = Q_{k+1,k+2}, \quad H_T = I + E_{k+1,k+2}$$

with $Q_{k+1,k+2}$ the permutation matrix for $(k+1\ k+2)$ and $a_{i,k} \in \mathbb{Z}/2$. (All of the handleslides specified in Axiom 4.2(3) with lower lift on S_k are collected into the H_S matrix; this is possible since each $h_{i,k}$ commutes with Q.) Thus, for $i \neq k$ we compute

$$\begin{aligned}(pH_S QH_T j)(S_i) &= \begin{cases} (pH_S QH_T)(S_i) = p(S_i) & \text{if } i < k, \\ (pH_S QH_T)(S_{i+2}) = p(S_{i+2}) & \text{if } i > k \end{cases} \\ &= S_i;\end{aligned}$$

For $i = k$, we obtain

$$\begin{aligned}(pH_S QH_T j)(S_k) &= (pH_S QH_T)(S_{k+2}) \\ &= pH_S Q(S_{k+1} + S_{k+2}) = pH_S(S_{k+2} + S_{k+1}) \\ &= p(S_{k+2}) = S_k.\end{aligned}$$

□

Let x_0 be a basepoint, belonging to a region $R_0 \subset M \setminus \Sigma_{\mathcal{C}}$.

Corollary 4.8 (1) *The homology* $H(\mathcal{C}_{x_0}) := H(V(R_0), d_0)$ *is independent of the choice of* x_0 *and* R_0.

(2) *The continuation maps induce a well-defined antihomomorphism*

$$\Phi_{\mathcal{C},x_0}\colon \pi_1(S, x_0) \to \mathrm{GL}(H(\mathcal{C}_{x_0})), \quad [\sigma] \mapsto H(f(\sigma)).$$

Proof This follows from Proposition 4.7(1)–(3). □

We refer to $H(\mathcal{C}_{x_0})$ as the *fiber homology* of $\mathcal{C}$ at x_0, and $\Phi_{\mathcal{C},x_0}$ as the *monodromy representation.*

Remark 4.9 Although we have only defined $H(\mathcal{C}_{x_0}, \Phi_{\mathcal{C}})$ for $x_0 \in M \setminus \Sigma_{\mathcal{C}}$, it is standard that a representation of the fundamental group at any point $x_0 \in M$ of a connected space extends to a local system of vector spaces, well-defined up to isomorphism. In this way, the representation $\Phi_{\mathcal{C},x_0}$ is defined up to isomorphism for arbitrary $x_0 \in M$.

4.3 Generating families and MC2Fs

Proposition 4.10 *If the Legendrian L has a tame-at-infinity generating family F, then it has a 0–graded Morse complex 2–family, $\mathcal{C}$. Moreover:*

(1) *If F is linear at infinity, then we can take $\mathcal{C}$ to have vanishing fiber homology, $H(\mathcal{C}_{x_0}) = \{0\}$.*

(2) *If the domain of F is a trivial bundle over M, then we can take $\mathcal{C}$ to have trivial monodromy representation.*

Proof Let $F\colon E \to \mathbb{R}$ be a generating family for $L \subset J^1M$ with fiber N. In an open set $U \subset M$ above which E is trivialized, we can consider F as a 2–parameter family of smooth functions, $\{f_m\colon N \to \mathbb{R}\}_{m\in U}$. As discussed in [16, pages 22–23], after generic small perturbation there is a stratification $M = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ given by the critical points and values of the f_m. In the codimension-0 $\mathcal{F}_0$ stratum, all critical points are nondegenerate and critical values are distinct. The codimension-1 $\mathcal{F}_1$ stratum is the union of parameter values with a single birth-death or two nondegenerate points with a common critical value. The codimension-2 $\mathcal{F}_2$ stratum has six types of singularities: a unique swallowtail point and five various configurations of transverse intersections of the codimension-1 strata. The set $\mathcal{F}_1 \cup \mathcal{F}_2$ is the base projection $\Sigma = \Pi_B(\Sigma_{\text{cusp}} \cup \Sigma_{\text{cr}} \cup \Sigma_{\text{st}})$ of the singular set of L, made of the cusp loci, crossing loci, their various intersections and the swallowtail points.

A sheet of $\Pi_F(L)$ that lies above $U \subset M$ corresponds to a family of nondegenerate critical points q_m of f_m for $m \in U$ whose Morse indices $i_{\text{mo}}(q_m)$ are locally constant. Seen this way, the Morse index of critical points provides a $\mathbb{Z}$–valued Maslov potential on L. This implies $m(L) = 0$ and gives the grading on vector spaces for which the 0–graded requirements in Definition 4.1 are satisfied. Similarly, the locally well-defined relative Morse index of two such families of critical points equals the difference in Maslov potentials of the two corresponding sheets.

We review several properties of the stable and unstable manifolds of critical points that can be arranged following [16]. In order to produce the simplest behavior near cusps and swallowtail points, it is useful to have the property that all nondegenerate critical points have $1 \leq i_{\text{mo}}(p) \leq n-1$, where $n = \dim N$. This condition holds after stabilizing F via the quadratic form $Q(\mu_1, \mu_2) = \mu_1^2 - \mu_2^2$. When forming ascending and descending manifolds in the noncompact, but tame-at-infinity setting, we use gradient-like vector fields that agree outside of a compact set with the Euclidean gradient of the linear or quadratic function that F is equal to at infinity.

Following [16], there exists a 2–family, $\{g_m, V_m\}_{m \in M}$, of metrics g_m and gradient-like vector fields V_m (on the fibers of E) for the functions f_m, such that the following hold:

(1) For all $m \in \mathcal{F}_0$ and $p_m \in \mathrm{Crit}(f_m)$, the stable and unstable manifolds $W^s(p_m)$ and $W^u(p_m)$ vary smoothly with (m, p_m), ie the fiberwise stable and unstable manifolds of sheets of L are smooth manifolds.

(2) For all m near the points in $\mathcal{F}_1$ with a pair of "near birth-death" points p_m^+ and p_m^- with $i_{\mathrm{mo}}(p_m^+) - i_{\mathrm{mo}}(p_m^-) = 1$, the manifolds $W^u(p_m^+)$ and $W^s(p_m^-)$ intersect transversely at an intermediary level-set in one point.

(3) For all $m \in M$ and $p_m, q_m \in \mathrm{Crit}(f_m)$ with locally well-defined relative Morse index, $i_{\mathrm{mo}}(p_m) - i_{\mathrm{mo}}(q_m)$, equal to 1, 0, or -1, the unions (over m) of $W^s(p_m)$ and $W^u(q_m)$ are in general position.

(4) Outside of arbitrary small disk neighborhoods, $N(e_{\mathrm{st}}^0)$, of the swallowtail points, all the birth/death points are *independent*. An independent birth/death is one in which the stable (resp. unstable) manifolds of the newly born pair of points do not intersect the unstable (resp. stable) manifolds of the other critical points.

These items follow from [16, Theorem 3.1 on page 42, pages 52–53 and 62–63, and Chapter IV, Section 2, Part (C)].

We now translate these items into the language of Definition 4.1 to construct a Morse complex 2–family. Consider a pair of families of nondegenerate critical points p_m, q_m. If $i_{\mathrm{mo}}(p_m) - i_{\mathrm{mo}}(q_m) = -1$, then the set of $m \in M$ such that $W^u(p_m) \cap W^s(q_m) \neq \varnothing$ is a set of points which we use to define H_{-1}. If $i_{\mathrm{mo}}(p_m) - i_{\mathrm{mo}}(q_m) = 0$, then the set of $m \in M$ such that $W^u(p_m) \cap W^s(q_m) \neq \varnothing$ is a collection of curves in general position which we use to define H outside of $\bigcup N(e_{\mathrm{st}}^0)$. Both H_{-1} and H have natural upper and lower lifts to L specified by the image of the critical points $p_m, q_m \in \Sigma_F$ under the diffeomorphism $i_F \colon \Sigma_F \to L$. (Notation as in Section 2.2.) As in [16, Chapter IV, Section 2, Part (C), page 147], the intersection with $\partial N(e_{\mathrm{st}}^0)$ of handleslide arcs with lifts on the swallowtail sheets is as specified by Axiom 4.2(3), where the differential d_0 is the differential from the Morse complex of the f_m outside the swallowtail region.

We complete the definition of H by connecting these handleslide endpoints to the swallowtail point. As a technical point, the number of (i,k)–handleslide arcs only agrees with $\langle d_0 S_k, S_i\rangle$ mod 2; if necessary, we can connect any extra endpoints in pairs.

We now assign differentials d_ν to components R_ν of $\mathcal{F}_0 \setminus (H_{-1} \cup H) = M \setminus \Sigma_{\mathcal{C}}$. First, consider regions outside of $\bigcup N(e^0_{\mathrm{st}})$. We can assume that for generic $m \in R_\nu$, the gradient-like vector field V_m of f_m is Morse–Smale. We can then define d_ν as the Morse codifferential, which counts positive flows of V_m between critical points of relative Morse index 1. See Remark 4.6. This differential is independent of the choice of $m \in R_\nu$, since any other such $m' \in R_\nu$ can be connected to m by a path in R_ν along which the Morse–Smale condition holds except at finitely many points where two flowlines between the same pair of critical points of the f_m appear or disappear. This does not change d_ν. Finally, note that there is a unique way to assign differentials in $\bigcup N(e^0_{\mathrm{st}})$ so that Axiom 4.3 holds. (If necessary, see Propositions 6.1 or 6.2 below.)

We now verify that Axioms 4.2 and 4.3 follow from known Cerf theory, subject to the convention-reversing modification in Remark 4.6. That all endpoints for handleslide arcs are as in Axiom 4.2 is established over the course of [16, Chapter IV], which needs a complete treatment of 2–parameter families of functions and gradient-like vector fields for its invariance proof of the (Morse) K–theoretic Wh_2 pseudo-isotopy invariant. Endpoints as in Axiom 4.2(1) are discussed in [16, Chapter II, Section 1, page 89]. Endpoints as in Axiom 4.2(2) appear in the "exchange relation"; see [16, Chapter IV, Section 2, Part (A), page 131]. Near swallowtail points, Axiom 4.2(3) follows from the "dovetail relation"; see [16, Chapter IV, Section 2, Part (C), page 147].

