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Ends of Schreier graphs of hyperbolic groups

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We study the number of ends of a Schreier graph of a hyperbolic group. Let G be a hyperbolic group and let H be a subgroup of G. In general, there is no algorithm to compute the number of ends of a Schreier graph of the pair (G, H). However, assuming that H is a quasiconvex subgroup of G, we construct an algorithm.

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1 Introduction

The theory of ends was introduced by H Freundenthal for topological spaces [12]. The ends are often depicted as the connected components of some boundary of the space, where each end indicates a distinct way to move to infinity within the space. Later, Freudenthal extended his concept to finitely generated groups [13]. This theory was well studied in the middle of the last century.

For instance, H Hopf [18, Section 2.11] showed that finitely generated groups have 0, 1, 2 or infinitely many ends and, later, J Stallings [30, 4.A.6] proved that finitely generated groups have more than one end if and only if they split either as a free product with amalgamation or as an HNN extension over a finite group.

Meanwhile, A Borel adds a new perspective to the theory of ends by studying ends of a group relatively to a subgroup [4]. If G is a finitely generated group and H a subgroup, the *number of relative ends* of the pair (G, H) is the number of ends of a corresponding Schreier graph, that is the quotient of a Cayley graph of G under the action of H. Later, C Houghton [19] and then P Scott [28] pursued this work.

The theory of ends remains of great interest. In 1999, V Gerasimov studied connectedness of the boundary of hyperbolic groups with an algorithmic approach and obtained the following theorem:

Theorem 1.1 [14] *There is an algorithm that, given a finite presentation of a hyperbolic group, computes the number of ends of this group.* □

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The question of an analogous result for relative ends of a pair of groups arises naturally. Surprisingly, the answer isn't straightforward. Indeed, we prove the following result:

Theorem 1.2 *There exist pairs of groups (G, H), where G is a hyperbolic group, for which it is impossible to decide algorithmically if the pair has 0, 2 or infinitely many relative ends.*

Let us take a look at one of the few significant examples that can be clearly outlined. Consider a closed oriented surface S of genus $g \geq 2$ endowed with a hyperbolic metric and denote by $G = \pi_1(S)$ the surface group. Consider also a subsurface Σ of S with totally geodesic boundary. The universal cover of the surface S with basepoint in Σ is $\mathbb{H}^2$. In this space, a connected component of the inverse image of Σ is a convex polygon $\mathcal{C}$ having infinitely many sides (see Figure 1). If H denotes the fundamental group of Σ, the quotient $\mathcal{C}/H$ is isomorphic to Σ.

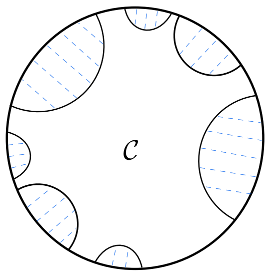

Figure 1: The convex set $\mathcal{C}$ in $\mathbb{H}^2$ and a projection onto $\mathcal{C}$

Note that there exists an orthogonal H–equivariant projection of $\mathbb{H}^2$ onto the convex polygon $\mathcal{C}$; see Bridson and Haefliger [6, II.2]. The space formed by taking the quotient of the hyperbolic plane $\mathbb{H}^2$ by the group H is composed of a convex core Σ with a funnel attached to each connected component of its boundary (see Figure 2). This space inherits from $\mathbb{H}^2$ a projection onto its convex core, which, extended to infinity, identifies the connected components of the boundary of Σ to the ends of the quotient space.

So the number of relative ends of the pair (G, H) is equal to the number of connected components of the boundary of the convex core Σ.

In light of this example, detecting the number of relative ends of a pair of groups may still be possible in specific cases. Recall that the number of relative ends of a pair of groups is essentially the number of ends of some quotient space. It appears

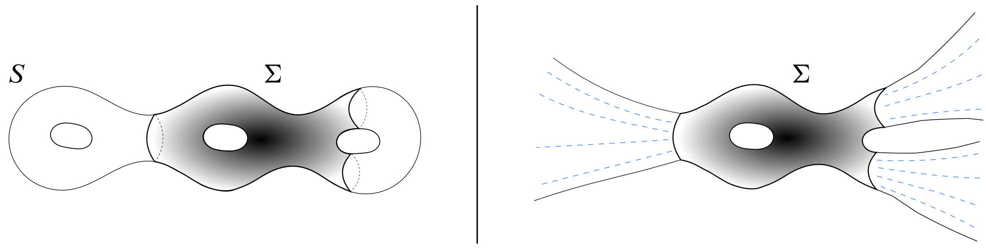

Figure 2: Example of a closed surface S of genus 3 and a subsurface Σ with totally geodesic boundary

that for the pairs of groups referred to in Theorem 1.2, the subgroup is usually not quasiconvex (Remark 3.3). So we restrict ourselves to the study of the quotient of a proper hyperbolic geodesic metric space X by a quasiconvex-cocompact group H of isometries of X. The quotient space X/H has good properties as it is hyperbolic and satisfies a geodesic extension property. A careful study of the ends of this quotient space X/H will lead us to the main result of this paper, namely:

Theorem 1.3 *There exists an algorithm to compute the number of ends of a one-ended hyperbolic group relatively to a quasiconvex subgroup.*

To study the ends of the former quotient space X/H, we identify a convex core and fix a basepoint $\overline{x}_0$ in it. Given the previous example of the surface group, the geometry of the boundary of the convex core may help us determine the number of ends of X/H. So we determine a number R_0 such that the sphere $S(\overline{x}_0, R_0)$ contains the convex core and we define an equivalence relation $\sim$ on this sphere (Definition 5.6). In Section 5, we establish a bijection between the set of ends of X/H and the set of equivalence classes for the relation $\sim$ on $S(\overline{x}_0, R_0)$, via what comes to be known as their shadows (Definition 5.9). Therefore, the number of ends of X/H is equal to the number of equivalence classes on $S(\overline{x}_0, R_0)$. Applying this result to group theory, the number of relative ends of a one-ended hyperbolic group and a quasiconvex subgroup is equal to the number of equivalence classes of the sphere $S(\overline{x}_0, R_0)$ in the corresponding quotient space. For this reason, the construction of an algorithm to compute the number of these equivalence classes leads to Theorem 1.3.

Section 2 gathers definitions and properties dealing with hyperbolicity and quasiconvexity and recalls the graph-theoretic approach of the theory of ends. In Section 3, Markov properties and Rips construction are combined to prove that, in general, there is no algorithm to compute the number of relative ends of a pair of groups. Section 4 presents the quotient spaces and some of their interesting properties, including a new proof of the hyperbolicity of this space. Section 5 is devoted to the study of the ends of the quotient spaces, where we relate connected components of the boundary of a convex core to the ends. Finally, Section 6 presents an algorithm to compute the number of relative ends of a hyperbolic group and a quasiconvex subgroup.

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2 Background

This section introduces the notations and gathers basic facts on hyperbolic geometry. For more details, the reader can refer to [16] but also [9; 15; 6].

2.1 Hyperbolic spaces and groups

Recall some definitions of the original paper of M Gromov [16].

Let (X, d) be a metric space. A *geodesic* in X is an isometric embedding of an interval of $\mathbb{R}$ in X. The metric space X is called *geodesic* if every two points in X can be joined by a geodesic. For x_0 a basepoint in X, the *Gromov product* of two points $x, y \in X$ at x_0 is given by

$$\langle x, y\rangle_{x_0} = \tfrac{1}{2}(d(x_0, x) + d(x_0, y) - d(x, y)).$$

A geodesic metric space X is *hyperbolic* if there exists a real number $\delta \geq 0$ such that for all points $x, y, z \in X$,

$$\langle x, y\rangle_{x_0} \geq \min\{\langle x, z\rangle_{x_0}, \langle y, z\rangle_{x_0}\} - \delta.$$

It turns out that a different choice of basepoint only changes the value of the constant δ, as explained in [9, page 2].

There are various equivalent definition of hyperbolicity (see [16, Section 6.3; 9, 1.3.6]). Thus, a geodesic metric space is also *hyperbolic* if there exists $\delta \geq 0$ such that each side of a geodesic triangle is contained in the δ–neighbourhood of the union of the two other sides. Such triangles are called *δ–thin*.

Moreover, Section 8 of [16] provides a classification of isometries of hyperbolic spaces. In particular, *hyperbolic isometries* of a hyperbolic space X are elements h such that for any point $x \in X$, the map $n \mapsto h^n x$ is a quasi-isometric embedding of $\mathbb{Z}$ into X.

Remark 2.1 We use here the convention of I Kapovich and N Benakli [21, 2.22]: a group acts *geometrically* on a geodesic space if it acts by isometries, cocompactly and properly discontinuously (for any compact K in the space, there is a finite number of elements g of this group such that K intersects $g \cdot K$). In particular, the Švarc–Milnor lemma [32; 25] asserts that if a group acts geometrically on a proper geodesic metric space then the orbit map is a quasi-isometry.

Recall that a metric space is *proper* if every closed ball is compact. A proper hyperbolic geodesic metric space X can be compactified by attaching its boundary ∂X. To do so, say that two geodesic rays c_1 and c_2 in X are *equivalent* if there exists $K > 0$ such that $d(c_1(t), c_2(t)) \leq K$ for all $t \geq 0$. Then the *boundary* ∂X of X is the set of equivalence classes of geodesic rays in X for this relation. The space $X \cup \partial X$ is endowed with the induced topology from X (see eg Chapter 7 of [15]). For a broad overview on boundaries of hyperbolic spaces we refer the reader to Kapovich and Benakli [21].

According to [9, 10.6.6], every hyperbolic isometry fixes two points on the boundary ∂X, which are attractive and repulsive points for the action of this isometry on ∂X. Moreover, in a δ–hyperbolic space, geodesic triangles with vertices on the boundary are 4δ–thin, as proven eg in [22, 3.9]; these triangles are often called *ideal triangles*.

The approximating tree technique provided by the following lemma is often hidden behind powerful results (see Proposition 2.5 and Theorem 4.3) as it transcribes properties from trees (where $\delta = 0$) to hyperbolic spaces.

