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Action dimension of lattices in Euclidean buildings

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We show that if a discrete group Γ acts properly and cocompactly on an n–dimensional, thick, Euclidean building, then Γ cannot act properly on a contractible $(2n-1)$–manifold. As an application, if Γ is a torsion-free S–arithmetic group over a number field, we compute the minimal dimension of contractible manifold that admits a proper Γ–action. This partially answers a question of Bestvina, Kapovich, and Kleiner.

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1 Introduction

The *action dimension* of a discrete group Γ, denoted by $\operatorname{actdim}(\Gamma)$, is the minimal dimension of contractible manifold M that Γ acts on properly and discontinuously. If Γ is torsion-free, then M/Γ is aspherical, and hence a model for the classifying space $B\Gamma$. Therefore, $\operatorname{actdim}(\Gamma)$ is the minimal dimension of such a manifold model. The *geometric dimension* of Γ is the minimal dimension of any model for $B\Gamma$. If we assume that Γ admits a model $B\Gamma$ which is a finite CW–complex, a theorem of Stallings [21] implies that $\operatorname{actdim}(\Gamma)$ is bounded above by twice the geometric dimension.

The motivation for this paper is a conjecture by Davis and Okun [14], which claims that nontrivial L^2–Betti numbers of a discrete group should provide a lower bound for action dimension. This generalizes an older conjecture of Singer concerning vanishing of the L^2–cohomology of the universal covers of closed aspherical manifolds.

Action dimension conjecture *If the i^{th} L^2–Betti number of Γ is nontrivial, then* $\operatorname{actdim}(\Gamma) \geq 2i$.

For a nice introduction to L^2–cohomology, we refer the reader to Eckmann [17]. If Γ is the fundamental group of a closed Riemannian manifold, then the L^2–Betti numbers of Γ provide a dimension to the space of square-summable harmonic forms on the universal cover. A positive solution to the action dimension conjecture has several nice implications; in particular, it implies a classical conjecture of Hopf and Thurston that the Euler characteristic of a closed, aspherical $2n$–manifold has sign $(-1)^n$.

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Though both L^2–cohomology and action dimension seem difficult to compute, the conjecture has been verified for many important classes of groups, such as lattices in Lie groups (Bestvina and Feighn [6]), mapping class groups (Despotovic [15], McMullen [20], Gromov [18]), $\mathrm{Out}(F_n)$ (Bestvina, Kapovich and Kleiner [7]), and most Artin groups (Avramidi, Davis, Okun and Schreve [4], Davis and Leary [12], Davis and Huang [11]).

In this paper, we show the conjecture holds for groups that act properly and cocompactly on thick Euclidean buildings. The L^2–Betti numbers of such groups are nontrivial (and concentrated) in the dimension of the building, and we show that their action dimension is greater than or equal to twice this dimension.

Theorem 1.1 *If Γ is a cocompact lattice in a thick, n–dimensional Euclidean building, then $\mathrm{actdim}(\Gamma) \geq 2n$. If Γ is torsion-free, then $\mathrm{actdim}(\Gamma) = 2n$.*

The second statement follows immediately from Stallings' theorem, since the geometric dimension of such Γ will be n. Our main application of this theorem is to compute the action dimension of S–arithmetic groups over number fields. These groups act properly on the product of a symmetric space and a Euclidean building. Bestvina and Feighn [6] showed that lattices in connected semisimple Lie groups have action dimension equal to the dimension of the symmetric space, and it was conjectured in [7] that the action dimension of S–arithmetic groups was equal to the dimension of the symmetric space plus twice the dimension of the Euclidean building. We confirm the conjecture in Section 6.

To prove Theorem 1.1, we use the *obstructor dimension* method introduced by Bestvina, Kapovich, and Kleiner in [7], which surprisingly is what *all* of the above computations of action dimension rely on. This involves finding subcomplexes of the visual boundary of a Euclidean building that are hard to embed into Euclidean space. In fact, a cohomological obstruction to embedding into Euclidean space due to van Kampen must be nontrivial. The visual boundary of a Euclidean building admits the structure of a spherical building, and the following corollary of our method may be of independent interest.

Theorem 1.2 *Let $\mathcal{B}$ be a finite, k–dimensional, spherical building, and fix a chamber $C \in \mathcal{B}$. Let $\mathrm{Opp}(C)$ denote the subcomplex of chambers opposite to C. If $H_k(\mathrm{Opp}(C), \mathbb{Z}_2) \neq 0$, then $\mathcal{B}$ does not embed into $\mathbb{R}^{2k}$.*

Therefore, the only way to embed these spherical buildings into Euclidean space is by appealing to general position. This was shown earlier by Tancer and Worwerk [22] for type A_n and certain type B_n buildings. In the general case, we do not know of a thick spherical building which does not have a chamber C with $H_k(\mathrm{Opp}(C), \mathbb{Z}_2) \neq 0$.

In [4], the action dimension of certain right-angled Artin groups was computed. The key tool was a computation of the van Kampen obstruction for certain simplicial complexes called *octahedralizations*. Our computation for spherical buildings relies on finding embedded octahedralizations based on $\mathrm{Opp}(C)$ inside the spherical building.

This paper is structured as follows. In Sections 2 and 3, we review the obstructor methods of [7] and the computation of the van Kampen obstruction of the octahedralization. In Section 4, we review spherical buildings and compute their van Kampen obstruction. In Section 5, we show the action dimension conjecture for lattices in Euclidean buildings, and in Section 6, we compute the action dimension of S–arithmetic groups.

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2 The van Kampen obstruction

The first cohomological obstruction to embedding a simplicial complex in $\mathbb{R}^n$ was introduced by van Kampen [19]. In this section we review this obstruction and describe how a coarsening of this obstruction gives a lower bound for the action dimension.

Let K be a simplicial complex, and let $\Delta \subset K \times K$ denote the simplicial neighborhood of the diagonal:

$$\Delta := \{(\sigma, \tau) \mid \sigma, \tau \in K,\ \sigma \cap \tau \neq \varnothing\}.$$

The complement $K \times K - \Delta$ admits a free $\mathbb{Z}_2$–action by flipping the factors. Let $\mathcal{C}(K) = (K \times K - \Delta)/\mathbb{Z}_2$ be the *simplicial configuration space of* K. So, $\mathcal{C}(K)$ is the space of unordered pairs of disjoint simplices of K:

$$\mathcal{C}(K) := \{\{\sigma, \tau\} \mid \sigma, \tau \in K,\ \sigma \cap \tau = \varnothing\}.$$

Definition 2.1 Let K be a k–dimensional simplicial complex, and let $f\colon K \to \mathbb{R}^n$ be a general position map. The images of a pair of disjoint simplices σ and τ with $\dim\sigma + \dim\tau = n$ intersect under f in a finite number of points. The $\mathbb{Z}_2$*–valued van Kampen obstruction* $\mathrm{vk}^n_{\mathbb{Z}/2}(K) \in H^n(\mathcal{C}(K), \mathbb{Z}_2)$ is defined by

$$\mathrm{vk}^n_{\mathbb{Z}/2}(\{\sigma,\tau\}) = |f(\sigma) \cap f(\tau)| \mod 2.$$

One can show that the class of this cocycle does not depend on f, which implies that if $\mathrm{vk}^n_{\mathbb{Z}/2}(K) \neq 0$, then K does not embed into $\mathbb{R}^n$. In this case, we say K is an n*–obstructor*. In fact, such a K cannot embed into any contractible n–manifold; see [7, Proposition 5].

