A note on knot concordance

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We use classical techniques to answer some questions raised by Daniele Celoria about almost-concordance of knots in arbitrary closed 3–manifolds. We first prove that, given $Y^3 \neq S^3$, for any nontrivial element $g \in \pi_1(Y)$ there are infinitely many distinct smooth almost-concordance classes in the free homotopy class of the unknot. In particular we consider these distinct smooth almost-concordance classes on the boundary of a Mazur manifold and we show none of these distinct classes bounds a PL–disk in the Mazur manifold, but all the representatives we construct are topologically slice. We also prove that all knots in the free homotopy class of $S^1 \times \mathrm{pt}$ in $S^1 \times S^2$ are smoothly concordant.

0 Introduction

We consider manifolds that are smooth and oriented. Let Y be a closed, connected, oriented 3–manifold. A *knot k* in Y is an isotopy class of a smooth embedding $S^1 \hookrightarrow Y$. Two knots k_1 and k_2 are said to be *concordant* if there is a smooth proper embedding of an annulus $F\colon S^1 \times [0, 1] \hookrightarrow Y \times [0, 1]$ such that its boundary is $\partial F(S^1 \times [0, 1]) = k_1 \times \{0\} \cup (-k_2) \times \{1\}$, where $(-k_2)$ is the same knot k_2 with the reversed orientation. If we allow F to have only finitely many singular points, all of which are cones over knots, then k_1 and k_2 are called *PL–concordant*. We call these knots *singular concordant* if we allow F to be an immersion instead of an embedding. Two knots are singular concordant if and only if they are freely homotopic. One can see this fact by using the immersion theorems and general position arguments (these can be found in Hirsch [9]) on the trace of homotopy.

Concordance is an equivalence relation $\sim$ on the set of oriented knots in Y. The set of equivalence classes is denoted by;

$$\mathcal{C}(Y) = \{\text{oriented knots in } Y\}/\!\sim.$$

Concordant knots k_1 and k_2 are freely homotopic, hence they are homologous. In [3], Daniele Celoria defines the concept of almost-concordance of knots. Two knots k_1

and k_2 in Y are said to be *almost-concordant* if there are $k_1', k_2' \subset S^3$ such that $k_1 \# k_1' \sim k_2 \# k_2'$, and this is expressed by $k_1 \stackrel{.}{\sim} k_2$. Like concordance, almost-concordance is an equivalence relation, and it implies free homotopy of knots.

We denote almost-concordance classes by $\widetilde{\mathcal{C}}(Y)$. More generally,

$$\mathcal{C}_\gamma(Y) := \mathcal{K}_\gamma(Y)/\sim, \quad \widetilde{\mathcal{C}}_\gamma(Y) := \mathcal{K}_\gamma(Y)/\stackrel{.}{\sim},$$

where $\mathcal{K}_\gamma(Y)$ is the free homotopy class of a knot γ in Y.

Theorem 1 *Given a closed 3–manifold Y, for a nontrivial element $g \in \pi_1(Y)$ we can construct infinitely many distinct almost-concordance classes in the free homotopy class of the unknot. If $h \notin \{g, g^{-1}\}$, then the almost-concordance classes constructed using g and h (as in Figure 3) are disjoint.*

A question raised in [3] is: *Does there exists a pair (Y, m) such that $C_m(Y)$ is finite?* Theorem 2 provide a positive answer.

Theorem 2 *All knots in the free homotopy class of $S^1 \times \mathrm{pt}$ in $S^1 \times S^2$ are smoothly concordant, ie $|\mathcal{C}_x(S^1 \times S^2)| = 1$, where x represents $S^1 \times \mathrm{pt}$ in $S^1 \times S^2$.*

After this paper was posted, a similar result to Theorem 2 appeared in Davis, Nagel, Park and Ray [4]. In Friedl, Nagel, Orson and Powell [7], there are also related results to the above theorems in the topological category.

1 Wall's self-intersection number, a concordance invariant for null-homotopic knots and the proof of Theorem 1

There are many approaches to knot concordance problem; here we focus on one of the classical techniques. This technique is based on Wall's intersection number [13]. The application of this idea to knot concordance was studied in [12] by Schneiderman.

