On some adjunctions in equivariant stable homotopy theory

Po Hu
Igor Kriz
Petr Somberg

We investigate certain adjunctions in derived categories of equivariant spectra, including a right adjoint to fixed points, a right adjoint to pullback by an isometry of universes, and a chain of two right adjoints to geometric fixed points. This leads to a variety of interesting other adjunctions, including a chain of six (sometimes seven) adjoints involving the restriction functor to a subgroup of a finite group on equivariant spectra indexed over the trivial universe.

1 Introduction

In equivariant stable homotopy theory, we study, for a compact Lie group G, generalized cohomology theories which are stable under suspensions by one-point compactifications of finite-dimensional G–representations. Such theories are represented in the derived category DG–$\mathcal{U}$–spectra, where $\mathcal{U}$ is the complete universe (see Lewis, May, Steinberger and McClure [9]), ie a G–inner product space of countable dimension containing infinitely many copies of all irreducible G–representations. Certain functors come up naturally when studying these theories: for example, the *fixed point functor* $(?)^G$, which is used in calculating homotopy groups, and also the *geometric fixed point functor* Φ^G, which was quite important in the work of Hu and Kriz [7] on real-oriented $\mathbb{Z}/2$–spectra, as well as, later, in the solution by Hill, Hopkins and Ravenel [6] of the Kervaire invariant 1 problem.

The left derived functor of the functor Φ^G has a right adjoint, which again has a right adjoint. (Throughout this paper, we will focus on the derived context, so this language will often be omitted.) In a recent paper [1], Balmer, Dell'Ambrogio and Sanders investigated a general framework in which certain "geometric functors" between tensor triangulated categories have two right adjoints. In fact, they proved that under suitable assumptions (see Section 2 below), only three possibilities arise, namely a chain of

three, five or infinitely many adjoints on both sides. Geometric fixed points, in the case of a complete universe, satisfy the assumptions of [1], and in this context (see also Balmer and Sanders [2]), it seemed interesting to look at this example more closely.

In a recent paper [12], Sanders investigates another example of the 3–adjunction [1], namely "inflation", ie the fixed G–spectrum, indexed over the complete universe, associated with a spectrum X. In fact, Sanders introduces a beautiful formalism which enables an abstract treatment of the Adams isomorphism. The right adjoint of the inflation functor on the level of derived categories is the fixed point functor $(?)^G$ of a spectrum indexed over the complete universe. Again, by the work of [1], this functor has an additional right adjoint on derived categories, which the authors of the present paper also observed independently [8] in connection with their work on spectral Lie algebras. Unlike the case of geometric fixed points, this functor, however, is much harder to describe, and even now remains somewhat mysterious.

Inspecting this example more closely suggests looking beyond the case of a complete universe. The reason is that the fixed point functor $(?)^G$ on the derived category $DG\text{–}\mathscr{U}\text{–}$ spectra really is a composition of two functors, the first of which is pullback i^* via the inclusion $i\colon \mathscr{U}^G \to \mathscr{U}$ from the trivial universe $\mathscr{U}^G \cong \mathbb{R}^\infty$. On spectra indexed over the trivial universe (which represent generalized cohomology theories only stable with respect to ordinary suspensions), geometric and ordinary fixed points are the same thing.

A natural question then arises: Does the pullback i^* with respect to an isometry of universes also have a right adjoint on the level of derived categories? Are the observations of the previous paragraphs also true for noncomplete universes? The answer to the first question is yes, as is, for the most part, the answer to the second question. It is important to note, however, that we are now leaving the world of the assumptions of [1] since, for spectra indexed over a noncomplete universe, the important assumption of [1] that compact objects be strongly dualizable precisely fails for those triangulated categories.

Since inflation is a case of functoriality with respect to change of groups (the case of a surjection), what about restriction, ie the case of an injection of groups? In this case, for complete universes, we have the well-known Wirthmüller isomorphism [9], which also was a part of the inspiration for [1] as well as, for example, Fausk, Hu and May [4]. What happens in the case of noncomplete universes? It turns out that in this case, which does not satisfy the assumptions of [1], we always have a chain of four adjoints. However, in the case of a finite group and the trivial universe, we show that

there is, in fact, a chain of six adjoints, and in the case of a finite abelian group and the trivial subgroup for the trivial universe, there is a chain of seven adjoints. In special cases, this can be worked out quite explicitly. We also have counterexamples showing that, in general, these chains of adjunctions extend no further.

In a closely related case of the endofunctor of smashing with a finite spectrum, we again have a chain of infinitely many adjunctions on both sides in the case of a complete universe. Generally, we always have a chain of three adjoints, and for the case of a finite group and the trivial universe, we have a chain of five adjoints (six adjoints in the case of a primary cyclic group). In both cases, these chains extend no further in general. These endofunctors, in fact, give us, at least in principle, a description of the right adjoint to pullback along an isometry of universes.

The purpose of the present paper is to treat these situations as completely as we are, at the moment, able, both in terms of positive statements and counterexamples, since they appear to be important for the foundations of equivariant stable homotopy theory. Here is a more detailed description of the situations we consider:

(A) **Restriction** It is a tradition from group cohomology to separate pullback with respect to a homomorphism of groups into the case when the homomorphism is injective (restriction) and surjective (inflation). For equivariant spectra too, the two cases behave somewhat differently, and for this reason, we too treat them separately. The forgetful functor $\mathrm{res} = \mathrm{res}_H^G \colon DG\text{–}\mathscr{U}\text{–spectra} \to DH\text{–}\mathscr{U}\text{–spectra}$, where $H \subseteq G$ is a closed subgroup of a compact Lie group G and $\mathscr{U}$ is *any G–universe* (not necessarily complete), is well known to have a left adjoint $G \ltimes_H ?$ and a right adjoint $F_H[G, ?)$. We show that $F_H[G, ?)$ also has a *right adjoint* Ξ_H^G. The left projection formula in the sense of [1, (3.11)] is well known to hold, but we show that the right projection formula in the sense of [1, (2.16)] is false in general. It is well known that if $\mathscr{U}$ is the complete G–universe, then the left and right adjoints to res_H^G are "shifts" of each other, and hence the chain of adjunctions described extends to an infinite chain of adjunctions on both sides. This is the Wirthmüller isomorphism. However, we show by example that for a general universe, $G \ltimes_H (?)$ may not have a left adjoint, and Ξ_H^G may not have a right adjoint. Thus, we have a chain of four adjoints in general. We show however that in the case when $\mathscr{U} = \mathbb{R}^\infty$ is the "trivial" universe and G is finite, $G \ltimes_H ?$ has two more adjoints to the left (thus giving a chain of a total of six adjoints), and when G is abelian and finite, $G \ltimes_H ?$ has three adjoints to the left (thus giving a chain of seven adjoints). In both cases, we have examples showing that this chain of adjoints may not extend any further.

(B) **Smashing with a finite spectrum** Let X be a retract of a finite cell G–$\mathscr{U}$–spectrum for any universe $\mathscr{U}$. The functor $X \wedge ?: DG$–$\mathscr{U}$–spectra $\to DG$–$\mathscr{U}$–spectra then has a right adjoint $F(X, ?)$. We show that this functor has a further right adjoint $R(X, ?)$. In the case when $\mathscr{U}$ is a complete universe, $F(X, ?) = DX \wedge ?$, and hence $R(X, ?) = X \wedge ?$. However, we show that in general, $R(X, ?)$ does not have a right adjoint. On the other hand, we show that $X \wedge ?$ always has two left adjoints (leading to a chain of at least five adjoints), and for $G = \mathbb{Z}/p$ it has exactly three (leading to a chain of six adjoints).

(C) **Change of universe** For an isometry of G–universes $i: \mathscr{U} \to \mathscr{V}$ for a compact Lie group G, the universe change functor $i^*: DG$–$\mathscr{V}$–spectra $\to DG$–$\mathscr{U}$–spectra is well known to have a left adjoint, which we denote by $i_\sharp$. We prove that it also has a right adjoint, which we denote by i_*. (In [9], $i_\sharp$ was denoted by i_*. However, in all sorts of contexts of sheaf theory, i_* is always the right adjoint, which is why we use the alternate notation.) We show that in general, $i_\sharp$ does not have a left adjoint, and i_* does not have a right adjoint. The right projection formula [1, (2.16)] is false. We have a chain of three adjoints in this case.

