A May-type spectral sequence for higher topological Hochschild homology

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Given a filtration of a commutative monoid A in a symmetric monoidal stable model category $\mathcal{C}$, we construct a spectral sequence analogous to the May spectral sequence whose input is the higher order topological Hochschild homology of the associated graded commutative monoid of A, and whose output is the higher order topological Hochschild homology of A. We then construct examples of such filtrations and derive some consequences: for example, given a connective commutative graded ring R, we get an upper bound on the size of the THH–groups of E_∞–ring spectra A such that $\pi_*(A) \cong R$.

1 Introduction

Suppose $A = F_0 A \supseteq F_1 A \supseteq F_2 A \supseteq \cdots$ is a filtered augmented k–algebra, where k is a field. In his 1964 PhD thesis, J P May [33] sets up a spectral sequence with input $\mathrm{Ext}^{*,*}_{E_0^* A}(k, k)$ which converges to $\mathrm{Ext}^*_A(k, k)$. Here $E_0^* A = \bigoplus_{n \geq 0} F_n A / F_{n+1} A$ is the associated graded algebra of the filtration of A.

In the present paper, we construct an analogous spectral sequence for topological Hochschild homology and its "higher order" generalizations (as in Pirashvili [37]). Given a filtered E_∞–ring spectrum A, we construct a spectral sequence

$$\text{(1)} \qquad\qquad E^1_{*,*} \cong \text{THH}_{*,*}(E^*_0 A) \Rightarrow \text{THH}_*(A).$$

Here $E^*_0 A$ is the *associated graded E_∞–ring spectrum of A*; part of our work in this paper is to define this "associated graded E_∞–ring spectrum", and prove that it has good formal properties and useful examples (eg Whitehead towers; see (5), below).

More generally, we construct, given any connective generalized homology theory E_* (see Definition 3.4.1) and any simplicial finite set $X_\bullet$, a spectral sequence

$$\text{(2)} \qquad\qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes E^*_0 A) \Rightarrow E_*(X_\bullet \otimes A).$$

We recover spectral sequence (1) as a special case of (2) by letting $E_* = \pi_*$ and letting $X_\bullet$ be a simplicial model for the circle S^1.

We formulate a definition (see Definition 3.1.1) of a "filtered E_∞–ring spectrum" which is sufficiently well behaved that we can actually construct a spectral sequence of the form (2), identify its E^1– and E^∞–terms and prove its multiplicativity and convergence properties. Actually our constructions and results work in a somewhat wider level of generality than commutative ring spectra: we fix a symmetric monoidal stable model category $\mathcal{C}$ satisfying some reasonable hypotheses (spelled out in Running Assumptions 2.0.3 and 2.0.4), and we work with filtered commutative monoid objects in $\mathcal{C}$. In the special case where $\mathcal{C}$ is the category of symmetric spectra in pointed simplicial sets, in the sense of Hovey, Shipley, and Smith [26] and Schwede [42], the commutative monoid objects are equivalent to E_∞–ring spectra. Our hope is that our framework is sufficiently general that an interested reader could also apply it to monoidal model categories of equivariant, motivic, and/or parametrized spectra.

The main difficulty in constructing the spectral sequence (2) at this level of generality is identifying the $E^1_{*,*}$–page. We refer to the theorem identifying the $E^1_{*,*}$–page as the *fundamental theorem of the May filtration*, and briefly it can be described in words using the slogan "higher order Hochschild homology commutes with passage to the associated graded commutative ring spectrum". This theorem does not follow easily by categorical properties, and in fact the bulk of Section 3.3 consists of a proof of this theorem.

We observe in Appendix A that with the right adjustments, one can construct a version of spectral sequence (2) with coefficients in a filtered symmetric A–bimodule M:

$$(3) \qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes (E^*_0 A, E^*_0 M)) \Rightarrow E_*(X_\bullet \otimes (A, M)),$$

and as a special case,

$$(4) \qquad E^1_{*,*} \cong E_{*,*} \mathrm{THH}(E^*_0 A, E^*_0 M) \Rightarrow E_* \mathrm{THH}(A, M).$$

Some of the most important cases of filtered commutative ring spectra, or filtered commutative monoid objects in general, are those which arise from Whitehead towers: given a cofibrant connective commutative monoid in symmetric spectra, we construct a filtered commutative monoid

$$(5) \qquad A = \tau_{\geq 0} A \leftarrow \tau_{\geq 1} A \leftarrow \tau_{\geq 2} A \leftarrow \cdots,$$

where each map is a cofibration in $\mathcal{C}$ and the induced map $\pi_n(\tau_{\geq m} A) \to \pi_n(\tau_{\geq m-1} A)$ is an isomorphism if $n \geq m$, and $\pi_n(\tau_{\geq m} A) \cong 0$ if $n < m$. While the homotopy type of $\tau_{\geq m} A$ is very easy to construct, it takes us some work to construct a sufficiently rigid *multiplicative* model for the Whitehead tower (5); this is the content of Theorem 4.2.1.

If $\mathcal{C}$ is the category of symmetric spectra in pointed simplicial sets, then the associated graded ring spectrum of the Whitehead tower (5) is the generalized Eilenberg–Mac Lane ring spectrum $H\pi_*(A)$ of the graded ring $\pi_*(A)$. Consequently we get a spectral sequence

$$(6) \qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes H\pi_* A) \Rightarrow E_*(X_\bullet \otimes A),$$

and as a special case,

$$(7) \qquad E^1_{*,*} \cong \mathrm{THH}_{*,*}(H\pi_* A) \Rightarrow \mathrm{THH}_*(A).$$

Many explicit computations are possible using spectral sequence (7) and its generalizations with coefficients in a bimodule (see Appendix A). For example, in [1], the first author uses these spectral sequences to compute topological Hochschild homology of the algebraic K–theory spectra of a large class of finite fields.

In the present paper, in lieu of explicit computations using our new spectral sequences, we point out that the mere existence of these spectral sequences implies an upper bound on the size of the topological Hochschild homology groups of a ring spectrum: namely, if R is a graded-commutative ring and $X_\bullet$ is a simplicial finite set and E_* is a generalized homology theory, then for any E_∞–ring spectrum A such that $\pi_*(A) \cong R$,

$E_*(X_\bullet \otimes A)$ is a subquotient of $E_*(X_\bullet \otimes HR)$. Here we write HR for the generalized Eilenberg–Mac Lane spectrum with $\pi_*(HR) \cong R$ as graded rings.

Consequently, in Theorem 5.2.1 we arrive at the slogan "the topological Hochschild homology of A is bounded above by the topological Hochschild homology of $H\pi_*(A)$". This lets us extract information about the topological Hochschild homology of E_∞–ring spectra A from information depending *only on the ring $\pi_*(A)$ of homotopy groups of A*. We demonstrate how to apply this idea in Theorem 5.2.6 and its corollaries, by working out the special case where $R = \mathbb{Z}_{(p)}[x]$ for some prime p, with x in positive grading degree $2n$. We get, for example, that for any p–local finite-type E_∞–ring spectrum A such that $\pi_*(A) \cong \mathbb{Z}_{(p)}[x]$, the Poincaré series of the mod p topological Hochschild homology $(S/p)_*(\mathrm{THH}(A))$ satisfies the inequality

$$\sum_{i \geq 0} \big(\dim_{\mathbb{F}_p}(S/p)_*(\mathrm{THH}(A))\big)t^i \leq \frac{(1 + (2p-1)t)(1 + (2n+1)t)}{(1 - 2nt)(1 - 2pt)},$$

where we interpret $\leq$ as in Definition 5.2.2. Furthermore, if p does not divide n, then $\mathrm{THH}_{2i}(A) \cong 0$ for all i congruent to $-p$ modulo n such that $i \leq pn - p - n$, and $\mathrm{THH}_{2i}(A) \cong 0$ for all i congruent to $-n$ modulo p such that $i \leq pn - p - n$. In particular, $\mathrm{THH}_{2(pn-p-n)}(A) \cong 0$. If p divides n, then $\mathrm{THH}_i(A) \cong 0$, unless i is congruent to -1, 0, or 1 modulo $2p$.

As a specific example, consider the p–local Adams summand ℓ, which satisfies the property that p does not divide $n = p - 1$ for any prime p. The theorem then states that $\mathrm{THH}_{2i}(\ell) \cong 0$ for all i congruent to -1 modulo $p - 1$ and all i congruent to 1 modulo p such that $i \leq p^2 - 3p + 1$, which agrees with the computation of $\mathrm{THH}_*(\widehat{\ell}_p)$ in these degrees due to Angeltveit, Hill, and Lawson [4, Theorem 2.6].

In Section 4.1, we further the development of filtered objects in a model category satisfying Running Assumptions 2.0.3 and 2.0.4 along with Running Assumption 4.1.2. This theory is used to construct multiplicative Whitehead towers as cofibrant decreasingly filtered commutative monoids in symmetric spectra in pointed simplicial sets (Theorem 4.2.1). In Appendix B, we discuss the Bousfield–Kan spectral sequence (Theorem B.1.4) in order to address a technical lemma (Lemma B.2.3) needed to construct the multiplicative Whitehead tower.

There is some precedent for spectral sequence (1): when A is a filtered commutative ring (rather than a filtered commutative ring spectrum), M Brun constructed a spectral sequence of the form (1) in the paper [11]. In [12], Brun also studies THH of filtered ring

spectra, but using the technology of FSPs rather than our general approach, and stops short of producing the THH–May spectral sequence or the multiplicative Whitehead filtration. In Theorem 2.9 of the preprint [3], V Angeltveit remarks that a version of spectral sequence (1) exists for commutative ring spectra by virtue of a lemma in [11] on associated graded FSPs of filtered FSPs; filling in the details to make this spectral sequence have the correct E^1–term, E^∞–term, convergence properties, and multiplicativity properties takes a lot of work, and even aside from the substantially greater level of generality of the results in the present paper (allowing $X_\bullet \otimes A$ and not just $S^1 \otimes A$, working with commutative monoids in symmetric monoidal model categories rather than any particular model for ring spectra, working with coefficient bimodules as in Appendix A), we think it is valuable to add these very nontrivial details to the literature.

2 Conventions and running assumptions

Conventions 2.0.1 By convention, the "cofiber of $f\colon X \to Y$" will mean that f is a cofibration and we are forming the pushout $Y \amalg_X 0$ in the given pointed model category. Also, by convention, we will write Y/X as shorthand for the cofiber of $f\colon X \to Y$.

Conventions 2.0.2 By convention, given a coproduct $\coprod_{i \in I} X_i$ of objects X_i in a cocomplete category $\mathcal{A}$, we will refer to the map $X_j \hookrightarrow \coprod_{i \in I} X_i$ for $j \in I$, given by definition of the coproduct, as the inclusion map. When $\mathcal{A}$ is a subcategory of $\mathcal{B}$, we will also refer to the evident functor $\mathcal{A} \hookrightarrow \mathcal{B}$ as an inclusion. Since one use of inclusion refers to map between objects in a category $\mathcal{A}$ and the second use of inclusion refers to a functor between categories, no confusion should arise.

We will write $\mathrm{Comm}(\mathcal{C})$ for the category of commutative monoid objects in a symmetric monoidal category $\mathcal{C}$, we will write $s\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$, and we will write $\wedge$ for the symmetric monoidal product in a symmetric monoidal category $\mathcal{C}$, since the main example we have in mind is the category of symmetric spectra, in which the symmetric monoidal product is the smash product. We will write $S^1_\bullet$ throughout for the minimal simplicial model for the circle $\Delta[1]/\delta\Delta[1]$, which is important for defining topological Hochschild homology.

Running Assumption 2.0.3 Throughout, let $\mathcal{C}$ be a left proper stable model category equipped with the structure of a symmetric monoidal model category in the sense of [43], satisfying the following axioms: A model structure (necessarily unique) on $\mathrm{Comm}(\mathcal{C})$ exists in which weak equivalences and fibrations are created by the forgetful functor $\mathrm{Comm}(\mathcal{C}) \to \mathcal{C}$. The forgetful functor $\mathrm{Comm}(\mathcal{C}) \to \mathcal{C}$ commutes with geometric realization of simplicial objects and sends cofibrant objects to cofibrant objects. Geometric realization of simplicial cofibrant objects in $\mathcal{C}$ commutes with the monoidal product; ie if $X_\bullet$ and $Y_\bullet$ are simplicial cofibrant objects of $\mathcal{C}$, then the canonical comparison map

$$|X_\bullet \wedge Y_\bullet| \to |X_\bullet| \wedge |Y_\bullet|$$

is a weak equivalence in $\mathcal{C}$. We will say a class of morphisms $\mathcal{M}$ in $\mathcal{C}$ is called *retractile* if whenever a composite $A \xrightarrow{f} B \xrightarrow{g} C$ is in $\mathcal{M}$ then $A \xrightarrow{f} B$ is in $\mathcal{M}$. We additionally assume that $\mathcal{C}$ comes equipped with a class of morphisms, which we will refer to as *levelwise cofibrations*, which are closed under composition, and such that the cofibrations in $\mathcal{C}$ are contained in the class of levelwise cofibrations, and such that the class of levelwise cofibrations is retractile. We also assume that if X is a cofibrant object in $\mathcal{C}$ and f is a levelwise cofibration in $\mathcal{C}$, then $X \wedge f$ is a levelwise cofibration in $\mathcal{C}$.

Here are a few immediate consequences of these assumptions about $\mathcal{C}$: Since being cofibrantly generated is part of the definition of a monoidal model category in [43], $\mathcal{C}$ is cofibrantly generated and hence can be equipped with functorial factorization systems. We assume that a choice of functorial factorization has been made and we will use it implicitly whenever a cofibration-acyclic-fibration or acyclic-cofibration-fibration factorization is necessary. Smashing with any given object is a left adjoint, hence preserves colimits. Smashing with any given cofibrant object is a left Quillen functor, hence preserves cofibrations and acyclic cofibrations, and by Ken Brown's lemma, preserves weak equivalences between cofibrant objects.

Since $\mathcal{C}$ is assumed left proper, a homotopy cofiber of any map $f: X \to Y$ between cofibrant objects in $\mathcal{C}$ can be computed by factoring f as $f = f_2 \circ f_1$ with $f_1: X \to \tilde{Y}$ a cofibration and $f_2: \tilde{Y} \to Y$ an acyclic fibration, and then taking the pushout of the following diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\;f_1\;} & \tilde{Y} \\
\downarrow & & \\
0 & &
\end{array}
$$

In particular, if f is already a cofibration, the pushout map $Y \to Y \amalg_X 0$ is a homotopy cofiber of f.

The added assumption that $\mathcal{C}$ comes equipped with a retractile class of morphisms called the levelwise cofibrations was necessary to resolve a technical issue pointed out to the authors by Birgit Richter and Ayelet Lindenstrauss. We recognize that it complicates the running assumptions, however it is satisfied in the main example of interest (as discussed below) and provides the THH–May spectral sequence with some desired properties. The levelwise cofibrations are used only in two places in the paper, in Observation 3.2.2 and in Remark 4.2.2.

Running Assumption 2.0.4 In addition to Running Assumption 2.0.3, we assume our model category $\mathcal{C}$ satisfies the following condition: a map $X_\bullet \to Y_\bullet$ in the category of simplicial objects in $\mathcal{C}$ is a Reedy cofibration between Reedy-cofibrant objects whenever the following all hold:

(1) The object X_n in $\mathcal{C}$ is cofibrant for each n.

(2) Each of the degeneracy maps $s_i: X_n \to X_{n+1}$ and $s_i: Y_n \to Y_{n+1}$ is a levelwise cofibration in $\mathcal{C}$.

(3) Each induced map $X_n \to Y_n$ is a cofibration in $\mathcal{C}$.

A consequence of this assumption is that the geometric realization of a map of simplicial objects in $\mathcal{C}$ satisfying items (1), (2), and (3) is a cofibration.

The main motivating example of a category $\mathcal{C}$ satisfying Running Assumption 2.0.3 is the category of symmetric spectra in pointed simplicial sets sSet$_*$, denoted by Sp$_{\text{sSet}_*}$, equipped with the positive flat stable model structure. In this case, Comm$(\mathcal{C})$ is then the category of commutative ring spectra and it is known to be equivalent to the category of E_∞–ring spectra; see [44, Corollary 4.8]. The existence of the desired

model structure on $\mathrm{Comm}(\mathcal{C})$ is proven in [43, Theorem 4.1]. The fact that the forgetful functor $\mathrm{Comm}(\mathcal{C}) \to \mathcal{C}$ commutes with geometric realization in the positive flat stable model structure on $\mathcal{C}$ is a consequence of [23, Theorem 1.6]. The fact that there exists a model structure on $\mathrm{Comm}(\mathcal{C})$ created by the forgetful functor and that the forgetful functor preserves cofibrations with cofibrant source is a consequence of [44, Theorem 5.7]. The category $\mathrm{Sp}_{\mathrm{sSet}_*}$ with the positive flat stable model structure satisfies Running Assumption 2.0.4, as the authors prove in [2]. To see that the levelwise cofibrations in symmetric spectra of pointed simplicial sets are retractile, note that the levelwise cofibrations are simply the levelwise cofibrations of pointed simplicial sets, hence levelwise monomorphisms of pointed simplicial sets, and monomorphisms are retractile in any category. If $f\colon A \to B$ is a flat cofibration in symmetric spectra then the pushout product $f \,\square\, -$ preserves levelwise cofibrations, so in particular all flat cofibrations are levelwise cofibrations and for any flat cofibrant object X, the functor $X \wedge -$ preserves levelwise cofibrations (by [42, Proposition 2.8], for example).

3 Construction of the spectral sequence

3.1 Filtered commutative monoids and associated graded commutative monoids

Definition 3.1.1 By a *cofibrant decreasingly filtered object in* $\mathcal{C}$ we mean a sequence of cofibrations in $\mathcal{C}$

$$\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0$$

such that each object I_i is cofibrant.

Definition 3.1.2 By a *cofibrant decreasingly filtered commutative monoid in* $\mathcal{C}$ we mean a cofibrant decreasingly filtered object $\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0$ in $\mathcal{C}$, and for every pair of natural numbers $i, j \in \mathbb{N}$, a map in $\mathcal{C}$

$$\rho_{i,j}\colon I_i \wedge I_j \to I_{i+j},$$

and a map $\eta\colon \mathbb{1} \to I_0$, satisfying the axioms listed below. For the sake of listing the axioms concisely, it will be useful to have the following notation: if $i' \leq i$, we will write $f_i^{i'}\colon I_i \to I_{i'}$ for the composite

(8) $$f_i^{i'} = f_{i'+1} \circ f_{i'+2} \circ \cdots \circ f_{i-1} \circ f_i.$$

Here are the axioms we require:

Compatibility For all $i, j, i', j' \in \mathbb{N}$ with $i' \leq i$ and $j' \leq j$, the following diagram commutes:

$$
\begin{array}{ccc}
I_i \wedge I_j & \xrightarrow{\rho_{i,j}} & I_{i+j} \\
{\scriptstyle f_i^{i'} \wedge f_j^{j'}} \downarrow & & \downarrow {\scriptstyle f_{i+j}^{i'+j'}} \\
I_{i'} \wedge I_{j'} & \xrightarrow{\rho_{i',j'}} & I_{i'+j'}
\end{array}
$$

Commutativity For all $i, j \in \mathbb{N}$, the diagram

$$
\begin{array}{ccc}
& I_i \wedge I_j & \\
{\scriptstyle \chi_{I_i, I_j}} \downarrow & & \searrow {\scriptstyle \rho_{i,j}} \\
I_j \wedge I_i & \xrightarrow{\rho_{j,i}} & I_{i+j}
\end{array}
$$

commutes, where $\chi_{I_i, I_j} \colon I_i \wedge I_j \xrightarrow{\cong} I_j \wedge I_i$ is the symmetry isomorphism in $\mathcal{C}$.

Associativity For all $i, j, k \in \mathbb{N}$, the following diagram commutes:

$$
\begin{array}{ccc}
I_i \wedge I_j \wedge I_k & \xrightarrow{\mathrm{id}_{I_i} \wedge \rho_{j,k}} & I_i \wedge I_{j+k} \\
{\scriptstyle \rho_{i,j} \wedge \mathrm{id}_{I_k}} \downarrow & & \downarrow {\scriptstyle \rho_{i,j+k}} \\
I_{i+j} \wedge I_k & \xrightarrow{\rho_{i+j,k}} & I_{i+j+k}
\end{array}
$$

Unitality For all $i \in \mathbb{N}$, the diagram

$$
\begin{array}{ccc}
& \mathbb{1} \wedge I_i & \\
{\scriptstyle \eta \wedge \mathrm{id}_{I_i}} \downarrow & & \searrow {\scriptstyle \cong} \\
I_0 \wedge I_i & \xrightarrow{\rho_{0,i}} & I_i
\end{array}
$$

commutes, where the map marked $\cong$ is the (left-)unitality isomorphism in $\mathcal{C}$.

Cofibrancy in degree 0 The commutative monoid I_0 is cofibrant in $\mathrm{Comm}(\mathcal{C})$.

Cofibrancy of degree-0 quotient The composite map $S \to I_0 \to I_0/I_1$ is a levelwise cofibration in $\mathcal{C}$.

Note that in the last condition we do not require that I_0/I_1 be cofibrant in $\mathrm{Comm}(\mathcal{C})$, but only that the map $S \to I_0 \to I_0/I_1$ be a levelwise cofibration in $\mathcal{C}$. We hope that Observation 3.2.2 and Remark 3.2.3 will be helpful to the reader who is wondering about the role of this "cofibrancy of degree-0 quotient" condition.

Remark 3.1.3 If $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, then I_0 is a cofibrant commutative monoid in $\mathcal{C}$, with multiplication map $\rho_{0,0}\colon I_0 \wedge I_0 \to I_0$ and unit map $\eta\colon \mathbb{1} \to I_0$. The objects I_i for $i > 0$ do not receive commutative monoid structures from the structure of $I_\bullet$, but instead play a role analogous to that of the nested sequence of powers of an ideal in a commutative ring. If we neglect the commutativity axiom from Definition 3.1.2, then the special case $\cdots \xrightarrow{\mathrm{id}} I \xrightarrow{\mathrm{id}} I \to R$ of Definition 3.1.2 recovers the definition of a Smith ideal. Leaving the commutativity axiom in Definition 3.1.2 intact, we recover the notion of a commutative Smith ideal, as studied in [44].

Definition 3.1.4 Suppose $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$. We shall say that $I_\bullet$ is *Hausdorff* if $\operatorname{holim}_n I_n \simeq 0$. We shall say that $I_\bullet$ is *finite* if there exists some $n \in \mathbb{N}$ such that $f_m\colon I_m \to I_{m-1}$ is a weak equivalence for all $m > n$.

Remark 3.1.5 Definition 3.1.2 has the advantage of concreteness, but if we are willing to temporarily neglect the cofibrancy assumption in Definition 3.1.2, then there is an equivalent, more concise definition of a decreasingly filtered commutative monoid. Observe that the data of a decreasingly filtered commutative monoid is the same as the data of a lax symmetric monoidal functor $I_\bullet\colon \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$, where $\mathbb{N}^{\mathrm{op}}$ is the opposite category of $\mathbb{N}$, viewed as a partially ordered set, and equipped with a symmetric monoidal structure with addition as the symmetric monoidal product and 0 as the unit.

Recall that due to Day [14], the full subcategory of lax symmetric monoidal functors in $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ is equivalent to the category $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ of commutative monoid objects in the symmetric monoidal category $(\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}, \otimes_{\mathrm{Day}}, \mathbb{1}_{\mathrm{Day}})$, where $\otimes_{\mathrm{Day}}$ is the (unenriched) Day convolution symmetric monoidal product and $\mathbb{1}_{\mathrm{Day}}$ is the unit of this symmetric monoidal product (see Day [14] for these constructions). In sum, specifying the data of a decreasingly filtered commutative monoid, without the cofibrancy condition, is the same as specifying an object in $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$.

Remark 3.1.5 does not address the cofibrancy conditions needed for an object in the category $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ to be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ in the sense of Definition 3.1.1. We will discuss this in detail in Section 4.1.

Definition 3.1.6 (the associated graded monoid) Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$. By $E_0^* I_\bullet$, the *associated graded commutative monoid of $I_\bullet$*, we mean the graded commutative monoid object in $\mathcal{C}$ defined as follows:

As an object of $\mathcal{C}$,

$$E_0^* I_\bullet = \coprod_{n \in \mathbb{N}} I_n / I_{n+1}.$$

The unit map $\mathbb{1} \to E_0^* I_\bullet$ is the composite $\mathbb{1} \xrightarrow{\eta} I_0 \to I_0/I_1 \hookrightarrow E_0^* I_\bullet$.

The multiplication on $E_0^* I_\bullet$ is given as follows. Since the smash product commutes with colimits, hence with coproducts, to specify a map $E_0^* I_\bullet \wedge E_0^* I_\bullet \to E_0^* I_\bullet$ it suffices to specify a component map

$$\nabla_{i,j} \colon I_i/I_{i+1} \wedge I_j/I_{j+1} \to E_0^* I_\bullet$$

for every $i, j \in \mathbb{N}$. We define such a map $\nabla_{i,j}$ as follows: first, we have the commutative square

$$
\begin{array}{ccc}
I_{i+1} \wedge I_j & \xrightarrow{\rho_{i+1,j}} & I_{i+j+1} \\
{\scriptstyle f_{i+1} \wedge \mathrm{id}_{I_j}} \downarrow & & \downarrow {\scriptstyle f_{i+j+1}} \\
I_i \wedge I_j & \xrightarrow{\rho_{i,j}} & I_{i+j}
\end{array}
$$

so, since the vertical maps are cofibrations by Definition 3.1.2, we can take vertical cofibers to get a map

$$\widetilde{\nabla}_{i,j} \colon I_i/I_{i+1} \wedge I_j \to I_{i+j}/I_{i+j+1},$$

which is well defined by Running Assumption 2.0.3.

