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Algebraic ending laminations and quasiconvexity

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We explicate a number of notions of algebraic laminations existing in the literature, particularly in the context of an exact sequence

$$1 \to H \to G \to Q \to 1$$

of hyperbolic groups. These laminations arise in different contexts: existence of Cannon–Thurston maps; closed geodesics exiting ends of manifolds; dual to actions on $\mathbb{R}$–trees.

We use the relationship between these laminations to prove quasiconvexity results for finitely generated infinite-index subgroups of H, the normal subgroup in the exact sequence above.

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1 Introduction

1.1 Statement of results

The main results in this paper establish that for certain naturally occurring distorted (in the sense of Gromov [21]) hyperbolic subgroups H of hyperbolic groups G, many quasiconvex subgroups K of H are in fact quasiconvex in the larger hyperbolic group G. The following, one of the main theorems of this paper, illustrates this.

Theorem 1.1 (see Theorems 4.7 and 5.14) *Let*

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of hyperbolic groups, where H is either a free group or a (closed) surface group and Q is convex cocompact in outer space or Teichmüller space, respectively (for the free group, we assume further that Q is purely hyperbolic). Let K be a finitely generated infinite-index subgroup of H. Then K is quasiconvex in G.

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The (original motivating) case where H is a closed surface group and $Q=\mathbb{Z}$ in Theorem 1.1 was dealt with by Scott and Swarup [48]. The more general case of H a closed surface group (and no further restrictions on Q) was recently obtained by Dowdall, Kent and Leininger [15] using different methods. In [16], the preprint of which appeared shortly after a first version of the present paper was made public, Dowdall and Taylor use the methods of [17] on convex cocompact purely hyperbolic subgroups of $\mathrm{Out}(F_n)$ to give a substantially different proof of Theorem 1.1 when H is free.

For the statement of our next theorem, some terminology needs to be introduced. A Teichmüller geodesic ray r ($\subset \mathrm{Teich}(S)$) is said to be *thick* (see Minsky [32; 33; 34] and Rafi [45]) if r lies in the thick part of Teichmüller space, ie there exists $\epsilon > 0$ such that for all $x \in r$, the length of the shortest closed geodesic (or equivalently, injectivity radius for closed surfaces) on the hyperbolic surface S_x corresponding to $x \in \mathrm{Teich}(S)$ is bounded below by ϵ. It follows (from Masur and Minsky [31], Minsky [34] and Rafi [45]) that the projection of r to the curve complex is a parametrized quasigeodesic, and the universal curve U_r over r (associating S_x to x and equipping the resulting bundle with an infinitesimal product metric) has a hyperbolic universal cover $\widetilde{U}_r$ [33; 34]. To emphasize this hyperbolicity we shall call these geodesic rays *thick hyperbolic rays*. We shall refer to $\widetilde{U}_r$ as the *universal metric bundle* (of hyperbolic planes) over r.

Analogously, we define a geodesic ray r in Culler–Vogtmann outer space cv_n [13] to be *thick hyperbolic* if

(1) r projects to a parametrized quasigeodesic in the free factor complex $\mathcal{F}_n$,

(2) the bundle of trees X over r (thought of as a metric bundle; see Mj and Sardar [42] and Section 2.2 below) is hyperbolic.

In this case too, we shall refer to X as the universal metric bundle (of trees) over r.

Theorem 1.2 (see Theorems 4.6 and 5.15) *Let r be a thick hyperbolic quasi-geodesic ray either*

(1) *in* $\mathrm{Teich}(S)$ *for S a closed surface of genus greater than* 1*, or*

(2) *in the outer space* cv_n *corresponding to* F_n*.*

Let X be the universal metric bundle of hyperbolic planes or trees (respectively) over r. Let H denote respectively $\pi_1(S)$ or F_n, and let $i\colon H \to X$ denote an orbit map. Let K be a finitely generated infinite-index subgroup of H. Then $i(K)$ is quasiconvex in X.

The following theorem generalizes the closed surface cases of Theorems 1.1 and 1.2 to surfaces with punctures.

Theorem 1.3 (see Theorems 6.4 and 6.1) *Let $H = \pi_1(S^h)$ for S^h a hyperbolic surface of finite volume. Let r be a thick hyperbolic ray in Teichmüller space $\mathrm{Teich}(S^h)$, and let $r_\infty \in \partial\,\mathrm{Teich}(S^h)$ be the limiting surface ending lamination. Let X denote the universal metric bundle over r minus a small neighborhood of the cusps, and let $\mathcal{H}$ denote the horosphere boundary components. Let K be a finitely generated infinite-index subgroup of H. Then any orbit of K in X is relatively quasiconvex in $(X, \mathcal{H})$.*

Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures, and let $H_1, \dots, H_n$ be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h. Let

$$1 \to H \to G \to Q \to 1 \qquad \text{and} \qquad 1 \to H_i \to N_G(H_i) \to Q \to 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}$, $i = 1, \dots, n$.

Let K be a finitely generated infinite-index subgroup of H. Then K is relatively quasiconvex in G.

The first part of the second statement in Theorem 1.3 is from [42]. The relative quasiconvexity part of the second statement (which requires relative hyperbolicity as its framework) is what is new.

1.2 Techniques

The main technical tool used to establish the above theorems is the theory of laminations. A guiding motif that underlies much of this paper is that the directions of maximal distortion for a hyperbolic group H acting on a hyperbolic metric space X are encoded in a lamination. Hence if the set of such laminations supported on a subgroup K of H is empty, we should expect that the subgroup K is undistorted in X, or equivalently, quasiconvex in X. Unfortunately, there are a number of competing notions of laminations existing in the literature, and they do not all serve the same purpose. To make this philosophy work therefore, we need to investigate the relationships between these different kinds of laminations.

The weakest notion is that of an *algebraic lamination* for a hyperbolic group H: an H–invariant, flip-invariant, closed subset

$$\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H) \setminus \Delta,$$

where Δ denotes the diagonal in $\partial H \times \partial H$; see Bestvina, Feighn and Handel [4], Coulbois, Hilion and Lustig [9], Kapovich and Lustig [26; 27] and Mitra [35].

Several classes of algebraic laminations have come up in the study of automorphisms of hyperbolic groups, especially free and surface groups:

(1) The dual lamination $\Lambda_{\mathbb{R}}$ arising from an action of H on an $\mathbb{R}$–tree; see Bestvina, Feighn and Handel [4], Coulbois, Hilion and Lustig [9; 10], Kapovich and Lustig [26] and Thurston [50]. See Definition 3.18, which allows us to make sense of this for the action of any hyperbolic group H on an $\mathbb{R}$–tree.

(2) The ending lamination Λ_{EL} or Λ_{GEL} arising from closed geodesics exiting an end of a 3–manifold [50] (see also [35] for an algebraization of this concept). In the (group-theoretic) context of this paper, Λ_{EL} or Λ_{GEL} is defined using Gromov–Hausdorff limits following [35] rather than projectivized measured lamination space as in [50]. Thus, Λ_{EL} or Λ_{GEL} may be intuitively described as Hausdorff limits of closed curves whose geodesic realizations exit an end. For normal hyperbolic subgroups H of hyperbolic groups G, we have $\Lambda_{\mathrm{EL}} = \Lambda_{\mathrm{GEL}}$ [35].

(3) The Cannon–Thurston lamination Λ_{CT} arising in the context of the existence of a Cannon–Thurston map; see Cannon and Thurston [8] and Mitra [35].

Note that the above three notions make sense in the rather general context of a hyperbolic group H. These different kinds of laminations play different roles:

(1) The dual lamination $\Lambda_{\mathbb{R}}$ often has good mixing properties like arationality [50] or minimality (see Coulbois, Hilion and Reynolds [12]) or the dual notion of indecomposability for the dual $\mathbb{R}$–tree (see Guirardel [22]).

(2) The Cannon–Thurston laminations Λ_{CT} play a role in determining quasiconvexity of subgroups; see Mitra [37] and Scott and Swarup [48]. See Lemma 3.4 below.

(3) The above two quite different contexts are mediated by ending laminations Λ_{EL} in the following sense. Theorem 3.10 [35] equates Λ_{EL} with Λ_{CT} in the general context of hyperbolic normal subgroups of hyperbolic groups. The relationship between Λ_{EL} and $\Lambda_{\mathbb{R}}$ has not been established in this generality. It is known however for surface groups (see [34] and [35]) and free groups (see Dowdall, Kapovich and Taylor [14]) in the context of convex cocompact subgroups of the mapping class group or $\mathrm{Out}(F_n)$. It is this state of the art with respect to the relationship between Λ_{EL} and $\Lambda_{\mathbb{R}}$ that forces us to restrict ourselves to surface groups and free groups in this paper.

We give a few forward references to indicate how Λ_{EL} mediates between Λ_{CT} and $\Lambda_{\mathbb{R}}$ and also sketch the strategy of proof of the main results. It is easy to see that in various natural contexts the collection of ending laminations Λ_{EL} or Λ_{GEL} are contained in the collection of Cannon–Thurston laminations Λ_{CT} (Proposition 3.13) as well as in the dual laminations $\Lambda_{\mathbb{R}}$ (Proposition 5.8). Further, the (harder) reverse containment of Λ_{CT} in Λ_{EL} has been established in a number of cases (Theorem 3.10 from [35] for instance). What remains is to examine the reverse containment of $\Lambda_{\mathbb{R}}$ in Λ_{EL} in order to complete the picture. This is the subject of [14] and [27] in the context of free groups and [39] by Mj in the context of surface Kleinian groups.

What kicks in after this are the mixing properties of $\Lambda_{\mathbb{R}}$ established by various authors. Arationality of ending laminations for surface groups was established in [50], and arationality in a strong form for free groups was established by Bestvina and Reynolds [5], Coulbois, Hilion and Reynolds [12], Guirardel [22] and Reynolds [46; 47]. It follows from these results that Λ_{CT} is arational in a strong sense — no leaf of Λ_{CT} is contained in a finitely generated infinite-index subgroup K of H for various specific instances of H. Quasiconvexity of K in G (or more generally some hyperbolic metric bundle X) then follows from Lemma 3.4. Accordingly, each of the Sections 4, 5 and 6 has two subsections each: one establishing arationality and the second combining arationality along with the general theory of Section 3 to prove quasiconvexity.

2 Cannon–Thurston maps and metric bundles

2.1 Cannon–Thurston maps

Suppose that H is a hyperbolic subgroup of a hyperbolic group G or that H is a group acting properly on a hyperbolic metric space X. Let Γ_H and Γ_G denote Cayley graphs of H and G with respect to finite generating sets. Assume that the finite generating set for H is contained in that of G. Let $\widehat{\Gamma}_H$, $\widehat{\Gamma}_G$ and $\widehat{X}$ denote the Gromov compactifications. Further, let ∂H, ∂G and ∂X denote the boundaries [20]. (It is a fact that the boundaries $\partial\Gamma_H$ and $\partial\Gamma_G$ of the corresponding Cayley graphs are independent of the finite generating sets chosen; hence we use the symbols ∂H and ∂G).

Definition 2.1 Let H be a hyperbolic subgroup of a hyperbolic group G (resp. acting properly on a hyperbolic metric space X). Let Γ_H and Γ_G denote Cayley graphs of H and G as above.

Let $i\colon \Gamma_H \to \Gamma_G$ (resp. $i\colon \Gamma_H \to X$) denote the inclusion map (resp. an orbit map of H extended by means of geodesics over edges).

A Cannon–Thurston map for the pair (H,G) (resp. (H,X)) is said to exist if there exists a continuous extension of i to $\hat{\imath}\colon \widehat{\Gamma}_H \to \widehat{\Gamma}_G$ (resp. $\hat{\imath}\colon \widehat{\Gamma}_H \to \widehat{X}$). The restriction $\partial i\colon \partial H \to \partial G$ (resp. $\partial i\colon \partial H \to \partial X$) of $\hat{\imath}$ is then called the Cannon–Thurston map for the pair (H,G) (resp. (H,X)).