Axiom 4.3(1) is immediate, since when passing the crossing locus through a point m that is disjoint from handleslides, swallowtail, or cusp points, the Morse complex remains unchanged, except for the ordering of generators by critical value. Axiom 4.3(2) is a well-known result [20, Section 7]. Axiom 4.3(3) follows from items (2) and (4) of the list of properties for the stable and unstable manifolds of the critical points (see earlier in this proof).

Thus, we have produced an MC2F, $\mathcal{C}$, from a tame-at-infinity generating family. It remains to establish (1) and (2) in the statement of the proposition.

For (1), observe that the fiber homology $H(\mathcal{C}_{x_0})$ is the cohomology of the Morse complex of f_{x_0} (the restriction of F to the fiber above x_0). Assuming F is linear at infinity, f_{x_0} has the form

$$f_{x_0}\colon E'_{x_0} \times \mathbb{R}^k \to \mathbb{R},$$

where E'_{x_0} is the (compact) fiber of E' above x_0, and agrees with a nonzero linear function $l\colon \mathbb{R}^k \to \mathbb{R}$ outside of a compact set. We can split $\mathbb{R}^k \cong \ker l \oplus \mathbb{R}$, and by compactifying the $\ker l$ factor, we can extend f_{x_0} to a smooth function

$$f_{x_0}\colon E'_{x_0} \times S^{k-1} \times \mathbb{R} \to \mathbb{R}$$

that (i) is proper and (ii) agrees with the projection to the $\mathbb{R}$ factor outside of a compact set. This extension does not change the Morse complex of f_{x_0}, and in this setting the Morse complex computes the relative cohomology of $(f \le T, f \le -T)$, where $T \gg 0$; see for instance [20]. Since

$$(f \le T, f \le -T) = \big(E'_{x_0} \times S^{k-1} \times (-\infty, T], E'_{x_0} \times S^{k-1} \times (-\infty, -T]\big),$$

it follows that $H(\mathcal{C}_{x_0}) = \{0\}$.

To prove (2), assume $E \to M$ is the trivial bundle $M \times N$. (By the tame-at-infinity assumption, $N = N' \times \mathbb{R}^k$ with N' compact.) Let σ be a loop in M, generic with respect to the base projection of the singular set. The induced generating family on S^1 (with trivial bundle domain $S^1 \times N$), call it F_{S^1}, extends to a tame-at-infinity generating family on D^2 (with domain $D^2 \times N$). (This is because the subset of $C^\infty(N, \mathbb{R})$ consisting of those functions agreeing with a fixed linear or quadratic function on $\mathbb{R}^k$ at infinity is contractible.) Taking the extension of F_{S^1} to $D^2 \times N$ to be sufficiently generic, the transversality condition in the definition of generating families will hold and the front projection of the resulting Legendrian on J^1D^2 will be generic. This Legendrian is equipped with an MC2F, $\mathcal{C}'$, such that the continuation map for $\mathcal{C}'$ associated to the S^1 boundary loop of D^2 agrees with the continuation map for σ. By Proposition 4.7(3), this continuation map induces the identity map on homology (since it is chain homotopic to the continuation map for a constant loop). □

5 From MC2F to CHD

In this section, we show how to construct a CHD, and hence an augmentation, from an MC2F. A key technical point in associating a CHD to an MC2F is to allow continuation maps to be associated to the edges of a compatible polygonal decomposition for L. This is not immediate from Section 4.2 since edges may be contained in the singular set of L, but is accomplished by shifting 0–cells and 1–cells off of the singular set. See Figure 8 for a summary.

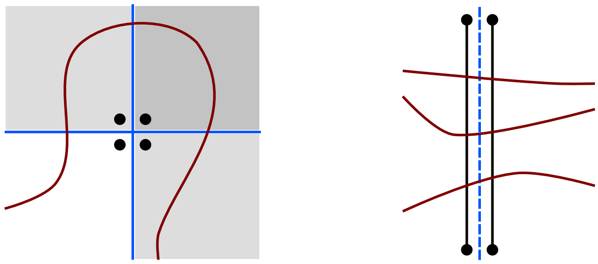

Figure 8: Given an MC2F for L, differentials $d(e^0_\beta \to e^0_\beta)$: $V(e^0_\beta) \to V(e^0_\beta)$ are defined by shifting vertices e^0_β into bordering 2–cells (left). Continuation maps $f(e^1_\beta \to e^1_\beta)$ are assigned to 1–cells by a similar shift (right). The choice of shift is nonunique and well-definedness is verified in Propositions 5.2 and 5.3. In these propositions, we also see that the associated boundary differentials and boundary maps (as in Section 3.4) have similar interpretations via shifting.

5.1 Continuation maps associated to edges of a compatible cell decomposition

Let $\mathcal{E}$ be a compatible polygonal decomposition for L satisfying Convention 2.3.

Definition 5.1 An MC2F is *nice* with respect to $\mathcal{E}$ if:

(1) The handleslide sets are transverse to the 1–skeleton of $\mathcal{E}$ except at swallowtail points which may be endpoints of handleslide arcs (as in Axiom 4.2(3)).

(2) In a neighborhood of each upward swallowtail point, the (i,k)–handle slide arcs (as in Axiom 4.2(3)) are contained in the corner labeled S, while both the S and T corners contain a $(k+1,k+2)$–handleslide arc. A similar condition is imposed at downward swallowtail points.

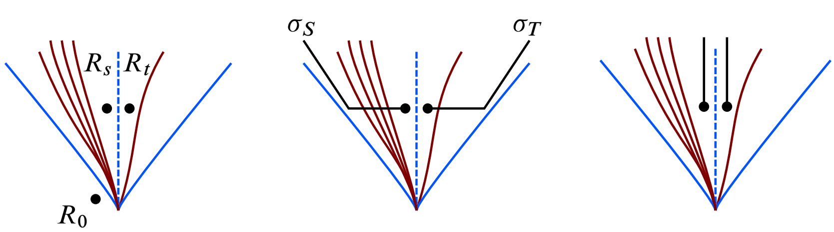

Figure 9: The regions R_s, R_t, and R_0 at a swallowtail point used to define the $d(e^0_{\mathrm{st}} \to e^d_\alpha)$ (left), the paths that define $f(e^1_S \to e^2_S)$ and $f(e^1_T \to e^2_T)$ (center) and $f(e^1_{\mathrm{cr}} \to e^d_\alpha)$ (right)

Let $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ be an MC2F that is nice with respect to $\mathcal{E}$. Recall that the d_ν are differentials on $V(R_\nu)$, where $\{R_\nu\}$ is the set of connected components of $M \setminus \Sigma_{\mathcal{C}}$ (with $\Sigma_{\mathcal{C}}$ the union of the handleslide sets of $\mathcal{C}$ and the singular set of L).

Using $\mathcal{C}$, we now associate to each appearance of a vertex e^0_β in the closure of another cell, $e^0_\beta \xrightarrow{j} e^d_\alpha$ (notation as in Section 3.4), a differential

$$d(e^0_\beta \to e^d_\alpha)\colon V(e^d_\alpha) \to V(e^d_\alpha).$$

We proceed according to whether or not e^0_β is a swallowtail point:

- **Assuming e^0_β is not a swallowtail point** Choose a component R_ν whose closure contains a neighborhood of e^0_β in $\overline{e^d_\alpha}$. The sheets $L(e^d_\alpha)$ are identified with a subset of $L(R_\nu)$ in the usual way, so that

$$L(R_\nu) = L(e^d_\alpha) \sqcup L_{\mathrm{cusp}} \tag{5-1}$$

with the sheets in L_{cusp} meeting in pairs at cusp edges above e^d_α. Axiom 4.3(2)–(3) imply that in the resulting direct sum $V(R_\nu) = V(e^d_\alpha) \oplus V_{\mathrm{cusp}}$ the $V(e^d_\alpha)$ component is a subcomplex of $(V(R_\nu), d_\nu)$. Thus, we can define

$$d(e^0_\beta \to e^d_\alpha) = d_\nu|_{V(e^d_\alpha)}.$$

- **Assuming e^0_β is a swallowtail point** When e^d_α is one of e^2_S, e^2_T, or e^1_{cr} we identify $L(e^d_\alpha)$ with $L(R_s)$, $L(R_t)$, or $L(R_t)$ respectively, where R_t (resp. R_s) is the region that borders the crossing locus on the side labeled T (resp. S). Take the corresponding d_t or d_s for $d(e^0_\beta \to e^d_\alpha)$. For any other e^d_α, the sheets $L(e^d_\alpha)$ are identified bijectively with $L(R_0)$, where R_0 is the region that borders e^0_β from outside the swallowtail region; the resulting isomorphism $V(e^d_\alpha) \cong V(R_0)$ allows us to put $d(e^0_\beta \to e^d_\alpha) = d_0$. See Figure 9.

Proposition 5.2 (1) *The differentials $d(e^0_\beta \to e^d_\alpha)$ are well defined.*

(2) *For any $e^0_\beta \to e^d_\alpha$, the differential $d(e^0_\beta \to e^d_\alpha)$ is the boundary differential associated to $d_\beta \colon = d(e^0_\beta \to e^0_\beta)$* (as in Section 3.4).

Proof Well-definedness is only in question in the nonswallowtail case. Suppose that R_ν and R_μ are two regions that border the cell e^d_α at the vertex e^0_β. (For $d = 1$ there could be two such regions, for $d = 0$ there may be many. See Figure 8 for a concrete example.) We can get from R_ν to R_μ by passing through a sequence of 1–cells with a common endpoint at e^0_β. Thus, we can assume without loss of generality that R_ν

and R_μ share such a 1–cell in their boundary. Moreover, if that 1–cell is a cusp edge we may assume the two cusp sheets exist above R_μ but not above R_ν.

The splitting from (5-1) defines an inclusion $i_\nu\colon V(e^d_\alpha) \to V(R_\nu)$ and projection $p_\nu\colon V(R_\nu) \to V(e^d_\alpha)$, and analogous maps i_μ and p_μ are defined for R_μ. We need to show that $D_\nu = D_\mu$, where

$$D_\nu = p_\nu \circ d_\nu \circ i_\nu \quad \text{and} \quad D_\mu = p_\mu \circ d_\mu \circ i_\mu.$$

Either (2) or (3) of Axiom 4.3 (depending if the 1–cell where R_ν and R_μ meet is a crossing or a cusp) provides a chain map $h\colon (V(R_\nu), d_\nu) \to (V(R_\mu), d_\mu)$. It is clear from the definitions that $h \circ i_\nu = i_\mu$, and $p_\nu = p_\mu \circ h$, so the equality $D_\nu = D_\mu$ follows in a routine manner.