Example Suppose $K_{3,3}$ is the *Kuratowski graph*, the join of 3 points with 3 points. There is a class in $H_2(\mathcal{C}(K_{3,3}), \mathbb{Z}_2)$ which consists of all unordered pairs of disjoint simplices. By mapping $K_{3,3}$ into $\mathbb{R}^2$ with some general position map and counting intersections, it is easy to see that $\mathrm{vk}^2_{\mathbb{Z}/2}(K_{3,3})$ evaluates nontrivially on this class.

We will now use the van Kampen obstruction to give a lower bound for the action dimension. This requires the following definition, due to Bestvina:

Definition 2.2 A $\mathcal{Z}$*–structure* on a group Γ is a pair $(\widetilde{X}, Z)$ of spaces satisfying the following four axioms:

- $\widetilde{X}$ is a Euclidean retract.
- $X = \widetilde{X} - Z$ admits a covering space action of Γ with compact quotient.
- Z is a $\mathcal{Z}$–set in $\widetilde{X}$, ie there exists a homotopy $\widetilde{X} \times [0,1] \to \widetilde{X}$ such that H_0 is the identity and $H_t(X) \subset X$ for all $t > 0$.
- The collection of translates of a compact set in X forms a null-sequence in $\widetilde{X}$, ie for every open cover $\mathcal{U}$ of $\widetilde{X}$ all but finitely many translates are contained in a single element of U.

A space Z is a *boundary* of Γ if there is a $\mathcal{Z}$–structure $(\widetilde{X}, Z)$ on Γ. For example, if Γ acts properly and cocompactly on a CAT(0) space X, then compactifying X by the visual boundary ∂X gives a $\mathcal{Z}$–structure on Γ.

Theorem 2.3 [7] *Suppose Z is a boundary of a group Γ, and that K is an embedded n–obstructor in Z. Then* $\mathrm{actdim}(\Gamma) \geq n+2$.

Heuristically, if Γ admits a $\mathcal{Z}$–structure and acts properly on a contractible $(n+1)$–manifold M, then there would be an injective boundary map $Z \to \partial M$. Since M is contractible, ∂M should be an n–sphere. This would contradict K being an n–obstructor, and hence prove Theorem 2.3. Of course, these optimistic statements are false in general, and the proof of Theorem 2.3 requires much more work.

Example Let $\Gamma = F_2 \times F_2$ be the direct product of two finitely generated free groups. Then Γ acts properly and cocompactly on a product of trees, whose visual boundary is a product of Cantor sets. This contains the graph $K_{3,3}$, so by Theorem 2.3, $\mathrm{actdim}(\Gamma) = 4$. More generally, Bestvina, Kapovich, and Kleiner showed that the n–fold product of free groups has $\mathrm{actdim} = 2n$, using the fact that the n–fold join of 3 points has $\mathrm{vk}^{2n-2}_{\mathbb{Z}/2} \neq 0$.

When we compute the action dimension of S–arithmetic groups over number fields, we use the slightly more general concept of *obstructor dimension*, denoted by $\mathrm{obdim}(\Gamma)$. For simplicity, we only give the definition for type VF groups. We lose no generality since S–arithmetic groups over number fields are virtually of finite type.

Let K be a simplicial complex, and let $\mathrm{Cone}(K) := K \times [0,\infty)/(K \times 0)$ denote the infinite cone on K. Given a triangulation of $\mathrm{Cone}(K)$, we set every edge to have length 1 and equip $\mathrm{Cone}(K)$ with the induced path metric.

Definition 2.4 Let X be a proper metric space and K a simplicial complex. A map $h\colon \mathrm{Cone}(K) \to X$ is *expanding* if for every σ and τ in K with $\sigma \cap \tau = \varnothing$, the images $\mathrm{Cone}(\sigma)$ and $\mathrm{Cone}(\tau)$ *diverge*, ie for every $D > 0$ there exists $t \in \mathbb{R}^+$ such that $h(\sigma \times [t,\infty])$ and $h(\tau \times [t,\infty])$ are distance $> D$ apart in X.

Definition 2.5 Let Γ be a discrete group, and assume Γ acts properly and cocompactly by isometries on a contractible proper metric space X. Then $\mathrm{obdim}(\Gamma)$ is the maximal $n+2$ such that there is a proper expanding map $h\colon \mathrm{Cone}(K) \to X$ with K an n–obstructor.

The following is the main theorem of [7].

Theorem 2.6 $$\mathrm{obdim}(\Gamma) \leq \mathrm{actdim}(\Gamma).$$

Finally, in Section 6 we need the product lemma for obstructor dimension, which follows immediately from the join lemma in [7].

Lemma 2.7 *Suppose X is a proper cocompact contractible Γ–complex, and that $f\colon \mathrm{Cone}(J)\times\mathrm{Cone}(K)\to X$ is a proper expanding map. If J is an n–obstructor and K is an m–obstructor, then $\mathrm{obdim}(\Gamma)\geq n+m+2$.*

3 Octahedralizations

In this section, we recall the definition of a certain simplicial complex with nontrivial van Kampen obstruction. This complex was used in [4] to give lower bounds for the action dimension of right-angled Artin groups. We will show that this complex also provides lower bounds for the action dimension of groups acting properly and cocompactly on Euclidean buildings.

Given a finite set V, let $\Delta(V)$ denote the full simplex on V and let $O(V)$ denote the boundary complex of the octahedron on V. In other words, $O(V)$ is the simplicial complex with vertex set $V\times\{\pm1\}$ such that a subset $\{(v_0,\varepsilon_0),\dots,(v_k,\varepsilon_k)\}$ of $V\times\{\pm1\}$ spans a k–simplex if and only if its first coordinates $v_0,\dots,v_k$ are distinct. Projection onto the first factor $V\times\{\pm1\}\to V$ induces a simplicial projection $p\colon O(V)\to\Delta(V)$. We will denote the vertices $(v,+1)$ and $(v,-1)$ by v^+ and v respectively, and the simplices $(\sigma,1)$ and $(\sigma,-1)$ by σ^+ and σ respectively,

Any finite simplicial complex L with vertex set V is a subcomplex of $\Delta(V)$. The *octahedralization* $O(L)$ of L is the inverse image of L in $O(V)$:

$$O(L) := p^{-1}(L)\subset O(V).$$

We also will say that $O(L)$ is the result of "doubling the vertices of L". In particular, an n–simplex (the n–fold join of a point) in L becomes an n–octahedron (the n–fold join of two points) in $O(L)$. Also, inclusions of simplices induce inclusions of octahedra in a canonical way. We will usually assume that L is a flag complex, which means that if the 1–skeleton of a simplex is in L, then the entire simplex is in L.