Let k be a null-homotopic knot in Y; consider a singular concordance of k to the unknot u — after capping the unknot with a disk, we get a proper immersion of a disk $D \looparrowright Y \times I$ with $k = \partial D$. Let p be a transverse self-intersection of the immersion D;

then any small neighbourhood of p looks like two surfaces intersecting at p. These surfaces are called *sheets*. The self-intersection number of k, defined as Wall's self-intersection number of D, takes its value in the group ring $\mathbb{Z}[\pi_1 Y]$. To define this self-intersection number we first fix a path from the basepoint y_0 of $Y \times I$ to a basepoint of the immersed disk D, called a *whisker* of D. Now $g_p \in \pi_1(Y, y_0)$ is defined in the following way: it is a loop starting from y_0 going to the basepoint of D using the whisker, then to the self-intersection point p of D, then changing the sheet at the intersection point going back to the basepoint of D, and finally to y_0 using the whisker. Then

$$\mu(k) := \mu(D) = \sum_p \text{sign}(p) \cdot g_p \in \mathbb{Z}[\pi_1 Y].$$

Since D is simply connected, the loop g_p does not depend on the path we choose while travelling on D as long as it stays away from self-intersection points. The value of $\text{sign}(p)$ is $+1$ if the orientation of $Y \times I$ at p matches with the orientation induced from sheets of D at p, and it is -1 otherwise. After fixing the whisker there is still an indeterminacy coming from the choice of the first sheet. Altering this choice changes the loop from g_p to g_p^{-1}. Also, self-intersection points coming from cusp homotopies give elements which are trivial in $\pi_1(Y)$. Since we are interested in a homotopy invariant, we also quotient out these elements, arriving at the abelian group

$$\widetilde{\Lambda} := \frac{\mathbb{Z}[\pi_1 Y]}{\{g - g^{-1} \mid g \in \pi_1(Y)\} \oplus \mathbb{Z}[1]}.$$

Here $\mathbb{Z}[1]$ is the abelian subgroup generated by the trivial element of $\pi_1(Y)$. Homotopy invariance in the above discussion follows from the following two propositions.

Proposition 3 (from Chapter 1.6 of [6]) *A homotopy between immersions of a surface in a 4–manifold is homotopic to a composition of homotopies, each of which is a regular homotopy or a cusp homotopy in some ball, or the inverse of a cusp homotopy.*

Proposition 4 (from Chapter 1.7 of [6]) *Intersection numbers and reduced self-intersection numbers in $\widetilde{\Lambda}$ are invariant under homotopy rel boundary. The $\mathbb{Z}[1]$ component of the self-intersection number is invariant under regular homotopy, and conversely two immersions of a sphere or disk which are homotopic rel boundary, and have the same framed boundary, are regularly homotopic rel boundary if and only if the $\mathbb{Z}[1]$ components of the self-intersection numbers are equal.*

Now we state and prove Schneiderman's knot concordance invariant.
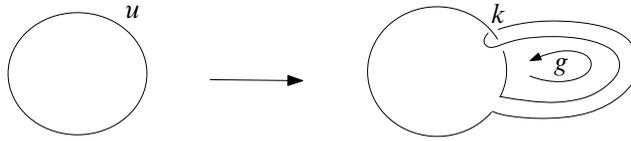
Figure 1: $\mu(k) = g$

Theorem 5 [12] *The map*

$$\mu: \mathcal{C}_1(Y) \to \widetilde{\Lambda}, \quad k \mapsto \mu(k),$$

is well defined and onto.

Proof We recall the proof from [12].

Well defined Let D and D' be singular null-concordances of a knot k, taking a singular sphere $S = D \cup_k D' \subset Y \times I$ gives $S \in \pi_2(Y \times I) = \pi_2(Y)$. By [8, Proposition 3.12], there exists a disjoint collection of embedded 2–spheres generating $\pi_2(Y)$ as a $\pi_1(Y)-$module. Tubing these generators together in $Y \times I$, we get an embedded sphere in $Y \times I$. This implies

$$\mu(S) = 0 = \mu(D) - \mu(D'),$$

therefore $\mu(k)$ doesn't depend on D.