(D) **Inflation** For a compact Lie group G, a G–universe $\mathscr{U}$, and an onto homomorphism of compact Lie groups $G \to J = G/H$ for a closed normal subgroup H of G, we have the functor $\inf = \inf_G^J = {}_\mathscr{U}\inf_G^J: \mathscr{U}^H$–$J$–spectra $\to \mathscr{U}$–G–spectra. This functor is most universal when $\mathscr{U}$ is an H–fixed universe, since in general we have

$$(1) \qquad\qquad {}_\mathscr{U}\inf_G^J = i_\sharp \circ {}_{\mathscr{U}^H}\inf_G^J,$$

where $i: \mathscr{U}^H \to \mathscr{U}$ is the inclusion. In the case when $\mathscr{U}$ is an H–fixed universe, $\inf_G^J$ has a left adjoint $?/H$ and a right adjoint $(?)^H$ which has a right adjoint $\widetilde{E[H]} \wedge \inf_G^J$, which has a further right adjoint $F(\widetilde{E[H]}, ?)^H$. No further right or left adjoints exist, so this is a chain of five adjoints. Further, the left and right projection formulas [1, (3.11), (2.16)] hold and $\widetilde{E[H]}$ is the dualizing object in the sense of [1], even though the assumptions of [1] are not satisfied. If $\mathscr{U}$ is an arbitrary universe, then in general, ${}_\mathscr{U}\inf_G^J$ has no left adjoint, and its right adjoint $(?)^H$ has a right adjoint $i_*(\widetilde{E[H]} \wedge \inf_G^J)$, which in general has no right adjoint, so this is a chain of three adjoints. The right projection formula is true. In the case when $\mathscr{U}$ is a complete universe, this case satisfies the assumptions of [1].

(E) **Geometric fixed points** For a compact Lie group G and any G–universe $\mathscr{U}$, and $J = G/H$ for a closed normal subgroup $H \subseteq G$, we may consider the functor $\Phi^H = {}_\mathscr{U}\Phi^H = (\widetilde{E[H]} \wedge ?)^H: DG$–$\mathscr{U}$–spectra $\to DJ$–$\mathscr{U}^H$–spectra. This functor is

most universal when $\mathscr{U}$ is the largest universe with given $\mathscr{U}^H$ (up to isomorphism), since for a general embedding $i\colon \mathscr{U} \to \mathscr{V}$, where $\mathscr{V}^H \cong \mathscr{U}^H$, we have

$$\mathscr{U}\Phi^H = \mathscr{V}\Phi^H \circ i_\sharp.$$

In the case when $\mathscr{U}$ is a complete universe, the assumptions of [1] are satisfied, and a 3–duality therefore holds. In other words, $\mathscr{U}\Phi^H$ has a right adjoint, which again has a right adjoint, and the projection formula holds. This adjunction in general extends no further, so this is a case of 3–duality in the sense of [1]. If $\mathscr{U}$ is not a complete universe, we still have a 3–duality, and the projection formula still holds.

2   The main results

Let us begin by reviewing the setup of Balmer, Dell'Ambrogio and Sanders [1]. In the greatest generality, they talk about triangulated categories. A triangulated category $\mathscr{T}$ is called *compactly generated* if it has coproducts, and has a set of compact objects $\mathscr{G}$ which generate $\mathscr{T}$. To generate means that if for $x \in \mathrm{Obj}(\mathscr{T})$, for every $z \in \mathscr{G}$, $\mathscr{T}(z, x) = 0$, then $x = 0$. An object x of $\mathscr{T}$ is called compact if $\mathscr{T}(x, ?)$ sends coproducts in $\mathscr{T}$ to coproducts of abelian groups.

In this paper, we consider the derived categories of G–$\mathscr{U}$–spectra, where G is a compact Lie group, and $\mathscr{U}$ is a G–universe. These categories are compactly generated, where the generators are (integral) suspensions of suspension spectra of orbits (by closed subgroups). These spectra generate essentially by definition of the derived category; see [9; 3]. The fact that these generators are compact is widely known and widely used, but since we could not locate a proof in the literature, we present one in the appendix.

The authors of [1] use the following two facts to construct adjoint functors:

Lemma 2.1 [1, Corollary 2.3] *Let $F\colon \mathscr{T} \to \mathscr{S}$ be an exact (triangle-preserving) functor between triangulated categories, where $\mathscr{T}$ is compactly generated. Then*

(a) *F has a right adjoint if and only if it preserves coproducts,*

(b) *F has a left adjoint if and only if it preserves products.* □

Lemma 2.2 [1, Lemma 2.5] *Let* $F\colon \mathscr{T} \to \mathscr{S}$ *be left adjoint to* $G\colon \mathscr{S} \to \mathscr{T}$, *where* F *and* G *are exact functors between triangulated categories, and* $\mathscr{T}$ *is compactly generated. Then* F *preserves compact objects if and only if* G *preserves coproducts.* ☐

The authors of [1] consider a functor $f^*\colon \mathscr{T} \to \mathscr{S}$ and investigate patterns of adjunction of the forms

$$(2) \qquad\qquad \Big\downarrow f^* \quad \Big\uparrow f_* \quad \Big\downarrow f^{(1)}$$

and

$$(3) \qquad\qquad \Big\uparrow f_{(1)} \quad \Big\downarrow f^* \quad \Big\uparrow f_* \quad \Big\downarrow f^{(1)} \quad \Big\uparrow f_{(-1)}$$

Their assumption is that $f^*\colon \mathscr{T} \to \mathscr{S}$ is an exact functor between compactly generated tensor triangulated categories which preserves the symmetric monoidal structure, and preserves coproducts. They additionally assume that both in $\mathscr{T}$ and $\mathscr{S}$, compact objects are strongly dualizable. Under these assumptions, they prove that (2) always occurs, and additionally, one has the *right projection formula* stating that we have an isomorphism

$$(4) \qquad\qquad x \wedge f_*(y) \xRightarrow{\cong} f_*(f^*(x) \wedge y),$$

where (4) is the canonical morphism. (We write the symmetric monoidal structure as $\wedge$, since in the topological contexts we discuss, it is always the smash product.) The dualizing object by definition is $f^{(1)}(\mathbb{1})$, where $\mathbb{1}$ is the unit of the symmetric monoidal structure (in all our cases, this is the sphere spectrum S in the appropriate category). Then the authors of [1] prove that the additional right adjoint $f_{(-1)}$ exists if and only if the additional left adjoint $f_{(1)}$ exists, leading to the (3) scenario. Additionally, if that happens, they prove the *left projection formula*

$$(5) \qquad\qquad f_{(1)}(f^*(x) \wedge y) \xRightarrow{\cong} x \wedge f_{(1)}(y),$$

where (5) is, again, the canonical morphism.

The authors of [1] also prove that if either $f_{(1)}$ admits a left adjoint or $f_{(-1)}$ admits a right adjoint, then (3) extends to an infinite chain of adjunctions on both sides.

The category DG–$\mathscr{U}$–spectra for a compact Lie group G and a universe $\mathscr{U}$ is always compactly generated, where the compact generators are (de)suspensions of suspension spectra of orbits. Further, compact objects are strongly dualizable when $\mathscr{U}$ is a complete

universe (ie contains representatives of all isomorphism classes of finite-dimensional G–representations). However, compact objects are not strongly dualizable in general, notably when G is nontrivial and $\mathscr{U}$ is the trivial universe, containing only copies of the trivial representation. Therefore, the conclusions of [1] do not, strictly speaking, apply to most of our situations. Nevertheless, Lemmas 2.1 and 2.2 have the immediate:

Corollary 2.3 *If a functor* $f^*\colon \mathscr{T} \to \mathscr{S}$ *is an exact functor, where* $\mathscr{T}, \mathscr{S}$ *are compactly generated triangulated categories, preserves compact objects and coproducts, then it has a right adjoint* f_* *which in turn has another right adjoint* $f^{(1)}$, *ie the scenario* (2) *occurs.*

Proof By Lemma 2.1, a right adjoint f_* exists, and by Lemma 2.2, it preserves coproducts. Additionally, since $\mathscr{T}$ is compactly generated, distinguished triangles can be tested by long exact sequences on morphism groups from compact objects, so f_* preserves distinguished triangles by adjunction and by the fact that f^* preserves compact objects. Therefore, the additional right adjoint $f^{(1)}$ exists by Lemma 2.2. $\square$

Comment When a triangulated category is compactly generated, compact objects are precisely objects of the smallest thick subcategory generated by the compact generators. Therefore, for the functor f^* to preserve compact objects, it is sufficient to show that it sends the compact generators to compact objects.

