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Framed cobordism and flow category moves

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Framed flow categories were introduced by Cohen, Jones and Segal as a way of encoding the flow data associated to a Floer functional. A framed flow category gives rise to a CW complex with one cell for each object of the category. The idea is that the Floer invariant should take the form of the *stable homotopy type* of the resulting complex, recovering the Floer cohomology as its singular cohomology. Such a framed flow category was produced, for example, by Lipshitz and Sarkar from the input of a knot diagram, resulting in a stable homotopy type generalising Khovanov cohomology.

We give moves that change a framed flow category without changing the associated stable homotopy type. These are inspired by moves that can be performed in the Morse–Smale case without altering the underlying smooth manifold. We posit that if two framed flow categories represent the same stable homotopy type then a finite sequence of these moves is sufficient to connect the two categories. This is directed towards the goal of reducing the study of framed flow categories to a combinatorial calculus.

We provide examples of calculations performed with these moves (related to the Khovanov framed flow category), and prove some general results about the simplification of framed flow categories via these moves.

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1 Introduction

A *framed flow category* is, roughly speaking, a way of encoding the flow data associated to a Morse–Smale vector field or, more generally, the flow data of a Floer functional. Objects of the category correspond to critical points while the morphism set between any two objects has the structure of a framed manifold with corners corresponding to the space of flowlines between the objects.

A framed flow category gives rise to a CW complex with one cell for each object of the category, and the attaching maps determined by the morphism spaces. When the

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framed flow category arises from a Floer functional, the (shifted) cohomology of the CW complex agrees with the corresponding Floer cohomology. The idea is that the whole CW complex (up to suspension and desuspension) should be considered as the Floer invariant, instead of just its cohomology. In other words, the Floer invariant should take the form of a *stable homotopy type*.

In this paper we construct moves that change a framed flow category without changing the associated stable homotopy type. These are inspired by moves that can be performed in the Morse–Smale situation without altering the underlying smooth manifold. We posit that if two framed flow categories represent the same stable homotopy type then a finite sequence of these moves is sufficient to connect the two categories. This is directed towards the goal of reducing the study of framed flow categories (and their associated stable homotopy types) to combinatorics.

We show in Theorem 6.2 that, after a finite sequence of our moves, we can arrange for the associated cochain complex to the framed flow category to be in primary Smith normal form. This immediately gives Corollary 6.3. This corollary implies, for example, that if two framed flow categories each represent the same Moore space, then the categories are indeed connected by a finite sequence of moves.

In a later paper we shall go further, developing the combinatorics of the moves and proving results based now on the classifications due to Chang, and Baues and Hennes, of stable homotopy types of small cohomological width.

1.1 Plan of the paper

In Section 2, we give a condensed recap of the necessary definitions of a framed flow category. This is based on the formulation by Lipshitz and Sarkar [10], and is intended as a quick-reference guide for the rest of the paper. In Section 3 we introduce an analogue for framed flow categories of the operation of *handle sliding* in a handlebody decomposition of a smooth manifold. In Section 4 we similarly introduce an analogue of the *Whitney trick*.

Handle sliding, the extended Whitney trick, and handle cancellation (already introduced by Jones, Lobb and Schütz [7]) are each moves that can be performed on framed flow categories while preserving the associated stable homotopy type. We give an example of their application in Section 5. Recently, Lipshitz and Sarkar [10] have constructed a framed flow category from the input of a knot diagram. This gives a stable homotopy type whose cohomology recovers Khovanov cohomology (a knot invariant intimately connected with Floer cohomology). Our example is drawn from a particular quantum degree ($q = 21$) of the Khovanov stable homotopy type of the disjoint union of three right-handed trefoils and exploits the "matched diagram" techniques of Jones, Lobb and Schütz [6].

Finally, in Section 6 we prove general results about simplification of framed flow categories. Essentially, these results can be taken as saying that each framed flow category is move-equivalent to a framed flow category realising the simplest possible cochain complex giving the correct cohomology.

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2 Framed flow categories

The definition of a flow category was originally given by Cohen, Jones and Segal [4] but the version we are working with is based on the definition by Lipshitz and Sarkar [10]; the equivalence of the two approaches is shown in [10, Proposition 3.27].

Before we can give the definition of a flow category, we will need a sharpening of the concept of a smooth manifold with corners. This refinement was originally defined by Jänich [5] and was further developed by Laures [8]. We refer the reader to these sources for a fuller treatment.

We write the half-line as $\mathbb{R}_+ = [0, \infty)$.

Definition 2.1 Let M be a smooth m–dimensional manifold with corners. For each $p \in M$ there is an integer $l(p)$ such that an open neighbourhood of p is diffeomorphic to $(\mathbb{R}_+)^l \times \mathbb{R}^{m-l}$, with p corresponding to $0 \in (\mathbb{R}_+)^l \times \mathbb{R}^{m-l}$. We say p is a *corner point of codimension* l. The closure of any connected component of the set $\{p \in M \mid l(p) = 1\}$ is called a *face* of M, and the union of the faces is the *boundary* ∂M of M.

Now suppose each $p \in M$ belongs to precisely $l(p)$ faces. In this case, a *smooth m–dimensional $\langle n \rangle$–manifold* is a manifold with corners M, together with an ordered n–tuple of faces $(\partial_1 M, \dots, \partial_n M)$, which satisfy the following conditions:

- $\partial_1 M \cup \cdots \cup \partial_n M = \partial M$.
- $\partial_i M \cap \partial_j M$ is a face of $\partial_i M$ and $\partial_j M$ for all $i \neq j$.

In this case, for any nonempty subset $J \subset \{1, \dots, n\}$, we will write
$$\partial_J M = \bigcap_{j \in J} \partial_j M.$$

Now we introduce some more notation. Let n be a nonnegative integer and let $\boldsymbol{d} = (d_0, \dots, d_n)$ be an $(n+1)$–tuple of nonnegative integers. We write
$$\mathbb{E}^{\boldsymbol{d}} = \mathbb{R}^{d_0} \times \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{d_n},$$
$$\mathbb{E}^{\boldsymbol{d}}[a:b] = \mathbb{E}^{(d_a, \dots, d_{b-1})} \quad \text{for } 0 \le a < b \le n+1.$$

We introduce a standard subdivision of the boundary of $\mathbb{E}^{\boldsymbol{d}}$ into the following faces:
$$\partial_i \mathbb{E}^{\boldsymbol{d}} = \mathbb{R}^{d_0} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}^{d_{i-1}} \times \{0\} \times \mathbb{R}^{d_i} \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{d_n}$$
for $i = 1, \dots, n$, so that $\mathbb{E}^{\boldsymbol{d}}$ is a $(d_0 + \cdots + d_n + n)$–dimensional $\langle n \rangle$–manifold.

Definition 2.2 A *neat immersion (embedding)* of a smooth m–dimensional $\langle n \rangle$–manifold M is a smooth immersion (embedding) $\iota\colon M \looparrowright \mathbb{E}^{\boldsymbol{d}}$ such that $\iota^{-1} \partial_i \mathbb{E}^{\boldsymbol{d}} = \partial_i M$ for all i and, for any $I \subset J \subset \{1, \dots, n\}$, the intersection $\partial_I M \cap \partial_J \mathbb{E}^{\boldsymbol{d}} = \partial_J M$ is perpendicular.

Any compact m–dimensional $\langle n \rangle$–manifold admits a neat embedding with respect to some $\mathbb{E}^{\boldsymbol{d}}$; see Laures [8, 2.1.7]. Moreover, a neat immersion can always be improved to a neat embedding by perturbation, possibly after increasing the integers in the tuple $\boldsymbol{d}$.

Definition 2.3 A *flow category* $\mathscr{C}$ is a category with finitely many objects, and equipped with a function $|\cdot|\colon \mathrm{Ob}(\mathscr{C}) \to \mathbb{Z}$, called the *grading*, satisfying the following:

(1) $\mathrm{Hom}(x, x) = \{\mathrm{id}\}$ for all $x \in \mathrm{Ob}(\mathscr{C})$ and, for $x \neq y \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}(x, y)$ is a smooth, compact $(|x| - |y| - 1)$–dimensional $\langle |x| - |y| - 1 \rangle$–manifold, which we denote by $\mathcal{M}(x, y)$. (By convention, a manifold of dimension < 0 is the empty set.)

(2) For $x, y, z \in \mathrm{Ob}(\mathscr{C})$ with $|z| - |y| = m$, the composition map restricts to
$$\circ\colon \mathcal{M}(z, y) \times \mathcal{M}(x, z) \to \mathcal{M}(x, y),$$
which is an embedding into $\partial_m \mathcal{M}(x, y)$. Furthermore,
$$\circ^{-1}(\partial_i \mathcal{M}(x, y) \cap \partial_m \mathcal{M}(x, y)) = \begin{cases} \partial_i \mathcal{M}(z, y) \times \mathcal{M}(x, z) & \text{for } i < m, \\ \mathcal{M}(z, y) \times \partial_{i-m} \mathcal{M}(x, z) & \text{for } i > m. \end{cases}$$

(3) For $x \neq y \in \mathrm{Ob}(\mathscr{C})$, the map $\circ$ induces a diffeomorphism

$$\partial_i \mathcal{M}(x,y) \cong \coprod_{z,\,|z|=|y|+i} \mathcal{M}(z,y) \times \mathcal{M}(x,z).$$

The manifold $\mathcal{M}(x,y)$ is called the *moduli space from x to y*, and we set $\mathcal{M}(x,x) = \varnothing$.

Definition 2.4 Let $\mathscr{C}$ be a flow category and write

$$k = \min\{|x| : x \in \mathrm{Ob}(\mathscr{C})\}, \quad n = \max\{|x| : x \in \mathrm{Ob}(\mathscr{C})\} - k.$$

Suppose $\boldsymbol{d} = (d_k, \ldots, d_{n+k})$ is an $(n+1)$–tuple of nonnegative integers. A *neat immersion (resp. neat embedding)* ι of a flow category $\mathscr{C}$ relative to $\boldsymbol{d}$ is a collection of neat immersions (resp. neat embeddings) $\iota_{x,y} \colon \mathcal{M}(x,y) \looparrowright \mathbb{E}^{\boldsymbol{d}}[|y| : |x|]$ such that for all objects x, y, z and points $(p,q) \in \mathcal{M}(x,z) \times \mathcal{M}(z,y)$,

$$\iota_{x,y}(q \circ p) = (\iota_{z,y}(q), 0, \iota_{x,z}(p)).$$

If ι is a neat immersion of a flow category $\mathscr{C}$ relative to $\boldsymbol{d}$, then a *framing* φ of $(\mathscr{C}, \iota)$ is a collection of immersions

$$\varphi_{x,y} \colon \mathcal{M}(x,y) \times [-\varepsilon, \varepsilon]^A \looparrowright \mathbb{E}^{\boldsymbol{d}}[|y| : |x|], \quad A = d_{|y|} + \cdots + d_{|x|-1},$$

extending $\iota_{x,y}$, such that for all objects x, y, z and points $(p,q) \in \mathcal{M}(x,z) \times \mathcal{M}(z,y)$,

$$\varphi_{x,y}(q \circ p, t_1, t_2, \ldots, t_A) = (\varphi_{z,y}(q, t_1, t_2, \ldots, t_B), 0, \varphi_{x,z}(p, t_{B+1}, \ldots, t_A)),$$

where $B = d_{|y|} + \cdots + d_{|z|-1}$.

For another $(n+1)$–tuple $\boldsymbol{d}' = (d'_k, \ldots, d'_{n+k})$ of nonnegative integers, we write $\boldsymbol{d}' \geq \boldsymbol{d}$ when $d'_j \geq d_j$ for each $j = k, \ldots, n+k$. We note that a (framed) neat immersion of a flow category relative to some $\boldsymbol{d}$ can be improved to a (framed) neat *embedding* relative to some $\boldsymbol{d}' \geq \boldsymbol{d}$ by perturbation.

Definition 2.5 A *framed flow category* is a triple $(\mathscr{C}, \iota, \varphi)$, where $(\mathscr{C}, \iota)$ is a flow category with neat immersion, relative to some $\boldsymbol{d}$, and φ is a framing for $(\mathscr{C}, \iota)$.

One can associate to a framed flow category $(\mathscr{C}, \iota, \varphi)$ a geometric realisation as a finite cell complex $|\mathscr{C}|_{\iota,\varphi} = |\mathscr{C}|$; see [10, Definition 3.23] for precise details of the version we are using. The construction of $|\mathscr{C}|$ requires several auxiliary choices, one such being an integer C such that $C > \pm|x|$ for all $x \in \mathrm{Ob}(\mathscr{C})$. To construct $|\mathscr{C}|$, one takes for each object x of $\mathscr{C}$ a $(C+m)$–dimensional cell $\mathcal{C}(x)$, where $|x| = m$. These cells are glued together using the data of the framed moduli spaces (essentially via the Thom construction).

The homotopy type of $|\mathscr{C}|$ depends on the framed embedding (ι, φ), but modifying (ι, φ) by an isotopy results in a homeomorphic cell complex; see [10, Lemma 3.25]. Modifying the other choices involved in constructing $|\mathscr{C}|$, including possibly increasing $\boldsymbol{d}$ to $\boldsymbol{d}' \geq \boldsymbol{d}$, will result, up to homotopy, in some precise number of reduced suspensions of $|\mathscr{C}|$, which are compensated for by modifying C; see [10, Definition 3.2.3]. Thus the following is a well-defined topological object arising from a framed flow category, and independent of choice of $\boldsymbol{d}$.

Definition 2.6 For a framed flow category $(\mathscr{C}, \iota, \varphi)$ and any object x of $\mathscr{C}$, let C be the difference between the dimension of the associated cell $\mathcal{C}(x) \subset |\mathscr{C}|_{\iota,\varphi}$ and the grading of x (this C is independent of x). The *stable homotopy type of* $(\mathscr{C}, \iota, \varphi)$ is the desuspended suspension spectrum $\mathcal{X}(\mathscr{C}) := \Sigma^{-C}\Sigma^{\infty}|\mathscr{C}|_{\iota,\varphi}$, considered up to homotopy of spectra.

