The fundamental group of locally standard $T$–manifolds

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We calculate the fundamental group of locally standard $T$–manifolds under the assumption that the principal $T$–bundle obtained from the free $T$–orbits is trivial. This family of manifolds contains nonsingular toric varieties which may be non-compact, quasitoric manifolds and toric origami manifolds with coörientable folding hypersurface. Although the fundamental groups of the above three kinds of manifolds are well-studied, we give a uniform and simple method to generalize the formulas of their fundamental groups.

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1 Introduction

Torus manifolds, introduced by A Hattori and M Masuda [5], are a generalization of toric manifolds, compact nonsingular toric varieties. A torus manifold is a $2n$–dimensional, closed, connected, orientable, smooth manifold $M$ with an effective smooth action of an $n$–dimensional torus $T \cong (S^1)^n$ such that the fixed-point set $M^T \neq \emptyset$. A torus manifold $M$ is called locally standard if every point of $M$ has a $T$–invariant open neighborhood equivariantly diffeomorphic to a $T$–invariant open set of a faithful representation space of $T$. In this case the orbit space $M/T$ is a nice manifold with corners. For more details about locally standard torus manifolds, readers can see V Buchstaber and T Panov’s nice book [1]. If a torus manifold is locally standard, $q_*: \pi_1(M) \to \pi_1(M/T)$ is an isomorphism, where $q_*$ is induced by the quotient map $q: M \to M/T$. This was proved by Wiemeler [10] and Yoshida [11]. If we allow $M$ to be noncompact, nonorientable and $M^T = \emptyset$, ie $M$ is a locally standard $T$–manifold, $q_*$ may fail to be an isomorphism. Nonsingular toric varieties (see Fulton [4] or Cox, Little and Schenck [2]), quasitoric manifolds (see Davis and Januszkiewicz [3]) and toric origami manifold with coörientable folding hypersurfaces (see Holm and Pires [7]) are typical examples of locally standard $T$–manifolds. We use the identification $N := \text{Hom}(S^1, T) = H_1(T) = H_2(BT)$ and let $M_i$ for $i = 1, \ldots, m$

$^1$Here the locally standard $T$–manifold $M$ is slightly different from the definition by Buchstaber and Panov [1] since we allow the manifold to be noncompact and nonorientable, but we require the orbit space $M/T$ to have finitely many faces.
be the characteristic manifolds of \( M \), ie \( M_i \) is a connected codimension-2 submanifold of \( M \) fixed pointwise by a circle subgroup \( S_i \subset T \). Thus, \( M_i \) determines a primitive element \( v_i \in N \) up to sign such that \( S_i = v_i(S^1) \). Let \( \hat{N} \) be the sublattice of \( N \) generated by \( v_1, \ldots, v_m \). Inspired by the arguments of Kuwata, Masuda and Zeng [9] and T Holm and A R Pires’s results in [8], we can use a simple method to generalize the known results to locally standard \( T \)–manifolds whose free \( T \)–orbits is a trivial \( T \)–bundle. The following is our main result (see Theorem 2.2).

**Theorem** Let \( M \) be a locally standard \( T \)–manifold such that the free part of the \( T \)–action is a trivial torus bundle. Then

\[
\pi_1(M) \cong \pi_1(M/T) \times N/\hat{N}.
\]

2 The fundamental group of locally standard \( T \)–manifolds

In this section we will deduce the formula for \( \pi_1(M) \) based on some arguments in [9] and a general position lemma in [6].

We set up some notation. Let \( M \) be a locally standard \( T \)–manifold of dimension \( 2n \) and \( q: M \to M/T =: Q \) be the quotient map. We assume that the principal \( T \)–bundle obtained from the free \( T \)–orbits is trivial.

We use the identification

\[
N := \text{Hom}(S^1, T) = H_1(T) = H_2(BT).
\]

Let \( M_i \) for \( i = 1, \ldots, m \) be the characteristic manifolds of \( M \) and \( Q_i := M_i/T \), so \( Q_i \) is a facet of \( Q \). Let \( v_i \) be a primitive element of \( N = \text{Hom}(S^1, T) \) such that \( v_i(S^1) \) fixes \( M_i \) pointwise (\( v_i \) is uniquely determined up to sign). Let \( \hat{N} \) be the sublattice of \( N \) generated by \( v_1, \ldots, v_m \).

All homology groups are taken with \( \mathbb{Z} \) coefficients unless otherwise stated.

Let \( Q^{(n-2)} \) be the union of all \((n-2)\)–faces of \( Q \). Let \( Q' \) be a “small closed tubular neighborhood” of \( Q^{(n-2)} \subset Q \) and let \( M' = q^{-1}(Q') \).