Theorem 2.2 [36] *Let G be a hyperbolic group and let H be a hyperbolic normal subgroup of G. Then a Cannon–Thurston map exists for the pair (H,G).*

2.2 Metric bundles

To state a theorem analogous to Theorem 2.2 in the more general geometric (not necessarily group-invariant) setting of a metric bundle, some material needs to be summarized from [42].

Definition 2.3 Suppose (X,d) and (B,d_B) are geodesic metric spaces; let $c \geq 1$ be a constant, and let $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a function. We say that X is an *(f,c)–metric bundle* over B if there is a surjective 1–Lipschitz map $p\colon X \to B$ such that the following conditions hold:

(1) For each point $z \in B$, the preimage $F_z := p^{-1}(z)$ is a geodesic metric space with respect to the path metric d_z induced from X. The inclusion maps $i\colon (F_z,d_z) \to X$ are uniformly metrically proper as measured by f, ie $d(i(x),i(y)) \leq N$ implies that $d_z(x,y) \leq f(N)$ for all $z \in B$ and $x,y \in F_z$.

(2) Suppose $z_1,z_2 \in B$ with $d_B(z_1,z_2) \leq 1$, and let γ be a geodesic in B joining them. Then for any point $x \in F_z$ for $z \in \gamma$, there is a path in $p^{-1}(\gamma)$ of length at most c joining x to both F_{z_1} and F_{z_2}. It follows that there exists $K = K(f,c) \geq 1$ such that the following holds: Suppose $z_1,z_2 \in B$ with $d_B(z_1,z_2) \leq 1$ and let γ be a geodesic in B joining them. Let $\phi\colon F_{z_1} \to F_{z_2}$ be any map such that for all $x_1 \in F_{z_1}$, there is a path of length at most c in $p^{-1}(\gamma)$ joining x_1 to $\phi(x_1)$. Then ϕ is a K–quasi-isometry.

We now describe the two kinds of metric bundles that will concern us in this paper. First, let r be a geodesic (or more generally a quasigeodesic) ray in $\mathrm{Teich}(S)$ for S a closed surface of genus greater than 1. Then the universal bundle over $\mathrm{Teich}(S)$

restricted to r, say U_r, has a natural metric. Through any point $x \in S_z$, the fiber over $z \in \mathrm{Teich}(S)$, there is a canonical isometric lift of r. By declaring these lifts to be orthogonal to S_z at every such point x equips U_r with the natural metric. The universal cover of U_r with the lifted metric is then the required metric bundle over r. The fiber F_z over z is the universal cover of S_z.

The (unprojectivized) Culler–Vogtmann outer space corresponding to F_n will be denoted by cv_n [13] and its boundary by $\partial \mathrm{cv}_n$. We describe the metric bundle over a ray r in cv_n. For our purposes we shall require that r is a folding path [3]. It is proved in [3, Proposition 2.5] that for any $z \in \mathrm{cv}_n$, there is a point z' at uniformly bounded distance from z such that a geodesic ray starting at z may be constructed as a concatenation of a geodesic segment from z to z' followed by a folding path starting at z'. Thus, for our purposes, up to changing the initial point of r by a uniformly bounded amount, we might as well assume that r is a folding path. The universal (marked) graph bundle over cv_n restricted to r, say U_r, is, as before, equipped with a natural metric by lifting r isometrically to geodesic rays through points in fibers. The universal cover of this bundle of graphs with the lifted metric is the metric bundle over r in this situation. Note that since folding paths define maps between fibers over two points in a natural way, the resulting metric bundle comes canonically equipped with an action of a free group acting fiberwise.

The next theorem establishes the existence of a Cannon–Thurston map in this setting:

Theorem 2.4 [42, Theorem 5.3] *Let r be one of the following:*

(1) *a thick hyperbolic quasigeodesic ray in* $\mathrm{Teich}(S)$ *for S a closed surface of genus greater than* 1,

(2) *a folding path in the outer space* cv_n *corresponding to F_n.*

Let X be the universal metric bundle of hyperbolic planes or trees (respectively) over r, and suppose that X is hyperbolic. Let H denote respectively $\pi_1(S)$ or F_n. Then the pair (H, X) has a Cannon–Thurston map.

The paper [42] deals with a somewhat more general notion (referred to in that work as a metric graph bundle) than the one covered by Definition 2.3. However, for the purposes of this paper, it suffices to consider the more restrictive notion of a metric bundle given by 2.3. Theorem 2.4 in the form that we shall apply it will require only the restricted notion of Definition 2.3.

3 Laminations

An *algebraic lamination* [4; 9; 26; 27; 35] for a hyperbolic group H is an H–invariant, flip-invariant, nonempty closed subset

$$\mathcal{L} \subseteq \partial^2 H = (\partial H \times \partial H) \setminus \Delta,$$

where Δ is the diagonal in $\partial H \times \partial H$ and the flip is given by $(x, y) \sim (y, x)$. Here ∂H is equipped with the Gromov topology, and $\partial^2 H$ with the subspace topology of the product topology. (Note that in [20], the notation $\partial^2 H$ is reserved for $(\partial H \times \partial H \setminus \Delta)/\sim$. We prefer to use the notation here as we shall generally be dealing with bi-infinite geodesics rather than unordered pairs of points on ∂H.) Various classes of laminations exist in the literature and in this section, we describe three such classes that arise naturally.

3.1 Cannon–Thurston laminations

In this section we shall define laminations in the context of a hyperbolic group H acting properly on a hyperbolic metric space X. For instance, X could be a Cayley graph of a hyperbolic group G containing H. We choose, as before, a generating set of H, and in the case when X is a Cayley graph of a hyperbolic supergroup G, we assume that the generating set of H is extended to one of G, ensuring a natural inclusion map $i\colon \Gamma_H \to \Gamma_G$. Choosing a basepoint $*$, the orbit map from the vertex set of H to X which sends h to $h*$ will be denoted by i. Further, i is extended to the edges of Γ_H by sending them to geodesic segments in X. The laminations we consider in this section go back to [35] and correspond intuitively to (limits) of geodesic segments in H whose geodesic realizations in X live outside large balls about a basepoint.

We recall some basic facts and notions; see [35; 37]. If λ is a geodesic segment in Γ_H, a *geodesic realization* λ^r, of λ, is a geodesic in X joining the endpoints of $i(\lambda)$.

Let $\{\lambda_n\}_n \subset \Gamma_H$ be a sequence of geodesic segments such that $1 \in \lambda_n$ and $\lambda_n^r \cap B(n) = \varnothing$, where $B(n)$ is the ball of radius n around $i(1) \in X$. Take all bi-infinite subsequential limits of pairs of endpoints of all such sequences $\{\lambda_i\}$ (in the product topology on $\widehat{\Gamma}_H \times \widehat{\Gamma}_H$) and denote this set by $\mathcal{L}_0$. Let t_h denote left translation by $h \in H$.

Definition 3.1 The Cannon–Thurston prelamination $\Lambda_{\rm CT} = \Lambda_{\rm CT}(H, X)$ is given by

$$\Lambda_{\rm CT} = \big\{\{p, q\} \in \partial^2 H : p, q \text{ are the endpoints of } t_h(\lambda) \text{ for some } \lambda \in \mathcal{L}_0\big\}.$$

For this definition of Λ_{CT}, one does not need the existence of a Cannon–Thurston map. However, this Λ_{CT} is not yet a lamination as closedness is not guaranteed (as was pointed out to us by the referee); hence the expression prelamination. In the presence of a Cannon–Thurston map, Λ_{CT} is indeed a lamination, and we have an alternate description of Λ_{CT} as follows.

Definition 3.2 Suppose that a Cannon–Thurston map exists for the pair (H,X). Define

$$\Lambda^1_{\mathrm{CT}} = \big\{\{p,q\} \in \partial^2 H : \hat{\imath}(p) = \hat{\imath}(q)\big\}.$$

Lemma 3.3 [37] *If a Cannon–Thurston map exists, $\Lambda_{\mathrm{CT}} = \Lambda^1_{\mathrm{CT}}$ is a lamination.*

Note that closedness of Λ_{CT} follows from continuity of the Cannon–Thurston map. The following lemma characterizes quasiconvexity in terms of Λ_{CT}.

Lemma 3.4 [37] *H is quasiconvex in X if and only if $\Lambda_{\mathrm{CT}} = \varnothing$.*

We shall be requiring a generalization of Lemma 3.4 to relatively hyperbolic groups [20; 18; 7]. Let H be a relatively hyperbolic group, hyperbolic relative to a finite collection of parabolic subgroups $\mathcal{P}$. The relatively hyperbolic (or Bowditch) boundary $\partial(H,\mathcal{P}) = \partial_r H$ of the relatively hyperbolic group $(H,\mathcal{P})$ was defined by Bowditch [7]. The collection of bi-infinite geodesics $\partial^2_r H$ is given by $(\partial_r H \times \partial_r H) \setminus \Delta$ as usual. The existence of a Cannon–Thurston map in this setting of a relatively hyperbolic group H acting on a relatively hyperbolic space $(X,\mathcal{H})$ was investigated in [6; 38; 41]. Such an H acts in a *strictly type-preserving* manner on a relatively hyperbolic space $(X,\mathcal{H})$ if the stabilizer $\mathrm{Stab}_H(Y)$ for any $Y \in \mathcal{H}$ is equal to a conjugate of an element of $\mathcal{P}$ and if each conjugate of an element of $\mathcal{P}$ stabilizes some $Y \in \mathcal{H}$. The notion of the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}} = \Lambda_{\mathrm{CT}}(H,X)$ is defined as above to be the set of pairs of distinct points $\{x,y\} \in \partial^2_r H$ identified by the Cannon–Thurston map. The proof of Lemma 3.4 from [37] directly translates to the following in the relatively hyperbolic setup. We refer the reader to [25] for the definition of relative quasiconvexity.

Lemma 3.5 *Suppose that the relatively hyperbolic group $(H,\mathcal{P})$ acts in a* ***strictly type-preserving*** *manner on a relatively hyperbolic space $(X,\mathcal{H})$ such that the pair (H,X) has a Cannon–Thurston map. Let $\Lambda_{\mathrm{CT}} = \Lambda_{\mathrm{CT}}(H,X)$. Then any orbit of H is relatively quasiconvex in X if and only if $\Lambda_{\mathrm{CT}} = \varnothing$.*

Remark 3.6 We include an observation as to what happens when we pass to quasiconvex or relatively quasiconvex subgroups. Let K (resp. $(K, \mathcal{P}_1)$) be a quasiconvex (resp. relatively quasiconvex) subgroup of a hyperbolic (resp. relatively hyperbolic) group H (resp. $(H, \mathcal{P})$). Then the boundary ∂K (resp. $\partial_r K$) embeds in ∂H (resp. $\partial_r H$). This induces an embedding of $\partial^2 K$ (resp. $\partial_r^2 K$) in $\partial^2 H$ (resp. $\partial_r^2 H$).

It therefore follows that if H acts geometrically on a hyperbolic metric space X (resp. $(H, \mathcal{P})$ acts in a strictly type-preserving manner on a relatively hyperbolic space $(X, \mathcal{H})$) such that the pair (H, X) has a Cannon–Thurston map, then the pair (K, X) has a Cannon–Thurston map given by a composition of the embedding of ∂K into ∂H (resp. $\partial_r K$ into $\partial_r H$) followed by the Cannon–Thurston map from ∂H to ∂X (resp. $\partial_r H$ into $\partial_r X$). Further,

$$\Lambda_{\mathrm{CT}}(K, X) = \Lambda_{\mathrm{CT}}(H, X) \cap \partial^2 K \quad (\text{resp. } \Lambda_{\mathrm{CT}}(K, X) = \Lambda_{\mathrm{CT}}(H, X) \cap \partial_r^2 K),$$

where the intersection is taken in $\partial^2 H$ (resp. $\partial_r^2 H$).

