Algebraic & Geometric Topology 18 (2018) 3363–3402

The universal quantum invariant and colored ideal triangulations

SAKIE SUZUKI

The Drinfeld double of a finite-dimensional Hopf algebra is a quasitriangular Hopf algebra with the canonical element as the universal R–matrix, and one can obtain a ribbon Hopf algebra by adding the ribbon element. The universal quantum invariant of framed links is constructed using a ribbon Hopf algebra. In that construction, a copy of the universal R–matrix is attached to each crossing, and invariance under the Reidemeister III move is shown by the quantum Yang–Baxter equation of the universal R–matrix. On the other hand, the Heisenberg double of a finite-dimensional Hopf algebra has the canonical element (the S–*tensor*) satisfying the pentagon relation. In this paper we reconstruct the universal quantum invariant using the Heisenberg double, and extend it to an invariant of equivalence classes of *colored ideal triangulations* of 3–manifolds up to *colored moves*. In this construction, a copy of the S–tensor is attached to each tetrahedron, and invariance under the *colored Pachner* $(2,3)$ *moves* is shown by the pentagon relation of the S–tensor.

16T25, 57M27, 81R50

Published: 18 October 2018 DOI: 10.2140/agt.2018.18.3363

1 Introduction

The universal quantum invariant (see Lawrence [21; 22] and Ohtsuki [28]) associated to a ribbon Hopf algebra is an invariant of framed tangles in a cube which has the universal property over Reshetikhin–Turaev invariants [29]. The relationship between the universal quantum invariant and 3–dimensional, global, topological properties of tangles is not well understood, mainly because of the 2–dimensional definition using link diagrams. In this paper, we give a reconstruction of the universal quantum invariant using *colored ideal triangulations* of tangle complements and give an extension of the universal quantum invariant to an invariant of equivalence classes of colored ideal triangulations of 3–manifolds up to *colored moves*. We expect that our framework will become a new method to study the quantum invariants in a 3–dimensional way.

1.1 Reconstruction and extension of the universal quantum invariant

In the theory of quantum groups there are two doubles of a finite-dimensional Hopf algebra A. One is the *Drinfeld double* $D(A)$ and the other is the *Heisenberg double* $H(A)$. They are both isomorphic to $A^* \otimes A$ as vector spaces.

The Drinfeld double $D(A)$ is a quasitriangular Hopf algebra with a canonical element $R \in D(A)^{\otimes 2}$ as the universal R–matrix, which satisfies the quantum Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12};$$

see eg Drinfel'd [11] and Majid [24; 25]. One can obtain a ribbon Hopf algebra $D(A)^\theta$ by adding the ribbon element θ. In what follows we assume that the universal quantum invariant is associated to $D(A)^\theta$ for a finite-dimensional Hopf algebra A.

The Heisenberg double $H(A)$ is a generalization of the Heisenberg algebras; see Kapranov [15], Lu [23] and Semenov, Tian and Shansky [30]. Baaj and Skandalis [1] and Kashaev [18] showed that a canonical element $S \in H(A)^{\otimes 2}$, which we call the S*–tensor*, satisfies the pentagon relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}.$$

Kashaev [18] also constructed an algebra embedding $\phi\colon D(A) \to H(A) \otimes H(A)^{\mathrm{op}}$ such that the image of the universal R–matrix is a product of four variants of the S–tensor:

$$\phi^{\otimes 2}(R) = S''_{14}S_{13}\widetilde{S}_{24}S'_{23} \in (H(A) \otimes H(A)^{\mathrm{op}})^{\otimes 2}, \tag{1-1}$$

where S', S'' and $\widetilde{S}$ are the images of S by maps constructed from the antipode; see Theorem 3.4.

The situation (1-1) reminds us of the situation of an *octahedral triangulation* (see Cho, Kim and Kim [10], Weeks [33] and Yokota [34]) of the complement of a link in $S^3 \setminus \{\pm\infty\}$, where an octahedron consisting of four tetrahedra is associated to each crossing of a link diagram.[1] Actually, corresponding to the formula (1-1), Kashaev [17] constructed the R–matrix consisting of four *quantum dilogarithms* defined by Faddeev and Kashaev [12], and gave a link invariant. Baseilhac and Benedetti [5] also constructed the R–matrix consisting of four quantum dilogarithms, each of which is associated to tetrahedron in a singular triangulation of a 3–manifold, and they recovered Kashaev's R–matrix. Hikami and Inoue [13; 14] constructed the R–matrix consisting of four *mutations* in a cluster algebra. Here a mutation is associated to a flip of triangulated surface, where a flip is obtained by attaching a tetrahedron to the surface. They also recovered Kashaev's R–matrix up to a gauge transformation.

In this context, it is natural to ask if we can reconstruct the universal quantum invariant of a tangle using an octahedral triangulation of its complement, where a copy of the S–tensor is associated to each tetrahedron in the octahedral triangulation.

The answer is yes, and in this paper we give such a reconstruction. Here, we would like to stress that we can construct the universal quantum invariant using the S–tensor by simply rewriting the universal R–matrix by (variants of) the S–tensor using $\phi^{\otimes 2}$. However, an important result is that we give a way to relate a copy of the S–tensor to an ideal tetrahedron in an octahedral triangulation, and a way to read these copies of the S–tensor to obtain the universal quantum invariant. The framework of the above reconstruction enables us to extend the universal quantum invariant to an invariant for *colored singular triangulations* of 3–manifolds up to *colored moves*.[2]

1.2 Universal quantum invariant as a state-sum invariant with weights in a noncommutative ring

Let us explain the nature of the coloring on a singular triangulation from a viewpoint of state-sum constructions.