In the scenarios described in the introduction, the functors res_H^G of case (A), $X \wedge ?$ of case (B), $i_\sharp$ of case (C), $\inf_G^J$ of case (D) and $_{\mathscr{U}}\Phi^G$ of case (E) are all exact functors which preserve compact generators and coproducts, so Corollary 2.3 applies if we take these functors for f^*. In other words, a right adjoint f_* exists which, in turn, has again a right adjoint $f^{(1)}$. Interestingly, the functors $f^{(1)}$ in this case do not appear to have been noticed in most cases. In this note, we will consider some examples.

Before that, however, let us discuss the (3) scenario, ie the case of a chain of five adjoints. It turns out that under our weaker assumptions, it is false that the existence of one of the adjoints $f_{(-1)}$, $f_{(1)}$ would imply the existence of the other. Nevertheless, the existence of the functors $f_{(1)}$ and $f_{(-1)}$ can still be tested using the following:

Corollary 2.4 *If* $f^*\colon \mathscr{T} \to \mathscr{S}$ *is an exact functor between compactly generated triangulated categories which preserves coproducts and compact objects. Then the functor* $f^{(1)}$ *has a right adjoint if and only if* f_* *preserves compact objects, and* f^* *has a left adjoint if and only if it preserves products.*

Proof The first statement follows from Lemmas 2.2 and 2.1(a), and the second statement follows from Lemma 2.1(b). ☐

In case (A) and (B) of the introduction, the functor $f_{(1)}$ exists, but in general the functor $f_{(-1)}$ does not. In case (C), in general neither $f_{(1)}$ nor $f_{(-1)}$ exists. In case (D) for a trivial universe, both $f_{(1)}$ and $f_{(-1)}$ exist, and in cases (E) for the complete universe, neither $f_{(1)}$ nor $f_{(-1)}$ exists in general.

Additionally, in cases (A), (C), (D) and (E), the functor f^* preserves symmetric monoidal structure, and hence we can ask about the projection formulas (4) and (5). In case (A), the formula (5) holds, but the formula (4) is false in general. In case (C), the formula (4) is false in general. In case (D) for a trivial universe, the formula (5) is false in general, and the formula (5) is true for the trivial universe but false in general. In case (E), the formula (4) for a complete universe holds (this follows from Theorem 2.15 of [1], since the assumptions in this case are satisfied). For a general universe, the formula (4) is false in this case.

We now turn to discussing each case of the introduction individually in more detail.

2.1 Inflation for the case of an H–fixed universe

We discuss this part of case (D) first, since it will be used in our other discussions. Here we have a normal subgroup H of G, $J = G/H$, and $f^* = \inf_G^J \colon DJ\text{–}\mathscr{U}\text{–spectra} \to DG\text{–}\mathscr{U}\text{–spectra}$, where $\mathscr{U} = \mathscr{U}^H$ is an H–trivial G–universe. Unless $H = \{e\}$, the assumptions of [1] are never satisfied, because the suspension spectrum of the orbit $G/\{e\}$ is not strongly dualizable in $DG\text{–}\mathscr{U}\text{–spectra}$. Nevertheless, we have the "5–scenario" of [1]. A left adjoint of f^* is $f_{(1)} = \,?/H$ (more precisely its left derived functor, ie ? should be a cell spectrum). On the other hand, a right adjoint is $f_* = \,?^H$, which in turn has a right adjoint $f^{(1)} = (\inf_G^J(?)) \wedge \widetilde{E[H]}$, where $E[H] = E\mathscr{F}[H]$ is the classifying space of the family $\mathscr{F}[H]$ of closed subgroups of G which H is not subconjugate to. (Note that since H is normal, this is just the family of all subgroups not containing H.) Here $\widetilde{X}$ means the unreduced suspension of a G–space X. To see this, mapping a $G\text{–}\mathscr{U}$–cell spectrum X to $\inf_G^J Y \wedge \widetilde{E[H]}$, where Y is a $J\text{–}\mathscr{U}$–cell spectrum, the cells of X with isotropy superconjugate to H (which cannot be attached to cells with isotropy not superconjugate to H, since $\mathscr{U}$ is H–fixed) must map to Y, while there is no obstruction to mapping any other cells to $\inf_G^J Y \wedge \widetilde{E[H]}$; a similar argument applies to homotopies. The functor $f^{(1)}$ has a right adjoint $f_{(-1)} = F(\widetilde{E[H]}, ?)^H$. It is also worth noting that not only the functor f^* but

also the functor f_* is strongly symmetric monoidal, while $f^{(1)}$ preserves the smash product but not the unit.

The left projection formula (5) states that

$$(6) \qquad (\inf_G^J(X) \wedge Y)/H \xrightarrow{\sim} X \wedge (Y/H),$$

which is true. In effect, both sides clearly preserve homotopy cofibers, so it suffices to consider the case when $X = J/K_+$ for some closed subgroup $K \subseteq J$. Then letting $\tilde{K}$ be the inverse image of K via the projection $G \to J$, $\inf_G^J(X) = G/\tilde{K}_+$. Now again by preservation of homotopy cofibers, we may also assume that $Y = G/\Gamma_+$ for some closed subgroup $\Gamma \subseteq G$, at which point (6) follows from the analogous consideration on spaces (G–orbits). (We will show below in Section 2.5 that for a G–universe $\mathcal{U}$ which is not H–fixed, inflation does not have a left adjoint, so the left projection formula makes no sense.)

The right projection formula (4) states that

$$X \wedge (Y^H) \xrightarrow{\sim} (\inf_G^J(X) \wedge Y)^H,$$

which is also true by a similar induction. (In fact, the right projection formula is true even without assuming that $\mathcal{U}$ is H–fixed; see Section 2.5 below.)

The dualizing object in this case is the G–$\mathcal{U}$–suspension spectrum of $\widetilde{E[H]}$. The functor $f^{(1)}$ does not generally preserve compact objects since the dualizing object is not compact. (For example for $G = H = \mathbb{Z}/2$, and the trivial universe, it suffices to show that $E\mathbb{Z}/2_+$ is not compact, which, since $?/H$ preserves compact objects, reduces to showing that $B\mathbb{Z}/2_+$ is not a compact spectrum. This is well known and also follows, for example, from Lemma A.1 of the appendix.) Thus, $f_{(-1)}$ in general does not have a right adjoint.

On the other hand, $f_{(1)}$ does not in general preserve products. Again, for $G = H = \mathbb{Z}/2$ and the trivial universe, consider the countable product of the spectra $E\mathbb{Z}/2_+ \wedge K$ (where K is, say, the fixed K–theory $\mathbb{Z}/2$–spectrum). The countable product of these spectra still has trivial fixed points, so the canonical map

$$E\mathbb{Z}/2_+ \wedge \prod_{\mathbb{N}} K \to \prod_{\mathbb{N}} (E\mathbb{Z}/2_+ \wedge K)$$

is an equivalence. Applying $?/(\mathbb{Z}/2)$ and using (6), we get

$$B\mathbb{Z}/2_+ \wedge \prod_{\mathbb{N}} K \xrightarrow{\sim} \left(\prod_{\mathbb{N}} (E\mathbb{Z}/2_+ \wedge K) \right) \Big/ \mathbb{Z}/2 \to \prod_{\mathbb{N}} (B\mathbb{Z}/2_+ \wedge K),$$

where the second map is the map which would be an equivalence if $f_{(1)}$ preserved products, while the composition is (33) of the appendix, which is not an equivalence by Lemma A.1. Thus, in general, $f_{(1)}$ does not have a left adjoint.