Now we have the commutative diagram in Figure 1 in which the columns are cofiber sequences. So we have a factorization of the composite map $\widetilde{\nabla}_{i,j} \circ (\mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1})$ through the zero object, by Running Assumption 2.0.3. Thus, we have the commutative square

$$
\begin{array}{ccc}
I_i/I_{i+1} \wedge I_{j+1} & \xrightarrow{\hspace{2cm}} & 0 \\
{\scriptstyle \mathrm{id}_{I_i/I_{i+1}} \wedge f_{j+1}} \downarrow & & \downarrow \\
I_i/I_{i+1} \wedge I_j & \xrightarrow{\ \widetilde{\nabla}_{i,j}\ } & I_{i+j}/I_{i+j+1}
\end{array}
$$

and, taking vertical cofibers, a map

$$I_i/I_{i+1} \wedge I_j/I_{j+1} \to I_{i+j}/I_{i+j+1},$$

which we compose with the inclusion map $I_{i+j}/I_{i+j+1} \hookrightarrow E_0^* I_\bullet$ to produce our desired map $\nabla_{i,j} \colon I_i/I_{i+1} \wedge I_j/I_{j+1} \to E_0^* I_\bullet$. (Note that all these maps are defined in the model category $\mathcal{C}$, not just in $\mathrm{Ho}(\mathcal{C})$.)
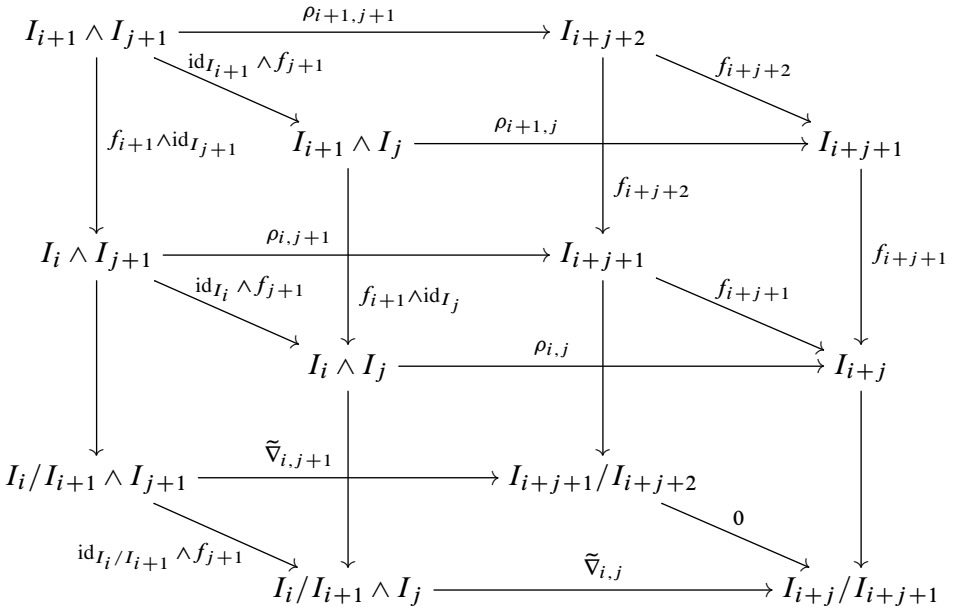
Figure 1

This produces a multiplication map $E_0^* I_\bullet \wedge E_0^* I_\bullet \to E_0^* I_\bullet$ that, together with the unit map $\mathbb{1} \to E_0^* I_\bullet$, satisfies the necessary commutativity, associativity, and unitality conditions to make $E_0^* I_\bullet$ a commutative monoid object in $\mathcal{C}$, by construction.

3.2 Tensoring and pretensoring over simplicial sets

We will write f Sets for the category of finite sets. First we introduce the *pretensor product*, which is merely a convenient notation for the well-known "Loday construction" of [29]:

Definition 3.2.1 We define a functor

$$- \widetilde{\otimes} -: sf \, \mathrm{Sets} \times \mathrm{Comm}\, \mathcal{C} \to s \, \mathrm{Comm}\, \mathcal{C},$$

which we call the *pretensor product*, as follows. If $X_\bullet$ is a simplicial finite set and A a commutative monoid in $\mathcal{C}$, the simplicial commutative monoid $X_\bullet \widetilde{\otimes} A$ is given by:

For all $n \in \mathbb{N}$, the n–simplex object $(X_\bullet \widetilde{\otimes} A)_n = \coprod_{x \in X_n} A$ is a coproduct, *taken in* $\mathrm{Comm}(\mathcal{C})$, of copies of A, with one copy for each n–simplex $x \in X_n$. Recall that the categorical coproduct in $\mathrm{Comm}(\mathcal{C})$ is the smash product $\wedge$.

For all positive $n \in \mathbb{N}$ and all $0 \leq i \leq n$, the ith face map

$$d_i \colon (X_\bullet \widetilde{\otimes} A)_n \to (X_\bullet \widetilde{\otimes} A)_{n-1}$$

is given on the component corresponding to an n–simplex $x \in X_n$ by the map

$$A \to \coprod_{y \in X_{n-1}} A$$

which is inclusion of the coproduct factor corresponding to the $(n{-}1)$–simplex $\delta_i(x)$.

For all positive $n \in \mathbb{N}$ and all $0 \leq i \leq n$, the ith degeneracy map

$$s_i \colon (X_\bullet \widetilde{\otimes} A)_n \to (X_\bullet \widetilde{\otimes} A)_{n+1}$$

is given on the component corresponding to an n–simplex $x \in X_n$ by the map

$$A \to \coprod_{y \in X_{n+1}} A$$

which is inclusion of the coproduct factor corresponding to the $(n{+}1)$–simplex $\sigma_i(x)$.

We have defined the pretensor product on objects; it is then defined on morphisms in the evident way.

We define the *tensor product*

$$- \otimes - \colon sf\,\mathrm{Sets} \times \mathrm{Comm}\,\mathcal{C} \to \mathrm{Comm}\,\mathcal{C}$$

as the geometric realization of the pretensor product:

$$X_\bullet \otimes A = |X_\bullet \widetilde{\otimes} A|.$$

It is easy to check that $X_\bullet \widetilde{\otimes} A$ is indeed a simplicial object in $\mathrm{Comm}(\mathcal{C})$.

When $\mathcal{C}$ is the category of symmetric spectra, the tensor product $X_\bullet \otimes A$ agrees with the tensoring of commutative ring spectra over simplicial sets. (This is proven in [35], although using (an early incarnation of) S–modules [20], rather than symmetric spectra; the symmetric monoidal Quillen equivalence of S–modules and symmetric spectra, as in [41], then gives us the same result in symmetric spectra.) The same is true when E is a commutative S–algebra and $\mathcal{C}$ is the category of E–modules. In fact, the tensor product defined in Definition 3.2.1 agrees with the tensoring over simplicial sets in every case of a symmetric monoidal model category whose category of commutative monoids is tensored over simplicial sets that is known to the authors.

In particular, if $X_\bullet$ is the minimal simplicial model for the circle $S^1_\bullet$, then $S^1_\bullet \widetilde{\otimes} A$ is the cyclic bar construction on A, and hence (by the main result of [35]) $S^1_\bullet \otimes A$ agrees with the topological Hochschild homology ring spectrum $\mathrm{THH}(A, A)$.

For other simplicial sets, $X_\bullet \otimes A$ is regarded as a generalization of topological Hochschild homology, eg as "higher order Hochschild homology" in [37].

Observation 3.2.2 Suppose that $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid, in the sense of Definition 3.1.2, and suppose $X_\bullet$ is a simplicial finite set. The assumptions made in Definition 3.1.2, particularly the "cofibrancy of degree-0 quotient" assumption, together with Running Assumption 2.0.4, ensure that $X_\bullet \widetilde{\otimes} E^*_0 I_\bullet$ is Reedy-cofibrant as a simplicial object of $\mathcal{C}$ (but *not* as a simplicial object of $\mathrm{Comm}(\mathcal{C})$). The argument is as follows: Each $I_{n+1} \to I_n$ is a cofibration in $\mathcal{C}$, so the pushout map $0 \to I_n/I_{n+1}$ is a cofibration, so each I_n/I_{n+1} is cofibrant. So $X_m \widetilde{\otimes} E^*_0 I_\bullet$ is a coproduct of cofibrant objects for each m, hence $X_m \widetilde{\otimes} E^*_0 I_\bullet$ is cofibrant for each m. So, if we know that each degeneracy map in $X_\bullet \widetilde{\otimes} E^*_0 I_\bullet$ is a levelwise cofibration (in the sense of Running Assumption 2.0.3), then $X_\bullet \widetilde{\otimes} E^*_0 I_\bullet$ is Reedy-cofibrant in $\mathcal{C}^{\Delta^{\mathrm{op}}}$ by Running Assumption 2.0.4. The degeneracy maps in $X_\bullet \widetilde{\otimes} E^*_0 I_\bullet$ are smash products of coproducts of copies of the map $0 \to I_n/I_{n+1}$ for $n > 0$ and copies of the composite map $S \to I_0 \to I_0/I_1$, so the "cofibrancy of degree-0 quotient" condition from Definition 3.1.2 is exactly what is necessary to ensure that the degeneracy maps are levelwise cofibrations and hence that $X_\bullet \widetilde{\otimes} E^*_0 I_\bullet$ is Reedy-cofibrant in $\mathcal{C}^{\Delta^{\mathrm{op}}}$.

Remark 3.2.3 Because of Observation 3.2.2, we claim that when $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid, the spectral sequence of Definition 3.4.2 that we will construct using the pretensor product has good homotopical properties, *even though* the assumptions in Definition 3.1.2 are not enough to guarantee that $E^*_0 I_\bullet$ is cofibrant in $\mathrm{Comm}(\mathcal{C})$. For the sake of the spectral sequence of Definition 3.4.2 (ie the central motivating construction in this paper), it is enough to know that $X_\bullet \otimes E^*_0 I_\bullet$ has the correct homotopy type, ie that generalized homologies of $X_\bullet \otimes E^*_0 I_\bullet$ are computable from those of $\pi_*(E^*_0 I_\bullet)$ by the usual methods (eg the Bökstedt spectral sequence) one uses in order to compute THH or its higher-order variants.

For that purpose, it is enough to know that $X_\bullet \widetilde{\otimes} A$ is Reedy-cofibrant *as a simplicial object of $\mathcal{C}$*, not necessarily as a simplicial object of $\mathrm{Comm}(\mathcal{C})$. This is because Reedy-cofibrancy of $X_\bullet \widetilde{\otimes} A$ as a simplicial object of $\mathcal{C}$ is enough to give us a Bousfield–Kan-type spectral sequence with E_2–term the homology of the alternating sign chain complex obtained by applying a generalized homology theory E_* to $X_\bullet \widetilde{\otimes} A$, and

which converges to $E_*|X_\bullet \widetilde{\otimes} A| = E_*(X_\bullet \otimes A)$ (the Bökstedt spectral sequence is the special case $E = H\mathbb{F}_p$); and it is enough to tell us that the geometric realization $|X_\bullet \widetilde{\otimes} A| = X_\bullet \otimes A$ is a model for the homotopy colimit of the functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ given by $X_\bullet \widetilde{\otimes} A$. See [10, Chapter XII] for a classical account of these ideas; we also provide some details and discussion in the present paper in Theorem B.1.5.

More is true, however: Suppose that $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid, and suppose that we are not satisfied by $E_0^* I_\bullet$'s lack of cofibrancy in $\mathrm{Comm}(\mathcal{C})$. Let $cE_0^* I_\bullet$ be a cofibrant commutative monoid in $\mathcal{C}$ and let $cE_0^* I_\bullet \to E_0^* I_\bullet$ be a weak equivalence in $\mathrm{Comm}(\mathcal{C})$. Then

$$(9) \qquad\qquad X_\bullet \widetilde{\otimes} c E_0^* I_\bullet \to X_\bullet \widetilde{\otimes} E_0^* I_\bullet$$

is a Reedy weak equivalence (in $\mathrm{Comm}(\mathcal{C})^{\Delta^{\mathrm{op}}}$ as well as in $\mathcal{C}^{\Delta^{\mathrm{op}}}$) whose domain and codomain are both Reedy-cofibrant in $\mathcal{C}^{\Delta^{\mathrm{op}}}$, so (9) induces a weak equivalence (in $\mathcal{C}$) on geometric realizations, by the famous Theorem D in Reedy's thesis [39]. So, as long as $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid, the homotopy type of $X_\bullet \otimes E_0^* I_\bullet$ in $\mathcal{C}$ is not affected by the failure of $E_0^* I_\bullet$ to be cofibrant in $\mathrm{Comm}(\mathcal{C})$.

3.3 The fundamental theorem of the May filtration

The fundamental theorem of the May filtration relies on the following lemma.

Lemma 3.3.1 *Suppose I and J are objects of $\mathcal{C}$ and $f\colon I' \to I$ and $g\colon J' \to J$ are cofibrations. Suppose I, J, I', and J' are cofibrant. Let $P = (I \wedge J') \amalg_{I' \wedge J'} (I' \wedge J)$ denote the pushout (which, by Running Assumption 2.0.3, is a homotopy pushout). Let $f \,\square\, g\colon P \to I \wedge J$ denote the canonical map given by the universal property of the pushout, known as the pushout product. Then $f \,\square\, g$ is a cofibration by the pushout product axiom in the definition of a monoidal model category, as in [43], and the cofiber of $f \,\square\, g$ is isomorphic to $(I/I') \wedge (J/J')$. So the following sequence is a cofiber sequence:*

$$(10) \qquad\qquad P \xrightarrow{\,f\,} I \wedge J \to (I/I') \wedge (J/J').$$

Proof This lemma occurs as Lemma 4.7 in May [34] and its proof is easily generalized to a general model category $\mathcal{C}$ satisfying Running Assumption 2.0.3. $\qquad\square$

We now define some categories and functors that will be important for Definition 3.3.2. If S is a finite set, we will equip the set $\mathbb{N}^S$ of functions from S to $\mathbb{N}$ with the

L^1–norm, that is, $|x| = \sum_{s \in S} x(s)$, and with the strict direct product order, that is, $x \leq y$ in $\mathbb{N}^S$ if and only if $x(s) \leq y(s)$ for all $s \in S$. If $T \xrightarrow{f} S$ is a function between finite sets, let $\mathbb{N}^T \xrightarrow{\mathbb{N}^f} \mathbb{N}^S$ be the function of partially ordered sets defined by

$$(\mathbb{N}^f(x))(s) = \sum_{\{t \in T : f(t) = s\}} x(t).$$

If S is a finite set, for each $n \in \mathbb{N}$ we will let $\mathcal{D}_n^S$ be the subposet of $\mathbb{N}^S$ consisting of all functions $x \in \mathbb{N}^S$ such that $|x| \geq n$. If $T \xrightarrow{f} S$ is a function between finite sets, let $\mathcal{D}_n^T \xrightarrow{\mathcal{D}_n^f} \mathcal{D}_n^S$ be the function of partially ordered sets defined by restricting $\mathbb{N}^f$ to $\mathcal{D}_n^T$.

For each $x \in \mathbb{N}^S$ and each $n \in \mathbb{N}$, let $\mathcal{D}_{n;x}^S$ denote the following subposet of $\mathbb{N}^S$:

(11) $$\mathcal{D}_{n;x}^S = \{y \in \mathbb{N}^S : y \geq x \text{ and } |y| \geq n + |x|\}.$$

So, for example, $\mathcal{D}_{n;\vec{0}}^S = \mathcal{D}_n^S$, where $\vec{0}$ is the constant zero function. If $T \xrightarrow{f} S$ is a function between finite sets and $x \in \mathbb{N}^T$ and $n \in \mathbb{N}$, let $\mathcal{D}_{n;x}^T \xrightarrow{\mathcal{D}_{n;x}^f} \mathcal{D}_{n;\mathcal{D}_n^f(x)}^S$ be the function of partially ordered sets defined by restricting $\mathbb{N}^f$ to $\mathcal{D}_{n;x}^T$.

Let S be a finite set and let n be a nonnegative integer. We write $\mathcal{E}_{n,k}^S$ for the set

(12) $$\mathcal{E}_{n,k}^S = \{x \in \{0, 1, \ldots, n\}^S : |x| \geq k\}.$$

When $n = k$, we simply write $\mathcal{E}_n^S$ for this partially ordered set. The definition of $\mathcal{E}_n^S$ is natural in S in the following sense: if $T \xrightarrow{f} S$ is a injective map of finite sets, then $\mathbb{N}^f$ naturally restricts to a function $\mathcal{E}_n^T \to \mathcal{E}_n^S$.

Definition 3.3.2 Suppose $I_\bullet$ is a cofibrant decreasingly filtered object in $\mathcal{C}$ and suppose S is a finite set. We will let $\mathcal{F}^S(I_\bullet) : (\mathbb{N}^S)^{\mathrm{op}} \to \mathcal{C}$ be the functor sending x to the smash product $\bigwedge_{s \in S} I_{x(s)}$, and defined on morphisms in the apparent way, and we will let $\mathcal{F}_n^S(I_\bullet) : (\mathcal{D}_n^S)^{\mathrm{op}} \hookrightarrow (\mathbb{N}^S)^{\mathrm{op}} \xrightarrow{\mathcal{F}^S(I_\bullet)} \mathcal{C}$ be the functor which is the composite of $\mathcal{F}^S(I_\bullet)$ with the inclusion of $(\mathcal{D}_n^S)^{\mathrm{op}}$ into $(\mathbb{N}^S)^{\mathrm{op}}$ as a subcategory.

If $x \in \mathcal{D}_n^S$, we will write $\mathcal{F}_{n;x}^S(I_\bullet)$ for the restriction of the diagram $\mathcal{F}^S(I_\bullet)$ to $\mathcal{D}_{n;x}^S$, that is, $\mathcal{F}_{n;x}^S(I_\bullet)$ is the composite $\mathcal{F}_{n;x}^S(I_\bullet) : (\mathcal{D}_{n;x}^S)^{\mathrm{op}} \hookrightarrow (\mathbb{N}^S)^{\mathrm{op}} \xrightarrow{\mathcal{F}^S(I_\bullet)} \mathcal{C}$. Finally, let $\mathcal{M}_n^S(I_\bullet)$ denote the colimit given by $\mathcal{M}_n^S(I_\bullet) = \mathrm{colim}(\mathcal{F}_n^S(I_\bullet))$ in $\mathcal{C}$. By the natural inclusion of $\mathcal{D}_n^S$ into $\mathcal{D}_{n-1}^S$ as a subcategory, we now have a sequence in $\mathcal{C}$:

(13) $$\cdots \to \mathcal{M}_3^S(I_\bullet) \to \mathcal{M}_2^S(I_\bullet) \to \mathcal{M}_1^S(I_\bullet) \to \mathcal{M}_0^S(I_\bullet) \cong \bigwedge_{s \in S} I_0.$$

We refer to the functor $\mathbb{N}^{\mathrm{op}} \to \mathcal{C}$ given by sending n to $\mathcal{M}_n^S(I_\bullet)$ as the *May filtration* on $\bigwedge_{s \in S} I_0$.

The May filtration is functorial in S in the following sense: if $T \xrightarrow{f} S$ is a map of finite sets, we have a functor $\mathcal{D}_n^f : \mathcal{D}_n^T \to \mathcal{D}_n^S$ given by

$$(\mathcal{D}_n^f(x))(s) \mapsto \sum_{\{t \in T : f(t) = s\}} x(t)$$

and a map of diagrams from $\mathcal{F}_n^T(I_\bullet)$ to $\mathcal{F}_n^S(I_\bullet)$ given by sending the object $\mathcal{F}_n^T(I_\bullet)(x) = \bigwedge_{t \in T} I_{x(t)}$ to the object $\mathcal{F}_n^S(I_\bullet)(\mathcal{D}_n^f(x)) = \bigwedge_{s \in S} I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$ by the map

$$\bigwedge_{t \in T} I_{x(t)} \to \bigwedge_{s \in S} I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$$

given as the smash product, across all $s \in S$, of the maps

$$\bigwedge_{\{t \in T : f(t) = s\}} I_{x(t)} \to I_{\Sigma_{\{t \in T : f(t) = s\}} x(t)}$$

given by multiplication via the maps ρ of Definition 3.1.2.

To really make Definition 3.3.2 precise, we should say in which order we multiply the factors $I_{x(t)}$ using the maps ρ; but the purpose of the associativity and commutativity axioms in Definition 3.1.2 is that any two such choices commute, so any choice of order of multiplication will do.

Definition 3.3.3 (the May filtration) If $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ and $X_\bullet$ a simplicial finite set, by the *May filtration on* $X_\bullet \widetilde{\otimes} I_0$ we mean the functor $\mathcal{M}^{X_\bullet}(I_\bullet) : \mathbb{N}^{\mathrm{op}} \to \mathcal{C}^{\Delta^{\mathrm{op}}}$ given by sending a natural number n to the simplicial object $\mathcal{M}_n^{X_\bullet}(I_\bullet)$ of $\mathcal{C}$, with $\mathcal{M}_n^{X_i}(I_\bullet)$ defined as in Definition 3.3.2, and with face and degeneracy maps defined as follows: The face map $d_i : \mathcal{M}_n^{X_j}(I_\bullet) \to \mathcal{M}_n^{X_{j-1}}(I_\bullet)$ is the colimit of the map of diagrams $\mathcal{F}_n^{X_j}(I_\bullet) \to \mathcal{F}_n^{X_{j-1}}(I_\bullet)$ induced, as in Definition 3.3.2, by $\delta_i : X_j \to X_{j-1}$. The degeneracy map $s_i : \mathcal{M}_n^{X_j}(I_\bullet) \to \mathcal{M}_n^{X_{j+1}}(I_\bullet)$ is the colimit of the map of diagrams $\mathcal{F}_n^{X_j}(I_\bullet) \to \mathcal{F}_n^{X_{j+1}}(I_\bullet)$ induced, as in Definition 3.3.2, by $\sigma_i : X_j \to X_{j+1}$.

Remark 3.3.4 The associative, commutative, and unital multiplications on the objects I_i, via the maps ρ of Definition 3.1.2, also yield (by taking smash products of the maps ρ) associative, commutative, and unital multiplication natural transformations

(14) $$\mathcal{F}_m^S(I_\bullet) \wedge \mathcal{F}_n^S(I_\bullet) \to \mathcal{F}_{m+n}^S(I_\bullet),$$

hence, on taking colimits, associative, commutative, and unital multiplication maps

$$\mathcal{M}_m^S(I_\bullet) \wedge \mathcal{M}_n^S(I_\bullet) \to \mathcal{M}_{m+n}^S(I_\bullet),$$

ie the functor $\mathbb{N}^{\mathrm{op}} \to \mathcal{C}$ sending n to $\mathcal{M}_n^S(I_\bullet)$ is a cofibrant decreasingly filtered commutative monoid, in the sense of Definition 3.1.2. Note furthermore that if $f\colon T \to S$ is a map of finite sets, then the induced maps $\mathcal{F}_m^T(I_\bullet) \to \mathcal{F}_m^S(I_\bullet)$ commute with the multiplication maps (14), and so $\mathcal{M}_\bullet^T(I_\bullet) \to \mathcal{M}_\bullet^S(I_\bullet)$ is a map of cofibrant decreasingly filtered commutative monoids.

Consequently, for any simplicial finite set $X_\bullet$, we have that $\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)$ is a simplicial object in the category of cofibrant decreasingly filtered commutative monoids in $\mathcal{C}$. Since geometric realization commutes with the monoidal product in $\mathcal{C}$ by our running assumptions on $\mathcal{C}$, this in turn implies that the geometric realization $|\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)|$ of $\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ by Running Assumption 2.0.4. It can easily be shown that $\mathcal{M}_n^{X_\bullet}(I_\bullet)$ satisfies item (2) in Running Assumption 2.0.4 for each n since I_0 is assumed cofibrant as an object in $\mathcal{C}$. Items (1) and (3) of Running Assumption 2.0.4 are satisfied by definition of $\mathcal{M}_n^{X_\bullet}(I_\bullet)$ and by definition of the maps $\mathcal{M}_n^{X_\bullet}(I_\bullet) \to \mathcal{M}_{n-1}^{X_\bullet}(I_\bullet)$. Therefore, the commutative monoid $|\mathcal{M}_\bullet^{X_\bullet}(I_\bullet)|$ in $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$. (We remind the reader that, by the main theorem of the authors' paper [2], an example of a setting in which Running Assumption 2.0.4 holds is the category of symmetric spectra equipped with the positive flat stable model structure.)

Definition 3.3.5 Suppose $n \in \mathbb{N}$. Let $\iota\colon \mathcal{D}_{n+1}^S \hookrightarrow \mathcal{D}_n^S$ be the canonical inclusion of categories. We will write $\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)$ for the left Kan extension of $\mathcal{F}_{n+1}^S(I_\bullet)\colon (\mathcal{D}_{n+1}^S)^{\mathrm{op}} \to \mathcal{C}$ along ι^{op}; ie if we write $\mathrm{Kan}\colon \mathcal{C}^{(\mathcal{D}_{n+1}^S)^{\mathrm{op}}} \to \mathcal{C}^{(\mathcal{D}_n^S)^{\mathrm{op}}}$ for the left adjoint of the functor $\mathcal{C}^{(\mathcal{D}_n^S)^{\mathrm{op}}} \to \mathcal{C}^{(\mathcal{D}_{n+1}^S)^{\mathrm{op}}}$ induced by ι, then $\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet) = \mathrm{Kan}(\mathcal{F}_{n+1}^S(I_\bullet))$. We have a canonical map $c\colon \widetilde{\mathcal{F}}_{n+1}^S(I_\bullet) \to \mathcal{F}_n^S(I_\bullet)$, by the universal property of this Kan extension.

Remark 3.3.6 It is a elementary exercise in combinatorics to show that there are $\binom{n+\#(S)}{n}$ elements in the set $\{x \in \mathbb{N}^S : |x| = n\}$, where $\#(S)$ is the cardinality of S. This set indexes a coproduct in the following lemma.