Lemma 2.2 [16, Section 6.1] *Let (X, d) be a δ–hyperbolic space, $x_0, x_1, \dots, x_n$ be a set of $n+1$ points of $X \cup \partial X$ and Y be the union of n geodesic segments joining x_0 to other points. Assume that $2n \leq 2^k + 1$. There exist a simplicial tree (T, d_T) and a continuous map $f \colon X \to T$ such that:*

1. $d(u, v) - 2k\delta \leq d_T(f(u), f(v)) \leq d(u, v)$ *for all* $u, v \in Y$.
2. *The restriction of f to each segment is isometric.* □

Hyperbolic spaces also satisfy the following geodesic extension property:

Proposition 2.3 [3, 3.1] *Let X be a geodesic proper δ–hyperbolic space with basepoint x_0 and $\#\partial X \geq 2$. Let G be a group acting geometrically on X. There exists a constant $\mu \geq 0$ such that for any point x in X, there is a geodesic ray emanating from x_0 passing within distance μ from x.* □

The constant μ is called a *constant of geodesic extension* for X.

Remark 2.4 If x is a point in X and $c\colon \mathbb{R}_{\geq 0} \to X$ is a geodesic ray emanating from x_0 and passing within distance μ from x, then [6, III.H, Proposition 1.15] gives $d\big(x, c(d(x_0, x))\big) \leq 2\mu + 2\delta$ with δ a hyperbolicity constant for X.

A finitely generated group G is *hyperbolic* if, for a finite generating system $\mathcal{S}$ of G, the Cayley graph $\Gamma_{\mathcal{S}}(G)$ is hyperbolic. This definition turns out to be independent on the choice of the finite generating set as a change of generating set produces a Cayley graph quasi-isometric to the first (see [16, Section 2.3.E]). A hyperbolic group G acts naturally by left multiplication on its Cayley graph $\Gamma_{\mathcal{S}}(G)$ and this action continuously extends to an action of G on the boundary of $\Gamma_{\mathcal{S}}(G)$; then the *boundary* of the hyperbolic group G is the boundary of $\Gamma_{\mathcal{S}}(G)$. As a quasi-isometry between hyperbolic spaces induces a homeomorphism between their boundary (see [9, 3.2.2]), this definition does not depend on the generating set up to homeomorphism.

2.2 Quasiconvexity

Let (X, d) be a geodesic proper metric space. A subset Y of X is *quasiconvex* if there exists a constant $\varepsilon \geq 0$ such that all geodesics joining two points of Y in X are contained in $Y^{+\varepsilon} = \{x \in X \mid d(x, Y) \leq \varepsilon\}$, the ε–neighbourhood of Y. According to [16, Section 7.3.A], if X is a δ–hyperbolic space and $Y \subset X$ is an ε–quasiconvex subset, then $Y^{+\rho}$ is 2δ–quasiconvex, for all $\rho \geq \varepsilon$. For instance, every quasigeodesic is a quasiconvex set according to the theorem of stability [16, Section 7.2].

The concept of quasiconvexity extends naturally to groups: a subgroup H of a finitely generated group G is *quasiconvex* if for some (and all) finite generating set $\mathcal{S}$ of G, H is a quasiconvex subset of $\Gamma_{\mathcal{S}}(G)$.

Quasiconvex subgroups of hyperbolic groups have interesting properties: for instance, a quasiconvex subgroup of a hyperbolic group is in turn finitely generated, hyperbolic and so finitely presentable (see Chapter 10 of [9]).

If the space X is δ–hyperbolic, recall that the limit set of a subgroup $H \leq \mathrm{Isom}(X)$ is the set $\Lambda H = \overline{H \cdot x} \cap \partial X$ of accumulation points in ∂X of an orbit of a point

$x \in X$ under the action of H. Then the *weak convex hull* of the limit set of H, denoted by $C(\Lambda H)$, is the union of all bi-infinite geodesics with endpoints in ΛH. In particular, $C(\Lambda H)$ is 8δ–quasiconvex.

The group H is *quasiconvex-cocompact* if it acts geometrically on $C(\Lambda H)$.

Let X be a geodesic proper metric space and Y a subset of X. Let $\eta \geq 0$. An *η–projection* is a map $\pi\colon X \to Y$ such that every point $x \in X$ satisfies $d(x, y) \leq d(x, Y) + \eta$. A fundamental property of projections on quasiconvex sets is the following contraction property:

Proposition 2.5 [8] *Let X be a δ–hyperbolic geodesic space and Y an ε–quasi-convex subset of X. Let $\pi\colon X \to Y$ be an η–projection. For all points $x, x' \in X$, we have*

$$d(\pi(x), \pi(x')) \leq \max\{\tau, 2\tau + d(x, x') - d(x, \pi(x)) - d(x', \pi(x'))\},$$

where $\tau = 12\delta + 2\varepsilon + 2\eta$. Furthermore, we also have

$$d(\pi(x), \pi(x')) \leq \tau + d(x, x'). \qquad \square$$

If H is a group acting on X, it will be convenient to assume that ε–projections on H–invariant sets are also H–equivariant, as in the proof of Proposition 4.1 for example.

2.3 Ends and relative ends

This part recalls the graph-theoretic approach used to define ends of finitely generated group in [6, pages 144–148].

A map between two topological spaces is *proper* if the inverse image of any compact set by this map is also a compact set. Two proper rays c and c' in a topological space X are *converging to the same end* of X if for every compact set $K \subset X$ there exists an integer N such that $c([N, +\infty))$ and $c'([N, +\infty))$ are contained in the same path-connected component of $X \setminus K$. It defines an equivalence relation on proper rays. The set of these equivalence classes form the *set of ends* of the topological space X, denoted by $\mathrm{Ends}(X)$. As a quasi-isometry between geodesic proper spaces induces a homeomorphism between their sets of ends, the set of ends of a finitely generated group is the set of ends of a Cayley graph of this group. The *number of ends* of a finitely generated group G is denoted by $e(G)$.

Let G be a group generated by a finite set $\mathcal{S}$ and let H be a subgroup of G. The *Schreier graph* $\Gamma_{\mathcal{S}}(G, H)$ of G with respect to H, also called the relative Cayley graph, is the quotient of $\Gamma_{\mathcal{S}}(G)$ under the left action of H.

Remark 2.6 If H is a normal subgroup of G, the Schreier graph $\Gamma_{\mathcal{S}}(G, H)$ is exactly the Cayley graph of the group G/H associated with $\mathcal{S}$. Indeed, in this case, the cosets of H in G are the orbits of elements of G under the action of H.

As already mentioned in the introduction, the *number of relative ends of the pair* (G, H) *associated with* $\mathcal{S}$ is the number of ends of the Schreier graph $\Gamma_{\mathcal{S}}(G, H)$. The aim of the following work is to give an algorithm to determine the number of relative ends of a pair of groups.

3 The quasiconvex hypothesis

Computing the number of relative ends of a pair of groups happens to be more difficult than expected. The following theorem states that, in general, there is no algorithm to compute the number of relative ends of a pair of groups.

Theorem 3.1 *We can construct a pair of groups* (G, H) *for which it is impossible to decide algorithmically if*

(1) G/H *is finite;*

(2) *the pair* (G, H) *has two relative ends;*

(3) *the pair* (G, H) *has infinitely many relative ends.*

The proof of this theorem mainly relies on the Rips construction given in [27]. Given a finite presentation of a group Q, the Rips construction furnishes a hyperbolic group G (in fact a small cancellation group) and a subgroup H of G for which the number of relative ends depends on properties of the original group presentation that are not recursively recognizable, namely Markov properties.

As explained in [23, IV.4], a *Markov property* of a finitely presented group is a property $\mathcal{P}$ for which there exist two finitely presented groups, G_+ and G_-, such that G_+ satisfies $\mathcal{P}$ and G_- cannot be embedded in any finitely presented group which satisfies $\mathcal{P}$. A large range of well-known properties of finitely presented groups are Markov properties (see also [24] for more details). The following Markov properties are used in the proof of Theorem 3.1:

(1) "Being the trivial group" (ie the group reduced to a single element) with $G_+ = 1$ and $G_- = \mathbb{Z}_2$ as witnesses.

(2) "Being a group of cardinality 2" with $G_+ = \mathbb{Z}_2$ and $G_- = \mathbb{Z}_5$.

(3) "Being of cardinality less than or equal to 2" with $G_+ = \mathbb{Z}_2$ and $G_- = \mathbb{Z}_5$.

The main result about Markov properties is the Adyan–Rabin theorem [1; 26]. It states that Markov properties are not recursively recognizable. This will be a key point in the proof of Theorem 3.1.

We will also need the following lemma:

Lemma 3.2 *Let A be a finitely presented group. Let $Q = A * \mathbb{Z}_2$.*

(1) *The group Q is finite if and only if the group A is the trivial group.*

(2) *The group Q has 2 ends if and only if the group A is of cardinality 2.*

(3) *The group Q has infinitely many ends if and only if the cardinality of A is greater than 2.*

Proof Firstly, by Hopf's theorem, $e(Q) = 0$ if and only if the group Q is finite. But by definition, the free product of two groups is finite if and only if one of them is the trivial group. This implies that the group $Q = A * \mathbb{Z}_2$ is finite if and only if the group A is the trivial group. Then $e(Q) = 0$ if and only if A is the trivial group.

The second point of this result comes from P Scott and T Wall [29, Theorem 5.12], which indicates that $e(Q) = 2$ if and only if Q splits into an HNN extension $A*_C$ or in a free product with amalgamation $A *_C B$, where C is finite and $|A/C| = |B/C| = 2$. By comparison with $Q = A * \mathbb{Z}_2$, the group C is here the trivial group, $B = \mathbb{Z}_2$ and A is a group of cardinal 2. Thus, the group Q has two ends if and only if $Q = \mathbb{Z}_2 * \mathbb{Z}_2$.

The point (1) and Stallings' theorem [30, 4.A.6] imply that Q has infinitely many ends if and only if the cardinality of A is greater than 2. □

Proof of Theorem 3.1 Let A be a finitely presented group. By Adyan and Rabin's theorem, there is no algorithm to decide if A satisfies the properties:

(1) "Being the trivial group".