[1] Throughout this paper we consider only topological ideal triangulations, and we do not consider geometric structures on them.

[2] The universal quantum invariant of a tangle is an isotopy invariant, while the extended universal quantum invariant of the complement of the tangle is *not* a topological invariant. That is because there is a *canonical* coloring for the complement of a tangle, and the universal quantum invariant of a tangle is equal to the extended universal quantum invariant of its complement with the canonical coloring.

One can obtain a state-sum invariant of tangles and 3–manifolds by associating a $6j$*–symbol* to each tetrahedron in a triangulation of a 3–manifold, where the values of the $6j$–symbol on colors on the edges of a tetrahedron give a weight of the state sum; see Ocneanu [27] and Turaev and Viro [32].

In the context of hyperbolic geometry, there are several attempts to construct a state-sum invariant of hyperbolic links and hyperbolic 3–manifolds such that, to each tetrahedron, one associates Faddeev and Kashaev's quantum dilogarithm, and the values of them on the cross ratio moduli of hyperbolic ideal triangulation give weights of the state sum. It seems that the first relation between quantum state sums and hyperbolic geometry is by Kashaev [16]. For an odd integer $N > 1$, he proposed a state sum for triangulations of pairs (M, L) of a closed oriented 3–manifold M and a link L in M using the cyclic $6j$–symbol $R(p, q, r)$ of the Borel subalgebra of $U_q(\mathrm{sl}_2)$. He also showed that $R(p, q, r)$ is obtained from certain operators S and $\Psi_{p,q,r}$ on $\mathbb{C}^N \otimes \mathbb{C}^N$, where S satisfies a certain pentagon relation and $\Psi_{p,q,r}$ satisfies a version of the quantum dilogarithm identity. A semiclassical limit of this identity gives Rogers's identity for Euler's dilogarithm, and this fact seems to lead Kashaev [19] to his famous conjecture about the relationship between his invariant and the hyperbolic volumes of link complements.

Murakami and Murakami [26] showed that Kashaev's R–matrix is conjugate (up to scalar multiplication) to that of the *colored Jones polynomial* J_N with $q = \exp \frac{2\pi i}{N}$ and an N–dimensional irreducible representation of $U_q(\mathrm{sl}_2)$. This result also showed that, in the case of links in the three-sphere, the Kashaev state sums lead to well-defined invariants. Murakami and Murakami's construction could be seen as a state-sum invariant with a weight associated to a crossing, consisting of four quantum dilogarithms.

Baseilhac and Benedetti [2; 3; 4; 5] constructed *quantum hyperbolic invariants* (QHI) for triples (M, L, r), where M is a compact oriented 3–manifold, L is a nonempty link in M, and r is a flat principal bundle over M with structure group $\mathrm{PSL}(2, \mathbb{C})$. These invariants are obtained by adapting and generalizing the constructions of Kashaev, and in the case where M is the three-sphere and r is the trivial flat bundle, they recovered the Kashaev invariants. In [6], they reorganized QHI as invariants for tuples (M, L, r, κ), where κ is a family of cohomological classes called weights. In this version, the QHI are defined by state sums, where tensors called matrix dilogarithms (related to the cyclic $6j$–symbols $R(p, q, r)$) are associated to tetrahedra in a singular triangulation. The arguments of the matrix dilogarithms are certain special systems of N^{th} roots of hyperbolic shape parameters on the tetrahedra, encoding the flat bundle r and the weights κ.

On the other hand, the universal quantum invariant could be seen as a state-sum invariant with weights being tensors of a ribbon Hopf algebra; a weight is associated to each fundamental tangle (see Figure 2), in particular, a copy of the universal R–matrix is associated to each crossing, and one takes products of the weights in the order following the orientations of strands of a tangle (see Section 2.2 for the precise definition). We would like to apply this framework to a state-sum construction using triangulations; ie our motto (framework) is:

> *Using an element S satisfying a pentagon relation in a (noncommutative) algebra, construct a state-sum invariant of* 3*–manifolds by associating a copy of S to each tetrahedron of a (singular) triangulation.*

The state-sum invariants using $6j$–symbols (resp. quantum dilogarithms) could be treated in this framework as functions, rather than as their values in $\mathbb{C}$, on colors on edges of tetrahedra (resp. cross ratio moduli of ideal tetrahedra [6]), and we expect to obtain those invariants from the universal quantum invariant naturally keeping this framework.

In the above framework one does not need to fix colors on the edges of a tetrahedron or cross ratio modulus of an ideal tetrahedron, and for the proof of invariance of state sums, instead of the pentagon identity of $6j$–symbols or of quantum dilogarithms, one would work with an algebraic pentagon relation. Moreover, we expect that such an invariant involves combinatorial information of a triangulation in its noncommutative algebra structure, including the consistency and the completeness conditions of ideal triangulations when we fix cross ratio moduli.