2.2 Restriction

Let $f^* = \mathrm{res}_H^G \colon DG\text{–}\mathscr{U}\text{–spectra} \to DH\text{–}\mathscr{U}\text{–spectra}$ be the forgetful functor, where $\mathscr{U}$ is any G–universe. We have a left adjoint $f_{(1)} = G \ltimes_H ?$ and a right adjoint $f_* = F_H[G, ?)$. It is well known [9] that the left projection formula (5) holds. By Corollary 2.3, f_* always has another right adjoint, $f^{(1)}$. When $\mathscr{U}$ is the complete universe, the assumptions of [1] are satisfied, of course, and in fact the classical Wirthmüller isomorphism [9] asserts that we have an infinite chain of adjunctions. To get a feel for what the functor $f^{(1)}$ is like in general, let us consider an example.

Let us consider the case when $G = \mathbb{Z}/2$, $H = \{e\}$, and $\mathscr{U} = \mathbb{R}^\infty$ is the trivial universe. In this case, we can construct a cofibration sequence for an $\{e\}$–spectrum X

$$(7) \qquad \mathbb{Z}/2_+ \wedge X \to F(\mathbb{Z}/2_+, X) \to \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X$$

as follows: One has $f^* f_*(X) = X \times X$ since, nonequivariantly, G is a two-point set, and the first map (7) is the adjoint to the canonical map $X \to X \times \{*\} \to X \times X$. The first morphism (7) is an equivalence nonequivariantly by stability (as it is the canonical morphism from the coproduct to the product of two copies of X), but the source has trivial fixed points, while the target has fixed points X (embedded diagonally). Thus, the cofiber in (7) has fixed points X and is trivial nonequivariantly.

Now recall that the category $DG\text{–}\mathbb{R}^\infty\text{–spectra}$ (naive G–spectra) for G finite is equivalent to the diagram derived category of functors from the orbit category $\mathscr{O}_G$ into spectra. This folklore fact about $G\text{–}\mathbb{R}^\infty$–spectra is proved the same way as for G–spaces: For G–spaces, the forgetful functor U from $\mathscr{O}_G^{\mathrm{Op}}$–spaces to G–spaces (by taking $X \mapsto X_{G/\{e\}}$) is left adjoint to the functor Ψ, where for a G–space X, we put $\Psi(X)(G/H) = X^H$ (the right Kan extension). If we take objectwise equivalences on $\mathscr{O}_G^{\mathrm{Op}}$–spaces, and let Ψ create equivalences on G–spaces, it is formal that the left derived functor LU is an inverse to the total derived functor $D\Psi$ on derived categories.

Now since the first map (7) is an equivalence nonequivariantly, and the first term has trivial fixed points, the homotopy cofiber of that map is trivial nonequivariantly and has fixed points X. Thus (see [9, Section II.8]) it maps into $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X$, and

the map induces an equivalence on fixed points as well as nonequivariantly, and thus, is an equivalence.

Now (7) can be written in the framework (2) as

$$
(8) \qquad\qquad f_{(1)}(X) \to f_*(X) \to \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X.
$$

By the results of Section 2.1, the last term of the cofibration sequence (8) is in fact the right adjoint to the functor $(?)^{\mathbb{Z}/2}$ on derived categories.

In fact, while it is not crucial for what follows, it is interesting to realize that for $X = S$, the cofibration (7) can be realized geometrically by stabilizing the canonical cofibration sequence

$$
(9) \qquad\qquad S^n \vee S^n \to S^n \times S^n \to S^n \wedge S^n,
$$

where all terms are given the $\mathbb{Z}/2$–equivariant structure with the generator of $\mathbb{Z}/2$ switching factors. The last term is equivariantly homeomorphic to $S^{n+n\alpha}$, where α is the 1–dimensional real sign representation. Desuspending n times and taking a colimit with $n \to \infty$ (using the fact that $S^{\infty\alpha}$ is a model for $\widetilde{E\mathbb{Z}/2}$), the connecting map of (9), after stabilization, gives a model of the connecting map of (7) for $X = S$.

The connecting map of (9) can then be described as a map

$$
(10) \qquad\qquad S^{n+n\alpha} \to \mathbb{Z}/2_+ \wedge S^{n+1}
$$

given as the (trivial) suspension of the following map: In S^{2n-1}, consider an embedding of $S^{n-1} \times S^{n-1}$, thus splitting S^{2n-1} into two solid tori $S^{n-1} \times D^n$. Consider the $\mathbb{Z}/2$–equivariant structure where the generator swaps the two factors of $S^{n-1} \times S^{n-1}$, and the two solid tori. Then we get a map into $S^n \vee S^n$ by collapsing $S^{n-1} \times S^{n-1}$ to a point, and projecting each solid torus $S^{n-1} \times D^n$ to D^n (with the boundary collapsed to a point). This map is equivariant when we consider on $S^n \vee S^n$ the $\mathbb{Z}/2$–equivariant structure where the generator swaps factors.

We can see that the connecting map is nontrivial by observing that applying $(?)/(\mathbb{Z}/2)$ to the first morphism (7), we obtain the canonical map

$$
(11) \qquad\qquad X \to \mathrm{Sp}^2(X),
$$

which does not split for $X = S$ by considering Steenrod operations; see [11; 10].

Let us study the cofibration sequence (7) in more detail. The second morphism in (7) was constructed using obstruction theory. More explicitly, we have

$$(12) \qquad\qquad F(\mathbb{Z}/2_+, X)^{\mathbb{Z}/2} = X = (\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X)^{\mathbb{Z}/2},$$

and $\mathrm{res}_{\{e\}}^{\mathbb{Z}/2}(\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X)$ is contractible, and hence for a cell $\{e\}$–spectrum X, the (nonequivariant) space $\mathscr{Q}_X$ of $\mathbb{Z}/2$–equivariant morphisms

$$F(\mathbb{Z}/2_+, X) \to \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X$$

extending the morphism (12), is contractible. Now for each $\alpha \in \mathscr{Q}_X$, the composite morphism

$$(13) \qquad\qquad \mathbb{Z}/2_+ \wedge X \xrightarrow{\iota} F(\mathbb{Z}/2_+, X) \xrightarrow{\alpha} \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X,$$

where ι is the first morphism of (7), is null-homotopic (since the source has trivial fixed points), and for the same reason, the space $\mathscr{S}_X$ of pairs (α, h), where h is a null-homotopy of (13), is also contractible. Note that specifying an element of $\mathscr{S}_X$ is equivalent to specifying a morphism of $\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectra

$$(14) \qquad\qquad C(\iota) \to \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X$$

extending the identification (12) (where C denotes the mapping cone). Thus, the space of such morphisms is contractible. Further, every such morphism (14) is a $\mathbb{Z}/2$–equivariant equivalence.

Similarly, one also sees that the space of morphisms

$$(15) \qquad\qquad \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X \to C(\iota)$$

which are the identity on fixed points (using the identification (12)) is contractible, and that each of those morphisms is an equivalence. Even more generally, if we denote the morphism ι of (13) more specifically by ι_X, then for any morphism of $\{e\}$–spectra

$$(16) \qquad\qquad f\colon X \to Y,$$

the space of all morphisms

$$\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X \to C(\iota_Y)$$

which restrict to f on fixed points is contractible. This implies, in particular, that the morphism (15), and hence its composition

$$(17) \qquad\qquad \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X \to \Sigma\mathbb{Z}/2_+ \wedge X$$

with the canonical natural morphism

$$C(\iota) \to \Sigma\mathbb{Z}/2_+ \wedge X,$$

is natural in the derived category. (See [9, Section II.8] for further discussion of this method.)

Now the natural transformation (17) in $D\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectra has a right adjoint

$$(18) \qquad \Omega(\operatorname{res}_{\{e\}}^{\mathbb{Z}/2} Z) \to F(\widetilde{E\mathbb{Z}/2}, Z)^{\mathbb{Z}/2}$$

(here Z is a $\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectrum, and (18) is a morphism of $\{e\}$–spectra). Continuing (17) to the right, we get a sequence of functors in $D\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectra

$$\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} X \to \Sigma\mathbb{Z}/2_+ \wedge X \to \Sigma F(\mathbb{Z}/2_+, X),$$

which is a cofibration sequence objectwise. Therefore, we have a right adjoint sequence

$$(19) \qquad \Omega F(\widetilde{E\mathbb{Z}/2}, Z) \to \Omega(\operatorname{res}_{\{e\}}^{\mathbb{Z}/2} Z) \to F(\widetilde{E\mathbb{Z}/2}, Z)^{\mathbb{Z}/2},$$

where the composition of the two maps (19) is 0, since the adjoint of the 0 morphism is 0. Additionally, however, (19) induces a long exact sequence on homotopy groups by the adjunction, and thus is a cofibration sequence objectwise.