To check (2) in the nonswallowtail case, we may assume that the same region R_ν is used in defining $d_\beta = d(e^0_\beta \to e^0_\beta)$ and $d(e^0_\beta \to e^d_\alpha)$. The sheets of $L(e^d_\alpha)$ not identified with sheets of $L(e^0_\beta)$ occur in pairs that meet at a cusp above e^0_β. From (2) and (3) of Axiom 4.3, it follows that $d(e^0_\beta \to e^d_\alpha)$ takes $S_b \mapsto S_a$ for each such pair of cusping sheets (with S_a the upper of the two sheets) and agrees with d_β on the span of $L(e^0_\beta) \subset L(e^d_\alpha)$. Thus, $d(e^0_\beta \to e^d_\alpha)$ is indeed related to d_β precisely as in the boundary differential construction of Section 3.4.

In the swallowtail case, $d_{\mathrm{st}} = d(e^0_{\mathrm{st}} \to e^0_{\mathrm{st}})$ is the differential d_0 from the component R_0 outside the swallowtail region, and this is the same as $d(e^0_{\mathrm{st}} \to e^d_\alpha)$ and the boundary differential for all neighboring e^d_α except for e^2_S, e^2_T, and e^1_{cr}. In Section 3.4, the associated boundary differential for $e^0_{\mathrm{st}} \to e^2_T$ is defined as $\widehat{d}_T = h_{k+1,k+2} d_{k,k+1} h_{k+1,k+2}$, where $d_{k,k+1} = d_0 \oplus d_{\mathrm{cusp}}$ using the isomorphism $V(e^2_T) = V(e^0_{\mathrm{st}}) \oplus V_{\mathrm{cusp}}$; the splitting arises from identifying $L(e^0_{\mathrm{st}})$ with the subset $\{S_1, \dots, \widehat{S}_k, \widehat{S}_{k+1}, \dots, S_n\} \subset L(e^2_T)$, and $d_{\mathrm{cusp}} S_{k+1} = S_k$. (Subscripts indicate ordering above e^2_T.) To see that this $\widehat{d}_T$ agrees with $d(e^0_{\mathrm{st}} \to e^2_T) = d_t$, travel from the region R_0 to R_t by passing first through the e^1_T cusp edge and then across the $h_{k+1,k+2}$ handleslide arc that appears in the T half of the swallowtail region; according to (3) and (1) of Axiom 4.3, the differential from the MC2F will change first from d_0 to $d_{k,k+1}$ and then to $h_{k+1,k+2} d_{k,k+1} h_{k+1,k+2}$ when we arrive at R_t; therefore, we have $d_t = \widehat{d}_T$. Next, apply Axiom 4.3(2) and the definition of $\widehat{d}_S$ from (3-3) to see that

$$d(e^0_{\mathrm{st}} \to d^2_S) = d_s = Q d_t Q^{-1} = Q \widehat{d}_T Q^{-1} = \widehat{d}_S.$$

Finally, note that for e^1_{cr} the boundary differential and $d(e^0_{\mathrm{st}} \to e^1_{\mathrm{cr}})$ are defined to agree with d_t and $\widehat{d}_T$ respectively. □

Suppose that the 1–cell e^1_β has initial and terminal vertices e^0_- and e^0_+. For each inclusion $e^1_\beta \to \overline{e^d_\alpha}$ as an edge, we associate a morphism

$$f(e^1_\beta \to e^d_\alpha)\colon (V(e^d_\alpha), d(e^0_- \to e^d_\alpha)) \to (V(e^d_\alpha), d(e^0_+ \to e^d_\alpha)).$$

In the case when $e^1_\beta = e^d_\alpha$, we refer to $f_\beta := f(e^1_\beta \to e^1_\beta)$ as the *continuation map* for the edge e^1_β.

We proceed according to whether or not e^1_β has an endpoint at a swallowtail:

- **Assuming e^1_β has no endpoints at swallowtails** Choose a neighboring 2–cell e^2_γ containing e^d_α in its closure. (When $e^d_\alpha = e^1_\beta$, there are two choices; when $d = 2$, $e^2_\gamma = e^d_\alpha$.) Shift e^1_β slightly to a path σ contained in the interior of a collar neighborhood $e^1_\beta \times [0,\epsilon) \subset \overline{e^2_\gamma}$ that is disjoint from H_{-1} and such that $e^0_\pm \times [0,\epsilon)$ is disjoint from H. Let R_- and R_+ denote the components that contain the shifts of e^0_- and e^0_+. The continuation map
$$f(\sigma)\colon (V(R_-), d_-) \to (V(R_+), d_+)$$
is well defined by Proposition 4.7(6). As usual, we can split $L(e^2_\gamma) = L(e^d_\alpha) \sqcup L_{\mathrm{cusp}}$. We can assume σ does not intersect handleslide arcs from H with endpoint lifts on sheets of L_{cusp} (as in Observation 4.4(4) these arcs are not allowed to reach the cusp edge). Then $f(\sigma)$ respects the decomposition $V(R_-) = V(R_+) = V(e^2_\gamma) = V(e^d_\alpha) \oplus V_{\mathrm{cusp}}$ and we define $f(e^1_\beta \to e^2_\alpha)$ as the component

(5-2) $$f(\sigma) = f(e^1_\beta \to e^d_\alpha) \oplus \mathrm{id}\colon V(e^d_\alpha) \oplus V_{\mathrm{cusp}} \to V(e^d_\alpha) \oplus V_{\mathrm{cusp}}.$$

- **Assuming e^1_β has an endpoint at a swallowtail, e^0_{st}** In view of Convention 2.3, the endpoint at the swallowtail e^0_{st} must be the initial point of e^1_S, e^1_T, and e^1_{cr}. In the case of e^1_{cr}, the $f(e^1_{\mathrm{cr}} \to e^d_\alpha)$ are defined as above. For e^1_S, define $f(e^1_S \to e^2_S) = f(\sigma_S)$ for a path σ_S that starts in R_s near the swallowtail point, runs perpendicularly across the handleslide arcs in the S corner of the swallowtail region, and then runs parallel to e^1_S (remaining on the side of e^1_S where the cusp sheets exist). For other e^d_α, define $f(e^1_S \to e^d_\alpha)$ to be a continuation map for a path that is a shift of e^1_S to the outside of the swallowtail region.

Define the $f(e^1_T \to e^d_\alpha)$ similarly. See Figure 9.

Proposition 5.3 (1) *The morphisms $f(e^1_\beta \to e^d_\alpha)$ are well defined.*

(2) *For any $e^1_\beta \to e^d_\alpha$, the morphism $f(e^1_\beta \to e^d_\alpha)$ is the boundary map associated to f_β* (as in Section 3.4).

Proof We only need to verify well-definedness when $e^d_\alpha = e^1_\beta$. Then there are two competing shifts, σ_a and σ_b, of e^1_β into the two neighboring cells e^2_a and e^2_b. Since H is transverse to e^1_β, assuming σ_a and σ_b are sufficiently close to e^1_β there will be a bijection between the sequence of handleslide arcs appearing along the paths σ_a and σ_b; specifically, the bijection identifies the endpoints of the components of the intersection of H with $e^1_\beta \times [-\epsilon, \epsilon]$. Moreover, above σ_a and σ_b the endpoint lifts of these handleslides belong to the subsets $i_a(L(e^1_\beta)) \subset L(e^2_a)$ and $i_b(L(e^1_\beta)) \subset L(e^2_b)$, and agree in $L(e^1_\beta)$. Thus, the $V(e^1_\beta)$ component of the continuation maps f_a and f_b agree, as required.

For (2), we need to show that for $e^1_\beta \to e^2_\gamma$, the map $f(e^1_\beta \to e^2_\gamma)$ is the boundary morphism for $f(e^1_\beta \to e^1_\beta)$. In the nonswallowtail case or in the case of a swallowtail with $e^1_\beta = e^1_{\mathrm{cr}}$, this is clear from the definition of boundary morphism and (5-2).

In the swallowtail with $e^1_\beta = e^1_X$ for $X = S$ or T, we have

$$f(e^1_X \to e^2_X) = f(\sigma_X) = f(\sigma_0 * \sigma_1) = f(\sigma_1) \circ f(\sigma_0) = \big(f(e^1_X \to e^1_X) \oplus \mathrm{id}_{V_{\mathrm{cusp}}}\big) \circ H_X,$$

where we decomposed $\sigma_X = \sigma_0 * \sigma_1$. Here, σ_0 is the part of σ_X that starts at R_s or R_t and crosses all of the handleslide arcs that end at the X corner of e^0_{st}, and σ_1 is the remaining portion of σ_X that runs parallel to e^1_X. The map H_X is as defined in (3-4). That $f(\sigma_0)$ agrees with H_X is a consequence of the arrangement of handleslide arcs at e^0_{st} specified by Definition 5.1(2). □

5.2 Constructing a CHD from an MC2F

Definition 5.4 We say that a CHD $\mathcal{D} = (\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\})$ for $\mathcal{E}$ and a nice MC2F $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ *agree on the* 1*–skeleton* if for every 0–cell, e^0_α, and every 1–cell, e^1_β,

$$d_\alpha = d(e^0_\alpha \to e^0_\alpha) \quad \text{and} \quad f_\beta = f(e^1_\beta \to e^1_\beta), \tag{5-3}$$

where $d(e^0_\alpha \to e^0_\alpha)$ and $f(e^1_\beta \to e^1_\beta)$ denote the differentials and continuation maps associated to 0–cells and 1–cells by $\mathcal{C}$.