Fix a simplex Δ in L, and let $D_\Delta(L)$ denote the full subcomplex of $O(L)$ containing L and Δ^+. We say that $D_\Delta(L)$ is L doubled over Δ. See Figure 1 for the example of L an n–cycle.

In [4], the van Kampen obstruction of $D_\Delta(L)$ was calculated.

Theorem 3.1 *If L is a k–dimensional flag simplicial complex, then*

$$\mathrm{vk}^{2k}_{\mathbb{Z}/2}(D_\Delta(L))\neq 0 \iff H_k(L,\mathbb{Z}_2)\neq 0.$$

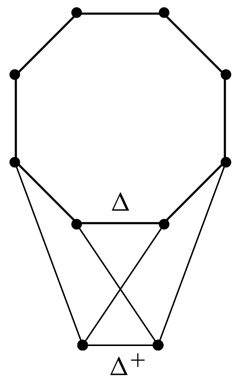

Figure 1: If L is an n–cycle, then $D_\Delta(L)$ is a subdivided $K_{3,3}$.

4 Spherical buildings

We will only give a brief description of buildings; the reader is encouraged to refer to [3] for complete details.

Definition 4.1 A *Coxeter group* (W, S) is a group generated by involutions $s_i \in S$, with the only other relations being that every pair of elements s_i and s_j generates a dihedral group (perhaps D_∞). In other words, (W, S) has a presentation

$$\langle s_1, s_2, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

with exponents $m_{ij} \in \mathbb{N} \cup \infty$ such that $m_{ii} = 1$.

For example, proper cocompact reflection groups acting on $\mathbb{S}^n$, $\mathbb{E}^n$ and $\mathbb{H}^n$ are all examples of Coxeter groups. We will assume from now on that $|S| < \infty$.

A *mirror structure over* S on a space X is a family of subspaces $\{X_s\}_{s \in S}$ indexed by S. If (W, S) is a Coxeter system and X has a mirror structure over S, let $S(x) := \{s \in S \mid x \in X_s\}$. Let $W_{S(x)}$ be the Coxeter subgroup generated by $S(x)$. Now, define an equivalence relation $\sim$ on $W \times X$ by $(w, x) \sim (w', y)$ if and only if $x = y$ and $w^{-1}w' \in W_{S(x)}$. Let $\mathcal{U}(W, X)$ denote the quotient space

$$\mathcal{U}(W, X) = (W \times X)/\sim,$$

which is called the *basic construction*. There is a natural W–action on $W \times X$ which respects the equivalence relation, and hence descends to an action on $\mathcal{U}(W, X)$. One can think of $\mathcal{U}(W, X)$ as pasting together copies of X with the exact gluing given by the Coxeter group.

Definition 4.2 Let (W,S) be a Coxeter group, and let Δ be a simplex of dimension $|S|-1$. We can label the codimension-one faces of Δ by the elements of S. The set of codimension-one faces is a mirror structure on Δ. The space $\Phi_W := \mathcal{U}(W,\Delta)$ is called the *Coxeter complex* of (W,S).

Definition 4.3 A *building* is a simplicial complex with a distinguished set of subcomplexes called *apartments*. This collection of apartments satisfies the following axioms:

(1) Each apartment is isomorphic to a Coxeter complex.

(2) For any two simplices, there is an apartment containing both of them.

(3) If two apartments contain simplices σ and τ, then there is an isomorphism between these apartments fixing $\sigma \cap \tau$ pointwise.

The top-dimensional simplices of a building are called *chambers*. The codimension-one simplices are called *panels*. A building is *thick* if each panel is contained in at least 3 chambers.

If W is finite, the building is called *spherical*, since each apartment is homeomorphic to a sphere. In a spherical Coxeter complex, there is a well-defined notion of *opposite* chambers. Note that every chamber σ in a Coxeter complex corresponds to a Coxeter group element w_σ. Two chambers σ and τ are opposite if they correspond to Coxeter group elements w_σ, w_τ such that $w_\sigma = w_0 w_\tau$, where w_0 is the longest element of W with respect to the standard generating set S. Opposition naturally extends to spherical buildings; in fact, two chambers σ and τ are opposite in one apartment if and only if they are opposite in each apartment containing them both.

Two chambers in an apartment are opposite if and only if they are on opposite sides of every wall in that apartment, where a *wall* is a fixed-point set inside an apartment of a reflection of W. One can analogously define *opposition of simplices*: α and β are opposite in an apartment if they are contained in the same walls and separated by every wall that does not contain them both, and opposite in a building if they are opposite in each apartment which contains them.

A spherical building is *right-angled* if all apartments are isomorphic to an n–octahedron, which is the Coxeter complex of the right-angled Coxeter group $W \cong (\mathbb{Z}/2)^{n+1}$. In an octahedron, the vertices v and v^+ are opposite. More generally, two simplices σ and τ in an octahedron are opposite if and only if $p(\sigma) = p(\tau)$ and $\sigma \cap \tau = \varnothing$.

Example Let $\mathcal{P}$ denote the poset of proper nonempty subspaces of $\mathbb{R}^n$. Let $\mathrm{Flag}(\mathcal{P})$ denote the associated flag complex, ie the vertices of $\mathrm{Flag}(\mathcal{P})$ consist of elements of $\mathcal{P}$ and simplices of $\mathrm{Flag}(\mathcal{P})$ correspond to chains of inclusions between these subspaces. $\mathrm{Flag}(\mathcal{P})$ has a natural structure of a spherical building, where apartments correspond to choosing a basis of $\mathbb{R}^n$ and then considering all subspaces generated by proper subsets of these basis vectors. In this case, the Coxeter group is the symmetric group S_n, which acts by permuting the basis vectors. The Coxeter complex can be identified with the barycentric subdivision of the $(n-2)$–simplex. A chamber is a maximal chain $[v_0] \subset [v_0, v_1] \subset \cdots \subset [v_0, v_1, \ldots, v_{n-2}]$. Two chambers C and C' are opposite in this building precisely when any two subspaces in C and C', respectively, are in general position.