**Lemma 2.1** [9, Proposition 3.1] \( H_1(M) \cong H_1(M \setminus \text{Int } M') \cong H_1(Q) \oplus N/\hat{N} \).

**Proof** Note that \( M' \) is homotopy equivalent to \( q^{-1}(Q^{(n-2)}) \) and \( q^{-1}(Q^{(n-2)}) \) is a finite union of codimension-4 manifolds, so \( H_1(M) \cong H_1(M \setminus \text{Int } M') \) by Lemma 5.3.
and Remark 5.4 in [6]. For the calculation of $H_1(M \setminus \text{Int} M')$ we can use the same argument as in [9] as follows. Let $Q^0 := (\text{Int} Q \cap (Q \setminus Q'))$ and $Q^1$ be the intersection of $Q \setminus Q'$ and a small open neighborhood of $Q \setminus \text{Int} Q$ in $Q$. Since

$$q^{-1}(Q^0) \simeq Q \times T, \quad q^{-1}(Q^1) \simeq \bigcup_{i=1}^m (Q_i \times T/v_i(S^1)),$$

the Mayer–Vietoris exact sequence for the triple $(M \setminus M', q^{-1}(Q^0), q^{-1}(Q^1))$ yields the exact sequence

$$\cdots \rightarrow \bigoplus_{i=1}^m H_1(Q_i \times T) \xrightarrow{f_1} H_1(Q \times T) \oplus \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)) \rightarrow H_1(M \setminus M')$$

$$\rightarrow \bigoplus_{i=1}^m H_0(Q_i \times T) \xrightarrow{f_0} H_0(Q \times T) \oplus \bigoplus_{i=1}^m H_0(Q_i \times T/v_i(S^1)).$$

As is easily seen, $f_0$ is injective, so

$$H_1(M \setminus M') \cong \text{coker } f_1. \quad (2)$$

We write $f_1$ as $(\psi_1, \varphi_1)$ according to the decomposition of the target space. Since

$$\varphi_1: \bigoplus_{i=1}^m H_1(Q_i \times T) \rightarrow \bigoplus_{i=1}^m H_1(Q_i \times T/v_i(S^1)),$$

which is $f_1$ composed with the projection on the second factor, is surjective, one has

$$\text{coker } f_1 \cong H_1(Q \times T)/\psi_1(\ker \varphi_1). \quad (3)$$

Since $H_1(Y \times T) = H_1(Y) \oplus H_1(T)$ for any topological space $Y$, elements in $\ker \varphi_1$ are of the form $(c_1v_1, \ldots, c_mv_m)$ with integers $c_i$. It follows that

$$H_1(Q \times T)/\psi_1(\ker \varphi_1) \cong H_1(Q) \oplus \hat{N}/\hat{N}. \quad (4)$$

Thus the lemma follows from (2), (3) and (4). \qed

Next we consider the fundamental group $\pi_1(M)$. Since the complement of the set of principal $T$–orbits $q^{-1}(\text{Int} Q)$ is of codimension 2, the inclusion map $\iota$ from $q^{-1}(\text{Int} Q)$ to $M$ induces an epimorphism

$$\iota_*: \pi_1(q^{-1}(\text{Int} Q)) \rightarrow \pi_1(M).$$
by Lemma 5.3 in [6] and, since the principal $T$–bundle $q^{-1}(\text{Int } Q) \to \text{Int } Q$ is assumed to be trivial and $\text{Int } Q$ is homotopy equivalent to $Q$, the above epimorphism can be regarded as

$$\iota_*: \pi_1(Q) \times \pi_1(T) \to \pi_1(M).$$

We note that the kernel of $\iota_*$ is contained in the second factor $\pi_1(T)$ because the composition $q_* \circ \iota_*$, where $q_*: \pi_1(M) \to \pi_1(Q)$, agrees with the projection on the first factor.

Note that a principal $T$–orbit near $M_i$ deforms to a $T$–orbit in $M_i$, which means that the element $v_i \in \text{Hom}(S^1, T) = N$ maps to zero in $\pi_1(M)$ via $\iota_*$. Since this holds for any $i$, one can conclude that the subgroup $\hat{N}$ of $N$ generated by $v_i$ for all $i$ lies in the kernel of $\iota_*$. Therefore, $\iota_*$ in (5) induces an epimorphism

$$\psi: \pi_1(Q) \times N/\hat{N} \to \pi_1(M),$$

where the composition $q_* \circ \psi$ is the projection on the first factor $\pi_1(Q)$, so that the kernel of $\psi$ is contained in $N/\hat{N}$.

**Theorem 2.2**  The homomorphism $\psi$ in (6) is an isomorphism.

**Proof**  The map $\psi$ in (6) induces an epimorphism

$$H_1(Q) \times ((N/\hat{N})/ \text{ker } \psi) \to H_1(M).$$

By Lemma 2.1, we have

$$H_1(M) \cong H_1(Q) \oplus N/\hat{N}.$$

This together with (7) implies that $\text{ker } \psi$ is trivial and hence $\psi$ is an isomorphism.

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