Lemma 3.3.7 *Let $I_\bullet$ be a cofibrant decreasingly filtered object in $\mathcal{C}$ and let S be a finite set. Then the cofiber of the map* (see Definition 3.3.5 *for the definition of the map c*)

$$\mathrm{colim}(\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)) \xrightarrow{\mathrm{colim}\,c} \mathrm{colim}(\mathcal{F}_n^S(I_\bullet)),$$

computed in $\mathcal{C}$, is isomorphic to the coproduct in $\mathcal{C}$

$$\coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\left(\bigwedge_{s \in S} I_{x(s)} \right) / (\mathrm{colim}\, \mathcal{F}_{1;x}^S(I_\bullet)) \right).$$

This isomorphism is natural in the variable S.

Proof One knows that the left Kan extension of $\mathcal{F}_{n+1}^S(I_\bullet)$ agrees with $\mathcal{F}_{n+1}^S(I_\bullet)$ wherever both are defined: Given $x \in \mathcal{D}_{n+1}^S$, the pointwise formula for a Kan extension gives us that $\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)(x)$ is the colimit, over all $y \to x$ in $\mathcal{D}_{n+1}^S$, of $\mathcal{F}_{n+1}^S(I_\bullet)(y)$. Since the identity map on x is already in $\mathcal{D}_{n+1}^S$, we get that $\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)(x) \cong \mathcal{F}_{n+1}^S(I_\bullet)(x)$. Hence

$$\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)(x) \cong \mathcal{F}_{n+1}^S(I_\bullet)(x) = \bigwedge_{s \in S} I_{x(s)}$$

for all $x \in \mathcal{D}_{n+1}^S \subseteq \mathcal{D}_n^S$. The elements of $\mathcal{D}_n^S$ which are *not* in $\mathcal{D}_{n+1}^S$ are those x such that $|x| = n$, and by the pointwise formula for a Kan extension (see eg [31]) one knows that, for all x such that $|x| = n$, we have an isomorphism of $\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)(x)$ with the colimit of the values of $\mathcal{F}_{n+1}^S(I_\bullet)$ over those elements of $(\mathcal{D}_{n+1}^S)^{\mathrm{op}}$ which map to x, ie $\mathrm{colim}(\mathcal{F}_{1,x}^S(I_\bullet))$, the colimit of the values of $\mathcal{F}_{n+1}^S(I_\bullet)$ over $(\mathcal{D}_{1,x}^S)^{\mathrm{op}} \subseteq (\mathcal{D}_{n+1}^S)^{\mathrm{op}}$.

For each $x \in \mathcal{D}_n^S$, the map $c(x)$ can be shown to be a cofibration by iterated use of the pushout product axiom, so the cofiber of $c(x)$ is a homotopy cofiber. By the previous paragraph the levelwise cofiber $\mathrm{cof}\, c \colon (\mathcal{D}_n^S)^{\mathrm{op}} \to \mathcal{C}$ of the natural transformation c is given as follows:

$$(\mathrm{cof}\, c)(x) \cong \begin{cases} 0 & \text{if } |x| > n, \\ (\mathcal{F}_n^S(I_\bullet))/(\mathrm{colim}(\mathcal{F}_{1,x}^S(I_\bullet))) & \text{if } |x| = n. \end{cases}$$

Hence, on taking colimits, we have

$$\mathrm{cof}\, \mathrm{colim}\, c \cong \mathrm{colim}\, \mathrm{cof}\, c = \coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\left(\bigwedge_{s \in S} I_{x(s)} \right) / (\mathrm{colim}\, \mathcal{F}_{1;x}^S(I_\bullet)) \right),$$

as claimed. $\square$

Lemma 3.3.8 *Suppose S is a finite set and suppose $Z_{s,1} \to Z_{s,0}$ is a cofibration for each $s \in S$. Suppose the objects $Z_{s,1}$ and $Z_{s,0}$ are cofibrant for every s. Let $\mathcal{G}_S \colon (\mathcal{E}_1^S)^{\mathrm{op}} \to \mathcal{C}$ be the functor given on objects by $\mathcal{G}_S(x) = \bigwedge_{s \in S} Z_{s,x(s)}$, and given on morphisms in the obvious way.*

Then the smash product $\bigwedge_{s \in S} Z_{s,0} \to \bigwedge_{s \in S}(Z_{s,0}/Z_{s,1})$ of the cofiber projections $Z_{s,0} \to Z_{s,0}/Z_{s,1}$ fits into a cofiber sequence:

$$\mathrm{colim}\, \mathcal{G}_S \to \bigwedge_{s \in S} Z_{s,0} \to \bigwedge_{s \in S}(Z_{s,0}/Z_{s,1}).$$

Proof If the cardinality of S is one, the statement of the lemma is true by the definition of a cofiber.

The case in which the cardinality of S is two is precisely Lemma 3.3.1, already proven.

For the case in which the cardinality of S is greater than two, we introduce a notation we will need to use: let $\mathcal{PO}$ denote the category indexing pushout diagrams, ie

$$(15) \qquad \mathcal{PO} = \left(\begin{array}{c} [1'] \\ \swarrow \qquad \searrow \\ [1] \qquad\qquad [0] \end{array} \right),$$

the symbols $[1']$, $[1]$, and $[0]$ each representing an object, and the arrows each representing a morphism. We observe that $\mathcal{PO}$ and $\mathcal{E}_1^S$ are not arbitrary small categories but are in fact partially ordered sets; this simplifies some of the arguments we will give in the rest of the proof.

Suppose that the cardinality of S is greater than two. Choose an element $s_0 \in S$. We will write S' for the complement $S' = \{s \in S : s \neq s_0\}$ of s_0 in S. Define objects X_1', X_2', Y_1', and Y_2' in $\mathcal{C}$ as follows: $Y_1' = \operatorname{colim} \mathcal{G}_{S'}$, $X_1' = \bigwedge_{s \in S'} Z_{s,0}$, $Y_2' = Z_{s_0,1}$, and $X_2' = Z_{s_0,0}$. Now we apply the statement of the lemma in the (already proven, above) case $S = \{1, 2\}$, using X_1', X_2', Y_1', and Y_2' in place of $Z_{1,0}$, $Z_{2,0}$, $Z_{1,1}$, and $Z_{2,1}$, to obtain a cofiber sequence

$$(16) \qquad \operatorname{colim} \mathcal{B} \to \bigwedge_{s \in S} Z_{s,0} \to \bigwedge_{s \in S} (Z_{s,0}/Z_{s,1}),$$

where $\mathcal{B}$ is the functor $\mathcal{PO} \to \mathcal{C}$ given by

$$\mathcal{B}([1']) = (\operatorname{colim} \mathcal{G}_{S'}) \wedge Z_{s_0,1},$$
$$\mathcal{B}([1]) = \left(\bigwedge_{s \in S'} Z_{s_0,0} \right) \wedge Z_{s_0,1},$$
$$\mathcal{B}([0]) = (\operatorname{colim} \mathcal{G}_{S'}) \wedge Z_{s_0,0}.$$

By Lemma 3.3.1, we know that the map $\operatorname{colim} \mathcal{G}_{S'} \to \bigwedge_{s \in S'} Z_{s,0}$ is a cofibration in the case $S = \{1, 2\}$. Since $\operatorname{colim} \mathcal{B}$ is constructed as a pushout, the pushout product axiom ensures that the map $\operatorname{colim} \mathcal{B} \to \bigwedge_{s \in S} Z_{s,0}$ is also a cofibration as long as $\operatorname{colim} \mathcal{G}_{S'} \to \bigwedge_{s \in S'} Z_{s,0}$ is a cofibration. It suffices to show that $\operatorname{colim} \mathcal{B} \cong \operatorname{colim} \mathcal{G}_S$. This will show that the map $\operatorname{colim} \mathcal{G}_S \to \bigwedge_{s \in S} Z_{s,0}$ is a cofibration and allow us to identify the cofiber, thus completing the induction on the cardinality of the set S. We can reindex, to describe $\operatorname{colim} \mathcal{B}$ as the colimit of a larger diagram $\mathcal{H} \colon (\mathcal{E}_1^{S'})^{\mathrm{op}} \times \mathcal{PO} \to \mathcal{C}$ given by

$$(x, [1']) \mapsto \left(\bigwedge_{s \in S'} Z_{s,x(s)} \right) \wedge Z_{s_0,1},$$
$$(x, [1]) \mapsto \left(\bigwedge_{s \in S'} Z_{s,0} \right) \wedge Z_{s_0,1},$$
$$(x, [0]) \mapsto \left(\bigwedge_{s \in S'} Z_{s,x(s)} \right) \wedge Z_{s_0,0}.$$

We have a functor $P\colon (\mathcal{E}_1^{S'})^{\mathrm{op}} \times \mathcal{PO} \to (\mathcal{E}_1^{S})^{\mathrm{op}}$ given by

$$(x, [1'])(s) = \begin{cases} x(s) & \text{if } s \neq s_0, \\ 1 & \text{if } s = s_0, \end{cases}$$

$$(x, [1])(s) = \begin{cases} 0 & \text{if } s \neq s_0, \\ 1 & \text{if } s = s_0, \end{cases}$$

$$(x, [0])(s) = \begin{cases} x(s) & \text{if } s \neq s_0, \\ 0 & \text{if } s = s_0. \end{cases}$$

Now we claim that the canonical map $\operatorname{colim} \mathcal{H} \to \operatorname{colim} \mathcal{G}_S$ given by P is an isomorphism in $\mathcal{C}$. We define a functor $I\colon (\mathcal{E}_1^S)^{\mathrm{op}} \to (\mathcal{E}_1^{S'})^{\mathrm{op}} \times \mathcal{PO}$ by

$$x \mapsto \begin{cases} (I'(x), [1']) & \text{if } x(s_0) = 1, \\ (I'(x), [0]) & \text{if } x(s_0) = 0, \end{cases}$$

where $I'\colon (\mathcal{E}_1^S)^{\mathrm{op}} \to (\mathcal{E}_1^{S'})^{\mathrm{op}}$ is the functor given by restriction, ie $I'(x)(s) = x(s)$ for $s \in S'$. Now we observe some convenient identities:

$$(\mathcal{G}_S \circ P)(x, j) = \begin{cases} \left(\bigwedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 1} & \text{if } j = [1'], \\ \left(\bigwedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 0} & \text{if } j = [0], \\ \left(\bigwedge_{s \in S'} Z_{s, 0}\right) \wedge Z_{s_0, 1} & \text{if } j = [1] \end{cases}$$

$$= \mathcal{H}(x, j),$$

$$(\mathcal{H} \circ I)(x) = \begin{cases} \left(\bigwedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 1} & \text{if } x_{s_0} = 1, \\ \left(\bigwedge_{s \in S'} Z_{s, x(s)}\right) \wedge Z_{s_0, 0} & \text{if } x_{s_0} = 0 \end{cases}$$

$$= \mathcal{G}_S(x).$$

We conclude that P and I give mutually inverse maps between $\operatorname{colim} \mathcal{G}_S$ and $\operatorname{colim} \mathcal{H}$, ie $\operatorname{colim} \mathcal{G}_S \cong \operatorname{colim} \mathcal{H}$, and hence $\operatorname{colim} \mathcal{G}_S \cong \operatorname{colim} \mathcal{B}$, as desired. So from cofiber sequence (16), we have a cofiber sequence

$$\operatorname{colim} \mathcal{G}_S \to \bigwedge_{s \in S} Z_{s, 0} \to \bigwedge_{s \in S} (Z_{s, 0} / Z_{s, 1}),$$

as desired.

From inspection of the colimit diagrams one sees that the cofiber sequence (16) does not depend on the choice of $s_0 \in S$, and naturality in S follows. $\qquad\square$

Lemma 3.3.9 *Let S be a finite set, let n be a positive integer, and let $x \in \mathbb{N}^S$. Let $\mathcal{E}_n^S$ and $\mathcal{D}_{n;x}^S$ be the partially ordered sets defined in equations* (11) *and* (12). *Let $J_{n;x}^S\colon \mathcal{E}_n^S \to \mathcal{D}_{n;x}^S$ be the functor (ie morphism of partially ordered sets) defined by*

$$(J_{n;x}(y))(s) = x(s) + y(s).$$

Then $J_{n;x}$ has a right adjoint. Consequently $J_{n;x}$ is a cofinal functor; ie for any functor F defined on $\mathcal{D}_{n;x}^S$ such that the limit $\lim F$ exists, the limit $\lim(F \circ J_{n;x}^S)$ also exists, and the canonical map $\lim(F \circ J_{n;x}^S) \to \lim F$ is an isomorphism.

Proof We construct the right adjoint explicitly. Let $K_{n;x}^S \colon \mathcal{D}_{n;x}^S \to \mathcal{E}_n^S$ be the functor defined by

$$(K_{n;x}^S(y))(s) = \min\{n, y(s) - x(s)\}.$$

(We remind the reader that every element $y \in \mathcal{D}_{n;x}^S$ has the property that $y(s) \geq x(s)$ for all $s \in S$, so $y(s) - x(s)$ will always be nonnegative.)

Now suppose $z \in \mathcal{E}_n^S$ and $y \in \mathcal{D}_{n;x}^S$. Then $z \leq K_{n;x}^S(y)$ if and only if $z(s) \leq K_{n;x}^S(y)(s)$ for all $s \in S$; ie $z \leq K_{n;x}^S(y)$ if and only if $z(s) \leq \min\{n, y(s) - x(s)\}$ for all $s \in S$. By the definition of $\mathcal{E}_n^S$, we have $z(s) \leq n$ for all $s \in S$. Hence $z \leq K_{n;x}^S(y)$ if and only if $z(s) \leq y(s) - x(s)$ for all $s \in S$; ie $z \leq K_{n;x}^S(y)$ if and only if $x(s) + z(s) \leq y(s)$ for all $s \in S$; ie $z \leq K_{n;x}^S(y)$ if and only if $J_{n;x}^S(z) \leq y$. Hence $\hom_{\mathcal{E}_n^S}(z, K_{n;x}^S(y))$ is nonempty if and only if $\hom_{\mathcal{D}_{n;x}^S}(J_{n;x}^S(z), y)$ is nonempty. Because $\mathcal{E}_n^S$ and $\mathcal{D}_{n;x}^S$ are partially ordered sets and hence their hom-sets are either nonempty or have only a single element, we now have a (natural) bijection

$$\hom_{\mathcal{E}_n^S}(z, K_{n;x}^S(y)) \cong \hom_{\mathcal{D}_{n;x}^S}(J_{n;x}^S(z), y),$$

which is exactly what we are looking for: $J_{n;x}^S$ is left adjoint to $K_{n;x}^S$.

For the fact that having a right adjoint implies cofinality, see Mac Lane [31, Section IX.3]. (Mac Lane handles the equivalent dual case.) □

Theorem 3.3.10 (Fundamental Theorem of the May filtration) *Let $I_\bullet$ be a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, and let $X_\bullet$ be a simplicial finite set. Then the associated graded commutative monoid $E_0^*|\mathcal{M}^{X_\bullet}(I_\bullet)|$ of the geometric realization of the May filtration is weakly equivalent, as a commutative graded monoid, to the tensoring $X_\bullet \otimes E_0^* I_\bullet$ of $X_\bullet$ with the associated graded commutative monoid of $I_\bullet$:*

$$E_0^*|\mathcal{M}^{X_\bullet}(I_\bullet)| \simeq X_\bullet \otimes E_0^* I_\bullet.$$

Proof We must compute the filtration quotients

$$(17) \qquad |\mathcal{M}_n^{X_\bullet}(I_\bullet)|/|\mathcal{M}_{n+1}^{X_\bullet}(I_\bullet)| \cong |\mathcal{M}_n^{X_\bullet}(I_\bullet)/\mathcal{M}_{n+1}^{X_\bullet}(I_\bullet)|.$$

(We have isomorphism (17) because $|\mathcal{M}_{n+1}^{X_\bullet}(I_\bullet)| \to |\mathcal{M}_n^{X_\bullet}(I_\bullet)|$ is a cofibration, by Remark 3.3.4.)

We compute the filtration quotients as follows. First, we claim that there exists, for any finite set S and for all $n \in \mathbb{N}$, a cofiber sequence

$$(18) \quad \mathrm{colim}(\mathcal{F}_{n+1}^S(I_\bullet)) \to \mathrm{colim}(\mathcal{F}_n^S(I_\bullet)) \to \coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\bigwedge_{s \in S} (I_{x(s)} / I_{1+x(s)}) \right)$$

in $\mathcal{C}$, natural in S. We have already defined (in Definition 3.3.2) how $\mathcal{F}_n^S$ is natural, ie functorial in S; by taking the obvious coproduct of quotients, this naturality in S induces a naturality in S on the terms $\coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\bigwedge_{s \in S} (I_{x(s)} / I_{1+x(s)}) \right)$ appearing in (18). The claim that (18) is a cofiber sequence implies that

$$(19) \quad |\mathcal{M}_n^{X_k}(I_\bullet)| / |\mathcal{M}_{n+1}^{X_k}(I_\bullet)| \cong \coprod_{\{x \in \mathbb{N}^{X_k} : |x| = n\}} \left(\bigwedge_{s \in X_k} (I_{x(s)} / I_{1+x(s)}) \right),$$

and naturally implies the necessary naturality with respect to the face and degeneracy maps.

We now show that the cofiber sequence (18) exists. First, by the universal property of the Kan extension from Lemma 3.3.7, the cofiber of the map $\mathrm{colim}(\mathcal{F}_{n+1}^S(I_\bullet)) \to \mathrm{colim}(\mathcal{F}_n^S(I_\bullet))$ agrees with the cofiber of the map $\mathrm{colim}(\widetilde{\mathcal{F}}_{n+1}^S(I_\bullet)) \to \mathrm{colim}(\mathcal{F}_n^S(I_\bullet))$. By Lemma 3.3.7, this cofiber is the coproduct

$$\coprod_{\{x \in \mathbb{N}^S : |x| = n\}} \left(\left(\bigwedge_{s \in S} I_{x(s)} \right) / (\mathrm{colim}\, \mathcal{F}_{1;x}^S(I_\bullet)) \right).$$

In Lemma 3.3.9, we showed that the functor $J_{1;x}$ is cofinal, hence that the comparison map of colimits

$$\mathrm{colim}(\mathcal{F}_{1;x}(I_\bullet) \circ J_{1;x}) \to \mathrm{colim}(\mathcal{F}_{1;x}(I_\bullet))$$

is an isomorphism. (We here have a colimit, not a limit as in the statement of Lemma 3.3.9, since $\mathcal{F}_{1;x}(I_\bullet)$ is a *contravariant* functor on $\mathcal{D}_{1;x}^S$. Of course Lemma 3.3.9 still holds in this dual form.)

Now Lemma 3.3.8 identifies the cofiber

$$\left(\bigwedge_{s \in S} I_{x(s)} \right) / (\mathrm{colim}(\mathcal{F}_{1;x}(I_\bullet) \circ J_{1;x}))$$

with $\bigwedge_{s \in S} (I_{x(s)} / I_{1+x(s)})$, as desired. So we have our cofiber sequence of the form (18).

All isomorphisms in the lemmas we have invoked in this proof are natural in S, with the exception of the isomorphisms from Lemmas 3.3.9 and 3.3.8 which directly involve $\mathcal{E}_1^S$, only because we did not specify in Lemma 3.3.8 how $\mathcal{G}_S$ is functorial in S. In the

present proof, $\mathcal{G}_S$ is $\mathcal{F}_{1;x}(I_\bullet) \circ J_{1;x}$, and the cofinality of $J_{1;x}$ together with the fact that $K_{1;x} \circ J_{1;x} = \mathrm{id}_{\mathcal{E}_1^S}$ implies, on inspection of the colimit diagrams, that the isomorphism

$$\mathrm{colim}\,\mathcal{G}_S = \mathrm{colim}(\mathcal{F}_{1;x}(I_\bullet) \circ J_x) \cong \mathrm{colim}(\mathcal{F}_{1;x}(I_\bullet))$$

is natural in S; details are routine and left to the reader. We conclude that the cofiber sequence (18) is indeed natural in S.

Now we have the sequence of simplicial commutative monoids in $\mathcal{C}$

$$\mathcal{M}_0^{X_\bullet}(I_\bullet) \leftarrow \mathcal{M}_1^{X_\bullet}(I_\bullet) \leftarrow \mathcal{M}_2^{X_\bullet}(I_\bullet) \leftarrow \cdots,$$

and geometric realization commuting with cofibers together with the isomorphism (19) implies that the comparison map

$$(20) \qquad\qquad X_\bullet \otimes E_0^* I_\bullet \to E_0^* |\mathcal{M}^{X_\bullet}(I_\bullet)|$$

of objects in $\mathcal{C}$ is a weak equivalence. Hence the comparison map (20) in $\mathrm{Comm}(\mathcal{C})$ must also be a weak equivalence, since the weak equivalences in $\mathrm{Comm}(\mathcal{C})$ are created by the forgetful functor $\mathrm{Comm}(\mathcal{C}) \to \mathcal{C}$, by Running Assumption 2.0.3. □

3.4 Construction of the topological Hochschild–May spectral sequence

Definition 3.4.1 By a *connective generalized homology theory on $\mathcal{C}$* we shall mean the choice, for each integer n, of a functor $E_n \colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ab}$ satisfying these axioms:

Exactness For each integer n and each distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\mathrm{Ho}(\mathcal{C})$, the following sequence of abelian groups is exact:

$$\cdots \to E_n(Y) \to E_n(Z) \to E_n(\Sigma X) \to E_n(\Sigma Y) \to E_n(\Sigma Z) \to \cdots.$$

Additivity For each integer n and each collection of objects $\{X_i\}_{i\in I}$ in $\mathrm{Ho}(\mathcal{C})$, the following canonical map of abelian groups is an isomorphism:

$$\coprod_{i\in I} E_n(X_i) \to E_n\left(\coprod_{i\in I} X_i\right).$$

Connectivity of the unit object We have $E_n(\mathbb{1}) \cong 0$ for all $n < 0$.

Connectivity of smash products Suppose that X and Y are objects of $\mathcal{C}$, and that A and B are nonnegative integers such that $E_n(X) \cong 0$ for all $n < A$, and $E_n(Y) \cong 0$ for all $n < B$. Then $E_n(X \wedge Y) \cong 0$ for all $n < A + B$.

Clearly, Definition 3.4.1 is just a formulation, in a general pointed model category, of (the triangulated category form of) the Eilenberg–Steenrod axioms, from [19], for a generalized homology theory with connective coefficients. The "connectivity of smash products" axiom is easily proven anytime one has an E–homology Künneth spectral sequence in $\mathcal{C}$, which is the case in any of the usual models for the stable homotopy category.

Definition 3.4.2 If $I_\bullet$ is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, $X_\bullet$ is a simplicial finite set, and E_* is a connective generalized homology theory on $\mathcal{C}$, then by the *topological Hochschild–May spectral sequence for* $X_\bullet \widetilde{\otimes} I_\bullet$ we mean the spectral sequence obtained by applying E_* to the tower of cofiber sequences in $\mathcal{C}$

(21)
$$
\begin{array}{ccc}
\cdots \longrightarrow |\mathcal{M}_1^{X_\bullet}(I_\bullet)| & \longrightarrow & |\mathcal{M}_0^{X_\bullet}(I_\bullet)| \\
\downarrow & & \downarrow \\
|\mathcal{M}_1^{X_\bullet}(I_\bullet)|/|\mathcal{M}_2^{X_\bullet}(I_\bullet)| & & |\mathcal{M}_0^{X_\bullet}(I_\bullet)|/|\mathcal{M}_1^{X_\bullet}(I_\bullet)|
\end{array}
$$

That is, it is the spectral sequence of the exact couple

$$
\begin{array}{ccc}
D^1_{*,*} & \longrightarrow & D^1_{*,*} \\
& \nwarrow \quad \swarrow & \\
& E^1_{*,*} &
\end{array}
$$

where $E^1_{*,*} = \bigoplus_{i,j} H_i |\mathcal{M}_j^{X_\bullet}(I_\bullet)|/|\mathcal{M}_{j+1}^{X_\bullet}(I_\bullet)|$ and $D^1_{*,*} = \bigoplus_{i,j} H_i |\mathcal{M}_j^{X_\bullet}(I_\bullet)|$.

Lemma 3.4.3 (connectivity conditions) *Let E_* be a connective generalized homology theory on $\mathcal{C}$. Suppose that there exist objects Z and E of $\mathcal{C}$ such that $E_*(-)$ is naturally isomorphic to $[\Sigma^* Z, E \wedge -]$. Let*

(22) $$\cdots \to Y_2 \to Y_1 \to Y_0$$

be a sequence in $\mathcal{C}$, and suppose that $E_n(Y_i) \cong 0$ for all $n < i$. Then

$$[\Sigma^n Z, \operatorname{holim}_i(E \wedge Y_i)] \cong 0$$

for all n. If we instead suppose that A is a nonnegative integer and that $X_\bullet$ is a simplicial object of $\mathcal{C}$ such that $E_n(X_i) \cong 0$ for all $n < A$ and all i, then $E_n(|X_\bullet|) \cong 0$ for all $n < A$.