Since all finitely generated infinite-index subgroups K of free groups and surface groups are quasiconvex (resp. relatively quasiconvex), this applies, in particular, when H is a free group or a surface group.

3.2 Algebraic ending laminations

In [35], the first author gave a different, more group theoretic description of ending laminations motivated by Thurston's description in [50]. Thurston's description uses a transverse measure which is eventually forgotten [29; 5], whereas the approach in [35] uses Hausdorff limits and is purely topological in nature. We rename the ending laminations of [35] *algebraic ending laminations* to emphasize the difference.

Thus some of the topological aspects of Thurston's theory of ending laminations were generalized to the context of normal hyperbolic subgroups of hyperbolic groups and used to give an explicit description of the continuous boundary extension $\hat{\imath}\colon \widehat{\Gamma}_H \to \widehat{\Gamma}_G$ occurring in Theorem 2.2.

Let

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of finitely presented groups where H, G and hence Q (from [43]) are hyperbolic. In this setup one has algebraic ending laminations (defined below) naturally parametrized by points in the boundary ∂Q of the quotient group Q.

Corresponding to every element $g \in G$ there exists an automorphism of H taking h to $g^{-1}hg$ for $h \in H$. Such an automorphism induces a bijection ϕ_g of the vertices of Γ_H. This gives rise to a map from Γ_H to itself, sending an edge $[a,b]$ linearly to a shortest edge-path joining $\phi_g(a)$ to $\phi_g(b)$.

Fix $z \in \partial Q$, and let $[1,z)$ be a geodesic ray in Γ_Q starting at the identity 1 and converging to $z \in \partial Q$. Let σ be a single-valued quasi-isometric (qi) section of Q into G. The existence of such a qi-section σ was proved by Mosher [43]. Let z_n be the vertex on $[1,z)$ such that $d_Q(1,z_n) = n$, and let $g_n = \sigma(z_n)$.

Next, fix $h \in H$. A geodesic segment $[a,b] \subset \Gamma_H$ will be called a *free homotopy representative* (or shortest representative in the same conjugacy class) of h, if

1. $a^{-1}b$ is conjugate to h *in* H,
2. the length of $[a,b]$ is shortest amongst all such conjugates of h in H.

Let $\mathcal{L}_0^h$ be the (H–invariant) collection of all free homotopy representatives of h in Γ_H. Intuitively, $\mathcal{L}_0^h$ can be thought of as the collection of all geodesic segments in Γ_H that are lifts of shortest closed geodesics in Γ_H/H in the same conjugacy class as h (in the setting of a closed manifold of negative curvature, these would be geodesic segments that are path-lifts of the unique closed geodesic in the free homotopy class of a closed loop corresponding to h). Identifying equivalent geodesics (ie geodesics sharing the same set of endpoints) in $\mathcal{L}_0^h$ one obtains a subset L_0^h of (ordered) pairs of points in $\widehat{\Gamma}_H$. Next, let $\mathcal{L}_n^h$ be the (H–invariant) collection of all free homotopy representatives of $\phi_{g_n^{-1}}(h)$ ($= g_nhg_n^{-1}$) in Γ_H. Again, identifying equivalent geodesics in $\mathcal{L}_n^h$ one obtains a subset L_n^h of (ordered) pairs of points in $\widehat{\Gamma}_H$.

See Figure 1, where the long vertical arrow on the right depicts the geodesic ray $[1,z)$ in Γ_Q. We assume that h is chosen to be a free homotopy representative of itself. The corresponding path is assumed to lie in the translate (or alternately, coset) $g_n\Gamma_H$. Then $g_nhg_n^{-1}$ is a path starting and ending in Γ_H and we pass to its free homotopy representative in Γ_H to get an element of $\mathcal{L}_n^h$. It is important to note that elements of $\mathcal{L}_n^h$ are geodesics in Γ_H, but not in Γ_G. What we are intuitively doing here is looking at a closed loop σ_h based at g_n in Γ_G/H corresponding to h and sitting over $z_n \in [1,z)$. We then concatenate in order:

1. Start with a path σ_n from 1 to g_n. The word in G corresponding to σ_n is g_n.
2. This is followed by σ_h. The word in G corresponding to σ_h is h.
3. This is followed by $\overline{\sigma}_n$ (the "opposite" path to σ_n). The word in G corresponding to $\overline{\sigma}_n$ is g_n^{-1}.

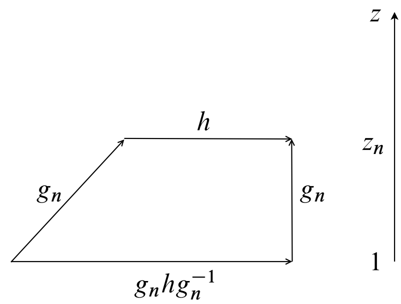

Figure 1

This gives a loop based at $1 \in \Gamma_G/H$, and after "homotoping" it back to Γ_H/H and "tightening", we get a free homotopy representative.

Definition 3.7 The intersection with $\partial^2 H$ of the union of all subsequential limits (in the product topology on $\widehat{\Gamma}_H \times \widehat{\Gamma}_H$) of $\{L_n^h\}$ is denoted by Λ_{0z}^h. It is clear that Λ_{0z}^h and $\Lambda_{0z}^{h^{-1}}$ are related by the flip.

The *algebraic ending prelamination* corresponding to $z \in \partial\Gamma_Q$ is given by

$$\Lambda_{\mathrm{EL}}^z = \bigcup_{h \in H} \Lambda_{0z}^h.$$

The *algebraic ending lamination* corresponding to $z \in \partial\Gamma_Q$ is given by the closure $\overline{\Lambda_{\mathrm{EL}}^z}$.

We indicate the slight modification to the above definition necessary to make it work for a hyperbolic metric bundle X over a ray $[0,\infty)$, with fibers universal covers of (metric) surfaces or graphs as in Section 2.2. One prefers to think of the vertex spaces as corresponding to integers and edge spaces corresponding to intervals $[n-1,n]$, where $n \in \mathbb{N}$. Let $\sigma\colon [0,\infty) \to X$ be a qi-section [42, Proposition 2.10] through the identity element in the fiber H_0 over 0. The fiber H_n over n is acted upon cocompactly by a (surface or free) group H. Further, thickness of the ray guarantees that the quotient of each fiber by H is of *uniformly* bounded diameter. Each H_n contains a preferred set of points (vertices) given by the H–orbit of $\sigma(n)$. For $n \in \mathbb{N}$ and $x \in H\sigma(n)$, there exists a unique H–translate σ_x of $\sigma([0,\infty))$ through x. Since H_n/H is of uniformly bounded diameter (independent of n) it makes sense to consider the (H–invariant) collection $\mathcal{L}_n^D$ of all free homotopy representatives of $\sigma([0,n])[x_n,y_n]\overline{h_n\sigma([0,n])}$, where $x_n = \sigma(n)$, $y_n \in H\sigma(n)$, $d_X(x_n,y_n) \le D$ and $\overline{h_n\sigma([0,n])}$ denotes the translate of $\sigma([0,n])$ through y_n with reverse orientation. As before, this gives a subset L_n^D

of (ordered) pairs of points in $\widehat{H}_0$. The intersection with $\partial^2 H_0$ of the union of all subsequential limits of $\{L_n^D\}$ is denoted by Λ^D. Note that Λ^D is invariant under the flip here.

Definition 3.8 The *algebraic ending prelamination* corresponding to $z = r(\infty)$ is given by

$$\Lambda_{\mathrm{EL}}^z = \bigcup_{D \in \mathbb{N}} \Lambda^D,$$

and the *algebraic ending lamination* is given by the closure $\overline{\Lambda_{\mathrm{EL}}^z}$. We let

$$\Lambda_{\mathrm{EL}} = \overline{\Lambda_{\mathrm{EL}}^z}.$$

(Here the superscript z is initially used for the sake of consistency with the notation in Definition 3.7 and then dropped to be consistent with Definition 3.9.)

In Definition 3.7 above we have followed [35]. As was pointed to us by the referee, the fact that we are choosing free homotopy representatives and shortest representatives implies that we are in fact applying $\phi_{g_n^{-1}}$ to the conjugacy class $[h]$ rather than h itself. However, once we have applied $\phi_{g_n^{-1}}$ to $[h]$, we need to choose shortest representatives and their cyclic permutations in order to extract subsequential limits. We have made the choice here so that we can quote Theorem 3.10 directly from [35]. Further, the generalization to Definition 3.8 becomes natural with this choice.

Note also that Λ_{0z}^h and Λ^D are indeed closed as we are taking all subsequential limits. However, closedness may be destroyed when we take the union over all h (or D); hence the term prelamination. By Theorem 3.10, Λ_{EL}^z is actually a lamination in the context we are interested in.

We explain the convention of using $\phi_{g_n^{-1}}$ in the motivating case of the cover of a hyperbolic 3–manifold fibering over the circle [50] corresponding to the fiber S. The group Q is $\mathbb{Z}$ here and the fiber over n is denoted by $S \times \{n\}$. Here h should be thought of as (a lift of) a bounded length curve σ on $S \times \{n\}$. Also $\phi_{g_n^{-1}}(h)$ may be thought of in this case as (a lift of) the closed geodesic on $S \times \{0\}$ freely homotopic to σ. The ending lamination in this situation is obtained by taking limits of such closed geodesics in a suitable topology (which is not important for us here).

Definition 3.9 The *algebraic ending lamination* Λ_{EL} for the triple (H, G, Q) is defined by

$$\Lambda_{\mathrm{EL}} = \Lambda_{\mathrm{EL}}(H, G, Q) = \bigcup_{z \in \partial \Gamma_Q} \overline{\Lambda_{\mathrm{EL}}^z}.$$

It follows from [35] that Λ_{EL} is in fact closed and hence an algebraic lamination in our sense. The main theorem of that work equates Λ_{EL} and Λ_{CT}.

Theorem 3.10 [35] $\Lambda_{\mathrm{EL}}(H, G, Q) = \Lambda_{\mathrm{EL}} = \Lambda_{\mathrm{CT}} = \Lambda_{\mathrm{CT}}(H, G)$.

We shall be needing a slightly modified version of Theorem 3.10 later, when we consider hyperbolic metric bundles X over rays $[0, \infty)$, with fibers universal covers of (metric) surfaces or graphs as in Section 2.2. We note here that the proof in [35] goes through in this case, too, with small modifications. We outline the steps of that proof here and indicate the technical modifications from [42].

Proposition 3.11 *Let X be a hyperbolic metric bundle over a ray $[0, \infty)$, with fibers universal covers of (metric) surfaces or graphs as in Section 2.2 and let H be the associated surface or free group. Let Λ_{EL} denote the algebraic ending lamination from Definition 3.8. Then $\Lambda_{\mathrm{EL}} = \Lambda_{\mathrm{CT}} = \Lambda_{\mathrm{CT}}(H, X)$.*

Sketch of proof The proof of Lemma 3.5 in [35] goes through directly establishing that $\Lambda_{\mathrm{EL}} \subset \Lambda_{\mathrm{CT}}$.

The crucial technical tool after this is the construction of a ladder. The corresponding construct in the metric bundle context is given in Section 2.2 of [42] and generalizes the construction in [36; 35]. Quasiconvexity of ladders when the metric bundle is hyperbolic is now established by Theorem 3.2 of [42].