When we use a (singular) triangulation, we do not have a canonical order on the set of weights on tetrahedra in the triangulation. Thus we need to fix an order; then we naturally come to a notion of the colored singular triangulation:[3] Each tetrahedron is attached to two strands, and strands are connected globally in the triangulation. Then a copy of the S–tensor is associated to the two strands of each tetrahedron, and

[3] The notion of colored singular triangulations can be interpreted by the notion of branchings; see Baseilhac and Benedetti [6; 7], Benedetti and Petronio [8; 9] and Remark 6.1. In this paper we keep the former one since it is defined combinatorially and fit to our purpose. When one would like to see geometric properties of state sums, then the latter one would make more sense.

we can read the copies of the S–tensor in the order following the orientations of the strands. Corresponding to the *Pachner* $(2,3)$ *move* and the $(0,2)$ *move* of singular triangulations, we define *colored Pachner* $(2,3)$ *moves* and *colored* $(0,2)$ *moves* of colored singular triangulations. The extension of the universal quantum invariant is an invariant of colored singular triangulations up to certain *colored moves*. In this paper these strands first arise from a tangle diagram, and then we consider strands more generally in singular triangulations of topological spaces.

Organization

Section 2 is devoted to the definition of the universal quantum invariant associated to a ribbon Hopf algebra. In Section 3 we recall the Drinfeld double $D(A)$ and the Heisenberg double $H(A)$ of a finite-dimensional Hopf algebra A, where the universal R–matrix in $D(A)^{\otimes 2}$ and the S–tensor in $H(A)^{\otimes 2}$ satisfy the quantum Yang–Baxter equation and the pentagon equation, respectively. We also recall from [18] how these elements are related via an embedding of $D(A)$ into $H(A) \otimes H(A)^{\mathrm{op}}$. In Section 4 we give a reconstruction of the universal quantum invariant using the Heisenberg double. In Section 5 we define colored diagrams and extend the universal quantum invariant to an invariant of colored diagrams up to colored moves. Sections 6 and 7 are devoted to 3–dimensional descriptions of the reconstruction and the extension of the universal quantum invariant. In Section 6 we define colored singular triangulations of topological spaces. The universal quantum invariant can be considered as an invariant of the colored singular triangulations. In Section 7 we define colored ideal triangulations of tangle complements[4] arising from octahedral triangulations, which have been studied in eg [10; 34] in the context of the hyperbolic geometry.

Acknowledgements

This work was partially supported by JSPS KAKENHI Grant Number 15K17539. The author is deeply grateful to Kazuo Habiro and Tomotada Ohtsuki for helpful advice and encouragement. She would like also to thank Stéphane Baseilhac, Riccardo Benedetti, Naoya Enomoto, Stavros Garoufalidis, Rei Inoue, Rinat Kashaev, Akishi Kato, Seonhwa Kim, Thang Le and Yuji Terashima for their helpful discussions and comments.

[4]For links, this construction corresponds to the branched triangulations defined in [5, Section 2.3] without walls.

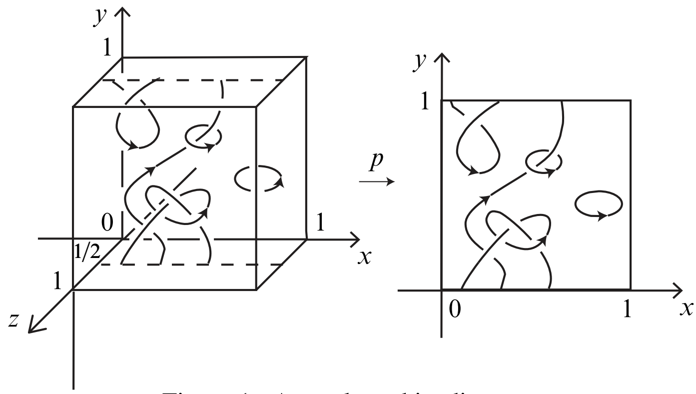

Figure 1: A tangle and its diagram

2 Universal quantum invariant

In this paper, a *tangle* means a proper embedding in a cube $[0,1]^3$ of a compact, oriented 1–manifold, whose boundary points are on the two parallel lines $[0,1]\times\{0,1\}\times\left\{\frac{1}{2}\right\}$. A *tangle diagram* is obtained from the projection $p\colon (x,y,z)\mapsto (x,y,0)$ to the (x,y)–plane; see Figure 1. A *framed* tangle is a tangle equipped with a trivialization of its normal tangent bundle, which is presented in a diagram by the blackboard framing.

2.1 Ribbon Hopf algebras

Let $(A,\eta_A,m_A,\varepsilon_A,\Delta_A,\gamma_A)$ be a finite-dimensional Hopf algebra over a field k, with k–linear maps

$$\eta_A\colon k\to A,\quad \varepsilon_A\colon A\to k,\quad m_A\colon A\otimes A\to A,\quad \Delta_A\colon A\to A\otimes A,\quad \gamma_A\colon A\to A,$$

which are called *unit*, *counit*, *multiplication*, *comultiplication*, and *antipode*, respectively. We will omit the subscript A of each map above when there is no confusion.

For distinct integers $1\le j_1,\dots,j_m\le l$ and $x=\sum x_1\otimes\cdots\otimes x_m\in A^{\otimes m}$, we write

(2-1) $$x^{(l)}_{j_1\dots j_m}=\sum (x_1)_{j_1}\cdots(x_m)_{j_m}\in A^{\otimes l},$$

where $(x_i)_{j_i}$ denotes the element in $A^{\otimes l}$ obtained by placing x_i on the j_i^{th} tensorand, ie

$$(x_i)_{j_i}=1\otimes\cdots\otimes x_i\otimes\cdots\otimes 1,$$

where x_i is at the $j_i{}^{\text{th}}$ position. For example, for $x=\sum x_1\otimes x_2\otimes x_3$, we have $x^{(3)}_{312}=\sum x_2\otimes x_3\otimes x_1$. Abusing the notation, we will omit the superscript of $x^{(l)}_{j_1\dots j_m}$ and write $x_{j_1\dots j_m}$.