4.1 Convexity

In order to compute the van Kampen obstruction of a spherical building $\mathcal{B}$, we will embed $D_\Delta(\mathrm{Opp})$ into $\mathcal{B}$ and apply Theorem 3.1. In order to define this embedding, we need to briefly recall convexity in spherical buildings. In order to avoid too much terminology, we use definitions that are a result of theorems in [3].

A wall in a Coxeter complex Φ_W separates Φ_W into two halfspaces called *roots*. These roots are permuted by the reflection in W corresponding to the wall. A subcomplex Σ of Φ_W is *convex* if it is an intersection of roots. For example, any intersection of walls is a convex subcomplex. The *convex hull* of two simplices $\sigma, \tau \in \Sigma$ is the intersection of all roots containing both, and by definition is contained in every convex subcomplex containing σ and τ.

In a building $\mathcal{B}$, there is an analogous notion of convex subcomplexes and convex hulls. We will only need the fact that the convex hull of two simplices in a building is the same as the convex hull of the two simplices in any apartment that contains them both. For example, suppose C is a chamber in $\mathcal{B}$, and σ, τ are two chambers in $\mathrm{Opp}(C)$. Suppose that Σ is the convex hull in $\mathcal{B}$ of $\sigma \cap \tau$ and its opposite simplex $(\sigma \cap \tau)^o$ in C. If A_σ and A_τ are the corresponding apartments in $\mathcal{B}$, then by the above, Σ is contained in both A_σ and A_τ, and coincides with the convex hull of $\sigma \cap \tau$ and $(\sigma \cap \tau)^o$ in both apartments. In each apartment, the convex hull is precisely the intersection of all walls in that apartment containing $\sigma \cap \tau$ and $(\sigma \cap \tau)^o$.

4.2 Bending homeomorphisms

Let $\mathcal{B}$ be an n–dimensional spherical building. Let A be an apartment in $\mathcal{B}$, fix a chamber C in A, and let $\{H_1, H_2, \ldots, H_{n+1}\}$ be the set of walls of A that intersect C

in a codimension-one face. If O^n is an n–octahedron, then analogously we fix an n–simplex C' in O^n and consider the set of walls $\{H'_1, H'_2, \dots, H'_{n+1}\}$ that intersect C' in a codimension-one face. Let $f\colon O^n \to A$ be a "bending" homeomorphism which satisfies the following conditions:

(1) f maps C' to C.

(2) $f\bigl(\bigcap_k H'_{i_k}\bigr) = \bigcap_k H_{i_k}$ for any subset of $\{H'_i\}$.

More generally, any homeomorphism between Coxeter complexes satisfying (2) is said to be *wall-preserving* (where again the Coxeter groups are not necessarily the same). Now, fix a chamber $\Delta^+ \in \mathcal{B}$, and let $\mathrm{Opp}(\Delta^+)$ be the simplicial complex of opposite simplices. For each chamber $\sigma \in \mathrm{Opp}(\Delta^+)$, there is a unique apartment A_σ in $\mathcal{B}$ containing Δ^+ and σ. We now fix a chamber σ and a bending homeomorphism $f\colon O(\sigma) \to A_\sigma$, as above, which sends $\sigma \to \sigma$ and $\sigma^+ \to \Delta^+$.

There is also a retraction map $\Psi\colon \mathcal{B} \to A_\sigma$ which fixes Δ^+. This retraction is constructed by gluing together the various isomorphisms $\psi_\tau\colon A_\tau \to A_\sigma$ which are guaranteed by axiom (3) of a building, where A_τ is another apartment containing Δ^+. In particular, by the proof of [3, Proposition 4.33], ψ_τ and $\psi_{\tau'}$ can be made to agree on $A_\tau \cap A_{\tau'}$. Furthermore, each ψ_τ is wall-preserving, as ψ_τ preserves the bijection between elements of W and chambers in A_τ and A_σ which sends the identity to Δ^+.

The map Ψ restricts to a retraction $\mathrm{Opp}(\Delta^+) \to \sigma$, and hence induces a retraction $O(\Psi)\colon O(\mathrm{Opp}(\Delta^+)) \to O(\sigma)$. Note that for any chamber $\tau \subset \mathrm{Opp}(\Delta^+)$, the restriction $O(\Psi)|_{O(\tau)}$ is also a wall-preserving homeomorphism, which sends τ^+ to σ.

We assemble the homeomorphisms $\psi_\tau^{-1}\colon A_\sigma \to A_\tau$ and $f \circ O(\Psi)|_{O(\tau)}\colon O(\tau) \to A_\sigma$ to get a map $F\colon O(\mathrm{Opp}(\Delta^+)) \to \mathcal{B}$. Specifically, define

$$F(x) = \psi_\tau^{-1} \circ f \circ O(\Psi)(x) \quad \text{if } x \in O(\tau) \subset O(\mathrm{Opp}(\Delta^+)).$$

Lemma 4.4 *F is a well-defined map.*

Proof The map F is illustrated in Figure 2. We need to show that if τ and τ' are chambers in $\mathrm{Opp}(\Delta^+)$ which intersect, and $x \in O(\tau) \cap O(\tau')$, then

$$\psi_\tau^{-1} \circ f \circ O(\Psi)(x) = \psi_{\tau'}^{-1} \circ f \circ O(\Psi)(x).$$

Since ψ_τ and $\psi_{\tau'}$ agree on $A_\tau \cap A_{\tau'}$, it suffices to show that $f \circ O(\Psi)(x)$ is in $\Psi(A_\tau \cap A_{\tau'})$.