The proof of aperiodicity of ending laminations established in Section 4.1 of [35] uses only the group structure of the fiber (but not of the total space) and hence goes through with $\sigma([0, n])$ replacing the quasigeodesic $[1, g_n]$.

The final ingredient in the proof is the fact that qi-sections coarsely separate ladders (Lemma 4.8 in Section 4.2 of [35]). The proof is the same in the case of metric bundles.

With all these ingredients in place, the proof of Theorem 4.11 of [35] now goes through in the more general context of metric bundles to establish that $\Lambda_{\mathrm{EL}} = \Lambda_{\mathrm{CT}}$. □

3.2.1 Surface ending laminations It is appropriate to explicate at this juncture the relation between the ending laminations introduced by Thurston in [50, Chapter 9], which we call *surface ending laminations* henceforth, and the algebraic ending laminations we have been discussing. This will be particularly relevant when we deal with surface Kleinian groups, where the surface has punctures. Work of several authors including [33; 29; 6; 39] explore related themes.

The Thurston boundary $\partial\,\mathrm{Teich}(S)$ consists of projectivized measured laminations on S. Let r be a thick hyperbolic geodesic ray in Teichmüller space $\mathrm{Teich}(S)$, where S is a surface possibly with punctures. Then, by a result of Masur [30], it has a unique ideal point $r_\infty \in \partial\,\mathrm{Teich}(S)$ corresponding to a uniquely ergodic lamination. Let $\Lambda_{\mathrm{EL}}(r_\infty)$ be the geodesic lamination underlying r_∞. Let X_0 be the universal curve over r. Let X_1 denote X_0 with a small neighborhood of the cusps removed. Minsky proves [33] that X_1 is (uniformly) bi-Lipschitz homeomorphic to the convex core minus (a small neighborhood of) cusps of the unique simply degenerate hyperbolic 3–manifold M with conformal structure on the geometrically finite end given by $r(0) \in \mathrm{Teich}(S)$ and ending lamination of the simply degenerate end given by $\Lambda_{\mathrm{EL}}(r_\infty)$. The convex core of M is denoted by Y_0 and let Y_1 denote Y_0 with a small neighborhood of the cusps removed. Thus X_1 and Y_1 are (uniformly) bi-Lipschitz homeomorphic. Let X denote the universal cover of X_1 and $\mathcal{H}$ its collection of boundary horospheres. Then X is (strongly) hyperbolic relative to $\mathcal{H}$. Let $H = \pi_1(S)$ regarded as a relatively hyperbolic group, hyperbolic rel. cusp subgroups. The relative hyperbolic (or Bowditch) boundary $\partial_r H$ of the relatively hyperbolic group is still the circle (as when S is closed) and $\partial_r^2 H$ is defined as $(\partial_r H \times \partial_r H) \setminus \Delta$ as usual. The existence of a Cannon–Thurston map in this setting of a relatively hyperbolic group H acting on a relatively hyperbolic space $(X, \mathcal{H})$ has been proven in [6] (see also [38]).

The *diagonal closure* $\mathrm{Diag}(\mathcal{L})$ of a surface lamination $\mathcal{L}$ is an algebraic lamination given by the transitive closure of the relation defined by $\mathcal{L}$ on $\partial^2 H$. The *closed diagonal closure* $\mathcal{L}^d$ of a surface lamination $\mathcal{L}$ is an algebraic lamination given by the *closure in* $\partial^2 H$ of the transitive closure of the relation defined by $\mathcal{L}$ on $\partial^2 H$. When S is closed, each complementary ideal polygon of $\mathcal{L}$ has finitely many sides; so the closed diagonal closure $\mathcal{L}^d$ agrees with the diagonal closure $\mathrm{Diag}(\mathcal{L})$ and comprises the original lamination $\mathcal{L}$ along with the union of these diagonals (which are allowed to intersect). For a punctured surface S^h however, it is not enough just to take the transitive closure of the relation defined by $\mathcal{L}$. In this case, the fundamental group H is free and equals that of a compact core S^K of S^h (ie a compact submanifold of S^h whose inclusion induces a homotopy equivalence). The lamination thought of as a subset of $\widetilde{S}^K$, now has a complementary domain with infinitely many (bi-infinite) sides (the so-called crown domain) one of which corresponds to a lift $\widetilde{\sigma}$ of a boundary component σ of S^K. The transitive closure of $\mathcal{L}$ *does not* include the boundary points of $\widetilde{\sigma}$ in particular. However, the closure (in $\partial^2 H$) of the transitive closure of $\mathcal{L}$ captures all these, and is also closed under the transitive closure operation. We shall return to this later when dealing with punctured surfaces.

Theorem 3.12 [33; 6] *Let r be a thick hyperbolic geodesic in* $\mathrm{Teich}(S)$*, and let $\Lambda_{\mathrm{EL}}(r_\infty)$ denote its endpoint in $\partial\,\mathrm{Teich}(S)$ regarded as a surface lamination. Let X be the universal cover of X_1. Then $\Lambda_{\mathrm{CT}}(H,\widetilde{M}) = \Lambda_{\mathrm{CT}}(H,(X,\mathcal{H})) = \Lambda_{\mathrm{EL}}(r_\infty)^d$.*

Note that Theorem 3.12 holds both for closed surfaces as well as surfaces with finitely many punctures.

3.2.2 Generalized algebraic ending laminations The setup of a normal hyperbolic subgroup of a hyperbolic subgroup is quite restrictive. Instead we could consider H acting geometrically on a hyperbolic metric space X. Let $Y = X/H$ denote the quotient. Let $\{\sigma_n\}$ denote a sequence of free homotopy classes of closed loops in Y (these necessarily correspond to conjugacy classes in H) such that the geodesic realizations of $\{\sigma_n\}$ in Y exit all compact sets. Then subsequential limits of all such sequences define again an algebraic lamination, which we call a *generalized algebraic ending lamination* and denote by Λ_{GEL} $(=\Lambda_{\mathrm{GEL}}(H,X))$.

Then Lemma 3.5 of [35] (or Proposition 3.1 of [39] or Section 4.1 of [40]) gives:

Proposition 3.13 *If the pair (H,X) has a Cannon–Thurston map, then*

$$\Lambda_{\mathrm{GEL}}(H,X) \subset \Lambda_{\mathrm{CT}}(H,X).$$

3.3 Laminations dual to an $\mathbb{R}$–tree

We recall some of the material from [1, Section 3.1] on convergence of a sequence $\{(X_i,*_i,\rho_i\}$ of based H–spaces for H a fixed group.

An *H–space* is a pair (X,ρ), where X is a metric space and $\rho\colon H \to \mathrm{Isom}(X)$ is a homomorphism. Equivalently, it is an *action* of H on X by isometries. Let d_X denote the metric on X. A triple $(X,*,\rho)$ (for $*\in X$) is a *based H–space* if (X,ρ) is an H–space and $*$, also called the basepoint, is not a global fixed point under the action of H.

The space of all nonzero pseudometrics (or distance functions) on H, equipped with the compact-open topology is denoted by $\mathcal{D}$ (the condition that $*$ is not a global fixed point guarantees that $\mathcal{D}$ is nonempty). Note that an element of $\mathcal{D}$ is a nonnegative real valued function on $H \times H$. Assume that H acts on $H \times H$ diagonally and on $[0, \infty)$ trivially. Let $\mathcal{ED} \subset \mathcal{D}$ denote the subspace of H–equivariant pseudometrics under this action. Projectivizing $\mathcal{ED}$ (using the scaling action and passing to the quotient), we obtain the *projectivized equivariant distance functions* denoted by $\mathcal{PED}$. A pseudometric on H is said to be δ–hyperbolic if the associated metric space is δ–hyperbolic (the equivalence class of the identity element is taken to be the basepoint).

A based H–space $(X, *, \rho)$ induces an equivariant pseudometric $d = d_{(X,*,\rho)}$ on H by defining $d(g,h) := d_X(\rho(g)(*), \rho(h)(*))$. If the stabilizer of $*$ under the induced action is trivial, then H can, as usual, be identified with the orbit of $*$. This gives an induced metric $d_{(X,*,\rho)}$ on H.

Definition 3.14 [1] A sequence $(X_i, *_i, \rho_i)$, $i = 1, 2, \dots$ of based H–spaces *converges* to the based H–space $(X, *, \rho)$ if $[d_{(X_i,*_i,\rho_i)}] \to [d_{(X,*,\rho)}]$ in $\mathcal{PED}$. We denote this by $\lim_{i\to\infty}(X_i, *_i, \rho_i) = (X, *, \rho)$.

Theorem 3.15 [1, Theorem 3.3] *Let $(X_i, *_i, \rho_i)$ be a convergent sequence of based H–spaces such that*

(1) *there exists $\delta \geq 0$ such that each X_i is δ hyperbolic,*

(2) *there exists $h \in H$ such that the sequence $d_i = d_{X_i}(*_i, \rho_i(h)(*))$ is unbounded.*

Then there is a based H–tree $(T, *)$ (without global fixed points) and an isometric action $\rho\colon H \to \mathrm{Isom}(T)$ such that $(X_i, *_i, \rho_i) \to (T, *, \rho)$.

Note that convergence of $(X_i, *_i, \rho_i)$ (in terms of projective length functions in $\mathcal{PED}$) forces uniqueness of the projectivized length function. In particular, if there is an h' such that the growth rates $d'_i = d_{X_i}(*_i, \rho_i(h')(*))$ are much greater than d_i (more than linear), there would not be an action of H on the limit space as h' would be forced to translate $*$ by an infinite distance *after projectivizing*. Thus implicitly, the hypothesis of Theorem 3.15 selects out the maximal growth rate of the d_i's and scales by this.

Definition 3.16 For a convergent sequence $(X_i, *_i, \rho_i)$ as in Theorem 3.15, we define a dual algebraic lamination as follows: Let h_i be any sequence such that

$$\frac{d_{(X_i,*_i,\rho_i)}(1, h_i)}{d_i} \to 0.$$

The collection of all limits of $(h_i^{-\infty}, h_i^{\infty})$ in $\partial^2 H$ will be called the dual ending lamination corresponding to the sequence $(X_i, *_i, \rho_i)$ and will be denoted by $\Lambda_{\mathbb{R}}\{(X_i, *_i, \rho_i)\}$.

Next, let $1 \to H \to G \to Q \to 1$ be an exact sequence of hyperbolic groups. As in Section 3.2 let $z \in \partial Q$ and let $[1, z)$ be a geodesic ray in Γ_Q; let σ be a single-valued quasi-isometric section of Q into G. Let z_i be the vertex on $[1, z)$ such that $d_Q(1, z_i) = i$ and let $g_i = \sigma(z_i)$. Now, let $X_i = \Gamma_H$, $*_i = 1 \in \Gamma_H$ and $\rho_i(h)(*) = \phi_{g_i^{-1}}(h)(*)$. With this notation the following proposition is immediate from Definition 3.7:

Proposition 3.17 $\Lambda^z_{\mathrm{EL}} \subset \Lambda_{\mathbb{R}}\{(X_i, *_i, \rho_i)\}.$

An alternative description can be given directly in terms of the action on the limiting $\mathbb{R}$–tree in Theorem 3.15 as follows. The ray $[1, z) \subset Q$ defines a graph X_z of spaces where the underlying graph is a ray $[0, \infty)$ with vertices at the integer points and edges of the form $[n-1, n]$. All vertex and edge spaces are abstractly isometric to Γ_H. Let $e_n = {g_{n-1}}^{-1} g_n$. The edge space to vertex space inclusions are given by the identity to the left vertex space and by ϕ_{e_n} to the right. We call X_z the *universal metric bundle* over $[1, z)$ (though it depends on the qi-section σ of Q used as well). Hyperbolicity of X_z is equivalent to the *flaring* condition of Bestvina–Feighn [2] as shown for instance in [42] in the general context of metric bundles.