For k–modules V, W, we define the symmetry map

$$\tau_{V,W}\colon V\otimes W\to W\otimes V,\quad a\otimes b\mapsto b\otimes a. \tag{2-2}$$

A quasitriangular Hopf algebra $(A,\eta,m,\varepsilon,\Delta,\gamma,R)$ is a Hopf algebra $(A,\eta,m,\varepsilon,\Delta,\gamma)$ with an invertible element $R\in A^{\otimes 2}$, called *the universal R–matrix*, such that

$$\Delta^{\mathrm{op}}(x)=R\Delta(x)R^{-1}\quad\text{for } x\in A,\quad (\Delta\otimes 1)(R)=R_{13}R_{23},\quad (1\otimes\Delta)(R)=R_{13}R_{12},$$

where $\Delta^{\mathrm{op}}=\tau_{A,A}\circ\Delta$.

A ribbon Hopf algebra $(A,\eta,m,\varepsilon,\Delta,\gamma,R,\boldsymbol{r})$, see eg [17], is a quasitriangular Hopf algebra $(A,\eta,m,\varepsilon,\Delta,\gamma,R)$ with a central, invertible element $\boldsymbol{r}\in A$, called the *ribbon element*, such that

$$\boldsymbol{r}^2=u\gamma(u),\quad \gamma(\boldsymbol{r})=\boldsymbol{r},\quad \varepsilon(\boldsymbol{r})=1,\quad \Delta(\boldsymbol{r})=(R_{21}R)^{-1}(\boldsymbol{r}\otimes\boldsymbol{r}),$$

where

$$u=\sum S(\beta)\alpha, \tag{2-3}$$

with $R=\sum\alpha\otimes\beta$.

2.2 Universal quantum invariant for framed tangles

In this section, we recall the universal quantum invariant [28; 21; 22] for framed tangles associated to a ribbon Hopf algebra $(A,\eta,m,\varepsilon,\Delta,\gamma,R,\boldsymbol{r})$.

Let $T=T_1\cup\cdots\cup T_n$ be an n–component, framed, ordered tangle.

Set

$$N=\mathrm{Span}_k\{ab-ba\mid a,b\in A\}\subset A.$$

For $i=1,\dots,n$, let

$$A_i=\begin{cases}A & \text{if } \partial T_i\neq\varnothing,\\ A/N & \text{if } \partial T_i=\varnothing.\end{cases}$$

We define the universal quantum invariant $J(T)\in A_1\otimes\cdots\otimes A_n$ in three steps as follows. We follow the notation in [31].

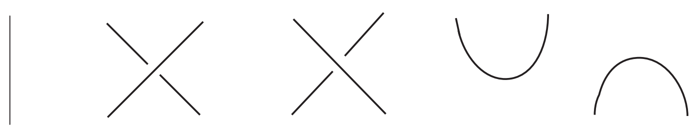

Figure 2: Fundamental tangles, where the orientation of each strand is arbitrary

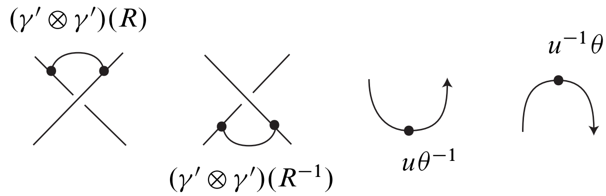

Figure 3: How to place labels on the fundamental tangles

Step 1 (choose a diagram) We choose a diagram D of T which is obtained by pasting, horizontally and vertically, copies of the fundamental tangles depicted in Figure 2.

Step 2 (attach labels) We attach labels on the copies of the fundamental tangles in the diagram, following the rule described in Figure 3, where each γ' should be replaced with γ if the string is oriented upwards, and with the identity otherwise. We do not attach any label to the other copies of fundamental tangles, ie to a straight strand and to a local maximum or minimum oriented from right to left.

Step 3 (read the labels) We define the i^{th} tensorand of $J(D)$ as the product of the labels on the i^{th} component of D, where the labels are read off along T_i reversing the orientation, and written from left to right. Here, if T_i is a closed component, then we choose arbitrary point p_i on T_i and read the label from p_i. The labels on the crossings are read as in Figure 4.

As is well known [28], $J(T) := J(D)$ does not depend on the choice of the diagram and the basepoints p_i, and thus defines an isotopy invariant of tangles.