Using stability, we have a natural sequence of right adjoints on the level of derived categories:

$$(20) \qquad F(\widetilde{E\mathbb{Z}/2}, Z)^{\mathbb{Z}/2} \to f^{(1)}(Z) \to f^*(Z),$$

where Z is a $\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectrum. It is worth noting that it is not obvious how to conclude this directly on the level of triangulated categories: We may define a distinguished triangle of functors

$$A \to B \to C \to \Sigma A$$

as a sequence of natural transformations which is a distinguished triangle on every object. However, it is then not obvious that if A and one of the functors B or C have a right or left adjoint, so does the third. Additionally, even if this occurs, ie A, B and C have right (or left) adjoints A', B' and C', it is then not obvious that the adjoint triangle

$$C' \to B' \to A' \to \Sigma C'$$

is distinguished. (This seems, in fact, like an interesting problem.)

This is the reason why a modern algebraic topologist seldom works fully in the triangulated category directly, and always, implicitly or explicitly, has the underlying "point set-level" (ie nonderived) category of spectra in mind. We shall see more examples of this technique below.

Note that from the cofibration sequence (20) it follows that in this case $f^{(1)}$ does not have a right adjoint, since f_* does but $F(\widetilde{E\mathbb{Z}/2}, ?)^{\mathbb{Z}/2}$ does not, as already shown in Section 2.1. In effect, by Corollary 2.4, it suffices to observe that in (8), f_* does not preserve compact objects, since $f_{(1)}$ preserves them and $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}}?$ does not (since $F(\widetilde{E\mathbb{Z}/2}, ?)^{\mathbb{Z}/2}$ has no derived right adjoint).

However, using the same method, from formula (7), we see that inductively, $\mathbb{Z}/2_+ \wedge ?$ has as many left adjoints as the last term of the cofibration $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}}?$. They key point is that the homotopy fiber F functor from the category of morphisms of (nonderived) spectra has a left adjoint, namely the homotopy cofiber (ie mapping cone) functor C. Additionally, for a morphism $f\colon X \to Y$ in the nonderived category of spectra, if we denote by $i\colon Y \to Cf$ the canonical natural morphism, then we have a canonical natural morphism $X \to Fi$, which is an equivalence if X, Y are cell. Similar comments apply to G–$\mathcal{U}$–spectra for any G, $\mathcal{U}$. Thus, starting from a homotopy cofiber sequence of functors on the nonderived level, we may replace it by a homotopy fiber sequence on the nonderived level, which has a left adjoint, which is a homotopy cofiber sequence. Additionally, if the functors in question preserve homotopy, we get a corresponding adjunction on the derived level. Provided two of the left adjoint functors have again left adjoints on the nonderived level, we can iterate this procedure, starting at every stage on the nonderived level, and noting we have a corresponding adjunction on the left derived level. Applying this to (7), when $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}}?$ has no further iterated left adjoints, by Corollary 2.4, it does not preserve products, and hence the corresponding iterated left adjoint of $\mathbb{Z}/2_+ \wedge ?$ does not, as the remaining term of the (co)fiber sequence does. (More details could be given here, but below we will actually compute these functors explicitly.) As we showed in Section 2.1, the left derived functor of $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}}?$ has a sequence of a total of three left adjoints, and not more.

Thus, in this case, precisely the functors $f_{(-1)}$, $f_{(2)}$, $f^{(-2)}$ exist, leading to a chain of a total of seven adjoints. In fact, these functors can be described geometrically by taking successive left adjoints of (7): we have

$$f^{(-1)}X = \mathrm{res}_{\{e\}}^{\mathbb{Z}/2} X / X^{\mathbb{Z}/2}.$$

Additionally, $f_{(2)}Y$ is the fiber of the canonical morphism $\mathbb{Z}/2_+ \wedge Y \to \inf_{\mathbb{Z}/2}^{\{e\}}Y$, and $f^{(-2)}X$ is the cofiber of a morphism

$$X/(\mathbb{Z}/2) \to \operatorname{res}_{\{e\}}^{\mathbb{Z}/2} X / X^{\mathbb{Z}/2},$$

which is a variant of the transfer. Again, we see that this functor has no left adjoint.

One can also see that the right projection formula (4) fails for the case $G = \mathbb{Z}/2$, $H = \{e\}$ and the trivial universe. In effect, if this formula were true, it would say (putting $Y = S$) that for a $\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectrum X, the canonical morphism

(21) $$X \wedge F(\mathbb{Z}/2_+, S) \to F(\mathbb{Z}/2_+, \inf_{\mathbb{Z}/2}^{\{e\}}\operatorname{res}_{\{e\}}^{\mathbb{Z}/2} X)$$

is an equivalence. Since the analogous statement replacing $F(\mathbb{Z}/2_+, ?)$ by $\mathbb{Z}/2_+ \wedge ?$ holds, this is equivalent to the canonical morphism

(22) $$\widetilde{E\mathbb{Z}/2} \wedge X \to \widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}}\operatorname{res}_{\{e\}}^{\mathbb{Z}/2} X$$

being an equivalence (since $\widetilde{E\mathbb{Z}/2} \wedge X \sim \widetilde{E\mathbb{Z}/2} \wedge (X^{\mathbb{Z}/2})_{\text{fixed}}$, the canonical morphism (22) is induced by the canonical morphism $X^{\mathbb{Z}/2} \to E_{\{e\}}$, which is also (22) on fixed points). Thus, (22) is not an equivalence when X is not fixed, and hence neither is (21).

What in this example can be generalized? Let us specialize to the case of a finite group G and the trivial universe. (The main significance of the finiteness being that the orbit category is finite.) In this case, the cofiber sequence (7) generalizes to

(23) $$G \ltimes_H X \to F_H[G, X] \to \widetilde{E\mathscr{F}(H)} \wedge F_H[G, X],$$

where $\mathscr{F}(H)$ is the family of all subgroups of G subconjugate to H. The cofibration sequence (23) can be used to gain information on both left and right adjoints. Recall that for a subgroup K of G which is not subconjugate to H, we have

$$F_H[G, X]^K \sim \prod_{a \in K \backslash G / H} X^{H \cap a^{-1}Ka}.$$

Denote, as usual, by $N(K)$ the normalizer of K, and $W(K) = N(K)/K$. Recall that, for a general (not necessarily normal) subgroup K of G, we have a functor $(?)^K \colon DG$–$\mathbb{R}^\infty$–spectra $\to DW(K)$–$\mathbb{R}^\infty$–spectra. As a $W(K)$–spectrum,

(24) $$F_H[G, X]^K \sim \bigvee_{[a] \in N(K) \backslash G / H} F_{W(K, H)}[W(K), X^{H \cap a^{-1}Ka}),$$

where

$$W(K, H) = \frac{a^{-1} Ka \cdot H \cap a^{-1} N(K)a}{a^{-1} Ka}.$$

We can express $\widetilde{E\mathscr{F}(H)} \wedge F_H[G, X]$ as a finite homotopy (co)limit of G–$\mathbb{R}^\infty$–spectra of the form

$$G \ltimes_{N(K)} (F_H[G, X]^K),$$

where K is not subconjugate to H, so by (24) and (23), we can, in principle, inductively write down a model for the right adjoint of $F_H[G, X]$ (since $|W(K)| < |G|$).

On the other hand, formula (23) can also be used to construct two left adjoints to $G \ltimes_H X$. By induction, again, it suffices to prove that the functor $\widetilde{E\mathscr{F}[H]} \wedge ?$ has two left adjoints. This follows from the following general fact, which can be traced back to [9].

Proposition 2.5 *For any family $\mathscr{F}$ of subgroups of a finite group G, the functor*

$$(25) \qquad \widetilde{E\mathscr{F}} \wedge ?: DG\text{–}\mathbb{R}^\infty\text{–spectra} \to DG\text{–}\mathbb{R}^\infty\text{–spectra}$$

has two left adjoints.