Concordance invariance If $k_1, k_2 \in \mathcal{C}_1(Y)$ and $k_1 \sim k_2$, then $\mu(k_2) = \mu(C \cup D) = \mu(D) = \mu(k_1)$ where C is a concordance from k_1 to k_2 and D is the singular concordance of k_2.

Surjectivity To construct $\pm g \in \mathbb{Z}[\pi_1(Y)]$ start with an unknot u and push an arc from u around a loop representing $g \in \pi_1(Y)$ and create a $\pm$ clasp as in Figure 1. Iterating this process, one can get any desired element in $\mathbb{Z}[\pi_1(Y)]$ via connected summing of such knots.   □

Lemma 6 *For any knots $k \in \mathcal{K}_1(Y)$, $k' \subset S^3$ we have*

$$\mu(k \, \# \, k') = \mu(k).$$

This implies that $\mu: \widetilde{\mathcal{C}}_1(Y) \to \widetilde{\Lambda}$ is well defined and onto.

Proof We will construct a singular disk which will give us the desired result. By definition, k bounds a proper immersion of a disk $D \subset Y \times I$, and similarly k' bounds $D' \subset S^3 \times I$. Any band sum $D \,\#_b\, D'$ where the interior of b is away from k and k'

gives a proper immersion of a disk in $Y \times I$ bounded by $k \# k'$. Take the basepoint and the whisker of D as a basepoint and a whisker for $D \#_b D'$, so

$$\mu(D \#_b D') = \mu(D) + \beta\mu(D')\beta^{-1},$$

where $\beta \in \pi_1(Y)$ is determined by the band b and the whisker. On the other hand, $\pi_1(S^3) = 1$ and D' lies entirely in $S^3 \times I$, therefore $\beta\mu(D')\beta^{-1} = 0 \in \widetilde{\Lambda}$, hence

$$\mu(D \#_b D') = \mu(D) \quad \text{and} \quad \mu(k \#_b k') = \mu(k). \qquad \square$$

This observation implies that Schneiderman's concordance invariant μ is also an almost-concordance invariant on freely null-homotopic knots.

Proof of Theorem 1 By Theorem 5 and Lemma 6, $\mu \colon \widetilde{\mathcal{C}}_1(Y) \to \widetilde{\Lambda}$ is well defined, onto, and is an almost-concordance invariant on null-homotopic knots. For every nontrivial element $g \in \pi_1(Y)$ the target space $\widetilde{\Lambda}$ contains a subgroup isomorphic to $\mathbb{Z}$ generated by g. $\qquad \square$

Example 1 Let W^4 be a Mazur manifold as in Figure 2. There are various ways to see that the boundary is not the 3–sphere. Its fundamental group is known to be nontrivial [10].
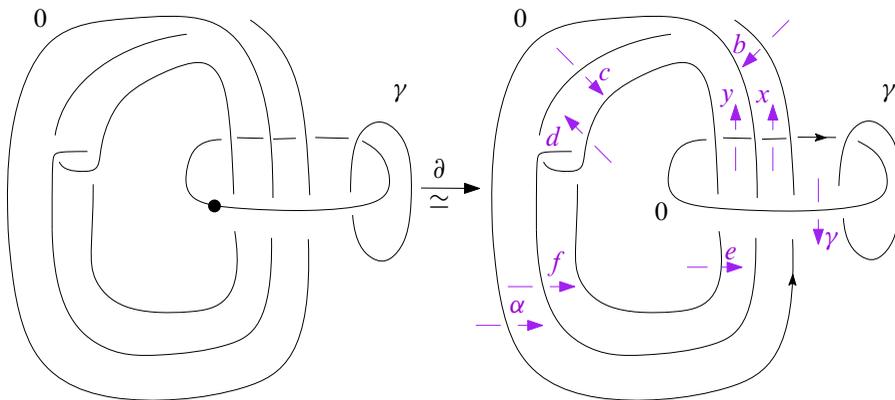