(2) "Being a group of cardinality 2".

(3) "Being of cardinality less than or equal to 2".

Then, by Lemma 3.2, there is no algorithm to decide if

(1) Q is finite;

(2) Q has 2 ends;

(3) Q has infinitely many ends.

Now denote by G and H groups associated to Q by the Rips construction. So the number of ends of Q is equal to the number of relative ends of the pair (G, H). Indeed, the Rips construction for the group Q furnishes a short exact sequence

$$1 \to H \to G \to Q \to 1,$$

where G is a hyperbolic group and H is a finitely presented subgroup of G. In particular, the group G/H is isomorphic to Q. Therefore the Schreier graph of the pair (G, H) is exactly the Cayley graph of Q according to the same generating set. Thus the number of relative ends of the pair (G, H) is equal to the number of ends of Q. In particular, there is no algorithm to decide if

(1) G/H is finite;

(2) the pair (G, H) has two relative ends;

(3) the pair (G, H) has infinitely many relative ends. □

It is possible to give an explicit finite group presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$ for which we cannot decide if the group defined is the trivial group (see [7] for instance). This example provides a group Q from which the Rips construction produces explicitly a hyperbolic group G and a subgroup H such that G/H is finite. And so this is an example for case (1) of the theorem. Likewise, with the presentations $\langle \mathcal{S}, b \mid \mathcal{R}, b^2 \rangle$ and $\langle \mathcal{S}, b \mid \mathcal{R}, b^3 \rangle$, we can explicit hyperbolic groups and a subgroup such that the number of relative ends is two or infinite.

Remark 3.3 In each case, the subgroup H raised by the Rips construction is a normal subgroup of the hyperbolic group G. If such a subgroup were quasiconvex then it would necessarily be finite or of finite index in G (see Proposition 3.9 in [2]). Added to the well-known properties of quasiconvex subgroups of hyperbolic group, this observation tends to consider this particular class of subgroups.

4 Quotient spaces

4.1 Description of quotient spaces

From now on, let X be a proper hyperbolic geodesic metric space and let G be a group acting geometrically on X. Let H be a quasiconvex-cocompact group of isometries of X. Denote by $C(\Lambda H)$ the weak convex hull of the limit of X and fix a point $x_0 \in C(\Lambda H)$. Let $\delta_X \geq 0$ be a constant large enough for X to be δ_X–hyperbolic and any orbit of H in $C(\Lambda H)$ to be δ_X–quasiconvex.

For every point x in X, denote by $\overline{x}$ the point in X/H equal to the orbit of x under the action of H. The quotient space X/H is endowed by a natural distance inherited from X: for all points x and y of X,

$$d_{X/H}(\overline{x}, \overline{y}) = \inf_{h \in H} d_X(x, hy) = d_X(x, H \cdot y).$$

(Distances' subscripts will be omitted when it is clear from the context.)

As H is quasiconvex-cocompact, the action of H on its weak convex hull $C(\Lambda H)$ is geometric and so the quotient $C(\Lambda H)/H$, called the *convex core*, is contained in a closed ball of X/H.

4.2 Hyperbolicity

Proposition 4.1 *There is a constant α such that for every hyperbolic element $h \in H$ and every point $x \in X$ with $d(x, H \cdot x_0) \geq \alpha$, we have $d(x, hx) \geq 100\delta_X$.*

Proof Let h be a hyperbolic element of H. As h is torsion-free, we may assume that h is also a primitive element (that is, there is no element k in the group and no integer $n \neq 0, 1, -1$ such that $h = k^n$). Denote by T the set of elements $x \in X$ such that $d(x, hx) < 100\delta_X$. To complete the proof, we intend to find a constant α such that the Hausdorff distance between T and the orbit $H \cdot x_0$ is less than α.

To do so, set $D := \operatorname{diam}(X/G)$, the diameter of the quotient of X by the action of G. So every element of T is a distance less than D from every orbit for the action of G. For $x \in T$ and $g \in G$, we have $d(x, gx_0) \leq D$ and $d(x, hx) < 100\delta_X$ and so

$$d(x_0, g^{-1}hgx_0) = d(gx_0, hgx_0) \leq 2D + 100\delta_X.$$

Note that for all $x \in X$, the set $\{k \in G \mid d(x, kx) < 2D + 100\delta_X\}$ is finite. Therefore, there exists a finite subset $\{g_1, \dots, g_r\}$ of G such that if $d(x_0, g^{-1}hgx_0) < 2D + 100\delta_X$ for some $g \in G$, then $g_i^{-1}hg_i = g^{-1}hg$ for some $i \in \{1, \dots, r\}$. This implies that $gg_i^{-1} \in C_G(h)$, the centralizer of h in G. Since $\langle h \rangle$ is of finite index in $C_G(h)$, choose right coset representatives $k_1, \dots, k_s$ of $\langle h \rangle$ in the centralizer $C_G(h)$. It follows that $gg_i^{-1} = h^n k_j$ for some $j \in \{1, \dots, s\}$ and some integer n. Then we have

$$d(gx_0, h^n x_0) = d(h^n k_j g_i x_0, h^n x_0) \leq \max\{d(x_0, k_v g_u x_0) \mid 1 \leq u \leq r, 1 \leq v \leq s\}.$$

Thus, $d(x, H \cdot x_0) \leq D + \max\{d(x_0, k_v g_u x_0) \mid 1 \leq u \leq r,\ 1 \leq v \leq s\}$ for all $x \in T$. So the Hausdorff distance between T and the orbit $H \cdot x_0$ is less than $\alpha := D + \max\{d(x_0, k_v g_u x_0) \mid 1 \leq u \leq r,\ 1 \leq v \leq s\}$. □

Remark 4.2 When the space X is a Cayley graph of the hyperbolic group G, we can find a more straightforward expression for α using Proposition 2.5:

- When the translation length of h is greater than $16\delta_X$, the inequality given by Proposition 2.5 implies that if $d(x, hx) < 100\delta_X$, we have $d(x, H \cdot x_0) < 132\delta_X$.
- Otherwise, we use Proposition 3.1 of [10]. It gives an integer $n_0 = (b_{8\delta_X})^2!$ (where $b_{8\delta_X}$ is the number of elements of G of length less than $8\delta_X$) such that, if h^- and h^+ are respectively repulsive and attractive fixed points for h, then h^{n_0} fixes a geodesic line joining h^- to h^+. So, by taking a multiple of n_0 if necessary, the first case applies to h^{n_0}.

Thus, when X is a Cayley graph, Proposition 4.1 holds with $\alpha := (132 + 100 n_0)\delta_X$.

The following theorem was stated in [16, 5.3] without proof. Later, it has been proved by Kapovich [20] and by R Foord [11], both unpublished.

Theorem 4.3 *Let X be a proper hyperbolic geodesic metric space and G be a group acting geometrically on X. Let H be quasiconvex-cocompact torsion-free group of isometries of X. Then X/H is a hyperbolic space.*

Proof We intend to find a constant $\delta_{X/H}$ such that all points $\bar{x}$, $\bar{y}$ and $\bar{z}$ in X/H satisfy

$$(\star) \qquad \langle \bar{y}, \bar{z} \rangle_{\bar{x}_0} \geq \min\{\langle \bar{y}, \bar{x} \rangle_{\bar{x}_0}, \langle \bar{x}, \bar{z} \rangle_{\bar{x}_0}\} - \delta_{X/H}.$$

To do so, set $\rho := \mathrm{diam}(C(\Lambda H)/H) + \alpha + \delta_X$, where $\mathrm{diam}(C(\Lambda H)/H)$ is the diameter of the convex core $C(\Lambda H)/H$ and α is the constant arising in Proposition 4.1. Fix three points $\bar{x}$, $\bar{y}$ and $\bar{z}$ in X/H.

Firstly, assume that $\min\{\langle \bar{y}, \bar{x} \rangle_{\bar{x}_0}, \langle \bar{x}, \bar{z} \rangle_{\bar{x}_0}\} > \rho$. Let x be a lift of $\bar{x}$ in X satisfying $d(x_0, x) = d(\bar{x}_0, \bar{x})$. Denote by y and z respective lifts of $\bar{y}$ and $\bar{z}$ such that $d(x, y) = d(\bar{x}, \bar{y})$ and $d(x, z) = d(\bar{x}, \bar{z})$. By definition of the Gromov product, these points satisfy

$$\langle y, x \rangle_{x_0} \geq \langle \bar{y}, \bar{x} \rangle_{\bar{x}_0} > \rho \quad \text{and} \quad \langle x, z \rangle_{x_0} \geq \langle \bar{x}, \bar{z} \rangle_{\bar{x}_0} > \rho.$$

Then, as X is δ_X–hyperbolic, the following inequality holds:

$$\langle y, z\rangle_{x_0} \geq \min\{\langle \overline{y}, \overline{x}\rangle_{\overline{x}_0}, \langle \overline{x}, \overline{z}\rangle_{\overline{x}_0}\} - \delta_X.$$

Since $\rho \geq \delta_X$, the set $(H \cdot x_0)^{+\rho}$ is $2\delta_X$–quasiconvex and there exists a δ_X–projection $\pi\colon X \to (H \cdot x_0)^{+\rho}$ on this set.

Let x_1 denote the point on a geodesic segment $[x_0, x]$ at distance ρ from x_0. Apply the approximating tree lemma, Lemma 2.2, to a geodesic triangle Δ with vertices x, x_1 and $\pi(x)$: there exists a simplicial tree T endowed with a simplicial metric d_T such that (for convenience, we denote by the same letter points in Δ and their image in T) $d(u,v) - 4\delta_X \leq d_T(u,v) \leq d(u,v)$ for all $u, v \in \Delta$ and $d_T(x_1, \pi(x)) = d(x_1, \pi(x))$.

