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Alternating links have representativity 2

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We prove that if L is a nontrivial alternating link embedded (without crossings) in a closed surface $F \subset S^3$, then F has a compressing disk whose boundary intersects L in no more than two points. Moreover, whenever the surface is incompressible and ∂–incompressible in the link exterior, it can be isotoped to have a standard tube at some crossing of any reduced alternating diagram.

57M25; 57M50

1 Introduction

Following Ozawa [10], the *representativity* $r(L)$ of a link $L \subset S^3$ is

$$r(L) = \max_{F \in \mathcal{F}_L} \min_{X \in \mathcal{X}_F} |\partial X \cap L|,$$

where $\mathcal{F}_L$ is the set of positive genus, closed surfaces $F \subset S^3$ containing L, and a "closed surface" is compact and connected without boundary; $\mathcal{X}_F$ is the set of compressing disks for F in S^3; and $|\partial X \cap L|$ is the number of connected components (ie points) of $\partial X \cap L$. That is, letting $r(F, L) := \min_{X \in \mathcal{X}_F} |\partial X \cap L|$ for each $F \in \mathcal{F}_L$, the representativity of L is $r(L) = \max_{F \in \mathcal{F}_L} r(F, L)$. This notion extends the earlier notion of representativity from graph theory. In 2011, Pardon [13] applied representativity (although he did not use this term) to answer a question posed by Gromov [4] in 1983 regarding knot distortion. The distortion of an embedded circle γ in $\mathbb{R}^3$ is defined to be

$$\delta(\gamma) = \sup_{p,q \in \gamma} \frac{d_\gamma(p,q)}{d_{\mathbb{R}^3}(p,q)},$$

where d_γ is arclength along γ and $d_{\mathbb{R}^3}$ is Euclidean distance in $\mathbb{R}^3$. Gromov asked whether there exists a uniform upper bound on distortion for all isotopy classes of knots, or at least for torus knots. Specifically, Gromov asked, does every isotopy class of knots have a representative with distortion less than, say, 100? To answer this question, Pardon showed that every knot isotopy class K satisfies

$$\delta(K) := \min_{\text{representatives } \gamma \text{ of } K} \delta(\gamma) \geq \tfrac{1}{160} r(K),$$

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where $r(K)$ denotes representativity. In particular, since the representativity of any (p,q)–torus knot is $r(T_{p,q}) = \min\{p,q\}$ (more to the point and easier to check is $r(T_{p,q}) \geq \min\{p,q\}$), so that $\delta(T_{p,q}) \to \infty$ as $p,q \to \infty$, Pardon was able to answer Gromov's question in the negative. Current work of Blair, Campisi, Taylor and Tomova provides a lower bound for $\delta(K)$ in terms of bridge distance and bridge number [2]; they construct an infinite family of knots for which their bound is arbitrarily stronger than Pardon's by applying our main result to certain highly twisted plat projections; see Johnson and Moriah [5].

By showing that bridge number is an upper bound for representativity, Ozawa [11] proves that 2–bridge knots L have representativity $r(L) = 2$. This result and his results on the representativity of algebraic knots and 3–stranded pretzel knots in [12] lead Ozawa to conjecture that $r(L) = 2$ holds for alternating knots in general (see [11, Conjecture 4]). Our main result confirms Ozawa's conjecture:

Main theorem *Every nontrivial, nonsplit alternating link L has representativity $r(L) = 2$.*

The main theorem implies that the only alternating torus *links* are the 2–braids $T_{2,q}$, again since $r(T_{p,q}) = \min\{p,q\}$. Murasugi first proved this fact in the case of knots, using the Alexander polynomial (see [9, Theorem 3.2]); Menasco and Thistlethwaite later provided a geometric proof (see [8, Corollary 1.2]).

Corollary 1.0.1 *The only alternating torus links are the 2–braids $T_{2,q}$.*

Thank you to Colin Adams for pointing out the following additional corollary:

Corollary 1.0.2 *If F is a closed surface in S^3 and L is a hyperbolic alternating link with $L \subset F$, then $F \setminus L$ is not isotopic to a totally geodesic surface in the hyperbolic structure on $S^3 \setminus L$.*

Proof Let F be a closed surface in S^3 containing a hyperbolic alternating link L, with X a compressing disk for F in S^3 realizing $|\partial X \cap L| = r(F, L)$. Suppose that, under the covering $\mathbb{H}^3 \to S^3 \setminus L$, the surface $F \setminus L$ lifts to a union $\widetilde{F}$ of geodesic planes in $\mathbb{H}^3$. Consider a component $\widetilde{X}_0$ of the lift of X, whose boundary consists of $|\partial X \cap L|$ ideal points and as many arcs on $\widetilde{F}$. Because X is a compressing disk for F, $\partial \widetilde{X}_0$ cannot lie entirely on one geodesic plane, and each of its arcs must have distinct (ideal) endpoints. Yet, from the main theorem, $\partial \widetilde{X}_0$ consists of at most two arcs, and two geodesic planes in $\mathbb{H}^3$ share at most one point of tangency on $\partial \mathbb{H}^3$. $\square$

To prove the main theorem, we employ the crossing ball structures introduced by William Menasco [6; 7]. Roughly, the idea is to insert a ball C_t at each crossing of a given diagram D on S^2, to perturb L to lie on $(S^2 \setminus C) \cup \partial C$, where $C = \bigsqcup C_t$, and then to isotope a given closed surface $F \supset L$ (fixing L and the crossing ball structure $S^2 \cup C$) so as to minimize its intersections with C and S^2 away from L. We show that whenever F is essential (incompressible and ∂–incompressible in the link exterior $S^3 \setminus \mathrm{int}(\nu L)$), there exists an isotopy of F which produces a standard tube near some crossing (see Figure 5). This crossing tube lemma not only provides a compressing disk for F whose boundary intersects L in two points; it also provides a possible inductive move, in the tradition of Adams and Kindred [1] and Gabai [3], albeit one still awaiting application.

Crossing tube lemma *Given a nontrivial, reduced alternating diagram of a link L and a closed, essential surface $F \supset L$, there exists an isotopy of F after which some crossing has a standard tube.*

Outline Section 2 describes the crossing ball setup in more detail; Section 3 develops convenient technical moves; Section 4 establishes several lemmas; Section 5 proves the main theorem. Everything in Sections 2, 3 and 4.1 works regardless of alternatingness and is intended to be a useful reference.

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2 Initial setup

2.1 Link diagrams and crossing balls

A link diagram $D \subset S^2$ is the image of an immersion of one or more circles in $S^2 \subset S^3$ in which all self-intersections are double points at which the two intersecting arcs are transverse in S^2 and are labeled with opposite normal directions relative to S^2. Thus, a link diagram D can also be seen as a smoothly embedded 4–valent graph in S^2 with over-under information at each vertex. An embedding of the underlying link L can be obtained by perturbing the two arcs of D near each crossing point in the indicated normal directions.

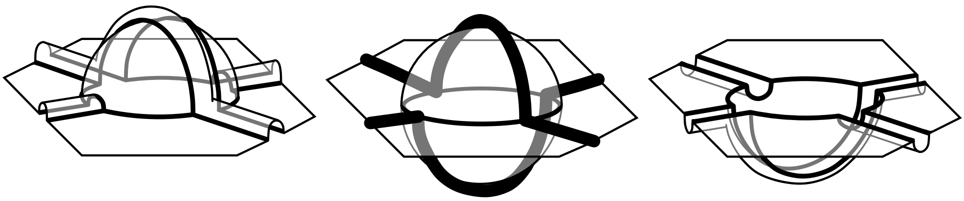

Figure 1: A link near a crossing ball (center) with S^+ (left) and S^- (right)

A link diagram D is called *alternating* if, for each edge of D (seen as a 4–valent graph), the crossing points at its two ends are labeled with opposite normal directions. A link is called alternating if it has an alternating diagram. A link diagram D is called *reduced* if it lacks *nugatory* crossings, which means that every crossing point is incident to four distinct components of $S^2 \setminus D$. A link L is called *split* if $S^3 \setminus L$ is reducible, ie if there is an embedded 2–sphere in $S^3 \setminus L$ which does not bound a ball in $S^3 \setminus L$.

Let D be a diagram of a link L (later, we will assume further that D is reduced and alternating) with crossing points c_t for $t = 1, \dots, n$. Insert small, mutually disjoint (closed) *crossing balls* C_t centered at the respective crossing points c_t. Denote $C = \bigsqcup_{t=1}^n C_t$. Construct an embedding of L in $(S^2 \setminus C) \cup \partial C$ by perturbing the two arcs in which D intersects each crossing ball C_t in their indicated normal directions from $S^2 \cap C_t$ to ∂C_t, while fixing $D \cap S^2 \setminus \mathrm{int}(C)$.