Proposition 5.5 *Let $\mathcal{E}$ be a compatible polygonal decomposition for L. For any nice ρ–graded MC2F $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$, there exists a ρ–graded CHD $\mathcal{D} = (\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\})$ such that $\mathcal{C}$ and $\mathcal{D}$ agree on the* 1*–skeleton.*

Proof Use (5-3) to define $\{d_\alpha\}$ and $\{f_\beta\}$. The requirements of Definition 3.6(1)–(2) are easily seen to hold. In particular, Proposition 5.2 shows that the $\{f_\beta\}$ have the

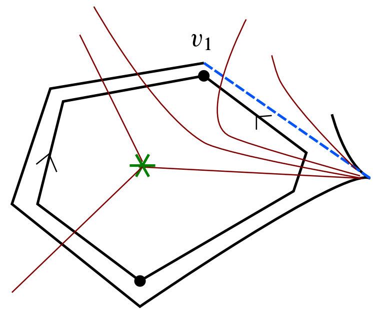

Figure 10: The homotopy operators K_γ relate the continuation maps associated to paths σ_a and σ_b that trace the boundary of e^2_γ from v_0 to v_1. The pictured 2–cell has a swallowtail point at its right-most vertex.

correct complexes for their domains and codomains, and Proposition 4.7(4) shows that the $f_\beta - \mathrm{id}$ is strictly upper triangular with degree $0 \bmod \rho$.

It remains to construct the homotopy operators $\{K_\lambda\}$. For a given 2–cell, e^2_γ, recall the chain isomorphisms from Definition 3.6(3), written as $\widehat{f}_j^{\eta_j} \circ \cdots \circ \widehat{f}_1^{\eta_1}$ and $\widehat{f}_m^{\eta_m} \circ \cdots \circ \widehat{f}_{j+1}^{\eta_{j+1}}$. Using Proposition 5.3(2), the definition of the $f(e^1_\beta \to e^2_\gamma)$, and Proposition 4.7(5)–(6), we compute

$$\begin{aligned}\widehat{f}_j^{\eta_j} \circ \cdots \circ \widehat{f}_1^{\eta_1} &= f(e^1_j \to e^2_\gamma)^{\eta_j} \circ \cdots \circ f(e^1_1 \to e^2_\gamma)^{\eta_1} \\ &= f(\sigma_1^{\eta_1} * \cdots * \sigma_j^{\eta_j}) = f(\sigma_a),\end{aligned}$$

where the σ_i, $1 \le i \le j$, are appropriate shifts into e^2_γ of the 1–cells e^1_i, $1 \le i \le j$, that occur around one half of the boundary of e^2_γ traversed from v_0 to v_1. The concatenation $\sigma_a = \sigma_1^{\eta_1} * \cdots * \sigma_j^{\eta_j}$ is then a shift of this half of the boundary of e^2_γ into its interior. Similarly, $\widehat{f}_m^{\eta_m} \circ \cdots \circ \widehat{f}_{j+1}^{\eta_{j+1}} = f(\sigma_b)$, where σ_b is a shift of the other half of the boundary of e^2_γ. Since σ_a and σ_b are path homotopic in the interior of e^2_γ, Proposition 4.7(6) gives the existence of the required (strictly upper triangular) homotopy operator K_γ. See Figure 10. □

6 From CHD to MC2F

We next establish the construction, converse to that of the previous section, of an MC2F from a CHD. Loosely, this can be viewed as a 2–dimensional analog of factoring an upper triangular matrix into a product of elementary matrices. After observing that this completes the proofs of Theorem 1.1, we use the connection between CHDs and MC2Fs to associate continuation maps to augmentations. In Proposition 6.8, we observe that properties of these continuation maps can obstruct the existence of linear-at-infinity generating families as well as generating families with trivial bundles as their domain.

6.1 Lemmas for constructing MC2Fs

When constructing MC2Fs it is convenient to begin by specifying the handleslide sets H and H_{-1}, and then check that the required differentials $d_\nu\colon R_\nu \to R_\nu$ can be constructed, satisfying Axiom 4.3. We record in Propositions 6.1–6.3 several cases in which the existence of the differentials is automatic. See Figure 11.

Proposition 6.1 *Let $L \subset J^1M$ have an MC2F $\mathcal{C}$ defined near the boundary of a disk $D \subset M$ such that $D \cap \Sigma_{\mathrm{cusp}} = \varnothing$, where Σ_{cusp} is the base projection of cusp edges. Suppose that the handleslide set H of $\mathcal{C}$ is extended over D so that*

- *there are no superhandleslide points in D, and*
- *Axiom 4.2 holds.*

Then there is a unique way to assign differentials d_ν to the regions of $D \setminus \Sigma_{\mathcal{C}}$ so that $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ is an MC2F over D.

Proof Let $f\colon (D, \partial D) \to ([0,1], \{0\})$ be a Morse function with a single critical point that is an absolute maximum at a point $x_0 \in D \setminus \Sigma_{\mathcal{C}}$ with $f(x_0) = 1$, and such that the restriction of f to $\Sigma_{\mathcal{C}}$ is Morse. It suffices to show how to extend the assignment of differentials $\{d_\nu\}$ from $f^{-1}([0, a-\delta])$ to $f^{-1}([0, a+\delta])$ when $f^{-1}(\{a\})$ contains a single point p that is a codimension-2 (in M) point of $\Sigma_{\mathcal{C}}$ or a critical point of f restricted to the 1–dimensional strata of $\Sigma_{\mathcal{C}}$. Since there are no swallowtails, cusps, or superhandleslides in D, we only need to consider

(a) critical points (maxima/minima) of f restricted to a crossing or handleslide arc,

(b) transverse intersections of two crossing and/or handleslide arcs, and

(c) triple points of $\Pi_F(L)$.

Parametrize a neighborhood N of p by $[-\delta, \delta] \times [-\delta, \delta]$ so that $f(x_1, x_2) = a + x_2$, and all crossings/handleslides exit N along $x_2 = \pm\delta$. Let $R_\pm$ denote the regions of $f^{-1}([0, a+\delta]) \setminus \Sigma_{\mathcal{C}}$ that contain the boundaries $x_1 = \pm\delta$. Differentials for $R_\pm$ and for all regions in $f^{-1}([0, a]) \setminus \Sigma_{\mathcal{C}}$ are already specified at the bottom of N, where $x_2 = -\delta$. At $x_2 = +\delta$, as x_1 increases from $-\delta$ to $+\delta$, we pass through a sequence of regions

$R_0, R_1, \dots, R_n$ with $R_0 = R_-$ and $R_n = R_+$. Since we already have a differential on R_0, Axiom 4.3 specifies a unique way to assign differentials to $R_1, \dots, R_{n-1}$. We just need to verify that the differential on R_{n-1} is related to the one already specified on $R_n = R_+$ as required in Axiom 4.3. This amounts to the statement that the continuation map associated to the paths from R_- to R_+ at $x_2 = -\delta$ and $x_2 = +\delta$ agree, and this has already been observed in the proof of Proposition 4.7. □

Proposition 6.2 *Suppose that near a swallowtail point p for a Legendrian $L \subset J^1M$, an arbitrary upper triangular differential d_0 is assigned to the complement of the swallowtail region, and handleslide arcs, H, as required in Axiom 4.2(3), are placed within the swallowtail region. Then there exists a unique way to assign differentials d_ν within the swallowtail region to extend d_0 and H to an MC2F $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ defined near p.*

Proof As usual we consider the case of an upward swallowtail point involving sheets k, $k+1$, and $k+2$. Let R_0 be the region with two fewer sheets. Suppose that as we pass through the swallowtail region from one cusp edge to the other the regions $R_1, \dots, R_r$ appear in order. Passing from R_0 into R_1, the differential d_1 is specified by d_0 via Axiom 4.3(3); passing from R_i to R_{i+1} for $1 \le i \le r-1$, the differential d_{i+1} is specified by Axiom 4.3(1)–(2). Finally, when passing from R_r back into R_0, it is important to have that d_r and d_0 are related as in Axiom 4.3(3), ie we need $d_r = d_0 \oplus d_{k,k+1}$, where $d_{k,k+1}S_{k+1} = S_k$. The net effect of passing from R_1 to R_r is to conjugate the differential $d_1 = d_0 \oplus d_{k,k+1}$ by $H_S \circ Q \circ H_T$, where Q interchanges S_{k+1} and S_{k+2} and the maps H_S and H_T are as in (3-4). Thus, the required equation is

$$(d_0 \oplus d_{k,k+1}) \circ (H_S \circ Q \circ H_T) = (H_S \circ Q \circ H_T) \circ (d_0 \oplus d_{k,k+1}).$$

This is straightforward to verify with a direct computation. Alternatively, observe that if d_0 has matrix A, then in the notation of Lemma 3.5 the matrix of $d_0 \oplus d_{k,k+1}$ is $\widehat{A}_{k,k+1}$. The matrices A_S and A_T considered in that lemma have $A_S Q = Q A_T$ (by (3-3)), and so using (3-6) we compute

$$\widehat{A}_{k,k+1} H_S Q H_T = H_S A_S Q H_T = H_S Q A_T H_T = H_S Q H_T \widehat{A}_{k,k+1}. \qquad \square$$

Suppose that an MC2F $\mathcal{C}'$ for $L \subset J^1M$ has been defined on a subsurface $M' \subset M$ with nonempty boundary. Let $D \subset (M' \setminus \Sigma_{\mathcal{C}'})$ be a half-open disk with $\partial D \subset \partial M'$. Suppose that L has n sheets above D, and let d_0 denote the differential assigned to D by $\mathcal{C}'$.