If x' denotes a projection of x on the segment $[x_1, \pi(x)]$ in T, the approximating tree lemma gives

$$\begin{aligned} d_T(x, x') &\geq d(x, x') - 4\delta_X \\ &\geq d(x, (H \cdot x_0)^{+\rho}) - 6\delta_X \quad \text{(since } (H \cdot x_0)^{+\rho} \text{ is } 2\delta_X\text{–quasiconvex)} \\ &\geq d(x, \pi(x)) - 7\delta_X \qquad \text{(since } \pi \text{ is a } \delta_X\text{–projection on } (H \cdot x_0)^{+\rho}) \\ &\geq d_T(x, \pi(x)) - 7\delta_X. \end{aligned}$$

And so $d_T(\pi(x), x') = d_T(x, \pi(x)) - d_T(x, x') \leq 7\delta_X$. A similar computation with x_1 instead of $\pi(x)$ also gives $d_T(x_1, x') = d_T(x, x_1) - d_T(x, x') \leq 7\delta_X$. Then $d(x_1, \pi(x)) = d_T(x_1, \pi(x)) \leq 14\delta_X$ and so

$$d(x_0, \pi(x)) \leq \rho + 14\delta_X.$$

The contraction property of projections on quasiconvex sets indicates that we are in one of two cases:

- $d(\pi(x), \pi(y)) \leq 18\delta_X$ and the above inequality induces $d(x_0, \pi(y)) \leq \rho + 32\delta_X$.
- $d(\pi(x), \pi(y)) \leq 36\delta_X + d(x,y) - d(x, \pi(x)) - d(y, \pi(y))$, which is equivalent to

  $$\begin{aligned} d(x,y) &\geq d(\pi(y), x_0) - d(x_0, \pi(x)) + d(x, x_0) - d(x_0, \pi(x)) + d(y, \pi(y)) \\ &\qquad - 36\delta_X \\ &\geq d(\pi(y), x_0) + d(x, x_0) + d(y, \pi(y)) - 2\rho - 64\delta_X. \end{aligned}$$

  The choice of the lifts y and z and the definition of a δ_X–projection imply that

  $$d(\overline{x}, \overline{y}) \geq d(\pi(y), x_0) + d(\overline{x}, \overline{x}_0) + d(\overline{y}, \overline{x}_0) - 3\rho - 64\delta_X.$$

  Then the triangular inequality for $d(\overline{x}, \overline{y})$ induces $d(x_0, \pi(y)) \leq 3\rho + 64\delta_X$.

As π is a δ_X–projection, the point y satisfies

$$d(\overline{y},\overline{x}_0)-\rho\le d(y,\pi(y))\le d(\overline{y},\overline{x}_0)-\rho+\delta_X.$$

In every instance, we have

$$d(x_0,y)\le d(x_0,\pi(y))+d(\pi(y),y)\le 3\rho+64\delta_X+d(\overline{x}_0,\overline{y})-\rho+\delta_X.$$

Replacing y by z in the previous lines gives $d(x_0,z)\le 2\rho+65\delta_X+d(\overline{x}_0,\overline{z})$. Then the Gromov product of y and z at x_0 is equal to

$$\begin{aligned}\langle y,z\rangle_{x_0}&=\tfrac{1}{2}(d(x_0,y)+d(x_0,z)-d(y,z))\\&\le\tfrac{1}{2}(d(\overline{x}_0,\overline{y})+d(\overline{x}_0,\overline{z})-d(\overline{y},\overline{z}))+2\rho+65\delta_X\\&\le\langle\overline{y},\overline{z}\rangle_{\overline{x}_0}+2\rho+65\delta_X.\end{aligned}$$

This implies that if $\min\{\langle\overline{y},\overline{x}\rangle_{\overline{x}_0},\langle\overline{x},\overline{z}\rangle_{\overline{x}_0}\}>\rho$, the points $\overline{x}$, $\overline{y}$ and $\overline{z}$ satisfy the inequality ($\star$).

Furthermore, if $\min\{\langle\overline{y},\overline{x}\rangle_{\overline{x}_0},\langle\overline{x},\overline{z}\rangle_{\overline{x}_0}\}\le\rho$ then

$$\min\{\langle\overline{y},\overline{x}\rangle_{\overline{x}_0},\langle\overline{x},\overline{z}\rangle_{\overline{x}_0}\}-(2\rho+65\delta_X)<0.$$

As the Gromov product $\langle\overline{y},\overline{z}\rangle_{\overline{x}_0}$ is always positive, the inequality ($\star$) is still satisfied. Therefore, we can conclude that the space X/H is $\delta_{X/H}$–hyperbolic with $\delta_{X/H}:=2(\mathrm{diam}(C(\Lambda H)/H)+\alpha+\varepsilon)+65\delta_X$. □

From now on, we may also assume that the group H is torsion-free.

4.3 Covering and geodesic extension property

Proposition 4.4 *The space $X\setminus(H\cdot x_0)^{+\delta_{X/H}}$ is a covering space of the complement of the closed ball $\overline{B}(\overline{x}_0,\delta_{X/H})$ in X/H.*

Proof In light of Proposition 4.1 and Theorem 4.3, for every element $h\in H$ and every point $x\in X$ with $d(x,H\cdot x_0)>\delta_{X/H}$, there exists $d>0$ such that the closed ball $\overline{B}(\overline{x}_0,d)$ never intersects its translation $h\cdot\overline{B}(\overline{x}_0,d)$. By a classical result of topology (see [17, 1.40]), the quotient map from $X\setminus(H\cdot x_0)^{+\delta_{X/H}}$ to $(X\setminus(H\cdot x_0)^{+\delta_{X/H}})/H$ is then a covering map. Then, by definition of the distance in X/H, the quotient $(H\cdot x_0)^{+\delta_{X/H}}/H$ is equal to the ball $\overline{B}(\overline{x}_0,\delta_{X/H})$. Therefore, $X\setminus(H\cdot x_0)^{+\delta_{X/H}}$ is locally homeomorphic to the complement of $\overline{B}(\overline{x}_0,\delta_{X/H})$ in X/H. □

This proposition allows to prove that the quotient space X/H also satisfies the geodesic extension property of Proposition 2.3.

Corollary 4.5 *If δ_X is also a constant of geodesic extension for X, there exists $\mu \geq 0$ such that for every point $\overline{x}$ in X/H, there is a geodesic ray emanating from $\overline{x}_0$ passing within distance μ from x.*

Proof By Theorem 4.3, there exists some constant $\delta_{X/H}$ such that X/H is $\delta_{X/H}$–hyperbolic. Consider a point $\overline{x}$ in X/H in the complement of $B(\overline{x}_0, \delta_{X/H})$ and denote by x a lift of $\overline{x}$ in X such that $d(x_0, x) = d(\overline{x}_0, \overline{x})$. By assumption, there exists a geodesic ray $c\colon \mathbb{R}_{\geq 0} \to X$ emanating from x_0 passing within distance δ_X from x.

Using the covering map described in Proposition 4.4, the ray c furnishes a geodesic ray $\overline{c}\colon [\delta_{X/H}+1, +\infty) \to X/H$ emanating from the point $\overline{c(\delta_{X/H}+1)}$ and passing within distance δ_X from $\overline{x}$. This ray together with a geodesic ray emanating from $\overline{x}_0$ with the same endpoint at infinity and a geodesic segment $[\overline{x}_0, \overline{c(\delta_{X/H}+1)}]$ form an ideal triangle in X/H. As ideal triangles are $4\delta_{X/H}$–thin in X/H, the point $\overline{x}$ within distance $4\delta_{X/H} + \delta_X$ from a geodesic ray emanating from $\overline{x}_0$. So $4\delta_{X/H} + \delta_X$ is a suitable constant of geodesic extension for X/H. □

By replacing the constant $\delta_{X/H}$ by $4\delta_{X/H} + \delta_X$ if necessary, we can always assume that X/H satisfies the geodesic extension property with $\delta_{X/H}$.

5 Ends of quotient spaces

5.1 Boundary

By Theorem 4.3, the quotient space X/H is hyperbolic. This means that we can use results on hyperbolic spaces to obtain information on its ends. In particular, the following result relates the set of ends to the boundary of hyperbolic spaces:

Proposition 5.1 [15, 5.17] *Let X be a proper hyperbolic geodesic metric space. The natural map from the boundary ∂X of X to the set of ends* $\mathrm{Ends}(X)$ *of X is continuous and surjective and the fibres are connected components of ∂X.* □

Therefore, studying the boundary of the quotient space can give us information on its space of ends.

Proposition 5.2 *If H acts properly discontinuously on X then the quotient map $\partial X \setminus \Lambda H \to \partial X/H$ is a covering map.*

Proof According to the theorem of M Coornaert proven in [8], if the group H acts properly discontinuously on X, it also acts properly discontinuously on $\partial X \setminus \Lambda H$. Moreover, as H is torsion-free, its action is also free. This implies that for every point $u \in \partial X \setminus \Lambda H$, there exists an open set U containing u that does not intersect $h \cdot U$ for every nontrivial element $h \in H$. By a classical result of topology (see [17, I.3, Exercise 23]), it follows that the quotient map $\partial X \setminus \Lambda H \to (\partial X \setminus \Lambda H)/H$ is a covering map. The equality $(\partial X \setminus \Lambda H)/H = \partial X/H$ arises naturally. □

Proposition 5.3 *If ∂X is connected then $\partial X/H$ is locally path connected.*

The proof of this proposition relies on results discussed in detail in Section 6.

Proof A theorem proven independently by B Bowditch [5] and G A Swarup [31] asserts that if ∂X is connected then it has no global cut-point. Moreover, Section 3 of [3] indicates that if ∂X has no global cut-point, it is locally path connected. The combination of these results with Proposition 5.2 completes the proof. □

Although this result isn't sufficient to understand the set of ends of X/H, it brings into view the interesting work of M Bestvina and G Mess.

5.2 Large Bestvina–Mess condition

In the paper [3], Bestvina and Mess give a condition ($\ddagger_M$) on X that implies the local connectedness of the boundary of X. For our purpose, we are going to consider a slight variation of their condition. For $M > K > 0$, set:

($\ddagger_{M,K}$) There exists an integer $L > 0$ such that for every $R \geq K + 2\delta_X$ and for all points $x \in S(x_0, R)$ and $y \in S(x_0, R)^{+K}$ such that $d(x, y) \leq M$, there is a path of length less than or equal to L joining x to y in the complement of $\overline{B}(x_0, R - K - 2\delta_X)$.