Figure 2: A homology sphere, Wirtinger presentation

By using the Wirtinger presentation we describe the fundamental group:

$$\pi_1(Y) = \{\gamma, \alpha \mid \gamma^2\alpha\gamma^{-1}\alpha\gamma^{-1}\alpha^{-1}\gamma\alpha^{-1}\gamma\alpha^{-1}\gamma^{-1}\alpha\gamma^{-1}\alpha = 1,$$
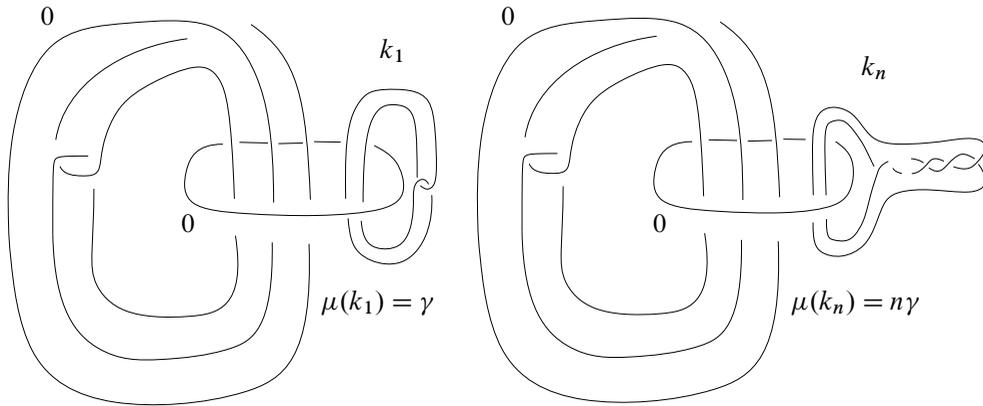$$\gamma^{-1}\alpha\gamma^{-1}\alpha^2\gamma\alpha\gamma^{-2}\alpha^3 = 1\}.$$

Figure 3: Distinct almost-concordant families

Notice that setting $\gamma = 1$ in this presentation would make this group trivial, hence γ is a nontrivial element of $\pi_1(Y)$. To construct an example corresponding to Theorem 1, take an unknot and push an arc along a nontrivial loop γ; we get Figure 3, left. Obviously $\mu(k_1) = \gamma^{\pm} \in \widetilde{\Lambda}$ is nontrivial. Hence, it is not almost-concordant to the unknot. On the other hand, by iterating this process (ie increasing the number of twists) we can construct infinitely many null-homotopic knots k_n with distinct μ invariant in the homology sphere; see Figure 3, right.

2   Proof of Theorem 2

Proof of Theorem 2   First we introduce a (genus zero) cobordism move to a knot k, which starts with k and ends with a two-component link, consisting of the knot obtained from k by changing one of its crossings union a small linking circle, as shown in
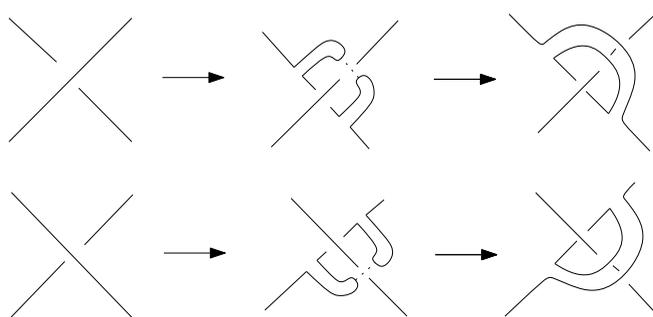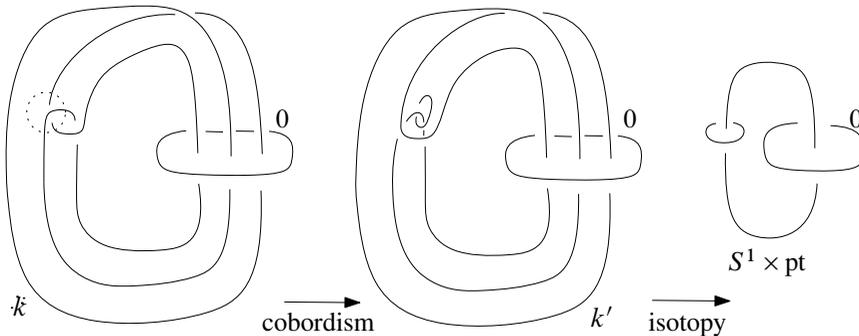