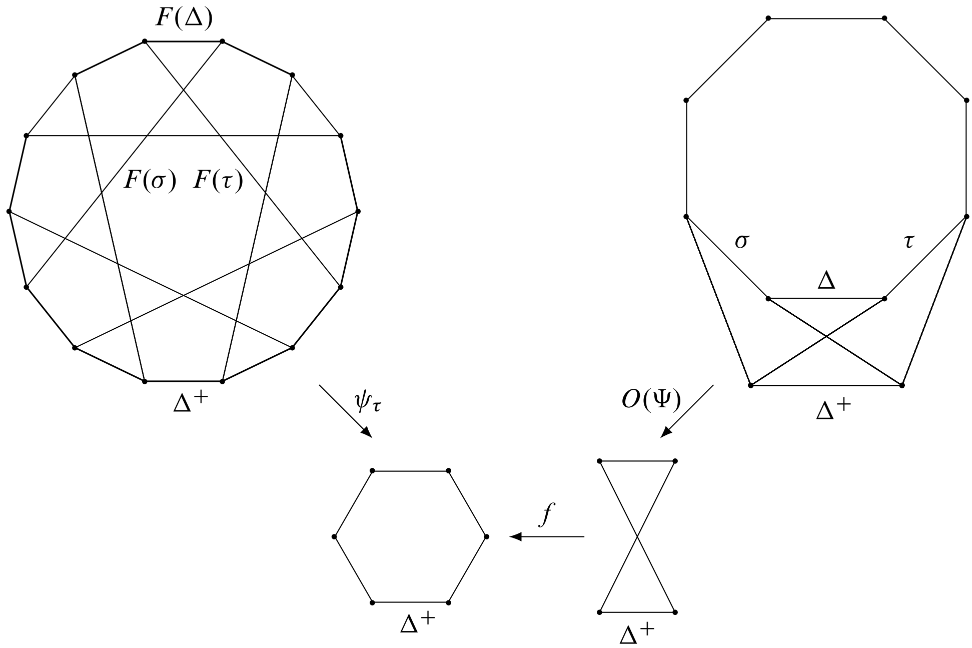

Figure 2: The bending map $F = \psi_\tau^{-1} \circ f \circ O(\Psi)$

In Section 4.1, we noted that $A_\tau \cap A_{\tau'}$ contains the intersection of all walls containing $(\tau \cap \tau')$ and its opposite simplex in Δ^+. Since ψ_τ and $\psi_{\tau'}$ are wall-preserving, $\Psi(A_\tau \cap A_{\tau'})$ contains the intersection of all walls in A_σ containing $\Psi(\tau \cap \tau')$ and its opposite simplex in Δ^+.

Since $x \in O(\tau \cap \tau')$, it is contained in each wall of $O(\tau)$ and $O(\tau')$ that contains $(\tau \cap \tau')$ and its opposite simplex $(\tau \cap \tau')^+$. Therefore, $O(\Psi)(x)$ is contained in each wall of $O(\sigma)$ that contains $\Psi(\tau \cap \tau')$ and $\Psi(\tau \cap \tau')^+$. Therefore, $f \circ O(\Psi)(x)$ is contained in the intersection of all walls in A_σ which contain $\Psi(\tau \cap \tau')$ and its opposite simplex in Δ^+, which by the above guarantees it is in $\Psi(A_\tau \cap A_{\tau'})$. $\square$

Now, we will show that F restricts to an embedding on $D_\Delta(\mathrm{Opp}(\Delta^+))$, for some choice of a chamber Δ in $\mathrm{Opp}(\Delta^+)$. To do this, we need to be more precise about the image of simplices under F. See Figure 3 for the image of F in the case of the Fano plane.

For a finite Coxeter group (W, S), we choose a chamber C in the Coxeter complex which corresponds to the identity element, and identify the vertices of that chamber with the elements of S. For $s \in S$, we say that the *s–wall* in the Coxeter complex is the unique wall containing every other vertex in C.

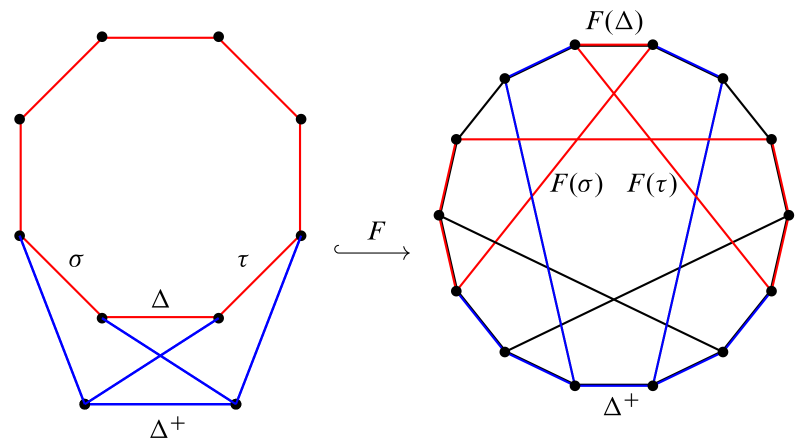

Figure 3: Embedding $D_\Delta(\mathrm{Opp}(\Delta^+))$ into the Fano plane. The red cycle is the opposite complex of Δ^+.

Definition 4.5 Let (W,S) be a Coxeter group, and let $w \in W$. Then

$$\mathrm{In}(w) = \{s \in S \mid \ell(ws) < \ell(w)\},$$

where $\ell(w)$ denotes the length of w in W with respect to the generating set S.

$\mathrm{In}(w)$ is precisely the set of letters with which a reduced expression for w can end. Note that $\mathrm{In}(w^{-1})$ is precisely the elements $s \in S$ such that $\ell(sw) < \ell(w)$, ie it consists of w which are separated from the identity chamber by the s–wall.

Definition 4.6 Let $\alpha \in D_\Delta(\mathrm{Opp}(\Delta^+))$ be a simplex. Let $V^+(\alpha)$ denote the vertices of α contained in Δ^+ and let $V(\alpha)$ denote the vertices of α contained in $\mathrm{Opp}(\Delta^+)$.

For a chamber σ in $\mathrm{Opp}(\Delta^+)$, identify $O(\sigma)$ as the Coxeter complex of $(\mathbb{Z}/2)^{n+1}$, and identify Δ^+ with the identity chamber. A chamber $\alpha \subset O(\sigma)$ corresponds to an element w with $\mathrm{In}(w^{-1}) = V(\alpha)$, ie w is separated from the identity by each s–wall for $s \in V(\sigma)$. Since the homeomorphisms f, $O(\Psi)$ and $\Psi|_{A_\tau}$ are wall-preserving and preserve Δ^+, we have the following lemma.

Lemma 4.7 *For $\tau \in \mathrm{Opp}(\Delta^+)$ and α a chamber in $O(\tau)$, the image $F(\alpha)$ is a union of chambers in A_τ corresponding to Coxeter group elements w with $\mathrm{In}(w^{-1}) = V(\alpha)$.*

Now, we can prove our main theorem of this section.

Theorem 4.8 *F restricted to $D_\Delta(\mathrm{Opp}(\Delta^+))$ is an embedding.*

Proof It is obvious that F restricted to each simplex is a homeomorphism. Therefore, it suffices to show that disjoint top dimensional simplices are mapped disjointly under F. Note that this is obviously true for chambers in $\mathrm{Opp}(\Delta^+)$. Let σ, τ be chambers in $\mathrm{Opp}(\Delta^+)$, and let α and β be disjoint chambers in $O(\sigma) \cap D_\Delta(\mathrm{Opp}(\Delta^+))$ and $O(\tau) \cap D_\Delta(\mathrm{Opp}(\Delta^+)$, respectively. We must show that $F(\alpha) \cap F(\beta) = \varnothing$. Again, we identify Δ^+ with the identity chamber of W.

If α contains a vertex s^+ in Δ^+, the chambers in the image of $F(\alpha)$ correspond to Coxeter group elements w with $s \notin \mathrm{In}(w^{-1})$. On the other hand, since $s^+ \notin \beta$, the chambers in $F(\beta)$ correspond to elements with $s \in \mathrm{In}(w^{-1})$. Therefore, any intersection between α and β must occur on the s–wall of the respective apartments.