Call each resulting component of $L \cap S^2$ an *edge* of L — note that $L \cap S^2 = L \cap D = D \setminus \mathrm{int}(C)$ — and call each component of $L \cap \partial C$ an *overpass* or an *underpass* of L, according to which side of S^2 it lies on. Near each crossing, this looks like Figure 1, center.

2.2 The regular neighborhood νL

Let $\nu L \subset S^3$ be a closed regular neighborhood of the link L, viewed as (the total space of) a disk bundle $\pi\colon \nu L \to L$ for which the restrictions $\pi|_{\partial C}$, $\pi|_{S^2 \setminus \mathrm{int}(C)}$ are also bundle maps onto their images (each with fiber a closed interval). Thus, each component of νL intersects $(S^2 \setminus C) \cup \partial C$ in an annular neighborhood of its core.

Let B^+ and B^- be the two components into which $S^2 \cup C \cup \nu L$ cuts S^3; ie $B^\pm$ are the closures of the components of $S^3 \setminus (S^2 \cup C \cup \nu L)$. Let $S^+ = \partial B^+$ and $S^- = \partial B^-$. Near each crossing, S^+ and S^- appear as in Figure 1, left and right. As a quick exercise, check that $S^+ \cup S^- = (S^2 \setminus (C \cup \nu L)) \cup \partial(C \cup \nu L)$, that $S^+ \cap S^- = S^2 \setminus \mathrm{int}(C \cup \nu L)$, and that, for each point $p \in L \cap S^2 \cap \partial C$, the boundary

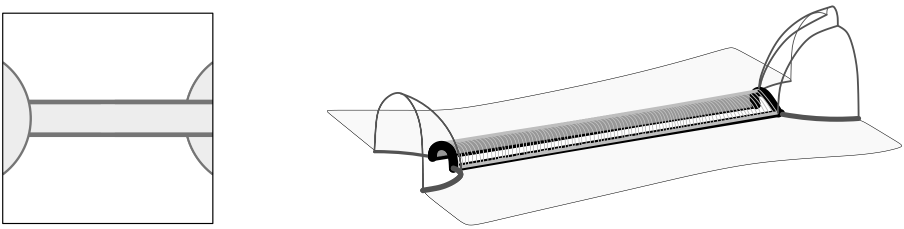

Figure 2: Two views of (the upper half of) an edge of $\partial\nu L$ from an alternating link diagram

of the disk $\pi^{-1}(p)$ is a meridian on $\partial\nu L$ which consists of an arc of $\partial\nu L\cap\partial C\cap S^{\pm}$ glued to an arc of $\partial\nu L\cap S^{\mp}\setminus C$ at the two points of $\pi^{-1}(p)\cap\partial\nu L\cap S^{+}\cap S^{-}$.

Use this setup to extend the terminology of edges, overpasses, and underpasses from L to $\partial\nu L$ as follows. Call each component of $\pi^{-1}(L\cap\partial C)\cap\partial\nu L\cap S^{+}$ an *overpass* of $\partial\nu L$, call each component of $\pi^{-1}(L\cap\partial C)\cap\partial\nu L\cap S^{-}$ an *underpass* of $\partial\nu L$ and call each component of $\pi^{-1}(L\cap S^2)\cap\partial\nu L$ an *edge* of $\partial\nu L$. Each edge of $\partial\nu L$ is a cylinder that intersects S^2 in two arcs of $\partial\nu L\cap S^{+}\cap S^{-}$.

This terminology gives $\partial\nu L$ the following cell decomposition (see Figures 1 and 2). The 0–cells are the points of $\partial\nu L\cap S^2\cap\partial C$, eight on the boundary of each crossing ball. There are several types of (closures of) 1–cells:

- arcs of $\partial\nu L\cap S^{+}\cap S^{-}$, two running along each edge *of* $\partial\nu L$;
- arcs of $\partial\nu L\cap\partial C\cap S^{+}$, two along each overpass of $\partial\nu L$;
- arcs of $\partial\nu L\cap\partial C\cap S^{-}$, two along each underpass of $\partial\nu L$; and
- arcs of $\pi^{-1}(L\cap S^2\cap\partial C)\cap S^{\pm}$, eight near each crossing ball C_t:
  - four in $\partial\nu L\cap\partial C_t$,
  - two in $S^{+}\cap\partial\nu L\setminus\partial C$, joining the overpass of $\partial\nu L$ at C_t with edges of $\partial\nu L$, and
  - two in $S^{-}\cap\partial\nu L\setminus\partial C$, joining the underpass of $\partial\nu L$ at C_t with edges of $\partial\nu L$.

Finally, the 2–cells' closures are overpasses of $\partial\nu L$, underpasses of $\partial\nu L$, components of $\partial\nu L\cap C$ and bands obtained by cutting each edge of $\partial\nu L$ along the two arcs in which it intersects S^2.

2.3 Essentiality and minimal complexity for the closed surface $F\supset L$

So far, the link $L\subset(S^2\setminus C)\cup\partial C$ follows a link diagram $D\subset S^2$; νL is a closed regular neighborhood of L in S^3, seen as a disk bundle $\pi\colon\nu L\to L$; and $B^{\pm}$ are the components into which $S^2 \cup C \cup \nu L$ cuts S^3, with $S^\pm = \partial B^\pm$. Now also let F be any closed surface in S^3 of positive genus that contains L. (Recall that a *closed surface* is assumed to be compact and connected without boundary.) Fixing $L \subset F$, S^2 and C, isotope F so that

- F is transverse to S^+ and S^-;
- the restriction $\pi|_F$ is a bundle map (so that $F \cap \nu L$ is a regular neighborhood of L in F and each component of $F \cap \partial \nu L$ is the image of a section of $\pi\colon \nu L \to L$); and
- $F \cap C \cap \partial \nu L = \varnothing$ (so that F intersects each overpass and underpass of $\partial \nu L$ in precisely two arcs).

Perform such isotopy so as to *minimize lexicographically* the numbers of components of $F \cap \partial C \setminus \nu L$ and of $F \cap S^+ \cap S^-$, ie to minimize the *complexity*

$$(|F \cap \partial C \setminus \nu L|, |F \cap S^+ \cap S^-|),$$

where bars count connected components. Note that the first and last conditions listed above ensure that $F \cap \partial C \setminus \nu L$ will consist only of simple closed curves, since the endpoints of any arc of $F \cap \partial C \setminus \nu L$ would lie on $\partial C \cap \partial \nu L$.

The surface F is said to be *compressible* in the link exterior $E = S^3 \setminus \operatorname{int}(\nu L)$ if there is a disk $X \subset E$ with $X \cap F = \partial X$ a simple closed curve (a "circle") that does not bound a disk in F. The surface F is compressible in E if and only if $r(F, L) = 0$. In particular, every split link L has representativity $r(L) = 0$. Indeed, given a closed surface F that contains a split link L, isotope F in the link exterior E so that F intersects the splitting sphere Q transversally in the smallest possible number of components. Because F is connected, it must intersect Q, and any innermost circle γ of $F \cap Q$ bounds a disk in Q which is a compressing disk for F, by the minimality of $F \cap Q$.

The surface F is said to be *∂–compressible* in the link exterior E if there is an arc α on $\partial \nu L$ which is parallel to F through E — say, through a disk X, with β denoting the parallel arc $\partial X \cap F$ — but not through $\partial \nu L$. In this case, α is also parallel through a bigon $Y \subset \nu L$ to an arc $\alpha' \subset F \cap \nu L$ which intersects L in one point. Gluing X and Y along α produces a compressing disk Z for F in S^3 whose boundary $\partial Z = \alpha' \cup \beta$ intersects L in one point. Thus, if F is ∂–compressible in E, then $r(F, L) \le 1$, and conversely.

In particular, the trivial knot L has representativity $r(L) = 1$, since any positive genus, closed surface containing L must be compressible or ∂–compressible. A closed surface

$F \supset L$ is called *essential* if it is neither compressible nor ∂–compressible in the link exterior.