Suppose now that the sequence $\{(X_i, *_i, \rho_i)\}$, with $X_i = \Gamma_H$, $*_i = 1 \in \Gamma_H$ and $\rho_i(h)(*) = \phi_{g_i^{-1}}(h)(*)$, converges as a sequence of H–spaces to an H–action on an $\mathbb{R}$–tree $T = T(\{X_i, *_i, \rho_i\})$. Generalizing the construction of Coulbois, Hilion and Lustig [10; 11] to the hyperbolic group H we have the following notion of an algebraic lamination (contained in $\partial^2 H$) dual to T. The translation length in T will be denoted by l_T.

Definition 3.18 Let

$$L_\epsilon(T) = \overline{\{(g^{-\infty}, g^{\infty}) : l_T(g) < \epsilon\}},$$

where $\bar{A}$ denotes the closure of A. Define

$$\Lambda_{\mathbb{R}}\{(X_i, *_i, \rho_i)\} = \Lambda_{\mathbb{R}}(T) = \bigcap_{\epsilon > 0} L_\epsilon(T).$$

4 Closed surfaces

4.1 Arationality

Establishing arationality of Λ_{CT} for surface laminations arising out of a thick hyperbolic ray or an exact sequence of hyperbolic groups really involves identifying the algebraic Cannon–Thurston lamination Λ_{CT} with (the original) geodesic laminations introduced by Thurston [50]. To distinguish them from algebraic laminations, we shall refer to geodesic laminations on surfaces as surface laminations. The results of this subsection (though not the next subsection) hold equally for S compact or finite-volume noncompact.

A surface lamination $\mathcal{L} \subset S$ is *arational* if it has no closed leaves. It is called *filling* if it intersects every essential nonperipheral closed curve on the surface and *minimal* if it equals the closure of any of its leaves. Note that for an arational minimal lamination, the complement consists of ideal polygons. Adjoining some (nonintersecting) diagonals, we can still obtain an arational lamination, which is however no longer minimal. However, from an arational lamination we can obtain a unique arational minimal lamination by throwing away such diagonal leaves. Being filling is equivalent to saying that all complementary components of $\mathcal{L}$ are either topological disks or once punctured disks. Note that every filling lamination is automatically arational. We say that a bi-infinite geodesic l in $\widetilde{S}$ is *carried by a subgroup* $K \subset H$ $(= \pi_1(S))$ if both endpoints of l lie in the limit set $\Lambda_K \subset \partial\widetilde{S}$. A surface lamination $\mathcal{L} \subset S$ is *strongly arational* if no leaf of $\mathcal{L}$ or a diagonal in a complementary ideal polygon is carried by a finitely generated infinite-index subgroup K of H. The next lemma holds for both compact and noncompact hyperbolic surfaces of finite volume.

Lemma 4.1 *Any minimal arational geodesic lamination $\mathcal{L}_0$ on a finite-volume complete hyperbolic surface S is strongly arational.*

Proof We assume that S is equipped with a complete finite-volume hyperbolic metric and suppose that $\mathcal{L}_0$ is a minimal arational geodesic lamination. Consider a finitely generated infinite-index subgroup K of H. By the LERF (locally extended residually finite) property of surface groups [49], there exists a finite-sheeted cover S_1 of S such that K is a geometric subgroup of $\pi_1(S_1)$, ie it is the fundamental group of an embedded incompressible subsurface Σ of S_1 with geodesic boundary α_1. Let $\mathcal{L}_1$ be the lift of $\mathcal{L}_0$ to S_1.

We now show that $\mathcal{L}_1$ is minimal arational. Since $\mathcal{L}_0$ is arational, and leaves of $\mathcal{L}_1$ are lifts of leaves of $\mathcal{L}_0$, arationality of $\mathcal{L}_1$ follows. Let l be any leaf of $\mathcal{L}_1$ and $\overline{l}$ the closure of l in $\mathcal{L}_1$. Note that there are no diagonal leaves in $\mathcal{L}_1$ as such a leaf would have to come from a diagonal leaf in $\mathcal{L}_0$. If $\overline{l} \neq \mathcal{L}_1$, then $\mathcal{L}_1$ is not minimal and must contain a closed leaf l'. Since we have already shown that $\mathcal{L}_1$ is arational, this is a contradiction. Hence, all leaves of $\mathcal{L}_1$ are dense in $\mathcal{L}_1$; ie $\mathcal{L}_1$ is minimal as well.

In fact, any diagonal in a complementary ideal polygon of $\mathcal{L}_1$ is forward asymptotic to a leaf of $\mathcal{L}_1$ and is therefore also dense in $\mathcal{L}_1$; in particular it intersects α_1. Hence no leaf of $\mathcal{L}_1$, nor a diagonal in a complementary ideal polygon, is carried by Σ. The result follows. □

Theorem 4.2 [29] *The boundary $\partial\,\mathrm{CC}(S)$ of the curve complex $\mathrm{CC}(S)$ consists of minimal arational geodesic surface laminations.*

The following theorem may be taken as a definition of convex cocompactness for subgroups of the mapping class group of a surface with (at most) finitely many punctures.

Theorem 4.3 [19; 28; 23] *A subgroup Q of $\mathrm{MCG}(S)$ is convex cocompact if and only if some (any) orbit of Q in the curve complex $\mathrm{CC}(S)$ is qi-embedded.*

Recall that the Thurston boundary $\partial\,\mathrm{Teich}(S)$ of Teichmüller space is the space of projectivized measured laminations. The following theorem gives us the required strong arationality result.

Theorem 4.4 *Let S be a complete hyperbolic surface of finite volume. Let r be a thick hyperbolic ray in Teichmüller space $\mathrm{Teich}(S)$ and let $r_\infty \in \partial\,\mathrm{Teich}(S)$ be the limiting surface lamination. Then r_∞ is strongly arational.*

In particular if Q is a convex cocompact subgroup of $\mathrm{MCG}(S)$ and r is a quasi-geodesic ray in Q starting at $1 \in Q$, then its limit r_∞ in the boundary $\partial\,\mathrm{CC}(S)$ of the curve complex is strongly arational.

Proof Recall that for a thick hyperbolic ray r in $\mathrm{Teich}(S)$, we have $r_\infty \in \partial\,\mathrm{CC}(S)$. For the second statement of the theorem, ∂Q embeds as a subset of $\partial\,\mathrm{CC}(S)$ by Theorem 4.3 and hence the boundary point $r_\infty \in \partial\,\mathrm{CC}(S)$ as well.

By Theorem 4.2, r_∞ is an arational minimal lamination. Hence by Lemma 4.1, r_∞ is strongly arational. □

4.2 Quasiconvexity

We now turn to closed surfaces. Let

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of hyperbolic groups with $H = \pi_1(S)$ for a closed hyperbolic surface S. Then Q is convex cocompact [19] and its orbit in both $\mathrm{Teich}(S)$ and $\mathrm{CC}(S)$

are quasiconvex. By Theorem 3.10 $\Lambda_{\mathrm{EL}}(H,G,Q)=\Lambda_{\mathrm{EL}}=\Lambda_{\mathrm{CT}}=\Lambda_{\mathrm{CT}}(H,G)$. Further, $\Lambda_{\mathrm{EL}}=\bigcup_{z\in\partial Q}\overline{\Lambda^z_{\mathrm{EL}}}$. Recall that $\overline{\Lambda^z_{\mathrm{EL}}}$ denotes the algebraic ending lamination corresponding to z, and $\Lambda_{\mathrm{EL}}(z)$ denotes the surface ending lamination corresponding to z. By Theorem 3.12, $\overline{\Lambda^z_{\mathrm{EL}}}=\Lambda_{\mathrm{EL}}(z)^d$. We combine all this as follows.

Theorem 4.5 [33; 35] *If*

$$1\to H\to G\to Q\to 1$$

is an exact sequence with Q convex cocompact and $H=\pi_1(S)$ for a closed surface S of genus greater than 1, and

$$z=r_\infty\in\partial Q\subset\partial\,\mathrm{CC}(S),$$

then any lift of $[1,z)$ to $\mathrm{Teich}(S)$ is thick hyperbolic. Further,

$$\Lambda_{\mathrm{CT}}(H,G)=\bigcup_{z\in\partial Q}\Lambda_{\mathrm{EL}}(z)^d.$$

We are now in a position to prove the main theorems of this section.

Theorem 4.6 *Let $H=\pi_1(S)$ for S a closed surface of genus greater than 1. Let r be a thick hyperbolic ray in Teichmüller space $\mathrm{Teich}(S)$ and let $r_\infty\in\partial\,\mathrm{Teich}(S)$ be the limiting surface ending lamination. Let X denote the universal metric bundle over r. Let K be a finitely generated infinite-index subgroup of H. Then any orbit of K in X is quasiconvex.*

Proof By Theorem 4.4, the lamination r_∞ is strongly arational. Hence no leaf or diagonal of r_∞ is carried by K. By Theorem 3.12, the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H,X)=\Lambda_{\mathrm{EL}}(r_\infty)^d$. Hence no leaf of $\Lambda_{\mathrm{CT}}(H,X)$ is carried by K. By Lemma 3.4 and Remark 3.6, any orbit of K in X is quasiconvex in X. □

The next theorem was proven by Dowdall, Kent and Leininger [15, Theorem 1.3] using different methods.

Theorem 4.7 *Let*

$$1\to H\to G\to Q\to 1$$

be an exact sequence of hyperbolic groups with $H=\pi_1(S)$ (S closed) and Q convex cocompact. Let K be a finitely generated infinite-index subgroup of H. Then K is quasiconvex in G.

Proof As in the proof of Theorem 4.6, the lamination $\Lambda_{\mathrm{EL}}(z)$ is strongly arational for each $z \in \partial Q \subset \partial\,\mathrm{CC}(S)$ (where we identify the boundary of Q with the boundary of its orbit in $\mathrm{CC}(S)$). Hence for all $z \in \partial Q$, no leaf of $\Lambda_{\mathrm{EL}}(z)^d$ is carried by K. By Theorem 4.5,

$$\Lambda_{\mathrm{CT}}(H,G) = \Lambda_{\mathrm{EL}}(H,G) = \bigcup_{z\in\partial Q} \Lambda_{\mathrm{EL}}(z)^d.$$

Hence no leaf of $\Lambda_{\mathrm{CT}}(H,G)$ is carried by K. By Lemma 3.4 and Remark 3.6, K is quasiconvex in G. □

5 Free groups

For the purposes of this section, $H = F_n$ is free.

5.1 Arationality

Recall that the (unprojectivized) Culler–Vogtmann outer space corresponding to F_n is denoted by cv_n and its boundary by $\partial\mathrm{cv}_n$. The points of $\partial\mathrm{cv}_n$ correspond to very small actions of F_n on $\mathbb{R}$–trees.

Definition 5.1 [22] An $\mathbb{R}$–tree $T \in \partial\mathrm{cv}_n$ is said to be *indecomposable* if for any nondegenerate segments I and J contained in T, there exist finitely many elements $g_1,\dots,g_n \in F_n$ such that

(1) $I \subset \bigcup_{i=1,\dots,n} g_i J$,

(2) $g_i J \cap g_{i+1} J$ is a nondegenerate segment for any $i = 1,\dots,n-1$.

Dual to $T \in \partial\mathrm{cv}_n$ is an algebraic lamination $\Lambda_{\mathbb{R}}(T)$ defined as follows (which we had generalized to Definition 3.18 for general hyperbolic groups):

Definition 5.2 [10; 11] Let $L_\epsilon(T) = \overline{\{(g^{-\infty}, g^{\infty}) \mid l_T(g) < \epsilon\}}$. Define $\Lambda_{\mathbb{R}}(T) := \bigcap_{\epsilon>0} L_\epsilon(T)$.