Proof Recall that the category DG–$\mathbb{R}^\infty$–spectra is equivalent to the diagram derived category of $\mathcal{O}_G^{\mathrm{Op}}$–spectra (ie with objectwise equivalences), where $\mathcal{O}_G$ is the orbit category of G. We can identify a family $\mathscr{F}$ with a full subcategory of $\mathcal{O}_G$ on the subgroups which belong to the family, and the corresponding *cofamily* $\widetilde{\mathscr{F}}$ with the full subcategory of $\mathcal{O}_G$ on all the remaining subgroups.

For an $\mathscr{F}^{\mathrm{Op}}$–spectrum X, a G–$\mathbb{R}^\infty$–spectrum $\widetilde{E\mathscr{F}} \wedge X$ is well defined, and if we denote this functor by ϕ_*, it is right adjoint to the forgetful functor ϕ^* from G–$\mathbb{R}^\infty$–spectra ($= \mathcal{O}_G^{\mathrm{Op}}$–spectra) to $\widetilde{\mathscr{F}}^{\mathrm{Op}}$–spectra. This functor then has a left adjoint $\phi_\sharp = \mathcal{O}_G^{\mathrm{Op}} \ltimes_{\widetilde{\mathscr{F}}} ?$. The left adjoint to (25) is $\phi_\sharp \phi^*$.

To show that a further left adjoint exists, by Corollary 2.4, it suffices to show that the functor $\phi_\sharp \phi^*$ preserves products. This follows from the following result. $\qquad\square$

Lemma 2.6 *Let $\mathscr{C}$ be a finite category in which every endomorphism has an inverse, and let F be a contravariant functor from $\mathscr{C}$ to the category of finite sets. Suppose that $\mathscr{C}(x, x)$ acts freely (from the right) on $F(x)$ for every $x \in \mathrm{Obj}(\mathscr{C})$. Then the functor*

$$(26) \qquad B_\wedge(F_+, \mathscr{C}_+, ?): \mathscr{C}\text{–spectra} \to \text{Spectra}$$

preserves products.

Proof In the case when $\mathscr{C}$ is a group, under our assumptions, the functor (26) is just a "sum of finitely many copies", so it preserves products. In the general case, it arises from the corresponding functors for the automorphism groups of $\mathscr{C}$ by a finite homotopy colimit over a poset, so the conclusion follows from stability. $\qquad\square$

It is not difficult to give an example of an inclusion of finite group $H \subset G$ where the functor $G \ltimes_H ?$ from H–$\mathbb{R}^\infty$–spectra to G–$\mathbb{R}^\infty$–spectra does not have three left adjoints. Let $G = \mathbb{Z}/4$ and $H = \mathbb{Z}/2$. In this case, (23) becomes

$$(27) \qquad (\mathbb{Z}/4) \ltimes_{\mathbb{Z}/2} X \to F_{\mathbb{Z}/2}[\mathbb{Z}/4, X) \to \widetilde{E[\mathbb{Z}/4]} \wedge X^{\mathbb{Z}/2},$$

so again by induction, the existence of three left adjoints to $(\mathbb{Z}/4) \ltimes_{\mathbb{Z}/2} ?$ would be equivalent to the existence of three left adjoints to $\widetilde{E[\mathbb{Z}/4]} \wedge X^{\mathbb{Z}/2}$. We know however that the left adjoint to that functor is $\inf_{\mathbb{Z}/2}^{\{e\}}(?)^{\mathbb{Z}/4}$, whose left adjoint, in turn, is $\inf_{\mathbb{Z}/4}^{\{e\}}(?/(\mathbb{Z}/2))$. This functor does not preserve products, since $?/(\mathbb{Z}/2)$ does not, and $\inf_{\mathbb{Z}/4}^{\{e\}}$ has a left inverse $?^{\mathbb{Z}/4}$, which preserves products. Thus, $\inf_{\mathbb{Z}/4}^{\{e\}}(?/(\mathbb{Z}/2))$ has no left adjoint, as claimed.

On the other hand, if G is finite abelian, the functor $G \ltimes ?$ from spectra to G–$\mathbb{R}^\infty$–spectra does have three left adjoints (and hence $\mathrm{res}^G_{\{e\}}$ has four left adjoints, leading to a chain of seven adjoints). Again, it suffices to show that the rightmost term of the cofibration sequence (23), which in this case is $\widetilde{EG} \wedge F[G, ?)$, has three left adjoints. Again, we can model $\widetilde{EG}$ as a finite homotopy colimit of $\widetilde{E[H]}$ for subgroups $\{e\} \neq H \subseteq G$, and therefore it suffices to show that $\widetilde{E[H]} \wedge F[G, ?) \sim \widetilde{E[H]} \wedge F[G, ?)^H$ has three left adjoints. However, for a spectrum X, $F[G, X)^H$ is a sum of (finitely many) copies of X, so we are reduced to showing that $\widetilde{E[H]} \wedge X$ from spectra to G–$\mathbb{R}^\infty$–spectra has three left adjoints. But the left adjoint to that functor is $(\mathrm{res}^G_H ?)^H$, which we already know has two left adjoints.

2.3 Smashing with a finite spectrum

Here we are not dealing with symmetric monoidal functors, so there is no discussion of projection formulas. The functor $X \wedge ?$ has a right adjoint $F(X, ?)$ for any spectrum X, which passes on to derived categories. Additionally, by Corollary 2.4, on derived categories, $F(X, ?)$ has an additional right adjoint if and only if $X \wedge ?$ preserves compact objects, which happens if and only if X is itself compact. If $\mathscr{U}$ is a complete universe, we of course have $F(X, ?) = DX \wedge ?$, so there is an infinite chain of adjunctions in both directions.

Note that if X is of the form G/H_+, where H is a closed subgroup of G, on the level of derived categories, $X \wedge ?$ is isomorphic to $G \ltimes (\mathrm{res}_H^G ?)$, so we already know that it has two right adjoints. In the case when X is finite, therefore, the cellular filtration on X gives a filtration on the second right adjoint, where the associated graded pieces can, in principle, be described by the methods of Section 2.2.

We can show the right adjoint to the endofunctor $F(\mathbb{Z}/2_+, ?)$ in $D\mathbb{Z}/2$–$\mathbb{R}^\infty$–spectra has no right adjoint. In effect, considering Corollary 2.4, if $F(\mathbb{Z}/2_+, ?)$ preserved compact objects, $F(\mathbb{Z}/2_+, \inf_{\mathbb{Z}/2}^{\{e\}} ?)$ (ie the middle term of (7)) would. So, since the first term of (7) preserves compactness, it would follow that $\widetilde{E\mathbb{Z}/2} \wedge \inf_{\mathbb{Z}/2}^{\{e\}} ?$ preserves compactness, which we already showed is not the case.

Regarding left adjoints, it follows from what we showed in Section 2.2 that the endofunctor $G \ltimes (\mathrm{res}_H^G ?)$, and hence the endofunctor $X \wedge ?$ for X a finite spectrum, in DG–$\mathbb{R}^\infty$–spectra with G finite has two left adjoints (again, we consider the functors on the nonderived level, and take strictly functorial homotopy (co)fibers, working inductively on the number of cells of X), thus leading to a chain of five adjoints involving $X \wedge ?$. Furthermore, for $G = \mathbb{Z}/p$, since the only orbits are trivial and $\mathbb{Z}/p$, also by the results of Section 2.2, we have a third left adjoint, leading to a chain of six adjoints.

In the case when $G = \mathbb{Z}/2$ and $\mathscr{U} = \mathbb{R}^\infty$, we can show that no further left adjoints exist. In effect, by the results of Section 2.2, the first left adjoint to $\mathbb{Z}/2 \ltimes \mathrm{res}_{\{e\}}^{\mathbb{Z}/2} X$ is $\mathbb{Z}/2 \ltimes (\mathrm{res}_{\{e\}}^{\mathbb{Z}/2} X / X^{\mathbb{Z}/2})$, and hence the second left adjoint is the fiber of the canonical morphism

$$(28) \qquad \mathbb{Z}/2 \ltimes (\mathrm{res}_{\{e\}}^{\mathbb{Z}/2} X / X^{\mathbb{Z}/2}) \to \inf_{\mathbb{Z}/2}^{\{e\}} (\mathrm{res}_{\{e\}}^{\mathbb{Z}/2} X / X^{\mathbb{Z}/2}).$$

Since the first one of these functors has two left adjoints, it suffices to show that the second one does not. Now the left adjoint to the second functor (28) is the cofiber of

$$\mathbb{Z}/2 \ltimes (X/(\mathbb{Z}/2)) \to \inf_{\mathbb{Z}/2}^{\{e\}} (X/(\mathbb{Z}/2)),$$

which does not preserve products (for example, applying $\mathrm{res}_{\{e\}}^{\mathbb{Z}/2}$, we get $X/(\mathbb{Z}/2)$ again, which we already showed does not preserve products). Thus, we have a chain of precisely six adjoints in this case.