This condition also characterizes the local connectedness of one-ended hyperbolic spaces. More specifically, the proof of Bestvina and Mess in [3, 3.2] also gives literally the following result:

Proposition 5.4 *If there exist constants M and K with $M \geq 4K + 18\delta_X + 6$ such that X satisfies the condition ($\ddagger_{M,K}$) then the boundary of X is locally connected.* □

Now we give a Bestvina–Mess condition for X/H. For $M > 0$, set:

($\dagger_M$) There exists an integer $L > 0$ such that for every $R \geq \max\{M + \delta_{X/H}, 8\delta_{X/H}\}$ and for all points $\bar{x}, \bar{y} \in S(\bar{x}_0, R)$ and $d(\bar{x}, \bar{y}) \leq M$, there is a path of length less than or equal to L joining $\bar{x}$ to $\bar{y}$ in the complement of $\bar{B}(\bar{x}_0, R - 8\delta_{X/H})$.

The following result outlines a link between the condition ($\ddagger_{M,K}$) for X and the condition ($\dagger_M$) for X/H:

Proposition 5.5 *If there exists a constant $M > 4\delta_{X/H}$ such that X satisfies the condition ($\ddagger_{M,4\delta_{X/H}}$) then the quotient space X/H satisfies the condition ($\dagger_M$).*

Proof Let us fix $M > 4\delta_{X/H}$ such that ($\ddagger_{M,4\delta_{X/H}}$) is satisfied by X. Consider points $\bar{x}$ and $\bar{y}$ in X/H such that $d(\bar{x}_0, \bar{x}) = d(\bar{x}_0, \bar{y}) = R \geq \max\{M + \delta_{X/H}, 8\delta_{X/H}\}$ and $d(\bar{x}, \bar{y}) \leq M$. Denote by x a lift of $\bar{x}$ in X satisfying $d(x_0, x) = d(\bar{x}_0, \bar{x})$. As $R > M + \delta_{X/H}$, lift $\bar{B}(\bar{x}, M)$ to $\bar{B}(x, M)$ using the covering map described in Proposition 4.4. By lifting a geodesic in this ball joining $\bar{x}$ to $\bar{y}$, we obtain a lift y of $\bar{y}$ in the ball $\bar{B}(x, M)$ such that $d(x, y) = d(\bar{x}, \bar{y})$. We are going to prove that these points x and y satisfy the conditions of ($\ddagger_{M,4\delta_{X/H}}$).

Let π be a δ_X–projection from X to the $2\delta_X$–quasiconvex thickened orbit of x_0, $(H \cdot x_0)^{+\delta_{X/H}}$. In particular, y satisfies

$$d(\bar{y}, \bar{x}_0) - \delta_{X/H} \leq d(y, \pi(y)) \leq d(\bar{y}, \bar{x}_0) - \delta_{X/H} + \delta_X.$$

The contraction property of projections on quasiconvex sets (Proposition 2.5) asserts that we have one of two cases:

- $d(\pi(x), \pi(y)) \leq 18\delta_X$ and we have the upper bound on $d(x_0, y)$

$$\begin{aligned} d(x_0, y) &\leq d(x_0, \pi(x)) + d(\pi(x), \pi(y)) + d(\pi(y), y) \\ &\leq \delta_{X/H} + 18\delta_X + d(\bar{x}_0, \bar{y}) + \delta_X \\ &\leq 2\delta_{X/H} + d(x_0, x). \end{aligned}$$

- $d(\pi(x), \pi(y)) \leq 36\delta_X + d(x, y) - d(x, \pi(x)) - d(y, \pi(y))$, which implies that

$$\begin{aligned} d(x, y) &\geq d(\pi(x), \pi(y)) + d(x, \pi(x)) + d(y, \pi(y)) - 36\delta_X \\ &\geq d(x_0, \pi(y)) + d(x, x_0) + d(y, \pi(y)) - 2d(x_0, \pi(x)) - 36\delta_X. \end{aligned}$$

  Then our choice of x and y yields that

$$d(\bar{x}, \bar{y}) \geq d(x_0, \pi(y)) + d(\bar{x}_0, \bar{x}) + d(\bar{x}_0, \bar{y}) - 3\delta_{X/H} - 36\delta_X.$$

The triangle inequality for $d(\bar{x},\bar{y})$ gives $d(x_0,\pi(y))\le 3\delta_{X/H}+36\delta_X$. Therefore, we obtain

$$d(x_0,y)\le d(x_0,\pi(y))+d(\pi(y),y)\le 4\delta_{X/H}+d(x_0,x).$$

In either case, x and y satisfy $d(x_0,x)=R$, $|d(x_0,x)-d(x_0,y)|\le 4\delta_{X/H}$ and $d(x,y)\le M$. As X satisfies $(\ddagger_{M,4\delta_{X/H}})$, there is an integer $L>0$ and a path of length less than or equal to L joining x to y in the complement of $\bar{B}(x_0,R-4\delta_{X/H}-2\delta_X)$.

Now, we show that the image of this path in X/H stays in the complement of $\bar{B}(\bar{x}_0,R-8\delta_{X/H})$. To do so, consider a point $z\in X$ on this path. The above arguments also show that the image $\bar{z}$ of z in X/H satisfies $d(x_0,z)\le 3\delta_{X/H}+37\delta_X+d(\bar{x}_0,\bar{z})$ and this implies that $d(\bar{x}_0,\bar{z})\ge R-8\delta_{X/H}$. Then the image of the path in X/H forms a path of length less than or equal to L joining $\bar{x}$ to $\bar{y}$ in the complement of $\bar{B}(\bar{x}_0,R-8\delta_{X/H})$. □

5.3 From sphere to shadows

From now on, fix an integer $M\ge 43\delta_{X/H}+4$ and set $R_0=M+\delta_{X/H}$. In the text below, paths are supposed to be of finite length and $S(\bar{x}_0,R_0)$ will denote the sphere around $\bar{x}_0$ of radius R_0.

In light of the Bestvina–Mess conditions, the following equivalence relation arises naturally:

Definition 5.6 Two points on $S(\bar{x}_0,R_0)$ are *equivalent* if there exists a path joining them in the complement of the open ball $B(\bar{x}_0,R_0-3\delta_{X/H})$.

Denote by $\sim$ this equivalence relation on the sphere $S(\bar{x}_0,R_0)$ and $[\bar{x}]$ the class of a point $\bar{x}$ under this relation. In order to link these equivalence classes with the whole quotient space X/H, define a projection on $\bar{B}(\bar{x}_0,R_0)$.

First, note that the thickened orbit $(H\cdot x_0)^{+R_0}$ is $2\delta_X$–quasiconvex since R_0 is greater than the quasiconvexity constant of the orbit which is δ_X. So consider an H–equivariant δ_X–projection on this thickened orbit, namely $\pi_0\colon X\to(H\cdot x_0)^{+R_0}$. Given π_0, define a projection in X/H, that is, a map $\pi_0'\colon X/H\to\bar{B}(\bar{x}_0,R_0)$ such that:

- For every point $\bar{x}$ in the complement of $\bar{B}(\bar{x}_0,R_0)$, denote by x a lift of $\bar{x}$ in X. The map π_0' sends $\bar{x}$ to $\overline{\pi_0(x)}$, the image of $\pi_0(x)$ in X/H.
- The map π_0' is the identity on the closed ball $\bar{B}(\bar{x}_0,R_0)$.

Remark 5.7 The definition of π_0' does not depend on the choice of the lift of $\overline{x}_0$. Indeed, if x' is another lift of $\overline{x}$ in X, there exists an element h in H such that $x' = hx$. Then, by H–equivariance of π_0, we have that

$$d(\overline{\pi_0(x)}, \overline{\pi_0(x')}) = \min_{h\in H} d(\pi_0(x), h\pi_0(h'x)) = \min_{h\in H} d(\pi_0(x), hh'\pi_0(x)) = 0.$$

The projection $\pi_0'(\overline{x})$ is also equal to $\overline{\pi_0(x')}$.

Proposition 5.8 *For every geodesic ray $c\colon \mathbb{R}_{\geq 0} \to X/H$ emanating from $\overline{x}_0$ and for every integer $R \geq R_0$, there exists a path joining $c(R_0)$ to $\pi_0'(c(R))$ in the complement of $B(\overline{x}_0, R_0 - 10\delta_X)$.*

Proof Let $(r_i)_{1\leq i\leq n}$ be a finite sequence of integers such that $R_0 = r_1 \leq r_2 \leq \cdots \leq r_n = R$ and $r_{i+1} - r_i \leq \delta_X$ for all $i \in \{1, \dots, n-1\}$. Denote by x_1 a lift of $c(r_1)$ in X. Then, for each $i > 1$, denote by x_i a lift of $c(r_i)$ in X satisfying $d(x_{i-1}, x_i) \leq \delta_X$. By Remark 5.7, $d\big(\pi_0'(c(r_i)), \pi_0'(c(r_{i+1}))\big) = d(\overline{\pi_0(x_i)}, \overline{\pi_0(x_{i+1})})$ for all $i \in \{1, \dots, n-1\}$. Now apply the contraction property of projections on quasiconvex sets (Proposition 2.5) to each pair (x_i, x_{i+1}): for all $i \in \{1, \dots, n-1\}$,

$$d(\overline{\pi_0(x_i)}, \overline{\pi_0(x_{i+1})}) \leq d(\pi_0(x_i), \pi_0(x_{i+1})) \leq 18\delta_X + d(x_i, x_{i+1}) < 20\delta_X.$$

As π_0' is the identity on $S(\overline{x}_0, R_0)$, we also have that $\pi_0'(c(R_0)) = \overline{x}_1 = c(R_0)$. Now consider the path obtained by concatenating a geodesic segment $[c(R_0), \overline{\pi_0(x_1)}]$ with geodesic segments $[\overline{\pi_0(x_i)}, \overline{\pi_0(x_{i+1})}]$ for $i \in \{1, \dots, n-1\}$ and $[\overline{\pi_0(x_n)}, \pi_0'(c(R))]$. This path joins $c(R_0)$ to $\pi_0'(c(R))$ and stays in the complement of the open ball $B(\overline{x}_0, R_0 - 10\delta_X)$. □

As Theorem 4.3 gives $\delta_{X/H}$ much greater than $65\delta_X$, Proposition 5.8 indicates that $c(R_0) \sim \pi_0'(c(R))$ and it induces the following definition:

Definition 5.9 A *shadow* of an equivalence class for $\sim$ is the inverse image of this class under π_0'.