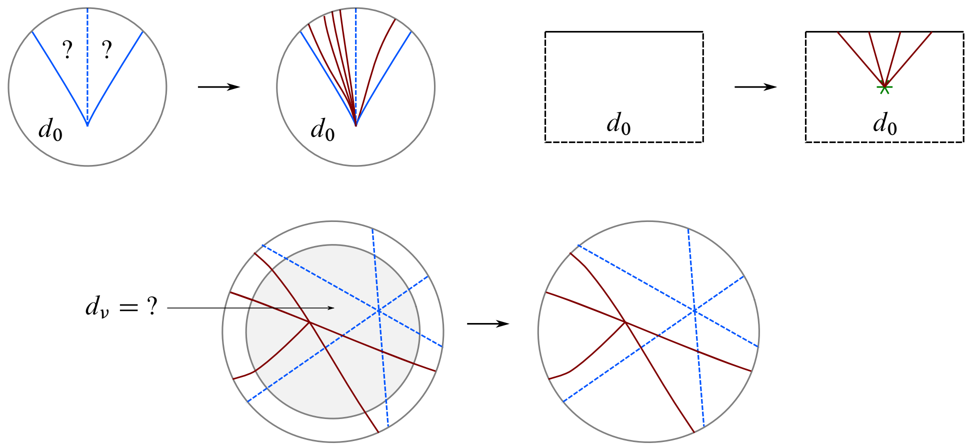

Figure 11: Tools for constructing MC2Fs (clockwise from top left): determining differentials near swallowtail points (Proposition 6.2); adding superhandleslide points (Proposition 6.3); and extending the assignment of differentials $\{d_\nu\}$ over the interior of a disk disjoint from Σ_{cusp} (Proposition 6.1)

Proposition 6.3 *Suppose we place, for some $1 \leq i < j \leq n$, an (i, j)–superhandleslide point p in the interior of D, and add handleslide arcs in D from p to ∂D as specified by Axiom 4.2(2) using the differential d_0. Then there is a unique way to assign differentials $\{d_\nu\}$ in D to produce an MC2F, $\mathcal{C}$, that agrees with $\mathcal{C}'$ outside of D.*

Proof Again, Axiom 4.3 gives a unique way to assign differentials as we pass from R_0, the unbounded region of D, through the sequence of new regions $R_1, \dots, R_r$ created by the handleslides with endpoints at p (see Figure 11). We need to verify that Axiom 4.3 holds when we pass from R_r back to R_0, ie that the composition of the handleslide maps associated to the sequence of arcs coming out of p commutes with d_0. For an (i, j)–superhandleslide, the matrix for this composition of handleslide maps is

$$H = I + D_0 E_{i,j} + E_{i,j} D_0,$$

where D_0 is the matrix of d_0, and we compute

$$D_0 H = D_0 + D_0 E_{i,j} D_0 = H D_0. \qquad \square$$

6.2 Constructing an MC2F from a CHD

Let $\mathcal{E}$ be a compatible polygonal decomposition for a Legendrian $L \subset J^1 M$.

Proposition 6.4 *For any ρ–graded CHD $\mathcal{D} = (\{d_\alpha\}, \{f_\beta\}, \{K_\gamma\})$ for $(L, \mathcal{E})$, there exists a nice ρ–graded MC2F $\mathcal{C} = (\{d_\nu\}, H, H_{-1})$ such that $\mathcal{D}$ and $\mathcal{C}$ agree on the 1–skeleton.*

Proof The proof consists of three steps where we define $\mathcal{C}$ in a neighborhood of 0–cells and then extend over the 1– and 2– skeletons.

Step 1 (defining $\mathcal{C}$ in a neighborhood of the 0–skeleton) Let $N_0 \subset M$ consist of a union of small disks, $N_0 = \cup_\alpha N(e^0_\alpha)$, centered at the 0–cells of $\mathcal{E}$. Given e^0_α, we define $\mathcal{C}$ on $N(e^0_\alpha)$ as follows:

- **When e^0_α is not a swallowtail point** We do not introduce any handleslide arcs in $N(e^0_\alpha)$, so we just need to define differentials d_ν: $V(R_\nu) \to V(R_\nu)$ for each of the regions $R_\nu \subset N(e^0_\alpha)$ in the complement of the singular set of L. For such a R_ν, we use the usual splitting

$$V(R_\nu) = V(e^0_\alpha) \oplus V_{\mathrm{cusp}} \quad \text{and put} \quad d_\nu = d_\alpha \oplus d_{\mathrm{cusp}}.$$

It is easy to check that Axiom 4.3 holds.

- **When e^0_α is a swallowtail point** Take the differential $d_0 := d_\alpha$ for the region R_0 outside the swallowtail region. Next, add handleslide arcs as specified by Axiom 4.2(3), positioned in the S and T corners as in Definition 5.1(2). By Proposition 6.2, there exists a unique way to define the differentials d_ν for the components R_ν of $N(e^0_\alpha) \setminus \Sigma_{\mathcal{C}}$ within the swallowtail region.

Step 2 (extending $\mathcal{C}$ to a neighborhood of the 1–skeleton) Let N_1 be the union of N_0 with small tubular neighborhoods, $N(e^1_\beta)$, of each 1–cell. (In particular, at each swallowtail point e^0_{st}, the $N(e^1_L)$, $N(e^1_R)$, and $N(e^1_{\mathrm{cr}})$ should meet the boundary of the disk neighborhood $\partial N(e^0_{\mathrm{st}})$ along an arc that is disjoint from the handleslide set of $N(e^0_\alpha)$.) Given e^1_β, we now extend $\mathcal{C}$ over $N(e^1_\beta) \setminus N_0$. Begin by labeling the sheets of $L(e^1_\beta)$ as $S_1, S_2, \dots, S_n$, and factor f_β into a product of handleslide maps

$$f_\beta = h_{i_r, j_r} \circ \cdots \circ h_{i_1, j_1}. \tag{6-1}$$

(Such a factorization exists by the usual Gauss–Jordan elimination algorithm.) In $N(e^1_\beta) \setminus N_0$, we then place a sequence of r corresponding handleslide arcs that run across $N(e^1_\beta)$ perpendicularly to e^1_β; following the orientation of e^1_β, the lower and upper lifts of the l^{th} arc are the sheets above $N(e^1_\beta)$ that continuously extend S_{i_l}, S_{j_l}.

Starting from the neighborhood of e^0_- where differentials for $\mathcal{C}$ are already defined and following the orientation of e^1_β there is a unique way to assign differentials $\{d_\nu\}$ to the regions of $N(e^1_\beta)\setminus\mathcal{C}$ so that Axiom 4.3 holds. Moreover, the factorization (6-1) shows that when the disk neighborhood of e^0_+ is reached the differentials match the previously defined differentials from Step 1.

It is clear at this point that $\mathcal{C}$ agrees with $\mathcal{D}$ on the 1–skeleton.

Step 3 (extending $\mathcal{C}$ to the interior of 2–cells) Given a 2–cell e^2_γ, we currently have $\mathcal{C}$ defined in a collar neighborhood, $U\subset\overline{e^2_\gamma}$, of $\partial\overline{e^2_\gamma}$. Let $C=(\partial U)\cap e^2_\gamma$, ie C is a closed curve that is the one boundary component of U belonging to the interior of e^2_γ. Let $w_0, w_1\in C$ denote points on $\partial N(e^0_{v_i})\cap C$ corresponding to the initial and terminal vertices, v_0 and v_1, of e^2_γ. In the case v_i is a swallowtail point where the S or T corner appears in e^2_γ, place w_i on the e^1_{cr} side of the handleslide arcs that meet $\partial N(e^0_{\mathrm{st}})$. There are two arcs σ_a and σ_b oriented from w_0 to w_1 and such that $C=\sigma_a\cup\sigma_b$. Along these arcs a sequence of handleslides from N_1 meet C transversally, and by construction the continuation maps are

$$f(\sigma_a)=\widehat{f}_j^{\eta_j}\circ\cdots\circ\widehat{f}_1^{\eta_1}\quad\text{and}\quad f(\sigma_b)=\widehat{f}_m^{\eta_m}\circ\cdots\circ\widehat{f}_{j+1}^{\eta_{j+1}},$$

where we follow the notation of Definition 3.6. (This uses that at any swallowtail vertices of e^2_γ, the handleslide arcs with endpoints on $\partial N(e^0_{\mathrm{st}})$ produce the factor of H_X that is required in the definition of boundary map for the edges e^1_X with $X=S$ or T.)

The homotopy operator $K_\gamma\colon (V(e^2_\gamma),\widehat{d}_{v_0})\to(V(e^2_\gamma),\widehat{d}_{v_1})$ from $\mathcal{D}$ then satisfies

$$f(\sigma_a)-f(\sigma_b)=d_{w_1}K_\gamma+K_\gamma d_{w_0},$$

where the differentials d_{w_i} are from $\mathcal{C}$ at the regions R_{w_i} bordering the w_i; they agree with the boundary differentials $\widehat{d}_{v_i}$ written above with the domain and codomain of K_γ (by Proposition 5.2). Moreover, postcomposing both sides with $(f(\sigma_b))^{-1}$ leads to the equation

$$f(C)-\mathrm{id}=dK+Kd, \tag{6-2}$$

where we orient C as $\sigma_a*\sigma_b^{-1}$; K is the upper triangular homotopy operator given by $K=(f(\sigma_b))^{-1}K_\gamma$; and $d=d_{w_0}$.

For convenience, in the following we parametrize $\overline{e^2_\gamma}$ by $I^2=[0,1]\times[0,1]$ with coordinates $(x_1,x_2)\in I^2$. Moreover, we assume that N_1 (the current domain of definition of $\mathcal{C}$) is an ϵ–neighborhood of $\partial(I^2)$, and has its boundary curve C oriented clockwise. Furthermore, we assume all handleslides arcs in $N_1\cap\overline{e^2_\gamma}$ appear near the left-hand boundary in $[0,\epsilon]\times(\epsilon,1-\epsilon)$. Note that the differential assigned by $\mathcal{C}$ to the common region bordered by the top, bottom and right side of $\partial(I^2)$ is $d=d_{w_0}$.

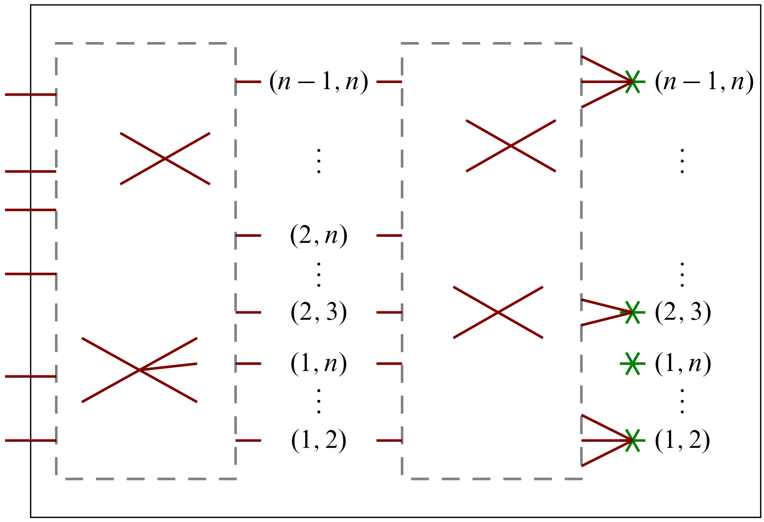

Figure 12: Extending $\mathcal{C}$ over the interior of e^2_γ

To complete the proof, we extend $\mathcal{C}$ over the remainder of I^2. The approach is pictured schematically in Figure 12. We will use the following terminology: We say that the handleslide set H is *lexicographically ordered* along an oriented path σ if the indices of upper and lower lifts, (i,j), of handleslide arcs that intersect σ are weakly increasing along σ with respect to lexicographical order. We say that two handleslide arcs *commute* if the indices of their lifts, (i_1,j_1) and (i_2,j_2), satisfy $j_1 \neq i_2$ and $i_1 \neq j_2$.