In the case when $G = \mathbb{Z}/4$, $\mathscr{U} = \mathbb{R}^\infty$ and $X = (\mathbb{Z}/4)/(\mathbb{Z}/2)_+$, we can show that a third left adjoint does not exist, thus leading to a chain of precisely five adjoints. In effect, by (27), it suffices, again, to work with the endofunctor $\widetilde{E[\mathbb{Z}/4]} \wedge (\mathrm{res}_{\mathbb{Z}/2}^{\mathbb{Z}/4} X)^{\mathbb{Z}/2}$

instead. Its left adjoint is $\mathbb{Z}/4 \ltimes_{\mathbb{Z}/2} \inf_{\mathbb{Z}/2}^{\{e\}}(X^{\mathbb{Z}/4})$, which we can represent as the fiber of the morphism

$$F_{\mathbb{Z}/2}[\mathbb{Z}/4, \inf_{\mathbb{Z}/2}^{\{e\}}(X^{\mathbb{Z}/4})) \to \widetilde{E[\mathbb{Z}/4]} \wedge (\inf_{\mathbb{Z}/2}^{\{e\}}(X^{\mathbb{Z}/4}))^{\mathbb{Z}/2}.$$

The second functor is $\widetilde{E[\mathbb{Z}/4]} \wedge X^{\mathbb{Z}/4}$, which we already know has two left adjoints, so it suffices to show that $F_{\mathbb{Z}/2}[\mathbb{Z}/4, \inf_{\mathbb{Z}/2}^{\{e\}}(X^{\mathbb{Z}/4}))$ does not. In effect, its left adjoint is $\inf_{\mathbb{Z}/4}^{\{e\}}((\mathrm{res}_{\mathbb{Z}/2}^{\mathbb{Z}/4}X)/(\mathbb{Z}/2))$. This functor does not preserve products: since $\inf_{\mathbb{Z}/4}^{\{e\}}$ has a left inverse $\mathrm{res}_{\{e\}}^{\mathbb{Z}/4}$ which preserves products, it suffices to show that $(\mathrm{res}_{\mathbb{Z}/2}^{\mathbb{Z}/4}X)/(\mathbb{Z}/2)$ does not preserve products, but that follows from the same example as before.

2.4 Change of universe

Let $i\colon \mathscr{U} \to \mathscr{V}$ be an isometry of G–universes. Then recall [9, Section II.1] that a G–$\mathscr{V}$–spectrum Y as an object of the derived category can be described by describing the G–$\mathscr{U}$–spectra $i^*\Sigma^V Y$, where V runs through finite-dimensional G–representations contained in $\mathscr{V}$ (and it suffices to consider those representations V which do not have irreducible summands in $\mathscr{U}$). Thus, to describe i_*, it suffices to describe $i^*\Sigma^V i_*$, which is right adjoint to

$$(29) \qquad i^*\Sigma^{-V} i_\sharp X = \Omega^V i^* i_\sharp X = \mathop{\mathrm{hocolim}}_{W} \Omega^{W+V} \Sigma^W X,$$

where, again, W runs through subrepresentations of $\mathscr{V}$ with no irreducible summand contained in $\mathscr{U}$. But we know that Ω^W has right adjoint $R^W = R(S^W, ?)$, so the right adjoint to (29) is

$$(30) \qquad i^*\Sigma^V i_* X = \mathop{\mathrm{holim}}_{W} \Omega^W R^{W+V} X.$$

From this point of view, we have a description of the functor i_*.

Regarding additional adjoints, in general, $i_\sharp$ does not preserve products, and thus does not have a left adjoint. To see this, let us consider again the case $G = \mathbb{Z}/2$, where $\mathscr{U} = \mathbb{R}^\infty$ is the trivial universe and $\mathscr{V}$ is the complete universe. We will show that the functor

$$(i^*(i_\sharp(\inf_{\mathbb{Z}/2}^{\{e\}})))^{\mathbb{Z}/2}\colon D\mathrm{Spectra} \to D\mathrm{Spectra}$$

does not preserve products, which is sufficient, since the functors $(?)^{\mathbb{Z}/2}$, i^* and $\inf_{\mathbb{Z}/2}^{\{e\}}$ do preserve products. In effect, it is well known that we have a cofibration sequence

$$B\mathbb{Z}/2_+ \wedge X \to (i^*(i_\sharp(\inf_{\mathbb{Z}/2}^{\{e\}}X)))^{\mathbb{Z}/2} \to X$$

(see eg [9, Section II.7] or, even, much more explicitly, and generally, Section V.11), where the third term preserves products, and again, the connecting map is natural (it is, in fact, in this case 0), so it suffices to prove that the first term does not, which is Lemma A.1.

For the same reason, i^* does not preserve compact objects (since the functors $(?)^{\mathbb{Z}/2}$ and $\inf_{\mathbb{Z}/2}^{\{e\}}$ do), so i_* does not have a right adjoint.

The right projection formula (4), for $x = \mathbb{Z}/2_+$ and $y = S$ (where $G = \mathbb{Z}/2$, $\mathscr{U}$ is the trivial universe and $\mathscr{V}$ is the complete universe), would say that

$$\mathbb{Z}/2_+ \wedge i^*(S) \sim i^*(\mathbb{Z}/2_+).$$

We know that the right-hand side has fixed points by the Wirthmüller isomorphism, while the left-hand side does not. Therefore, (4) is false.

2.5 Inflation: the general case

By formula (1), the general case of the inflation reduces to the case of an H–fixed universe, and change of universes. However, it remains to resolve the question of how many adjoints we have, and the question of a projection formula.

In the case when $G = \mathbb{Z}/2$ and $\mathscr{U}$ is the complete universe, we have, again, for a spectrum X, a cofibration sequence

$$B\mathbb{Z}/2_+ \wedge X \to (\inf_{\mathbb{Z}/2}^{\{e\}} X)^{\mathbb{Z}/2} \to X,$$

thus showing that $(?)^{\mathbb{Z}/2}$ does not preserve compact objects, since $\inf_{\mathbb{Z}/2}^{\{e\}}$ does. Therefore, the right adjoint to $(?)^{\mathbb{Z}/2}$ in this case does not have a further right adjoint by Corollary 2.4. On the other hand, this situation actually satisfies the assumptions of [1], and therefore, we also know that the functor $\inf_{\mathbb{Z}/2}^{\{e\}}$ does not have a left adjoint.

We also know from [1] that in the case of inflation from a complete universe to a complete universe, the right projection formula (4) is satisfied. However, it turns out to be true in general, which is curious, since it is false for change of universe. In effect, in the general case, the right projection formula asserts an equivalence

$$(31) \qquad\qquad X \wedge (Y^H) \xrightarrow{\sim} ((\inf_G^J X) \wedge Y)^H.$$

Since both sides are stable under desuspensions by finite subrepresentations of $\mathscr{U}^H$, it suffices to consider the case when X is a J–space. In that case, however (when applied to a cell spectrum Y), the Vth space of both sides for V fixed under the action of H is the colimit of $\Omega^W(X \wedge Y_{V+W})$ over finite subrepresentations W of $\mathscr{U}$.

2.6 Geometric fixed points

As already remarked, the most universal case is the case of a complete universe. In that case, the assumptions of [1] are satisfied, so we know that there are two right adjoints, and the right projection formula (4) holds. This turns out to be the case in general. In fact, the geometric fixed point functor coincides with the fixed point functor in the case of an H–fixed universe, so this case also generalizes the rightmost three functors of the chain of five adjoints for inflation in the case when $\mathscr{U}$ is H–fixed. By the same arguments, then, one shows that in general, the right adjoint to Φ^H is

$$(32) \qquad \widetilde{E[H]} \wedge \inf_G^J(?),$$

which, in turn, has the right adjoint $(F(\widetilde{E[H]},?))^H$. It is easy to see that in the case of a complete universe with $G = \mathbb{Z}/2$ and $H = \{e\}$, (32) does not preserve compact objects, since $\widetilde{E\mathbb{Z}/2}$ is not compact (as $E\mathbb{Z}/2_+$ is not). Therefore, the second right adjoint to $\Phi^{\mathbb{Z}/2}$ does not have an additional right adjoint in this case, and by [1], $\Phi^{\mathbb{Z}/2}$ does not have a left adjoint. Thus, in this case, we have the "3–scenario" of [1].