Figure 4: Crossing change

Figure 5: An example of crossing change

Figure 4. Let k be a knot freely homotopic to $k' = S^1 \times \mathrm{pt}$ in $S^1 \times S^2$; one can go from k to k' by finitely many crossing changes and isotopies. Change all the necessary crossings of k by the cobordism described above. Notice that for every crossing change, we get a small linking circle to the resulting knot. See Figure 5 as an example. It is obvious from Figure 6 that all those small circles which link k' bound disks in $S^1 \times S^2$ disjoint from k'. We accomplish this by sliding over the 0–framed circle. By capping with disks these unknots, we get a concordance from k to k' in $S^1 \times S^2$. □
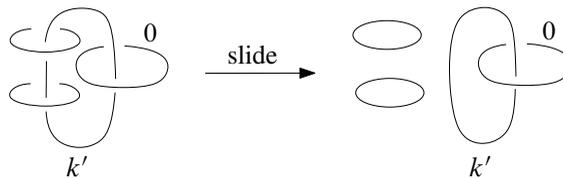

Figure 6: Sliding and capping with disks

3 PL–slice

The notion of almost-concordance is same as the PL–concordance in $Y \times I$. Indeed, if k_1 and k_2 are PL–concordant then we may assume without loss of generality, the concordance has only one singular point which locally looks like a cone over a knot k. It is easy to see $k_1 \# -k$ is smoothly concordant to k_2 by removing a ball around the cone point and connecting two boundary components by removing neighbourhood of an arc lying on the concordance connecting k_1 to k. On the other hand, if we have an almost concordance between k_1 and k_2, ie $k_1 \# k'_1$ is concordant to $k_2 \# k'_2$, then push the local knots inside the 4–manifold and take the cone over the knots in some local ball to get a PL–concordance. Basically this tells us the family of knots we construct
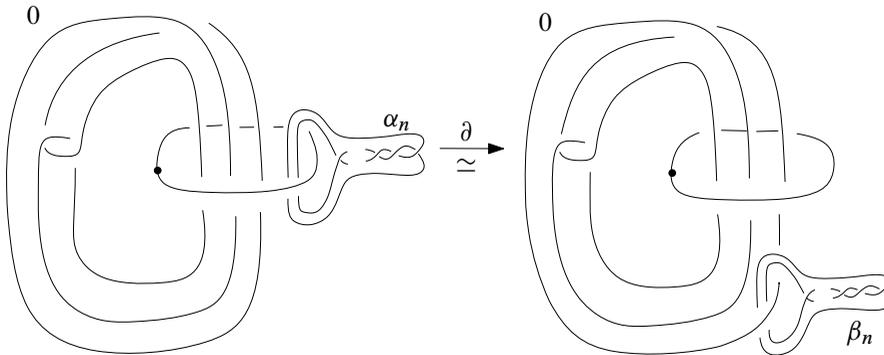
Figure 7: Boundary diffeomorphism

in Example 1, in particular in Figure 3, cannot bound a PL–disk in the collar of the manifold but it can still bound in a 4–manifold which Y bounds.

Next we see that none of these family members α_n in Figure 7 bounds a PL–disk in the Mazur manifold W^4.

Here we imitate Akbulut [1]. Observe that W^4 is a Stein domain by [5]. Consider the boundary diffeomorphism which takes α_n to β_n as in Figure 7, using $0 \leftrightarrow \bullet$ exchange and symmetry of the link surgery diagram of Mazur manifold. The knot β_n is smoothly slice. To see that α_n is not slice we use the adjunction inequality as in [2]. Let $F \subset W^4$ be a properly embedded oriented surface in a Stein domain such that $k = \partial F \subset \partial W^4$ is a Legendrian knot with respect to the induced contact structure.

Let f denote the framing of k induced from the trivialization of the normal bundle of F; then

$$-\chi(F) \geq (\mathrm{tb}(k) - f) + |\mathrm{rot}(k)|.$$

Recall that the rotation number $\mathrm{rot}(k)$ and the Thurston–Bennequin number $\mathrm{tb}(k)$ are given by the formulae

$$\mathrm{rot}(k) = \tfrac{1}{2}(\text{number of "downward" cusps} - \text{number of "upward" cusps}),$$

$$\mathrm{tb}(k) = \mathrm{bb}(k) - c(k),$$

where $\mathrm{bb}(\alpha)$ is the blackboard framing (or writhe) of the front projection of k and $c(k)$ is the number of right cusps.