3 Handle slides in framed flow categories

In this section, we will define the framed flow category analogue for the process in handlebody theory of sliding one i–handle over another i–handle. This results in a new framed flow category giving rise to the same stable homotopy type.

Theorem 3.1 *If $(\mathscr{C}_S, \iota_S, \varphi_S)$ is the result of a* handle slide *in $(\mathscr{C}, \iota, \varphi)$, then there is a stable homotopy equivalence*

$$\mathcal{X}(\mathscr{C}_S) \simeq \mathcal{X}(\mathscr{C}).$$

We first describe a method for handle sliding, on which we base our flow category analogue. Given a handlebody H which has i–handles h_1 and h_2, form a new handlebody H' by attaching an i–handle h_3 and a cancelling $(i+1)$–handle g, attached as follows. The attaching sphere for h_3 will be glued along the sphere embedded in H formed by taking an embedded connected sum of slightly pushed-off copies of the attaching spheres of h_1 and h_2. The attaching sphere of g will be glued to intersect the belt spheres of each of our three i–handles precisely once positively each (and with no other i–handle interactions).

There are then exactly three i–handles against which g can be cancelled. The handlebodies which are the effects of cancelling g against h_1, h_2 and h_3 are homeomorphic to, respectively, the effect of h_1 slid over h_2, the effect of h_2 slid over h_1, and the original handlebody H. The reason for describing handle sliding with such an emphasis on handle cancellation is that it will allow us to build on the flow category handle cancellation technique described in the earlier paper of Jones, Lobb and Schütz [7].

Let $\mathscr{C}$ denote a framed flow category $(\mathscr{C}, \iota, \varphi)$ with respect to an $(n+1)$–tuple of nonnegative integers $(d_k, \dots, d_{n+k})$. Suppose x and y are objects of $\mathscr{C}$ with grading i. To mimic the geometric process above, we will introduce a pair of cancelling objects e and f in degrees i and $i+1$, respectively. Morphisms will be introduced such that cancelling f against e returns $\mathscr{C}$, and cancelling f against x returns a new category $\mathscr{C}_S$, which we will call the *effect of sliding x over y*. It is proved in [7, Theorem 2.17] that handle cancellation in flow categories preserves the stable homotopy type of a framed flow category, and hence the stable homotopy types $\mathcal{X}(\mathscr{C})$ and $\mathcal{X}(\mathscr{C}_S)$ will agree.

The main task of this section therefore will be to correctly describe, embed, and frame the *intermediate flow category* obtained by introducing e and f. This is carried out in Sections 3.1 and 3.2, with Theorem 3.1 deduced in Section 3.3. We end with a discussion of how handle sliding affects the framings of 1–dimensional moduli spaces in Section 3.4.

3.1 Sweeping through an arc

We begin with a technical construction that we will later use to embed and frame the additional moduli spaces that we will need to add to $\mathscr{C}$ to construct the intermediate flow category.

Suppose for this subsection that M^m is a smooth m–dimensional $\langle n\rangle$–manifold. Then $M\times[0,1]$ has the structure of a smooth $(m+1)$–dimensional $\langle n+1\rangle$–manifold, given by setting $\partial_{n+1}(M\times[0,1]) = M\times\{0,1\}$ and $\partial_i(M\times[0,1]) = \partial_i M\times[0,1]$ for $i\le n$.

Write $\boldsymbol{d} = (d_k, \dots, d_{n+k})$ and suppose there is a neat immersion $\iota\colon M \looparrowright \mathbb{E}^{\boldsymbol{d}}$. We can always assume, possibly after increasing d_{n+k} by 1 and arranging this to be the first coordinate of $\mathbb{R}^{d_{n+k}}$, that postcomposing ι with the projection to the first coordinate of $\mathbb{R}^{d_{n+k}} \subset \mathbb{E}^{\boldsymbol{d}}$ is the constant map to the origin. Denote by $R\colon \mathbb{E}^{\boldsymbol{d}} \to \mathbb{E}^{\boldsymbol{d}}$ the reflection in the plane through the origin of $\mathbb{E}^{\boldsymbol{d}}$, which is perpendicular to the first coordinate axis of $\mathbb{R}^{d_{n+k}} \subset \mathbb{E}^{\boldsymbol{d}}$. Note that $R\circ\iota = \iota$ because of our assumption that ι does not extend into the first coordinate direction of $\mathbb{R}^{d_{n+k}}$. Suppose ι admits a framing $\varphi\colon M\times[-\varepsilon,\varepsilon]^A \looparrowright \mathbb{E}^{\boldsymbol{d}}$, with $A = d_k + \dots + d_{n+k}$. Then $-\varphi := R\circ\varphi$ determines another possible framing for $R\circ\iota = \iota$ in $\mathbb{E}^{\boldsymbol{d}}$.

We now sweep ι through a semicircular arc in two new dimensions.

Lemma 3.2 *For real numbers $u \in \mathbb{R}$ and $v \in \mathbb{R}_+$, let*

$$\gamma_{u,v}(t) = \big(v\sin(\pi t), v(u-\cos(\pi t))\big) \in \mathbb{R}_+ \times \mathbb{R}.$$

Then the function

$$\iota'\colon M \times [0,1] \looparrowright \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R}, \quad \iota'(p,t) := (\iota(p), \gamma_{u,v}(t)),$$

is a neat immersion.

Proof Denote by $N^{(l)}$ the set of corner points of codimension l in a manifold with corners N, and write $N^{(0)} = N \setminus \partial N$. Then, for $l > 0$,

$$(M \times [0,1])^{(l)} = (M^{(l-1)} \times \{0,1\}) \sqcup (M^{(l)} \times (0,1)).$$

But

$$\iota'(M^{(l-1)} \times \{0,1\}) \subset (\mathbb{E}^{\boldsymbol{d}})^{(l-1)} \times \{0\} \times \mathbb{R}$$

and

$$\iota'(M^{(l)} \times (0,1)) \subset (\mathbb{E}^{\boldsymbol{d}})^{(l)} \times (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R},$$

so that both components of $(M \times [0,1])^{(l)}$ are immersed in $(\mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R})^{(l)}$, as required. Moreover, as ι was a neat immersion and the arc $\gamma_{u,v}$ meets the $\mathbb{R}$–axis perpendicularly, ι' satisfies the final condition of Definition 2.2, so that ι' is a neat immersion, as required. □

We now provide several choices of framing for ι'.

Lemma 3.3 *The framings on the boundary given by*

Version (1) $\varphi_1\colon M \times \{0,1\} \times [-\varepsilon,\varepsilon]^{\boldsymbol{A}} \times [-\varepsilon,\varepsilon] \looparrowright \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R}$,

$$\varphi_1(p,0,\boldsymbol{q},r) = (\varphi(p,\boldsymbol{q}), 0, r + v(u-1)),$$
$$\varphi_1(p,1,\boldsymbol{q},r) = (\varphi(p,\boldsymbol{q}), 0, -r + v(u+1)),$$

Version (2) $\varphi_2\colon M \times \{0,1\} \times [-\varepsilon,\varepsilon]^{\boldsymbol{A}} \times [-\varepsilon,\varepsilon] \looparrowright \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R}$,

$$\varphi_2(p,0,\boldsymbol{q},r) = (\varphi(p,\boldsymbol{q}), 0, -r + v(u-1)),$$
$$\varphi_2(p,1,\boldsymbol{q},r) = (-\varphi(p,\boldsymbol{q}), 0, -r + v(u+1)),$$

extend, respectively, to the following different framings for the whole of ι':

$$\varphi_1^s, \varphi_1^n \varphi_2^s, \varphi_2^n\colon M \times [0,1] \times [-\varepsilon,\varepsilon]^{\boldsymbol{A}+1} \looparrowright \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R}.$$

Proof To build the required framing φ_1^s, take the immersion

$$\varphi' \colon M \times [0,1] \times [-\varepsilon, \varepsilon]^{\boldsymbol{A}} \looparrowright \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+ \times \mathbb{R}, \quad (p, t, \boldsymbol{q}) \mapsto (\varphi(p, \boldsymbol{q}), \gamma_{u,v}(t)),$$

and extend to an immersion φ_1^s of $M \times [0,1] \times [-\varepsilon, \varepsilon]^{\boldsymbol{A}} \times [-\varepsilon, \varepsilon]$ by simply sending the new framing coordinate to a normal neighbourhood to $\gamma_{u,v}$ in $\mathbb{R}_+ \times \mathbb{R}$ in such a way that on the boundary we have φ_1 as required.

To build the required framing $\varphi_1^{\boldsymbol{n}}$, we add an extra twist into the framing along the interval. To be more precise, note that $\mathbb{E}^{\boldsymbol{d}}$ has one coordinate direction $\mathbb{R}$ which is flipped by the reflection R. We will refer to this coordinate as e_1, the $\mathbb{R}_+$ coordinate as $\overline{e}$, and to the last coordinate as e_2. For a fixed $p \in M$ the curve $\alpha(t) := \iota'(p, t)$ has at $t = \frac{1}{2}$ its tangent vector in the e_2 coordinate. We now change the framing φ_1^s to $\varphi_1^{\boldsymbol{n}}$ near $t = \frac{1}{2}$ by rotating 360° around the e_2–axis; compare Figure 1, left.

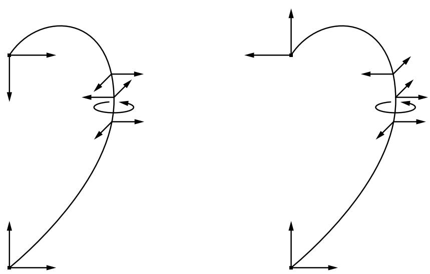


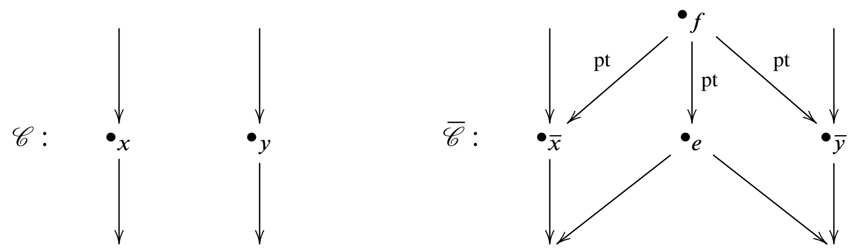
Figure 1: A framing of the path α using a full rotation on the left and half a rotation on the right

To extend the framing φ_2 we use the same coordinates e_1, $\overline{e}$ and e_2, but only rotate 180° around the e_2–axis near $t = \frac{1}{2}$ as in Figure 1, right. Since this is only a half rotation, it matters in which direction we rotate. We declare φ_2^s to be the framing where the first vector starting with e_1 points halfway into the $\overline{e}$–direction, and $\varphi_2^{\boldsymbol{n}}$ to be the framing where the first vector starting with e_1 points halfway into the $-\overline{e}$–direction. □

Remark 3.4 If M is a point which is framed as $\varepsilon \in \{+, -\}$, then φ_1 describes a framing $(\varepsilon e_1, e_2)$ at $t = 0$ and a framing $(\varepsilon e_1, -e_2)$ at $t = 1$, while φ_2 describes a framing $(\varepsilon e_1, -e_2)$ at $t = 0$ and a framing $(-\varepsilon e_1, -e_2)$ at $t = 1$. The framings φ_1^s, $\varphi_1^{\boldsymbol{n}}$, φ_2^s and $\varphi_2^{\boldsymbol{n}}$ describe all the possible framings of an interval up to deformations, with the given fixed endpoints. Given the choice of the *coherent system of paths* described in [11, Lemma 3.1], one can check that φ_1^s describes the standard path if and only if $\varepsilon = -$, while φ_2^s describes the standard path if and only if $\varepsilon = +$.

3.2 The intermediate flow category

We return to the framed flow category $(\mathscr{C}, \iota, \varphi)$ with respect to an $(n+1)$–tuple $(d_k, \dots, d_{n+k})$ of nonnegative integers. Suppose x and y are objects of $\mathscr{C}$ with grading i. We now add the objects e and f to $\mathscr{C}$, along with some new moduli spaces. Schematically, the old and the new morphisms are given by the following diagrams (with grading indicated by horizontal levels):



Specifically, the morphisms and moduli spaces are given as follows:

Definition 3.5 The *intermediate flow category* $\overline{\mathscr{C}}$ is the flow category with objects

$$\mathrm{Ob}(\overline{\mathscr{C}}) = \{\overline{a} \mid a \in \mathrm{Ob}(\mathscr{C})\} \cup \{e, f\},$$

where $|e| = i$ and $|f| = i+1$, and whose moduli spaces are given by

$$\mathcal{M}(e, \overline{b}) = \mathcal{M}(x, b) \sqcup \mathcal{M}(y, b),$$
$$\mathcal{M}(f, \overline{b}) = \begin{cases} \text{pt} & \text{if } \overline{b} = \overline{x}, e, \overline{y}, \\ \mathcal{M}(e, \overline{b}) \times [0,1] & \text{otherwise.} \end{cases}$$

We have $\mathcal{M}(\overline{a}, e) = \varnothing = \mathcal{M}(\overline{a}, f)$ and, in all other cases, $\mathcal{M}(\overline{a}, \overline{b}) = \mathcal{M}(a, b)$.

It will be helpful for the reader in understanding subsequent arguments to note that for the moduli spaces

$$\mathcal{M}(f, \overline{b}) = \mathcal{M}(e, \overline{b}) \times [0,1] = (\mathcal{M}(x, b) \times [0,1]) \sqcup (\mathcal{M}(y, b) \times [0,1]),$$

we have in mind the boundary identifications

$$\mathcal{M}(x, b) \times \{0\} = \mathcal{M}(e, b) \times \mathcal{M}(f, e),$$
$$\mathcal{M}(x, b) \times \{1\} = \mathcal{M}(x, b) \times \mathcal{M}(f, x),$$
$$\mathcal{M}(y, b) \times \{0\} = \mathcal{M}(y, b) \times \mathcal{M}(f, y),$$
$$\mathcal{M}(y, b) \times \{1\} = \mathcal{M}(e, b) \times \mathcal{M}(f, e).$$

Remark 3.6 Because of the various possible framing extensions in Lemma 3.3, it is possible to choose different framings for $\mathcal{M}(f,\overline{b})$. Different framing choices will not affect the stable homotopy type, which remains the stable homotopy type of $\mathscr{C}$, but they will lead to different formulae in Proposition 3.11 below. We will now make specific framing choices which we will carry with us throughout the paper, and will thus determine the values in Proposition 3.11.