"Being in the same shadow" is clearly an equivalence relation on points in the complement of $B(\overline{x}_0, R_0)$, for which shadows are equivalence classes. Denote by $\mathscr{S}(\overline{x})$ the shadow of the class $[\overline{x}]$ and $\mathscr{S}_{X/H}$ the set of shadows in X/H.

Proposition 5.10 *The set $\mathscr{S}_{X/H}$ of shadows in X/H is in bijection with the set $S(\overline{x}_0, R_0)/\!\sim$ of equivalence classes for the relation $\sim$ on $S(\overline{x}_0, R_0)$.*

Proof Consider $f\colon S(\bar{x}_0, R_0)/\sim \to \mathscr{S}_{X/H}$ the map which sends each equivalence class to its shadow in X/H.

Let $\bar{x}$ and $\bar{y}$ be two points in $S(\bar{x}_0, R_0)$ such that $\mathscr{S}(\bar{x}) = \mathscr{S}(\bar{y})$. In particular, $\pi'_0(\mathscr{S}(\bar{x})) = \pi'_0(\mathscr{S}(\bar{y}))$. Moreover, by definition of π'_0, the class of $\bar{x}$ for the relation $\sim$ is $[\bar{x}] = \pi'_0(\mathscr{S}(\bar{x}))$ and the class of $\bar{y}$ for this relation is $[\bar{y}] = \pi'_0(\mathscr{S}(\bar{y}))$. Thus, the map f is one-to-one.

Let $\mathscr{S} \in \mathscr{S}_{X/H}$. By definition of the shadow, there is a point $\bar{x} \in S(\bar{x}_0, R_0)$ such that $\mathscr{S}$ is the inverse image of $[\bar{x}]$ under π'_0. Then the image of $[\bar{x}]$ in $\mathscr{S}_{X/H}$ is $\mathscr{S}$ and the map f is surjective.

To conclude, f is a bijection between the set of equivalence classes on $S(\bar{x}_0, R_0)$ and the set of shadows in X/H. □

5.4 From shadows to ends

In this subsection, we establish a link between shadows and ends of the quotient space X/H. Recall that $M \geq 43\delta_{X/H} + 4$ and $R_0 = M + \delta_{X/H}$ are fixed.

Lemma 5.11 *If X satisfies $(\ddagger_{M,4\delta_{X/H}})$, then for all integers $R \geq R_0$, any two points on $S(\bar{x}_0, R)$ in the same shadow are joined by a path in the complement of the open ball $B(\bar{x}_0, R - 8\delta_{X/H})$.*

We follow closely the arguments given by Bestvina and Mess in [3, 3.2].

Proof We proceed by induction on R. Consider two points $\bar{x}$ and $\bar{y}$ on $S(\bar{x}_0, R_0)$ in the same shadow. This means that $\pi'_0(\bar{x})$ and $\pi'_0(\bar{y})$ are joined by a path in the complement of $B(\bar{x}_0, R_0 - 3\delta_{X/H})$. But $\pi'_0(\bar{x}) = \bar{x}$ and $\pi'_0(\bar{y}) = \bar{y}$. So there exists a path joining $\bar{x}$ to $\bar{y}$ in the complement of $B(\bar{x}_0, R_0 - 3\delta_{X/H}) \supset B(\bar{x}_0, R_0 - 8\delta_{X/H})$.

Let $R \geq R_0$. Assume now that all pair of points on $S(\bar{x}_0, R)$ in the same shadow are joined by a path in the complement of $B(\bar{x}_0, R - 8\delta_{X/H})$. Consider two points $\bar{x}$ and $\bar{y}$ on $S(\bar{x}_0, R+1)$ in the same shadow. Denote by c_0 and c_1 two geodesic rays emanating from $\bar{x}_0$ passing within distance $\delta_{X/H}$ from $\bar{x}$ and $\bar{y}$, respectively.

To prove the inductive step, construct a sequence of points on $S(\bar{x}_0, R+1)$ that are sufficiently close to use that X/H satisfies $(\dagger_M)$ and to join $\bar{x}$ and $\bar{y}$ by a path in the complement of $B(\overline{x}_0, R+1-8\delta_{X/H})$. To do so, we prove that $c_0(R)$ and $c_1(R)$ are in the same shadow using

$$\pi_0'(c_0(R)) \sim \pi_0'(c_0(R+1)) \sim \pi_0'(\overline{x}) \sim \pi_0'(\overline{y}) \sim \pi_0'(c_1(R+1)) \sim \pi_0'(c_1(R)),$$

and then construct the sequence of points using the induction hypothesis for $c_0(R)$ and $c_1(R)$.

First of all, prove $\pi_0'(c_0(R+1)) \sim \pi_0'(\overline{x})$. Denote by z a lift of $c_0(R+1)$ and x' a lift of $\overline{x}$ such that $d(x',z) = d(c_0(R+1), \overline{x}) \le 4\delta_{X/H}$ (according to Remark 2.4). By the contraction property (Proposition 2.5), we obtain

$$\begin{aligned} d\big(\pi_0'(c_0(R+1)), \pi_0'(\overline{x})\big) = d(\overline{\pi_0(z)}, \overline{\pi_0(x')}) &\le d(\pi_0(z), \pi_0(x')) \\ &\le 18\delta_X + d(z,x') < 5\delta_{X/H}. \end{aligned}$$

Therefore, there exists a path joining points $\pi_0'(c_0(R+1))$ and $\pi_0'(\overline{x})$ in the complement of $B(\overline{x}_0, R_0-3\delta_{X/H})$. An analogous computation indicates that there exists also a path joining $\pi_0'(c_1(R+1))$ and $\pi_0'(\overline{y})$ in the complement of $B(\overline{x}_0, R_0-3\delta_{X/H})$. By Proposition 5.8, there is a path joining $\pi_0'(c_i(R))$ to $\pi_0'(c_i(R+1))$ in the complement of $B(\overline{x}_0, R_0-10\delta_X)$ for $i=0,1$. Moreover, as $\overline{x}$ and $\overline{y}$ are in the same shadow, there exists also a path joining $\pi_0'(\overline{x})$ and $\pi_0'(\overline{y})$ in the complement of $B(\overline{x}_0, R_0-3\delta_{X/H})$. By concatenating these paths, we obtain a path from $\pi_0'(c_0(R))$ to $\pi_0'(c_1(R))$ in the complement of $B(\overline{x}_0, R_0-3\delta_{X/H})$. And so the points $c_0(R)$ and $c_1(R)$ on the sphere $S(\overline{x}_0, R)$ are in the same shadow.

Figure 3 depicts the inductive step.

Now apply the induction hypothesis to $c_0(R)$ and $c_1(R)$: there exists a path joining $c_0(R)$ and $c_1(R)$ in the complement of $B(\overline{x}_0, R-8\delta_{X/H})$. Consider points $\overline{p}_0, \dots, \overline{p}_n$ on the path satisfying $\overline{p}_0 = c_0(R)$, $\overline{p}_n = c_1(R)$ and $d(\overline{p}_i, \overline{p}_{i+1}) \le \delta_X$ for all $i \in \{0, \dots, n-1\}$. Then, for all i, denote by c_i a geodesic ray emanating from $\overline{x}_0$ passing within distance $\delta_{X/H}$ from $\overline{p}_i$.

Denote by $\overline{q}_i$ a point of c_i at distance less than or equal to $\delta_{X/H}$ of $\overline{p}_i$. Then the distance between $\overline{q}_i$ and $\overline{x}_0$ is at least $R-9\delta_{X/H}$. Denote by $\overline{s}_i$ a point of c_i satisfying

$$\begin{cases} d(\overline{s}_i, \overline{x}_0) \ge R+1, \\ d(\overline{s}_i, \overline{q}_i) \le 1 + 9\delta_{X/H}. \end{cases}$$

For all $i \in \{0, \dots, n-1\}$, these points satisfy $d(\overline{s}_i, \overline{s}_{i+1}) \le 2 + 20\delta_{X/H} + \delta_X$. Consider a geodesic triangle formed by the subsegments $[\overline{x}_0, \overline{s}_i] \subset c_i$ and $[\overline{x}_0, \overline{s}_{i+1}] \subset c_{i+1}$ and

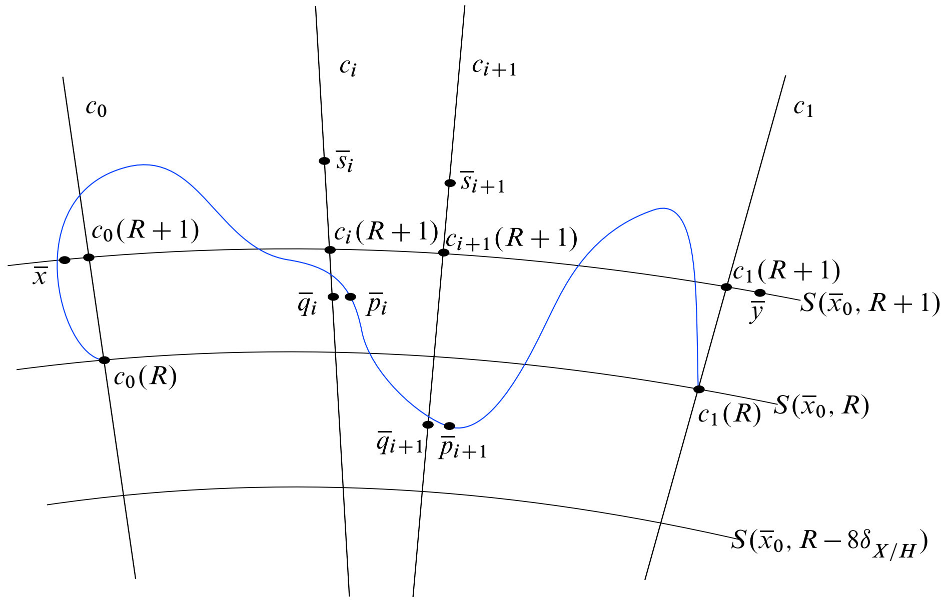

Figure 3: Joining $\overline{x}$ and $\overline{y}$ in the complement of $B(\overline{x}_0, R+1-8\delta_{X/H})$

a geodesic segment joining $\overline{s}_i$ and $\overline{s}_{i+1}$. According to [6, III.H.1.15], we obtain

$$d(c_i(R+1), c_{i+1}(R+1)) \leq 2(2+20\delta_{X/H}+\delta_X+\delta_{X/H}) \leq 4+43\delta_{X/H}.$$