Proof Since $\mathcal{C}$ is assumed stable, the homotopy limit $\operatorname{holim}_i Y_i$ is the homotopy fiber of the map

$$\prod_{n \in \mathbb{N}} Y_n \xrightarrow{\text{id} - T} \prod_{n \in \mathbb{N}} Y_n$$

in $\operatorname{Ho}(\mathcal{C})$, where T is the product of the maps $Y_n \to Y_{n-1}$ occurring in the sequence (22). For each object Z of $\mathcal{C}$, we then have the long exact sequence obtained by applying the functor $[\Sigma^* Z, E \wedge -]$ to the fiber sequence

$$\operatorname{holim}_i Y_i \to \prod_{n \in \mathbb{N}} Y_n \xrightarrow{\text{id} - T} \prod_{n \in \mathbb{N}} Y_n,$$

hence the Milnor exact sequence

$$0 \to R^1 \lim_i [\Sigma^{j+1} Z, E \wedge Y_i] \to [\Sigma^j Z, \operatorname{holim}_i E \wedge Y_i] \to \lim_i [\Sigma^j Z, E \wedge Y_i] \to 0.$$

The assumption that $[\Sigma^j Z, E \wedge Y_i] \cong 0$ for $j < i$ guarantees that the sequence

$$\cdots \to [\Sigma^j Z, E \wedge Y_2] \to [\Sigma^j Z, E \wedge Y_1] \to [\Sigma^j Z, E \wedge Y_0]$$

is eventually constant and zero for all j, hence both its limit and $R^1 \lim$ vanish for all j, hence $[\Sigma^j Z, \operatorname{holim}_i E \wedge Y_i] \cong 0$ for all j.

The Bousfield–Kan spectral sequence, ie the E–homology spectral sequence of the simplicial object $X_\bullet$, has input $E^1_{s,t} \cong \pi_s(E \wedge X_t)$ and converges to $E_{s+t}(|X_\bullet|)$, since $E_n(X_i) \cong 0$ for all $n < A$ and all i. The differential in this spectral sequence is of the form $d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}$, hence this spectral sequence has a nondecreasing upper vanishing curve at E^1, hence converges strongly. Triviality of $E^1_{s,t}$ for $s < A$ and $t < 0$ then gives us that $E_s(|X_\bullet|)$ vanishes for $s < A$. $\qquad\square$

Definition 3.4.4 If S is a finite set, recall that $\mathcal{D}^S_i$ is the partially ordered set of functions $f \colon S \to \mathbb{N}$ such that $|f| \geq i$. The category $(\mathcal{D}^S_i)^{\text{op}}$ has a cofinal subcategory $(\mathcal{E}^S_i)^{\text{op}}$ defined in (12). We give this cofinal subcategory a Reedy category structure by letting the degree function be $-|s|$, letting $((\mathcal{E}^S_i)^{\text{op}})_+$ contain all morphisms and $((\mathcal{E}^S_i)^{\text{op}})_-$ contain only identity morphisms.

Remark 3.4.5 Since the category $(\mathcal{E}^S_i)^{\text{op}}$ is a direct category by Definition 3.4.4, the Reedy model structure on the category of functors of the form $(\mathcal{E}^S_i)^{\text{op}} \to \mathcal{C}$, where $\mathcal{C}$ is a model category, agrees with the projective model structure.

Since $|x| < i \cdot (\#S)$, the target of the degree map on $(\mathcal{E}^S_i)^{\text{op}}$ is a (finite) ordinal, and this satisfies the requirement for a Reedy category.

Lemma 3.4.6 *Let S be a finite set, and for each integer i, let $\mathcal{D}_i^S$ and $(\mathcal{E}_i^S)^{op}$ be the categories defined immediately preceding Definition 3.3.2. For each $s \in S$ and each $n \in \mathbb{N}$, let $\mathcal{J}_{n+1}(s) \to \mathcal{J}_n(s)$ be a cofibration between cofibrant objects in $\mathcal{C}$, and let $\mathcal{J} \colon (\mathcal{D}_i^S)^{op} \to \mathcal{C}$ be the functor given by $\mathcal{J}(f) = \bigwedge_{s \in S} \mathcal{J}_{f(s)}(s)$. Equip $(\mathcal{E}_i^S)^{op}$ with the Reedy structure of Definition 3.4.4. Then the restriction of $\mathcal{J}$ to $(\mathcal{E}_i^S)^{op}$ is Reedy-cofibrant.*

Furthermore, if N is an integer and we additionally assume that E_* is a connective generalized homology theory (as defined in Definition 3.4.1) such that $E_m(\mathcal{J}(f)) \cong 0$ for all $m < N$ and $f \in (\mathcal{D}_i^S)^{op}$, then $E_m(\mathrm{colim}\,\mathcal{J}) \cong 0$ for all $m < N$ and all integers i.

Proof Choose an object $f \in (\mathcal{E}_i^S)^{op}$. Then let $(\mathcal{E}_{1;f}^S)^{op}$ be the punctured overcategory of f; ie $(\mathcal{E}_{1;f}^S)^{op}$ is the full subcategory of the overcategory $(\mathcal{E}_i^S)^{op} \downarrow f$ generated by all objects other than id_f. The latching object $L_f(\widetilde{\mathcal{J}})$ is simply the colimit of $\widetilde{\mathcal{J}}$ restricted to $((\mathcal{E}_{1;f}^S)^{op})_+ = (\mathcal{E}_{1;f}^S)^{op}$, and in $(\mathcal{E}_{1;f}^S)^{op}$ we have the cofinal subcategory consisting of all functions $g \colon S \to \mathbb{N}$ such that $g \neq f$ and such that $f(s) \leq g(s) \leq f(s) + 1$ for all $s \in S$.

The composite map

$$\mathrm{colim}(\widetilde{\mathcal{J}})^{op} \xrightarrow{\cong} L_f(\widetilde{\mathcal{J}}) \to \widetilde{\mathcal{J}}(f)$$

is precisely the pushout product of the maps $\{\mathcal{J}_{f(s)+1}(s) \to \mathcal{J}_{f(s)}(s)\}_{s \in S}$ and therefore it is a cofibration by the pushout product axiom. Hence, $\widetilde{\mathcal{J}}$ is Reedy-cofibrant.

Since $\widetilde{\mathcal{J}}$ is also projectively cofibrant by Remark 3.4.5, $\mathrm{colim}\,\widetilde{\mathcal{J}}$ is weakly equivalent to $|\mathrm{sr}(\widetilde{\mathcal{J}})|$ by Theorem B.1.2. Since the category $(\mathcal{E}_i^S)^{op}$ is cofinal in $(\mathcal{D}_i^S)^{op}$,

$$E_*(\mathrm{colim}\,\mathcal{J}) \cong E_*(\mathrm{colim}\,\widetilde{\mathcal{J}}).$$

Together, these two facts imply that $E_*(\mathrm{colim}(\mathcal{J}))$ is concentrated in degrees $\geq N$ by Theorem B.1.5. $\qquad\square$

Lemma 3.4.7 *Suppose E_* is a connective generalized homology theory as defined in Definition 3.4.1, and $\mathcal{M}_i^S(I_\bullet)$ is the ith degree of the May filtration for a finite set S and a cofibrant decreasingly filtered commutative monoid $I_\bullet$ as defined in Definition 3.1.2. Then, if $E_m(I_i) \cong 0$ for all $m, i \in \mathbb{N}$ such that $m < i$, then $E_m(\mathcal{M}_i^S(I_\bullet)) \cong 0$ for all $m, i \in \mathbb{N}$ such that $m < i$.*

Proof This is immediate from Lemma 3.4.6 and the connectivity hypotheses in Definition 3.4.1. $\qquad\square$

Theorem 3.4.8 Suppose $I_\bullet$ is a Hausdorff cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$, $X_\bullet$ is a simplicial finite set, and E_* is a connective generalized homology theory on $\mathcal{C}$. Suppose $E_*(-) \cong [\Sigma^* Z, -\wedge E]$ for some objects Z and E in $\mathcal{C}$. Suppose the following connectivity axiom: $E_m(I_n) \cong 0$ for all $m < n$.

Then the topological Hochschild–May spectral sequence is strongly convergent, its differential satisfies the graded Leibniz rule, and its input and output and differential are as follows:

$$E^1_{s,t} \cong E_{s,t}(X_\bullet \otimes E_0^* I_\bullet) \Rightarrow E_s(X_\bullet \otimes I_0), \quad d^r \colon E^r_{s,t} \to E^r_{s-1,t+r}.$$

Proof It is standard (see eg the section on Adams spectral sequences in [8]) that the E–homology spectral sequence of a tower of cofiber sequences of the form (21) converges to $E_*(|\mathcal{M}_0^{X_\bullet}(I_\bullet)|)$ as long as $[\Sigma^* Z, \operatorname{holim}_i(E \wedge |\mathcal{M}_i^{X_\bullet}(I_\bullet)|)]$ is trivial. By Lemma 3.4.7 $E_m(|\mathcal{M}_i^{X_\bullet}(I_\bullet)|) \cong 0$ for all $m < i$, so by Lemma 3.4.3,

$$[\Sigma^n Z, \operatorname{holim}_i(E \wedge |\mathcal{M}_i^{X_\bullet}(I_\bullet)|)] \cong 0$$

for all n, as desired. Hence the spectral sequence converges to $E_*(|\mathcal{M}_0^{X_\bullet}(I_\bullet)|)$ and by Theorem 3.3.10 this is isomorphic to $E_*(X_\bullet \otimes I_0)$.

That the differential has the stated bidegree is a routine and easy computation in the spectral sequence of a tower of cofiber sequences. The sequence

$$\cdots \to |\mathcal{M}_2^{X_\bullet}(I_\bullet)| \to |\mathcal{M}_1^{X_\bullet}(I_\bullet)| \to |\mathcal{M}_0^{X_\bullet}(I_\bullet)|$$

is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ as observed in Remark 3.3.4 and therefore, in particular, it produces a "pairing of towers" in the sense of [16] and therefore by [16, Proposition 5.1] the differentials in the spectral sequence satisfy a graded Leibniz rule.

Strong convergence is also standard: the connectivity axiom, the "connectivity of smash products" axiom from Definition 3.4.1, and Lemma 3.4.7 together imply that our spectral sequence has a nondecreasing upper vanishing curve already at the E^1–term, so the spectral sequence converges strongly.   □

Remark 3.4.9 Another construction of our THH–May spectral sequence

$$(23) \qquad\qquad E^1_{*,*} \cong E_{*,*}(X_\bullet \otimes E_0^* I_\bullet) \Rightarrow E_*(X_\bullet \otimes I_0)$$

is possible using the Day convolution product. This construction is conceptually cleaner,

but it does not, to our knowledge, simplify the process of proving that the resulting spectral sequence has the correct input term, output term and convergence properties.

Recall from Remark 3.1.5 that a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ (without the cofibrancy assumption) is the same data as an object in $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\operatorname{op}}}$. A cofibrant object in $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\operatorname{op}}}$ with the projective model structure is, by Lemma 4.1.9, a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ (including that cofibrancy assumption).

Now fix a simplicial finite set $X_\bullet$ and a cofibrant commutative monoid object I in $\mathcal{C}^{\mathbb{N}^{\operatorname{op}}}$. Hence, we can form the pretensor product $X_\bullet \widetilde{\otimes} I_\bullet$, which is a simplicial object in $\operatorname{Comm}\mathcal{C}^{\mathbb{N}^{\operatorname{op}}}$. For example, if $X_\bullet$ is the usual minimal simplicial model for the circle $S^1_\bullet$, then $X_\bullet \widetilde{\otimes} I_\bullet$ is the cyclic bar construction using the Day convolution as the tensor product:

$$S^1_\bullet \widetilde{\otimes} I = \left(I \rightrightarrows I \otimes_{\mathrm{Day}} I \Rrightarrow I \otimes_{\mathrm{Day}} I \otimes_{\mathrm{Day}} I \cdots \right).$$

Since I is a functor $\mathbb{N}^{\operatorname{op}} \to \mathcal{C}$, we will write $I(n)$ for the evaluation of this functor at a nonnegative integer n. (If we instead think of I as a decreasingly filtered commutative monoid, as in most of the rest of this paper, we would write I_n instead of $I(n)$.) We write $(S^1_\bullet \widetilde{\otimes} I)(i)$ for the simplicial object in $\mathcal{C}$

$$(S^1_\bullet \widetilde{\otimes} I)(i) = \left(I(i) \rightrightarrows (I \otimes_{\mathrm{Day}} I)(i) \Rrightarrow (I \otimes_{\mathrm{Day}} I \otimes_{\mathrm{Day}} I)(i) \cdots \right).$$

Applying geometric realization to $(S^1_\bullet \widetilde{\otimes} I)(i)$, we get a cofibrant decreasingly filtered object in $\mathcal{C}$ (given Running Assumption 2.0.4)

$$|(S^1_\bullet \widetilde{\otimes} I)(0)| \leftarrow |(S^1_\bullet \widetilde{\otimes} I)(1)| \leftarrow |(S^1_\bullet \widetilde{\otimes} I)(2)| \leftarrow \cdots,$$

and the spectral sequence obtained by applying a generalized homology theory E_* to this cofibrant decreasingly filtered object in $\mathcal{C}$ is precisely the spectral sequence (23), the spectral sequence constructed and considered throughout this paper. (It is an easy exercise in unwinding definitions to check that this spectral sequence agrees with the one constructed in Definition 3.4.2, but to verify that the resulting spectral sequence has the expected input term, output term, and convergence properties amounts to exactly the same proofs already found in this paper which aren't expressed in terms of Day convolution.)

4 Decreasingly filtered commutative ring spectra

4.1 Convenient model structures on functor categories

For this section we will need some additional assumptions on our model category $\mathcal{C}$ satisfying Running Assumption 2.0.3. These are not used to construct the spectral sequence, but they are used to construct a large class of examples of decreasingly filtered commutative ring spectra in Section 4.2. Recall the symbol $\square$ defined in Lemma 3.3.1.

Definition 4.1.1 [44, Definition 3.4] We say a model category $\mathcal{A}$ satisfies the strong commutative monoid axiom if, for any cofibration h (respectively, any acyclic cofibration h), the map $h^{\square n}/\Sigma_n$ is a cofibration (respectively, an acyclic cofibration) for each integer $n \geq 1$.

Running Assumption 4.1.2 We assume that $\mathcal{C}$ satisfies the strong commutative monoid axiom. We also assume that $\mathcal{C}$ is locally presentable as a category. Additionally, we assume $\mathcal{C}$ is a right proper model category.

For example, symmetric spectra in the positive flat stable model structure satisfy Running Assumption 4.1.2; see Section 4.2.

Remark 4.1.3 If $\mathcal{C}$ is a model category satisfying Running Assumption 2.0.3 and Running Assumption 2.0.4, then $\mathcal{C}$ is cofibrantly generated and locally presentable and therefore it is combinatorial in the sense of Jeff Smith (see Dugger [15, Definition 2.1]).

We will also make use of categories enriched in a model category $\mathcal{C}$ satisfying Running Assumption 2.0.3. See Kelly [28], for a good treatment of enriched category theory at the level of generality needed for this paper.

Given a category $\mathcal{B}$ enriched over $\mathcal{C}$, we will also discuss the category of $\mathcal{C}$–functors on $\mathcal{B}$, written $\mathcal{C}^{\mathcal{B}}$. It can be shown (see Kelly [28] for example) that $\mathcal{C}^{\mathcal{B}}$ can again be equipped with a $\mathcal{C}$–enriched category structure, but we do not use that property here and we will write $\mathcal{C}^{\mathcal{B}}$ for the underlying category of enriched functors and natural transformations.

Definition 4.1.4 Suppose $\mathcal{C}$ satisfies Running Assumption 2.0.3. Let $\mathcal{N}^{\mathrm{op}}$ be the $\mathcal{C}$–enriched category with objects $\mathrm{ob}\,\mathcal{N}^{\mathrm{op}} = \mathbb{N}$ and morphism objects

$$\mathcal{N}^{\mathrm{op}}(n,m) = \begin{cases} \mathbb{1} & \text{if } n \geq m, \\ 0 & \text{if } n < m, \end{cases}$$

where $\mathbb{1}$ is the unit in $\mathcal{C}$ and 0 is the zero object of $\mathcal{C}$. We equip $\mathcal{N}^{\mathrm{op}}$ with a symmetric

monoidal product using the usual addition $+$, so that 0 is the unit of the symmetric monoidal structure on $\mathcal{N}^{\mathrm{op}}$. This makes $\mathcal{N}^{\mathrm{op}}$ a symmetric monoidal $\mathcal{C}$–category.

Remark 4.1.5 In Remark 3.1.5 we discussed the theorem, due to Day [14], that the category of lax symmetric monoidal functors in $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ is equivalent to the category $\mathrm{Comm}\,\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$. In the setting of categories enriched in $\mathcal{T}op_*$, based, compactly generated, weak-Hausdorff spaces, this is proven in [32, Proposition 22.1]. In the setting of categories enriched in a model category $\mathcal{C}$ satisfying Running Assumption 2.0.3 the same proof yields the desired result that the category of lax symmetric monoidal functors in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ is equivalent to the category of commutative monoids in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ with the enriched Day convolution symmetric monoidal product.

The following truncated versions of the category $\mathcal{N}^{\mathrm{op}}$ will also be useful for constructing decreasingly filtered commutative ring spectra.

Definition 4.1.6 We will define a category enriched in a category $\mathcal{C}$ satisfying Running Assumptions 2.0.3 and 4.1.2 called J_n^{op}. The objects in J_n are the subset $J_n \subset \mathbb{N}$ of all natural numbers consisting of all $i \in \mathbb{N}$ such that $i \leq n$. We then define J_n^{op} as a $\mathcal{C}$–subcategory of $\mathcal{N}^{\mathrm{op}}$ so that $J_n^{\mathrm{op}}(i, j) = \mathcal{N}^{\mathrm{op}}(i, j)$ whenever $i, j \leq n$. We give this category the structure of a symmetric monoidal $\mathcal{C}$–category $(J_n, \dot{+}, 0)$ by letting $i \dot{+} j = \min\{i + j, n\}$.

We now discuss the cofibrancy conditions needed on an object in $\mathrm{Comm}\,\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ in order to produce a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ in the sense of Definition 3.1.2. This will be used to construct a certain class of filtered commutative ring spectra in Theorem 4.2.1. In Proposition 4.1.7, we use a definition from [27]: an object X in a pointed monoidal model category is *virtually cofibrant* if $(0 \to X) \,\square\, -$ preserves cofibrations and preserves acyclic cofibrations.

Proposition 4.1.7 *Suppose $\mathcal{C}$ is a model category satisfying Running Assumptions 2.0.3 and 2.0.4 and $\mathcal{B}$ is a small symmetric monoidal category enriched in $\mathcal{C}$ with virtually cofibrant function spaces [27, Definition 2.2.12]. Then there exists a model structure on the category of $\mathcal{C}$–functors $\mathcal{C}^{\mathcal{B}}$, called the **projective model structure**, where a fibration (respectively, weak equivalence) is a natural transformation $\eta\colon F \to G$ between functors $F, G \in \mathcal{C}^{\mathcal{B}}$ such that $\eta_X\colon F(X) \to G(X)$ is a fibration (respectively, weak equivalence) for each object X in $\mathcal{B}$. In addition, because $\mathcal{C}$ satisfies Running Assumption 2.0.3, $\mathcal{C}^{\mathcal{B}}$ is a symmetric monoidal model category under Day convolution and it satisfies the Schwede–Shipley monoid axiom [43].*

Proof Since $\mathcal{C}$ is assumed to be combinatorial, the projective model structure exists on $\mathcal{C}^{\mathcal{B}}$ and it is also cofibrantly generated, by [27, Proposition 2.2.13]. Since $\mathcal{C}$ satisfies Running Assumptions 2.0.3 and 4.1.2, [27, Propositions 2.2.15 and 2.2.16] imply the second part of the proposition. $\square$

Remark 4.1.8 We will only apply Proposition 4.1.7 in the case where $\mathcal{B}$ is either the category $\mathcal{N}^{\mathrm{op}}$ of Definition 4.1.4 or J_n^{op} of Definition 4.1.6. It is easy to check that the categories $\mathcal{N}^{\mathrm{op}}$ and J_n^{op} have virtually cofibrant function spaces because the unit object $\mathbb{1}$ is always virtually cofibrant in a symmetric monoidal model category by the unit axiom.

Lemma 4.1.9 *Let $\mathcal{C}$ satisfy Running Assumptions 2.0.3 and 4.1.2 and let $\mathcal{N}^{\mathrm{op}}$ be the category defined in Definition 4.1.4. Let $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ be the category of $\mathcal{C}$–functors $\mathcal{N}^{\mathrm{op}} \to \mathcal{C}$, equipped with the projective model structure. Let P be a cofibrant object in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$. Then, for all $n \in \mathbb{N}$, the object $P(n)$ of $\mathcal{C}$ is cofibrant, and the morphism $P(n+1) \to P(n)$ is a cofibration in $\mathcal{C}$.*

Proof This is a consequence of [22, Lemma 3.1]. $\square$

The following definition is from [27, Proposition 2.2.13], and it will be useful for describing the generating cofibrations in the projective model structure on $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$.

Definition 4.1.10 Suppose $\mathcal{C}$ is a symmetric monoidal model category satisfying Running Assumptions 2.0.3 and 4.1.2. Each integer i corepresents a functor $\mathcal{N}^{\mathrm{op}} \to \mathcal{C}$, and we write $\mathcal{N}^{\mathrm{op}}(i, -)$ for this functor. Given a set of morphisms $\mathcal{M}$ in $\mathcal{C}$, we write $\mathcal{M} \otimes \mathcal{N}^{\mathrm{op}}$ for the collection of morphisms in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ which are of the form $\overline{f} \wedge \mathcal{N}^{\mathrm{op}}(i, -)$ for some integer i and some map $\overline{f}$ in $\mathcal{M}$.

Theorem 4.1.11 *Suppose $\mathcal{C}$ satisfies Running Assumptions 2.0.3 and 4.1.2 and $\mathcal{N}^{\mathrm{op}}$ is the category defined in Definition 4.1.4. Then the projective model structure on $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ satisfies the strong commutative monoid axiom.*

Proof By White [44, Lemma A.1], it suffices to check the strong commutative monoid axiom on the generating cofibrations. Due to [7, Proposition 4.5], a set of generating cofibrations in the projective model structure on $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ is the class of maps $\mathcal{J} \otimes \mathcal{N}^{\mathrm{op}}$, where $\mathcal{J}$ is a set of generating cofibrations of $\mathcal{C}$, and a set of generating acyclic cofibrations in the projective model structure is the set of maps $\mathbb{J} \otimes \mathcal{N}^{\mathrm{op}}$, where $\mathbb{J}$ is a set of generating acyclic cofibrations of $\mathcal{C}$ (see Definition 4.1.10 for the definition of the notation).

Since

$$\mathcal{N}^{\mathrm{op}}(i,-) \simeq \begin{cases} \mathbb{1} & \text{if } j \leq i, \\ 0 & \text{if } j > i, \end{cases}$$

a set of generating cofibrations for $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ consists of the set of the natural transformations $f \colon I \to J$ such that there is a map $\bar{f} \colon A \to B$ in $\mathcal{J}$ such that $f_j = \bar{f} \wedge \mathrm{id}_{\mathbb{1}} \colon A \wedge \mathbb{1} \to B \wedge \mathbb{1}$ for $j \leq i$ and $f_j = \bar{f} \wedge \mathrm{id}_0 \colon A \wedge 0 \to B \wedge 0$ for $j > i$. Similarly, a set of generating acyclic cofibrations consists of the natural transformations $f \colon I \to J$ such that there is some map $\bar{f} \colon A \to B$ in $\mathbb{J}$ such that $f_j = \bar{f} \wedge \mathrm{id}_{\mathbb{1}} \colon A \wedge \mathbb{1} \to B \wedge \mathbb{1}$ for $j \leq i$ and $f_j = \bar{f} \wedge \mathrm{id}_0 \colon A \wedge 0 \to B \wedge 0$ for $j > i$.

Let $h \colon I \to J$ be an map in $\mathcal{J} \otimes \mathcal{N}^{\mathrm{op}}$ (respectively, a map in $\mathbb{J} \otimes \mathcal{N}^{\mathrm{op}}$) with $\bar{h} \colon A \to B$ the corresponding map in $\mathcal{J}$ (respectively, $\mathbb{J}$). Then we need to prove that $h^{\square n}/\Sigma_n$ is a cofibration (respectively, an acyclic cofibration) in $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ with the projective model structure. The case $n = 1$ is vacuous and therefore we omit it.

In the case $n = 2$, we need to show that the map

$$h^{\square 2}/\Sigma_2 \colon \big((I \otimes_{\mathrm{Day}} J) \amalg_{I \otimes_{\mathrm{Day}} I} (J \otimes_{\mathrm{Day}} I)\big)/\Sigma_2 \to (J \otimes_{\mathrm{Day}} J)/\Sigma_2$$

is an (acyclic) cofibration in the projective model structure. To see this, note that $I \otimes_{\mathrm{Day}} I \cong (A \wedge A) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$, $J \otimes_{\mathrm{Day}} I \cong (B \wedge A) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$, $I \otimes_{\mathrm{Day}} J \cong (A \wedge B) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$, and $J \otimes_{\mathrm{Day}} J \cong (B \wedge B) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$. Therefore, the map $(h^{\square 2})/\Sigma_2$ is the map $(\bar{h}^{\square 2} \otimes \mathcal{N}^{\mathrm{op}}(2i,-))/\Sigma_2$ up to isomorphism. By definition of $\mathcal{N}^{\mathrm{op}}$, this is equivalent, up to isomorphism, to $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$, that is, $(h^{\square 2})/\Sigma_2$ is a composite of isomorphisms and the map $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$. Since $\mathcal{C}$ satisfies the strong commutative monoid axiom, $(\bar{h}^{\square 2})/\Sigma_2$ is a cofibration (respectively, acyclic cofibration). We claim that the map $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$ is also a cofibration (respectively, acyclic cofibration). We will give the argument that $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$ is a cofibration in the projective model structure when h is a cofibration; the argument that $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,-)$ is an acyclic cofibration when h is an acyclic cofibration will be omitted because it is essentially the same. We need to show that for any acyclic fibration $X \to Y$ in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ fitting into the diagram

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & \nearrow^{\;z} & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

there exists a lift $z \colon Z \to X$, where $W = \big((I \otimes_{\mathrm{Day}} J) \amalg_{I \otimes_{\mathrm{Day}} I} (J \otimes_{\mathrm{Day}} I)\big)/\Sigma_2$ and $Z = (J \otimes_{\mathrm{Day}} J)/\Sigma_2$. Since $((\bar{h}^{\square 2})/\Sigma_2) \wedge \mathcal{N}^{\mathrm{op}}(2i,m) \colon W_m \to Z_m$ is, up to isomorphism, the cofibration

$$\big((A \wedge B) \amalg_{B \wedge B} (B \wedge A)\big)/\Sigma_2 \to (B \wedge B)/\Sigma_2$$

when $m \leq 2i$, and it is, up to isomorphism, the map $0 \to 0$ when $m > 2i$, we can define the map $z \colon Z \to X$ to be the map $0 \to X_m$ when $m > 2i$, and we can define $z_{2i} \colon Z_{2i} \to X_{2i}$ to be the lift in the diagram

$$
\begin{array}{ccc}
W_{2i} & \longrightarrow & X_{2i} \\
\downarrow & {\scriptstyle z_{2i}} \nearrow & \downarrow \\
Z_{2i} & \longrightarrow & Y_{2i}
\end{array}
$$

which we know exists because $W_{2i} \to Z_{2i}$ is a cofibration and $X_j \to Y_j$ is an acyclic fibration for all j (because $X \to Y$ is an acyclic fibration in the projective model structure). For all $j < 2i$, we can then define z_j as the composite

$$Z_j = Z_{2i} \xrightarrow{z_{2i}} X_{2i} \to X_{2i-1} \to \cdots \to X_j.$$

There is no difficulty with the need for our lift maps to be compatible for various choices of j, since in the range $j \leq 2i$, the sequences $W_\bullet$ and $Z_\bullet$ are constant, and above this range, these sequences are zero! We have therefore defined a map $z_m \colon Z_m \to X_m$ for each $m \geq 0$ and the diagrams

$$
\begin{array}{ccc}
Z_m & \longrightarrow & X_m \\
\downarrow & & \downarrow \\
Z_{m-1} & \longrightarrow & X_{m-1}
\end{array}
$$

clearly commute for all $m \geq 0$ by the definition of z_m.