We need to be consistent with our choice when we vary the objects $\overline{b}$. However, since $\mathcal{M}(f,\overline{b})$ is the disjoint union of $\mathcal{M}(x,b)\times[0,1]$ and $\mathcal{M}(y,b)\times[0,1]$, it is enough to be consistent on each part. In particular, the framing choice we make on $\mathcal{M}(x,b)\times[0,1]$ will be different to the choice made on $\mathcal{M}(y,b)\times[0,1]$.

We now present a neat immersion and two choices of framing for $\overline{\mathscr{C}}$.

Proposition 3.7 *There exists a neat immersion and two framings* $(\overline{\iota},\overline{\varphi}(+))$ *and* $(\overline{\iota},\overline{\varphi}(-))$ *of* $\overline{\mathscr{C}}$*, relative to some tuple* $\boldsymbol{d}$*, which extend the neat embedding and framing* (ι,φ) *of* $\mathscr{C}$*.*

Proof We will deal with the $(\overline{\iota},\overline{\varphi}(+))$ version first and will suppress the "+" from the notation of $\overline{\varphi}(+)$ while we work on this case.

When $\overline{a}\neq e,f$, define the embeddings and framings $\overline{\iota}$ and $\overline{\varphi}$ of $\mathcal{M}(\overline{a},\overline{b})$ by the ι and φ as in $\mathscr{C}$.

Possibly after increasing the entry d_i of the tuple $(d_k,\dots,d_{n+k})$, we can and will assume that $d_i>0$. Call the possibly increased tuple $\boldsymbol{d}$. For ε small, define the embeddings $\overline{\iota}_{f,\overline{x}}$, $\overline{\iota}_{f,e}$ and $\overline{\iota}_{f,\overline{y}}$ as the points

$$(4\varepsilon,0,\dots,0),\ (0,0,\dots,0),\ (-4\varepsilon,0,\dots,0)\in\mathbb{R}^{d_i},$$

respectively. Extend these embeddings to framed embeddings $\overline{\varphi}_{f,\overline{x}}$, $\overline{\varphi}_{f,e}$ and $\overline{\varphi}_{f,\overline{y}}$ in the obvious way, by taking the product of ε neighbourhoods in each coordinate of $\mathbb{R}^{d_i}$. There are exactly two ways to frame an embedded point in any Euclidean space, which we will denote by "$+$" and "$-$". Declare $\overline{\varphi}_{f,\overline{x}}$ and $\overline{\varphi}_{f,\overline{y}}$ to both be $+$ framings and $\overline{\varphi}_{f,e}$ to be a $-$.

Fixing an object $\overline{b}$ with $|\overline{b}|<i$, we will now immerse and frame

$$\mathcal{M}(e,\overline{b})=\mathcal{M}(x,b)\sqcup\mathcal{M}(y,b).$$

To do this we will simply superimpose the embeddings and framings given by the ι and φ coming from $\mathscr{C}$. Note that this might introduce immersion points. Specifically, define the neat immersion

$$\bar{\iota}_{e,\bar{b}}\colon \mathcal{M}(e,\bar{b}) \looparrowright \mathbb{E}^{\boldsymbol{d}}[|\bar{b}|:|e|], \quad p \mapsto \begin{cases} \iota_{x,b}(p) & \text{if } p \in \mathcal{M}(x,b), \\ \iota_{y,b}(p) & \text{if } p \in \mathcal{M}(y,b). \end{cases}$$

And similarly, for $\varphi_{x,b}$ and $\varphi_{y,b}$, we obtain the framing $\bar{\varphi}_{e,\bar{b}}$.

We will now immerse and frame $\mathcal{M}(f,\bar{b})$, where $\bar{b} \neq \bar{x}, \bar{y}, e$. First apply the construction in Section 3.1, with version (1) of Lemma 3.3, to $\iota_{x,b}$. In the notation of that subsection set $(u,v) = (2\varepsilon, 2\varepsilon)$, and we obtain neat embeddings $\iota'_{x,b}$, $\varphi'_{x,b}$ and $\varphi^{n}_{x,b}$, each with codomain $\mathbb{E}^{\boldsymbol{d}}[|b|:i] \times \mathbb{R}_+ \times \mathbb{R}$ (recall as well that the construction in Section 3.1 potentially increased the d_{i-1} by 1 in order that the reflection R had the correct properties). Similarly, apply Section 3.1, with version (1) of Lemma 3.3 to $\iota_{y,b}$. In the notation of that subsection set $(u,v) = (-2\varepsilon, 2\varepsilon)$ and we obtain neat embeddings $\iota'_{y,b}$, $\varphi'_{y,b}$ and $\varphi^{s}_{y,b}$, each with codomain $\mathbb{E}^{\boldsymbol{d}}[|b|:i] \times \mathbb{R}_+ \times \mathbb{R}$. Notice that we use φ^{n}_{1} for (x,b), while we use φ^{s}_{1} for (y,b).

In order to use these functions to define a neat immersion $\bar{\iota}_{f,\bar{b}}$, we will need to increase the codomain of $\iota'_{x,b}$ and $\iota'_{y,b}$ by a factor of $\mathbb{R}^{d_i-1}$. We do so by simply setting the new $d_i - 1$ coordinates in the codomain of each neat embedding to 0. In an abuse of notation, use the same symbols for the new functions — ie from now on write

$$\iota'_{x,b}\colon \mathcal{M}(x,b) \times [0,1] \looparrowright \mathbb{E}^{\boldsymbol{d}}[|b|:|x|] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{d_i-1} = \mathbb{E}^{\boldsymbol{d}}[|\bar{b}|:|f|],$$
$$\iota'_{y,b}\colon \mathcal{M}(y,b) \times [0,1] \looparrowright \mathbb{E}^{\boldsymbol{d}}[|b|:|y|] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{d_i-1} = \mathbb{E}^{\boldsymbol{d}}[|\bar{b}|:|f|].$$

Using this, define a neat immersion of $\mathcal{M}(f,\bar{b})$,

$$\bar{\iota}_{f,\bar{b}}\colon \mathcal{M}(e,\bar{b}) \times [0,1] \looparrowright \mathbb{E}^{\boldsymbol{d}}[|\bar{b}|:|f|], \quad (p,q) \mapsto \begin{cases} \iota'_{x,b}(p,q) & \text{if } p \in \mathcal{M}(x,b), \\ \iota'_{y,b}(p,q) & \text{if } p \in \mathcal{M}(y,b). \end{cases}$$

To build a framing $\bar{\varphi}_{f,\bar{b}}$, we will need to modify *both* the domain and codomain of the framings $\varphi^{n}_{x,b}$ and $\varphi^{s}_{y,b}$. Specifically, we must extend the domains of these framings by a factor of $[-\varepsilon,\varepsilon]^{d_i-1}$, and we must extend the codomains by a factor of $\mathbb{R}^{d_i-1}$. We do this by simply extending $\varphi''_{x,b}$ and $\varphi''_{y,b}$ by the obvious inclusion $[-\varepsilon,\varepsilon]^{d_i-1} \subset \mathbb{R}^{d_i-1}$. Again we abuse notation and use the same symbols for the extended functions:

$$\varphi^{n}_{x,b}\colon \mathcal{M}(x,b) \times [0,1] \times [-\varepsilon,\varepsilon]^{\boldsymbol{A}} \looparrowright \mathbb{E}^{\boldsymbol{d}}[|b|:|x|] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{d_i-1},$$
$$\varphi^{s}_{y,b}\colon \mathcal{M}(y,b) \times [0,1] \times [-\varepsilon,\varepsilon]^{\boldsymbol{A}} \looparrowright \mathbb{E}^{\boldsymbol{d}}[|b|:|y|] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^{d_i-1},$$

where $A = d_{|\overline{b}|} + \cdots + d_{|f|-1}$. So, at last, we have a neat immersion

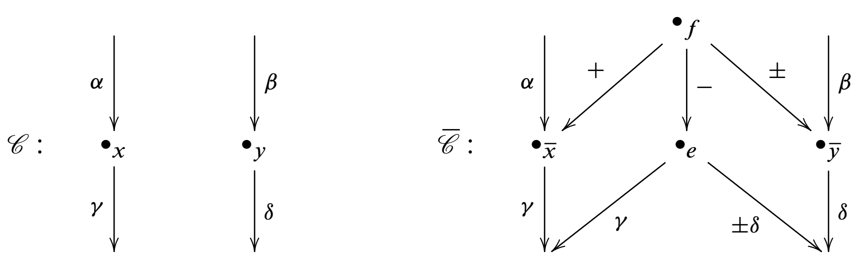
$$\overline{\varphi}_{f,\overline{b}}\colon \mathcal{M}(e,\overline{b}) \times [0,1] \times [-\varepsilon,\varepsilon]^{A} \looparrowright \mathbb{E}^{\boldsymbol{d}}[|\overline{b}| : |f|],$$
$$(p,q,\boldsymbol{t}) \mapsto \begin{cases} \varphi^{s}_{x,b}(p,q.\boldsymbol{t}) & \text{if } p \in \mathcal{M}(x,b), \\ \varphi^{n}_{y,b}(p,q,\boldsymbol{t}) & \text{if } p \in \mathcal{M}(y,b). \end{cases}$$

Note that the use of the construction from Section 3.1 is consistent with our choice of $+$ and $-$ framings $\overline{\varphi}_{f,\overline{x}}$, $\overline{\varphi}_{f,e}$ and $\overline{\varphi}_{f,\overline{y}}$ from earlier as points at either end of a sweeping arc had opposite framings. The $(\overline{\iota},\overline{\varphi}(+))$ case is completed.

Now we must describe the required modifications to work on the $(\overline{\iota},\overline{\varphi}(-))$ case. We will suppress the "$-$" notation as we work on this case.

The first difference from the $(\overline{\iota},\overline{\varphi}(+))$ case is that we declare $\overline{\varphi}_{f,\overline{x}}$ be a $+$ framing and both of $\overline{\varphi}_{f,\overline{y}}$ and $\overline{\varphi}_{f,e}$ to be $-$ framings. This, in turn, means that for each object b, we will use the framing φ^{s}_{2} from version (2) of Lemma 3.3 when we come to embed and frame the $\mathcal{M}(y,b)\times[0,1]$ component of $\mathcal{M}(f,\overline{b})$. In order to get into framing version (2) of this construction, we must endow the $\mathcal{M}(y,b)$ component of $\mathcal{M}(e,\overline{b})$ with the framing obtained by taking the image of $\varphi_{y,b}$ in $\mathbb{E}^{\boldsymbol{d}}[|b| : |y|]$ and performing the reflection R in the plane through the origin and perpendicular to the first coordinate axis of the $\mathbb{R}^{d_{i-1}}$ factor. Other than these modifications, the construction of the immersions and framings in the $(\overline{\iota},\overline{\varphi}(-))$ case proceeds exactly as the $(\overline{\iota},\overline{\varphi}(+))$ case. □

The $(\overline{\iota},\overline{\varphi}(\pm))$ cases are illustrated in the following diagram, where α, β, γ and δ denote framings and $-\delta$ represents the effect of reflecting δ by R as described above:



Proposition 3.8 *Cancelling e against f in $(\overline{\mathscr{C}},\overline{\iota},\overline{\varphi}(\pm))$, using the handle cancellation technique of* [7, Section 2.3]*, results in precisely $(\mathscr{C},\iota,\varphi)$.*

Proof Write $(\mathscr{C}_H,\iota_H,\varphi_H)$ for the result of cancelling e against f as described in [7, Section 2.3]. According to [7, Definition 2.9], the objects of $\mathscr{C}$ and of $\mathscr{C}_H$ are in one-to-one correspondence. Moreover, recall from [7, Section 2.3] that if a and b are objects of $\mathscr{C}$, then the moduli spaces of $\mathscr{C}_H$ are given by

$$\mathcal{M}(a,b) = \mathcal{M}(\overline{a},\overline{b}) \cup_g (\mathcal{M}(f,\overline{b}) \times \mathcal{M}(\overline{a},e)),$$

where g is a certain identification described generally in [7, Section 2.3]. However, when a is an object of $\mathscr{C}$, there are no nonempty moduli spaces $\mathcal{M}(\overline{a},e)$, so in fact the moduli spaces of $\mathscr{C}_H$ are the same as those of $\mathscr{C}$. This simplification, carried through the construction in [7, Theorem 2.17], actually results in a precise agreement of embeddings and framings, so that in fact $(\mathscr{C},\iota,\varphi) = (\mathscr{C}_H,\iota_H,\varphi_H)$. □

3.3 Handle sliding in framed flow categories

Definition 3.9 The framed flow category $(\mathscr{C}_S,\iota_S,\varphi_S(\pm))$, called the *effect of* $(\pm)$*–sliding x over y in* $(\mathscr{C},\iota,\varphi)$, is defined as the result of cancelling f against $\overline{x}$ in $(\overline{\mathscr{C}},\overline{\iota},\overline{\varphi}(\pm))$. We will often suppress the "$+$" and "$-$" from the notation.

The following lemma is a straightforward consequence of the constructions so far and the definition of handle cancellation given in [7, Definition 2.9].