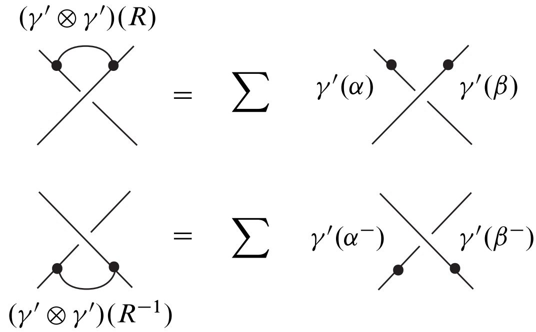

Figure 4: How to read the labels on crossings, where $R^{-1} = \sum \alpha^- \otimes \beta^-$

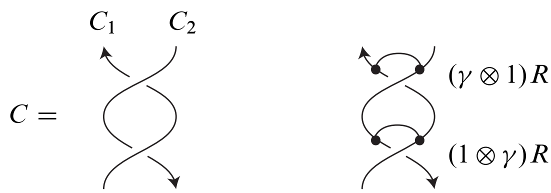

Figure 5: A tangle diagram C (left) and the label placed on C (right)

For example, for the tangle $C = C_1 \cup C_2$ shown in Figure 5, we have

$$J(C) = \sum \gamma(\alpha)\gamma(\beta') \otimes \alpha'\beta, \tag{2-4}$$

where $R = \sum \alpha \otimes \beta = \sum \alpha' \otimes \beta'$.

3 Drinfeld double and Heisenberg double

Let $(A, \eta, m, \varepsilon, \Delta, \gamma)$ be a finite-dimensional Hopf algebra. Let $A^* = \mathrm{Hom}_k(A, k)$. Define the pairing

$$\langle\ ,\ \rangle\colon A^* \otimes A \to k, \quad f \otimes x \mapsto f(x), \tag{3-1}$$

and extend it to

$$\langle\ ,\ \rangle\colon (A^*)^{\otimes n} \otimes A^{\otimes n} \to k$$

for $n \geq 1$ by

$$\langle f_1 \otimes \cdots \otimes f_n, x_1 \otimes \cdots \otimes x_n \rangle = \langle f_1, x_1 \rangle \cdots \langle f_n, x_n \rangle.$$

Then the dual Hopf algebra

$$A^* = (A^*, \eta_{A^*}{=}\varepsilon^*, m_{A^*}{=}\Delta^*, \varepsilon_{A^*}{=}\eta^*, \Delta_{A^*}{=}m^*, \gamma_{A^*}{=}\gamma^*)$$

is defined using the transposes of the morphisms of A, ie is defined uniquely by

$$\begin{aligned}
\langle \varepsilon^*(a), x \rangle &= a\varepsilon(x), && a \in k,\ x \in A,\\
\langle \Delta^*(f \otimes g), x \rangle &= \langle f \otimes g, \Delta(x) \rangle, && f, g \in A^*,\ x \in A,\\
\eta^*(f)a &= \langle f, \eta(a) \rangle, && f \in A^*,\ a \in k,\\
\langle m^*(f), x \otimes y \rangle &= \langle f, m(x \otimes y) \rangle, && f \in A^*,\ x, y \in A,\\
\langle \gamma^*(f), x \rangle &= \langle f, \gamma(x) \rangle, && f \in A^*,\ x \in A.
\end{aligned}$$

3.1 Drinfeld double and Yang–Baxter equation

For any finite-dimensional Hopf algebra with invertible antipode, the Drinfeld quantum double construction gives a quasitriangular Hopf algebra [11]. Here, we follow the notation in [20].

Let $(A, \eta, m, \varepsilon, \Delta, \gamma, \gamma^{-1})$ be a finite-dimensional Hopf algebra with invertible antipode, $A^{\mathrm{op}} = (A, \eta, m^{\mathrm{op}}, \varepsilon, \Delta, \gamma^{-1}, \gamma)$ the opposite Hopf algebra and $(A^{\mathrm{op}})^* = (A^*, \varepsilon^*, \Delta^*, \eta^*, (m^{\mathrm{op}})^*, (\gamma^{-1})^*, \gamma^*)$ the dual of the opposite Hopf algebra, where $m^{\mathrm{op}} = m \circ \tau_{A,A}$. For simplicity, we set

$$\overline{\gamma} = \gamma^{-1}.$$

Let $\Delta^{(0)} = \mathrm{id}$ and $\Delta^{(n)} = (\Delta \otimes 1^{\otimes n-1})\Delta^{(n-1)}$ for $n \geq 1$. In what follows, for $x \in A$ or $x \in A^*$, we use the notation

$$\Delta(x) = \Delta^{(1)}(x) = \sum x' \otimes x'' = \sum x^{(1)} \otimes x^{(2)},$$
$$(\Delta \otimes 1)\Delta(x) = \Delta^{(2)}(x) = \sum x' \otimes x'' \otimes x''' = \sum x^{(1)} \otimes x^{(2)} \otimes x^{(3)},$$
$$\Delta^{(n)}(x) = \sum x^{(1)} \otimes \cdots \otimes x^{(n+1)}$$

for $n \geq 3$. We have

$$(m^{\mathrm{op}})^*(f) = \Delta^{\mathrm{op}}(f) = \sum f'' \otimes f'$$

for $f \in (A^{\mathrm{op}})^*$.[5]

There is a unique left action

$$A \otimes (A^{\mathrm{op}})^* \to (A^{\mathrm{op}})^*, \quad a \otimes f \mapsto a \cdot f,$$

such that

$$\langle a \cdot f, x \rangle = \sum \langle f, \overline{\gamma}(a'') x a' \rangle,$$

for $a, x \in A$ and $f \in (A^{\mathrm{op}})^*$, which induces the left A–module coalgebra structure on $(A^{\mathrm{op}})^*$. Also, there is a unique right action

$$A \otimes (A^{\mathrm{op}})^* \to A, \quad a \otimes f \mapsto a^f,$$

such that

$$a^f = \sum f(\overline{\gamma}(a''') a') a''$$

for $a \in A$ and $f \in (A^{\mathrm{op}})^*$, which induces the right $(A^{\mathrm{op}})^*$–module coalgebra structure on A.