The same type of argument works for $n > 2$ giving

$$(h^{\square n})/\Sigma_n \cong ((\bar{h}^{\square n})/\Sigma_n) \wedge \mathcal{N}^{\mathrm{op}}(ni,-),$$

which is a cofibration by a proof essentially the same as the one above. Here the isomorphism is in the arrow category, but this is the same as saying $(h^{\square n})/\Sigma_n$ is a composite of isomorphisms and the map $((\bar{h}^{\square n})/\Sigma_n) \wedge \mathcal{N}^{\mathrm{op}}(ni,-)$, which is a cofibration, and therefore $(h^{\square n})/\Sigma_n$ is also a cofibration. $\qquad\square$

Lemma 4.1.12 *Let $\mathcal{C}$ be a model category satisfying Running Assumption 2.0.3 and Running Assumption 4.1.2 and let J_n^{op} be the category defined in Definition 4.1.6. Then the projective model structure on $\mathcal{C}^{J_n^{\mathrm{op}}}$ satisfies the strong commutative monoid axiom. The cofibrant objects in the projective model structure are exactly the objects $P \in \mathcal{C}^{J_n^{\mathrm{op}}}$ such that $P(i)$ is cofibrant in $\mathcal{C}$ for all $0 \leq i \leq n$ and $P(i) \to P(i-1)$ is a cofibration in $\mathcal{C}$.*

Proof The proof that $\mathcal{C}^{J_n^{\mathrm{op}}}$ satisfies the strong commutative monoid axiom is the same as the proof of Theorem 4.1.11 and therefore we omit it. The second part of this lemma is again a consequence of [22, Lemma 3.1], as the authors remark just after that lemma. $\square$

Remark 4.1.13 There are four different model structures that we use here, which are all commonly referred to as the projective model structure. The model category structure on $\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$ discussed in Proposition 4.1.7, the model structure created by the forgetful functor $U\colon \mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}} \to \mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$, the model structure on algebras over a commutative monoid I in $\mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$ created by the forgetful functor to $\mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$, and the model structure on modules over a commutative monoid I in $\mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$ created by the forgetful functor from I–modules (or equivalently symmetric I–bimodules) to $\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$. Each of these will be referred to as the projective model structure, and it should be clear from the context which of the four is meant. Each of these model structures also make sense when $\mathcal{N}^{\mathrm{op}}$ is replaced by J_n^{op}. Since the model structure on Sp is cofibrantly generated, each these projective model structures is cofibrantly generated, by Proposition 4.1.7, [44, Theorem 3.2], and [43, Theorem 4.1]. Consequently, as remarked in Hovey [24], we have functorial factorization in each of these model categories.

4.2 Whitehead towers

For this section, we will abbreviate and write Sp for the category of symmetric spectra of pointed simplicial sets with the positive flat stable model structure. This category satisfies Running Assumptions 2.0.3 and 2.0.4, as discussed in Section 2. This model category also satisfies Running Assumption 4.1.2: since it is combinatorial by Hovey, Shipley, and Smith [26], it satisfies the strong commutative monoid axiom by [44, Theorem 5.7], and it is also right proper by [26, Theorem 5.4.2]. The goal of this section is to produce a cofibrant decreasingly filtered commutative monoid in Sp as a specific multiplicative model for the Whitehead tower of a connective commutative monoid in Sp.

As a consequence of Lemma 4.1.9, a cofibrant object in the category $\operatorname{Comm} \operatorname{Sp}^{\mathcal{N}^{op}}$ equipped with the projective model structure is, in particular, a cofibrant decreasingly filtered commutative monoid in Sp (also see Remark 3.1.5).

Theorem 4.2.1 *Let R be a cofibrant connective commutative monoid in* Sp*. Then there exists a cofibrant decreasingly filtered commutative monoid in* Sp*,*

$$\cdots \to \tau_{\geq 2} R \to \tau_{\geq 1} R \to \tau_{\geq 0} R,$$

with structure maps

$$\rho_{i,j} \colon \tau_{\geq i} R \wedge \tau_{\geq j} R \to \tau_{\geq i+j} R,$$

such that $R \simeq \tau_{\geq 0} R$ and the map $\tau_{\geq n} R \to \tau_{\geq 0} R$ induces an isomorphism on π_k for $k \geq n$ and $\pi_k(\tau_{\geq n} R) \cong 0$ for $k < n$. This cofibrant decreasingly filtered commutative monoid in Sp *is denoted by $\tau_{\geq \bullet} R$.*

Proof We will prove the theorem by induction. First, we will consider R as a cofibrant object $\operatorname{Comm} \operatorname{Sp}^{J_0^{op}} = \operatorname{Comm} \operatorname{Sp}$ and we will produce a cofibrant object in $\operatorname{Comm} \operatorname{Sp}^{J_1^{op}}$ in projective model structure. To construct $\tilde{\tau}_{\geq 1} R$, we consider the map of commutative R–algebras $R \to H\pi_0 R$, constructed as in [6, Theorem 8.1]. This map is not usually a fibration (in fact it is a cofibration as discussed in Remark 4.2.2), so we factor it as a composite $R \to \tilde{\tau}_{\geq 0} R \to H\pi_0 R$ of an acyclic cofibration and a fibration in the category of commutative R–algebras in Sp. The resulting object $\tilde{\tau}_{\geq 0} R$ is a cofibrant commutative monoid in Sp equipped with a fibration $\tilde{\tau}_{\geq 0} R \to H\pi_0 R$ which is a map of $\tilde{\tau}_{\geq 0} R$–algebras.

We then define $\tilde{\tau}_{\geq 1} R$ to be the fiber of the map $\tilde{\tau}_{\geq 0} R \to H\pi_0 R$ in the category of $\tilde{\tau}_{\geq 0} R$–modules (equivalently symmetric $\tilde{\tau}_{\geq 0} R$–bimodules). The symmetric $\tilde{\tau}_{\geq 0} R$–bimodule structure produces an object $f_1 \colon \tilde{\tau}_{\geq 1} R \to \tilde{\tau}_{\geq 0} R$ in $\operatorname{Comm} \operatorname{Sp}^{J_1^{op}}$ with action maps $\rho_{i,j}$ for $i, j \in \operatorname{ob} J_1^{op}$ defined as follows: the map $\rho_{0,0} \colon \tilde{\tau}_{\geq 0} R \wedge \tilde{\tau}_{\geq 0} R \to \tilde{\tau}_{\geq 0} R$ is the multiplication map, the maps $\rho_{1,0}$ and $\rho_{0,1}$ are the right and left $\tilde{\tau}_{\geq 0} R$–module structure maps, and the map $\rho_{1,1}$ is the composite

$$\rho_{1,1} \colon \tilde{\tau}_{\geq 1} R \wedge \tilde{\tau}_{\geq 1} R \xrightarrow{\operatorname{id}_{\tilde{\tau}_{\geq 1} R} \wedge f_1} \tilde{\tau}_{\geq 1} R \wedge \tilde{\tau}_{\geq 0} R \xrightarrow{\rho_{1,0}} \tilde{\tau}_{\geq 1} R$$

or equivalently, because $\tilde{\tau}_{\geq 1} R$ is a symmetric $\tilde{\tau}_{\geq 0} R$–bimodule,

$$\rho_{1,1} \colon \tilde{\tau}_{\geq 1} R \wedge \tilde{\tau}_{\geq 1} R \xrightarrow{f_1 \wedge \operatorname{id}_{\tilde{\tau}_{\geq 1} R}} \tilde{\tau}_{\geq 0} R \wedge \tilde{\tau}_{\geq 1} R \xrightarrow{\rho_{0,1}} \tilde{\tau}_{\geq 1} R.$$

These maps are easily seen to satisfy the necessary associativity, commutativity, and

compatibility axioms. We then cofibrantly replace $\tilde{\tau}_{\geq 1} R \to \tilde{\tau}_{\geq 0} R$ in the projective model structure on $\mathrm{Comm\,Sp}^{J_1^{\mathrm{op}}}$ to produce an object $\tau_{\geq\bullet}^{\leq 1} R$ such that $R \simeq \tau_{\geq 0}^{\leq 1} R$, the map $\tau_{\geq 1}^{\leq 1} R \to \tau_{\geq 0}^{\leq 1} R$ induces an isomorphism on π_k for $k \geq 1$ and $\pi_k(\tau_{\geq 1}^{\leq 1} R) \cong 0$ for $k < 1$. Recall that the effect of cofibrantly replacing in this model structure is that the map is replaced by a cofibration and the objects are replaced by cofibrant objects without changing their homotopy type. This completes the base step in the induction. Note that since we have functorial factorization in Sp and $\mathrm{Comm\,Sp}^{J_1^{\mathrm{op}}}$, this construction is entirely functorial.

Now, for the inductive step: Suppose we can functorially construct, for an arbitrary $n \geq 1$, a cofibrant object $\tau_{\geq\bullet}^{\leq n-1}(R) \in \mathrm{ob\,Comm\,Sp}^{J_{n-1}^{\mathrm{op}}}$ such that $R \simeq \tau_{\geq 0}^{\leq n-1} R$, the map $\tau_{\geq j}^{\leq n-1} R \to \tau_{\geq 0}^{\leq n-1} R$ induces an isomorphism on π_k for $n-1 \geq k \geq j$ and $\pi_k(\tau_{\geq j}^{\leq n-1} R) \cong 0$ for $k < j$, and there is an acyclic cofibration $R \to \tau_{\geq 0}^{\leq n-1} R$. First, note that, due to Basterra [6, Theorem 8.1], we can construct a map of commutative R–algebras $R \to \gamma_{\leq n-1} R$, where $\gamma_{\leq n-1} R$ is a commutative R–algebra with the property that the map $R \to \gamma_{\leq n-1} R$ induces an isomorphism on π_k for $k \leq n-1$ and $\pi_k(\gamma_{\leq n-1} R) \cong 0$ for $k \geq n$. By the assumed functoriality of the construction of $\tau_{\geq\bullet}^{\leq n-1}(R)$, we get a map $\tau_{\geq\bullet}^{\leq n-1}(R) \to \tau_{\geq\bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$ of commutative $\tau_{\geq\bullet}^{\leq n-1}(R)$–algebras in $\mathrm{Comm\,Sp}^{J_{n-1}^{\mathrm{op}}}$. We first fibrantly replace $\tau_{\geq\bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$ in the category of commutative $\tau_{\geq\bullet}^{\leq n-1}(R)$–algebras to produce a fibrant object $\overline{\tau}_{\geq\bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$ which still receives a map from $\tau_{\geq\bullet}^{\leq n-1}(R)$. We then functorially factor the map $\tau_{\geq\bullet}^{\leq n-1}(R) \to \overline{\tau}_{\geq\bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$ as an acyclic cofibration followed by a fibration

$$\tau_{\geq\bullet}^{\leq n-1}(R) \to \tilde{\tau}_{\geq\bullet}^{\leq n-1}(R) \to \overline{\tau}_{\geq\bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$$

in the category of commutative $\tau_{\geq\bullet}^{\leq n-1}(R)$–algebras.

The following argument combines two ideas. As in the base step, we work in the category of symmetric $\tilde{\tau}_{\geq\bullet}^{\leq n-1}(R)$–bimodules throughout. This builds in the commutativity, associativity, unitality and compatibility of most of the structure maps $\rho_{i,j} \colon \tilde{\tau}_{\geq i}^{\leq n}(R) \wedge \tilde{\tau}_{\geq j}^{\leq n}(R) \to \tilde{\tau}_{\geq i+j}^{\leq n}(R)$, where $\tilde{\tau}_{\geq i}^{\leq n}(R) := \tilde{\tau}_{\geq i}^{\leq n-1}(R)$ for $i < n$, as we discuss below. Unfortunately, this alone does not build in the associativity of the maps $\rho_{i,j} \colon \tilde{\tau}_{\geq i}^{\leq n}(R) \wedge \tilde{\tau}_{\geq j}^{\leq n}(R) \to \tilde{\tau}_{\geq n}^{\leq n}(R)$ when $i,j < n$ and $i+j \geq n$. We thus build this into our definition of $\tilde{\tau}_{\geq n}^{\leq n} R$ as well. To combine these two ideas we must describe how to construct the associativity diagrams that we use in the category of $\tilde{\tau}_{\geq\bullet}^{\leq n-1}(R)$–bimodules.

Recall that a symmetric $\tilde{\tau}_{\geq\bullet}^{\leq n-1} R$–bimodule is a functor $X \colon J_{n-1}^{\mathrm{op}} \to \mathrm{Sp}$ along with

natural transformations

$$X \otimes_{\text{Day}} \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \to \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \quad \text{and} \quad \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \otimes_{\text{Day}} X \to \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$$

satisfying the usual associativity and commutativity axioms. Let F and G be constant symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–bimodules such that $F(\ell)$ is

$$\text{colim}_{i \dotplus j \geq n} \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R$$

for each $0 \leq \ell \leq n-1$ and $G(\ell)$ is

$$\text{colim}_{i \dotplus j \dotplus k \geq n} \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq k} R$$

for each $0 \leq \ell \leq n-1$. To make this precise we need to define the action maps, which are natural transformations

$$F \otimes_{\text{Day}} \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \to F, \quad \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \otimes_{\text{Day}} F \to F,$$
$$\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \otimes_{\text{Day}} G \to G, \quad G \otimes_{\text{Day}} \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \to G.$$

By the definition of Day convolution,

$$(F \otimes_{\text{Day}} \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R)(\ell) = \text{colim}_{a \dotplus b \geq \ell}(F(a) \wedge \widetilde{\tau}^{\leq n-1}_{\geq b} R),$$

and since F is constant,

$$\text{colim}_{a \dotplus b \geq \ell}(F(a) \wedge \widetilde{\tau}^{\leq n-1}_{\geq b} R) \cong \text{colim}_{i \dotplus j \geq n}(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R) \wedge \widetilde{\tau}^{\leq n-1}_{\geq 0} R.$$

Since smashing with a cofibrant object commutes with colimits, there is a weak equivalence

$$\text{colim}_{i \dotplus j \geq n}(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R) \wedge \widetilde{\tau}^{\leq n-1}_{\geq 0} R$$
$$\simeq \text{colim}_{i \dotplus j \geq n}(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq 0} R),$$

and we can use the maps $\rho_{j,0} \colon \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq 0} R \to \widetilde{\tau}^{\leq n-1}_{\geq j} R$ to define a map

$$\text{colim}_{i \dotplus j \geq n}(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq 0} R) \to \text{colim}_{i \dotplus j \geq n}(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R)$$

which provides a natural transformation $F \otimes_{\text{Day}} \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \to F$ in the evident way. The remaining action natural transformation for F is defined in an analogous way and the these two maps satisfy the commutativity and associativity axioms by the inductive hypothesis. We then define the two action natural transformations of G in the same way and they also satisfy the commutativity and associativity axioms by the inductive hypothesis.

Here is a sketch of our next steps in this proof: We will now define $\widetilde{\tau}^{\leq n}_{\geq n} R$ in such a way that it is automatically equipped with associative, commutative, and compatible maps

$$\rho_{i,j}: \widetilde{\tau}^{\leq n}_{\geq i} R \wedge \widetilde{\tau}^{\leq n}_{\geq j} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$$

for $i + j \geq n$, where $\widetilde{\tau}^{\leq n}_{\geq j} R$ is defined to be $\widetilde{\tau}^{\leq n-1}_{\geq j} R$ for $0 \leq j \leq n-1$. To build in the associativity of the structure maps $\rho_{i,j}: \widetilde{\tau}^{\leq n}_{\geq i} R \wedge \widetilde{\tau}^{\leq n}_{\geq j} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ for $0 < i, j < n$ and $i + j \geq n$, we will encode all of the necessary associativity diagrams into one pushout, where each object in the pushout is a colimit of a truncated directed cube diagram. We will then show that the map from this pushout to a certain given object is nullhomotopic and therefore factors through a single contractible object. We then define $\widetilde{\tau}^{\leq n}_{\geq n} R$ as the pullback of a diagram involving this contractible object. This is arranged so that we only need to make one choice of contractible object at each stage of the induction. We also work in the category of symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–bimodules throughout this process in order to encode the remaining associativity, commutativity, compatibility, and unitality diagrams.

Note that each map

$$\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \to \tau^{\leq n-1}_{\geq m} R \to \tau^{\leq n-1}_{\geq n-1}(\gamma_{\leq n-1} R)$$

is nullhomotopic for $i + j \geq n$ because $\pi_\ell(\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R) \cong 0$ for $\ell < n$ and $\pi_\ell(\tau_{\geq m}(\gamma_{\leq n-1} R)) \cong 0$ for $\ell > n-1$. We claim that the composite map from the Bousfield–Kan homotopy colimit (in the sense of our [Appendix B]) of the diagram

$$\mathrm{colim}_{i+j+k\geq n}\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq k} R \longrightarrow \mathrm{colim}_{i+j+k\geq n}\widetilde{\tau}^{\leq n-1}_{\geq i+j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq k} R$$

(24)
$$\downarrow$$
$$\mathrm{colim}_{i+j+k\geq n}\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j+k} R$$

to $\widetilde{\tau}^{\leq n-1}_{\geq m} R$, followed by the map $\widetilde{\tau}^{\leq n-1}_{\geq m} R \to \widetilde{\tau}^{\leq n-1}_{\geq m}(\gamma_{\leq n-1} R)$, is nullhomotopic as well. (Note that this will follow if the map from the diagram (24) to $\widetilde{\tau}^{\leq n-1}_{\geq n-1} R$ followed by the map $\widetilde{\tau}^{\leq n-1}_{\geq n-1} R \to \widetilde{\tau}^{\leq n-1}_{\geq n-1}(\gamma_{\leq n-1} R)$ is nullhomotopic since the composite of a nullhomotopic map with any other map is always nullhomotopic.)

To prove the claim above, we use the Bousfield–Kan spectral sequence of [Theorem B.1.4], which is discussed further in our self-contained [Appendix B]:

$$(R^s_E \lim_{d \in \mathcal{D}})(Z^t A(d)) \Rightarrow Z^{s+t} \text{ hocolim } A.$$

We will apply this spectral sequence when $\mathcal{D}$ is one of the small categories

$$\mathcal{P}_n^3 = \{(i,j,k) \in \mathbb{N}^3 : i \dot{+} j \dot{+} k \geq n \text{ and } 0 < i,j,k < n\}^{\mathrm{op}} \subseteq (\mathbb{N}^3)^{\mathrm{op}},$$
$$\mathcal{P}_n^2 = \{(\ell,m) \in \mathbb{N}^2 : \ell \dot{+} m \geq n \text{ and } 0 < \ell,m < n\}^{\mathrm{op}} \subseteq (\mathbb{N}^2)^{\mathrm{op}},$$

or the pushout category $\mathcal{PO}$, defined in (15). We will take Z to be $\tau_{\geq n-1}(\gamma_{\leq n-1} R)$. Let $H \colon \mathcal{P}_n^2 \to \mathcal{C}$ be the functor given by $H(i,j) = \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq j}^{\leq n-1} R$. A colimit of a functor on $\mathcal{P}_n^2$ can be written as an iterated pushout, so we can prove a vanishing result for $(R_E^s \lim_{d \in (\mathcal{P}_n^2)^{\mathrm{op}}})(Z^t H(d))$ for $s - t = 0$ by proving an appropriate vanishing result for $(R_E^s \lim_{d \in \mathcal{PO}^{\mathrm{op}}})(Z^t (\iota_* H)(d))$, where $\mathcal{PO}$ is the category indexing pushouts, as in (15), and $\iota_* H$ is the restriction of H along one of the inclusions $\iota \colon \mathcal{PO} \hookrightarrow \mathcal{P}_n^2$. By [17, Example 17.10] and our Theorem B.1.7, given a functor $\mathcal{F} \colon \mathcal{PO} \to \mathcal{C}$ and an object Y of $\mathcal{C}$, the map $[\mathrm{hocolim}\,\mathcal{F}, Y] \to \lim[\mathcal{F}, Y]$ is an isomorphism if $(R_E^1 \lim_{d \in \mathcal{PO}^{\mathrm{op}}})[\Sigma\mathcal{F}(d), Y] \cong [\Sigma\mathcal{F}(1'), Y]/([\Sigma\mathcal{F}(0), Y] + [\Sigma\mathcal{F}(1'), Y])$ vanishes.

So we carry out an induction: Let $\mathcal{I}_0 \colon \mathcal{PO} \to \mathcal{C}$ be the functor given by

$$\mathcal{I}_0(1') = \widetilde{\tau}_{\geq 2}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n-1}^{\leq n-1} R,$$
$$\mathcal{I}_0(0) = \widetilde{\tau}_{\geq 1}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n-1}^{\leq n-1} R,$$
$$\mathcal{I}_0(1) = \widetilde{\tau}_{\geq 2}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n-2}^{\leq n-1} R.$$

Vanishing of $[\Sigma\mathcal{I}_0(1'), \widetilde{\tau}_{\geq n-1}(\gamma_{\leq n-1} R)]$ and of $[\mathcal{I}_0(d), \widetilde{\tau}_{\geq n-1}(\gamma_{\leq n-1} R)]$ for $d = 0, 1$ is standard (by properties of maps from sufficiently connective spectra to sufficiently coconnective spectra), so $[\mathrm{hocolim}\,\mathcal{I}_0, Z]$ vanishes.

That was the initial step. For the inductive step, let $\mathcal{I}_j \colon \mathcal{PO} \to \mathcal{C}$ be the functor given by

$$\mathcal{I}_j(1') = \widetilde{\tau}_{\geq j+2}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n-j-1}^{\leq n-1} R,$$
$$\mathcal{I}_j(0) = \mathrm{hocolim}\,\mathcal{I}_{j-1},$$
$$\mathcal{I}_j(1) = \widetilde{\tau}_{\geq j+2}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n-j-2}^{\leq n-1} R.$$

and suppose we have already shown that $[\mathrm{hocolim}\,\mathcal{I}_{j-1}, Z]$ vanishes. Again,

$$[\Sigma\mathcal{I}_j(1'), \widetilde{\tau}_{\geq n-1}(\gamma_{\leq n-1} R)] \cong [\mathcal{I}_j(1), \widetilde{\tau}_{\geq n-1}(\gamma_{\leq n-1} R)] \cong 0$$

due to standard properties of maps from sufficiently connective spectra to sufficiently coconnective spectra, so $[\mathrm{hocolim}\,\mathcal{I}_j, Z]$ vanishes. Since the natural map

$$\mathrm{hocolim}\,\mathcal{I}_{n-2} \to \mathrm{hocolim}\,H$$

is an equivalence, the case $j = n - 2$ completes the induction.

Therefore, $\operatorname{hocolim} H \to \tilde{\tau}_{\geq n-1}^{\leq n-1}(\gamma_{\leq n-1} R)$ is nullhomotopic, and hence the composite

$$\operatorname{hocolim} H \to \tilde{\tau}_{\geq n-1}^{\leq n-1}(\gamma_{\leq n-1} R) \to \tilde{\tau}_{\geq n-1}^{\leq n-1}(\gamma_{\leq n-1} R)$$

is nullhomotopic.

Now we claim that the diagram $A \colon \mathcal{P}_n^3 \to \operatorname{Sp}$ given by

$$(i,j,k) \mapsto \tilde{\tau}_{\geq i}^{\leq n-1} R \wedge \tilde{\tau}_{\geq j}^{\leq n-1} R \wedge \tilde{\tau}_{\geq k}^{\leq n-1} R$$

is actually Reedy-cofibrant and hence, since $\mathcal{P}_n^3$ is a direct category (as in Remark 3.4.5), it is projectively cofibrant. To see this, it is sufficient to check that each object in the diagram is cofibrant, each map is a cofibration, and for each inclusion of a square-shaped diagram the map from the pushout of the upper left horn to the terminal vertex is cofibration by an elementary check of Reedy's conditions [39]. Each object in the diagram is cofibrant and each map in the diagram is a cofibration by definition. Also, one can easily check that for each inclusion of a square-shaped diagram the map from the pushout of the upper left horn to the terminal vertex is a cofibration by iterated use of the pushout product axiom (cf Lemma 3.4.6, where essentially the same result is proven in more detail). The fact that the canonical map from $|\operatorname{sr}(\mathcal{F})| \to \operatorname{colim} \mathcal{F}$ is a weak equivalence for projectively cofibrant A is Theorem B.1.2, which is a consequence of [21, Theorems 3.2 and 3.3].