Lemma 3.10 *The framed flow category $\mathscr{C}_S$ has objects*

$$\mathrm{Ob}(\mathscr{C}_S) = \{a' \mid a \in \mathrm{Ob}(\mathscr{C})\},$$

and moduli spaces given by

$$\begin{aligned}\mathcal{M}(x',b') &= \mathcal{M}(x,b) \sqcup \mathcal{M}(y,b),\\ \mathcal{M}(a',y') &= \mathcal{M}(a,x) \sqcup \mathcal{M}(a,y),\\ \mathcal{M}(a',x') &= \mathcal{M}(a,x).\end{aligned}$$

In all other cases,

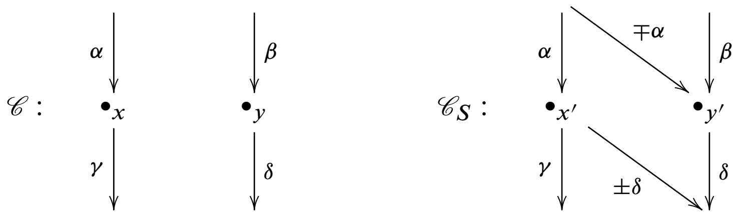
$$\mathcal{M}(a',b') = \mathcal{M}(a,b) \sqcup (\mathcal{M}(y,b) \times [0,1] \times \mathcal{M}(a,x)).$$

Proof The main thing to note is that after we cancel f against $\overline{x}$ in the intermediate flow category, we are left with object set $\{\overline{a} \mid a \in \mathrm{Ob}(\mathscr{C}) \setminus \{x\}\} \cup \{e\}$. Hence, the object $x' \in \mathrm{Ob}(\mathscr{C}_S)$ in the statement of the lemma is a relabelling of e. The rest of the lemma follows directly from [7, Definition 2.9]. Note that in the final cases of the lemma's statement, the formulation in [7, Definition 2.9] gives

$$\mathcal{M}(a',b') = \mathcal{M}(a,b) \cup_g \big((\mathcal{M}(x,b) \sqcup \mathcal{M}(y,b)) \times [0,1] \times \mathcal{M}(a,x)\big),$$

where g indicates that $\mathcal{M}(x,b) \times [0,1] \times \mathcal{M}(a,x)$ is glued to $\mathcal{M}(x,b) \times \mathcal{M}(a,x) \subset \mathcal{M}(a,b)$ along the $\mathcal{M}(x,b) \times \{1\} \times \mathcal{M}(a,x)$ end. The reader can compare with the comment after Definition 3.5 to see why it should be this end in particular. To obtain the form in the lemma's statement, we have contracted this cylinder. □

The following is a schematic for sliding x over y, where the framings determined by a $(\pm)$–slide are also shown:



The signs are calculated by following the construction through the handle cancellation in [7, Definition 2.9].

Now we are in a position to deduce the main result of this section.

Proof of Theorem 3.1 Define the intermediate framed flow category $(\overline{\mathscr{C}}, \bar{\iota}, \bar{\varphi}(\pm))$ as in Definition 3.5 and Proposition 3.7. By Proposition 3.8, the framed flow category obtained from cancelling e against f is precisely $(\mathscr{C}, \iota, \varphi)$. By sufficiently increasing $\boldsymbol{d}$ to some $\bar{\boldsymbol{d}}$, we may perturb the framed neat immersions $(\bar{\iota}, \bar{\varphi}(\pm))$ and (ι_S, φ_S) to framed neat embeddings and hence geometrically realise the CW complexes $|\overline{\mathscr{C}}|$ and $|\mathscr{C}_S|$. It is immediately clear from [7, Theorem 2.17] that there are homotopy equivalences

$$\mathcal{X}(\mathscr{C}) \simeq \mathcal{X}(\overline{\mathscr{C}}) \simeq \mathcal{X}(\mathscr{C}_S).$$ □

3.4 Framings of 1–dimensional moduli spaces

In order to compute cohomology operations such as Sq^2 we need to understand how the framings of the 1–dimensional moduli spaces behave after a handle slide. If we keep in mind the cancellation from the intermediate flow category $\overline{\mathscr{C}}$ to $\mathscr{C}_S$, we get

$$\mathcal{M}(a', x') = \mathcal{M}(f, e) \times \mathcal{M}(a, x),$$
$$\mathcal{M}(a', y') = \mathcal{M}(a, y) \sqcup (\mathcal{M}(f, y) \times \mathcal{M}(a, x)).$$

Furthermore,

$$\mathcal{M}(x', b') = \mathcal{M}(x, b) \sqcup \mathcal{M}(y, b),$$

but with the framing of $\mathcal{M}(y, b)$ depending on whether we do a $(+)$– or a $(-)$–handle slide.

Since $\mathcal{M}(f,e)$ is a negatively framed point, we get that the framing on $\mathcal{M}(a',x')$ corresponds to the old framing on $\mathcal{M}(a,x)$ by the framings in the cancelled category; compare [7]. Similarly, if we use the $(-)$–handle slide the framings on $\mathcal{M}(a',y')$ agree with the original framings on $\mathcal{M}(a,y)$ and $\mathcal{M}(a,x)$. In the case of the $(+)$–handle slide, however, the copy of $\mathcal{M}(a,x)$ in $\mathcal{M}(a',y')$ has an extra reflection in one coordinate.

Recall that interval components J in $\mathcal{M}(a,b)$ with $|a|=|b|+2$ can be framed in two ways (up to fixed boundary framing). The standard framings are described in [7, Section 3.2; 11, Section 3], and we write $\mathrm{fr}(J)\in\mathbb{Z}/2\mathbb{Z}$ with $\mathrm{fr}(J)=0$ if the framing corresponds to the standard choice, and $\mathrm{fr}(J)=1$ if not.

Similarly, if $|a|=|b|+1$ and $A\in\mathcal{M}(a,b)$ is a point, we write $\varepsilon_A\in\mathbb{Z}/2\mathbb{Z}$ for the framing sign of this point.

Proposition 3.11 *Let $\mathscr{C}$ be a framed flow category containing two objects x and y with the same grading, and let $\mathscr{C}_S$ be the framed flow category obtained from $\mathscr{C}$ by sliding x over y. Let a and b be objects in $\mathscr{C}$ with $|a|=|b|+2$.*

(1) *If $|a|=|x|+2$, then $\mathcal{M}(a',b')=\mathcal{M}(a,b)$ for all $b\neq y$, with the same framing. Furthermore,*

$$\mathcal{M}(a',y')=\mathcal{M}(a,y)\sqcup\mathcal{M}(a,x),$$

with components from $\mathcal{M}(a,y)$ identically framed. If $J\subset\mathcal{M}(a,x)$ is an interval component, write J' for the same component when viewed as a subset of $\mathcal{M}(a',y')$. Then:

(a) *If $\mathscr{C}_S$ is obtained by a $(-)$–handle slide, then*

$$\mathrm{fr}(J')=\mathrm{fr}(J).$$

(b) *If $\mathscr{C}_S$ is obtained by a $(+)$–handle slide, then*

$$\mathrm{fr}(J')=1+\mathrm{fr}(J).$$

(2) *If $|a|=|x|+1$, then*

$$\mathcal{M}(a',b')=\mathcal{M}(a,b)\sqcup\mathcal{M}(y,b)\times[0,1]\times\mathcal{M}(a,x)$$

and every component of $\mathcal{M}(a,b)$ is framed the same way. For each pair of points $(B,A)\in\mathcal{M}(y,b)\times\mathcal{M}(a,x)$, write $I_{B,A}$ the corresponding interval component in $\mathcal{M}(a',b')$.

(a) *If $\mathscr{C}_S$ is obtained by a $(-)$–handle slide, then*

$$\operatorname{fr}(I_{B,A}) = \varepsilon_B.$$

(b) *If $\mathscr{C}_S$ is obtained by a $(+)$–handle slide, then*

$$\operatorname{fr}(I_{B,A}) = 1 + \varepsilon_B.$$

(3) *If $|a| = |x|$ and $a \neq x$, then the framings of components in $\mathcal{M}(a', b')$ are the same as in $\mathcal{M}(a, b)$. For*

$$\mathcal{M}(x', b') = \mathcal{M}(x, b) \sqcup \mathcal{M}(y, b),$$

the framing values in the new moduli space agree with the framing values in the old moduli spaces.

Proof Assume $|a| = |x| + 2$. In $\mathscr{C}_S$ the object x' corresponds to e from the intermediate flow category, and the intervals in $\mathcal{M}(\overline{a}, e)$ are of the form $\{P\} \times J$, where $P \in \mathcal{M}(f, e)$ from the intermediate flow category with $\varepsilon_P = 1$, and J is an interval in the original $\mathcal{M}(a, x)$. By [7, Proposition 3.6.6], $\operatorname{fr}(\{P\} \times J) = \operatorname{fr}(J) + 1 + \varepsilon_P$. This means $\operatorname{fr}(\{P\} \times J) = \operatorname{fr}(J)$.

For $\mathcal{M}(a', y')$ the intervals in $\mathcal{M}(a, y)$ do not change, but we also get intervals of the form $\{B\} \times J$ with $B \in \mathcal{M}(f, \overline{y})$ and J in $\mathcal{M}(a, x)$. As before, $\operatorname{fr}(\{B\} \times J) = \operatorname{fr}(J) + 1 + \varepsilon_B$. Now if we use a $(-)$–slide, $\varepsilon_B = 1$, and, if we use a $(+)$–handle slide, $\varepsilon_B = 0$. Hence we get (1).

Now assume $|a| = |x| + 1$. In the intermediate flow category we get intervals J_C and J_B in $\mathcal{M}(f, \overline{b})$ coming from points $C \in \mathcal{M}(x, b)$ and $B \in \mathcal{M}(y, b)$. Because of our framing choices in Proposition 3.7 we get $\operatorname{fr}(J_C) = \varepsilon_C$; compare Remark 3.4. When passing to $\mathscr{C}_S$ the intervals $J_C \times \{A\}$ give a collar neighbourhood to an interval in $\mathcal{M}(a, b)$ with endpoint $(C, A) \in \mathcal{M}(x, b) \times \mathcal{M}(a, x)$. By [7, Propositions 3.7.5 and 3.7.6] the framing of the original interval does not change.

The intervals J_B are framed depending on whether we have a $(+)$– or a $(-)$–slide. In the case of a $(+)$–slide recall that we use φ_1^s of version (1) in Lemma 3.3. By Remark 3.4 we get $\operatorname{fr}(J_B) = 1 + \varepsilon_B$. Passing to $\mathscr{C}_S$ gives intervals $I_{B,A}$ for each $A \in \mathcal{M}(a, x)$ with the same framing value as J_B by [7, Proposition 3.7.6].

If we perform a $(-)$–slide, we use φ_2^s of version (2) in Lemma 3.3. Note at time 0 we have a framing $\overline{\varepsilon_B}$, and by Remark 3.4 we get the standard framing on the path if and only if the first vector of the framing points in the positive direction, that is, when $\varepsilon_B = 0$. This implies $\mathrm{fr}(I_{B,A}) = \varepsilon_B$.

Finally, assume that $|a| = |x|$. If $a \neq x$, then $\mathcal{M}(a', b') = \mathcal{M}(a, b)$ and the framings do not change. If $a = x$, we get $\mathcal{M}(x', b') = \mathcal{M}(x, b) \sqcup \mathcal{M}(y, b)$ and the framing of $\mathcal{M}(x, b)$ does not change. If we perform a $(+)$–slide, the framing of $\mathcal{M}(y, b)$ does not change either. In the case of a $(-)$–slide, the first coordinate of $\mathbb{R}^{d_{i-1}}$ is reflected. This means that the second coordinate of the framing of an interval is changing sign. However, flipping the sign of the second coordinate maps standard paths to standard paths; compare also [11, Lemma 3.3], although only half of this statement is proven there. Hence the framing does not change either. □

4 The extended Whitney trick in framed flow categories

In [8, Theorem 3.1.5], Laures extends the Pontryagin–Thom theorem to the setting of manifolds with corners using a suitably defined framed cobordism category. As the cell attachment maps in the Cohen–Jones–Segal construction are defined by a version of the Pontryagin–Thom collapse map, the homotopy classification of these attaching maps (hence the eventual homeomorphism type of the CW complex) is closely related to the framed cobordism classes of the moduli spaces in a flow category.

In this section we will reencode this relationship by defining a new technique for modifying framed flow categories, called *the extended Whitney trick* (compare with the Whitney trick of [7, Section 1]). Suppose $(\mathscr{C}, \iota, \varphi)$ is a framed flow category and, for some objects $x \neq y$ of $\mathscr{C}$, that $\mathcal{M}(x, y) = M$ is a manifold with boundary. Let W be a framed cobordism (rel boundary) between M and another manifold with boundary M'. We will show how to use W to define a new framed flow category $(\mathscr{C}_W, \iota_W, \varphi_W)$ in which M is replaced by M' (and the other moduli spaces in $\mathscr{C}$ are modified appropriately). Using the ideas of the Pontryagin–Thom theorem we will deduce the following:

Theorem 4.1 *If $(\mathscr{C}_W, \iota_W, \varphi_W)$ is the result of an* ***extended Whitney trick*** *in $(\mathscr{C}, \iota, \varphi)$, then there is a stable homotopy equivalence*

$$\mathcal{X}(\mathscr{C}_W) \simeq \mathcal{X}(\mathscr{C}).$$

4.1 Framed cobordism of manifolds with corners rel boundary

We will not need Laures's full machinery [8] of cobordism of manifolds with corners in the sequel. The eventual complicated interactions between the moduli spaces in a framed flow category mean that allowing an unrestricted framed cobordism in the sense of [8] becomes intractable. Instead we will work with the following:

Definition 4.2 Suppose M and M' are m–dimensional $\langle n\rangle$–manifolds with $\partial_i M = \partial_i M'$ for $i = 1, \dots, n$. An $(m+1)$–dimensional $\langle n+1\rangle$–manifold W is called a *cobordism rel boundary* between M and M' if $\partial_{n+1} W = M \sqcup M'$ and $\partial_i W = \partial_i M \times [0,1]$ for $i \neq n+1$.

Suppose an embedding $\tilde{\imath}\colon W \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times [0,1]$ of a cobordism rel boundary meets $\mathbb{E}^{\boldsymbol{d}} \times \{0,1\}$ orthogonally in $M \sqcup M'$ and induces neat embeddings

$$\tilde{\imath}|_M\colon M \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times \{1\}, \quad \tilde{\imath}|_{M'}\colon M' \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times \{0\}.$$

Then $\tilde{\imath}$ is an *embedded cobordism rel boundary* between the neat embeddings $(M', \tilde{\imath}|_{M'})$ and $(M, \tilde{\imath}|_M)$. Suppose furthermore that there exists a framing $\widetilde{\varphi}$ of such a $(W, \tilde{\imath})$ and that $\widetilde{\varphi}$ meets $\mathbb{E}^{\boldsymbol{d}} \times \{0,1\}$ orthogonally. Then the framing $\widetilde{\varphi}$ determines framings φ and φ' of $(M, \tilde{\imath}|_M)$ and $(M', \tilde{\imath}|_{M'})$, respectively. If such a $(W, \tilde{\imath}, \widetilde{\varphi})$ exists, it is called a *framed cobordism rel boundary* between $(M', \tilde{\imath}|_{M'}, \varphi')$ and $(M, \tilde{\imath}|_M, \varphi)$. $(M', \tilde{\imath}|_{M'}, \varphi')$ and $(M, \tilde{\imath}|_M, \varphi)$ are called *framed cobordant rel boundary* if there exists a framed cobordism rel boundary between them, possibly after enlarging the $\boldsymbol{d} \leq \boldsymbol{d}'$.