If α contains no other vertices of Δ^+, then the intersection of the image of α with the s–wall is in $\mathrm{Opp}(\Delta^+)$. Therefore, if α and β have nontrivial intersection, α must contain another vertex in Δ^+. Repeating this argument verbatim would eventually imply that $\alpha = \Delta^+$, which implies $F(\alpha) \cap F(\beta) = \varnothing$. □

4.3 Homology of Opp(C)

Now that we have embedded $D_\Delta(\mathrm{Opp}(\Delta^+))$ into our spherical building, we would like to verify the assumptions of Theorem 3.1. Therefore, we need to show that $\mathrm{Opp}(\Delta^+)$ is a flag complex with top-dimensional $\mathbb{Z}_2$–homology. The first statement follows straight from the definitions; we record it as a lemma.

Lemma 4.9 *For any chamber C in $\mathcal{B}$, the subcomplex* $\mathrm{Opp}(C)$ *is a flag complex.*

Proof We will show that $\mathrm{Opp}(C)$ is a full subcomplex. This means that if the vertex set of a simplex in $\mathcal{B}$ is contained in $\mathrm{Opp}(C)$, then the simplex itself is contained in $\mathrm{Opp}(C)$. It implies that $\mathrm{Opp}(C)$ is also a flag complex.

We prove the contrapositive: suppose σ is a simplex not in $\mathrm{Opp}(C)$, so there exists a wall such that σ and C lie on the same side. It follows that all the vertices of σ are not opposite to C. Therefore, if a simplex has vertex set in C, then the simplex is in C. □

On the other hand, it is not obvious to us that $H_k(\mathrm{Opp}(C), \mathbb{Z}_2) \neq 0$ for every thick spherical building. In the next lemma, we verify that this does occur for infinitely thick buildings, which suffices for our purposes.

Lemma 4.10 *If $\mathcal{B}$ has infinite thickness, then there exists a chamber C such that* $\mathrm{Opp}(C)$ *contains an apartment.*

Proof Let $A \subset \mathcal{B}$ be any apartment and Δ_0 be any chamber. If Δ_0 is opposite to all chambers in A, then we are done. If not, for each chamber A_i in A, there are only finitely many chambers that share a panel with Δ_0 and are closer to A_i. Therefore, we can "push" Δ_0 to a new chamber Δ_1 that is further from each A_i (or perhaps the same distance if Δ_0 is already opposite to A_i). Continuing in this way, we find a chamber C_0 opposite to each A_i, and therefore $A_i \in \mathrm{Opp}(C_0)$. □

Example If k is an infinite field and Flag($\mathcal{P}$) the associated spherical building, then it is easy to construct an apartment in Opp(C) for any chamber C. Any chamber C corresponds to a flag of subspaces $C = [v_0] \subset [v_0, v_1] \subset \cdots \subset [v_0, v_1, \dots, v_{n-2}]$. The opposite apartment is determined by choosing another basis in general position (eg choose all e_k to not lie in $[v_0, v_1, \dots, v_{n-2}]$).

Remark For the finite thickness case, there are some obvious cases where we can find cycles in Opp. If the thickness of each panel is odd, then Opp is itself a $\mathbb{Z}_2$–cycle. Note that this implies that Opp has nontrivial homology if the spherical building contains a spherical subbuilding of odd thickness. Also, for large enough thickness, a simple counting argument shows that Opp has top homology as it contains more n–simplices than $(n-1)$–simplices.

In general, we do not know if there always exist chambers such that Opp has top-dimensional homology. This may be subtle: in [2], Abramenko constructs examples of infinitely thick 1–dimensional buildings where for certain chambers Opp is a disjoint union of trees. However, other chambers inside these buildings have opposite complexes which contain apartments by Lemma 4.10. Note that higher rank spherical buildings have been fully classified by Tits [23]. It seems likely that one can construct top-dimensional cycles in the opposite complexes for each list in the classification, but the calculations were too hard for the author.

5 The action dimension conjecture for lattices in Euclidean buildings

The following theorem is classical. In [10], an alternate proof is given which also computes the L^2–cohomology of arbitrary buildings in terms of the weighted L^2–cohomology of an apartment.

Theorem 5.1 *If Γ acts properly and cocompactly on an n–dimensional Euclidean building, then the L^2–cohomology of Γ is concentrated in dimension n and is nontrivial if the building is thick.*

Therefore, the following confirms the action dimension conjecture in this case.

Theorem 5.2 *If Γ acts properly and cocompactly on a thick Euclidean building of rank n, then $\mathrm{actdim}(\Gamma) \geq 2n$. If Γ is torsion-free, $\mathrm{actdim}(\Gamma) = 2n$.*

Proof It is well known that the visual boundary of a thick Euclidean building admits the structure of a thick spherical building of rank $n-1$ with infinite thickness [3, Section 11.8]. By Theorem 2.3, Theorem 3.1, and Lemmas 4.8–4.10, it follows that $\mathrm{actdim}(\Gamma) \geq 2n$. If Γ is torsion-free, then equality follows from Stallings' theorem since the geometric dimension of Γ is n. □

5.1 More general buildings

For general buildings, the above strategy fails as we lose the spherical building at infinity. However, there are some specific examples where the action dimension is known. For example, Dymara and Osajda [16] have shown that the boundary of a thick, n–dimensional right-angled hyperbolic building is the universal Menger space (these only exist in low dimensions, as for $n > 4$ there are no right-angled Coxeter groups that act properly and cocompactly on $\mathbb{H}^n$). For non right-angled buildings, this was shown for $n = 2$ by Benakli [5]. Therefore, for cocompact lattices acting on such buildings the action dimension is at least twice the dimension of the building. It would be nice to extend Theorem 5.2 to all hyperbolic buildings, where we suspect the same bounds on action dimension should hold. The link of each vertex in a thick, locally finite, hyperbolic building $\mathcal{B}$ is a thick, finite, spherical building. The difficulty here is pushing this link in a compatible way to the building to get a proper expanding map $\mathrm{Cone(Lk)} \to \mathcal{B}$.

For general buildings, the geometric dimension of lattices can be less than the dimension of the building, so by Stallings' theorem, the action dimension must sometimes be less than twice the dimension of the building. For example, let L be an $(n-1)$–dimensional flag complex such that $H_{n-1}(L, \mathbb{Z}_2) = 0$. Form any graph product G_L such that every vertex group G_s is finite with $|G_s| > 2$. Then G_L acts properly and cocompactly on an n–dimensional thick, locally finite right-angled building X_L; see [13]. The geometric dimension of G_L is the same as the geometric dimension of the underlying Coxeter group, which has been explicitly computed and with these assumptions is less than n; see [9]. We conjecture that if Γ acts properly and cocompactly on a locally finite, thick building, then $\operatorname{actdim}(\Gamma) \geq 2\operatorname{gdim}(\Gamma)$.