The colimit of the diagram $A \colon \mathcal{P}_n^3 \to \operatorname{Sp}$ described above can be written as an iterated colimit over directed cubes with terminal vertex removed. So, by an induction totally analogous to the induction we just carried out using $\mathcal{I}_0, \mathcal{I}_1, \ldots$, the question of whether the map $|\operatorname{sr}(A)| \to \tilde{\tau}_{\geq m}(\gamma_{\leq n-1} R)$ is nullhomotopic, as in Question B.1.6, reduces to the case of Question B.1.6 for diagrams indexed by directed cubes with terminal vertex removed, which we handle in Lemma B.2.3. Let $\mathcal{D}$ be a subcategory of $\mathcal{P}_n^3$ isomorphic to a directed cube with terminal vertex removed. By Lemma B.2.3, it suffices to show that $(R_E^s \lim_{d \in \mathcal{D}})(Z^{-t} A(d))$ vanishes for $s - t = 0$ and $s = 0, 1, 2$ because $(R_E^s \lim_{d \in \mathcal{D}})(Z^{-t} A(d)) \cong 0$ for $s > 2$. The condition

$$(25) \qquad [\Sigma^t \tilde{\tau}_{\geq i}^{\leq n-1} R \wedge \tilde{\tau}_{\geq j}^{\leq n-1} R \wedge \tilde{\tau}_{\geq k}^{\leq n-1} R, \tilde{\tau}_{\geq m}^{\leq n-1}(\gamma_{\leq n-1} R)] \cong 0$$

for $t > -i - j - k + n - 1$ is sufficient to show that $(R_E^s \lim_{d \in \mathcal{D}})(Z^{-t} A(d))$ vanishes for $s - t = 0, 1$ and $s = 0, 1, 2$ when A is any of the truncated cubes in the iterative process and hence the maps

$$\operatorname{colim}_{i+j+k \geq n-1} \tilde{\tau}_{\geq i}^{\leq n-1} R \wedge \tilde{\tau}_{\geq j}^{\leq n-1} R \wedge \tilde{\tau}_{\geq k}^{\leq n-1} R \to \tilde{\tau}_{\geq m}^{\leq n-1} R \to \tilde{\tau}_{\geq m}^{\leq n-1}(\gamma_{\leq n-1} R)$$

are nullhomotopic for $0 \leq m \leq n-1$. Condition (25) is indeed satisfied, by standard properties of maps from sufficiently connected spectra to sufficiently coconnected spectra.

Finally, we conclude that the map from the diagram (24) to $\widetilde{\tau}_{\geq m}^{\leq n-1} R$ followed by the map $\widetilde{\tau}_{\geq m}^{\leq n-1} R \to \widetilde{\tau}_{\geq m}^{\leq n-1}(\gamma_{\leq n-1} R)$ is nullhomotopic, again using Example 17.10 in Dugger [17], along with the vanishing of each of

$$Z^1(\text{hocolim}_{i+j+k \geq m+1} \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq j}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq k}^{\leq n-1} R),$$

$$Z^0(\text{hocolim}_{i+j+k \geq m+1} \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq j+k}^{\leq n-1}),$$

$$Z^0(\text{hocolim}_{i+j+k \geq m+1} \widetilde{\tau}_{\geq i+j}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq k}^{\leq n-1} R).$$

We may also consider diagram (24) as a diagram in symmetric $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$–bimodules, and what we have shown is that the map of symmetric $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$–bimodules from the homotopy colimit of (24) to $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$ followed by the map $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \to \widetilde{\tau}_{\geq \bullet}^{\leq n-1}(\gamma_{\leq n-1} R)$ is nullhomotopic in the category of $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$–modules. The point is that we can therefore factor this map through a contractible $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$–module, which we denote by $C(n)_\bullet$.

The pullback of the diagram

(26)
$$\begin{array}{c} C(n)_\bullet \\ \downarrow \\ \widetilde{\tau}_{\geq \bullet}^{\leq n-1}(R) \longrightarrow \widetilde{\tau}_{\geq \bullet}^{\leq n-1}(\gamma_{\leq n-1} R) \end{array}$$

in the category of symmetric $\widetilde{\tau}_{\geq \bullet}^{\leq n-1}(R)$–bimodules is a functor $B_\bullet \colon J_{n-1}^{\text{op}} \to \text{Sp}$ along with structure maps

$$B_\bullet \otimes_{\text{Day}} \widetilde{\tau}_{\geq \bullet}^{\leq n-1}(R) \to B_\bullet \quad \text{and} \quad \widetilde{\tau}_{\geq \bullet}^{\leq n-1}(R) \otimes_{\text{Day}} B_\bullet \to B_\bullet.$$

(For the sake of consistency with our notation $\widetilde{\tau}_{\geq \bullet}^{\leq n-1}(R)$, the subscripted bullet in $B_\bullet$ indicates that $B_\bullet$ is a finite sequence $B_{n-1} \to B_{n-2} \to \cdots \to B_0$.) We observe that $B_{n-1} \simeq B_i$ for all $0 \leq i \leq n-1$, and there is a map $B_{n-1} \to \widetilde{\tau}_{\geq 0}^{\leq n-1} R$ inducing an isomorphism on π_k for $k \geq n$ and $\pi_k(B_{n-1}) \cong 0$ for $k < n$. We therefore define $\widetilde{\tau}_{\geq n}^{\leq n} R$ to be B_{n-1}.

Next we need to check that the structure maps $\rho_{i,j} \colon \widetilde{\tau}_{\geq i}^{\leq n} R \wedge \widetilde{\tau}_{\geq j}^{\leq n} R \to \widetilde{\tau}_{\geq n}^{\leq n} R$ for $i + j \geq n$, where $\widetilde{\tau}_{\geq i}^{\leq n} R$ is defined to be $\widetilde{\tau}_{\geq i}^{\leq n-1} R$ when $0 \leq i < n$, are unital, associative, and commutative. We use the symmetric $\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R$–bimodule structure to

produce associative and commutative structure maps $\rho_{i,n}\colon \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq n}^{\leq n} R \to \widetilde{\tau}_{\geq n}^{\leq n} R$ and $\rho_{n,i}\colon \widetilde{\tau}_{\geq n}^{\leq n-1} R \wedge \widetilde{\tau}_{\geq i}^{\leq n} R \to \widetilde{\tau}_{\geq n}^{\leq n} R$ for $0 \leq i < n$ as follows: We have the commutative diagram of natural transformations

$$
\begin{array}{ccc}
\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \otimes_{\mathrm{Day}} B & \longrightarrow & B \\
\downarrow & \nearrow & \\
B \otimes_{\mathrm{Day}} \widetilde{\tau}_{\geq \bullet}^{\leq n-1} R & &
\end{array}
$$

so by definition of Day convolution, there is, by evaluating at $n-1$, a commutative diagram

$$
\begin{array}{ccc}
\mathrm{colim}_{i \dotplus j \geq n-1} \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge B_j & \longrightarrow & B_{n-1} \\
\downarrow & \nearrow & \\
\mathrm{colim}_{\ell \dotplus m \geq n-1} B_\ell \wedge \widetilde{\tau}_{\geq m}^{\leq n-1} R & &
\end{array}
$$

in Sp. Since the map of colimits can equivalently be defined as the colimit of factor swap maps $\chi\colon \widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge B_j \to B_j \wedge \widetilde{\tau}_{\geq i}^{\leq n-1} R$, the commutativity of the diagram above implies the diagram

$$
\begin{array}{ccc}
\widetilde{\tau}_{\geq i}^{\leq n-1} R \wedge B_{n-1} & \longrightarrow & B_{n-1} \\
\downarrow & \nearrow & \\
B_{n-1} \wedge \widetilde{\tau}_{\geq i}^{\leq n-1} R & &
\end{array}
$$

commutes for all i such that $0 \leq i < n$.

We prove that $\rho_{n,i}$ and $\rho_{i,n}$ satisfy associativity with respect to the maps $\rho_{i,j}$ for i, $j < n$ by the same method. We will just describe one example of this, since the remaining examples are proven in the same way. By definition of B, there is commutative diagram of natural transformations

$$
\begin{array}{ccc}
\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \otimes_{\mathrm{Day}} \widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \otimes_{\mathrm{Day}} B & \longrightarrow & \widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \otimes_{\mathrm{Day}} B \\
\downarrow & & \downarrow \\
\widetilde{\tau}_{\geq \bullet}^{\leq n-1} R \otimes_{\mathrm{Day}} B & \longrightarrow & B
\end{array}
$$

so by definition of Day convolution, there is, by evaluating at $n-1$, a commutative

diagram

$$\mathrm{colim}_{i \dotplus j \dotplus k \geq n-1} \, \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge B_k \longrightarrow \mathrm{colim}_{i \dotplus j \dotplus k \geq n} \, \widetilde{\tau}^{\leq n-1}_{\geq i \dotplus j} R \wedge B_k$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathrm{colim}_{i \dotplus j \dotplus k \geq n-1} \, \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge B_{j \dotplus k} \longrightarrow B_{n-1}$$

Therefore, in particular, there is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge B_{n-1} & \longrightarrow & \widetilde{\tau}^{\leq n-1}_{\geq i \dotplus j} R \wedge B_{n-1} \\
\downarrow & & \downarrow \\
\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge B_{n-1} & \longrightarrow & B_{n-1}
\end{array}
$$

which proves the desired associativity diagram for B_{n-1}.

To define $\rho_{n,n} \colon \widetilde{\tau}^{\leq n}_{\geq n} R \wedge \widetilde{\tau}^{\leq n}_{\geq n} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$, we use the commutative diagram of symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–bimodules

(27)
$$
\begin{array}{ccc}
B \otimes_{\mathrm{Day}} B & \longrightarrow & B \otimes \widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \\
\downarrow & & \downarrow \\
\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R \otimes_{\mathrm{Day}} B & \longrightarrow & B
\end{array}
$$

which implies that there is a map

$$\mathrm{colim}_{i \dotplus j \geq n-1} \, B_i \wedge B_j \to B_{n-1}$$

by evaluating at $n - 1$ and therefore there is a map

$$B_{n-1} \wedge B_{n-1} \to B_{n-1},$$

which we then define to be $\rho_{n,n}$. This map is commutative by definition and it is associative with respect to all the other maps $\rho_{i,j}$ by the same argument used for associativity above because the diagram (27) is a diagram of symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–bimodules.

The maps $\rho_{i,j} \colon \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ for $i + j \geq n$ are compatible with all the maps $\rho_{i,j}$ for $i + j < n$ by definition of $\widetilde{\tau}^{\leq n}_{\geq n} R$ as the pullback in diagram (26). (Note that the forgetful functor from symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–modules to $\mathrm{Sp}^{J^{\mathrm{op}}_{n-1}}$ composed with the nth evaluation functor to Sp is a right adjoint and therefore it preserves limits.)

The maps $\rho_{i,n}\colon \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq n} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ and $\rho_{n,i}\colon \widetilde{\tau}^{\leq n-1}_{\geq n} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq i} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ are unital by definition of B as a symmetric $\widetilde{\tau}^{\leq n-1}_{\geq \bullet} R$–bimodule by the same kind of argument as given above for commutativity and associativity. We will say why the remaining maps $\rho_{i,j}\colon \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ for $i + j \geq n$ and $i, j < n$ are unital later.

We now prove associativity of the maps $\rho_{i,j}\colon \widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R \to \widetilde{\tau}^{\leq n-1}_{n} R$ for $0 < i, j < n$ and $i + j \geq n$. The diagrams

$$
\begin{array}{ccc}
\widetilde{\tau}^{\leq n}_{\geq i} R \wedge \widetilde{\tau}^{\leq n}_{\geq j} R \wedge \widetilde{\tau}^{\leq n}_{\geq k} R & \longrightarrow & \widetilde{\tau}^{\leq n}_{\geq i+j} R \wedge \widetilde{\tau}^{\leq n}_{\geq k} R \\
\downarrow & & \downarrow \\
\widetilde{\tau}^{\leq n}_{\geq i} R \wedge \widetilde{\tau}^{\leq n}_{\geq j+k} R & \longrightarrow & \widetilde{\tau}^{\leq n}_{\geq n} R
\end{array}
$$

commute for $i, j, k < n$ and $i + j + k \geq n$ by the definition of $\widetilde{\tau}^{\leq n}_{\geq n} R$ as the pullback (26). The maps $\rho_{i,j}\colon \widetilde{\tau}^{\leq n}_{\geq i} R \wedge \widetilde{\tau}^{\leq n}_{\geq j} R \to \widetilde{\tau}^{\leq n}_{\geq n} R$ are also commutative because the diagrams

$$
\begin{array}{ccc}
\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R & \longrightarrow & \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq i} R \\
& \searrow & \downarrow \\
& \widetilde{\tau}^{\leq n-1}_{\geq n-1} R & \longrightarrow \widetilde{\tau}^{\leq n-1}_{\geq n-1}(\gamma_{\leq n-1} R)
\end{array}
$$

commute for all i and j, and the diagrams

$$
\begin{array}{ccc}
\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R & \longrightarrow & \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq i} R \\
& \searrow & \downarrow \\
& C(n)_n & \longrightarrow \widetilde{\tau}^{\leq n-1}_{\geq n-1} \gamma_{\leq n-1} R
\end{array}
$$

commute for all i and j, so by the definition of B as the pullback of the diagram (26), the diagram

$$
\begin{array}{ccc}
\widetilde{\tau}^{\leq n-1}_{\geq i} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq j} R & \longrightarrow & \widetilde{\tau}^{\leq n-1}_{\geq j} R \wedge \widetilde{\tau}^{\leq n-1}_{\geq i} R \\
& \searrow & \downarrow \\
& & \widetilde{\tau}^{\leq n}_{\geq n} R
\end{array}
$$

commutes for $i, j < n$ and $i + j \geq n$.

The unitality of the maps $\rho_{i,j}$ for $i, j < n$ and $i + j \geq n$ follows by the induction hypothesis and a similar argument to the one above.

We have therefore produced an object in $\mathrm{Comm}\,\mathrm{Sp}^{J_n^{\mathrm{op}}}$. We then cofibrantly replace this object in $\mathrm{Comm}\,\mathrm{Sp}^{J_n^{\mathrm{op}}}$ to produce an object $\tau_{\geq\bullet}^{\leq n} R$. By induction, we can therefore produce an object in $\mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$ and then cofibrantly replace it to produce (by Remark 3.4.9) a cofibrant decreasingly filtered commutative monoid in Sp, denoted by $\tau_{\geq\bullet} R$, as desired. □

Remark 4.2.2 What we really produce in the proof above is a cofibrant commutative monoid in the category of functors $\mathrm{Comm}\,\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$, or by the equivalence of categories discussed in Remark 4.1.5, a lax symmetric monoidal functor in $\mathrm{Sp}^{\mathcal{N}^{\mathrm{op}}}$. Due to the triviality of our Sp–enrichment on $\mathcal{N}$, this is the same data as a lax symmetric monoidal functor in $\mathrm{Sp}^{\mathbb{N}^{\mathrm{op}}}$, and the cofibrancy condition produces a cofibrant decreasingly filtered commutative monoid in Sp as discussed in Definition 3.1.2.

We need to check that $\tau_{\geq\bullet} R$ satisfies the "cofibrancy of degree-0 quotient" condition from Definition 3.1.2. In order to see that the map $S \to \tau_{\geq 0} R / \tau_{\geq 1} R$ is a levelwise cofibration we observe, first of all, that the construction of the map $R \to H\pi_0 R$ is a cofibration because it is formed by attaching E_∞–cells to kill higher homotopy. This is proven in [20, IV.3.1] in the associative setting, but the same proof works in the commutative setting. We then observe that $\tau_{\geq 0} R$ is constructed so that the composite $S \to \tau_{\geq 0} R \to R \to H\pi_0 R$ is a cofibration of commutative monoids in Sp, hence also a levelwise cofibration in Sp. The map $\tau_{\geq 0} R \to H\pi_0 R$ factors through the projection $\tau_{\geq 0} R \to \tau_{\geq 0} R / \tau_{\geq 1} R$, so since levelwise cofibrations are retractile, the map $S \to \tau_{\geq 0} R / \tau_{\geq 1} R$ is also a levelwise cofibration. This proves the "cofibrancy of degree-0 quotient" condition, so $\tau_{\geq\bullet} R$ is a decreasingly filtered commutative monoid in symmetric spectra.

Example 4.2.3 Assume a prime $p > 2$ is fixed. Let j be a cofibrant replacement in $\mathrm{Comm}\,\mathrm{Sp}$, for the commutative ring spectrum $\hat{K}(\mathbb{F}_q)_p$, where q is a topological generator of $\mathbb{Z}_p^\times$. Then by Theorem 4.2.1, we produce a cofibrant decreasingly filtered commutative monoid in Sp, denoted by $j_{\geq\bullet}$. The associated graded commutative monoid $E_0^* j_{\geq\bullet}$ is

$$H\pi_0 j \vee \Sigma^{2p-3} H\pi_{2p-3} j \vee \Sigma^{4p-5} H\pi_{4p-5} j \vee \cdots$$

after forgetting the commutative monoid structure. We therefore denote the associated graded commutative monoid object by $H\pi_* j$. There is an isomorphism of graded rings $\pi_*(H\pi_* j) \cong \pi_*(j)$, but as we have seen $H\pi_* j$ is a generalized Eilenberg–Mac Lane spectrum so by taking the associated graded of this filtration of j we have

effectively killed off all the k–invariants of j. Note that by $H\pi_n j$ we mean an explicit model for the Eilenberg–Mac Lane spectrum constructed as the cofiber of a cofibration $j_{\geq n+1} \to j_{\geq n}$. This example is used to compute $V(1)_* \mathrm{THH}(j)$ in a paper by the first author [1].

Example 4.2.4 Let R be a commutative ring spectrum whose homotopy groups satisfy $\pi_k(R) \cong \hat{\mathbb{Z}}_p$ for $k = 0, n$ and $\pi_k(R) \cong 0$ otherwise. Then one can build

$$0 \to \Sigma^n H\hat{\mathbb{Z}}_p \to R$$

as a cofibrant decreasingly filtered commutative ring spectrum using Theorem 4.2.1, where $\Sigma^n H\hat{\mathbb{Z}}_p$ is again an explicit model for the Eilenberg–Mac Lane spectrum constructed in Theorem 4.2.1. Since one can construct a Postnikov truncation of a commutative ring spectrum as a commutative ring spectrum [6], we can produce an example of this type by considering the truncation of the connective p–complete complex K–theory spectrum

$$\Sigma^2 H\pi_2 ku_p \to ku_p^{\leq 2} \to H\pi_0 ku_p,$$

where the map $ku_p^{\leq 2} \to H\pi_0 ku_p$ is constructed as in [6, Theorem 8.1].

5 Applications

Let S/p be the mod p Moore spectrum and let R be a connective commutative ring spectrum. We now present two calculations: first, we calculate $(S/p)_* \mathrm{THH}(R)$ when R has the property that $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$, where $|x| > 0$; second, we provide a bound on topological Hochschild homology of R in terms of $\mathrm{THH}(H\pi_*(R))$ and we give an explicit bound in the case $\pi_*(R) \cong \mathbb{Z}_{(p)}[x]$, where $|x| = 2n$ for $n > 0$.

5.1 Topological Hochschild homology of Postnikov truncations

Let R be a commutative ring spectrum with the property that $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$, with $|x| > 0$. We will consider the THH–May spectral sequence

$$(S/p)_*(\mathrm{THH}(H\hat{\mathbb{Z}}_p \ltimes \Sigma^n H\hat{\mathbb{Z}}_p)) \Rightarrow (S/p)_*(\mathrm{THH}(R))$$

produced using the short filtration of a commutative ring spectrum R which was given in Example 4.2.4. First, we compute the input of the (S/p)–THH–May spectral sequence for this example.

Proposition 5.1.1 *Let p be an odd prime; then*

$$(S/p)_*(\mathrm{THH}(H\widehat{\mathbb{Z}}_p \ltimes \Sigma^n H\widehat{\mathbb{Z}}_p)) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1) \otimes_{\mathbb{F}_p} \mathrm{HH}_*(E(x)),$$

where $|x| = n$. The grading of $\mathrm{HH}_*(E(x))$ is given by the sum of the internal and homological gradings.

Proof Due to Bökstedt [9], there is an isomorphism

$$\pi_*(S/p \wedge \mathrm{THH}(H\widehat{\mathbb{Z}}_p)) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1).$$

Let $S \ltimes \Sigma^n S$ be the trivial split square-zero extension of S by $\Sigma^n S$. Then $H\mathbb{Z}$ and $S \ltimes \Sigma^n S$ are commutative S–algebras and there is an equivalence of commutative S–algebras $H\widehat{\mathbb{Z}}_p \ltimes \Sigma^n H\widehat{\mathbb{Z}}_p \simeq H\widehat{\mathbb{Z}}_p \wedge (S \ltimes \Sigma^n S)$. Since the functor $S^1_\bullet \otimes (-)$ commutes with coproducts in $\mathrm{Comm}\,\mathrm{Sp}$ by [35], there are equivalences

$$\mathrm{THH}(H\widehat{\mathbb{Z}}_p \ltimes \Sigma^n H\widehat{\mathbb{Z}}_p)) \simeq \mathrm{THH}(H\widehat{\mathbb{Z}}_p \wedge (S \ltimes \Sigma^n S))$$
$$\simeq \mathrm{THH}(H\widehat{\mathbb{Z}}_p) \wedge \mathrm{THH}(S \ltimes \Sigma^n S)$$

of commutative ring spectra. Since $S/p \wedge H\widehat{\mathbb{Z}}_p \simeq H\mathbb{F}_p$ and the spectrum $\mathrm{THH}(H\widehat{\mathbb{Z}}_p)$ is a $H\widehat{\mathbb{Z}}_p$–algebra, the spectrum $S/p \wedge \mathrm{THH}(H\widehat{\mathbb{Z}}_p)$ naturally has the structure of a $H\mathbb{F}_p$–module. Hence, there are isomorphisms

$$\pi_*(S/p \wedge \mathrm{THH}(H\widehat{\mathbb{Z}}_p) \wedge \mathrm{THH}(S \ltimes \Sigma^n S))$$
$$\cong \pi_*\big((S/p \wedge \mathrm{THH}(H\widehat{\mathbb{Z}}_p)) \wedge_{H\mathbb{F}_p} (H\mathbb{F}_p \wedge \mathrm{THH}(S \ltimes \Sigma^n S))\big)$$
$$\cong \pi_*(S/p \wedge \mathrm{THH}(H\widehat{\mathbb{Z}}_p)) \otimes_{\mathbb{F}_p} H\mathbb{F}_{p\,*}(\mathrm{THH}(S \ltimes \Sigma^n S)).$$

Now, we apply the Bökstedt spectral sequence

$$\mathrm{HH}_*(H\mathbb{F}_{p\,*}(S \ltimes \Sigma^n S)) \Rightarrow H_*(\mathrm{THH}(S \ltimes \Sigma^n S); \mathbb{F}_p),$$

whose input is isomorphic to $\mathrm{HH}_*(E(x))$. If $|x|$ is odd, then we have $\mathrm{HH}_*(E(x)) \cong E(x) \otimes_{\mathbb{F}_p} \Gamma(\sigma x)$, following from [13, Chapter 10, Theorem 6.1] and the standard fact $\mathrm{Tor}_*^{E(x)}(\mathbb{F}_p, \mathbb{F}_p) \cong \Gamma(\sigma x)$ (see [36] for details). If $|x|$ is even, then one easily computes

$$\mathrm{HH}_n(E(x)) \cong \begin{cases} E(x) & \text{if } * = 0, \\ \Sigma^{|x|(2i-1)}\mathbb{F}_p\{1\} & \text{if } n = 2i - 1, \\ \Sigma^{|x|(2i+1)}\mathbb{F}_p\{x\} & \text{if } n = 2i \end{cases}$$

for $i \geq 1$. There is an isomorphism of bigraded rings

$$\mathrm{HH}_{*,*}(E(x)) \cong E(x)[x_i, y_j : i \geq 1, j \geq 0]/\sim,$$

where the degrees are given by $|x_i| = (2i, 2|x|i + |x|)$ and $|y_j| = (2j + 1, 2j|x| + |x|)$, and the equivalence relation is the one that makes all products zero. (See Proposition 3.3 in [5] for the more general calculation of $\mathrm{HH}_*(\mathbb{F}_p[x]/x^h)$ when $|x| = 2n$, $n > 0$, and $(p, h) = 1$.) The representatives in the cyclic bar complex for x_i and y_j are $x^{\otimes 2i+1}$ and $1 \otimes x^{\otimes 2j+1}$, respectively. Whether $|x|$ is even or odd, the Bökstedt spectral sequence collapses, with no hidden multiplicative extensions, for bidegree reasons. $\square$

Corollary 5.1.2 (rigidity of THH mod p for Postnikov truncations) *Let R be a connective E_∞–ring spectrum with $\pi_*(R) \cong \hat{\mathbb{Z}}_p[x]/x^2$ and $\pi_i(R) \cong 0$ for $i \neq 0, k$. Suppose that*

$$p \not\equiv k + 1 \bmod 2k + 1.$$

Then the graded abelian group $\pi_*(S/p \wedge \mathrm{THH}(R))$ depends only on $\pi_*(R^{\leq 2k})$, ie only on p and k.

Proof The THH–May spectral sequence

$$(S/p)_{*,*}(\mathrm{THH}(H\hat{\mathbb{Z}}_p \ltimes \Sigma^{2k} H\hat{\mathbb{Z}}_p)) \Rightarrow (S/p)_*(\mathrm{THH}(R))$$

collapses since there are no possible differentials for bidegree reasons under the assumptions on k with respect to p. $\square$

Remark 5.1.3 Corollary 5.1.2 can be considered a rigidity theorem in the sense that $S/p \wedge \mathrm{THH}$ does not see the first Postnikov k–invariant in the cases given by the congruences above.