[5] In [20], Kassel uses the notation $\Delta^{\mathrm{op}}(f) = \sum f' \otimes f''$.

The Drinfeld double

$$D(A)=((A^{\mathrm{op}})^*\otimes A,\eta_{D(A)},m_{D(A)},\varepsilon_{D(A)},\Delta_{D(A)},\gamma_{D(A)},R)$$

is the quasitriangular Hopf algebra defined as the bicrossed product of A and $(A^{\mathrm{op}})^*$. Its unit, counit and comultiplication are given by those of $(A^{\mathrm{op}})^*\otimes A$; ie we have

$$\begin{aligned}\eta_{D(A)}(1) &= \quad \eta_{(A^{\mathrm{op}})^*\otimes A}(1) &&= 1\otimes 1,\\ \varepsilon_{D(A)}(f\otimes a) &= \varepsilon_{(A^{\mathrm{op}})^*\otimes A}(f\otimes a) &&= f(1)\varepsilon(a),\\ \Delta_{D(A)}(f\otimes a) &= \Delta_{(A^{\mathrm{op}})^*\otimes A}(f\otimes a) &&= \sum f''\otimes a'\otimes f'\otimes a''\end{aligned}$$

for $a\in A$ and $f\in (A^{\mathrm{op}})^*$. Its multiplication is given by

$$m_{D(A)}((f\otimes a)\otimes(g\otimes b))=\sum f(a'\cdot g'')\otimes a''^{g'}b=\sum fg(\overline{\gamma}(a''')?a')\otimes a''b \tag{3-2}$$

for $a,b\in A$ and $f,g\in (A^{\mathrm{op}})^*$, where the question mark ? denotes the place of the variable. Its antipode is given by

$$\gamma_{D(A)}(f\otimes a)=\sum \gamma(a'')\cdot\overline{\gamma}^*(f')\otimes \gamma(a')^{\overline{\gamma}^*(f'')}$$

for $a\in A$ and $f\in (A^{\mathrm{op}})^*$.

Fix a basis $\{e_a\}_{a\in\mathcal{I}}$ of A and its dual basis $\{e^a\}_{a\in\mathcal{I}}$ of A^*. The universal R–matrix is defined as the canonical element

$$R=\sum_a (1\otimes e_a)\otimes(e^a\otimes 1)\in D(A)\otimes D(A).$$

3.2 Heisenberg double and pentagon relation

Let A be a finite-dimensional Hopf algebra with an invertible antipode as in the previous section. The Heisenberg double

$$H(A)=(A^*\otimes A,\eta_{H(A)},m_{H(A)})$$

is the algebra with the unit $\eta_{H(A)}(1)=\eta_{A^*\otimes A}(1)=1\otimes 1$ and the multiplication

$$m_{H(A)}((f\otimes a)\otimes(g\otimes b))=\sum fg(?a')\otimes a''b \tag{3-3}$$

for $a,b\in A$ and $f,g\in (A^{\mathrm{op}})^*$.

Kashaev showed the following.

Theorem 3.1 [18] *The canonical element*

$$S=\sum_a (1\otimes e_a)\otimes(e^a\otimes 1)\in H(A)\otimes H(A)$$

satisfies the pentagon relation

$$S_{12}S_{13}S_{23}=S_{23}S_{12}\in H(A)^{\otimes 3}. \tag{3-4}$$

3.3 Drinfeld double and Heisenberg double

Let

$$H(A^*)=(A\otimes A^*,\eta_{H(A^*)},m_{H(A^*)})$$

be the Heisenberg double of the dual Hopf algebra A^* of A, where we identify $(A^*)^*$ and A in the standard way.

Set $A^{\mathrm{opcop}}=(A,\eta,m^{\mathrm{op}},\varepsilon,\Delta^{\mathrm{op}},\gamma,\gamma^{-1})$. We have the following lemma.

Lemma 3.2 *The algebras $H(A^*)$ and $H(A)^{\mathrm{op}}$ are isomorphic via the unique isomorphism $\Gamma\circ\tau$ such that*

$$\begin{aligned}\tau=\tau_{A^*,A}&\colon H(A^*)\to H(A^{\mathrm{opcop}})^{\mathrm{op}}, && x\otimes f\mapsto f\otimes x,\\ \Gamma=\bar{\gamma}^*\otimes\gamma&\colon H(A^{\mathrm{opcop}})^{\mathrm{op}}\to H(A)^{\mathrm{op}}, && f\otimes x\mapsto \bar{\gamma}^*(f)\otimes\gamma(x).\end{aligned}$$