Definition 5.3 [5] A leaf (p,q) of an algebraic lamination L is carried by a subgroup K of F_n if both p and q lie in the limit set of K.

Definition 5.4 A lamination L is called arational (resp. strongly arational) if no leaf of L is carried by a proper free factor of F_n (resp. by a proper finitely generated infinite-index subgroup of F_n).

A tree $T \in \partial\mathrm{cv}_n$ is called arational (resp. strongly arational) if $\Lambda_{\mathbb{R}}(T)$ is arational (resp. strongly arational).

The free factor complex $\mathcal{F}_n$ for F_n is a simplicial complex whose vertices are conjugacy classes A of free factors and simplices are chains $A_1 \subsetneq \cdots \subsetneq A_k$ of free factors.

Definition 5.5 [17; 24] A subgroup Q of $\mathrm{Out}(F_n)$ is said to be convex cocompact in $\mathrm{Out}(F_n)$ if some (and hence any) orbit of Q in $\mathcal{F}_n$ is qi-embedded.

A subgroup Q of $\mathrm{Out}(F_n)$ is said to be purely atoroidal if every element of Q is hyperbolic.

A geodesic or quasigeodesic (with respect to the Lipschitz metric) ray $[1,z)$ in outer space cv_n defines a metric bundle X_z where the underlying graph is a ray $[0,\infty)$ with vertices at the integer points and edges of the form $[n-1,n]$. As mentioned in Section 2.2, after moving the initial point of $[1,z)$ by a uniformly bounded amount, we can assume without loss of generality that $[1,z)$ is a folding path. Further, the $\mathbb{R}$–tree T_z corresponding to z, equipped with an F_n action, is exactly the tree encoded by $z \in \partial \mathrm{cv}_n$ (tautologically).

We refer the reader to [3, Section 2.4] for details on folding paths and geodesics in outer space The material relevant to this paper is efficiently summarized in [17, Section 2.7]. We call X_z the *universal metric bundle* over $[1,z)$. We shall be interested in two cases:

Case 1 $[1,z)$ is contained in a convex cocompact subgroup Q of $\mathrm{Out}(F_n)$, and for σ a qi-section [43], $\sigma([1,z))$ is identified with the corresponding quasigeodesic ray contained in an orbit of Q in cv_n. The universal metric bundle will (in this case) be considered over $\sigma([1,z))$.

Case 2 $[1,z)$ is a thick geodesic ray in cv_n, ie a geodesic ray projecting to a parametrized quasigeodesic in the free factor complex $\mathcal{F}_n$. As mentioned in the introduction, $[1,z)$ is said to be *thick hyperbolic* if, in addition, X_z is hyperbolic.

Remark 5.6 Case 1 is directly relevant to Theorem 5.14, while Case 2 pertains to Theorem 5.15. These two cases are logically independent, though the proofs are very similar.

We now describe the setup to be used in Proposition 5.8, which is extracted from the proof of Theorem 5.2 of [14]. Assume first that we are in Case 1. Since Q is convex cocompact, we may identify Q with an orbit in $\mathcal{F}_n$. This identification gives a Q–equivariant embedding of ∂Q into $\partial \mathcal{F}_n$. We identify ∂Q with its image under this embedding. Let $\mathcal{AT}$ consist of the projective classes of arational trees in $\partial \mathrm{cv}_n$. The authors of [14] give a natural map (following Bestvina–Reynolds [5]) $\partial\pi\colon \partial\mathcal{AT} \to \partial\mathcal{F}_n$ associating to each arational tree of $\partial \mathrm{cv}_n$ the corresponding point in the boundary of the free factor complex $\mathcal{F}_n$. Hence, by the identification of ∂Q with its image under the embedding into $\partial\mathcal{F}_n$, each point $z \in \partial Q$ corresponds to an equivalence class T_z of arational trees, where two such trees T_1 and T_2 are declared equivalent if their associated dual laminations $\Lambda_{\mathbb{R}}(T_1)$ and $\Lambda_{\mathbb{R}}(T_2)$ are the same.

Theorem 5.7 [14, Theorem 5.2] *For each $z \in \partial Q$, there exists $T_z \in \partial \mathrm{cv}_n$ which is free and arational such that $z \to \partial\pi(T_z)$ under the embedding of ∂Q into $\partial\mathcal{F}_n$ with the property that $\overline{\Lambda^z_{\mathrm{EL}}} = \Lambda_{\mathbb{R}}(T_z)$.*

A remark on a possible ambiguity that might arise from Theorem 5.7 as stated above is that $\Lambda_{\mathbb{R}}(T_z)$ depends on a choice of a tree lying in the fiber of $\partial\pi\colon \mathcal{AT} \to \partial\mathcal{F}_n$. However, as shown in [5] (see Theorem 5.12 below), the fiber consists of precisely the elements of the equivalence class mentioned above and hence $\Lambda_{\mathbb{R}}(T_z)$ is well defined independent of the choice.

We now turn to Case 2 and indicate briefly how the arguments of [14] go through in this case to prove the analogous statement, Proposition 5.8 below. Theorem 4.1 and Lemma 4.12 of [17] establish stability of $\mathcal{F}_n$–progressing quasigeodesics. While Lemma 5.5 of [14] is necessary to prove flaring for Case 1, flaring in Case 2 follows from hyperbolicity. (In fact it is shown in [42, Section 5.3, Proposition 5.8] that flaring is equivalent to hyperbolicity of X_z.) Also, thickness of the ray is by definition for Case 2. The crucial ingredient for [14, Theorem 5.2] is [14, Proposition 5.8], which, once [14, Propositions 5.5 and 5.6] are in place, makes no further use of the fact that $\sigma([1,z))$ comes from a ray in a convex cocompact Q (Case 1) but just that it is thick, stable and that the universal bundle over it satisfies flaring. Proposition 5.8 as stated below, now follows. Note that this part of the argument has nothing to do with identifying Λ_{EL} with Λ_{CT} (the latter is the content of Theorem 3.10 and Proposition 3.11).

Proposition 5.8 *Let $[1,z)$ be as in Case 2, and suppose that the universal metric bundle X_z is hyperbolic. Then $\Lambda_{\mathrm{EL}} = \Lambda_{\mathbb{R}}(T_z)$.*

Remark 5.9 A continuously parametrized version of the metric bundle described in Case 2 occurs in our context of folding paths in Culler–Vogtmann outer space cv_n converging to a point $z \in \partial \mathrm{cv}_n$. The same proof furnishes $\Lambda_{\mathrm{EL}} = \Lambda_{\mathbb{R}}(T_z)$ in the case of a continuously parametrized version of the metric bundle.

We collect together a number of theorems establishing mixing properties for F_n–trees.

Theorem 5.10 [46] *If T is a free indecomposable very small F_n–tree, then no leaf of the dual lamination $\Lambda_{\mathbb{R}}(T)$ is carried by a finitely generated subgroup of infinite index in F_n.*

Theorem 5.11 [47] *Let $T \in \partial \mathrm{cv}_n$. Then T is arational if and only if either*

(a) *T is free and indecomposable, or*

(b) *T is dual to an arational measured foliation on a compact surface S with one boundary component and with $\pi_1(S) = F_n$.*

Recall that $\mathcal{AT} \subset \partial \mathrm{cv}_n$ denotes the set of arational trees, equipped with the subspace topology. Define a relation $\sim$ on $\mathcal{AT}$ by $S \sim T$ if and only if $\Lambda_{\mathbb{R}}(S) = \Lambda_{\mathbb{R}}(T)$, and give $\mathcal{AT}/\sim$ the quotient topology.

Theorem 5.12 [5] *The space $\partial \mathcal{F}_n$ is homeomorphic to $\mathcal{AT}/\sim$. In particular, all boundary points of $\mathcal{F}_n$ are arational trees.*

Combining the above theorems we obtain the crucial mixing property we need (we refer the reader to the introduction for the definition of a thick hyperbolic ray).

Theorem 5.13 *Let r be a thick hyperbolic ray in outer space, and let $r_\infty \in \partial \mathrm{cv}_n$ be the limiting $\mathbb{R}$–tree. Then $\Lambda_{\mathbb{R}}(r_\infty)$ is strongly arational.*

In particular if Q is a convex cocompact purely hyperbolic subgroup of $\mathrm{Out}(F_n)$ and r is a quasigeodesic ray in Q starting at $1 \in Q$, then its limit r_∞ in the boundary $\partial \mathcal{F}_n$ of the free factor complex is strongly arational.

Proof By Theorem 5.12 every point in $\partial \mathcal{F}_n$ comes from an arational $\mathbb{R}$–tree. Hence r_∞ is arational.

Since r is hyperbolic, the metric bundle over r is hyperbolic by definition. In particular, the bundle satisfies the flaring condition [42, Proposition 5.8]. Hence every element of F_n has nonzero translation length on the limiting $\mathbb{R}$–tree r_∞, thus ruling out alternative (b) of Theorem 5.11. It follows from Theorem 5.11 that r_∞ is indecomposable free. It finally follows from Theorem 5.10 that r_∞ is strongly arational. Equivalently, $\Lambda_{\mathbb{R}}(r_\infty)$ is strongly arational.

Next, suppose that Q is a convex cocompact purely atoroidal subgroup of $\mathrm{Out}(F_n)$ and r a quasigeodesic ray in Q starting at $1 \in Q$. By Theorem 4.1 of [17], an orbit of Q is quasiconvex (in the strong symmetric sense). Then the limit point r_∞ of r lies in $\partial\mathcal{F}_n$ since the orbit map from Q to $\mathcal{F}_n$ is a qi-embedding and is therefore arational. Since Q is purely atoroidal quasiconvex, it follows from Corollary 5.3 of [14] that the lamination dual to the tree r_∞ cannot be carried by a surface with a puncture thus ruling out Theorem 5.11(b). Hence it is indecomposable free by Theorem 5.11. Again, from Theorem 5.10, r_∞ is strongly arational. □

5.2 Quasiconvexity

Theorem 5.14 *Let*

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of hyperbolic groups with $H = F_n$ and Q convex cocompact. Let K be a finitely generated infinite-index subgroup of H. Then K is quasiconvex in G.

Proof We first note that for each $z \in \partial Q \subset \partial\mathcal{F}_n$ (where we identify the boundary of Q with the boundary of its orbit in $\mathcal{F}_n$), the tree T_z is strongly arational by Theorem 5.13. In particular, no leaf of $\Lambda_{\mathbb{R}}(T_z)$ is carried by K. Hence for all $z \in \partial Q$, no leaf of $\Lambda_{\mathbb{R}}(T_z)$ is carried by K. By Theorem 5.7, the algebraic ending lamination $\overline{\Lambda^z_{\mathrm{EL}}} = \Lambda_{\mathbb{R}}(T_z)$. Further, by Theorem 3.10,

$$\Lambda_{\mathrm{CT}}(H,G) = \Lambda_{\mathrm{EL}}(H,G) = \bigcup_z \overline{\Lambda^z_{\mathrm{EL}}}.$$

Hence no leaf of $\Lambda_{\mathrm{CT}}(H,G)$ is carried by K. By Lemma 3.4 and Remark 3.6, K is quasiconvex in G. □

Theorem 5.15 *Let $H = F_n$, let r be a thick hyperbolic ray in outer space cv_n and let $r_\infty \in \partial\mathrm{cv}_n$ be the limiting $\mathbb{R}$–tree. Let X denote the universal metric bundle over r. Let K be a finitely generated infinite-index subgroup of H. Then any orbit of K in X is quasiconvex.*

Proof As in the proof of Theorem 5.14, the tree $T = r_\infty$ is strongly arational by Theorem 5.13. Hence no leaf of $\Lambda_{\mathbb{R}}(T)$ is carried by K. By Proposition 5.8 and Remark 5.9, for the algebraic ending lamination, we have

$$\Lambda_{\mathrm{EL}}(H,X) = \Lambda_{\mathrm{EL}} \subset \Lambda_{\mathbb{R}}(T).$$

Further, by Theorem 3.10 and Proposition 3.11, $\Lambda_{\mathrm{CT}}(H,X)=\Lambda_{\mathrm{EL}}(H,X)$. Hence no leaf of $\Lambda_{\mathrm{CT}}(H,X)$ is carried by K. By Lemma 3.4 and Remark 3.6, any orbit of K in X is quasiconvex in X. □

6 Punctured surfaces

For the purposes of this section S^h is a noncompact finite-volume hyperbolic surface and $H=\pi_1(S^h)$.