Corollary 5.1.4 *Let p be a prime such that $p \not\equiv 2 \bmod 3$. Then*

$$(28) \qquad \pi_*(S/p \wedge \mathrm{THH}(ku_p^{\leq 2})) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1) \otimes_{\mathbb{F}_p} \mathrm{HH}_*(E(x)),$$

up to multiplicative extensions in the THH–May E^∞–term (so, in particular, (28) is an isomorphism of graded abelian groups), where $|x| = 2$ and the degree of $\mathrm{HH}_*(E(x))$ in π_* is given by the sum of the internal and homological degree.

5.2 Upper bounds on the size of THH

Many explicit computations are possible using the THH–May spectral sequence, for example, the first author's computations of topological Hochschild homology of the algebraic K–theory of finite fields, in [1]. These computations are sufficiently lengthy that they merit their own separate paper.

In the present paper, in lieu of explicit computations using the THH–May spectral sequence, we point out that the mere existence of the THH–May spectral sequence implies an upper bound on the size of the topological Hochschild homology groups of a ring spectrum: namely, if R is a graded-commutative ring and $X_\bullet$ is a simplicial finite set and E_* is a generalized homology theory, then for any E_∞–ring spectrum A such that $\pi_*(A) \cong R$, we have that $E_*(X_\bullet \otimes A)$ is a subquotient of $E_*(X_\bullet \otimes HR)$. Here HR is the generalized Eilenberg–Mac Lane spectrum of the graded ring R.

In particular:

Theorem 5.2.1 *For all integers n and all connective E_∞–ring spectra A, the cardinality of $\mathrm{THH}_n(A)$ is always less than or equal to the cardinality of $\mathrm{THH}_n(H\pi_*(A))$.*

Below are more details in a more restricted class of examples, namely, the E_∞–ring spectra A such that $\pi_*(A) \cong \mathbb{Z}_{(p)}[x]$.

Definition 5.2.2 We put a partial ordering on power series with integer coefficients as follows: given $f, g \in \mathbb{Z}[\![t]\!]$, we write $f \leq g$ if and only if, for all nonnegative integers n, the coefficient of t^n in f is less than or equal to the coefficient of t^n in g.

Definition 5.2.3 Let A be a graded ring. We will say that a graded A–module M is *finite-type and free* if there exists a function $c\colon \mathbb{Z} \to \mathbb{N}$ and an isomorphism of graded A–modules
$$\coprod_{n \in \mathbb{Z}} (\Sigma^n A)^{\oplus c(n)} \xrightarrow{\ \cong\ } M.$$
We will say that M is *finite-type* if there exists an exact sequence of graded A–modules of the form $F_1 \to F_0 \to M \to 0$ with F_0 and F_1 both finite-type and free.

Lemma 5.2.4 *Let A be a Noetherian connective commutative graded ring. Then the collection of bounded-below finite-type graded A–modules is closed under taking kernels, cokernels, extensions, and tensor products over A. Consequently, every bounded-below finite-type graded A–module admits a resolution by bounded-below finite-type free graded A–modules, and if M and N are bounded-below finite-type graded A–modules, then so is $\mathrm{Tor}_{n,*}^A(M, N)$ for each nonnegative integer n, and furthermore, $\mathrm{Tor}_{s,t}^A(M, N)$ vanishes for all $s > t$.*

Proof This is a bit of elementary algebra and we leave the proof as an exercise. ☐

[Lemma 5.2.5](#) is surely not a new result:

Lemma 5.2.5 *Suppose that A is a connective E_∞–ring spectrum such that the ring $\pi_*(A)$ is commutative and Noetherian, and suppose that the graded $\pi_*(A)$–module $\pi_*(A \wedge A)$ is finite-type. Suppose that $X_\bullet$ is a simplicial finite set. Then $\pi_n(X_\bullet \otimes A)$ is a finitely generated $\pi_0(A)$–module for all n.*

Proof First, a quick induction: If we have already shown that the graded $\pi_*(A)$–module $\pi_*(A^{\wedge m})$ is connective and finite-type, then by Lemma 5.2.4, the input for the Künneth spectral sequence

$$
(29) \qquad E^2_{s,t} \cong \mathrm{Tor}^{\pi_*(A)}_{s,t}(\pi_*(A \wedge A), \pi_*(A^{\wedge m})) \Rightarrow \pi_{s+t}(A^{\wedge m+1}),
$$
$$
d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1},
$$

is first-quadrant and hence strongly convergent, and consists of a finite-type graded $\pi_*(A)$–module on each s–line. The differentials in spectral sequence (29) are $\pi_*(A)$–linear (see [20, Theorem IV.4.1], which is stated in terms of commutative S–algebras, but the linearity of the spectral sequence differentials is formal and works for any model of commutative ring spectra), so by Lemma 5.2.4, for all integers $r \geq 2$ we have that each s–line in the E^r–page of spectral sequence (29) is a finite-type graded $\pi_*(A)$–module. Any differential of length $> s$ supported on the s–line in spectral sequence (29) must be zero, but it is not impossible that the s–line could be hit by differentials of arbitrarily long length. Let $E^r_{s,*}$ denote the graded $\pi_*(A)$–module which is the s–line in the E^r–page of spectral sequence (29). Then, for each pair (s, t), there exists some N such that $E^r_{s,t} \cong E^{r+1}_{s,t}$ for all $r \geq N$. Consequently $E^\infty_{s,*}$ is the colimit of a sequence of graded $\pi_*(A)$–module surjections $E^{s+1}_{s,*} \to E^{s+2}_{s,*} \to \cdots$ with the property that, for each integer t, the sequence of $\pi_0(A)$–modules $E^{s+1}_{s,t} \to E^{s+2}_{s,t} \to \cdots$ is eventually constant. So the graded $\pi_*(A)$–module $E^\infty_{s,*}$ is finite-type.

Now $\pi_*(A^{\wedge m+1})$ admits a filtration whose filtration quotients are the rows in the E^∞–page of (29). For any fixed choice of integer N, we have that $\pi_*(A^{\wedge m+1})$ agrees in grading degrees $\leq N$ with a graded $\pi_*(A)$–module given by finitely many of the filtration quotients (eg the first $N + 1$ rows in the E^∞–page), due to the vanishing property in Lemma 5.2.4. So by Lemma 5.2.4, the graded $\pi_*(A)$–module $\pi_*(A^{\wedge m+1})$ is finite-type (and clearly connective), and we are ready to return to the inductive step.

So $\pi_*(A^{\wedge m})$ is a finite-type connective graded $\pi_*(A)$–module for each m. In particular, $\pi_n(A^{\wedge m})$ is a finitely generated $\pi_0(A)$–module for each n. Consequently in the Bousfield–Kan-type spectral sequence

$$
E^1_{s,t} \cong \pi_t(A^{\wedge \#(X_s)}) \Rightarrow \pi_{s+t}(X_\bullet \otimes A), \quad d^r \colon E^r_{s,t} \to E^r_{s-r,t+r-1}
$$

obtained by applying π_* to the simplicial ring spectrum $X_\bullet \widetilde{\otimes} A$ (here we are using the pretensor product of Definition 3.2.1), each bidegree is a finitely generated $\pi_0(A)$–module, and the spectral sequence is half-plane with exiting differentials, hence also strongly convergent by [8, Theorem 6.1]. Connectivity of each tensor power $A^{\wedge \#(X_s)}$ implies that only finitely many bidegrees in the E^∞–page contribute to each total degree in $\pi_*(X_\bullet \otimes A)$. Consequently $\pi_n(X_\bullet \otimes A)$ is a finitely generated $\pi_0(A)$–module for each integer n. $\square$

Theorem 5.2.6 *Let n be a positive integer, p a prime number, and let E be an E_∞–ring spectrum such that either $\pi_*(E) \cong \hat{\mathbb{Z}}_p[x]$ or $\pi_*(E) \cong \mathbb{Z}_{(p)}[x]$, with x in grading degree $2n$. Then the Poincaré series of the mod p topological Hochschild homology $(S/p)_*(\mathrm{THH}(E))$ satisfies the inequality*

$$\sum_{i \geq 0} \big(\dim_{\mathbb{F}_p}(S/p)_*(\mathrm{THH}(E))\big)t^i \leq \frac{(1 + (2p-1)t)(1 + (2n+1)t)}{(1 - 2nt)(1 - 2pt)}.$$

Proof It is a classical computation of Bökstedt (see [9]) that

$$(S/p)_*(\mathrm{THH}(H\hat{\mathbb{Z}}_p)) \cong (S/p)_*(\mathrm{THH}(H\mathbb{Z}_{(p)})) \cong E(\lambda_1) \otimes_{\mathbb{F}_p} P(\mu_1),$$

with λ_1 and μ_1 in grading degrees $2p - 1$ and $2p$, respectively.

Now we use the splitting theorem of Schwänzl, Vogt, and Waldhausen, [40, Lemma 3.1]: if K is a commutative ring, and W is a q–cofibrant S–algebra (ie, up to equivalence, an A_∞–ring spectrum), then there exists a weak equivalence of S–modules (not necessarily a weak equivalence of S–algebras!):

$$\mathrm{THH}(W \wedge HK) \simeq \mathrm{THH}(W) \wedge \mathrm{THH}(HK) \simeq (\mathrm{THH}(W) \wedge HK) \wedge_{HK} \mathrm{THH}(HK).$$

In our case, W is the free A_∞–algebra on a single $2n$–cell, and $K = \hat{\mathbb{Z}}_p$ or $\mathbb{Z}_{(p)}$. Hence $\mathrm{THH}(W) \wedge HK$ satisfies

$$(S/p)_*(\mathrm{THH}(W) \wedge HK) \cong (H\mathbb{F}_p)_*(\mathrm{THH}(W)) \cong P(x) \otimes_{\mathbb{F}_p} E(\sigma x),$$

by collapse of the Bökstedt spectral sequence for bidegree reasons. Hence

$$(S/p)_*(\mathrm{THH}(H\hat{\mathbb{Z}}_p[x])) \cong (S/p)_*(\mathrm{THH}(H\mathbb{Z}_{(p)}[x])) \cong E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x),$$

as a graded $\mathbb{F}_p$–vector space (but not necessarily as $\mathbb{F}_p$–algebras!), which has Poincaré series $\big((1 + (2p-1)t)(1 + (2n+1)t)\big)/\big((1 - 2nt)(1 - 2pt)\big)$. $\square$

Now we give a few amusing consequences of Theorem 5.2.6. Recall that a spectrum is said to be *finite-type* if it is weakly equivalent to a CW–spectrum with finitely many cells in each dimension; if a spectrum X is connective, then X is finite-type if and only if the graded $\mathbb{Z}$–module $H_*(X;\mathbb{Z})$ is finite-type. We will say that X is p–*local finite-type* if X is weakly equivalent to a p–local CW–spectrum with finitely many $S_{(p)}$–cells in each dimension. Again, if a p–local spectrum X is connective, then X is p–local finite-type if and only if the graded $\mathbb{Z}_{(p)}$–module $H_*(X;\mathbb{Z}_{(p)})$ is finite-type.

Corollary 5.2.7 *Let n be a positive integer, p a prime number, and let E be a p–local finite-type E_∞–ring spectrum such that $\pi_*(E) \cong \mathbb{Z}_{(p)}[x]$, with x in grading degree $2n > 0$. Then:*

- *If p does not divide n, then $\mathrm{THH}_{2i}(E) \cong 0$ for all i congruent to $-p$ modulo n such that $i \leq pn-p-n$, and $\mathrm{THH}_{2i}(E) \cong 0$ for all i congruent to $-n$ modulo p such that $i \leq pn - p - n$. In particular, $\mathrm{THH}_{2(pn-p-n)}(E) \cong 0$.*

- *If p divides n, then $\mathrm{THH}_i(E) \cong 0$, unless i is congruent to -1, 0, or 1 modulo $2p$.*

Proof Since we have assumed that E is p–local finite-type, each homology group $H_s(E;\pi_*(E))$ is a finite-type graded $\pi_*(E)$–module, and consequently each s–line in the Atiyah–Hirzebruch spectral sequence

(30)
$$E^2_{s,t} \cong H_s(E;\pi_t(E)) \Rightarrow \pi_{s+t}(E \wedge E),$$
$$d^r: E^r_{s,t} \to E^r_{s+r,t-r+1}$$

is a finite-type graded $\pi_*(E)$–module. Connectivity of E ensures strong convergence of the spectral sequence, as in [8, Theorem 12.2]. As the Atiyah–Hirzebruch spectral sequence is the spectral sequence obtained by applying E–homology to a CW–decomposition of E, an easy analysis of the spectral sequence of a tower of cofiber sequences shows that the spectral sequence differentials are graded $\pi_*(E)$–module morphisms. Consequently Lemma 5.2.4 implies that each s–line in the E^r–page of spectral sequence (30) is a finite-type graded $\pi_*(E)$–module, for each $r \geq 2$. Now an argument exactly like that used in the proof of Lemma 5.2.5 implies that each s–line in the E^∞–page of (30) is also a finite-type graded $\pi_*(E)$–module, and that $\pi_*(E \wedge E)$ is a finite-type graded $\pi_*(E)$–module. So we can make use of Lemma 5.2.5 later in this proof.

Now we split the proof into two cases, the case where $p \nmid n$ and the case where $p \mid n$:

Case 1 If p does not divide n, then the largest integer i such that the graded polynomial algebra $P(\mu_1, x)$ is trivial in grading degree $2i$ is $2(pn - p - n)$. (This is a standard exercise in elementary number theory. In schools in the United States it is often presented to students in a form like "What is the largest integer N such that you cannot make exactly $5N$ cents using only dimes and quarters?") Triviality of $P(\mu_1, x)$ in grading degree $2(pn - p - n)$ also implies triviality of $P(\mu_1, x)$ in grading degree $2(pn - p - n) - 2(p + n)$, hence the triviality of $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ in grading degree $2(pn - p - n)$, hence (multiplying by powers of x or μ_1) the triviality of $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ in all grading degrees $\leq 2(pn - p - n)$ which are congruent to $-2p$ modulo $2n$ or congruent to $-2n$ modulo $2p$.

So $(S/p)_{2i}(\mathrm{THH}(E))$ vanishes if $i \leq pn - p - n$ and $i \equiv -p$ modulo n or $i \equiv -n$ modulo p. The long exact sequence

$$\cdots \to (S/p)_{2i+1}(\mathrm{THH}(E)) \to \mathrm{THH}_{2i}(E) \xrightarrow{p} \mathrm{THH}_{2i}(E) \to (S/p)_{2i}(E) \to \cdots$$

then implies that $\mathrm{THH}_{2i}(E)$ is p–divisible. By Lemma 5.2.5, $\mathrm{THH}_{2i}(E)$ is a finitely generated $\pi_0(E)$–module. Since $\pi_0(E) \cong \mathbb{Z}_{(p)}$ is a PID, one knows its finitely generated modules explicitly, and the only finitely generated $\mathbb{Z}_{(p)}$–module which is p–divisible is the trivial module.

Case 2 If p divides n, then $E(\lambda_1, \sigma x) \otimes_{\mathbb{F}_p} P(\mu_1, x)$ is concentrated in grading degrees congruent to -1, 0, and 1 modulo $2p$. An argument exactly as in the previous part of this proof then shows that if i is not congruent to -1, 0, or 1 modulo $2p$, then $\mathrm{THH}_i(E)$ must be a p–divisible finitely generated $\mathbb{Z}_{(p)}$–module, hence is trivial. $\square$

Appendix A: The THH–May spectral sequence with filtered coefficients

In this appendix, we briefly describe how to generalize the THH–May spectral sequence to include filtered coefficients; ie given a pointed simplicial finite set $Y_\bullet$, a connective generalized homology theory E_* (as in Definition 3.4.1), a decreasingly filtered commutative monoid $I_\bullet$ in $\mathcal{C}$, and a decreasingly filtered $I_\bullet$–module $M_\bullet$ in $\mathcal{C}$ (see Definition A.1.1), there is a spectral sequence

$$(31) \qquad E^1_{*,*} = E_{*,*}(Y_\bullet \otimes (E_0 I_\bullet; E_0 M_\bullet)) \Rightarrow Y_\bullet \otimes (I_0; M_0),$$

where $Y_\bullet \otimes (I_0; M_0)$ is defined in Definition A.2.1 and $E_0 M_\bullet$ is defined in Remark A.1.2.

A.1 Filtered coefficients

Recall from Remark 3.1.5 and Section 4.1 that a cofibrant object in $\operatorname{Comm} \mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ (with the model structure created by the forgetful functor to $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$, where $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ is equipped with the projective model structure) is a cofibrant decreasingly filtered commutative monoid in $\mathcal{C}$ in the sense of Definition 3.1.2.

Definition A.1.1 Let $I_\bullet$ be a cofibrant object in $\operatorname{Comm} \mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$. By a cofibrant decreasingly filtered $I_\bullet$–module we mean a cofibrant object in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ with the structure of an $I_\bullet$–module (equivalently a symmetric $I_\bullet$–bimodule); ie there are natural transformations

$$\psi^\ell \colon I_\bullet \otimes_{\mathrm{Day}} M_\bullet \to M_\bullet \quad \text{and} \quad \psi^r \colon M_\bullet \otimes_{\mathrm{Day}} I_\bullet \to M_\bullet$$

satisfying the usual commutative diagrams for a module over a commutative monoid along with the commutativity of the diagram
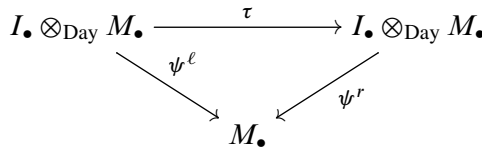
where τ is the factor swap map that we have by the definition of $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ as a symmetric monoidal category with respect to $\otimes_{\mathrm{Day}}$.

Remark A.1.2 There is an associated graded symmetric $E_0^* I_\bullet$–bimodule, $E_0^* M_\bullet$, which can be defined in a similar way to $E_0^* I_\bullet$; ie as an object in $\mathcal{C}$, it is

$$E_0^* M_\bullet = \bigvee_{i \geq 0} M_i / M_{i+1},$$

and to define it as a symmetric $E_0 I_\bullet$–bimodule we define maps

$$I_i / I_{i+1} \wedge M_j / M_{j+1} \to M_{i+j} / M_{i+j+1}$$

using the structure maps ψ^ℓ and ψ^r along with the structure maps $g_m \colon M_m \to M_{m-1}$ and the structure maps of $I_\bullet$ and then extend to a map

$$\left(\bigvee_{i \geq 0} I_i / I_{i+1} \right) \wedge \left(\bigvee_{i \geq 0} M_j / M_{j+1} \right) \to \bigvee_{i \geq 0} M_{i+j} / M_{i+j+1}$$

in the same way as in Definition 3.1.6.

A.2 Loday construction in the pointed setting

Definition A.2.1 For a finite pointed set S write $*_S$ for the basepoint of S and S' for $S - \{*_S\}$. Given a cofibrant commutative monoid R in a model category $\mathcal{C}$ satisfying Running Assumption 2.0.3 define a functor

$$f\mathrm{Set}_* \times R\text{–mod} \to R\text{–mod}$$

by sending (S, N) to $N \wedge \bigwedge_{s \in S} R$ and on morphisms by sending a based map of finite sets $S \to T$ to the composite map

$$N \wedge \bigwedge_{s \in S} R \simeq N \wedge \bigwedge_{s \in f^{-1}(*_T)} R \wedge \bigwedge_{s \in S - f^{-1}(*_T)} R \to X \wedge \bigwedge_{t \in T} R$$

defined as follows: the first map is the factor swap map and the second map is the smash product of the iterate of the (right) action map

$$N \wedge \bigwedge_{s \in f^{-1}(*_T)} R \to N$$

and the map

$$\bigwedge_{s \in S - f^{-1}(*_T)} R \to \bigwedge_{s \in S - f^{-1}(*_T)} R$$

as defined as in Definition 3.2.1. This functor naturally extends to a functor

$$(-) \widetilde{\otimes} (R; -): sf\mathrm{Set}_* \times R\text{–mod} \to sR\text{–mod}$$

by sending $(Y_\bullet, N)$ to the simplicial R–module with n–simplices

$$N \wedge \bigwedge_{y \in Y'_n} R$$

and using the functoriality of the functor $f\mathrm{Set}_* \times R\text{–mod} \to R\text{–mod}$ to define the face and degeneracy maps. We can therefore define

$$Y_\bullet \otimes (R; N) = |Y_\bullet \otimes (R; N)|.$$

Remark A.2.2 This construction is sufficiently general that given a cofibrant object $I_\bullet$ in $\mathrm{Comm}\,\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$ and a cofibrant decreasingly filtered $I_\bullet$–module $M_\bullet$ in $\mathcal{C}$, we can define an object in $\mathrm{Comm}^{\mathcal{N}^{\mathrm{op}}}$

$$Y_\bullet \otimes (I_\bullet; M_\bullet).$$

A.3 The fundamental theorem of the May filtration with coefficients

Recall that the fundamental theorem of the May filtration may be described using the slogan "higher order Hochschild homology commutes with passage to the associated graded commutative ring spectrum".

Definition A.3.1 Let $\mathcal{C}$ satisfy Running Assumptions 2.0.3 and 2.0.4. Given a pointed simplicial finite set $Y_\bullet$, a cofibrant object $I_\bullet$ in $\mathrm{Comm}\,\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$, and a cofibrant decreasingly filtered $I_\bullet$–module $M_\bullet$, we define *the May filtration of $M_0 \wedge \bigwedge_{y \in Y_0} I_0$* to be the cofibrant decreasingly filtered $I_\bullet$–module in $\mathcal{C}$: $Y_\bullet \otimes (I_\bullet; M_\bullet)$.

Remark A.3.2 Running Assumption 2.0.4 is needed to ensure that $Y_\bullet \otimes (I_\bullet; M_\bullet)$ is actually a *cofibrant* object in $\mathcal{C}^{\mathcal{N}^{\mathrm{op}}}$.

Theorem A.3.3 (Fundamental Theorem of the May filtration with coefficients) *There is an equivalence of $E_0^* I_\bullet$–modules*

$$E_0^*(Y_\bullet \otimes (I_\bullet \otimes M_\bullet)) \simeq Y_\bullet \otimes (E_0^* I_\bullet, E_0^* M_\bullet).$$

Proof To prove the theorem requires generalizing all of the definitions and lemmas from Section 3.3 to the pointed setting. The proofs of each of these generalizations of the lemmas from Section 3.3 follow in an evident way from the proofs that are already given. We therefore do not reprove them. The proof of the fundamental theorem of the May filtration also follows from the evident generalizations of the lemmas in Section 3.3 in the same way as the proof of Theorem 3.3.10. □

Remark A.3.4 The construction of the spectral sequence is exactly the same and therefore we do not discuss it here. To prove convergence and strong convergence of the spectral sequence we must prove the lemmas and theorems of Section 3.4 in the pointed setting, but these generalizations follow easily. We therefore produce a spectral sequence of the form (31) in the same way as before.

Appendix B: The Bousfield–Kan spectral sequence

B.1 The BKSS and nullhomotopic maps out of diagrams

Definition B.1.1 Given a small category $\mathcal{D}$, a category $\mathcal{A}$, and a functor $\mathcal{F}\colon \mathcal{D} \to \mathcal{A}$, we adopt this notation: we will write $\mathcal{D}^0$ for the set of objects of $\mathcal{D}$; we will write $\mathcal{D}^1$ for the set of morphisms in $\mathcal{D}$; if n is a positive integer, we will write $\mathcal{D}^n$ for the set of

composable ordered n–tuples of morphisms in $\mathcal{D}$; given a composable ordered n–tuple $d = (X_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} X_n)$ of morphisms in $\mathcal{D}$, we will write $\tilde{d}$ for X_n; we will write $\mathrm{sr}(\mathcal{F})$ for the *simplicial replacement of* $\mathcal{F}$, that is, the simplicial object of $\mathcal{A}$ given by

$$\coprod_{d\in\mathcal{D}^0} \mathcal{F}(\tilde{d}) \rightleftarrows \coprod_{d\in\mathcal{D}^1} \mathcal{F}(\tilde{d}) \underset{\longleftarrow}{\overset{\longleftarrow}{\rightleftarrows}} \coprod_{d\in\mathcal{D}^2} \mathcal{F}(\tilde{d}) \;\substack{\longleftarrow\\[-2pt]\rightleftarrows\\[-2pt]\longleftarrow} \cdots$$

with face and degeneracy maps induced by composition and inserting of an identity morphism, respectively, as operations on composable ordered tuples of morphisms in $\mathcal{D}$. (This construction is standard; see eg [10, Section XI.5.1].)

The following theorem seems to have had a long history: It seems as though it was understood in some form, at least in the setting of simplicial sets, by Bousfield and Kan when they wrote [10], and a proof is sketched in Corollary 9.8 and Proposition 9.11 of Dugger's unpublished notes [17], limited to the setting of topological spaces, but clearly using the technology of [18] which applies to much more general model categories. Finally, a really clear and general treatment can be derived from Theorems 3.2 and 3.3 of Gambino's paper [21], as explained in [21, Section 4]. Gambino's result only requires that $\mathcal{C}$ be a simplicial model category.

Theorem B.1.2 *Suppose that $\mathcal{D}$ is a small category such that the projective model structure on $\mathcal{C}^{\mathcal{D}}$ exists. Suppose that $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ is projectively cofibrant. Then the natural map $|\mathrm{sr}(\mathcal{F})| \to \operatorname{colim}\mathcal{F}$ in $\mathcal{C}$ is a weak equivalence.*

Theorem B.1.2 is our justification for the following notational convention: in this section and the following section, we will write hocolim for the Bousfield–Kan model for the homotopy colimit, as in [10]; that is, $\operatorname{hocolim}\mathcal{F} = |\mathrm{sr}(\mathcal{F})|$.