Remark 4.3 Examples of framed cobordisms rel boundary $W \subset \mathbb{E}^{\boldsymbol{d}'} \times [0,1]$ whose framed boundary $M \sqcup M'$ can be framed embedded in a smaller space $\mathbb{E}^{\boldsymbol{d}}$ are fairly common. This is why we have allowed the possibility of enlarging the ambient space in the final definition above. For instance, the standard generator of the framed cobordism group of closed 1–manifolds $\Omega_1^{\mathrm{fr}} \cong \mathbb{Z}/2\mathbb{Z}$ can be embedded, along with a framed normal neighbourhood, as $\varphi\colon S^1 \times D^2 \hookrightarrow \mathbb{R}^3$. But the minimum embedding dimension for a framed nullcobordism of $\varphi \sqcup \varphi$ is 5.

4.2 Pushing M out of the corner of X

We now make a slight digression into an easy but technical construction we will need later.

Suppose for this subsection that $(W, \tilde{\imath}, \widetilde{\varphi})$ is a framed embedded cobordism rel boundary between m–dimensional $\langle m\rangle$–manifolds (M', ι', φ') and (M, ι, φ), where ι and ι' are with respect to some $\boldsymbol{d} = (d_0, d_1, \dots, d_m)$. Denote by $U_\eta \subset (\mathbb{R}_+)^N$ the open ball at the origin with (small) radius $\eta > 0$, by $\overline{U}_\eta$ the corresponding closed ball and by

$H_\eta = \bar{U}_\eta \setminus U_\eta$. Suppose that

$$\iota_X\colon X \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times (\mathbb{R}_+)^N$$

is a neat embedding of an $(m+N)$–dimensional $\langle m+N\rangle$–manifold, with framing (X, ι_X, φ_X). Moreover, suppose that near the "corner" $\mathbb{E}^{\boldsymbol{d}} \times \mathbf{0}$ this embedding is

$$\iota_X(X) \cap \mathbb{E}^{\boldsymbol{d}} \times U_{2\eta} = \iota(M) \times U_{2\eta}$$

and that here the framing φ_X agrees with the framing

$$M \times U_{2\eta} \times [-\varepsilon, \varepsilon]^A \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times U_{2\eta}, \quad (p, q, \boldsymbol{t}) \mapsto (\varphi(p, \boldsymbol{t}), q),$$

where $A = d_0 + \cdots + d_m$. In particular, M forms part of the codimension-N boundary of X and the corresponding neighbourhood of $M \subset X$ in the corresponding corner of Euclidean space is a standard framed collar (in the precise sense we just described).

Later on we will need a mechanism to "push M out of the corner of X" and replace it with M'. Roughly speaking, this is achieved by glueing together the two spaces

$$X' = \iota_X(X) \setminus (M \times U_\eta), \quad Y = (M' \times \bar{U}_\eta) \cup_{M' \times H_\eta} \big((W \setminus (M' \times [0, \eta))) \times H_\eta\big),$$

and then embedding the result appropriately (compare Figure 2). But to ensure the smooth structures can be made to agree, we will need to use the standard technique of overlaying open collar neighbourhoods. Indeed, the concern for getting this technical detail right is the whole purpose of this technical digression.

Lemma 4.4 *With notation as above, the topological $(m+N)$–dimensional manifold with corners $X' \cup_{M \times H_\eta} Y$ is homeomorphic to a smooth $(m+N)$–dimensional $\langle m+N\rangle$–manifold Z, which has a neat embedding $Z \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times (\mathbb{R}_+)^N$ such that M' is in the corner $\mathbb{E}^{\boldsymbol{d}} \times \bar{U}_\eta$. Moreover, the embedding of Z can be framed compatibly with the framings of $(W, \tilde{\iota}, \tilde{\varphi})$ and of (X, ι_X, φ_X).*

In practice, we will only ever make use of the cases $N = 1, 2, 3$ but it is no extra effort to prove Lemma 4.4 in generality. Also note that if M is in a codimension-N corner then it is also in N codimension-$(N-1)$ corners. In fact the restriction of the construction in the proof to some choice of $(\mathbb{R}_+)^{N-1} \subset (\mathbb{R}_+)^N$ precisely reproduces the proof in this lower codimension. We will use this fact later.

Proof The topological manifold with corners Y already carries a smooth structure as the two components that are glued to make Y may be given their respective product

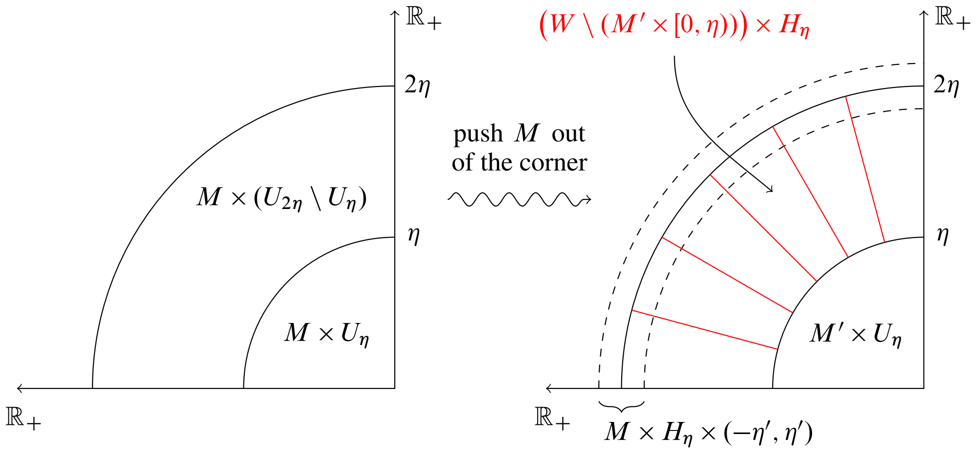


Figure 2: Example of pushing M out of the corner. Here $N = 2$ and $\mathbb{E}^{\boldsymbol{d}}$ is thought of as the direction perpendicular to the page. Each radial line is a copy of W (minus a collar at the M' boundary). Note the outer "shell" $M \times H_\eta \times (-\eta', \eta')$, after the push, could also be considered to be a copy of $M \times H_{2\eta} \times (-\eta', \eta')$ due to the radial dilation in the embedding ι_Y.

smooth structures, and near the glueing boundary $M' \times H_\eta$ these product smooth structures agree precisely with one another.

Any radial direction of $(\mathbb{R}_+)^N$ determines a copy of $\mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+$. We may embed $\iota_Y\colon Y \hookrightarrow \mathbb{E}^{\boldsymbol{d}} \times (\mathbb{R}_+)^N$ so that in any radial direction we have rescaled the embedding $\tilde{\iota}(W) \subset \mathbb{E}^{\boldsymbol{d}} \times [0,1] \subset \mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}_+$ by a factor of η. Choose another small $\eta' \ll \eta$. There is now a collar neighbourhood $\iota_Y(M \times H_\eta) \times (-\eta', 0]$ of the embedded boundary component $M \times H_\eta \subset Y$, where the collar direction is radial in $(\mathbb{R}_+)^N$. Similarly, the boundary component $M \times H_\eta \subset X'$ has a radial collar neighbourhood $M \times H_\eta \times [0, \eta') \subset Y$. Writing $(p,t) \sim \big(p, \frac{1}{2}(1-t)\big)$ for $p \in M \times H_{\eta'}$ and $t \in \big(0, \frac{1}{2}\eta'\big)$, define the identification space

$$Z = X' \sqcup Y/{\sim}.$$

The smooth structures on Y and X' can now be made compatible on this open collar overlay, so that Z has the structure of a smooth $\langle m+N \rangle$–manifold. Moreover, by radially dilating the embedding $\iota_X|_{X'}$ of X' and, combining with the embedding of Y, Z is seen to have a (neat) embedding in $\mathbb{E}^{\boldsymbol{d}} \times \mathbb{R}^N_+$ such that M' is in the corner, as required.

Now we turn to the framings. The framing of the corner embedding of $M' \times \overline{U}_\eta$ is given by extending the framing φ' of $\iota'\colon M' \hookrightarrow \mathbb{E}^{\boldsymbol{d}}$ trivially to the product. Similarly, we may frame the embedding of the product $\big(W \setminus (M' \times [0,\eta))\big) \times H_\eta$ by extending $\widetilde{\varphi}$. Note that these framings together form a framing of the embedded Y. Now, we assumed that φ_X agreed with φ near the corner $\mathbb{E}^{\boldsymbol{d}} \times \mathbf{0}$. Hence, to frame Z, we can simply glue the framing on X' coming from X to the framing on Y as they precisely agree on the collar overlap. □

4.3 The extended Whitney trick

Let $(\mathscr{C}, \iota, \varphi)$ be a framed flow category with respect to $\boldsymbol{d} = (d_k, \dots, d_{n+k})$. Suppose $\mathscr{C}$ has objects x and y with $|x| - |y| = m+1$ and write $(M, \iota, \varphi) = (\mathcal{M}(x,y), \iota_{x,y}, \varphi_{x,y})$. Suppose there is a framed cobordism rel boundary $(W, \tilde{\iota}, \widetilde{\varphi})$ between (M, ι, φ) and some other embedded, framed m–dimensional $\langle m \rangle$–manifold (M', ι', φ'). We will define a new framed flow category $(\mathscr{C}_W, \iota_W, \varphi_W)$ such that there is a homotopy equivalence $\mathcal{X}(\mathscr{C}) \simeq \mathcal{X}(\mathscr{C}_W)$. The definition is essentially a judicious use of cases $N = 1$ and $N = 2$ of Lemma 4.4 in order to push various moduli spaces out of various corners in a mutually compatible way.

Definition 4.5 With $(\mathscr{C}, \iota, \varphi)$ as above we will define $(\mathscr{C}_W, \iota_W, \varphi_W)$. First, we define the object set $\mathrm{Ob}(\mathscr{C}_W) = \{\overline{a} \mid a \in \mathrm{Ob}(\mathscr{C})\}$. The moduli spaces of $\mathscr{C}_W$ are given in cases (1)–(5) as follows:

(1) $\mathcal{M}(\overline{x}, \overline{y}) = M'$.

Cases (2) and (3) are applications of Lemma 4.4 with $N = 1$.

(2) If $a \in \mathrm{Ob}(\mathscr{C})$ with $\mathcal{M}(a,x) \neq \varnothing$, then there is an embedding

$$M \times \mathcal{M}(a,x) \subset \partial_{m+1} \mathcal{M}(a,y).$$

Take a standard collar neighbourhood $M \times \mathcal{M}(a,x) \times [0, 2\eta) \subset \mathcal{M}(a,y)$. Now that we have a standard collar, we may identify it with the $U_{2\eta} \subset \mathbb{R}_+$ of Section 4.2. We may now apply Lemma 4.4 to push $M \times \mathcal{M}(a,x)$ out of the corner of $(X, \iota_X) := (\mathcal{M}(a,y), \iota_{a,y})$ and replace it with $M' \times \mathcal{M}(a,x)$. We use the embedded cobordism rel boundary $(W \times \mathcal{M}(a,x), \tilde{\iota} \times \iota_{a,x})$. The moduli space $\mathcal{M}(\overline{a}, \overline{y})$ is defined as the resulting Z from application of the lemma.

(3) If $b \in \mathrm{Ob}(\mathscr{C})$ with $\mathcal{M}(y,b) \neq \varnothing$, then

$$\mathcal{M}(y,b) \times M \subset \partial_{|y|-|b|} \mathcal{M}(x,b);$$

we again take a standard collar neighbourhood $\mathcal{M}(y,b)\times M\times[0,2\eta)\subset\mathcal{M}(x,b)$. We may identify the collar with the $U_{2\eta}\subset\mathbb{R}_+$ of Section 4.2 and apply Lemma 4.4 to push $\mathcal{M}(y,b)\times M$ out of the corner of $(X,\iota_X):=(\mathcal{M}(x,b),\iota_{x,b})$, replacing it with $\mathcal{M}(y,b)\times M'$. We use the embedded cobordism rel boundary $(\mathcal{M}(y,b)\times W,\iota_{y,b}\times\tilde{\iota})$. The moduli space $\mathcal{M}(\overline{x},\overline{b})$ is defined as the resulting Z from application of the lemma.

Case (4) is an application of Lemma 4.4 with $N=2$.

(4) If $a,b\in\mathrm{Ob}(\mathscr{C})$ with both $\mathcal{M}(a,x),\mathcal{M}(y,b)\neq\varnothing$, then the following is a subset of the boundary $\partial\mathcal{M}(a,b)$:

$$\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)\subset\partial_{|y|-|b|}\mathcal{M}(a,b)\cap\partial_{|x|-|b|}\mathcal{M}(a,b).$$

Take two 1–dimensional collars neighbourhoods of $\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)$. The first collar is taken inside $\partial_{|x|-|b|}\mathcal{M}(a,b)$ and perpendicular to the boundary $\partial_{|y|-|b|}\mathcal{M}(a,b)$ while the second collar is taken vice versa. Note that the product of the collars then defines a standard open neighbourhood inside the full moduli space $\mathcal{M}(a,b)$:

$$\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)\times[0,2\eta)\times[0,2\eta)\subset\mathcal{M}(a,b).$$

We identify the two collar directions with $(\mathbb{R}_+)^2$ in Section 4.2 and note that the standard neighbourhood $U_{2\eta}\subset[0,2\eta)\times[0,2\eta)$ appears within the collar product. We may now apply Lemma 4.4 to push $\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)$ out of the corner of $(X,\iota_X):=(\mathcal{M}(a,b),\iota_{a,b})$ and replace it with $\mathcal{M}(y,b)\times M'\times\mathcal{M}(a,x)$. The embedded cobordism rel boundary we will use is $(\mathcal{M}(y,b)\times W\times\mathcal{M}(a,x),\iota_{y,b}\times\tilde{\iota}\times\iota_{a,x})$. The moduli space $\mathcal{M}(\overline{a},\overline{b})$ is defined as the resulting Z from application of the lemma.