- In $\left[\epsilon, \frac{1}{4}\right] \times I$, we extend the handleslide arcs from left to right, changing their vertical ordering as we go (observing Axiom 4.2), so that H becomes lexicographically ordered along $\left\{\frac{1}{4}\right\} \times I$ (as x_2 increases).

(This is possible: Start by extending the handleslide arcs that begin at $\{\epsilon\} \times I$ to $\left\{\frac{1}{4}\right\} \times I$, achieving the required permutation by factoring it into transpositions and interchanging adjacent handleslide arcs in a corresponding manner. With this initial step carried out, we return to any points where an (i,l)–handleslide arc crosses an (l,j)–handleslide arc for some $1 \leq i < l < j \leq n$, and for each such point, x, create a new (i,j)–handleslide arc with one endpoint at x and the other at an appropriate point on $\left\{\frac{1}{4}\right\} \times I$. Repeat this procedure inductively. Note that any (i,j)–handleslide arc created at the m^{th} step will have $i - j \geq m$, so that after finitely many steps the process is complete.)

For any $i < j$, let $\alpha_{i,j}$ be the number of (i,j)–handleslide arcs at $\left\{\frac{1}{4}\right\} \times I$. We can arrange that each $\alpha_{i,j}$ is either 0 or 1 since an adjacent pair of (i,j)–handleslide arcs with endpoints at $\{\frac{1}{4}\} \times I$ can be joined together into a single arc with a local maximum for the x_1–coordinate just before $x_1 = \frac{1}{2}$. The continuation map for $\{\frac{1}{4}\} \times I$ agrees with $f(C)$ (by Proposition 4.7(6)), and by definition is

$$f(C) = h_{n-1,n}^{\alpha_{n-1,n}} (h_{n-2,n}^{\alpha_{n-2,n}} h_{n-2,n-1}^{\alpha_{n-2,n-1}}) \cdots (h_{2,n}^{\alpha_{2,n}} \cdots h_{2,3}^{\alpha_{2,3}})(h_{1,n}^{\alpha_{1,n}} \cdots h_{1,2}^{\alpha_{1,2}}).$$

Observe that (due to the lexicographic ordering of subscripts) the matrix of this product is precisely

$$I + \sum_{i<j} \alpha_{i,j} E_{i,j},$$

so

$$\alpha_{i,j} = \langle (f - \mathrm{id}) S_j, S_i \rangle.$$

- In $[\frac{3}{4}, 1] \times I$, we start by placing in lexicographic order at $x_2 = \frac{7}{8}$ an (i, j)–superhandleslide point, for each $i < j$ with $\langle K S_j, S_i \rangle = 1$. In addition, we add handleslide arcs as specified by Axiom 4.2(2) running approximately horizontally from $\{\frac{7}{8}\} \times I$ to $\{\frac{3}{4}\} \times I$. As in Observation 4.4(1), we can always use the differential $d = d_{w_0}$ in determining which (if any) handleslide arcs need to appear with endpoint at a superhandleslide. It follows, at least mod 2, that the total number of (i, j)–handleslide arcs along $\{\frac{3}{4}\} \times I$ is

$$\langle (dK + Kd) S_j, S_i \rangle = \langle (f - \mathrm{id}) S_j, S_i \rangle = \alpha_{i,j}.$$

By Proposition 6.3, there is a unique way to assign differentials in $[\frac{3}{4}, 1] \times I$ to any new regions that are created by the handleslides ending at the new superhandleslide points.

- In $[\frac{1}{2}, \frac{3}{4}] \times I$, we extend the handleslide arcs from $x_1 = \frac{3}{4}$ to $x_1 = \frac{1}{2}$, arranging that the handleslides are lexicographically ordered at $\{\frac{1}{2}\} \times I$. Moreover, this can be done *without creating additional handleslide endpoints*.

(Assume inductively that the subset $X_{<m}$ of handleslide arcs that have their right endpoint at an (i, j)–superhandleslide point with $i < m$ have been extended to $\{\frac{1}{2}\} \times I$, where they appear in lexicographic order. To inductively complete the extension process, we need to extend the subset X_m of those handleslide arcs with right endpoint at an (m, j)–superhandleslide. Any such arc in X_m will be an (i', j')–handleslide for with $i' \le m < j \le j'$. Consequently, arcs in X_m commute with one another. At $x_1 = \frac{3}{4}$, all handleslide arcs from $X_{<m}$ appear below the arcs from X_m. Consequently, to extend a given (i', j')–handleslide arc from X_m appropriately, it will only need to cross (i'', j'')–handleslides from $X_{<m}$ having $i' \le i''$. In these cases the (i', j')

and (i'', j'') are such that the arcs commute since $i'' \le m < j'$ (because (i'', j'') has an endpoint at an (i, j)–superhandleslide with $i < m$) and $i' \le i'' < j''$.)

Since no new handleslide arcs were created, the number of (i, j)–handleslide arcs at $\left\{\frac{1}{2}\right\} \times I$ is still $a_{i,j}$ mod 2, and joining (i, j)–handleslide arcs together in pairs, we can assume the number of arcs is exactly $\alpha_{i,j}$.

- In $\left[\frac{1}{4}, \frac{1}{2}\right] \times I$, since handleslide arcs are lexicographically ordered along $x_1 = \frac{1}{4}$ and $x_1 = \frac{1}{2}$ and are in bijection (preserving (i, j)), we simply join the end points.

With the handleslide set complete, Proposition 6.1 shows that the differentials $\{d_\nu\}$ can be defined over $\left[\epsilon, \frac{3}{4}\right] \times I$. This completes the construction of $\mathcal{C}$. □

Theorem 1.1 that was stated in the introduction now follows easily.

Proof of Theorem 1.1 Proposition 3.7 shows the existence of a $\mathbb{Z}/2$–augmentation is equivalent to the existence of a CHD. Since a small perturbation can make any MC2F nice with respect to a given $\mathcal{E}$, Propositions 5.5 and 6.4 show that L has a CHD if and only if L has an MC2F. The statement about generating families then follows from Proposition 4.10. □

6.3 Monodromy representations for augmentations

Using Proposition 6.4, we can now associate a fiber homology space with monodromy representation to an augmentation.

Let $\mathcal{E}$ be a compatible polygonal decomposition for L, and let $\epsilon\colon (\mathcal{A}, \partial) \to (\mathbb{Z}/2, 0)$ be an augmentation of the corresponding cellular DGA. Let $e^0_\alpha \in \mathcal{E}$ be a 0–cell. Consider a small neighborhood $N(e^0_\alpha)$, and let $x_0 \in N(e^0_\alpha)$ be disjoint from the cusp/crossing locus; if e^0_α is a swallowtail point, we assume x_0 is outside the swallowtail region. Via Proposition 3.7, there is a unique CHD, $\mathcal{D}$, for $\mathcal{E}$ associated to ϵ. Then, using Proposition 6.4, there exists an MC2F $\mathcal{C}$ that agrees with $\mathcal{D}$ on the 1–skeleton. We can assume the handleslide set of $\mathcal{C}$ is disjoint from $N(e^0_\alpha)$, or the part of $N(e^0_\alpha)$ outside the swallowtail region in the case e^0_α is a swallowtail.

We define the *fiber homology* and *monodromy representation* of ϵ at x_0 by

$$H(\epsilon_{x_0}) := H(\mathcal{C}_{x_0}) \quad \text{and} \quad \Phi_{\epsilon,x_0} := \Phi_{\mathcal{C},x_0}\colon \pi_1(M, x_0) \to \mathrm{GL}(H(\epsilon_{x_0})).$$

(Recall $H(\mathcal{C}_{x_0})$ and $\Phi_{\mathcal{C},x_0}$ are defined in Corollary 4.8.)

Proposition 6.5 *For x_0 as above, $H(\epsilon_{x_0})$ and Φ_{ϵ,x_0} are well defined.*

Proof Since $\mathcal{C}$ and $\mathcal{D}$ agree on the 1–skeleton, the differential on $V(R_0)$ (where $R_0 \subset M \setminus \Sigma_{\mathcal{C}}$ and $x_0 \in R_0$) is determined by the differential d_α on $V(e^0_\alpha)$ from $\mathcal{D}$ via the boundary differential construction.

In addition, the continuation maps $f(\sigma)$ for paths σ that are shifts of a 1–cell e^1_β into bordering 2–cells are determined by the map f_β from $\mathcal{D}$ via the boundary map construction. Any $[\sigma] \in \pi_1(M, x_0)$ can be represented by a concatenation of such paths with some paths, τ_i, contained in the $N(e^0_\alpha)$. In the swallowtail case, the handleslide set, H, of $\mathcal{C}$ has a standard form in the S and T sides of the part of $N(e^2_\alpha)$ in the swallowtail region, while in other cases H is disjoint from $N(e^2_\alpha)$. Thus, we can take the τ_i to be independent of $\mathcal{C}$, so that $\Phi_{\mathcal{C},x_0}([\sigma])$ is determined by $\mathcal{D}$. □

Remark 6.6 As in Remark 4.9, although we have only defined $(H(\epsilon_{x_0}), \Phi_{\epsilon,x_0})$ near 0–cells, up to isomorphism there is a unique local system on all of M extending $(H(\epsilon_{x_0}), \Phi_{\epsilon,x_0})$.

Observation 6.7 (1) From Corollary 4.8, it follows that the isomorphism type of $H(\epsilon_{x_0})$ is independent of x_0.