6 S–arithmetic groups

In this section, we describe our main application of Theorem 5.2. Let k be a number field, ie a finite-degree extension of $\mathbb{Q}$. Let $G(k)$ be a semisimple linear algebraic group over k. For each finite place p let $\nu_p\colon k \to \mathbb{Z}$ denote the corresponding valuation. Let S be a set of finite places of k, including the set of infinite places. Let k_p be the completion of k with respect to the norm induced by ν_p. The ring of S–integers is defined by

$$\mathcal{O}_S := \{x \in k \mid \nu_p(x) \geq 0 \text{ for all } p \notin S\}.$$

A subgroup of $G(k)$ is an S–arithmetic subgroup if it is commensurable with $G(\mathcal{O}_S)$.

Example Let $S = \{p_1, p_2, \dots, p_m\}$ be a finite set of primes. Each prime determines a p–adic valuation $\mathbb{Q} \to \mathbb{Z}$ which sends $u \to n$ if $u = p^n x$ and neither the numerator nor the denominator of x is divisible by p. The ring of S–integers in this case is $\mathbb{Z}_S := \mathbb{Z}\bigl[\frac{1}{p_1}, \frac{1}{p_2}, \dots, \frac{1}{p_m}\bigr]$. An *$S$–arithmetic subgroup* Γ of $G(\mathbb{Q})$ is commensurable with $G(\mathbb{Z}_S)$.

For simplicity, we will pass to a torsion-free finite-index subgroup Γ of $G(\mathcal{O}_S)$, which always exists in this setting [8]. If S consists of only infinite places, then Γ is an arithmetic subgroup and acts on a symmetric space X_∞. If S has finite places, then for each finite place ν there is a thick Euclidean building X_ν associated to $G(k_\nu)$, and Γ acts properly on $X_\infty \times \prod_{\nu_p} X_{\nu_p}$. Furthermore, there is a partial compactification of X_∞ due to Borel and Serre, which we denote by X_∞^{BS}. By [8], the Γ–action extends to a proper and cocompact action on $X_\infty^{\mathrm{BS}} \times \prod_{\nu_p} X_{\nu_p}$. It will be important for us that if we fix a vertex v in $\prod_{\nu_p} X_{\nu_p}$ and restrict to the Γ–action on $\prod_{\nu_p} X_{\nu_p}$, the stabilizer of v in Γ is an arithmetic subgroup Γ_∞.

Bestvina and Feighn construct a proper, expanding map f from a coned $(\dim(X_\infty)-2)$–obstructor $\operatorname{Cone}(J)$ into X_∞ that is bounded distance from a Γ_∞–orbit. Theorem 5.2 gives a proper, expanding map g from a coned $\bigl(2\dim\bigl(\prod_{\nu_p} X_{\nu_p}\bigr)-2\bigr)$–obstructor $\operatorname{Cone}(K)$ into $\prod_{\nu_p} X_{\nu_p}$.

We take the product embedding into $X_\infty \times \prod_{\nu_p} X_{\nu_p}$ and then compose with the inclusion into the Borel–Serre partial compactification:

$$h := f \times g\colon \mathrm{Cone}(J) \times \mathrm{Cone}(K) \to X_\infty^{\mathrm{BS}} \times \prod_{\nu_p} X_{\nu_p}.$$

We will now show that h is proper and expanding with respect to a proper Γ–invariant metric on $X_\infty^{\mathrm{BS}} \times \prod_{\nu_p} X_{\nu_p}$. We first show that such a metric exists. Since X_∞^{BS} is a manifold with corners, it is separable and metrizable, and hence so is $X_\infty^{\mathrm{BS}} \times \prod_{\nu_p} X_{\nu_p}$. Therefore, the following general theorem applies.

Theorem 6.1 [1, Theorem 4.2] *Suppose X is σ–compact. If the action of Γ on X is proper and X is metrizable, then there is a Γ–invariant proper metric d_Γ on X that induces the topology of X.*

Suppose Γ acts by isometries on two metric spaces X and Y, and suppose the diagonal action of Γ on $X \times Y$ is proper. Suppose that X admits a partial compactification $\widehat{X}$ such that the Γ–action extends to a proper and cocompact action on $\widehat{X} \times Y$. Let d_X and d_Y respectively denote the left Γ–invariant metrics on X and Y, and let d_Γ be a left Γ–invariant metric on $\widehat{X} \times Y$ as in Theorem 6.1. We shall prove the following general theorem in the next section.

Theorem 6.2 *Suppose that there are proper expanding maps $f\colon \mathrm{Cone}(J) \to X$ and $g\colon \mathrm{Cone}(K) \to Y$. Let $(x, y) = (f(0), g(0))$, and suppose $\mathrm{Cone}(J)$ maps a bounded distance in the d_X–metric from the $\mathrm{Stab}_\Gamma(y)$–orbit of (x, y). For any Γ–invariant proper metric on $\widehat{X} \times Y$, the product map $h = f \times g\colon \mathrm{Cone}(J) \times \mathrm{Cone}(K) \to \widehat{X} \times Y$ is proper and expanding.*

By the above remarks and Theorem 2.6 this theorem implies the following:

Corollary 6.3 *If Γ is an S–arithmetic group over a number field, then the map $h\colon \mathrm{Cone}(J) \times \mathrm{Cone}(K) \to X_\infty^{\mathrm{BS}} \times \prod_{\nu_p} X_{\nu_p}$ defined as above is proper and expanding, and hence $\mathrm{actdim}(\Gamma) \geq \dim(X_\infty) + \sum_{i=1}^{|S|} 2\dim(X_\nu)$.*

6.1 Proof of Theorem 6.2

We first need the following two lemmas.