Proof We have

$$\begin{aligned}\tau(x\otimes f)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}\tau(y\otimes g)&=(f\otimes x)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}(g\otimes y)\\ &=(g\otimes y)\cdot_{H(A^{\mathrm{opcop}})}(f\otimes x)\\ &=\sum g\cdot_{(A^*)^{\mathrm{op}}}\langle f,?\cdot_{A^{\mathrm{op}}}y''\rangle\otimes y'\cdot_{A^{\mathrm{op}}}x\\ &=\sum\langle f,y''?\rangle g\otimes xy'\\ &=\sum\langle f',y''\rangle f''g\otimes xy'\\ &=\sum f''g\otimes xy'\langle f',y''\rangle\\ &=\sum\tau(xy'\langle f',y''\rangle\otimes f''g)\\ &=\sum\tau(x\langle ?f',y\rangle\otimes f''g)\\ &=\tau((x\otimes f)\cdot_{H(A^*)}(y\otimes g)),\end{aligned}$$

and

$$
\begin{aligned}
\Gamma(f\otimes x)\cdot_{H(A)^{\mathrm{op}}}\Gamma(g\otimes y) &= (\overline{\gamma}^*(f)\otimes\gamma(x))\cdot_{H(A)^{\mathrm{op}}}(\overline{\gamma}^*(g)\otimes\gamma(y))\\
&= (\overline{\gamma}^*(g)\otimes\gamma(y))\cdot_{H(A)}(\overline{\gamma}^*(f)\otimes\gamma(x))\\
&= \sum\overline{\gamma}^*(g)\langle\overline{\gamma}^*(f),?\gamma(y)'\rangle\otimes\gamma(y)''\gamma(x)\\
&= \sum\overline{\gamma}^*(g)\overline{\gamma}^*(f)'\langle\overline{\gamma}^*(f)'',\gamma(y)'\rangle\otimes\gamma(y)''\gamma(x)\\
&= \sum\overline{\gamma}^*(g)\overline{\gamma}^*(f'')\langle\overline{\gamma}^*(f'),\gamma(y'')\rangle\otimes\gamma(y')\gamma(x)\\
&= \sum(\overline{\gamma}^*\otimes\gamma)(\langle f',y''\rangle f''g\otimes xy')\\
&= \sum(\overline{\gamma}^*\otimes\gamma)(\langle f,y''?\rangle g\otimes xy')\\
&= \sum\Gamma(g\cdot_{(A^*)^{\mathrm{op}}}\langle f,?\cdot_{A^{\mathrm{op}}}y''\rangle\otimes y'\cdot_{A^{\mathrm{op}}}x)\\
&= \Gamma((g\otimes y)\cdot_{H(A^{\mathrm{opcop}})}(f\otimes x))\\
&= \Gamma((f\otimes x)\cdot_{H(A^{\mathrm{opcop}})^{\mathrm{op}}}(g\otimes y)),
\end{aligned}
$$

which completes the proof. □

Set

$$\phi(1\otimes e_a)=\sum 1\otimes e'_a\otimes 1\otimes\gamma(e''_a)\in H(A)\otimes H(A)^{\mathrm{op}}, \tag{3-5}$$

$$\phi(e^a\otimes 1)=\sum(e^a)''\otimes 1\otimes\overline{\gamma}^*((e^a)')\otimes 1\in H(A)\otimes H(A)^{\mathrm{op}}. \tag{3-6}$$

Kashaev [18] stated without proof that the Drinfeld double $D(A)$ can be realized as a subalgebra in the tensor product $H(A)\otimes H(A)^{\mathrm{op}}$ of the Heisenberg double $H(A)$ and its opposite algebra $H(A)^{\mathrm{op}}$ as follows.[6]

Theorem 3.3 [18] *There is a unique algebra homomorphism*

$$\phi\colon D(A)\to H(A)\otimes H(A)^{\mathrm{op}} \tag{3-7}$$

extending (3-5) *and* (3-6).

Proof We define $\phi\colon D(A)\to H(A)\otimes H(A)^{\mathrm{op}}$ by

$$\phi=m_{H(A)\otimes H(A)^{\mathrm{op}}}\circ((1\otimes\eta)^{\otimes 2}\otimes(\eta\otimes 1)^{\otimes 2})\circ(1\otimes\overline{\gamma}^*\otimes 1\otimes\gamma)\circ(\Delta^{\mathrm{op}}\otimes\Delta);$$

ie, for $f\in A^*$ and $x\in A$, we have

$$
\begin{aligned}
\phi(f\otimes x) &= \sum\langle\overline{\gamma}^*(f')'',\gamma(x'')'\rangle f''\otimes x'\otimes\overline{\gamma}^*(f')'\otimes\gamma(x'')''\\
&= \sum\langle f',x'''\rangle f'''\otimes x'\otimes\overline{\gamma}^*(f'')\otimes\gamma(x'').
\end{aligned}
$$

[6]In [18] Kashaev uses $H(A^*)$ instead of $H(A)^{\mathrm{op}}$.