As X satisfies the property $(\ddagger_{4+43\delta_{X/H}, 4\delta_{X/H}})$, there is a path joining $c_i(R+1)$ and $c_{i+1}(R+1)$ in the complement of $B(\overline{x}_0, R+1-8\delta_{X/H})$ for all $i \in \{0, \dots, n-1\}$. Moreover, given Remark 2.4, $d(c_0(R+1), \overline{x}) \leq 4\delta_{X/H}$ and $d(c_1(R+1), \overline{y}) \leq 4\delta_{X/H}$. Thus, there exists a path joining $\overline{x}$ to $\overline{y}$ in the complement of $B(\overline{x}_0, R+1-8\delta_{X/H})$.

To conclude, for all $R \geq R_0$, any two points on $S(\overline{x}_0, R)$ in the same shadow are joined by a path in the complement of $B(\overline{x}_0, R-8\delta_{X/H})$. □

This lemma extends to a more general result:

Proposition 5.12 *If X satisfies $(\ddagger_{M,4\delta_{X/H}})$ then any two points $\overline{x}, \overline{y} \in X/H$ in the same shadow are joined by a path in the complement of*

$$B(\overline{x}_0, \min\{d(\overline{x}_0, \overline{x}), d(\overline{x}_0, \overline{y})\} - 8\delta_{X/H}).$$

Proof Consider two points $\overline{x}$ and $\overline{y}$ in X/H in the same shadow. Set $R_x := d(\overline{x}_0, \overline{x})$ and $R_y := d(\overline{x}_0, \overline{y})$. By exchanging roles between $\overline{x}$ and $\overline{y}$ if necessary, we can assume that R_x is less than or equal to R_y and so $\min\{d(\overline{x}_0, \overline{x}), d(\overline{x}_0, \overline{y})\} = R_x$. Denote by c a geodesic ray emanating from $\overline{x}_0$ passing within distance $\delta_{X/H}$ from $\overline{y}$ and set $\overline{y}' := c(R_x)$. By Proposition 5.8 and as $d(\overline{y}, c) \leq \delta_{X/H}$, $\overline{y}$ and $\overline{y}'$ are in the same shadow. This implies that $\overline{x}$ and $\overline{y}'$ are in the same shadow. By applying Lemma 5.11, there exists a path joining $\overline{x}$ to $\overline{y}'$ in the complement of $B(\overline{x}_0, R_x - 8\delta_{X/H})$. Moreover, in light of Remark 2.4, $d(\overline{y}, c(R_y)) \leq 4\delta_{X/H}$ and so these points are joined by a path in the complement of $B(\overline{x}_0, R_y - 2\delta_{X/H})$.

The concatenation of these paths with the restriction of c to $[R_x, R_y]$ forms a path joining $\overline{x}$ to $\overline{y}$ in the complement of $B(\overline{x}_0, R_x - 8\delta_{X/H})$. □

Recall that Proposition 5.8 states that all points on a geodesic ray emanating from $\overline{x}_0$ at distance greater than or equal to R_0 from $\overline{x}_0$ are in the same shadow. Therefore, our relation $\sim$ extends to geodesic rays in X/H emanating from $\overline{x}_0$ in the following manner: two geodesic rays $c, c' \colon \mathbb{R}_{\geq 0} \to X/H$ emanating from $\overline{x}_0$ are "*in the same shadow*" if all points on c and c' at distance greater than or equal to R_0 from $\overline{x}_0$ are in the same shadow. Denote by $\mathscr{S}(c) := \mathscr{S}(c(R_0))$ the shadow defined by the class of $c(R_0)$.

Remark 5.13 There is a natural surjection from the set of geodesic rays emanating from $\overline{x}_0$ to the set of shadows in X/H.

Lemma 5.14 *Assume that X satisfies the property $(\ddagger_{M,4\delta_{X/H}})$. Two geodesic rays in X/H are in the same shadow if and only if they converge to the same end of X/H.*

Proof Consider two geodesic rays c and c' converging to the same end in X/H. This means that for all $R \geq R_0$, there is an integer $R' \geq R$ such that for all $r, r' \geq R'$, the points $c(r)$ and $c'(r')$ are in the same path connected component of the complement of $B(\overline{x}_0, R)$. Therefore, for all $r, r' \geq R'$, the points $c(r)$ and $c'(r')$ are joined by a path in the complement of $B(\overline{x}_0, R) \supset B(\overline{x}_0, R_0)$. By Proposition 5.8, the following equivalences are satisfied for all $r, r' \geq R'$:

$$\pi_0'(c(r)) \sim \pi_0'(c(R_0)) = c(R_0),$$
$$\pi_0'(c'(r')) \sim \pi_0'(c'(R_0)) = c'(R_0).$$

This means that there exist paths joining $\pi_0'(c(r))$ to $c(R_0)$ and $\pi_0'(c'(r'))$ to $c'(R_0)$ in the complement of $B(\overline{x}_0, R_0 - 8\delta_{X/H})$. The concatenation of these paths with the restrictions of rays $c([R_0, r])$ and $c'([R_0, r'])$ gives a path joining $\pi_0'(c(r))$ to

$\pi_0'(c'(r'))$ in the complement of $B(\overline{x}_0, R_0 - 8\delta_{X/H})$. By Proposition 5.8, this fact extends to every point far away from $\overline{x}_0$. Indeed, for all $r_1, r_2 \geq R_0$, we have

$$\pi_0'(c(r_1)) \sim c(R_0) \sim \pi_0'(c(r)) \sim \pi_0'(c'(r')) \sim c'(R_0) \sim \pi_0'(c'(r_2)).$$

Therefore, for all $r_1, r_2 \geq R_0$, there is a path joining $\pi_0'(c(r_1))$ to $\pi_0'(c'(r_2))$ in the complement of the open ball $B(\overline{x}_0, R_0 - 8\delta_{X/H})$, ie the geodesic rays c and c' are in the same shadow.

Now consider two geodesic rays $c, c'\colon \mathbb{R}_{\geq 0} \to X/H$ in the same shadow. By Proposition 5.12, for all $r, r' \geq R_0$, there is a path joining $c(r)$ to $c'(r')$ in the complement of $B(\overline{x}_0, \min\{r, r'\} - 8\delta_{X/H})$. This implies that for $R \geq R_0$, every point of $c([R, +\infty))$ is joined by a path in the complement of $B(\overline{x}_0, R - 8\delta_{X/H})$ to a point of $c'([R, +\infty))$. Therefore, these rays converge to the same end of X/H. □

Remark 5.15 The fact that two geodesic rays in X/H converging to a same end are in the same shadow holds even if X does not satisfy $(\ddagger_{M, 4\delta_{X/H}})$.

To sum up, if the space X satisfies the property $(\ddagger_{M, 4\delta_{X/H}})$ with $M \geq 43\delta_{X/H} + 4$, the number of shadows for geodesic rays in X/H is equal to the number of ends for X/H. This implies the following result:

Theorem 5.16 *Let $M \geq 43\delta_{X/H} + 4$ and $R_0 \geq M + \delta_{X/H}$. If X satisfies the property $(\ddagger_{M, 4\delta_{X/H}})$, then the set of ends of X/H is in bijection with the set of equivalence classes for the relation $\sim$ on the sphere $S(\overline{x}_0, R_0)$.*

Proof Let $f\colon \mathrm{Ends}(X/H) \to \mathscr{S}_{X/H}$ be a map which sends an end $\mathrm{end}(c)$ defined by a geodesic ray c to the shadow $\mathscr{S}(c)$.

Let c and c' be two geodesic rays emanating from $\overline{x}_0$ in X/H such that $\mathscr{S}(c) = \mathscr{S}(c')$. Lemma 5.14 implies then that c and c' converge to the same end; this means that $\mathrm{end}(c) = \mathrm{end}(c')$. Thus, f is one-to-one.

Let $\mathscr{S} \in \mathscr{S}_{X/H}$. By Remark 5.13, the map sending geodesic rays emanating from $\overline{x}_0$ to shadows in X/H is surjective. So there exists a geodesic ray c emanating from $\overline{x}_0$ such that $\mathscr{S} = \mathscr{S}(c)$. The image of $\mathrm{end}(c)$ under f is then exactly $\mathscr{S}$. Therefore, the map f is surjective.

So there is a bijection between the set of ends of X/H and the set of shadows in X/H. By Proposition 5.10, there is a bijection between the set of ends and the set of equivalence classes for $\sim$ on the sphere $S(\overline{x}_0, R_0)$. □

In light of Theorem 5.16, the following corollary is straightforward:

Corollary 5.17 *Let $M \geq 43\delta_{X/H} + 4$ and $R_0 \geq M + \delta_{X/H}$. If the space X satisfies the property $(\ddagger_{M,4\delta_{X/H}})$, then the number of ends of X/H is equal to the number of equivalence classes for the relation $\sim$ on the sphere $S(\overline{x}_0, R_0)$.* □

Remark 5.18 In particular, the number of equivalence classes on the sphere centred in $\overline{x}_0$ of radius R_0 is finite since it is bounded by the size of this sphere. In this case, the number of ends of X/H is necessarily finite.