(5) In all other cases define $\mathcal{M}(\overline{a},\overline{b})=\mathcal{M}(a,b)$. Note this includes the case of $c\in\mathrm{Ob}(\mathscr{C})$ with $\mathcal{M}(x,c)\neq\varnothing\neq\mathcal{M}(c,y)$.

Note that, as per the remark preceding the proof of Lemma 4.4, the construction in case (4) agrees with the constructions in cases (2) and (3) when considering the induced moduli spaces $\mathcal{M}(\overline{x},\overline{b})$ and $\mathcal{M}(\overline{a},\overline{y})$ in the boundary of $\mathcal{M}(\overline{a},\overline{b})$.

We now describe the embedding and framing (ι_W,φ_W). For case (1), we have $((\iota_W)_{\overline{x},\overline{y}},(\varphi_W)_{\overline{x},\overline{y}}):=(\iota',\varphi')$, and for case (5), $((\iota_W)_{\overline{a},\overline{b}},(\varphi_W)_{\overline{a},\overline{b}}):=(\iota_{a,b},\varphi_{a,b})$. In cases (2), (3) and (4), the embeddings and framings differ from those of $(\mathscr{C},\iota,\varphi)$ only in a small neighbourhood of the glueing regions. The difference is determined by the embedding and framing of Z which we defined in Lemma 4.4.

Finally we deduce the main result of this section.

Proof of Theorem 4.1 First, we will assume $\boldsymbol{d}$ has been enlarged enough that the framed cobordism rel boundary $(W, \tilde{\imath}, \tilde{\varphi})$ is an embedding in $\mathbb{E}^{\boldsymbol{d}}[|y| : |x|] \times [0,1]$. This does not affect the eventual stable homotopy type of the framed flow category.

Let C be the integer defined in Definition 2.6 (which is the same as that C defined in [10, Definition 3.23]). In the definition of the CW complex associated to a framed flow category [10, Definition 3.23], each object $a \in \mathrm{Ob}(\mathscr{C})$ is assigned a $(C+|a|)$–dimensional cell $\mathcal{C}(a)$. Now, for each $a \in \mathrm{Ob}(\mathscr{C})$, we will define a continuous map

$$F_a \colon [0,1] \times \partial\mathcal{C}(a) \to X^{C+|a|-1}, \quad (t,p) \mapsto F_{a,t}(p),$$

where X^i is defined inductively for increasing i by setting $X^0 = \{\mathrm{pt}\}$,

$$X^i = X^{i-1} \cup_{F_a} ([0,1] \times \mathcal{C}(a)),$$

and with the union taken over all a such that $|a| = i$. Furthermore, for each a, the map F_a will be defined such that $F_{a,0}$ and $F_{a,1}$ are the attaching maps for cells $\mathcal{C}(a)$ and $\mathcal{C}(\overline{a})$ in the CW complexes $|\mathscr{C}|$ and $|\mathscr{C}_W|$, respectively. As such, the space $X := \bigcup_i X^i$ can easily be seen to deformation retract onto each of $|\mathscr{C}|$ and $|\mathscr{C}_W|$, so that they are homotopy equivalent to one another.

It remains to define the F_a. Recall that in [10, Definition 3.23], a large real number R is chosen and $\mathcal{M}(a,b) \times \mathcal{C}(b)$ is embedded as

$$\mathcal{C}_b(a) = [0,R] \times [-R,R]^{d_k} \times \cdots \times [0,R] \times [-R,R]^{d_{|b|-1}} \times \{0\} \times \mathcal{C}_{b,1} \\ \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{|a|}} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{n+k-1}} \subset \partial\mathcal{C}(a),$$

where $\mathcal{C}_{b,1}$ is the subset of $\mathbb{E}^{\boldsymbol{d}}[|b| : |a|]$ given by the framed embedding of $\mathcal{M}(a,b)$. The cell attaching map for $\mathcal{C}(a)$ is given on $\mathcal{C}_b(a) \cong \mathcal{M}(a,b) \times \mathcal{C}(b)$ by the projection to $\mathcal{C}(b)$, and on $\partial\mathcal{C}(a) \setminus \bigcup_b \mathcal{C}_b(a)$ by mapping to the basepoint. The rough idea of the construction of F_a will be to take a product of this entire process with an interval $[0,1]$, and then apply Lemma 4.4 to appropriate moduli spaces in the corners at the $1 \in [0,1]$ end. Restricting this pushing out process to the end $1 \in [0,1]$, this will simply recover the definition of the moduli spaces of $\mathscr{C}_W$. But in the interior of the interval $[0,1]$ we will obtain an embedded cobordism between $\mathcal{C}_{b,1}$ and $\mathcal{C}_{\overline{b},1}$. The Pontryagin–Thom collapse can be applied to this entire cobordism and, collecting these collapses over all such b, we will obtain the required homotopy between the attaching maps of $\mathcal{C}(a)$ and $\mathcal{C}(\overline{a})$.

Precisely, for each $a, b \in \mathrm{Ob}(\mathscr{C})$ we will define a framed embedded cobordism between $\mathcal{C}_{b,1}$ and $\mathcal{C}_{\overline{b},1}$, giving a subset $\mathcal{C}_{b,W} \subset \mathbb{E}^{\boldsymbol{d}}[|b| : |a|] \times [0,1]$ which in turn defines an embedding

$$[0,R]\times[-R,R]^{d_k}\times\cdots\times[0,R]\times[-R,R]^{d_{|b|-1}}\times\{0\}\times\mathcal{C}_{b,W}\\ \times\{0\}\times[-\varepsilon,\varepsilon]^{d_{|a|}}\times\cdots\times\{0\}\times[-\varepsilon,\varepsilon]^{d_{n+k-1}}\subset[0,1]\times\partial\mathcal{C}(a),$$

whose domain is homeomorphic to $Z_{a,b}\times\mathcal{C}(b)$ for spaces $Z_{a,b}$ to be defined below. Roughly, these $Z_{a,b}$ will function as interpolations between $\mathcal{M}(a,b)$ and $\mathcal{M}(\overline{a},\overline{b})$. They will be constructed by pushing moduli space out of one end of a cylinder on the embedded $\mathcal{M}(a,b)$ using Lemma 4.4. Once the $Z_{a,b}$ are defined, we can define $F_a\colon [0,1]\times\partial\mathcal{C}(a)\to X^{C+|a|-1}$ as follows. On $Z_{a,b}\times\mathcal{C}(b)$, it is the projection to $\mathcal{C}(b)$, and on $[0,1]\times\partial\mathcal{C}(a)\setminus\bigcup_b(Z_{a,b}\times\mathcal{C}(b))$, we map to the basepoint. By construction, this collapse map will agree precisely with the collapse maps for cells of $\mathscr{C}$ and $\mathscr{C}_W$ on the $\{0\}$ and $\{1\}$ ends, respectively. At the $\{0\}$ end this will be obvious from the construction of the $Z_{a,b}$. At the $\{1\}$ end this will follow from the remark preceding the proof of Lemma 4.4.

To complete the proof we will now construct the required $Z_{a,b}$.

First, if a and b are as in case (5) of Definition 4.5 then we simply set $Z_{a,b}:=\mathcal{M}(a,b)\times[0,1]$. We define a subspace $\mathcal{C}_{b,W}=\mathcal{C}_{b,1}\times[0,1]\subset\mathbb{E}^{\boldsymbol{d}}[|b|:|a|]\times[0,1]$ and hence an embedding $Z_{a,b}\times\mathcal{C}(b)\subset[0,1]\times\partial\mathcal{C}(a)$.

We proceed in a series of case analyses based on (1), (3), (2) and (4) of Definition 4.5, in that order. The pushing out of the corner constructions will be essentially those of Definition 4.5, except in 1 codimension higher.

For the first nontrivial case, set $a=x$ and $b=y$, corresponding to (1) of Definition 4.5. Consider the embedding

$$\iota\times\mathrm{id}\colon M\times[0,1]\hookrightarrow\mathbb{E}^{\boldsymbol{d}}[|y|:|x|]\times[0,1].$$

Now use Lemma 4.4 and the framed cobordism W to push M out of the codimension-1 corner $\mathbb{E}^{\boldsymbol{d}}[|y|:|x|]\times\{1\}$ and replace it with M'. This determines a subset $\mathcal{C}_{b,W}$ of $[-R,R]^{d_{|y|}}\times[0,R]\times\cdots\times[0,R]\times[-R,R]^{d_{|x|-1}}\times[0,1]$. Moreover, this results in an embedding of $Z_{a,b}\times\mathcal{C}(y)$ in $[0,1]\times\partial\mathcal{C}(x)$ (here $Z_{a,b}:=Z$ refers to the result of applying Lemma 4.4).

For the next case, set $a=x$, $b\neq y$ and $\mathcal{M}(y,b)\neq\varnothing$, corresponding to (3) of Definition 4.5. Then we have

$$\mathcal{M}(y,b)\times M\subset\partial\mathcal{M}(x,b)\subset\mathbb{E}^{\boldsymbol{d}}[|b|:|y|]\times\{0\}\times\mathbb{E}^{\boldsymbol{d}}[|y|:|x|]$$

and in the normal direction to this boundary we will take a 2η open collar neighbourhood. Consider that the product

$$\mathcal{M}(y,b)\times M\times[0,2\eta)\times[0,1]\subset\mathcal{M}(x,b)\times[0,1]$$

has a copy of $\mathcal{M}(y,b)\times M$ in the corner $(\mathbb{E}^{\boldsymbol{d}}[|b|:|y|]\times\{0\}\times\mathbb{E}^{\boldsymbol{d}}[|y|:|x|])\times\{0\}\times\{1\}$. Using Lemma 4.4 and the framed cobordism $\mathcal{M}(y,b)\times W$, we push $\mathcal{M}(y,b)\times M$ out of the corner and replace it with $\mathcal{M}(y,b)\times M'$. Note that all embeddings were framed, so that the resultant $Z_{a,b}$, is a framed embedding determining a subset $\mathcal{C}_{b,W}$ of $[-R,R]^{d_{|b|}}\times[0,R]\times\cdots\times[0,R]\times[-R,R]^{d_{|x|-1}}\times[0,1]$ and hence an embedding of $Z_{a,b}\times\mathcal{C}(b)$ in $[0,1]\times\partial\mathcal{C}(x)$. Note that this agrees with the construction from the first case:

$$\mathcal{C}_{b,W}\cap[-R,R]^{d_{|y|}}\times[0,R]\times\cdots\times[0,R]\times[-R,R]^{d_{|x|-1}}\times[0,1]=\mathcal{C}_{y,W},$$

as each case used Lemma 4.4.

Next we must look at cells $a\neq x$. There are two such cases to consider. First suppose $a\neq x$, $b=y$ and that $\mathcal{M}(a,x)\neq\varnothing$, corresponding to (2) of Definition 4.5. We wish to deform the embedding of $M\times\mathcal{M}(a,x)$ appropriately. But this case proceeds identically to the case above where $a=x$, $b\neq y$ and $\mathcal{M}(y,b)\neq\varnothing$, so we omit the details.

Finally, there is the case that $a\neq x$, $b\neq y$ and $\mathcal{M}(a,x)\neq\varnothing\neq\mathcal{M}(y,b)$, corresponding to case (4) of Definition 4.5. This is treated by the same pushing out technique as before (but now we need to do it in codimension 3). Precisely, we must consider $\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)\subset\partial\mathcal{M}(a,b)$. But there is a copy of $\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)$ in the codimension-3 corner $(0,0,1)\in[0,2\eta)\times[0,2\eta)\times[0,1]$ of the cylinder on the 2–way collar neighbourhood

$$\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)\times[0,2\eta)\times[0,2\eta)\times[0,1]\subset\mathcal{M}(a,b)\times[0,1].$$

We push $\mathcal{M}(y,b)\times M\times\mathcal{M}(a,x)$ out of the corner using $\mathcal{M}(y,b)\times W\times\mathcal{M}(a,x)$ and we obtain $\mathcal{C}_{b,W}\cong Z_{a,b}\times\mathcal{C}(b)$ as in previous cases. This completes the case analysis. □

Example 4.6 We illustrate a homotopy from the proof Theorem 4.1 in the case where M and M' are 0–dimensional. Let M be a point, M' be 3 points and W be the 2–component cobordism from M to M' as in Figure 3. We suppose M is actually a corner of a larger space N, so that after the homotopy, M' is in the corner of this larger

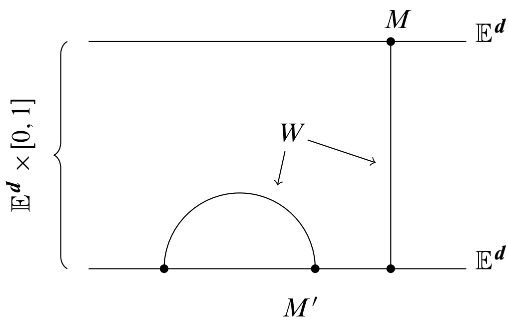


Figure 3: The cobordism W between M and M'

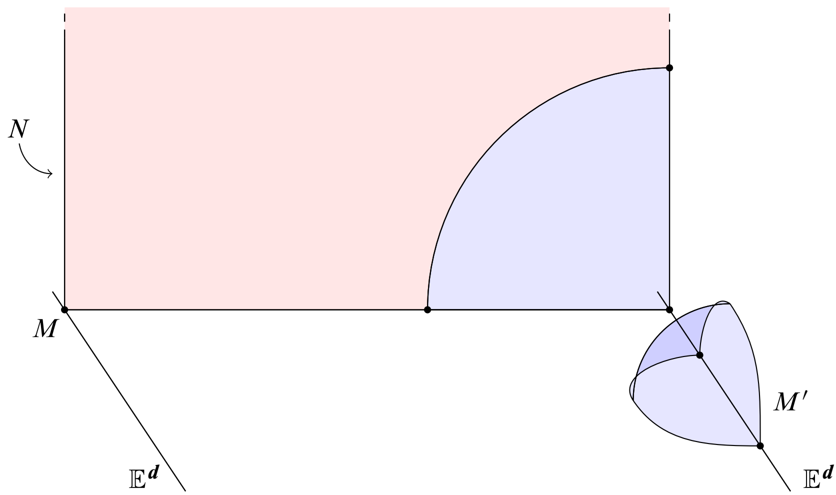


Figure 4: The cobordism Z

space. This is shown in Figure 4, where we have drawn N extending in the vertical axis, and drawn the framed cobordism Z constructed in Theorem 4.1 by pushing M out of the corner $N \times \{1\}$ of $N \times [0,1]$ using W.