Assume the curve α_n is slice, so $\chi(F) = 1$, $\mathrm{tb}(\alpha_n) = 2n - (2n - 1) = 1$, $\mathrm{rot}(\alpha_n) = 0$ and $f = 0$, so we have a contradiction: $-1 \geq 1$, and therefore α_n is not slice. The same argument as in [1, Theorem 1] shows α_n does not bound a PL–disk in W^4.

4 Topological slice

Here we show that the family of knots that we constructed in the previous example are all topologically slice and therefore they are all distinct elements in the almost-concordance class of topologically slice knots on the boundary of the Mazur manifold.

A knot k in a homology sphere Y has well-defined Alexander polynomial $\Delta_k(t)$ in $\mathbb{Z}[t^{\pm}]$. Let F be a Seifert surface of k in Y and X be the knot complement. Then

$$\Delta_k(t) := \det(tS - S^T),$$

where S is an associated Seifert matrix of the bilinear form η

$$\eta \colon H_1(F; \mathbb{Z}) \times H_1(F; \mathbb{Z}) \to \mathbb{Z}, \quad \eta(\alpha, \beta) = \mathrm{lk}(\alpha^+, \beta).$$

We adopt the convention that $\alpha^+ \in H_1(X - F)$ is the image of $\alpha \in H_1(F)$ via pushing α in the positive normal direction of F. As is seen in Figure 8, the Seifert surface F of k_n links the 0–framed knot. One of its generators x links that knot. In this case $\mathrm{lk}(x^+, x)$ is not a direct calculation, since we have to find a Seifert surface F_x (or F_{x^+}) of x (or x^+) to calculate $\mathrm{lk}(x^+, x)$. On the other hand, using the lemma below we can calculate the Seifert matrix easily.

Lemma 7 [11, Lemma 7.13] *Let $k \cup l$ be a boundary link* (ie knots k and l bound disjoint Seifert surfaces) *in a homology sphere Y, and Y' is a ± 1 surgery of Y along k. Then $\Delta_{l \subset Y}(t) = \Delta_{l' \subset Y'}(t)$, where $l' \subset Y'$ is the image of $l \subset Y$ under the surgery.*

Since α and k_n have disjoint Seifert surfaces — see Figure 8, left — we perform -1 surgery on α, and after some isotopy of k_n we get the right diagram. Therefore, for
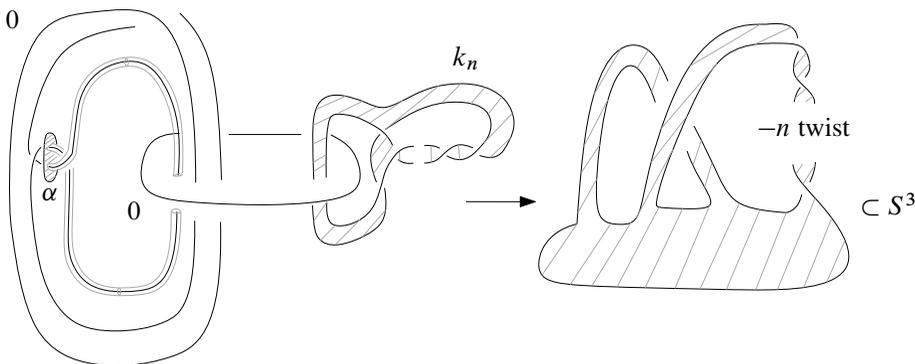

Figure 8: Alexander polynomial in homology sphere

the Seifert matrix $S = \begin{pmatrix} 0 & 1 \\ 0 & n \end{pmatrix}$ we have the corresponding Alexander polynomial

$$\Delta_{k_n \subset Y}(t) = \det(tS - S^T) = t \doteq 1.$$

Thanks to Freedman and Quinn [6, Theorem 11.7B], these knots are all topologically slice.

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