Any nontrivial *knot* L is contained in some positive genus, closed surface F with $r(F, L) = 2$, namely the interpolating surface for any incompressible Seifert surface for L, or more generally for any algebraically essential spanning surface for L. (A spanning surface V for L is an embedded unoriented surface with boundary $\partial V = L$ in S^3; the interpolating surface for a spanning surface V is the boundary of a regular neighborhood of V in the link exterior; and V is called algebraically essential if its interpolating surface is incompressible and ∂–incompressible in the link exterior.) Is this also true of nonsplit *links* L? Is $r(L) \geq 2$ *if* and only if L is nonsplit and nontrivial? This is true for all nonsplit, nontrivial links with an algebraically essential, *connected* spanning surface, using the interpolating surface. Does every nonsplit link have such a span?

At least in the alternating case, the answer is yes (in fact, *all* spanning surfaces for nonsplit alternating links are connected [1]). Thus, an alternating link L in a reduced alternating diagram D obeys

$$r(L) = 0 \iff L \text{ is split} \iff D \text{ is disconnected;}$$
$$r(L) = 1 \iff L \text{ is the unknot} \iff D \text{ is the trivial diagram.}$$

The main theorem states that whenever L is alternating, nontrivial and nonsplit, $r(L) = 2$.

2.4 A preliminary consequence of the initial setup

As in Sections 2.1–2.3, assume that a link $L \subset (S^2 \setminus C) \cup \partial C$ follows a link diagram $D \subset S^2$; that νL is a regular neighborhood of L, seen as a disk bundle $\pi\colon \nu L \to L$; that $B^{\pm}$ are the components of S^3–cut–along–$(S^2 \cup C \cup \nu L)$, with $S^{\pm} = \partial B^{\pm}$; and that a closed essential surface $F \supset L$ has been isotoped — subject to the conditions that $F \pitchfork S^{\pm}$, $\pi|_F$ is a bundle map and $F \cap C \cap \partial \nu L = \varnothing$ — so as to minimize its complexity $(|F \cap \partial C \setminus \nu L|, |F \cap S^+ \cap S^-|)$. This setup implies:

Proposition 2.4.1 *All components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks.*

Proof The $n+2$ components of $S^3 \setminus (S^+ \cup S^- \cup \nu L)$, namely $\mathrm{int}(B^+)$, $\mathrm{int}(B^-)$ and $\mathrm{int}(C_t \setminus \nu L)$ for $t = 1, \dots, n$, are all topological 3–balls, and their boundaries are the 2–spheres S^+, S^- and $\partial(C_t \setminus \nu L)$. These spheres intersect F transversally, hence in circles. Each such circle bounds a disk in the corresponding ball.

The incompressibility of F implies that each of these circles also bounds a disk in F. Minimal complexity then implies that this disk in F must lie entirely in the appropriate ball, as claimed. Specifically, each circle of $F \cap S^\pm$ bounds a disk of $F \cap B^\pm$, and each circle of $F \cap \partial C_t \setminus \nu L$ bounds a disk of $F \cap C_t \setminus \nu L$. $\square$

3 Technical conveniences

Throughout Section 3, maintain all setup from Sections 2.1–2.3, but replace the assumption that the complexity of F is minimized with the assumption that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks. Proposition 2.4.1 implies that this setting is more general than the initial setup.

3.1 Preliminary consequences in the broader setting: arcs and balls

Proposition 3.1.1 *All components of $F \cap \partial \nu L \cap S^\pm$, $F \cap \partial C \cap S^\pm$ and $F \cap S^+ \cap S^-$ are arcs.*

Proof Every component of νL contains an overpass and an underpass, or else L would be a split link, and F (assumed to be connected) would be compressible, contrary to assumption. Therefore, each component of $F \cap \partial \nu L$, which is (the image of) a section of $\pi\colon \nu L \to L$ by assumption, must intersect $S^+ \cap S^-$. Hence, no component of $F \cap \partial \nu L \cap S^\pm$ is a circle; instead, each must be an arc.

All components of $\partial C \cap S^+$, $\partial C \cap S^-$ and $S^+ \cap S^-$ are disks. If F intersected one of these disks in a circle, γ, then, since all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks, γ would bound disks of F in both components of $F \setminus (S^+ \cup S^- \cup \nu L)$ whose boundaries contain γ. This contradicts the assumption that F is connected. $\square$

Proposition 3.1.1 implies that the number of components of $F \cap S^+ \cap S^-$ equals half the number of points of $F \cap \partial(S^+ \cap S^-)$, which will be more convenient to count. Also, the conclusions of Propositions 2.4.1 and 3.1.1 together imply that $F \setminus \mathrm{int}(\nu L)$ has a cell decomposition in which the 0–cells are the points of $F \cap \partial(S^+ \cap S^-)$; the (closures of the) 1–cells are the arcs of $F \cap S^+ \cap S^-$, $F \cap \partial C \cap S^\pm$ and $F \cap \partial \nu L \cap S^\pm$; and the 2–cells are the components of $F \cap B^+$, $F \cap B^-$ and $F \cap C \setminus \nu L$.

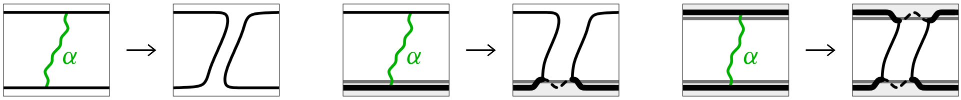

Figure 3: A bigon move pushes an arc $\beta \subset F \cap B^{\pm}$ past a parallel arc $\alpha \subset S^{\pm} \setminus \pi^{-1}(C)$, provided α is not parallel to F through $S^{\pm} \setminus C$ and $|\alpha \cap S^{+} \cap S^{-}| = 1$. There are three types of *bigon moves*, depending on how many endpoints of α are on $\partial \nu L$. The middle one, with one such endpoint, is also illustrated in Figure 4.

Proposition 3.1.2 *All components of $B^{\pm} \setminus F$ are 3–balls.*

Proof All components of $F \cap B^{\pm}$ are properly embedded disks, by assumption. Cutting a 3–ball along a collection of disjoint, properly embedded disks produces a collection of 3–balls. □

3.2 Bigon moves

Several proofs in Sections 4–5 will use an isotopy move which pushes an arc $\beta \subset F \cap B^{\pm}$ past a parallel arc $\alpha \subset S^{\pm} \setminus F$ and into $B^{\mp}$, through a disk Z with $Z \cap F = \beta \subset \partial Z = \alpha \cup \beta$. One type of this *bigon move* is illustrated in Figure 4; all three types are diagrammed in Figure 3. More precisely, a bigon move follows an arc $\alpha \subset S^{\pm}$ that

- intersects F precisely on its endpoints, which lie on the same circle γ of $F \cap S^{\pm}$;
- is not parallel in $S^{\pm} \setminus C$ to F;
- is disjoint from $\pi^{-1}(C)$, ie from crossing balls and over/underpasses; and
- intersects $S^{+} \cap S^{-}$ in exactly one component.

Think of α, which initially is not "part of the diagram", as a marker which joins two points that lie on the same circle γ of $F \cap S^{\pm}$ in a way that is "locally nonobvious", in the following sense. If R_0 is the component of $S^{+} \cap S^{-}$ that α intersects and

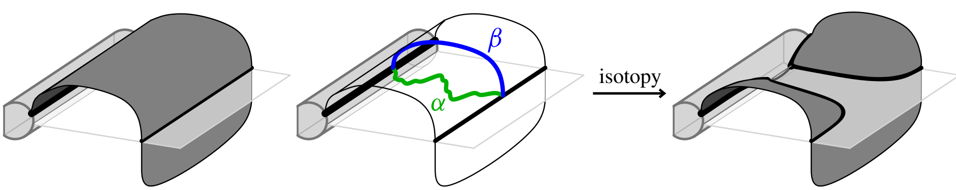

Figure 4: Bigon moves will often prove useful, even when they increase the complexity of F.

R is the union of R_0 with its incident edges of $\partial \nu L$, then $\alpha \subset R \cap S^{\pm} \subset S \pm \setminus C$; "locally nonobvious" refers to the fact that the endpoints of α lie on distinct arcs of $\gamma \cap R \cap S^{\pm}$, or else α would be parallel in $S^{\pm} \setminus C$ to F.

Because the circle γ bounds a disk $Y \subset F \cap B^{\pm}$, there is an arc $\beta \subset Y$ with the same endpoints as α. The circle $\alpha \cup \beta$ lies on the boundary of some component of $B^{\pm} \setminus F$, a ball, and therefore bounds a disk $Z \subset B^{\pm}$ whose interior is disjoint from F. The disk Z is a bigon in the sense that $\partial Z = \alpha \cup \beta$.

