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The fundamental group of locally standard T–manifolds

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We calculate the fundamental group of locally standard T–manifolds under the assumption that the principal T–bundle obtained from the free T–orbits is trivial. This family of manifolds contains nonsingular toric varieties which may be noncompact, quasitoric manifolds and toric origami manifolds with coörientable folding hypersurface. Although the fundamental groups of the above three kinds of manifolds are well-studied, we give a uniform and simple method to generalize the formulas of their fundamental groups.

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1 Introduction

Torus manifolds, introduced by A Hattori and M Masuda [5], are a generalization of toric manifolds, compact nonsingular toric varieties. A torus manifold is a $2n$–dimensional, closed, connected, orientable, smooth manifold M with an effective smooth action of an n–dimensional torus $T \cong (S^1)^n$ such that the fixed-point set $M^T \neq \varnothing$. A torus manifold M is called locally standard if every point of M has a T–invariant open neighborhood equivariantly diffeomorphic to a T–invariant open set of a faithful representation space of T. In this case the orbit space M/T is a nice manifold with corners. For more details about locally standard torus manifolds, readers can see V Buchstaber and T Panov's nice book [1]. If a torus manifold is locally standard, $q_*\colon \pi_1(M) \to \pi_1(M/T)$ is an isomorphism, where q_* is induced by the quotient map $q\colon M \to M/T$. This was proved by Wiemeler [10] and Yoshida [11]. If we allow M to be noncompact, nonorientable and $M^T = \varnothing$, ie M is a locally standard T–manifold,[1] q_* may fail to be an isomorphism. Nonsingular toric varieties (see Fulton [4] or Cox, Little and Schenck [2]), quasitoric manifolds (see Davis and Januszkiewicz [3]) and toric origami manifold with coörientable folding hypersurfaces (see Holm and Pires [7]) are typical examples of locally standard T–manifolds. We use the identification $N := \mathrm{Hom}(S^1, T) = H_1(T) = H_2(BT)$ and let M_i for $i = 1, \dots, m$

[1] Here the locally standard T–manifold M is slightly different from the definition by Buchstaber and Panov [1] since we allow the manifold to be noncompact and nonorientable, but we require the orbit space M/T to have finitely many faces.

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be the characteristic manifolds of M, ie M_i is a connected codimension-2 submanifold of M fixed pointwise by a circle subgroup $S_i \subset T$. Thus, M_i determines a primitive element $v_i \in N$ up to sign such that $S_i = v_i(S^1)$. Let $\widehat{N}$ be the sublattice of N generated by $v_1, \dots, v_m$. Inspired by the arguments of Kuwata, Masuda and Zeng [9] and T Holm and A R Pires's results in [8], we can use a simple method to generalize the known results to locally standard T–manifolds whose free T–orbits is a trivial T–bundle. The following is our main result (see Theorem 2.2).

Theorem *Let M be a locally standard T–manifold such that the free part of the T–action is a trivial torus bundle. Then*

$$\pi_1(M) \cong \pi_1(M/T) \times N/\widehat{N}.$$

2 The fundamental group of locally standard T–manifolds

In this section we will deduce the formula for $\pi_1(M)$ based on some arguments in [9] and a general position lemma in [6].

We set up some notation. Let M be a locally standard T–manifold of dimension $2n$ and $q\colon M \to M/T =: Q$ be the quotient map. We assume that the principal T–bundle obtained from the free T–orbits is trivial.

We use the identification

$$N := \operatorname{Hom}(S^1, T) = H_1(T) = H_2(BT).$$

Let M_i for $i = 1, \dots, m$ be the characteristic manifolds of M and $Q_i := M_i/T$, so Q_i is a facet of Q. Let v_i be a primitive element of $N = \operatorname{Hom}(S^1, T)$ such that $v_i(S^1)$ fixes M_i pointwise (v_i is uniquely determined up to sign). Let $\widehat{N}$ be the sublattice of N generated by $v_1, \dots, v_m$.

All homology groups are taken with $\mathbb{Z}$ coefficients unless otherwise stated.

Let $Q^{(n-2)}$ be the union of all $(n-2)$–faces of Q. Let Q' be a "small closed tubular neighborhood" of $Q^{(n-2)} \subset Q$ and let $M' = q^{-1}(Q')$.

Lemma 2.1 [9, Proposition 3.1] $H_1(M) \cong H_1(M \setminus \operatorname{Int} M') \cong H_1(Q) \oplus N/\widehat{N}$.