(2) Explicitly, the group $H(\epsilon_{x_0})$ is computed from ϵ as the homology of $(V(e^0_\alpha), d_\alpha)$, where

$$d_\alpha S_j = \sum_i \epsilon(a^\alpha_{i,j}) S_i.$$

The monodromy map $\Phi_{\epsilon,x_0}([\sigma])$ is computed by homotoping σ into a concatenation of 1–cells, $e^1_{\beta_1} * \cdots * e^1_{\beta_m}$; shifting each such 1–cell into the interior of a neighboring 2–cell (as in Section 5.1); and then connecting the endpoints with paths τ_i in the $N(e^0_\alpha)$. The resulting map has the form

$$\Phi_{\epsilon,x_0}([\sigma]) = f(\tau_m) \circ \widehat{f}^{\pm 1}_{\beta_m} \circ f(\tau_{m-1}) \circ \widehat{f}^{\pm 1}_{\beta_{m-1}} \circ \cdots \circ f(\tau_1) \circ \widehat{f}^{\pm 1}_{\beta_1} \circ f(\tau_0),$$

where each $\widehat{f}_{\beta_i}$ is obtained from the map f_{β_i} from $\mathcal{D}$ as in the boundary map construction. Except in the case of a swallowtail point, the $f(\tau_i)$ are simply compositions of the projection/inclusion maps, p and j, from cusp edges, and the permutation maps from crossings. At swallowtails, when τ_i connects an endpoint outside of the swallowtail region to one within the S (resp. T) region, the map $f(\tau_i)$ is

$$H_S \circ j \text{ or } p \circ H_S \quad (\text{resp. } H_T \circ j \text{ or } p \circ H_T)$$

depending on the orientation of τ_i.

We arrive at the following obstructions to particular types of generating families.

Proposition 6.8 (1) *If $H(\epsilon_{x_0}) \neq \{0\}$ for all augmentations ϵ, then L does not have a linear-at-infinity generating family.*

(2) *If Φ_ϵ is nontrivial for all augmentations ϵ, then L does not have a generating family whose domain is a trivial bundle over M.*

Proof This follows directly from the itemized statements in Proposition 4.10 and the definition of $(H(\epsilon_{x_0}), \Phi_{\epsilon,x_0})$. □

7 Examples

An easy corollary of Theorem 1.1 is that loose Legendrian surfaces [21] do not have generating families, since they do not have augmentations. In this section, we consider further examples, including Legendrians to which the more refined obstructions of Proposition 6.8 can be applied.

7.1 Treumann–Zaslow Legendrians

In [34], Treumann and Zaslow introduce an elegant class of Legendrian surfaces associated to trivalent graphs. For these surfaces, they study associated moduli spaces of constructible sheaves, and construct examples of nonexact Lagrangian fillings. In this section, we apply our approach to provide necessary and sufficient conditions for the existence of $\mathbb{Z}/2$–augmentations for this class of Legendrian surfaces.

Let $\Gamma \subset M$ be a trivalent graph. In [34, Section 2.1], a front projection called the *hyperelliptic wavefront* modeled on Γ is constructed[2] producing a Legendrian that we denote by $L_\Gamma \subset J^1M$. The base projection, $\Pi_B\colon L_\Gamma \to M$, is a 2–fold branched covering of M, with branch points at the vertices of Γ. Crossing arcs of the front projection sit above the edges of Γ; above each vertex of Γ, we have that $\Pi_F(L_\Gamma)$ matches a standard coordinate model in which three crossing arcs share a common endpoint. The front singularities that appear above vertices in $\Pi_F(L_\Gamma)$ are nongeneric, but appear with codimension 1 in the space of front projections as the D_4^- bifurcation of fronts; see [2]. A generic front for a surface Legendrian isotopic to L_Γ is obtained by replacing the singularities above vertices with the configuration of three swallowtail points pictured in Figure 13.

[2] Strictly speaking, [34] considers the case of $M = S^2$, but the construction works equally well to produce a front projection in J^1M for any surface M.

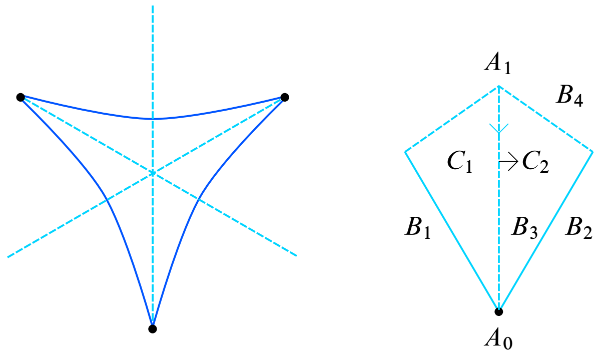

Figure 13: A generic base projection for L_Γ near vertices of Γ (left); there are three upward swallowtail points all placed on the upper sheet of L_Γ. Labeling for 0– and 1–cells used in the proof of Proposition 7.1 (right).

We refer to the components of $M \setminus \Gamma$ as *faces* of Γ, but note they do not need to be disks.

Proposition 7.1 *The Legendrian surface L_Γ has a $\mathbb{Z}/2$–augmentation if and only if every face of Γ has an even number of vertices.*

Proof In a neighborhood $N(v) \subset M$ of any vertex $v \in \Gamma$, an MC2F $\mathcal{C}$ can be constructed with handleslide set as pictured in Figure 14. The differentials $\{d_v\}$ are 0 in regions where L_Γ is 2–sheeted. Note that the endpoints of handleslide arcs are as required in Axiom 4.2: Endpoints at swallowtail points are as in Axiom 4.2(3), and occur between the pictured slices 1 and 2 as well as between slices 6 and 7 in the right half of Figure 14. When handleslide arcs intersect one another, an endpoint of a third handleslide arc occurs when appropriate according to Axiom 4.2(1); this

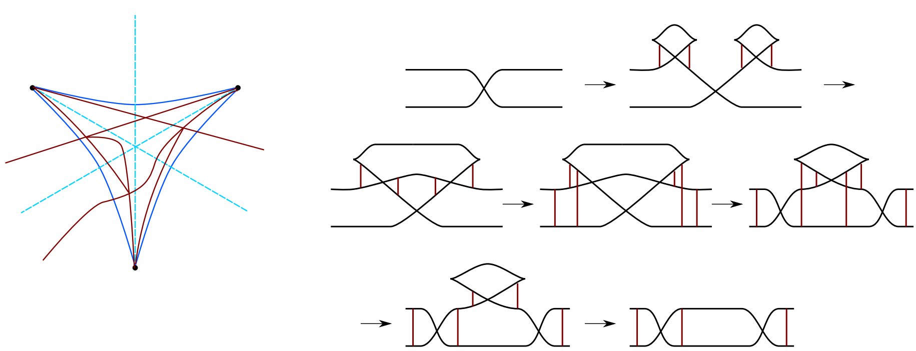

Figure 14: An MC2F for L_Γ near vertices of Γ (left) and slices of the front projection of L_Γ as x_2 decreases (right)

accounts for the two handleslides that appear at slice 4 as well as the handleslide that disappears between slices 5 and 6. Proposition 6.2 then defines differentials in neighborhoods of swallowtail points, and this assignment of differentials can be extended to a neighborhood of the cusp locus so that for regions bordering the cusp locus the only nonzero $d_v S_i$ is $d_v S_b = S_a$ with S_b and S_a the lower and upper sheets at the cusp edge. Finally, Proposition 6.1 extends the differentials $\{d_v\}$ over the remainder of $N(v)$.

Note that one handleslide arc enters each of the three faces adjacent to v. For any face F with an even number of vertices, it is then easy to extend $\mathcal{C}$ over F by connecting the handleslide arcs that exist near the vertices in pairs via paths in the interior of F.

It remains to show it is impossible to construct an MC2F if there is at least one face, F, with an odd number of vertices. For a vertex v of F, the neighborhood N of v above which L_Γ is 4–sheeted has a natural polygonal decomposition with six triangular 2–cells. Consider the third of N consisting of the two triangles with vertices at the swallowtail point that points into the face F, and label the cells of this region as in Figure 13 (right); number sheets above A_1 and B_3 as they are ordered above C_2. The choice of T and S corners at A_1 is unimportant since $T = S = I + E_{2,3}$.

Claim *Any augmentation* $\epsilon\colon \mathcal{A} \to \mathbb{Z}/2$ *satisfies* $\epsilon(b^1_{1,2}) + \epsilon(b^2_{1,2}) = 1$.

Proof of claim The differential $\partial B_3 = (A_0)_T (I + B_3) + (I + B_3) A_1$ is

$$\partial B_3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ & 0 & 0 & a^0_{1,2} \\ & & 0 & a^0_{1,2} \\ & & & 0 \end{bmatrix} (I + B_3) + (I + B_3) \begin{bmatrix} 0 & a^1_{1,2} & a^1_{1,3} & a^1_{1,4} \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}.$$

The $(2,3)$–entry of B_3 is 0 because of the crossing locus, so the top row of ∂B_3 is

$$\begin{bmatrix} 0 & 1 + a^1_{1,2} & 1 + a^1_{1,3} & a^1_{1,4} + b^3_{2,4} + b^3_{3,4} \end{bmatrix}.$$

The equation $\epsilon(\partial B_3) = 0$ then implies

$$\text{(7-1)} \qquad \epsilon(a^1_{1,2}) = 1; \quad \epsilon(a^1_{1,3}) = 1; \quad \epsilon(a^1_{1,4}) = \epsilon(b^3_{2,4}) + \epsilon(b^3_{3,4}).$$