6.1 Quasiconvexity for rays

Theorem 6.1 *Let $r\subset \mathrm{Teich}(S^h)$ be a thick geodesic ray and $r_\infty\in\partial\,\mathrm{Teich}(S^h)$ the limiting surface ending lamination. Let X denote the universal metric bundle over r minus a small neighborhood of the cusps, and let $\mathcal{H}$ denote the horosphere boundary components. Let K be a finitely generated infinite-index subgroup of H. Then any orbit of K in X is relatively quasiconvex in $(X,\mathcal{H})$.*

Proof The proof is slightly more involved than that of Theorem 4.6, the difficulty arising from the difference between the closed diagonal closure and the diagonal closure of a geodesic surface lamination $\mathcal{L}$. Recall that $\mathcal{L}^d$ denotes the *closure* of the diagonal closure of $\mathcal{L}$.

First, observe that if K corresponds to a parabolic subgroup, it is automatically relatively quasiconvex. Hence assume that K is not a parabolic subgroup. By Theorem 4.4 (which, recall, holds for punctured surfaces), the lamination r_∞ is strongly arational. Hence no leaf or diagonal of r_∞ is carried by K. By Theorem 3.12 (which, recall, holds for punctured surfaces as well), the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H,X)=\Lambda_{\mathrm{EL}}(r_\infty)^d$.

However, for a punctured surface, $\Lambda_{\mathrm{EL}}(r_\infty)^d$ does not equal the diagonal closure of $\Lambda_{\mathrm{EL}}(r_\infty)$ unlike the closed surface case. We shall analyze the difference shortly. Note also that r_∞ refers to the surface geodesic lamination living in S^h, whereas $\Lambda_{\mathrm{EL}}(r_\infty)$ refers to collections of pairs of points in $\partial^2_r H$ (or equivalently collections of bi-infinite geodesics in the universal cover of S^h). Let $\mathrm{Diag}(\Lambda_{\mathrm{EL}}(r_\infty))$ denote the diagonal closure of $\Lambda_{\mathrm{EL}}(r_\infty)$. For $i=1,\dots,l$, let $L_i=\langle a_i\rangle$ denote the peripheral cyclic subgroup of H generated by a_i. Let $z_1,\dots,z_l$ denote the corresponding punctures on S^h. There exists a unique connected component $D_i\subset S^h\setminus\Lambda_{\mathrm{EL}}(r_\infty)$ containing z_i. Such a D_i is called a *crown domain*. The boundary components of D_i are finitely many leaves of r_∞. In the universal cover each lift $\widetilde{D}_i$ of D_i is an infinite-sided polygon stabilized by a conjugate of a_i. Any ideal point of such a $\widetilde{D}_i$ shall be referred to as a *crown-tip*. With this terminology, $\mathrm{Diag}(r_\infty)$ consists of pairs of points (p,q) such that either

(1) (p,q) are endpoints of a leaf of r_∞ lifted to the universal cover of S^h, or

(2) there is a (necessarily finite) sequence $p = p_1, \dots, p_n = q$ of crown-tips such that (p_i, p_{i+1}) are endpoints of a leaf of r_∞ lifted to the universal cover.

Next, $\Lambda_{\mathrm{EL}}(r_\infty)^d \setminus \mathrm{Diag}(r_\infty)$ consists precisely of pairs of points (p,q) in $\partial_r H$ $(= S^1)$ such that p is a fixed point of a conjugate a_i^g of some parabolic a_i and q is a crown-tip on the boundary of the lift $\widetilde{D}_i$ stabilized by a_i^g. It follows that any bi-infinite geodesic in $\Lambda_{\mathrm{EL}}(r_\infty)^d \setminus \mathrm{Diag}(r_\infty)$ necessarily has one direction (the direction converging to the crown-tip) asymptotic to a lift of a leaf of r_∞. Since K is finitely generated, it is necessarily relatively quasiconvex in H. Hence, if K carries a leaf of $\Lambda_{\mathrm{EL}}(r_\infty)^d \setminus \mathrm{Diag}(r_\infty)$, we can translate such a leaf by larger and larger nonparabolic elements of K and pass to a limit to obtain a leaf of $\mathrm{Diag}(r_\infty)$ carried by K. This contradicts the fact (Theorem 4.4) that r_∞ (and hence $\mathrm{Diag}(r_\infty)$) is strongly arational. It follows that no leaf of $\Lambda_{\mathrm{CT}}(H,X)$ is carried by K. By Lemma 3.5 and Remark 3.6, any orbit of K in X is relatively quasiconvex in $(X,\mathcal{H})$. □

6.2 Quasiconvexity for exact sequences

Let

$$1 \to H \to G \to Q \to 1$$

be an exact sequence of relatively hyperbolic groups with $H = \pi_1(S^h)$ for a finite-volume hyperbolic surface S^h with finitely many peripheral subgroups $H_1, \dots, H_n$. Here Q is a convex cocompact subgroup of $\mathrm{MCG}(S^h)$, where MCG is taken to be the pure mapping class group, fixing peripheral subgroups (this is a technical point and is used only for expository convenience). Note that the normalizer $N_G(H_i)$ is then isomorphic to $H_i \times Q$ $(\subset G)$. The following characterizes convex cocompactness:

Proposition 6.2 [42, Proposition 5.17] *Let $H = \pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures, and let $H_1, \dots, H_n$ be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h. Let*

$$1 \to H \to G \to Q \to 1 \quad \textit{and} \quad 1 \to H_i \to N_G(H_i) \to Q \to 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}$, $i = 1, \dots, n$.

Conversely, if G is (strongly) hyperbolic relative to $\{N_G(H_i)\}$, $i = 1, \dots, n$, then Q is convex-cocompact.

Since Q is convex cocompact, its orbits in both $\mathrm{Teich}(S^h)$ and $\mathrm{CC}(S^h)$ are quasiconvex and qi-embedded [19; 28; 23]. Identify Γ_Q with a subset of $\mathrm{Teich}(S^h)$ by identifying the vertices of Γ_Q with an orbit $Q.o$ of Q and edges with geodesic segments joining the corresponding vertices.

Let X_0 be the universal curve over Γ_Q. Let X_1 denote X_0 with a small neighborhood of the cusps removed. Then X_1 is a union $\bigcup_{q \in \partial \Gamma_Q} X_q$, where X_q is a bundle over the quasigeodesic $[1, q)$ $(\subset \Gamma_Q \subset \mathrm{Teich}(S^h))$ with fibers hyperbolic surfaces diffeomorphic to S^h with a small neighborhood of the cusps removed. Minsky proved that

(1) the quasigeodesic $[1, q)$ stays a bounded distance from a geodesic in Teichmüller space ending at the point $q \in \partial\,\mathrm{Teich}(S^h))$ [34];

(2) the bundle X_q is (uniformly) bi-Lipschitz homeomorphic to the convex core minus (a small neighborhood of) cusps of the unique simply degenerate hyperbolic 3–manifold M with conformal structure on the geometrically finite end given by $o = 1.o \in \mathrm{Teich}(S^h)$ and ending lamination of the simply degenerate end given by $\Lambda_{\mathrm{EL}}(q)$ [33].

The convex core of M is denoted by Y_{q0}. Let Y_{q1} denote Y_{q0} with a small neighborhood of the cusps removed. Thus X_q and Y_{q1} are (uniformly) bi-Lipschitz homeomorphic. Let $\widetilde{X}_q$ denote the universal cover of X_q and $\mathcal{H}_q$ its collection of boundary horospheres. Then $\widetilde{X}_q$ is (strongly) hyperbolic relative to $\mathcal{H}_q$. Let $H = \pi_1(S^h)$ be thought of as a relatively hyperbolic group, hyperbolic relative to the cusp subgroups $\{H_i\}$, $i = 1, \dots, n$. The relative hyperbolic (or Bowditch) boundary $\partial_r H$ of the relatively hyperbolic group is still the circle (as when S is closed) and $\partial_r^2 H$ is defined as $(\partial_r H \times \partial_r H) \setminus \Delta$ as usual. The existence of a Cannon–Thurston map in this setting from the relative hyperbolic boundary of H to the relative hyperbolic boundary of $(\widetilde{X}_q, \mathcal{H}_q)$ has been proved in [6; 38]. Also, it is established in [6; 39] (see Theorem 3.12) that the Cannon–Thurston lamination for the pairs $(H, \widetilde{X}_q)$ is given by

$$\Lambda_{\mathrm{CT}}(H, \widetilde{X}_q) = \Lambda_{\mathrm{EL}}(q)^d,$$

where $\Lambda_{\mathrm{EL}}(q)^d$ denotes the closure of the diagonal closure of the ending lamination $\Lambda_{\mathrm{EL}}(q)$.

Next, by Proposition 6.2, G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}$, $i = 1, \dots, n$. Note that the inclusion of H into G is strictly type-preserving as an inclusion of relatively hyperbolic groups. The existence of a Cannon–Thurston map for the pair (H, G) is established in [44].

We shall require a generalization of Theorem 4.5 to punctured surfaces to obtain a description of $\Lambda_{\mathrm{CT}}(H, G)$. The description of the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H, G)$ for the pair (H, G) can now be culled from [35] and [44]. We shall give a brief sketch of the modifications necessary to the arguments of [35] so as to make them work in the present context. The crucial technical tool is the construction of a ladder, which we sketch now. As usual, fix finite generating sets of H and G such that the generating set of G contains that of H, thus giving a natural inclusion of Cayley graphs, Γ_H into Γ_G. Let Γ_Q denote the Cayley graph of Q with respect to the generating set given by the nontrivial elements of the quotient of the generating set of G.

Pal [44] proves the existence of a qi-section $\sigma\colon \Gamma_Q \to \Gamma_G$. Given any $a, b \in H$, we now look at geodesic segments $\lambda_q = [a\sigma(q), b\sigma(q)]$ in the coset $\Gamma_H \sigma(q) = \sigma(q)\Gamma_H$ (the equality of left and right cosets follows from normality of H) joining $a\sigma(q), b\sigma(q)$. Note that these are geodesics in the *intrinsic path-metric* on $\sigma(q)\Gamma_H$, which is isometric to Γ_H. The union $\bigcup_{q \in Q} \lambda_q$ is called a *ladder* corresponding to $[a, b] \subset \Gamma_H$. Note here that the ladder construction in [35] does not require hyperbolicity of G but only that of H. Since H is free in the present case, the construction of the ladder goes through.