Bigon move *Given an arc $\alpha \subset S^{\pm}$ satisfying the four conditions above, it is possible to isotope F near a parallel arc $\beta \subset F \cap B^{\pm}$ through a bigon Z past α into $B^{\mp}$.*

All bigon moves take place away from the crossing balls, and thus preserve both the number of components of $F \cap \partial C \setminus \nu L$ and the fact that each of these components bounds a disk of $F \cap C \setminus \nu L$. Not all bigon moves, however, preserve the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks. At least:

Lemma 3.2.1 *Performing a bigon move preserves the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks whenever $\alpha \cap \partial \nu L \neq \varnothing$ and whenever the complexity of F is minimized.*

Proof As noted, any bigon moves preserves the fact that all components of $F \cap C \setminus \nu L$ are disks. In order for a bigon move to upset the fact that all components of $F \cap B^+$ and $F \cap B^-$ are disks, a necessary condition is that the endpoints of α lie on the same circle of $F \cap S^{\pm}$ and also on the same circle of $F \cap S^{\mp}$. For this to be the case, both endpoints of α must lie in $S^+ \cap S^-$, as must all of α, since $\alpha \cap S^+ \cap S^-$ is connected by assumption. Thus, the condition $\alpha \cap \partial \nu L \neq \varnothing$ in the statement of the lemma suffices, as claimed.

Suppose instead that, with the complexity of F minimized, a bigon move follows an arc $\alpha \subset S^+ \cap S^-$ whose endpoints lie both on the same circle $\gamma = \partial Y$ of $F \cap S^{\pm}$ *and* on the same circle $\gamma' = \partial Y'$ of $F \cap S^{\mp}$. Letting Y and Y' denote the disks of $F \cap B^+$ and $F \cap B^-$ respectively bounded by γ and γ', there are arcs $\beta \subset Y$ and $\beta' \subset Y'$ with the same endpoints as α. Further, $\alpha \cup \beta$ and $\alpha \cup \beta'$ respectively bound disks $Z \subset B^{\pm}$ and $Z' \subset B^{\mp}$ whose interiors are disjoint from F. Gluing Z and Z' along α produces a disk $Z \cup Z'$ with boundary $\beta \cup \beta' \subset F \setminus \nu L$ whose interior is disjoint from F; since F is incompressible in the link exterior, $\beta \cup \beta'$ must bound a disk $X \subset F \setminus \nu L$. Since L is nonsplit, the 2–sphere $Z \cup Z' \cup X \subset S^3 \setminus \nu L$ bounds a 3–ball $W \subset S^3 \setminus \nu L$, through which X is parallel to $Z \cup Z'$.

Since the complexity of F is minimized, the disk X, like $Z \cup Z'$, must be disjoint from C and must intersect $S^+ \cap S^-$ in a single arc, δ. From $X \cap C = \varnothing$, it follows that $\partial W \cap C = \varnothing$. This and the fact that $W \cap \nu L = \varnothing$ imply that W is disjoint from $C \cup \nu L$, and in particular from $\partial(S^+ \cap S^-)$. Therefore, contrary to assumption, W intersects $S^+ \cap S^-$ in a single disk through which α is parallel to the arc $\delta \subset F$:

$$\partial(W \cap S^+ \cap S^-) = (\partial W \cap (S^+ \cap S^-)) \cup (W \cap \partial(S^+ \cap S^-)) = \alpha \cup \delta. \qquad \square$$

Note that a bigon move along α fixes the number of components of $F \cap S^+ \cap S^-$ — which equals half the number of points of $F \cap \partial(S^+ \cap S^-)$ — if and only if $\alpha \cap \partial \nu L = \varnothing$; otherwise, a bigon move increases this number, and thus the complexity of F. In particular:

Lemma 3.2.2 *If a sequence of bigon moves begins with the complexity of F minimized and if all bigon moves in this sequence along arcs disjoint from $\partial \nu L$ precede all other bigon moves in this sequence, then this sequence of bigon moves preserves the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks.*

Proof Because bigon moves along arcs disjoint from $\partial \nu L$ fix the complexity of F, all such moves in this sequence are performed while the complexity of F is still minimized. Satisfying the second sufficient condition from Lemma 3.2.1, these bigon moves preserve the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks. All remaining bigon moves follow arcs that intersect $\partial \nu L$, meeting the first sufficient condition from Lemma 3.2.1. Therefore, these moves too preserve the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks. $\square$

3.3 A key lemma

Several proofs in Sections 4–5 will use the following lemma:

Lemma 3.3.1 *If $\alpha \subset \partial \nu L \cap S^\pm$ is an arc whose endpoints $\partial\alpha = \alpha \cap F$ lie on distinct circles of $F \cap \partial \nu L$, then these endpoints also lie on distinct circles of $F \cap S^\pm$.*

Proof Let $\alpha \subset \partial \nu L \cap S^\pm$ be an arc whose endpoints $\partial\alpha = \alpha \cap F$ lie on the same circle γ of $F \cap S^\pm$. We claim that these endpoints must also lie on the same circle of $F \cap \partial \nu L$. Since γ bounds a disk $Y \subset F \cap B^\pm$, there is an arc $\beta \subset Y$ with the same endpoints as α. The circle $\alpha \cup \beta$ lies on the boundary of some component of $B^\pm \setminus F$, a ball, and therefore bounds a disk $Z \subset B^\pm$ whose interior is disjoint from F. The

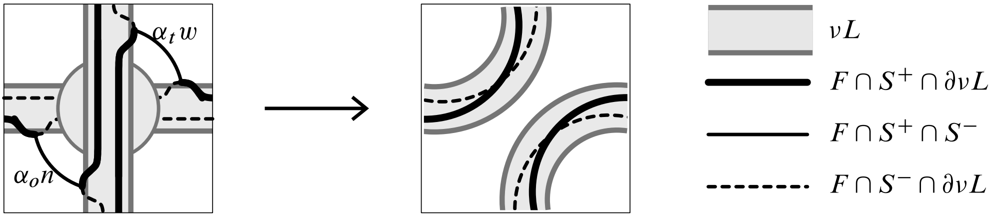

Figure 5: A *tube* near a crossing ball C_t features two arcs $\alpha_1, \alpha_2 \subset F \cap S^+ \cap S^-$ parallel through $S^+ \cap S^- \setminus F$ to C_t whose endpoints are (among) the points of $F \cap \partial\nu L \cap S^+ \cap S^-$ closest to C_t in each direction along $\partial\nu L$. In a minimal crossing link diagram, such a tube gives a compressing disk Z for F in S^3 with $|\partial Z \cap L| = 2$ (see Lemma 3.4.1), implying that $r(F, L) \leq 2$. Compressing F along Z changes νL and F as shown.

arc $\alpha \subset \partial\nu L$ is thus parallel in the link exterior through Z to F; ∂–incompressibility implies that α must also be parallel in $\partial\nu L$ to F, and in particular that the endpoints of α must lie on the same component of $F \cap \partial\nu L$, as claimed. □

The following special case is particularly noteworthy:

Lemma 3.3.2 *The two arcs of $F \cap \partial\nu L$ traversing each over/underpass lie on distinct circles of $F \cap S^\pm$.*

Proof This follows immediately from Lemma 3.3.1. □

3.4 Crossing tubes

Say that F has a standard *tube* near a crossing ball C_t if there are two arcs $\alpha_1, \alpha_2 \subset F \cap S^+ \cap S^-$ such that (1) for $r = 1, 2$, there is an isotopy of $(\alpha_r, \partial\alpha_r)$ through $(S^+ \cap S^- \setminus F, S^+ \cap S^- \cap \partial\nu L)$ to $(\partial C_t, S^+ \cap S^- \cap \partial\nu L \cap \partial C_t)$ — ie for $r = 1, 2$, α_r is parallel through $S^+ \cap S^-$ to C_t, allowing the endpoints to slide along $S^+ \cap S^- \cap \partial\nu L$ — and (2) the endpoints of α_1 and α_2 are also endpoints of the four arcs of $F \cap \partial\nu L \cap S^\pm$ that traverse the overpass and underpass at C_t — ie these endpoints are (among) the points of $F \cap \partial\nu L \cap S^+ \cap S^-$ closest to C_t in each of the four directions along $\partial\nu L$, in the sense of the disk bundle $\pi\colon \nu L \to L$.