5 The disjoint union of three trefoils

We make a sample calculation of a stable homotopy type using our framed flow category moves. The example framed flow category $\mathcal{C}_1$ that we shall start with is depicted in Figure 5. We shall show explicitly that $\mathcal{C}_1$ is equivalent via our flow category moves to the framed flow category $\mathcal{C}_5$ depicted in Figure 9. The form of $\mathcal{C}_5$ corresponds to the Baues–Hennes classification of low homological width stable homotopy types and this is discussed further towards the end of the section.

We choose to consider the framed flow category $\mathcal{C}_1$ since it arises in a particular context, namely as a framed flow category associated to the disjoint union of three trefoils, by the techniques developed by Jones, Lobb and Schütz [6].

Definition 5.1 There are eight objects in the category $\mathcal{C}_1$, depicted in Figure 5, each labelled by a 3–tuple. Any nonempty 0–dimensional moduli space consists of two points each, as shown. A letter "p" indicates that the point is framed positively, while a letter "m" indicates a negative framing.

The 1–dimensional moduli spaces are then given by four intervals each, with all of them framed 0. Each such interval of course has two endpoints, and these are given as follows:

$$\mathcal{M}(333,223) = \overset{233}{\underset{P_0\mathscr{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{323}{\underset{\widetilde{P}_0\mathscr{M}_0}{\dashv}} \quad \overset{233}{\underset{P_0\mathscr{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{323}{\underset{\widetilde{P}_1\mathscr{M}_0}{\dashv}} \quad \overset{233}{\underset{P_1\mathscr{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{323}{\underset{\widetilde{P}_0\mathscr{M}_1}{\dashv}} \quad \overset{233}{\underset{P_1\mathscr{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{323}{\underset{\widetilde{P}_1\mathscr{M}_1}{\dashv}}$$

$$\mathcal{M}(333,232) = \overset{233}{\underset{M_0\mathscr{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{P}_0\widetilde{\mathscr{P}}_0}{\dashv}} \quad \overset{233}{\underset{M_0\mathscr{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{P}_1\widetilde{\mathscr{P}}_0}{\dashv}} \quad \overset{233}{\underset{M_1\mathscr{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{P}_0\widetilde{\mathscr{P}}_1}{\dashv}} \quad \overset{233}{\underset{M_1\mathscr{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{P}_1\widetilde{\mathscr{P}}_1}{\dashv}}$$

$$\mathcal{M}(333,322) = \overset{323}{\underset{\widetilde{M}_0\mathscr{M}_0}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{M}_0\widetilde{\mathscr{P}}_0}{\dashv}} \quad \overset{323}{\underset{\widetilde{M}_0\mathscr{M}_1}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{M}_1\widetilde{\mathscr{P}}_0}{\dashv}} \quad \overset{323}{\underset{\widetilde{M}_1\mathscr{M}_0}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{M}_0\widetilde{\mathscr{P}}_1}{\dashv}} \quad \overset{323}{\underset{\widetilde{M}_1\mathscr{M}_1}{\vdash}}\overset{0}{\text{———}}\overset{332}{\underset{\overline{M}_1\widetilde{\mathscr{P}}_1}{\dashv}}$$

$$\mathcal{M}(233,222) = \overset{223}{\underset{p_0P_0}{\vdash}}\overset{0}{\text{———}}\overset{232}{\underset{\tilde{p}_0M_0}{\dashv}} \quad \overset{223}{\underset{p_0P_1}{\vdash}}\overset{0}{\text{———}}\overset{232}{\underset{\tilde{p}_1M_0}{\dashv}} \quad \overset{223}{\underset{p_1P_0}{\vdash}}\overset{0}{\text{———}}\overset{232}{\underset{\tilde{p}_0M_1}{\dashv}} \quad \overset{223}{\underset{p_1P_1}{\vdash}}\overset{0}{\text{———}}\overset{232}{\underset{\tilde{p}_1M_1}{\dashv}}$$

$$\mathcal{M}(323,222) = \overset{223}{\underset{p_0\widetilde{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_0\widetilde{M}_0}{\dashv}} \quad \overset{223}{\underset{p_0\widetilde{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_1\widetilde{M}_0}{\dashv}} \quad \overset{223}{\underset{p_1\widetilde{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_0\widetilde{M}_1}{\dashv}} \quad \overset{223}{\underset{p_1\widetilde{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_1\widetilde{M}_1}{\dashv}}$$

$$\mathcal{M}(332,222) = \overset{232}{\underset{\tilde{p}_0\overline{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_0\overline{M}_0}{\dashv}} \quad \overset{232}{\underset{\tilde{p}_0\overline{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_1\overline{M}_0}{\dashv}} \quad \overset{232}{\underset{\tilde{p}_1\overline{P}_0}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_0\overline{M}_1}{\dashv}} \quad \overset{232}{\underset{\tilde{p}_1\overline{P}_1}{\vdash}}\overset{0}{\text{———}}\overset{322}{\underset{\bar{p}_1\overline{M}_1}{\dashv}}$$

The 2–dimensional moduli space $\mathcal{M}(333,222)$ consists of four hexagons. It will follow from the Baues–Hennes classification [2] that the associated stable homotopy type is determined by the actions of Sq^1 and Sq^2. These operations are in turn determined by the framed moduli spaces of dimensions 0 and 1, so we will not keep track of the 2–dimensional moduli space.

For those less interested in the Khovanov stable homotopy type, the next proposition may be skipped. We focus on quantum degree $q=21$ of the Khovanov stable homotopy

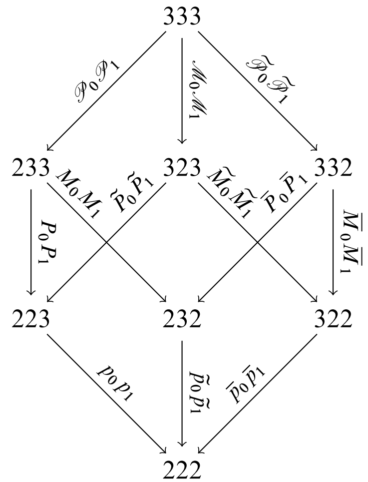


Figure 5: A subcategory $\mathcal{C}_1$ of $\mathscr{L}^{\mathrm{Kh}}(L)$

type of the disjoint union of three right-handed trefoils, since we know that there is a nontrivial Sq^3 in this degree. Indeed, the general formula of Lawson, Lipshitz and Sarkar [9] confirms the existence of the smash of Moore spaces

$$M(\mathbb{Z}/2\mathbb{Z},2)\wedge M(\mathbb{Z}/2\mathbb{Z},2)\wedge M(\mathbb{Z}/2\mathbb{Z},2)$$

as a wedge summand.

Proposition 5.2 *Let L be the disjoint union of three right-handed trefoils. The technique of [6] constructs a framed flow category $\mathscr{L}^{\mathrm{Kh}}(L)$. In quantum degree $q=21$, this framed flow category is the disjoint union of the framed flow category $\mathcal{C}_1$ described in Definition 5.1 with some other framed flow category.*

Proof The following calculation is similar to [7, Section 4.3]. In homological degree 9 we get exactly one object, which is based at $(3,3,3)$. The smoothing of this object consists of three circles, each of which is decorated with a $-$. The objects of homological degree 8 are based at $(2,3,3)$, $(3,2,3)$ and $(3,3,2)$. Again the smoothings are three circles, and one of them is decorated with a $+$. We therefore get 9 objects of degree 8. Note however that only three of those have nonempty moduli space with the object of degree 9, namely those where the $+$ corresponds to the position of the 2 in the base triple.

We will now only consider those objects that have nonempty moduli spaces with the object based at $(3,3,3)$, as these objects will give rise exactly to the product of

Moore spaces predicted above. We then get three objects of homological degree 7, based at $(2,2,3)$, $(2,3,2)$ and $(3,2,2)$, where the circle corresponding to the 3 is decorated $-$, and one object of homological degree 6 based at $(2,2,2)$ with all three circles decorated $+$. It is easy to see that these objects do not share nonempty moduli spaces with other objects in $\mathscr{L}^{\mathrm{Kh}}(L)$. □

We are now going to perform handle slides and Whitney tricks, simplifying the framed flow category $\mathcal{C}_1$ in a sequence of four propositions until we arrive at the "Baues–Hennes" category $\mathcal{C}_5$.

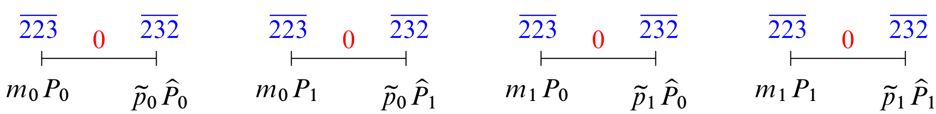
Proposition 5.3 *The framed flow category $\mathcal{C}_1$ is move-equivalent to a framed flow category $\mathcal{C}_2$, depicted in Figure 6. In this figure the 1–dimensional moduli space denoted by η is a nontrivially framed circle.*

Proof To begin, we slide 223 over 232 via a $(-)$–handle slide. We will write an overline above all objects to indicate the resulting flow category, but these overlines will disappear again in time for the next move. The affected moduli spaces are

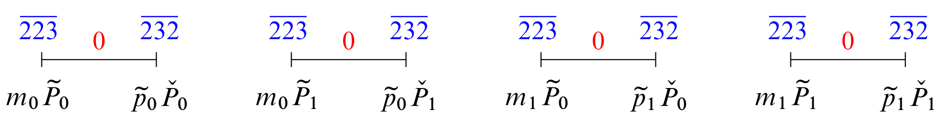
$$\mathcal{M}(\overline{333},\overline{232}) = \mathcal{M}(333,232) \sqcup \mathcal{M}(333,223)$$

with no change in framings by Proposition 3.11(a). Furthermore, both $\mathcal{M}(\overline{233},\overline{222})$ and $\mathcal{M}(\overline{323},\overline{222})$ get four new intervals, each framed 0 by Proposition 3.11(a). Note these new intervals correspond to $\mathcal{M}(232,222)\times\mathcal{M}(a,223)$ with $a=233$ or $a=323$. The points $\tilde{p}_0, \tilde{p}_1 \in \mathcal{M}(232,222)$ create new points $m_0, m_1 \in \mathcal{M}(\overline{223},\overline{222})$, $P_0, P_1 \in \mathcal{M}(233,223)$ create $\hat{P}_0, \hat{P}_1 \in \mathcal{M}(\overline{233},\overline{232})$, and $\tilde{P}_0, \tilde{P}_1 \in \mathcal{M}(323,223)$ create $\check{P}_0, \check{P}_1 \in \mathcal{M}(\overline{323},\overline{232})$.

The new intervals in $\mathcal{M}(\overline{233},\overline{222})$ are given by



and, similarly, in $\mathcal{M}(\overline{323},\overline{222})$ they are given by



We now perform the Whitney trick in $\mathcal{M}(\overline{223},\overline{222})$ with p_0, m_0 and with p_1, m_1. The result is that in $\mathcal{M}(\overline{233},\overline{222})$ and $\mathcal{M}(\overline{323},\overline{222})$ intervals are glued together. For example, the endpoint $p_0 P_0$ in the old $\mathcal{M}(233, 222)$ is identified with the endpoint $m_0 P_0$ in one of the new intervals. In fact, in each case two intervals are glued together to form a new interval. Furthermore, by [7, Proposition 3.3], the framing value of each new interval is 1. We then get

$$\mathcal{M}(\overline{233},\overline{222}) = \underset{\tilde p_0 \hat P_0}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\tilde p_0 M_0}{\overline{232}} \quad \underset{\tilde p_0 \hat P_1}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\tilde p_1 M_0}{\overline{232}} \quad \underset{\tilde p_1 \hat P_0}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\tilde p_0 M_1}{\overline{232}} \quad \underset{\tilde p_1 \hat P_1}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\tilde p_1 M_1}{\overline{232}}$$

$$\mathcal{M}(\overline{323},\overline{222}) = \underset{\tilde p_0 \check P_0}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar p_0 \widetilde M_0}{\overline{322}} \quad \underset{\tilde p_0 \check P_1}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar p_1 \widetilde M_0}{\overline{322}} \quad \underset{\tilde p_1 \check P_0}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar p_0 \widetilde M_1}{\overline{322}} \quad \underset{\tilde p_1 \check P_1}{\overline{232}} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar p_1 \widetilde M_1}{\overline{322}}$$

In the next step we again perform the Whitney trick, this time using $\hat P_0$, M_0 and $\hat P_1$, M_1 in $\mathcal{M}(\overline{233},\overline{232})$. Note that we now remove the overline from the objects. The effect on $\mathcal{M}(333, 232)$ is that the eight intervals turn into four similarly to the case above. More precisely, we get

$$\mathcal{M}(333,232) = \underset{\check P_0 \mathscr{M}_0}{323} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar P_0 \widetilde{\mathscr{P}}_0}{332} \quad \underset{\check P_1 \mathscr{M}_0}{323} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar P_1 \widetilde{\mathscr{P}}_0}{332} \quad \underset{\check P_0 \mathscr{M}_1}{323} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar P_0 \widetilde{\mathscr{P}}_1}{332} \quad \underset{\check P_1 \mathscr{M}_1}{323} \overset{1}{\vdash\!\!\!-\!\!\!-\!\!\!\dashv} \underset{\bar P_1 \widetilde{\mathscr{P}}_1}{332}$$

The moduli space $\mathcal{M}(233, 222)$ turns into a closed manifold. In fact, the outer intervals result in one circle each, and the inner two intervals are glued together along their endpoints to form a single circle. By [7, Proposition 3.4] all circles are labelled with 0, which means that each circle is *nontrivially* framed (compare [7] for framing conventions). Using the extended Whitney trick, we can reduce this to one nontrivially framed circle, which we denote by $\mathcal{M}(233, 222) = \eta$.