6 Application to group theory

In what follows, G is a hyperbolic group with connected boundary given by a finite presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$ and X is the Cayley graph of G with respect to $\mathcal{S}$. Moreover, assume that there exists a quasiconvex subgroup H of G and denote by x_0 a point in the weak convex hull $C(\Lambda H)$ of the limit set of H. In particular, the group H is a quasiconvex-cocompact group of isometries of X. Then the number of relative ends of the pair (G, H) is the number of ends of the associated Schreier graph X/H. Denote by δ_X a hyperbolicity constant for X which is a geodesic extension constant for X.

The aim of this section is to give an algorithm to compute the number of relative ends of the pair (G, H).

Here we go back over the results used in the proof of Proposition 5.3. Bestvina and Mess proved that if a hyperbolic space does not satisfy their condition then its boundary has a cut-point (see Proposition 3.3 of [3]). Their proof remains valid under condition $(\ddagger_{M,K})$ and gives the following result:

Proposition 6.1 *If there exist constants $M > K > 0$ such that X does not satisfy the condition $(\ddagger_{M,K})$ then the boundary of X contains a global cut-point.* □

A few years later, Bowditch and Swarup proved independently that there is no such cut-point in one-ended hyperbolic groups (see [5, 0.3; 31]).

Theorem 6.2 *If G is a one-ended hyperbolic group, then the boundary ∂G of G contains no global cut-point.* □

In light of Proposition 6.1 and Theorem 6.2, the following statement holds:

Corollary 6.3 *If G is a hyperbolic group with connected boundary, then any Cayley graph of G satisfies the condition $(\ddagger_{M,K})$ for all $M > K > 0$.* □

The following result arises easily from Corollaries 6.3 and 5.17 for the determination of relative ends.

Corollary 6.4 *Under our assumption on G and H, there exists a constant R_0 such that the number of relative ends of the pair (G, H) is equal to the number of equivalence classes on $S(\overline{x}_0, R_0)$ for the relation $\sim$.*

Proof As the group G has a connected boundary, Corollary 6.3 indicates that X satisfies the condition $(\ddagger_{M,K})$ for all $M > K > 0$. Furthermore, Corollary 5.17 implies that if $M \geq 43\delta_{X/H} + 4$ and $R_0 \geq M + \delta_{X/H}$, then the number of ends of the quotient space X/H is equal to the number of equivalence classes for the relation $\sim$ on the sphere $S(\overline{x}_0, R_0)$. □

To establish the existence of the sought algorithm, we need the following lemma:

Lemma 6.5 *Let $R_0 > 4\delta_{X/H} + \delta_X$. If there exists a path joining two points on the sphere $S(\overline{x}_0, R_0)$ in the complement of $\overline{B}(\overline{x}_0, R_0 - 3\delta_{X/H})$ then there exists an injective path joining these points in the annulus $A(\overline{x}_0, R_0 - 3\delta_{X/H}, R_0 + 10\delta_X(\#S)^{R_0})$.*

Proof Let $\overline{x}$ and $\overline{y}$ be two points on the sphere $S(\overline{x}_0, R_0)$ joined by a path of finite length in the complement of $\overline{B}(\overline{x}_0, R_0 - 3\delta_{X/H})$. We denote by $\overline{x} = \overline{p}_0, \overline{p}_1, \dots, \overline{p}_n = \overline{y}$ points on this path satisfying $d(\overline{p}_i, \overline{p}_{i+1}) \leq \delta_X$ for $i \in \{0, \dots, n\}$.

As $R_0 - 3\delta_{X/H} - \delta_X > \delta_{X/H}$, any closed ball of radius δ_X at $\overline{p}_i$ can be lifted in X using the covering map described in Proposition 4.4. Let x be a lift of $\overline{x} = \overline{p}_0$. Firstly, lift the ball $B(\overline{p}_0, \delta_X)$ into the ball $B(p_0, \delta_X)$. By construction, this ball contains a lift p_1 of $\overline{p}_1$ satisfying $d(p_0, p_1) \leq \delta_X$. By induction, we obtain lifts p_i of $\overline{p}_i$ still satisfying $d(p_i, p_{i+1}) \leq \delta_X$ for $i \in \{1, \dots, n-1\}$. Denote by $y = p_n$ the lift of $\overline{y} = \overline{p}_n$ obtained that way.

Denote by π_0 the δ_X–projection of X on the thickened orbit $(H \cdot x_0)^{+R_0}$ which is $2\delta_X$–quasiconvex. As $d(\overline{x}_0, \overline{x}) = d(\overline{x}_0, \overline{y}) = R_0$, we also have $d(H \cdot x_0, x) = d(H \cdot x_0, y) = R_0$ and then $\pi_0(x) = x$ and $\pi_0(y) = y$. Then, project the points p_i on $(H \cdot x_0)^{+R_0}$. For all $i \in \{0, \dots, n-1\}$, we have

$$\begin{aligned} d(\pi_0(p_i), \pi_0(p_{i+1})) &\leq d(p_i, p_{i+1}) + 18\delta_X \quad \text{(by Proposition 2.5)} \\ &\leq 19\delta_X. \end{aligned}$$

This implies that

$$\begin{cases} d(\overline{\pi_0(p_i)}, \overline{\pi_0(p_{i+1})}) \leq 19\delta_X & \text{for all } i \in \{1, \dots, n-1\}, \\ d(\overline{x}, \overline{\pi_0(p_1)}) \leq 19\delta_X, \\ d(\overline{\pi_0(p_{n-1})}, \overline{y}) \leq 19\delta_X. \end{cases}$$

Therefore, $\overline{x}$ is joined to $\overline{y}$ by a path formed by concatenating paths of length less than or equal to $19\delta_X$ joining the $\overline{\pi_0(p_i)}$ with each other. By suppressing some cycles on this path if necessary (some $\overline{\pi_0(p_i)}$ may be equal), we obtain a path of length less than or equal to $19\delta_X \times \#S(\overline{x}_0, R_0)$ joining $\overline{x}$ to $\overline{y}$ in the complement of $B(\overline{x}_0, R_0 - 10\delta_X)$.

By definition, the cardinality of the sphere $S(\overline{x}_0, R_0)$ is less than $(\#\mathcal{S})^{R_0}$. This implies that $\overline{x}$ and $\overline{y}$ are joined by an injective path of length less than or equal to $19\delta_X \times (\#\mathcal{S})^{R_0}$.

Finally, two points on the sphere $S(\overline{x}_0, R_0)$ joined by a path in the complement of the ball $\overline{B}(\overline{x}_0, R_0 - 3\delta_{X/H})$ are joined by a path in the open annulus

$$A(\overline{x}_0, R_0 - 3\delta_{X/H}, R_0 + 10\delta_X(\#\mathcal{S})^{R_0}). \qquad \square$$

Theorem 6.6 *Let G be a hyperbolic group with connected boundary given by a finite presentation $\langle \mathcal{S} \mid \mathcal{R} \rangle$. Let H be a quasiconvex subgroup of G. There exists an algorithm to compute the number of relative ends of the pair (G, H).*

Proof The group G is a hyperbolic group with connected boundary, so we can apply Corollary 6.4. For $M \geq 43\delta_{X/H} + 4$, there exists a constant $R_0 = M + \delta_{X/H}$ such that the number of relative ends of the pair (G, H) is equal to the number of equivalence classes for the relation $\sim$. Therefore, we have to determine whenever two points on $S(\overline{x}_0, R_0)$ are joined by a path in the complement of the open ball $B(\overline{x}_0, R_0 - 3\delta_{X/H})$.

Apply Lemma 6.5: two points on $S(\overline{x}_0, R_0)$ joined by a path in the complement of the ball $\overline{B}(\overline{x}_0, R_0 - 3\delta_{X/H})$ are joined by an injective path in the open annulus $A(\overline{x}_0, R_0 - 3\delta_{X/H}, R_0 + 10\delta_X(\#\mathcal{S})^{R_0})$. This means that drawing the ball

$$\overline{B}(\overline{x}_0, R_0 + 10\delta_X(\#\mathcal{S})^{R_0})$$

allows us to determine whenever two points on the sphere $S(\overline{x}_0, R_0)$ are joined by a path in the complement of $B(\overline{x}_0, R_0 - 3\delta_{X/H})$ and to split $S(\overline{x}_0, R_0)$ in equivalence classes for $\sim$.

- Draw the closed ball $\overline{B}(\overline{x}_0, R_0 + 10\delta_X(\#\mathcal{S})^{R_0})$. Pick up a point $\overline{x}_1$ at distance R_0 from $\overline{x}_0$ and look for every point of $S(\overline{x}_0, R_0)$ joined to $\overline{x}_1$ by a path staying in $A(\overline{x}_0, R_0 - 3\delta_{X/H}, R_0 + 10\delta_X(\#\mathcal{S})^{R_0})$. These points form the equivalence class $[\overline{x}_1]$. If this class contains every point of the sphere $S(\overline{x}_0, R_0)$, the pair (G, H) has one relative end. Otherwise, pick up a point $\overline{x}_2$ in $S(\overline{x}_0, R_0) \setminus [\overline{x}_1]$. Again, look for every point of $S(\overline{x}_0, R_0) \setminus [\overline{x}_1]$ joined to $\overline{x}_2$ by a path staying in $A(\overline{x}_0, R_0 - 3\delta_{X/H}, R_0 + 10\delta_X(\#\mathcal{S})^{R_0})$. If $[\overline{x}_2] = S(\overline{x}_0, R_0) \setminus [\overline{x}_1]$ then the pair (G, H) has 2 relative ends. Otherwise, pick up a point $\overline{x}_3$ on the sphere $S(\overline{x}_0, R_0)$ in the complement of $[\overline{x}_1]$ and $[\overline{x}_2]$ and so on. As the sphere $S(\overline{x}_0, R_0)$ contains a finite number of points, the algorithm stops at some point.

This procedure gives the number of equivalence classes for $\sim$ which is the number of relative ends of the pair (G, H). □

In light of Remark 4.2, if the value of δ_X is known, it is possible to determine explicitly every constant used in the algorithm.

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