Up to symmetry, this appears as in Figure 5 — the arcs α_1 and α_2 must be in opposite quadrants relative to C_t, not adjacent ones. The reason for this is that the only other possibility (see Figure 13, second from left) contradicts the essentiality of F in the link exterior, using Lemma 3.3.1, or more specifically Lemma 3.3.2.

A crossing tube sets up a surgery move on F and L as follows. Each of the two arcs α_1 and α_2 associated with a crossing tube near C_t has one endpoint on an edge of $\partial\nu L$ incident to the overpass at C_t and the other on an edge incident to the underpass at C_t. The two endpoints on edges incident to the overpass at C_t can be joined by an arc $\beta_1 \subset \partial\nu L \cap (S^- \cup C_t) \setminus F$; likewise, the two endpoints on edges incident to the underpass at C_t can be joined by an arc $\beta_2 \subset \partial\nu L \cap (S^+ \cup C_t) \setminus F$. The circle $\alpha_1 \cup \beta_1 \cup \alpha_2 \cup \beta_2$ bounds a disk X whose interior lies in $S^3 \setminus (F \cup \nu L)$. Both β_1 and β_2 are parallel through disks $Y_1, Y_2 \subset \nu L$ to arcs $\beta'_1, \beta'_2 \subset F$, each of which intersects L in a single point. Thus, the disk $Z := X \cup Y_1 \cup Y_2$ satisfies $Z \cap F = \partial Z = \alpha_1 \cup \beta'_1 \cup \alpha_2 \cup \beta'_2$ with $|\partial Z \cap L| = 2$. Moreover, the arcs β'_1 and β'_2 can be isotoped so that the two points in which they intersect L are the endpoints of the vertical crossing arc in C_t, one on the overpass and the other on the underpass.

The surgery move associated to the crossing tube near C_t (see Figure 5) consists of (1) cutting F along ∂Z, while cutting L at the two points of $\partial Z \cap L$, and (2) gluing in two parallel copies of the disk Z, while joining each pair of endpoints of L on the boundary of the glued-in copy of the disk Z with an arc in that disk. This surgery move is a compression of F in S^3 unless ∂Z bounds a disk in F, in which case the surgery move yields a surface with two components, one of them a sphere. The effect of the surgery move on L is the same as one of two possible "smoothings" near C_t, in the traditional sense from skein relations. After the surgery move, the resulting link is again embedded in the resulting surface.

Lemma 3.4.1 *Given a crossing tube in a minimal crossing diagram of a nonsplit link, the associated surgery move is a compression — the boundary of the surgery disk Z does not bound a disk in F.*

In particular, a crossing tube in a reduced alternating link diagram contains a "genuine" compressing disk Z for F in S^3 with $|\partial Z \cap L| = 2$. This lemma will round off the proof of the main theorem in Section 5.

Proof Let C_t be the crossing ball with the tube in question. Construct the disk Z as before, and suppose for contradiction that ∂Z bounds a disk Y in F. Since $|\partial Y \cap L| = |\partial Z \cap L| = 2$ and L is nonsplit, $Y \cap L$ consists of a single arc, call it δ. From $(Y \cup Z) \cap L = \delta$ it follows that the two 2–spheres on the boundary of a thin regular neighborhood of $Y \cup Z$ intersect L in a total of two points; hence, one of these

2–spheres is disjoint from L. Since L is nonsplit, this implies that the disks Y and Z are parallel through a ball W in S^3 *whose interior is disjoint from* L.

The arc $\delta = Y \cap L$ is parallel through the ball W to any arc δ' in Z that joins the overpass and underpass of L at C_t. Taking δ' to be the vertical arc in C_t that joins the overpass and underpass of L, isotope the arc $\delta \subset L$ to δ', while fixing the rest of L. This isotopy eliminates all crossings incident to δ, including the one at C_t, without creating any new ones. We assumed this was impossible. □

3.5 Height

In the broader setting of Section 3, construct graphs $G^{\pm}$ (they will be trees) whose vertices correspond to the components of $S^{\pm}$–cut–along–F and whose edges correspond to the components of $F \cap S^{\pm}$, such that the edge corresponding to each circle $\gamma \subset F \cap S^{\pm}$ joins the vertices corresponding to the two components of $S^{\pm}$–cut–along–F whose boundaries contain γ. Assign *heights* to the edges of $G^{\pm}$ and to the corresponding circles of $F \cap S^{\pm}$, as follows.

Leaves of $G^{\pm}$, corresponding to innermost circles of $F \cap S^{\pm}$, have height 0. Let $G_1^{\pm}$ be the graph obtained from $G^{\pm}$ by deleting all leaves; the edges of $G^{\pm}$ corresponding to the leaves of $G_1^{\pm}$, and their corresponding circles in $F \cap S^{\pm}$, have height 1. Recursively, define each $G_k^{\pm}$ to be the graph obtained from $G_{k-1}^{\pm}$ by deleting all leaves; the edges of $G^{\pm}$ corresponding to the leaves of $G_k^{\pm}$, and their corresponding circles in $F \cap S^{\pm}$, have height k. Thus, innermost circles of $F \cap S^{\pm}$ have height 0, noninnermost circles which enclose (to one side) innermost circles and no others have height 1, and so on.

Observation 3.5.1 *Either all circles of $F \cap S^{\pm}$ have height 0 or $F \cap S^{\pm}$ contains a circle with height 1.*

Figure 6 shows an example — a torus F containing the knot $L = T_{3,4}$ in an almost alternating diagram, with the associated graph G^+. (A reduced link diagram is called *almost alternating* if all but exactly four of its edges join an overpass to an underpass.) The torus is difficult to visualize directly, but one can verify its homeomorphism type by computing Euler characteristic.

The reader may also find the example in Figure 6 instructive going forward: as $r(L) = r(F, L) = 3 > 2$, the example illustrates the extent to which the arguments in the sequel hold, or fail, without the assumption of alternatingness. Namely, the configuration in Figure 6 satisfies the conclusions of all propositions and lemmas in Section 4.1, but escapes those from Section 4.2.

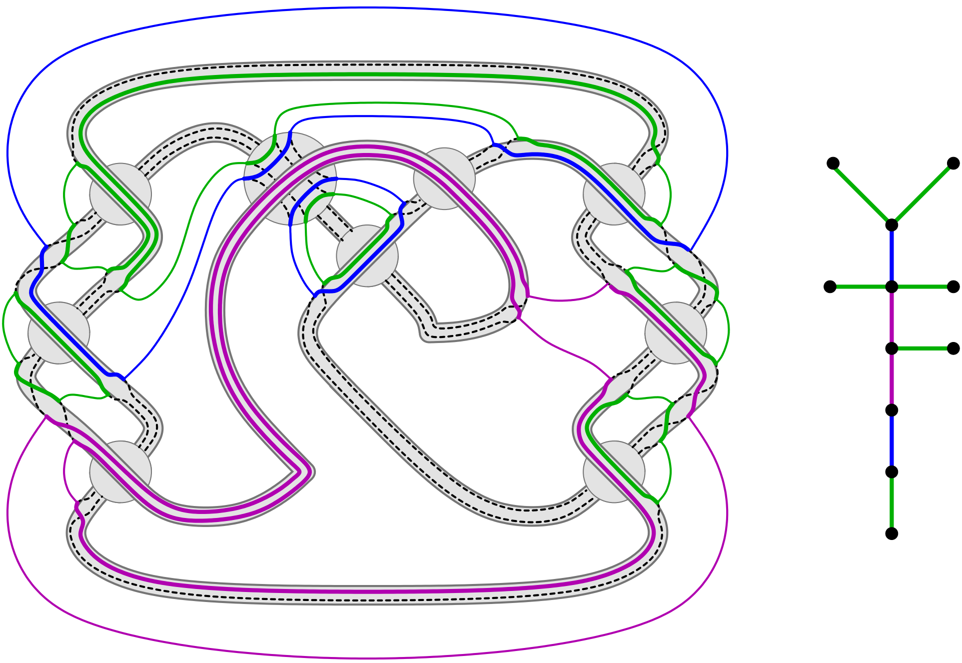

Figure 6: A torus F containing the knot $T_{3,4}$ in an almost alternating diagram. The edges of the graph G^+ correspond to the circles of $F \cap S^+$.