Lemma 6.4 *Suppose that (x_i, y_i) and (x_i', y_i') are a pair of sequences in $\widehat{X} \times Y$. If $d_Y(y_i, y_i') \to \infty$, then $d_\Gamma((x_i, y_i), (x_i', y_i')) \to \infty$.*

Proof By contradiction: suppose there exist subsequences $(x_i, y_i), (x_i', y_i')$ with $d_\Gamma((x_i, y_i), (x_i', y_i'))$ uniformly bounded. Since the Γ–action on $\widehat{X} \times Y$ is cocompact, there exist $\gamma_i \in \Gamma$ such that $(\gamma_i x_i, \gamma_i y_i)$, and hence $(\gamma_i x_i', \gamma_i y_i')$, are contained in a compact set K. We still have $d_Y(\gamma_i y_i, \gamma_i y_i') \to \infty$ since $d_Y(\gamma y_i, \gamma y_i') = d_Y(y_i, y_i')$. Since K projects to a compact set in Y, this is a contradiction. □

Lemma 6.5 *Let (x_i, y_i) and (x_i', y_i') in $X \times Y$ be a pair of sequences. Assume that y_i remains a bounded d_Y–distance from a basepoint y_0. Suppose that x_i and x_i' are bounded d_X–distance from the* $\mathrm{Stab}_\Gamma(y_0)$*–orbit of a basepoint x_0. If $d_X(x_i, x_i') \to \infty$, then $d_\Gamma((x_i, y_i), (x_i', y_i')) \to \infty$.*

Proof Assume for the sake of contradiction there exist subsequences (x_i, y_i) and (x_i', y_i') with $d_\Gamma((x_i, y_i), (x_i', y_i'))$ uniformly bounded. Choose γ_i in $\mathrm{Stab}_\Gamma(y_0)$ such that $d_X(\gamma_i x_i, x_0)$ is uniformly bounded. Since we assume the y_i are in a compact set of Y, we know that $d_Y(\gamma_i y_i, y_0)$ is uniformly bounded. The sequence $(\gamma_i x_i, \gamma_i y_i)$ is contained in a compact set in $X \times Y$, which implies $d_\Gamma((\gamma_i x_i, \gamma_i y_i), (x_0, y_0))$ is uniformly bounded. Therefore by assumption $d_\Gamma((\gamma_i x_i', \gamma_i y_i'), (x_0, y_0))$ is uniformly bounded.

We have $d_X(\gamma_i x_i', x_0) \to \infty$ since $d_X(\gamma_i x_i, \gamma_i x_i') \to \infty$ and $d_X(\gamma_i x_i, x_0)$ is uniformly bounded. Choose γ_i' in $\mathrm{Stab}_\Gamma(y_0)$ such that $d_X(\gamma_i' x_i', x_0)$ is uniformly bounded. Then the sequence $d_\Gamma((\gamma_i(\gamma_i')^{-1} x_0, y_0), (x_0, y_0))$ is uniformly bounded. Now since $d_X(x_i, x_i') \to \infty$, we have that the distance in the word metric of Γ between γ_i and γ_i' goes to ∞, which contradicts the properness of the Γ–action on $\widehat{X} \times Y$. □

Proof of Theorem 6.2 We show that $f \times g$ is proper and expanding. We will let (j, k) denote points in $\mathrm{Cone}(J) \times \mathrm{Cone}(K)$.

Proper Let $(f(j_0), g(k_0))$ be the image of the cone point. Suppose that $f \times g$ is not proper. Then there is a sequence (j_i, k_i) which leaves every compact set of $\mathrm{Cone}(J) \times \mathrm{Cone}(K)$ and is such that $h(j_i, k_i)$ is contained in a compact set C of $\widehat{X} \times Y$. Since C projects to a compact set in Y and g is proper, the k_i are contained in a compact subset of $\mathrm{Cone}(K)$. Therefore, the j_i leave every compact set in $\mathrm{Cone}(J)$, and since f is proper, this implies that $d_X(f(j_0), f(j_i)) \to \infty$, which contradicts Lemma 6.5.

Expanding Assume $\sigma \times \tau$ and $\sigma' \times \tau'$ are disjoint simplices in $\mathrm{Cone}(J) \times \mathrm{Cone}(K)$. Suppose $f \times g$ is not expanding. It follows that there are sequences (j_i, k_i) in

$\mathrm{Cone}(\sigma)\times\mathrm{Cone}(\tau)$ and (j_i', k_i') in $\mathrm{Cone}(\sigma')\times\mathrm{Cone}(\tau')$ which leave every compact set and have $d_\Gamma(h(j_i,k_i), h(j_i',k_i'))$ uniformly bounded. By Lemma 6.4, this implies that $d_Y(k_i,k_i')$ is uniformly bounded. Since g is expanding, this implies that k_i and k_i' are contained in a compact set. Therefore, j_i and j_i' leave every compact set, and since f is expanding we have $d_X(f(j_i), f(j_i')) \to \infty$, which contradicts Lemma 6.5. $\square$

Example Let $\Gamma = \mathrm{SL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$. In this case, Γ acts properly on $\mathbb{H}^2\times T$, where T is a trivalent tree. The obstructor complex we want to use is

$$\mathrm{Cone}(K_{3,3}) \cong \mathrm{Cone}(3\text{ points})\times\mathrm{Cone}(3\text{ points}).$$

In the trivalent tree, we map Cone(3 points) by choosing 3 disjoint rays emanating from a fixed base point. In $\mathbb{H}^2$, which we identify with the upper half-plane, we send Cone(3 points) to the orbit of i under the following matrices:

$$\left\{\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix} \;\middle|\; t\in\mathbb{R}\right\}\cup\left\{\begin{bmatrix}1 & 0\\ t & 1\end{bmatrix} \;\middle|\; t\in\mathbb{R}^+\right\}.$$

Unfortunately, the product embedding $\mathrm{Cone}(3\text{ points})\times\mathrm{Cone}(3\text{ points}) \to \mathbb{H}^2\times T$ cannot be used immediately. This map is obviously proper and expanding, and if the image was bounded distance from an $\mathrm{SL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)$–orbit we could take a neighborhood of the orbit to see that $\mathrm{obdim}\big(\mathrm{SL}_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)\big) = 4$. The difficulty here is that the image is *not* bounded distance from any orbit. It seems in this case that one could perturb the product embedding to stay within a bounded distance of an orbit, but for the general case we chose to keep our simple definition of the product embedding and lose the nice structure of the product metric on $\mathbb{H}^2\times T$ by passing to the Borel–Serre compactification.

Remark These obstructor methods also apply for certain S–arithmetic groups over function fields. Let K be the function field of an irreducible projective smooth curve C defined over a finite field $k := \mathbb{F}_q$. Let S be a finite nonempty set of (closed) points of C, and let $\mathcal{O}_S < K$ be the ring of functions that have no poles except possibly at points in S. If S is a single point p, it determines a valuation $\nu_p\colon K \to \mathbb{Z}$ which assigns to a function its order of vanishing at p.

If G is a linear algebraic group over K, then any group commensurable with $G(\mathcal{O}_S)$ is an S–arithmetic subgroup. Similarly, as in the number field case, to each S–arithmetic group there is a corresponding Euclidean building X_S, and in this case $G(\mathcal{O}_S)$ acts properly on X_S, with no symmetric space factor. In the general setting, the action is not cocompact, and we cannot conclude anything from Theorem 5.2. However, we do have the following theorem:

Theorem 6.6 *For K and S as above, if $G(\mathcal{O}_S)$ acts cocompactly on X_S, then* $\operatorname{actdim}(G(\mathcal{O}_S)) \geq 2\dim X_S$ *(this is true exactly when the K–rank of G is 0).*

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