In the case of a complete G–universe $\mathcal{U}$, we have the right projection formula by [1], but in fact, again, it is true in general: it asserts an equivalence (for cell spectra) of the form

$$X \wedge \widetilde{E[H]} \wedge \inf_J^G Y \xrightarrow{\sim} \Phi^H X \wedge \widetilde{E[H]} \wedge Y,$$

which holds for the same reason as in the case of an H–fixed universe.

Appendix

We record here some auxiliary results, which we consider known, but for which we could not find an easy reference.

Lemma A.1 *Let K denote* (*nonequivariant complex*) *periodic K–theory. Then the canonical morphism*

$$(33) \qquad B\mathbb{Z}/2_+ \wedge \left(\prod_{\mathbb{N}} K\right) \xrightarrow{\sim} \prod_{\mathbb{N}}(B\mathbb{Z}/2_+ \wedge K)$$

is not an equivalence.

Proof We are trying to show that in a particular case, Borel homology does not preserve products. Since Borel cohomology preserves products, it suffices to work

with Tate cohomology instead (recall the bottom row of the Tate diagram, which gives a cofibration sequence between Borel homology, Borel cohomology and Tate cohomology; for background on this, see [5]). Now the Borel cohomology of K has coefficients (in dimension 0) $\mathbb{Z} \oplus \mathbb{Z}_2$ where the Euler class maps the first summand to the second, so the Tate cohomology is $\mathbb{Q}_2$. Similarly, the Tate cohomology of a countable product of copies of K is $2^{-1}\big(\prod_{\mathbb{N}} \mathbb{Z}_2\big)$, the canonical map of which into $\prod_{\mathbb{N}} \mathbb{Q}_2$ is not an isomorphism (because of nonuniformity of denominators). □

Lemma A.2 *Let Y be a T_1–space and suppose we have an indexing set I, and for each $F \subset\subset I$ (meaning a finite subset) a subspace $Y_F \subseteq Y$ (with the induced topology) such that*

(1) $Y = \bigcup_{F \subset\subset I} Y_F$ *(with the colimit topology),*

(2) $Y_F \cap Y_G = Y_{F \cap G}$ *for $F, G \subset\subset I$,*

(3) $F \subseteq G \implies Y_F \subseteq Y_G$.

Suppose $f: X \to Y$ is a continuous map, where X is compact. Then there exists $F \subset\subset I$ such that $f(X) \subseteq Y_F$.

Proof If there exists no $F \subset\subset I$ with $F(X) \subseteq Y_F$, then by induction, we can construct sets

$$\varnothing = F_0 \subset F_1 \subset F_2 \subset \cdots \subset\subset I$$

and points

$$y_n \in (Y_{F_n} \cap f(X)) \smallsetminus Y_{F_{n-1}}$$

for $n = 1, 2, \dots$. (Once we have constructed F_{n-1}, our assumption $f(X) \nsubseteq Y_{F_{n-1}}$ implies there is a point $y_n \in f(X) \smallsetminus Y_{F_{n-1}}$. By assumption (1), $y_n \in Y_F$ for some $F \subset\subset I$, and we can take $F_n := F_{n-1} \cup F$.)

Claim If $G \subset\subset I$ and S is an infinite subset of $\{0, 1, 2, \dots\}$, then $\{y_s \mid s \in S\} \nsubseteq Y_G$.

Proof of claim To see this, select such a G and S. Then there is a natural number n such that

$$G \cap \left(\bigcup_n F_n\right) = G \cap F_n.$$

Assuming that $y_{n+1} \in Y_G$, we would have, by assumptions (2) and (3),

$$y_{n+1} \in Y_G \cap Y_{F_{n+1}} = Y_{G \cap F_{n+1}} = Y_{G \cap F_n} \subseteq Y_{F_n},$$

which is a contradiction. □

It follows from the claim and assumption (1) that for every n, the set

$$T_n = \{y_n, y_{n+1}, \dots\} \subseteq Y$$

is closed in Y. (Indeed, for any $F \subset\subset I$, the set $T_n \cap Y_F$ must be finite.) Hence

$$f^{-1}(T_1) \supseteq f^{-1}(T_2) \supseteq \cdots$$

are nonempty closed subsets of X, whereas

$$\bigcap f^{-1}(T_n) = f^{-1}\left(\bigcap T_n\right) = f^{-1}(\varnothing) = \varnothing,$$

contradicting the compactness of X. □

Corollary A.3 *Shift suspensions and desuspensions of suspension spectra of orbits are compact objects in the category $DG-\mathscr{U}$–spectra for any compact Lie group G and any universe $\mathscr{U}$.*

Proof Consider $G-\mathscr{U}$–cell spectra Z_i for $i \in I$. The assumptions of Lemma A.2 are satisfied first for a wedge of based spaces, and hence are also satisfied for

$$Y = \operatorname*{colim}_W \Omega^W \left(\bigvee_{i \in I} (Z_i)_{V+W}\right) \quad \text{and} \quad Y_F = \operatorname*{colim}_W \Omega^W \left(\bigvee_{i \in F} (Z_i)_{V+W}\right).$$

Now recall from [9, Sections I.2, I.3] that the spaces Y and Y_F are the constituent spaces of coproducts of the spectra Z_i over I and F. Additionally, maps from shift suspensions and desuspensions of suspension spectra of orbits into a spectrum are, by adjunction, the same thing as maps from suspensions of orbits into the constituent spaces, at which point we can apply Lemma A.2. □

References

[1] **P Balmer**, **I Dell'Ambrogio**, **B Sanders**, *Grothendieck–Neeman duality and the Wirth-müller isomorphism*, Compos. Math. 152 (2016) 1740–1776 MR

[2] **P Balmer**, **B Sanders**, *The spectrum of the equivariant stable homotopy category of a finite group*, Invent. Math. 208 (2017) 283–326 MR

[3] **A D Elmendorf**, **I Kriz**, **M A Mandell**, **J P May**, *Rings, modules, and algebras in stable homotopy theory*, Math. Surveys Monogr. 47, Amer. Math. Soc., Providence, RI (1997) MR

[4] **H Fausk**, **P Hu**, **J P May**, *Isomorphisms between left and right adjoints*, Theory Appl. Categ. 11 (2003) 107–131 MR

[5]   **J P C Greenlees**, **J P May**, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. 543, Amer. Math. Soc., Providence, RI (1995) MR

[6]   **M A Hill**, **M J Hopkins**, **D C Ravenel**, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. 184 (2016) 1–262 MR

[7]   **P Hu**, **I Kriz**, *Real-oriented homotopy theory and an analogue of the Adams–Novikov spectral sequence*, Topology 40 (2001) 317–399 MR

[8]   **P Hu**, **I Kriz**, **P Somberg**, *Derived representation theory and stable homotopy categorification of* sl_k, preprint (2016) Available at `http://www.math.lsa.umich.edu/~ikriz/drt16084.pdf`

[9]   **L G Lewis, Jr**, **J P May**, **M Steinberger**, **J E McClure**, *Equivariant stable homotopy theory*, Lecture Notes in Math. 1213, Springer (1986) MR

[10]   **R J Milgram**, *The homology of symmetric products*, Trans. Amer. Math. Soc. 138 (1969) 251–265 MR

[11]   **M Nakaoka**, *Cohomology mod p of the p–fold symmetric products of spheres*, J. Math. Soc. Japan 9 (1957) 417–427 MR

[12]   **B Sanders**, *The compactness locus of a geometric functor and the formal construction of the Adams isomorphism*, preprint (2016) arXiv

Department of Mathematics, Wayne State University
Detroit, MI, United States

Department of Mathematics, University of Michigan
Ann Arbor, MI, United States

Mathematical Institute of MFF UK
Praha 8, Czech Republic

po@math.wayne.edu,   kriz.igor@gmail.com,   somberg@karlin.mff.cuni.cz