4 Consequences of essentiality and minimal complexity

To review the setup and preliminary results, a link $L \subset (S^2 \setminus C) \cup \partial C$ follows a link diagram $D \subset S^2$ with n crossings; νL is a regular neighborhood of L, seen as (the total space) of a disk bundle $\pi\colon \nu L \to L$; balls $B^\pm$ are the closures of the two components of $S^3 \setminus (S^2 \cup C \cup \nu L)$, with $S^\pm = \partial B^\pm$ and $S^+ \cap S^- = S^2 \setminus \mathrm{int}(C \cup \nu L)$; and a closed surface F, which contains L and is *essential* (ie incompressible and ∂–incompressible in the link exterior, ie $r(F, L) \geq 2$), has been isotoped so that

- the restriction $\pi|_F$ is a bundle map;
- F is transverse to S^+, S^-;
- $F \cap C \cap \partial \nu L = \varnothing$;

and, subject to these conditions, the *complexity* of F is *minimized*, specifically

- the numbers of components of $F \cap \partial C \setminus \nu L$ and of $F \cap S^+ \cap S^-$ have lexicographically been minimized.

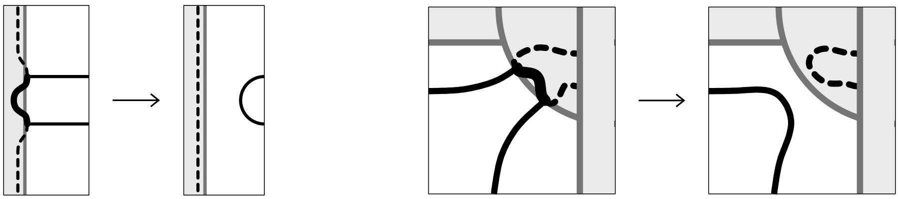

Figure 7: No arc of $F \cap \partial \nu L \cap S^{\pm}$ is parallel in $\partial \nu L$ to $S^+ \cap S^-$ (left) and no arc of $F \cap \partial C \cap S^{\pm}$ is parallel in ∂C to $S^+ \cap S^-$ (right).

This initial setup implies that all components of $F \cap B^+$, $F \cap B^-$ and $F \cap C$ are disks. In the more general setting where the assumption of minimal complexity is replaced with the assumption that $F \cap B^+$, $F \cap B^-$ and $F \cap C$ are comprised of disks, Section 3 established such technical conveniences as bigon moves, crossing tubes and height.

Section 4 delimits which local configurations are consistent with the initial setup, where F is essential and its complexity is minimized. Most results address the arcs of $F \cap \partial \nu L \cap S^{\pm}$, $F \cap \partial C \cap S^{\pm}$ and $F \cap S^+ \cap S^-$, and many extend to the case where F is incompressible but ∂–compressible in the link exterior. (One way to extend the proofs, roughly, is to isotope F so as to push any ∂–compressing disks into νL, and then to slide these disks along the link away from the local area under consideration.)

Several of the proofs require disrupting the condition of minimal complexity by creating new components of $F \cap S^+ \cap S^-$, usually through a sequence of bigon moves. There are several valid reasons to do this. In the proof of Lemma 4.1.6, a *temporary* increase in complexity enables the removal of a component of $F \cap \partial C \setminus \nu L$, lessening the complexity of F, a contradiction. One case in the proof of Lemma 4.2.4 increases complexity in order to reveal a ∂–compressing disk, thus contradicting essentiality. The other case in the proof of Lemma 4.2.4 and the proof of the crossing tube lemma in Section 5 both increase complexity in order to procure a crossing tube. In such cases, Lemma 3.2.1 will confirm that the bigon moves, while disrupting the condition of minimal complexity, preserve at least the fact that all components of $F \cap B^+$, $F \cap B^-$ and $F \cap C$ are disks, and thus the more general setting of Section 3. In particular, this will validate further bigon moves and applications of Lemma 3.3.1.

4.1 Local possibilities, regardless of alternatingness

Assume throughout Section 4.1 that D is a diagram of a nontrivial, nonsplit link L, and that L is contained in a closed surface $F \subset S^3$ (compact and connected without

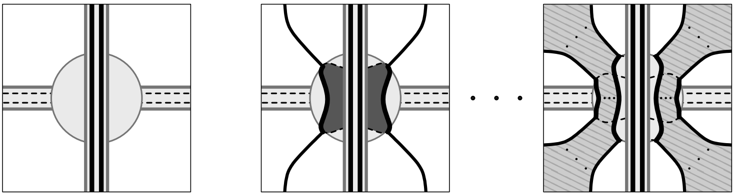

Figure 8: An arbitrary crossing ball: disjoint from F (left), intersecting F in a single component (center) and intersecting F in at least two components (right); see Lemma 4.1.3.

boundary). Establish all setup from Sections 2.1–2.3. Assume in particular that F is essential and its complexity $(|F \cap \partial C \setminus \nu L|, |F \cap S^+ \cap S^-|)$ has been minimized.

Proposition 4.1.1 *No arc of $F \cap \partial \nu L \cap S^{\pm}$ has both endpoints on the same component of $\partial \nu L \cap S^+ \cap S^-$.*

Proposition 4.1.2 *No arc of $F \cap \partial C \cap S^{\pm}$ has both endpoints on the same component of $\partial C \cap S^+ \cap S^-$.*

See Figure 7. Recall that $F \cap \partial \nu L \cap C = \varnothing$ by assumption.

Proof of Propositions 4.1.1 and 4.1.2 Any arc α_1 of $F \cap \partial \nu L \cap S^{\pm}$ with endpoints on the same component of $\partial \nu L \cap S^+ \cap S^-$ must be parallel through a disk $X \subset \partial \nu L \cap S^{\pm}$ to an arc $\beta_1 \subset \partial \nu L \cap S^+ \cap S^-$; but, then, isotoping F near α_1 through X past β_1 (Figure 7, left) would reduce the complexity of F, contrary to assumption.

Likewise, any arc α_2 of $F \cap \partial C \cap S^{\pm}$ with endpoints on the same component of $\partial C \cap S^+ \cap S^-$ must be parallel through a disk $Y \subset \partial C \cap S^{\pm}$ to an arc $\beta_2 \subset \partial C \cap S^+ \cap S^-$; but, then, isotoping F near α_2 through Y past β_2 (Figure 7, right) would reduce the complexity of F, contrary to assumption. □

Lemma 4.1.3 *Every component of $F \cap C \setminus \nu L$ is a disk whose boundary consists of four arcs, alternately on $S^+ \cap \partial C$ and $S^- \cap \partial C$, none of which is parallel in $\partial C \setminus \nu L$ to $\partial C \cap S^+ \cap S^-$.*

That is, each component of $F \cap C \setminus \nu L$ looks like a saddle, as in Figure 8, center, and each crossing ball looks like one of the pictures in Figure 8, depending on the number of components in which it intersects F.

Proof Lemma 4.1.3 is an immediate consequence of Propositions 2.4.1, 3.1.1 and 4.1.2. □

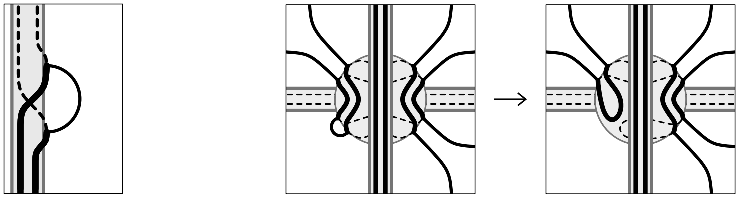

Figure 9: No arc of $F \cap S^+ \cap S^-$ is parallel in $S^+ \cap S^-$ to $\partial \nu L$ (left) or to ∂C (right).

Proposition 4.1.4 *No arc of $F \cap S^+ \cap S^-$ has both endpoints on the same component of $\partial \nu L \cap S^+ \cap S^-$.*

Proof If both endpoints of some arc of $F \cap S^+ \cap S^-$ were on the same component of $\partial \nu L \cap S^+ \cap S^-$ then, applying Proposition 4.1.1, an outermost such arc in $S^+ \cap S^-$ would appear as in Figure 9, left, contradicting the assumed ∂–incompressibility of F in the link exterior, eg by Lemma 3.3.1. □

Proposition 4.1.5 *No arc of $F \cap S^+ \cap S^-$ has both endpoints on the same crossing ball.*

Proof Suppose α_0 is an arc of $F \cap S^+ \cap S^-$ with both endpoints on the same crossing ball C_t, and assume that α_0 is outermost in $S^+ \cap S^-$, ie parallel through $S^+ \cap S^- \setminus F$ to C_t, as in Figure 9. Push F near α_0 through $S^+ \cap S^-$ past ∂C_t. This attaches two saddle-shaped disks in the interior of C_t, lessening the complexity of F, contrary to assumption. □

Lemma 4.1.6 *No arc $\alpha_0 \subset F \cap S^+ \cap S^-$ has one endpoint on a crossing ball and the other on an incident edge of $\partial \nu L$.*

Proof Suppose that α_0 is an arc of $F \cap S^+ \cap S^-$ with one endpoint on a crossing ball C_t and the other on an incident edge of $\partial \nu L$. Assume that α_0 is outermost in $S^+ \cap S^-$, ie parallel through $S^+ \cap S^- \setminus F$ to $\nu L \cup C$. Consider the circle of $F \cap \partial \nu L$ that contains an endpoint of α_0. Moving along this circle from that endpoint toward C_t, there is at most one more point on $S^+ \cap S^-$, by Proposition 4.1.1 and the assumption that α_0 is outermost. There are thus two cases up to symmetry (see Figure 10).