The result is the framed flow category $\mathcal{C}_2$ depicted in Figure 6. □

Proposition 5.4 *The framed flow category $\mathcal{C}_2$ is move-equivalent to the framed flow category $\mathcal{C}_3$ depicted in Figure 7. In Figure 7 we denote nontrivially framed circles by either ξ or η.*

Proof Starting from this category, we slide 322 over 232 with a $(-)$–handle slide. This introduces extra points M_0, M_1 in $\mathcal{M}(\overline{332},\overline{232})$ and $\check M_0$, $\check M_1$ in $\mathcal{M}(\overline{323},\overline{232})$, as well as m_0, m_1 in $\mathcal{M}(\overline{322},\overline{222})$. Similar to the last slide in Proposition 5.3, the moduli spaces $\mathcal{M}(\overline{333},\overline{232})$, $\mathcal{M}(\overline{323},\overline{222})$ and $\mathcal{M}(\overline{332},\overline{222})$ each acquire four new intervals.

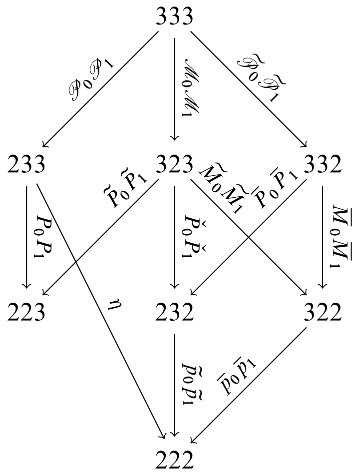


Figure 6: The framed flow category $\mathcal{C}_2$

Performing all obvious extended Whitney tricks (0–dimensional and 1–dimensional) now leads to the flow category indicated in Figure 7. One easily checks that each of $\mathcal{M}(333,232)$, $\mathcal{M}(323,222)$ and $\mathcal{M}(332,222)$ turns into three circles, and after extended Whitney tricks each contains one nontrivially framed circle. We denote the nontrivially framed circle in $\mathcal{M}(333,232)$ by ξ, mainly to follow the conventions of Baues and Hennes [2]. □

Proposition 5.5 *The framed flow category $\mathcal{C}_3$ is move-equivalent to the framed flow category $\mathcal{C}_4$ depicted in Figure 8.*

Proof Starting from Figure 7 we can perform two $(-)$–handle slides, sliding 323 over both 233 and 332. The moduli spaces $\mathcal{M}(\overline{333},\overline{223})$ and $\mathcal{M}(\overline{333},\overline{322})$ each consist of eight intervals, and $\mathcal{M}(\overline{323},\overline{222})$ contains exactly three nontrivially framed circles, which are together Whitney trick-equivalent to a single nontrivially framed circle. Note that in $\mathcal{M}(\overline{333},\overline{322})$ the framing of the four new intervals is different from the framing of the new intervals in $\mathcal{M}(\overline{333},\overline{223})$, but after performing the obvious Whitney tricks we get the flow category $\mathcal{C}_4$ depicted in Figure 8. □

Proposition 5.6 *The framed flow category $\mathcal{C}_4$ is move-equivalent to the framed flow category $\mathcal{C}_5$ depicted in Figure 9.*

Proof Finally, we slide 332 over 323, 233 over 323 and then 232 over 223 and 322. Again, using the extended Whitney trick on the 1–dimensional moduli spaces leads to the flow category $\mathcal{C}_5$ depicted in Figure 9. □

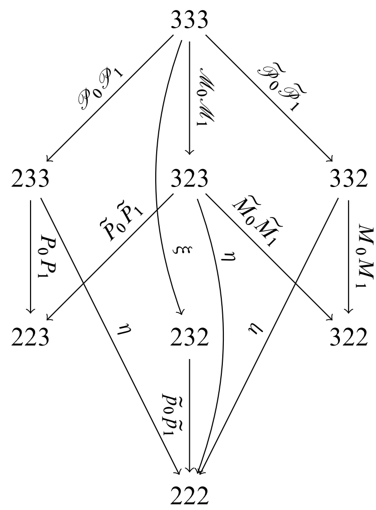


Figure 7: The framed flow category $\mathcal{C}_3$

The flow category $\mathcal{C}_5$ clearly has two wedge-summands $M(\mathbb{Z}/2\mathbb{Z}, 7)$ and the remaining four objects form a stable space with the property that $\mathrm{Sq}^2\,\mathrm{Sq}^1 = \mathrm{Sq}^1\,\mathrm{Sq}^2$ is nontrivial. It follows from the decomposition theorem of [2, Theorem 3.9] that the space corresponding to this category is the space called $X(\xi^2\eta_2, \mathrm{id})$, where $\mathrm{id}\colon V \to V$ is the identity of the 1–dimensional $\mathbb{Z}/2\mathbb{Z}$–vector space V.

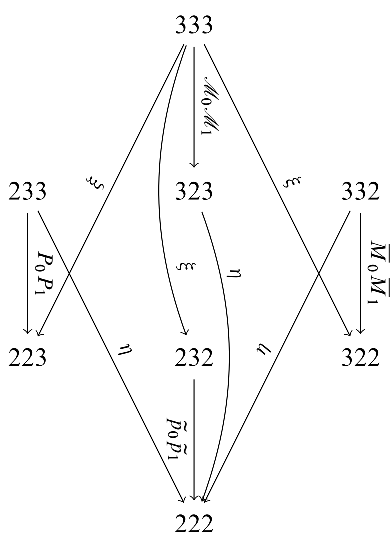


Figure 8: The framed flow category $\mathcal{C}_4$. Note that the associated cochain complex is in Smith normal form; see Section 6.

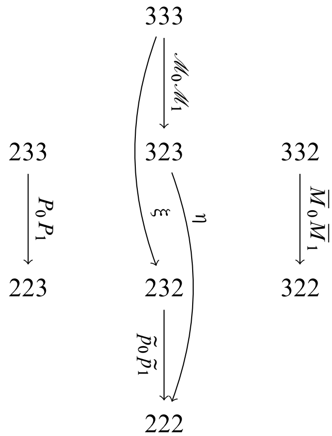


Figure 9: The flow category $\mathcal{C}_5$. The middle summand corresponds to a *special cyclic word* in the sense of Baues and Hennes [2].

We note that $(\xi^2\eta_2, \mathrm{id})$ is a *special cyclic word* in the sense of Baues and Hennes [2],[1] and the only indecomposable space in their list with $\mathrm{Sq}^2\,\mathrm{Sq}^1 = \mathrm{Sq}^1\,\mathrm{Sq}^2$ nontrivial.

6 A space level Smith normal form and further reductions

It is discussed in [10, Lemmas 3.25 and 3.26] that the stable homotopy type of $(\mathscr{C}, \iota, \varphi)$ is not affected by an isotopy of the framing (ι, φ) or by increasing $\boldsymbol{d}$ to $\boldsymbol{d}' \geq \boldsymbol{d}$. In addition to these basic modifications, we say the following are permissible *flow category moves*:

(1) Handle cancellation.

(2) Extended Whitney trick.

We remark that the handle slide of Section 3.3 is based on handle cancellation, so does not need to be included as an extra flow category move.

Definition 6.1 $(\mathscr{C}, \iota, \varphi)$ and $(\mathscr{C}', \iota', \varphi')$ are *directly move-equivalent* if $(\mathscr{C}', \iota', \varphi')$ is the effect of a flow category move on $(\mathscr{C}, \iota, \varphi)$ or vice versa. We say $(\mathscr{C}, \iota, \varphi)$ and $(\mathscr{C}', \iota', \varphi')$ are *move-equivalent* if they are related by a finite sequence of directly move-equivalent framed flow categories. Clearly, move-equivalence is an equivalence relation on the set of framed flow categories.

[1] In [2], the notation used is actually $\xi^1\eta_1$, while in [1] it is $\xi^2\eta_2$. We use the notation of [1].

We now consider a general method to simplify a framed flow category within its move-equivalence class.

Recall that a finite chain complex C_* of finitely generated projective modules over a principal ideal domain A can always be put into *Smith normal form.* That is, there is a basis $C_r \cong U_r \oplus V_r \oplus W_r$ such that the differentials $d_r\colon C_{r+1} \to C_r$ vanish on $V_r \oplus W_r$ and the matrix of $d_r|_{U_r}$ has the form

$$(0 \;\; D \;\; 0)^t,$$

where D is injective and diagonal with $D_{ii} \mid D_{i+1\,i+1}$. Writing $m(r) = \mathrm{rk}_A V_r$ and $n(r) = \mathrm{rk}_A W_r$, the Smith normal form basis presents the homology as

$$H_r(C;A) \cong A/D_{11}A \oplus \cdots \oplus A/D_{m(r)m(r)}A \oplus A^{n(r)}.$$

Note that some diagonal entries may be equal to 1.

If we are moreover allowed to add or remove cancelling A–module generators in adjacent homological degrees (which will correspond to elementary expansions or contractions of the matrices of d_r), the Smith normal form can be changed so that the diagonal entries of D are all prime powers. Elementary expansions and contractions result in chain homotopy equivalences, so the homology is not affected. In this changed form, the basis presents the unique primary decomposition of the homology modules. Call this modified type of basis the *primary Smith normal form.*

Recall that any framed flow category $(\mathscr{C}, \iota, \varphi)$ determines a based chain complex (C_*, d). The basis of C_r is given by the objects x of $\mathscr{C}$ with $|x| = r$. If $|x| = r$ and $|y| = r-1$ then the (x, y) entry in the matrix of d_{r-1} is given by the signed count of the points in $\mathcal{M}(x, y)$.

Theorem 6.2 *Any framed flow category $(\mathscr{C}, \iota, \varphi)$ is move-equivalent to some framed flow category whose based chain complex (C_*, d) is in primary Smith normal form and such that the number of points in any 0–dimensional moduli space is exactly the corresponding entry in the matrix of the differential d.*

Proof Suppose $(\mathscr{C}, \iota, \varphi)$ is a framed flow category. Any 0–dimensional moduli space $\mathcal{M}(x, y)$ consists of a certain number of positively framed points and a certain number of negatively framed points. By pairing a positive point with a negative point we may cancel them against each other using an extended Whitney trick. This will not affect any other 0–dimensional moduli spaces or the other components of $\mathcal{M}(x, y)$. By repeatedly doing this, we may assume that every nonempty 0–dimensional moduli space consists entirely of positively framed points or entirely of negatively framed points. Write this as a positive or negative integer $n_{x,y}$, and if $\mathcal{M}(x,y)$ is empty, set $n_{x,y}=0$. The chain complex of the framed flow category is then C_*, where C_n is freely generated by the objects x in grading level n and $d_{n-1}x=\Sigma_{|y|=n-1}n_{x,y}y$.

We wish to first put C_* into Smith normal form. To do so we will show that certain basis changes for the C_* can be realised via flow category moves. If $|x|=n$ then write r_x for the row vector corresponding to x in the matrix of d_n and c_x for the column vector in d_{n-1}. Now if $|x|=|y|$, then the reader may check that if we $(\pm)$–slide x over y, this does not affect r_x or c_y, but has the effect $r_y\mapsto r_y\mp r_x$ and $c_x\mapsto c_x\pm c_y$. Set $k=\min\{|x|\mid x\in \mathrm{Ob}(\mathscr{C})\}$. We may now perform the row and column operations required to put d_k into Smith normal form. As a result, $C_k\cong V_k\oplus W_k$ and $C_{k+1}\cong U_{k+1}\oplus V_{k+1}\oplus W_{k+1}$, as required. Next, observe that $\mathrm{im}(d_{k+1})\subset V_{k+1}\subset \ker d_k$, so that performing the handle slides required to put d_{k+1} into Smith normal form will not destroy the Smith normal form of d_k (as there are no 0–dimensional moduli spaces between V_{k+1} and C_k). We may now repeat this process on d_r, for increasing r, so that the whole of C_* is in Smith normal form.

To modify the flow category so that the chain complex is in primary Smith normal form, we only need to know that for each r, we can make elementary matrix expansions or contractions of d_r, using flow category moves. But to make a matrix expansion, we simply introduce a pair of objects x and y with $|x|=r+1$ and $|y|=r$ and with $\mathcal{M}(x,y)$ a single positively framed point. Clearly this has the required effect on the matrix of d_r, and x can be cancelled against y via handle cancellation, so introducing these points is a permissible flow category move. To make an elementary matrix contraction we perform a handle cancellation on the objects corresponding to the cancelling chain complex generators. □

Here is an easy corollary:

Corollary 6.3 *Suppose $(\mathscr{C}_1,\iota_1,\varphi_1)$ and $(\mathscr{C}_2,\iota_2,\varphi_2)$ are framed flow categories such that there is a stable homotopy equivalence $\mathcal{X}(\mathscr{C}_1)\simeq\mathcal{X}(\mathscr{C}_2)$ and that, for some $n\in\mathbb{Z}$ and all $k>0$, the reduced homology $\widetilde{H}_*(\mathscr{C}_1;\mathbb{Z}/k\mathbb{Z})$ is only supported in degrees n and $n+1$* (in particular this means the flow categories have the stable homotopy type of a wedge of Moore spaces). *Then $(\mathscr{C}_1,\iota_1,\varphi_1)$ and $(\mathscr{C}_2,\iota_2,\varphi_2)$ are move-equivalent.*

In fact we suggest that much more is true:

Conjecture 6.4 *If two framed flow categories determine the same stable homotopy type then they are move-equivalent to one another.*

We provide more evidence for this conjecture in a forthcoming paper [12], where we show that something similar to Corollary 6.3 is true but with the reduced homology now possibly supported in degrees n, $n+1$, $n+2$ and $n+3$. To do this, we will show how to use move equivalence to reduce the stable homotopy types of such flow categories to those associated to the Chang [3] and Baues–Hennes [2] homotopy classifications.

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