Definition B.1.3 Let $\mathcal{D}$ be a small category, and write Ab for the category of abelian groups. We will write E for the allowable class, in the sense of relative homological algebra (see [30, Chapter IX]), on the category of functors $\mathrm{Ab}^{\mathcal{D}}$, given as follows: a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathrm{Ab}^{\mathcal{D}}$ is in E if and only if the short exact sequence of abelian groups $0 \to \mathcal{F}'(d) \to \mathcal{F}(d) \to \mathcal{F}''(d) \to 0$ is split for all objects d of $\mathcal{D}$.

In the case where the underlying model category is simplicial sets (not a stable model category!), the spectral sequence of Theorem B.1.4 was constructed by Bousfield and Kan [10, Chapter XII]. It is a widespread bit of folklore that the construction also works in

more general model categories (see eg the discussion preceding [25, Proposition A.10]), so we give only a sketch of the proof of Theorem B.1.4.

In Theorem B.1.4, as everywhere in this paper, we let $\mathcal{C}$ be as in Running Assumption 2.0.3, but in the proof we sketch, we only use the assumptions that $\mathcal{C}$ is cocomplete, stable, closed and simplicial, and left proper.

Theorem B.1.4 (the Bousfield–Kan spectral sequence (BKSS) for generalized cohomology of a homotopy colimit) *Let $\mathcal{D}$ be a small category and let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor such that $F(d)$ is cofibrant for all $d \in \mathrm{ob}\,\mathcal{D}$. Let E be as in Definition B.1.3. Let X be a fibrant object of $\mathcal{C}$. Then there exists a spectral sequence*

$$(32) \qquad E^2_{s,t} \cong (R^s_E \lim_{d \in \mathcal{D}})([\Sigma^t F(d), X]) \Rightarrow [\Sigma^{t-s} \operatorname{hocolim} F, X]$$

which is strongly convergent if the functor $\lim\colon \mathrm{Ab}^{\mathcal{D}} \to \mathrm{Ab}$ *is of finite E–injective dimension (ie if there exists $N \in \mathbb{Z}$ such that $(R^s_E \lim)(\mathcal{F})$ vanishes for all $s > N$ and all $\mathcal{F} \in \mathrm{ob}\,\mathrm{Ab}^{\mathcal{D}}$).*

Sketch of proof It is routine to show that the cofibrancy assumption on each $F(d)$ is enough to imply that the simplicial object $\mathrm{sr}(F)$ of $\mathcal{C}$ is Reedy-cofibrant, and consequently that the geometric realization $|\mathrm{sr}(F)|$ is a homotopy colimit for F. Reedy-cofibrancy plays another role here, however: the latching comparison maps $L_n \mathrm{sr}(F) \to \mathrm{sr}(F)_n$ are each cofibrations in $\mathcal{C}$, and consequently (by Quillen's "condition SM7(b)", which holds for any closed simplicial model category by [38, Proposition 2.3]) the natural map $(L_n \mathrm{sr}(F) \otimes \Delta^n) \amalg_{L_n \mathrm{sr}(F) \otimes \delta\Delta^n} (\mathrm{sr}(F)_n \otimes \delta\Delta^n) \to \mathrm{sr}(F)_n \otimes \Delta^n$ is a cofibration in $\mathcal{C}$. By [39, Lemma 3.1], we can build $|\mathrm{sr}(F)|$ as a colimit $\operatorname{colim}_n |\mathrm{sr}(F)|_n$ of objects $|\mathrm{sr}(F)|_n$ given inductively by $|\mathrm{sr}(F)|_0 = \mathrm{sr}(F)_0$ and for each positive integer n a pushout square

$$(33)$$

$$
\begin{array}{ccc}
(L_n \mathrm{sr}(F) \otimes \Delta^n) \amalg_{L_n \mathrm{sr}(F) \otimes \delta\Delta^n} (\mathrm{sr}(F)_n \otimes \delta\Delta^n) & \longrightarrow & \mathrm{sr}(F)_n \otimes \Delta^n \\
\downarrow & & \downarrow \\
|\mathrm{sr}(F)|_{n-1} & \longrightarrow & |\mathrm{sr}(F)|_n
\end{array}
$$

Now since each $F(d)$ is assumed cofibrant, $\coprod_{d \in \mathcal{D}^0} F(d) = \mathrm{sr}(F)_0 = |\mathrm{sr}(F)|_0$ is also cofibrant. Consequently, when $n = 1$, square (33) is a pushout square in which the two objects on the left are cofibrant, and the top vertical map is a cofibration, so left properness of $\mathcal{C}$ implies that square (33) is also a homotopy pushout square, and that $|\mathrm{sr}(F)|_0 \to |\mathrm{sr}(F)|_1$ is a cofibration, hence that $|\mathrm{sr}(F)|_1$ is cofibrant.

That was the initial step in an induction. The inductive step works as follows: Since $0 \otimes \delta\Delta^n \cong 0 \otimes \Delta^n \cong 0$, another application of Quillen's condition SM7(b) gives us that, since $L_n \operatorname{sr}(F)$ is cofibrant (because it is a colimit of cofibrant objects), the map $L_n \operatorname{sr}(F) \otimes \delta\Delta^n \to L_n \operatorname{sr}(F) \otimes \Delta^n$ is a cofibration, and hence we have that $(L_n \operatorname{sr}(F) \otimes \Delta^n) \amalg_{L_n \operatorname{sr}(F) \otimes \delta\Delta^n} (\operatorname{sr}(F)_n \otimes \delta\Delta^n)$ is cofibrant, again using left properness of $\mathcal{C}$. So, if we have already shown that $|\operatorname{sr}(F)|_{n-1}$ is cofibrant, then the bottom horizontal map in (33) is a pushout of a cofibration, hence is a cofibration, hence $|\operatorname{sr}(F)|_n$ is cofibrant; and furthermore the square (33) is a homotopy pushout square, with horizontal cofiber $\Sigma^n \operatorname{sr}(F)_n$ by an argument dual to Proposition X.6.3 of [10]. By induction, we have a tower of cofiber sequences

$$
\begin{array}{ccccccc}
|\operatorname{sr}(F)|_0 & \longrightarrow & |\operatorname{sr}(F)|_1 & \longrightarrow & |\operatorname{sr}(F)|_2 & \longrightarrow & \cdots \\
\downarrow \cong & & \downarrow & & \downarrow & & \\
\operatorname{sr}(F)_0 & & \Sigma \operatorname{sr}(F)_1 & & \Sigma^2 \operatorname{sr}(F)_2 & &
\end{array}
$$

and, applying the generalized cohomology theory on $\operatorname{Ho}(\mathcal{C})$ represented by X, we get spectral sequence (32). □

Similarly, we have the following theorem:

Theorem B.1.5 (the Bousfield–Kan spectral sequence (BKSS) for generalized homology of a homotopy colimit) *Let $\mathcal{D}$ be a small category and let $F \colon \mathcal{D} \to \mathcal{C}$ be a functor such that $F(d)$ is cofibrant for all $d \in \operatorname{ob} \mathcal{D}$. Let E be as in Definition B.1.3. Let H_* be a connective generalized homology theory on $\mathcal{C}$, as defined in Definition 3.4.1. Then there exists a strongly convergent spectral sequence*

$$
E_{s,t}^2 \cong (L_s^E \operatorname{colim}_{d \in \mathcal{D}})(H_t(F(d))) \Rightarrow H_{s+t} \operatorname{hocolim} F.
$$

Proof The argument is essentially the same as in Theorem B.1.4, which is only a generalization of the proof of [10, XII.5.7]. Since $\operatorname{hocolim} F \cong |\operatorname{sr}(F)|$ is the cofiber of the map

$$
\coprod_{n \in \mathbb{N}} |\operatorname{sr}(F)|_n \xrightarrow{\ \mathrm{id} - T\ } \coprod_{n \in \mathbb{N}} |\operatorname{sr}(F)|_n,
$$

the exactness and additivity conditions in Definition 3.4.1 ensure that the homology of $\operatorname{hocolim} F$ is the colimit of the homologies of the finite stages $|\operatorname{sr}(F)|_n$; ie the spectral sequence computes $H_*(\operatorname{hocolim} F)$ as claimed. Strong convergence is automatic since this is a half-plane spectral sequence with exiting differentials, as in [8]. □

The rest of this section is occupied with an application of Theorem B.1.4. Here is the basic problem to be solved:

Question B.1.6 Suppose we are given a diagram of objects in our model category $\mathcal{C}$, and a map from the diagram to some particular object X. Suppose that the map from each of the objects in the diagram to X is nullhomotopic. Do all these maps from objects in the diagram to X factor through some homotopy colimit of the diagram, such that the map from the homotopy colimit to X is also nullhomotopic?

The answer to this question is not always yes: it depends on the diagram, and on X, and on the maps to X! The only tool we really need to answer Question B.1.6 is Theorem B.1.4, so while Question B.1.6 does not appear in [10] or in [39] or in any other reference we know of, nevertheless the question seems natural enough (and, in some situations, unavoidable enough) and its solution, in Theorem B.1.7, is a direct enough application of ideas from the 1970s that we do not claim that anything in this section is really "new".

Theorem B.1.7 *Let $\mathcal{C}$ be as in Running Assumption 2.0.3. Suppose that $\mathcal{D}$ is a small category, $F: \mathcal{D} \to \mathcal{C}$ is a functor such that $F(d)$ is cofibrant for each $d \in \mathrm{ob}\,\mathcal{D}$, X is a fibrant object of $\mathcal{C}$ and $\overline{X}: \mathcal{D} \to \mathcal{C}$ the constant X–valued functor on $\mathcal{C}$, and $\eta: F \to \overline{X}$ is a natural transformation such that $\eta(d)$ is nullhomotopic for each $d \in \mathrm{ob}\,\mathcal{D}$. Let E be as in Definition B.1.3, and suppose that the functor $\lim: \mathrm{Ab}^{\mathcal{D}} \to \mathrm{Ab}$ has finite E–injective dimension. If $(R_E^n \lim_{d \in \mathcal{D}})([\Sigma^n F(d), X]) \cong 0$ for all $n > 0$, then the resulting map $\mathrm{hocolim}\, F \to X$ is also nullhomotopic.*

Proof The terms in the E^2–page of spectral sequence (32) which can contribute to $[\Sigma^0 \mathrm{hocolim}\, F, X]$ in the E^∞–page are those of the form $(R_E^n \lim_{d \in \mathcal{D}})([\Sigma^n F(d), X])$. Consequently the vanishing hypothesis in the statement of the theorem ensures that the projection map $[\mathrm{hocolim}\, F, X] \to (R_E^0 \lim_{d \in \mathcal{D}})([F(d), X]) \cong \lim_{d \in \mathcal{D}}[F(d), X]$ is an isomorphism, and consequently that the map induced by η, which represents zero in $\lim_{d \in \mathcal{D}}[F(d), X]$, also represents zero in $[\mathrm{hocolim}\, F, X]$. $\qquad\square$

B.2 Application to truncated cubes

In this subsection, we prove a few lemmas on a particular special case of Theorem B.1.7 which arises in Section 4.2. The main result here is Lemma B.2.3, which we use in our proof of Theorem 4.2.1. As in Section 4.2, we will write Sp for the category of symmetric spectra in pointed simplicial sets with the positive flat stable model structure. We will continue to write Ab for the category of abelian groups.

Definition B.2.1 Let $\mathcal{D}$ be the small category with all objects and nonidentity morphisms drawn below:
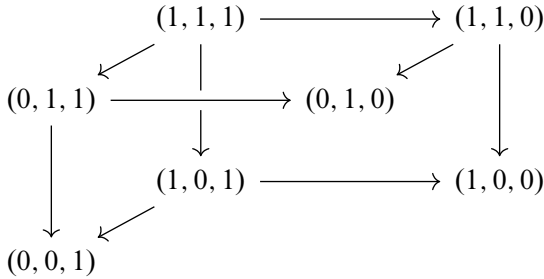
Lemma B.2.2 Let $\mathcal{F}: \mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$ be a functor. Then the following statements all hold:

- $R_E^0 \lim \mathcal{F} \cong \lim \mathcal{F}$ is a subgroup of $F(1,0,0) \oplus F(0,1,0) \oplus F(0,0,1)$.
- $R_E^1 \lim \mathcal{F}$ is a subquotient of
$$\mathcal{F}(1,1,1)^{\oplus 3} \oplus \mathcal{F}(1,1,0)^{\oplus 2} \oplus \mathcal{F}(1,0,1)^{\oplus 2} \oplus \mathcal{F}(0,1,1)^{\oplus 2}.$$
- $R_E^2 \lim \mathcal{F}$ is a subquotient of $\mathcal{F}(1,1,1)$.
- $R_E^n \lim \mathcal{F}$ vanishes for all $n > 2$.

Proof That $R_E^0 \lim \mathcal{F} \cong \lim \mathcal{F}$ is a standard consequence of lim being left exact. For the remaining statements, we first find all the relative cofree objects, ie the functors $\mathrm{Ab}^{\mathcal{D}^{\mathrm{op}}}$ that are in the image of the cofree functor $C: \mathrm{Ab}^{\times|\mathcal{D}^{\mathrm{op}}|} = \mathrm{Ab}^{\mathrm{ob}\,\mathcal{D}^{\mathrm{op}}} \to \mathrm{Ab}^{\mathcal{D}^{\mathrm{op}}}$, which is right adjoint to the functor $(\mathrm{ev}_d)_{d \in \mathcal{D}^{\mathrm{op}}}$. The relative cofree objects are the products of the diagrams of the form $C_i \in \mathrm{Ab}^{\mathcal{D}^{\mathrm{op}}}$, which we can think of as functors $C_{ijk}: \mathrm{Ab} \to \mathrm{Ab}^{\mathcal{D}^{\mathrm{op}}}$, as displayed in Figure 2.

Now we resolve $\mathcal{F}$ by relative cofree objects. We have a long exact sequence

(34) $$0 \to \mathcal{F} \to M_0 \to M_1 \to M_2 \to 0,$$

where the M_i are defined as follows:

$$M_0 = \coprod_{i,j,k} C_{ijk}(\mathcal{F}(i,j,k)),$$

$$M_1 = C_{011}(\mathcal{F}(1,1,1)) \oplus C_{101}(\mathcal{F}(1,1,1)) \oplus C_{110}(\mathcal{F}(1,1,1)) \oplus C_{001}(\mathcal{F}(1,0,1))$$
$$\oplus C_{100}(\mathcal{F}(1,1,0)) \oplus C_{010}(\mathcal{F}(0,1,1)) \oplus C_{010}(\mathcal{F}(1,1,0))$$
$$\oplus C_{100}(\mathcal{F}(0,1,1)) \oplus C_{100}(\mathcal{F}(1,0,1)),$$

$$M_2 = C_{001}(\mathcal{F}(1,1,1)) \oplus C_{010}(\mathcal{F}(1,1,1)) \oplus C_{100}(\mathcal{F}(1,1,1)).$$

$$C_{111} = \begin{bmatrix} & & X & \xleftarrow{\;\text{id}\;} & X \\ & \nearrow^{\text{id}} \;\text{id} & \uparrow & & \nearrow^{\text{id}} \uparrow \\ X & \xleftarrow{\quad} & & X & \quad\text{id} \\ \uparrow & & \text{id} & & \\ \text{id} & \quad X & \xleftarrow{\;\text{id}\;} & X & \\ & \nearrow^{\text{id}} & & & \\ X & & & & \end{bmatrix}$$

$$C_{101} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & 0 \\ & \nearrow^{\text{id}}\;0 & \uparrow & & \nearrow^{\text{id}} \uparrow \\ 0 & \xleftarrow{\quad} & & 0 & \quad\text{id} \\ \uparrow & & \text{id} & & \\ 0 & \quad X & \xleftarrow{\;\text{id}\;} & X & \\ & \nearrow^{\text{id}} & & & \\ X & & & & \end{bmatrix}$$

$$C_{011} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & 0 \\ & \nearrow^{0}\;\text{id} & \uparrow & & \nearrow^{0} \uparrow \\ X & \xleftarrow{\quad} & & X & \quad\text{id} \\ \uparrow & & \text{id} & & \\ \text{id} & \quad 0 & \xleftarrow{\;\text{id}\;} & 0 & \\ & \nearrow^{0} & & & \\ X & & & & \end{bmatrix}$$

$$C_{110} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & X \\ & \nearrow^{\text{id}}\;\text{id} & \uparrow & & \nearrow^{\text{id}} \uparrow \\ 0 & \xleftarrow{\quad} & & X & \quad\text{id} \\ \uparrow & & 0 & & \\ \text{id} & \quad 0 & \xleftarrow{\quad} & X & \\ & \nearrow^{\text{id}} & & 0 & \\ 0 & & & & \end{bmatrix}$$

$$C_{100} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & 0 \\ & \nearrow^{\text{id}}\;\text{id} & \uparrow & & \nearrow^{\text{id}} \uparrow \\ 0 & \xleftarrow{\quad} & & 0 & \quad 0 \\ \uparrow & & \text{id} & & \\ \text{id} & \quad 0 & \xleftarrow{\quad} & X & \\ & \nearrow^{\text{id}} & & 0 & \\ 0 & & & & \end{bmatrix}$$

$$C_{001} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & 0 \\ & \nearrow^{\text{id}}\;\text{id} & \uparrow & & \nearrow^{\text{id}} \uparrow \\ 0 & \xleftarrow{\quad} & & 0 & \quad\text{id} \\ \uparrow & & \text{id} & & \\ 0 & \quad 0 & \xleftarrow{\;\text{id}\;} & 0 & \\ & \nearrow^{0} & & & \\ X & & & & \end{bmatrix}$$

$$C_{010} = \begin{bmatrix} & & 0 & \xleftarrow{\;\text{id}\;} & 0 \\ & \nearrow^{\text{id}}\;\text{id} & \uparrow & & \nearrow^{0} \uparrow \\ 0 & \xleftarrow{\quad} & & X & \quad\text{id} \\ \uparrow & & 0 & & \\ \text{id} & \quad 0 & \xleftarrow{\;\text{id}\;} & 0 & \\ & \nearrow^{\text{id}} & & & \\ 0 & & & & \end{bmatrix}$$
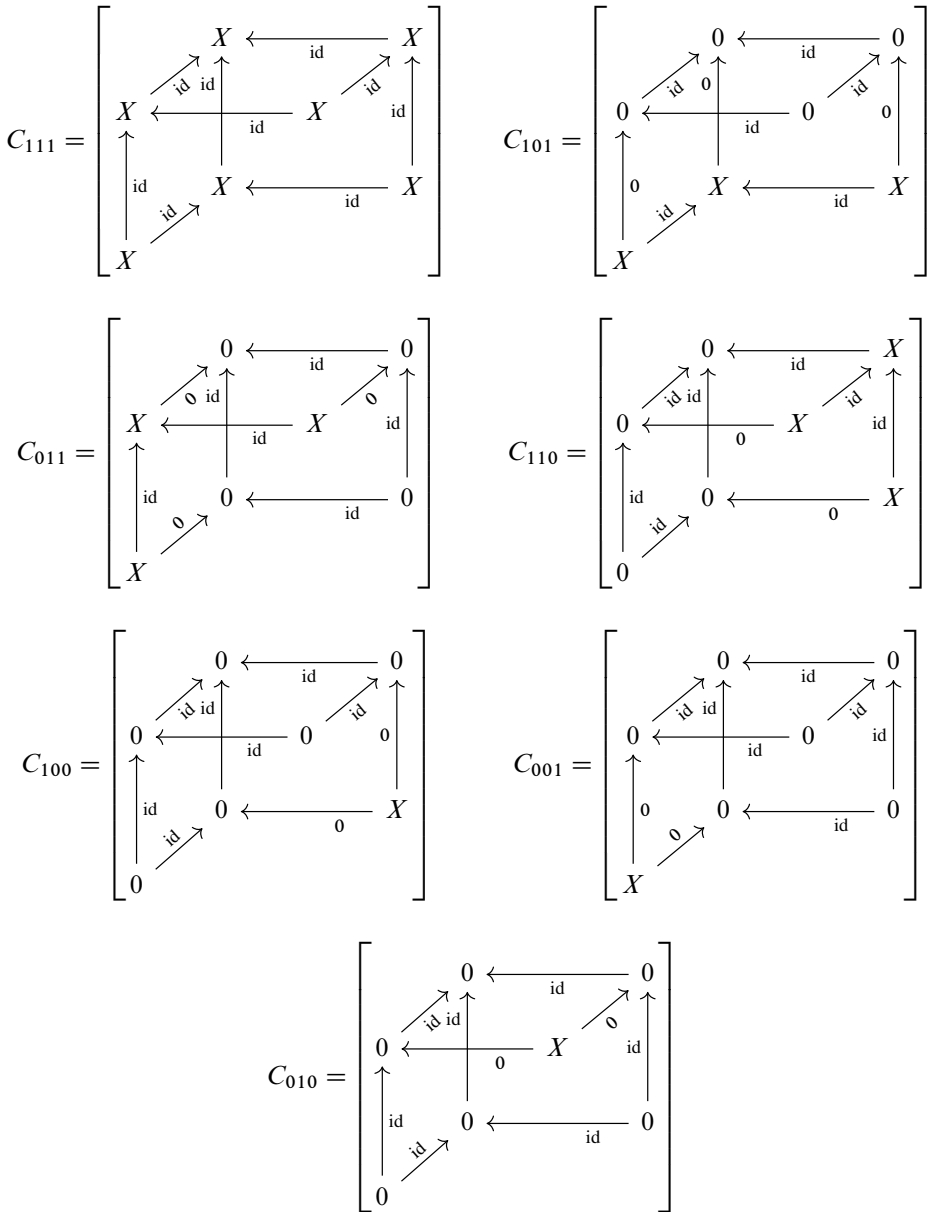
Figure 2

Clearly (34) is a resolution of $\mathcal{F}$ by E–projective objects in $\mathrm{Ab}^{\mathcal{D}^{\mathrm{op}}}$. For each object d of $\mathcal{D}$, each morphism in (34) has the property that its evaluation at d is the composite,

in Ab, of a split epimorphism followed by a split monomorphism. In other words, (34) is an E–resolution of $\mathcal{F}$, and we can use it to compute right E–derived functors. (See [30, Chapter IX] for these ideas from relative homological algebra.)

So, to compute $R_E^* \lim \mathcal{F}$, we can omit $\mathcal{F}$ from (34) and apply lim, producing the cochain complex

$$(35) \qquad\qquad 0 \to N_0 \to N_1 \to N_2 \to 0,$$

where the N_i are defined as follows:

$$N_0 = \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,1,0) \oplus \mathcal{F}(1,0,1) \oplus \mathcal{F}(0,1,1)$$
$$\oplus \mathcal{F}(1,0,0) \oplus \mathcal{F}(0,1,0) \oplus \mathcal{F}(0,0,1),$$

$$N_1 = \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,0,1) \oplus \mathcal{F}(1,1,0)$$
$$\oplus \mathcal{F}(0,1,1) \oplus \mathcal{F}(1,1,0) \oplus \mathcal{F}(0,1,1) \oplus \mathcal{F}(1,0,1),$$

$$N_2 = \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,1,1) \oplus \mathcal{F}(1,1,1).$$

Consequently $R_E^n \lim \mathcal{F}$ is a subquotient as claimed, and vanishes for all $n > 2$. (One could also compute the maps in (35) to get an explicit presentation of $R_E^1 \lim \mathcal{F}$ and $R_E^2 \lim \mathcal{F}$, but while interesting, this is unnecessary for the present paper.) □

Lemma B.2.3 *Suppose we have a commutative diagram in* Sp

(36)

and suppose there is a fibrant symmetric spectrum Z satisfying the conditions[1]

- $Z^{-1}(A_{(i,j,k)})$ vanishes whenever $i + j + k = 2$, and
- $Z^{-1}(A_{(1,1,1)})$ and $Z^{-2}(A_{(1,1,1)})$ vanish.

---

[1]Recall that $Z^{-m}(A_{(i,j,k)})$ is a standard notation for $[\Sigma^m A_{(i,j,k)}, Z]$.

Regard (36) *as a functor* $A\colon \mathcal{D} \to \mathrm{Sp}$. *Then the natural map of abelian groups* $[\mathrm{hocolim}\, A, Z] \to \lim_{d \in \mathcal{D}^{\mathrm{op}}}[A(d), Z]$ *is an isomorphism.*

In particular, if we make the additional assumption that $Z^0(A_{(i,j,k)})$ *vanishes whenever* $i + j + k = 1$, *then* $[\mathrm{hocolim}\, A, Z]$ *vanishes.*

Proof By Theorem B.1.7, the obstruction to $[\mathrm{hocolim}\, A, Z] \to \lim_{d \in \mathcal{D}^{\mathrm{op}}}[A(d), Z]$ being an isomorphism arises from the groups

$$(R_E^s \lim_{d \in \mathcal{D}})(Z^{-s}A(d))$$

for $s > 0$. By Lemma B.2.2, $(R_E^1 \lim_{d \in \mathcal{D}})(Z^{-1}A(d))$ vanishes as long as $Z^{-1}A_{i,j,k}$ vanishes for all triples (i, j, k) satisfying $i + j + k \geq 2$. Also by Lemma B.2.2, $(R_E^2 \lim_{d \in \mathcal{D}})(Z^{-2}A(d))$ vanishes as long as $Z^{-2}A_{1,1,1}$ vanishes. Finally, the group $(R_E^s \lim_{d \in \mathcal{D}})(Z^{-s}A(d))$ vanishes for all $s > 2$. This proves the first claim. If we additionally assume that $Z^0(A_{(i,j,k)})$ vanishes whenever $i + j + k = 1$, then $\lim_{d \in \mathcal{D}^{\mathrm{op}}}[A(d), Z]$ vanishes and consequently $[\mathrm{hocolim}\, A, Z]$ vanishes as well. □

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