In either case, begin with a sequence of (up to) three bigon moves, through the arcs labeled α_1, α_2, α_3 in Figure 10, in that order. Any of the arcs α_r can be parallel in

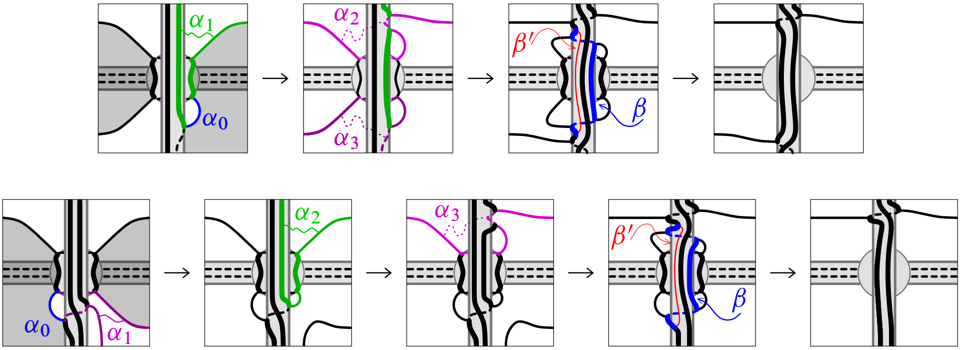

Figure 10: No arc α_0 of $F \cap S^+ \cap S^-$ has one endpoint on a crossing ball and the other on an incident edge of $\partial \nu L$.

$S^\pm \setminus C$ to F; in this case, omit the bigon move along α_r. In all cases, this sequence of bigon moves fits the hypotheses of Lemma 3.2.2 and thus preserves the fact that all components of $F \setminus (S^+ \cup S^- \cup \nu L)$ are disks.

Now an arc $\beta \subset \partial \nu L \cap F$ is parallel through a disk $Y \subset \partial \nu L$ with $Y \cap F = \beta$ to a second arc $\beta' \subset \partial \nu L \cap S^\pm$ with $\beta' \cap F = \partial \beta'$. This arc β' is parallel to an arc $\beta'' \subset F \cap B^\pm$ through a disk $Z \subset B^\pm$ with $Z \cap F = \beta''$. Further, β'' is parallel to β through a disk $X \subset F$, which contains an entire disk of $F \cap C \setminus \nu L$. Finally, the 2–sphere $X \cup Y \cup Z$ bounds a ball W in the link exterior.

Isotope (X, β) through (W, Y) to (Z, β'), while fixing $\beta'' = \partial X \cap \partial Z$. This removes the disk of $X \cap C$ and thus a component of $F \cap \partial C \setminus \nu L$. Since bigon moves always fix $|F \cap \partial C \setminus \nu L|$, this contradicts the initial assumption that the complexity of F was minimized. $\square$

4.2 Consequences of alternatingness

Maintain all setup from Section 4.1, with the additional assumption that D is alternating. That is, assume throughout Section 4.2 that D is a nontrivial, reduced alternating diagram of a link L, and that L is contained in a closed essential surface $F \subset S^3$ (compact and connected without boundary) whose complexity $(|F \cap \partial C \setminus \nu L|, |F \cap S^+ \cap S^-|)$ has been minimized.

Lemma 4.2.1 *Every edge of $\partial \nu L$ appears as in Figure 11.*

Proof This follows immediately from Propositions 3.1.1 and 4.1.1 and the alternatingness of D. $\square$

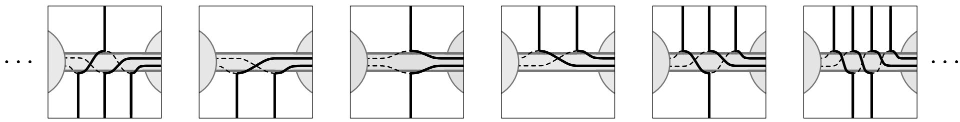

Figure 11: The types of edges of $\partial\nu L$ when D is alternating (see Lemma 4.2.1)

Proposition 4.2.2 *If γ is an innermost circle of $F\cap S^{\pm}$, then $\gamma\cap\partial C=\varnothing$.*

Proof Let γ_0 be an innermost circle of $F\cap S^+$ (without loss of generality), and suppose $\gamma_0\cap\partial C\neq\varnothing$. Then the disk $X\subset S^+\setminus F$ with $\partial X=\gamma_0$ intersects ∂C; let Y be a component of $X\cap\partial C$. There are now three cases, using Lemma 4.1.3: in one case (Figure 12, left), ∂Y contains two arcs of $\gamma_0\cap\partial C$, so that a bigon move yields an arc of $F\cap S^+\cap S^-$ with both endpoints on C_t, contradicting the minimality of $F\cap\partial C\setminus\nu L$ (recall Figure 9, right, and the proof of Proposition 4.1.5).

In the two remaining cases, ∂Y contains an arc of $\partial\nu L\cap\partial C_t$. This arc runs either along the boundary of the overpass of $\partial\nu L$ at C_t (right) or along the boundary of an edge of $\partial\nu L$ which is incident to the underpass at C_t (center). In either case, the arc is incident to an edge of $\partial\nu L$. Crucially, this edge of $\partial\nu L$ contains endpoints of $F\cap S^+\cap S^-$, by Lemma 4.2.1, which uses alternatingness. Moreover, in both cases, the closest of these endpoints to C_t along this edge of $\partial\nu L$ must lie on γ_0, since γ_0 has height 0. Therefore, in both these cases, a bigon move yields an arc of $F\cap S^+\cap S^-$ with one endpoint on C_t and the other on an incident edge of $\partial\nu L$. In the underpass case (center), this immediately contradicts the assumption of minimal complexity.

In the case that Y was incident to the overpass at C_t, the bigon move from Figure 12 increases complexity, but we can still perform the sequence of bigon moves from Figure 10, top (omitting any trivial ones as usual), to set up the final isotopy move from that sequence. This final move is valid here too, since the preceding sequence of bigon moves meets the conditions of Lemma 3.2.2. This final move eliminates a component of $F\cap C\setminus\nu L$. Since bigon moves always fix $|F\cap\partial C\setminus\nu L|$, this contradicts the initial assumption that the complexity of F was minimized. □

Proposition 4.2.3 *If γ is an innermost circle of $F\cap S^{\pm}$, so that γ bounds a disk $X\subset S^{\pm}\setminus F$, then at least one component of $\gamma\cap\partial\nu L$ traverses an over/underpass.*

Proof Lemma 4.2.1 implies that the only arcs of $F\cap\partial\nu L$ with endpoints on the same component of $\nu L\cap S^2\setminus C$ look like the arcs of this type in Figure 11, up to reflection — there is one far left in Figure 11, one second from right, and two far right.

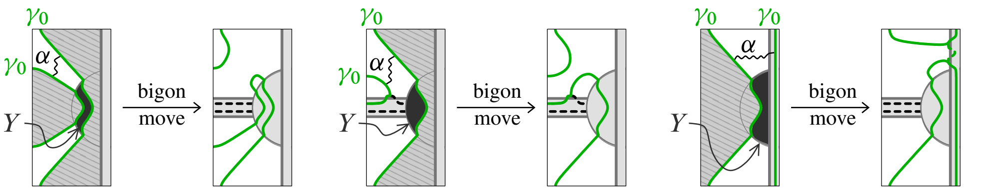

Figure 12: Every innermost circle $\gamma_0 \subset F \cap S^{\pm}$ is disjoint from ∂C (see Proposition 4.2.2).