Proof Note that M' is homotopy equivalent to $q^{-1}(Q^{(n-2)})$ and $q^{-1}(Q^{(n-2)})$ is a finite union of codimension-4 manifolds, so $H_1(M) \cong H_1(M \setminus \operatorname{Int} M')$ by Lemma 5.3

and Remark 5.4 in [6]. For the calculation of $H_1(M \setminus \operatorname{Int} M')$ we can use the same argument as in [9] as follows. Let $Q^0 := (\operatorname{Int} Q) \cap (Q \setminus Q')$ and Q^1 be the intersection of $Q \setminus Q'$ and a small open neighborhood of $Q \setminus \operatorname{Int} Q$ in Q. Since

$$q^{-1}(Q^0) \simeq Q \times T, \qquad q^{-1}(Q^1) \simeq \bigsqcup_{i=1}^{m} (Q_i \times T/v_i(S^1)),$$

$$q^{-1}(Q^0) \cap q^{-1}(Q^1) \simeq \bigsqcup_{i=1}^{m} (Q_i \times T), \quad q^{-1}(Q^0 \cup Q^1) = M \setminus M',$$

the Mayer–Vietoris exact sequence for the triple $(M \setminus M', q^{-1}(Q^0), q^{-1}(Q^1))$ yields the exact sequence

$$(1) \quad \cdots \to \bigoplus_{i=1}^{m} H_1(Q_i \times T) \xrightarrow{f_1} H_1(Q \times T) \oplus \bigoplus_{i=1}^{m} H_1(Q_i \times T/v_i(S^1)) \to H_1(M \setminus M') \\ \to \bigoplus_{i=1}^{m} H_0(Q_i \times T) \xrightarrow{f_0} H_0(Q \times T) \oplus \bigoplus_{i=1}^{m} H_0(Q_i \times T/v_i(S^1)).$$

As is easily seen, f_0 is injective, so

$$(2) \qquad H_1(M \setminus M') \cong \operatorname{coker} f_1.$$

We write f_1 as (ψ_1, φ_1) according to the decomposition of the target space. Since

$$\varphi_1 \colon \bigoplus_{i=1}^{m} H_1(Q_i \times T) \to \bigoplus_{i=1}^{m} H_1(Q_i \times T/v_i(S^1)),$$

which is f_1 composed with the projection on the second factor, is surjective, one has

$$(3) \qquad \operatorname{coker} f_1 \cong H_1(Q \times T)/\psi_1(\ker \varphi_1).$$

Since $H_1(Y \times T) = H_1(Y) \oplus H_1(T)$ for any topological space Y, elements in $\ker \varphi_1$ are of the form $(c_1 v_1, \dots, c_m v_m)$ with integers c_i. It follows that

$$(4) \qquad H_1(Q \times T)/\psi_1(\ker \varphi_1) \cong H_1(Q) \oplus N/\widehat{N}.$$

Thus the lemma follows from (2), (3) and (4). □

Next we consider the fundamental group $\pi_1(M)$. Since the complement of the set of principal T–orbits $q^{-1}(\operatorname{Int} Q)$ is of codimension 2, the inclusion map ι from $q^{-1}(\operatorname{Int} Q)$ to M induces an epimorphism

$$\iota_* \colon \pi_1(q^{-1}(\operatorname{Int} Q)) \to \pi_1(M)$$

by Lemma 5.3 in [6] and, since the principal T–bundle $q^{-1}(\operatorname{Int} Q) \to \operatorname{Int} Q$ is assumed to be trivial and $\operatorname{Int} Q$ is homotopy equivalent to Q, the above epimorphism can be regarded as

$$\iota_*\colon \pi_1(Q) \times \pi_1(T) \to \pi_1(M). \tag{5}$$

We note that the kernel of ι_* is contained in the second factor $\pi_1(T)$ because the composition $q_* \circ \iota_*$, where $q_*\colon \pi_1(M) \to \pi_1(Q)$, agrees with the projection on the first factor.

Note that a principal T–orbit near M_i deforms to a T–orbit in M_i, which means that the element $v_i \in \operatorname{Hom}(S^1, T) = N$ maps to zero in $\pi_1(M)$ via ι_*. Since this holds for any i, one can conclude that the subgroup $\widehat{N}$ of N generated by v_i for all i lies in the kernel of ι_*. Therefore, ι_* in (5) induces an epimorphism

$$\psi\colon \pi_1(Q) \times N/\widehat{N} \to \pi_1(M), \tag{6}$$

where the composition $q_* \circ \psi$ is the projection on the first factor $\pi_1(Q)$, so that the kernel of ψ is contained in $N/\widehat{N}$.

Theorem 2.2 *The homomorphism ψ in* (6) *is an isomorphism.*

Proof The map ψ in (6) induces an epimorphism

$$H_1(Q) \times ((N/\widehat{N})/\ker \psi) \to H_1(M). \tag{7}$$

By Lemma 2.1, we have

$$H_1(M) \cong H_1(Q) \oplus N/\widehat{N}.$$

This together with (7) implies that $\ker \psi$ is trivial and hence ψ is an isomorphism. □

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