If we consider the corresponding equation $\partial B'_3$, where B'_3 belongs to a different third of N, the location of the $a^1_{1,j}$ are permuted within the matrix A_1. For instance, in $\partial B'_3$, the top row of the A_1 matrix would become $\begin{bmatrix} 0 & a^1_{1,2} & a^1_{1,4} & a^1_{1,3} \end{bmatrix}$ or $\begin{bmatrix} 0 & a^1_{1,4} & a^1_{1,2} & a^1_{1,3} \end{bmatrix}$

depending on the choice of total ordering of sheets above B_3'. Thus, (7-1) for B_3' gives $\epsilon(a^1_{1,4}) = 1$ as well, so that the last equality of (7-1) is

$$1 = \epsilon(b^3_{2,4}) + \epsilon(b^3_{3,4}).$$

Now, considering the 2×2 block consisting of the 3rd and 4th rows and columns of ∂C_1 (resp. ∂C_2) gives the equation

$$\epsilon(b^3_{2,4}) + \epsilon(b^1_{1,2}) = 0 \quad (\text{resp. } \epsilon(b^3_{3,4}) + \epsilon(b^2_{1,2}) = 0),$$

so $\epsilon(b^1_{1,2}) + \epsilon(b^2_{1,2}) = 1$ as claimed. (For instance, in the equation

$$\partial C_2 = A_{v_1} C_2 + C_2 A_{v_0} + T(I + B_2)(I + B_4) + (I + B_3),$$

note that the $(3,4)$–entry is zero in each of $T = I + E_{2,3}$, B_4, $A_{v_1} C$, and $C A_{v_0}$, since sheets S_3 and S_4 cross above B_4 and the A and C matrices are both strictly upper triangular.) □

With the claim in hand, we note that if $\mathcal{A}$ had an augmentation then from Proposition 6.4, there would exist an MC2F agreeing with ϵ on the 1–skeleton and hence having, at each vertex of F, an odd number of handleslide arcs crossing into F through the 1–cells B_1 and B_2. Since no handleslides can enter F along the crossing arcs that run along the edges of Γ, and F has an odd number of vertices, this means that in total there are an odd number of handleslide arcs entering the 2–sheeted region above F. But, this is impossible since these arcs would have to meet in pairs in the interior of the 2–sheeted region of F. □

Remark 7.2 For 1–dimensional Legendrian knots, it is shown in [23] that the category of constructible sheaves from [32] is equivalent to a category whose moduli space of objects consists of augmentations up to DGA homotopy. A close connection between constructible sheaves and augmentations is expected in general.

Proposition 1.2 of [34] shows that $L_\Gamma \subset J^1 S^2$ has a constructible sheaf defined over $\mathbb{Z}/2$ if and only if the dual graph to Γ is 3–colorable. When Γ is 3–valent this condition is equivalent to every face of Γ having an even number of vertices (see [28, Theorem 2-5]), so our Proposition 7.1 is consistent with the expected connection between constructible sheaves and augmentations. A more extensive study of the DGAs for the Treumann–Zaslow fronts, including results about augmentations implying Proposition 6.8, is made in the recent work of Casals and Murphy [3].

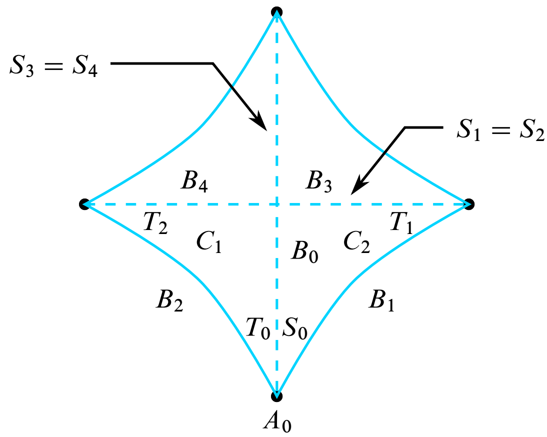

Figure 15: The resolution of a cone point, with labeling of cells and choice of S and T corners at swallowtail points as used in the proof of Proposition 7.3

7.2 The conormal of the unknot

The unit conormal bundle of the unknot is a Legendrian torus in the unit cotangent bundle $ST^*\mathbb{R}^3$ that, using a canonical contactomorphism $ST^*\mathbb{R}^3 \cong J^1S^2$, becomes a Legendrian $\Lambda_U \subset J^1S^2$. The front projection of Λ_U can be taken to be 2–sheeted with cone points at $(0,0,1)$ and $(0,0,-1)$ and no other singularities. A generic front diagram for Λ_U is obtained by perturbing the cone points to produce the configuration of four swallowtail points connected with cusps and crossings as pictured in Figure 15, as discussed in [10, Figure 7]. The four cusp arcs connect the middle two sheets labeled S_2 and S_3 above the cells inside the swallowtail region. The vertical (resp. horizontal) crossing arc is between sheets S_3 and S_4 (resp. sheets S_1 and S_2) and has its endpoints at two upward (resp. downward) swallowtail points. See [10] for more details.

Proposition 7.3 *The conormal of the unknot $\Lambda_U \subset J^1S^2$ does not have any linear-at-infinity generating family.*

Proof There is an obvious polygonal decomposition near the resolved cone point, and we label cells as in Figure 15. For any augmentation, the fiber homology $H(\epsilon_{x_0})$ can be computed from the complex associated to A_0, which is

$$V = \mathrm{Span}(S_1, S_2) \quad \text{with } dS_1 = 0,\ dS_2 = \epsilon(a^0_{1,2})S_1.$$

Thus, the result follows from Proposition 6.8(1) once we show that $\epsilon(a^0_{1,2}) = 0$ holds for any ϵ.

To this end, consider the differential of C_2, which (using initial and terminal vertices $v_0 = v_1 = A_0$) is

$$\partial C_2 = A_{v_0} C + C A_{v_1} + (I + B_0)(I + B_3)T_1(I + B_1)S_0 + I.$$

Since the matrices are upper triangular, the same equation holds when considering the upper-left 2×2 blocks; this 2×2 block is

$$\begin{bmatrix} 0 & \partial c^2_{1,2} \\ 0 & 0 \end{bmatrix} = 0 + 0 + (I + b^0_{1,2} E_{1,2})(I)(I + E_{1,2})(I)(I + a^0_{1,2} E_{1,2}) + I$$
$$= \begin{bmatrix} 0 & b^0_{1,2} + 1 + a^0_{1,2} \\ 0 & 0 \end{bmatrix}.$$

Thus, $\epsilon \circ \partial = 0$ implies

$$\epsilon(b^0_{1,2}) = 1 + \epsilon(a^0_{1,2}). \tag{7-2}$$

(The $(1,2)$–entry of the matrices B_3 and B_1 are zero because of a crossing and cusp arc respectively.)

Similarly, considering the upper-left 2×2 block of

$$\partial C_1 = A_{v_0} C + C A_{v_1} + (I + Q B_0 Q)(I + B_4) T_2 (I + B_2) T_0 + I$$

(the matrix $Q = Q_{3,4}$ is the permutation matrix for $(3\,4)$) gives

$$\epsilon(b^0_{1,2}) = 1. \tag{7-3}$$

Thus, the required equality $\epsilon(a^0_{1,2}) = 0$ follows from comparing (7-2) and (7-3). □

This result also follows from [6, Proposition 3.2], which proves a local obstruction for linear-at-infinity generating families for cone points. This alternative approach shows with a direct calculation that the fiber chain complex $\mathcal{C}_{x_0}$, where x_0 is the intersection of the two crossing loci in Figure 15, cannot be acyclic.

Remark 7.4 It is interesting to note that any generic 1–dimensional slice of Λ_U *does* admit a linear-at-infinity generating family. Indeed, pulling the front projection of Λ_U back along an immersion $f\colon S^1 \to S^2$ that is transverse to the base projection of the singular set produces a Legendrian $\Lambda_f \subset J^1 S^1$. The front projection of any Λ_f has a graded normal ruling obtained from taking all crossings to be switches, ie above the 4–sheeted region the middle two sheets are paired as are the outer two sheets. See [24] or [25] for a discussion of normal rulings in $J^1 S^1$; the proof of equivalence of the existence of graded normal rulings and linear-at-infinity generating families from [15] continues to hold in the J^1S^1 setting since away from crossings and cusps the generating families constructed in [15, Section 3] have a standard form depending only on the pairing of sheets.

Remark 7.5 As an alternative approach, the definition of MC2F and main results of this paper can all be extended to allow fronts with cone-point singularities using the extension of the cellular DGA to such fronts given in [26, Section 5.3]. The definition of MC2F for a Legendrian $L \subset J^1M$ with cone points has the following additions:

Let $R_\nu \subset M \setminus \Sigma_{\mathcal{C}}$ be a region that borders a cone point between sheets S_k and S_{k+1}. Then

(1) $\langle d_\nu S_{k+1}, S_k \rangle = 0$, and
(2) for any $i < k$ (resp. $k+1 < j$), there are $\langle d_\nu S_{k+1}, S_i \rangle$ (i,k)–handleslide arcs (resp. $\langle d_\nu S_j, S_k \rangle$ $(k+1, j)$–handleslide arcs) with endpoints at the cone point.

7.3 An example obstructing a trivial bundle domain

To illustrate the obstruction from Proposition 6.8(2), consider a nonseparating curve $\gamma \subset T^2$. There is a corresponding Legendrian $L_\gamma \subset J^1T^2$ with 2–sheeted front projection having a crossing arc above γ and no other crossings or cusps.

Proposition 7.6 *There is no tame generating family $F\colon E \to \mathbb{R}$ for L_γ whose domain is a trivial bundle over T^2.*

Proof To apply Proposition 6.8, we must show that any augmentation ϵ has nontrivial monodromy representation, Φ_{ϵ,x_0}. Let $\mathcal{C}$ be an MC2F that agrees with the corresponding CHD, $\mathcal{D} \leftrightarrow \epsilon$, on the 1–skeleton. Take x_0 to be slightly shifted off of γ, and σ a loop based at x_0 that intersects γ geometrically once just before its endpoint. The chain level continuation map for $\mathcal{C}$ has matrix of the form

$$f(\sigma) = Q(I + E_{1,2})^n = \begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix},$$

where n is the number of handleslide arcs that σ encounters and Q is the permutation matrix for $(1\,2)$. The differential from $\mathcal{C}$ at x_0 vanishes (via Observation 4.4(2)), so we conclude that $f(\sigma)$ induces a nonidentity map on homology, ie $\Phi_{\epsilon,x_0}([\sigma]) \neq 1$. □

Note that L_γ does have an obvious generating family whose domain is a nontrivial 2–fold cover of T^2.

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Department of Mathematical Sciences, Ball State University
Muncie, IN, United States

Department of Mathematics and Statistics, University of Massachusetts
Amherst, MA, United States

`rutherford@bsu.edu`, `sullivan@math.umass.edu`

Received: 5 April 2017 Revised: 17 December 2017