If such an arc lies on an innermost circle γ of $F \cap S^+$ (without loss of generality), then, since γ is innermost and $\gamma \cap \partial C = \varnothing$ by Proposition 4.2.2, γ must traverse the overpass at the crossing where the edge of $\partial \nu L$ containing this arc of γ meets an underpass. This is evident in Figure 11, using Lemma 3.3.1. $\square$

Lemma 4.2.4 *If both circles of $F \cap S^{\pm}$ traversing a given over/underpass have height 0 (to one side), ie are innermost, then F can be isotoped to have a standard tube near that crossing.*

Proof Let C_t be the ball at the crossing in question. By Lemma 3.3.2, the two arcs traversing the overpass (without loss of generality) at C_t lie on distinct circles, γ and γ_0, of $F \cap S^+$. Consider the two edges of $\partial \nu L$ incident to the underpass at C_t, each of which contains endpoints of $F \cap S^+ \cap S^-$, using Lemma 4.2.1. Of these endpoints, consider one (on each of these two edges of $\partial \nu L$) that is nearest to C_t along $\partial \nu L$. Up to symmetry, there are two cases, depending on which sides of D these two points lie on, relative to each other (see Figure 13).

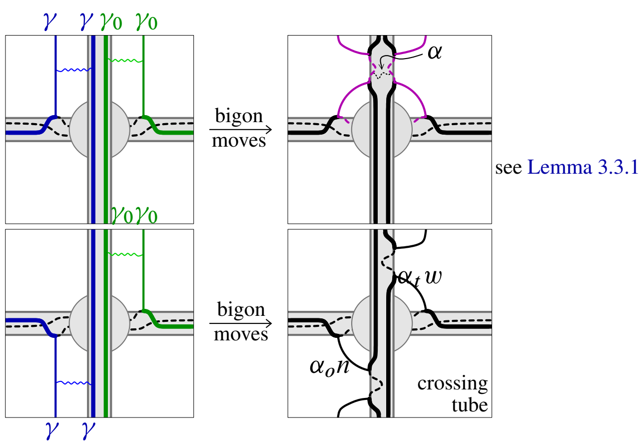

Figure 13: If both circles γ and γ_0 of $F \cap S^{\pm}$ traversing a given over/underpass, say at C_t, have height zero, then F can be isotoped to have a standard tube near C_t.

If these two points lie in adjacent quadrants near C_t (Figure 13, left), perform two bigon moves (unless the associated arc is parallel in $S^\pm \setminus C$ to F). Since each (possible) bigon move follows an arc with an endpoint on $\partial\nu L$, Lemma 3.2.1 implies that these moves preserve the fact that $S^+ \cup S^- \cup \nu L$ cuts F into disks. Moreover, these moves produce a diagram in which an arc $\alpha \subset \partial\nu L \cap S^-$ has endpoints on the same circle of $F \cap S^-$ but on distinct circles of $F \cap \partial\nu L$, which Lemma 3.3.1 states is impossible.

Therefore, these two points must lie in opposite quadrants $F \setminus D$ near C_t (Figure 13, right). In this case, a pair of bigon moves (to be omitted if trivial) immediately fashions a standard tube near C_t. Lemma 3.2.2 implies that these moves preserve the fact that $S^+ \cup S^- \cup \nu L$ cuts F into disks. □

5 Main results

Crossing tube lemma *Given a nontrivial, reduced alternating diagram of a link L and a closed, essential surface $F \supset L$, there exists an isotopy after which F has a standard tube near some crossing.*

Proof As in Section 2, let $L \subset (S^2 \setminus C) \cup \partial C$ follow a reduced alternating diagram, with νL a closed regular neighborhood of L seen as (the total space of) a disk bundle $\pi\colon \nu L \to L$, $B^\pm$ the components of $S^3 \setminus (S^2 \cup \mathrm{int}(C \cup \nu L))$ and $S^\pm = \partial B^\pm$. Let F be a closed, essential surface containing L. Fixing $L \subset F$, S^2 and C, isotope F — subject to the requirements that $F \pitchfork S^\pm$, $\pi|_F$ be a bundle map and $F \cap C \cap \partial\nu L = \varnothing$ — so as to minimize lexicographically the numbers of components of $F \cap \partial C \setminus \nu L$ and $F \cap S^+ \cap S^-$.

Consider $F \cap S^+$. If all circles have height 0, apply Lemma 4.2.4 at any overpass, and we are done. Otherwise, by Observation 3.5.1, there exists a circle γ_1 of $F \cap S^+$ with height 1. Let γ_0 be any (innermost) circle enclosed by γ_1. Apply Proposition 4.2.3 to consider an overpass which γ_0 traverses. Let γ denote the other circle of $F \cap S^+$ traversing this overpass. Note that $\gamma \neq \gamma_0$ by Lemma 3.3.1, or more specifically Lemma 3.3.2. If γ has height 0, then Lemma 4.2.4 completes the proof. Otherwise, γ must equal γ_1. See Figure 14.

Next, consider the circle γ' of $F \cap S^+$ from Figure 14, which must exist and be distinct from γ_0, due to Lemmas 4.2.1 and 3.3.1 and the assumption that γ_0 has height 0. If γ' also has height 0, then Lemma 3.3.1 implies that the edge of $\partial\nu L$ in question

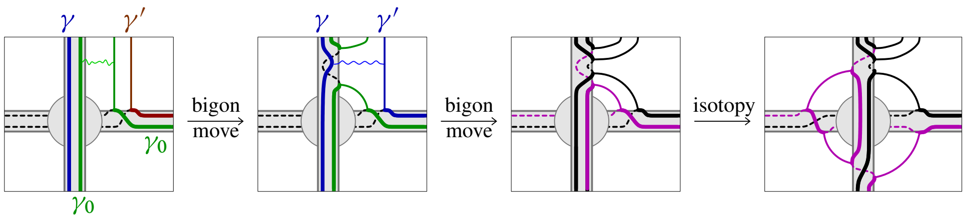

Figure 14: The final sequence of moves in the proof of the crossing tube lemma

must appear as in Figure 11, third from right; thus, γ_0 and γ' must traverse a common overpass, completing the proof, using Lemma 4.2.4. Otherwise, $\gamma' = \gamma_1 = \gamma$. This allows the sequence of isotopy moves shown in Figure 14, yielding the desired crossing tube. (Again, omit either isotopy move if the associated arc is parallel in $S^\pm \setminus C$ to F; Lemma 3.2.2 applies since each bigon move is along an arc with an endpoint on $\partial\nu L$.) □

Main theorem *Every nonsplit, nontrivial alternating link L has representativity $r(L) = 2$.*

Proof Given a closed surface F containing L and a reduced alternating diagram of L, apply the crossing tube lemma to obtain a standard tube at some crossing. Then apply Lemma 3.4.1 to conclude that the crossing tube contains a disk Z with $Z \cap F = \partial Z$, such that ∂Z intersects L in two points and does not bound a disk in F. □

References

[1] **C Adams**, **T Kindred**, *A classification of spanning surfaces for alternating links*, Algebr. Geom. Topol. 13 (2013) 2967–3007 MR

[2] **R Blair**, **M Campisi**, **S A Taylor**, **M Tomova**, *Distortion and the bridge distance of knots*, preprint (2017) arXiv

[3] **D Gabai**, *Genera of the alternating links*, Duke Math. J. 53 (1986) 677–681 MR

[4] **M Gromov**, *Filling Riemannian manifolds*, J. Differential Geom. 18 (1983) 1–147 MR

[5] **J Johnson**, **Y Moriah**, *Bridge distance and plat projections*, Algebr. Geom. Topol. 16 (2016) 3361–3384 MR

[6] **W Menasco**, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984) 37–44 MR

[7] **W Menasco**, *Determining incompressibility of surfaces in alternating knot and link complements*, Pacific J. Math. 117 (1985) 353–370 MR

[8] **W W Menasco**, **M B Thistlethwaite**, *Surfaces with boundary in alternating knot exteriors*, J. Reine Angew. Math. 426 (1992) 47–65 MR

[9] **K Murasugi**, *On the Alexander polynomial of the alternating knot*, Osaka Math. J. 10 (1958) 181–189 MR

[10] **M Ozawa**, *Non-triviality of generalized alternating knots*, J. Knot Theory Ramifications 15 (2006) 351–360 MR

[11] **M Ozawa**, *Bridge position and the representativity of spatial graphs*, Topology Appl. 159 (2012) 936–947 MR

[12] **M Ozawa**, *The representativity of pretzel knots*, J. Knot Theory Ramifications 21 (2012) art. id. 1250016 MR

[13] **J Pardon**, *On the distortion of knots on embedded surfaces*, Ann. of Math. 174 (2011) 637–646 MR

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