As in [35] (see also Definition 3.7 and Proposition 3.11), we assign to every boundary point $z \in \partial Q$ an algebraic ending lamination $\overline{\Lambda^z_{\mathrm{EL}}}$. Similarly (as in the hyperbolic case), for every z, there is a Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(z)$. The proof of the description of the ending lamination in [35] (using the ladder) now shows that the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H, G)$ for the pair (H, G) is the closure of the transitive closure of the union $\bigcup_{z \in \partial Q} \Lambda_{\mathrm{CT}}(z)$; see [39, Section 4.4]. We elaborate on this a bit. Recall that X_1 is a union $\bigcup_{q \in \partial \Gamma_Q} X_q$ and that the universal cover of X_1 is naturally quasi-isometric to G. Thus Γ_G can be thought of as a union (not disjoint) of the metric bundles over $[1, q)$, as q ranges over ∂Q. In fact if $P\colon G \to Q$ denotes projection, then $\widetilde{X}_q$ is quasi-isometric to $P^{-1}([1, q))$. The construction of the ladder and a coarse Lipschitz retract of Γ_G onto it then shows that a leaf of the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H, G)$ arises as a concatenation of at most two infinite rays, each of which lies in a leaf of the Cannon–Thurston lamination $\Lambda_{\mathrm{CT}}(H, P^{-1}([1, q)))$ for some q. Thus $\Lambda_{\mathrm{CT}}(H, G)$ is the closure of the transitive closure of the union $\bigcup_{q\in\partial Q}\Lambda_{\mathrm{CT}}(H,\widetilde{X}_q)$; ie

$$\Lambda_{\mathrm{CT}}(H,G)=\Bigl(\bigcup_{z\in\partial Q}\Lambda_{\mathrm{CT}}(z)\Bigr)^d.$$

We need to show that $\overline{\Lambda^z_{\mathrm{EL}}}=\Lambda_{\mathrm{CT}}(z)$. Lemma 3.5 of [35] goes through verbatim to show that $\overline{\Lambda^z_{\mathrm{EL}}}\subset\Lambda_{\mathrm{CT}}(z)$. It remains to show that $\Lambda_{\mathrm{CT}}(z)\subset\overline{\Lambda^z_{\mathrm{EL}}}$. But this is exactly the content of the main theorem of [6] (see also [38]).

We combine all this in the following.

Theorem 6.3 [33; 6; 35; 42] *Let $H=\pi_1(S^h)$ be the fundamental group of a surface with finitely many punctures, and let $H_1,\dots,H_n$ be its peripheral subgroups. Let Q be a convex cocompact subgroup of the pure mapping class group of S^h. Let*

$$1\to H\to G\to Q\to 1\qquad\text{and}\qquad 1\to H_i\to N_G(H_i)\to Q\to 1$$

be the induced short exact sequences of groups. Then G is strongly hyperbolic relative to the collection $\{N_G(H_i)\}$, $i=1,\dots,n$. Further, $\Lambda_{\mathrm{CT}}(H,G)=\bigl(\bigcup_{z\in\partial Q}\Lambda_{\mathrm{EL}}(z)\bigr)^d$.

We can now prove our last quasiconvexity theorem:

Theorem 6.4 *Let H and G be as in Theorem 6.3, and let K be a finitely generated infinite-index subgroup of H. Then K is relatively quasiconvex in G.*

Proof Without loss of generality, K is not contained in a parabolic subgroup of H (since then there is nothing to prove). As in the proof of Theorem 6.1, the lamination $\Lambda_{\mathrm{EL}}(q)$ is strongly arational for each $q\in\partial Q$. Hence, as in the proof of Theorem 6.1, no leaf of $\Lambda_{\mathrm{EL}}(q)^d$ is carried by K as q ranges over ∂Q. By Theorem 6.3, $\Lambda_{\mathrm{CT}}(H,G)$ is the closure of the transitive closure of $\bigcup_{z\in\partial Q}\Lambda_{\mathrm{EL}}(z)^d$. It follows (again as in the proof of Theorem 6.1) that no leaf of $\Lambda_{\mathrm{CT}}(H,G)$ is carried by K. By Lemma 3.5 and Remark 3.6, K is relatively quasiconvex in G. □

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References

[1] **M Bestvina**, *$\mathbb{R}$–trees in topology, geometry, and group theory*, from "Handbook of geometric topology" (R J Daverman, R B Sher, editors), North-Holland, Amsterdam (2002) 55–91 MR

[2] **M Bestvina**, **M Feighn**, *A combination theorem for negatively curved groups*, J. Differential Geom. 35 (1992) 85–101 MR

[3] **M Bestvina**, **M Feighn**, *Hyperbolicity of the complex of free factors*, Adv. Math. 256 (2014) 104–155 MR

[4] **M Bestvina**, **M Feighn**, **M Handel**, *Laminations, trees, and irreducible automorphisms of free groups*, Geom. Funct. Anal. 7 (1997) 215–244 MR

[5] **M Bestvina**, **P Reynolds**, *The boundary of the complex of free factors*, Duke Math. J. 164 (2015) 2213–2251 MR

[6] **B H Bowditch**, *The Cannon–Thurston map for punctured-surface groups*, Math. Z. 255 (2007) 35–76 MR

[7] **B H Bowditch**, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. 22 (2012) 1250016, 66 MR

[8] **J W Cannon**, **W P Thurston**, *Group invariant Peano curves*, Geom. Topol. 11 (2007) 1315–1355 MR

[9] **T Coulbois**, **A Hilion**, **M Lustig**, *Non-unique ergodicity, observers' topology and the dual algebraic lamination for $\mathbb{R}$–trees*, Illinois J. Math. 51 (2007) 897–911 MR

[10] **T Coulbois**, **A Hilion**, **M Lustig**, *$\mathbb{R}$–trees and laminations for free groups, I: Algebraic laminations*, J. Lond. Math. Soc. 78 (2008) 723–736 MR

[11] **T Coulbois**, **A Hilion**, **M Lustig**, *$\mathbb{R}$–trees and laminations for free groups, II: The dual lamination of an $\mathbb{R}$–tree*, J. Lond. Math. Soc. 78 (2008) 737–754 MR

[12] **T Coulbois**, **A Hilion**, **P Reynolds**, *Indecomposable F_N–trees and minimal laminations*, Groups Geom. Dyn. 9 (2015) 567–597 MR

[13] **M Culler**, **K Vogtmann**, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986) 91–119 MR

[14] **S Dowdall**, **I Kapovich**, **S J Taylor**, *Cannon–Thurston maps for hyperbolic free group extensions*, Israel J. Math. 216 (2016) 753–797 MR

[15] **S Dowdall**, **R P Kent, IV**, **C J Leininger**, *Pseudo-Anosov subgroups of fibered 3–manifold groups*, Groups Geom. Dyn. 8 (2014) 1247–1282 MR

[16] **S Dowdall**, **S J Taylor**, *The co-surface graph and the geometry of hyperbolic free group extensions*, J. Topol. 10 (2017) 447–482 MR

[17] **S Dowdall**, **S J Taylor**, *Hyperbolic extensions of free groups*, Geom. Topol. 22 (2018) 517–570 MR

[18] **B Farb**, *Relatively hyperbolic groups*, Geom. Funct. Anal. 8 (1998) 810–840 MR

[19] **B Farb**, **L Mosher**, *Convex cocompact subgroups of mapping class groups*, Geom. Topol. 6 (2002) 91–152 MR

[20] **M Gromov**, *Hyperbolic groups*, from "Essays in group theory" (S M Gersten, editor), Math. Sci. Res. Inst. Publ. 8, Springer (1987) 75–263 MR

[21] **M Gromov**, *Asymptotic invariants of infinite groups*, from "Geometric group theory, 2" (G A Niblo, M A Roller, editors), London Math. Soc. Lect. Note Ser. 182, Cambridge Univ. Press (1993) 1–295 MR

[22] **V Guirardel**, *Actions of finitely generated groups on $\mathbb{R}$–trees*, Ann. Inst. Fourier (Grenoble) 58 (2008) 159–211 MR

[23] **U Hamenstädt**, *Word hyperbolic extensions of surface groups*, preprint (2005) arXiv

[24] **U Hamenstädt**, **S Hensel**, *Stability in outer space*, preprint (2014) arXiv

[25] **G C Hruska**, **D T Wise**, *Packing subgroups in relatively hyperbolic groups*, Geom. Topol. 13 (2009) 1945–1988 MR

[26] **I Kapovich**, **M Lustig**, *Intersection form, laminations and currents on free groups*, Geom. Funct. Anal. 19 (2010) 1426–1467 MR

[27] **I Kapovich**, **M Lustig**, *Cannon–Thurston fibers for iwip automorphisms of F_N*, J. Lond. Math. Soc. 91 (2015) 203–224 MR

[28] **R P Kent, IV**, **C J Leininger**, *Shadows of mapping class groups: capturing convex cocompactness*, Geom. Funct. Anal. 18 (2008) 1270–1325 MR

[29] **E Klarreich**, *The boundary at infinity of the curve complex and the relative Teichmüller space*, preprint (1999) arXiv

[30] **H Masur**, *Two boundaries of Teichmüller space*, Duke Math. J. 49 (1982) 183–190 MR

[31] **H Masur**, **Y N Minsky**, *Geometry of the complex of curves, II: Hierarchical structure*, Geom. Funct. Anal. 10 (2000) 902–974 MR

[32] **Y N Minsky**, *Teichmüller geodesics and ends of hyperbolic 3–manifolds*, Topology 32 (1993) 625–647 MR

[33] **Y N Minsky**, *On rigidity, limit sets, and end invariants of hyperbolic 3–manifolds*, J. Amer. Math. Soc. 7 (1994) 539–588 MR

[34] **Y N Minsky**, *Bounded geometry for Kleinian groups*, Invent. Math. 146 (2001) 143–192 MR

[35] **M Mitra**, *Ending laminations for hyperbolic group extensions*, Geom. Funct. Anal. 7 (1997) 379–402 MR

[36] **M Mitra**, *Cannon–Thurston maps for hyperbolic group extensions*, Topology 37 (1998) 527–538 MR

[37] **M Mitra**, *On a theorem of Scott and Swarup*, Proc. Amer. Math. Soc. 127 (1999) 1625–1631 MR

[38] **M Mj**, *Cannon–Thurston maps and bounded geometry*, from "Teichmüller theory and moduli problems" (I Biswas, R S Kulkarni, S Mitra, editors), Ramanujan Math. Soc. Lect. Notes Ser. 10, Ramanujan Math. Soc., Mysore (2010) 489–511 MR

[39] **M Mj**, *Ending laminations and Cannon–Thurston maps*, Geom. Funct. Anal. 24 (2014) 297–321 MR

[40] **M Mj**, *Cannon–Thurston maps for Kleinian groups*, Forum Math. Pi 5 (2017) art. id. e1, 49 pages MR

[41] **M Mj**, **A Pal**, *Relative hyperbolicity, trees of spaces and Cannon–Thurston maps*, Geom. Dedicata 151 (2011) 59–78 MR

[42] **M Mj**, **P Sardar**, *A combination theorem for metric bundles*, Geom. Funct. Anal. 22 (2012) 1636–1707 MR

[43] **L Mosher**, *Hyperbolic extensions of groups*, J. Pure Appl. Algebra 110 (1996) 305–314 MR

[44] **A Pal**, *Relatively hyperbolic extensions of groups and Cannon–Thurston maps*, Proc. Indian Acad. Sci. Math. Sci. 120 (2010) 57–68 MR

[45] **K Rafi**, *Hyperbolicity in Teichmüller space*, Geom. Topol. 18 (2014) 3025–3053 MR

[46] **P Reynolds**, *On indecomposable trees in the boundary of outer space*, Geom. Dedicata 153 (2011) 59–71 MR

[47] **P Reynolds**, *Reducing systems for very small trees*, preprint (2012) arXiv

[48] **G P Scott**, **G A Swarup**, *Geometric finiteness of certain Kleinian groups*, Proc. Amer. Math. Soc. 109 (1990) 765–768 MR

[49] **P Scott**, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. 17 (1978) 555–565 MR

[50] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) Available at `http://msri.org/publications/books/gt3m`

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