The map ϕ is an algebra homomorphism as shown by

$$\begin{aligned}
\phi(1\otimes x)\phi(1\otimes y) &= \Big(\sum 1\otimes x'\otimes 1\otimes \gamma(x'')\Big)\cdot_{H(A)\otimes H(A)^{\mathrm{op}}}\Big(\sum 1\otimes y'\otimes 1\otimes \gamma(y'')\Big)\\
&= \sum 1\otimes x'y'\otimes 1\otimes \gamma(x'')\cdot_{A^{\mathrm{op}}}\gamma(y'')\\
&= \sum 1\otimes (xy)'\otimes 1\otimes \gamma((xy)'')\\
&= \phi((1\otimes x)\cdot_{D(A)}(1\otimes y)),\\
\phi(f\otimes 1)\phi(g\otimes 1) &= \Big(\sum f''\otimes 1\otimes \overline{\gamma}^*(f')\otimes 1\Big)\cdot_{H(A)\otimes H(A)^{\mathrm{op}}}\Big(\sum g''\otimes 1\otimes \overline{\gamma}^*(g')\otimes 1\Big)\\
&= \sum f''g''\otimes 1\otimes \overline{\gamma}^*(f')\cdot_{(A^*)^{\mathrm{op}}}\overline{\gamma}^*(g')\otimes 1\\
&= \sum (fg)''\otimes 1\otimes \overline{\gamma}^*((fg)')\otimes 1\\
&= \phi((f\otimes 1)\cdot_{D(A)}(g\otimes 1)),\\
\phi(f\otimes 1)\phi(1\otimes x) &= \Big(\sum f''\otimes 1\otimes \overline{\gamma}^*(f')\otimes 1\Big)\cdot_{H(A)\otimes H(A)^{\mathrm{op}}}\Big(\sum 1\otimes x'\otimes 1\otimes \gamma(x'')\Big)\\
&= \sum f''\otimes x'\otimes \langle \overline{\gamma}^*(f')'', \gamma(x'')'\rangle \overline{\gamma}^*(f')'\otimes \gamma(x'')''\\
&= \sum \langle f', x'''\rangle f'''\otimes x'\otimes \overline{\gamma}^*(f'')\otimes \gamma(x'')\\
&= \phi(f\otimes x)\\
&= \phi((f\otimes 1)\cdot_{D(A)}(1\otimes x)),
\end{aligned}$$

and

$$\begin{aligned}
&\phi(1\otimes x)\phi(f\otimes 1)\\
&\quad= \Big(\sum 1\otimes x^{(1)}\otimes 1\otimes \gamma(x^{(2)})\Big)\cdot_{H(A)\otimes H(A)^{\mathrm{op}}}\Big(\sum f^{(2)}\otimes 1\otimes \overline{\gamma}^*(f^{(1)})\otimes 1\Big)\\
&\quad= \sum \langle f^{(3)}, x^{(1)}\rangle f^{(2)}\otimes x^{(2)}\otimes \overline{\gamma}^*(f^{(1)})\otimes \gamma(x^{(3)})\\
&\quad= \sum \langle f^{(4)}, x^{(1)}\rangle \varepsilon(f^{(1)})\varepsilon(x^{(4)}) f^{(3)}\otimes x^{(2)}\otimes \overline{\gamma}^*(f^{(2)})\otimes \gamma(x^{(3)})\\
&\quad= \sum \langle f^{(4)}, x^{(1)}\rangle \langle f^{(1)}, \overline{\gamma}(x^{(5)})x^{(4)}\rangle f^{(3)}\otimes x^{(2)}\otimes \overline{\gamma}^*(f^{(2)})\otimes \gamma(x^{(3)})\\
&\quad= \sum \langle f^{(5)}, x^{(1)}\rangle \langle f^{(1)}, \overline{\gamma}(x^{(5)})\rangle \langle f^{(2)}, x^{(4)}\rangle f^{(4)}\otimes x^{(2)}\otimes \overline{\gamma}^*(f^{(3)})\otimes \gamma(x^{(3)})\\
&\quad= \phi\big(\langle f^{(1)}, \gamma(x^{(3)})\rangle \langle f^{(3)}, x^{(1)}\rangle f^{(2)}\otimes x^{(2)}\big)\\
&\quad= \phi\big((1\otimes x)\cdot_{D(A)}(f\otimes 1)\big),
\end{aligned}$$

where the fourth identity follows from $m^{\mathrm{op}}(1\otimes\overline{\gamma})\Delta = \eta\varepsilon$.

The map ϕ satisfies (3-5) and (3-6) as follows:

$$\phi(1\otimes e_a)=\sum\langle 1,e_a'''\rangle 1\otimes e_a'\otimes 1\otimes\gamma(e_a'')=\sum 1\otimes e_a'\otimes 1\otimes\gamma(e_a'')=\phi(1\otimes e_a),$$
$$\begin{aligned}\phi(e^a\otimes 1)&=\sum\langle (e^a)',1\rangle (e^a)'''\otimes 1\otimes(\overline{\gamma}^*(e^a)'')\otimes 1\\&=\sum (e^a)''\otimes 1\otimes\overline{\gamma}^*((e^a)')\otimes 1=\phi(e^a\otimes 1).\end{aligned}\qquad\square$$

Set

$$\begin{aligned}\hat{R}=\phi^{\otimes 2}(R)&=\sum_a \phi(1\otimes e_a)\otimes\phi(e^a\otimes 1)\\&=\sum 1\otimes e_a'\otimes 1\otimes\gamma(e_a'')\otimes(e^a)''\otimes 1\otimes\overline{\gamma}^*((e^a)')\otimes 1\in(H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 2}.\end{aligned}$$

Since $\phi^{\otimes 2}$ is an algebra homomorphism, the element $\hat{R}$ also satisfies the quantum Yang–Baxter equation

$$\hat{R}_{12}\hat{R}_{13}\hat{R}_{23}=\hat{R}_{23}\hat{R}_{13}\hat{R}_{12}, \tag{3-8}$$

where we use the notation (2-1) treating $H(A)\otimes H(A)^{\mathrm{op}}$ as one algebra. If we treat $H(A)\otimes H(A)^{\mathrm{op}}$ as the tensor of $H(A)$ and $H(A)^{\mathrm{op}}$, we have

$$\hat{R}_{1234}\hat{R}_{1256}\hat{R}_{3456}=\hat{R}_{3456}\hat{R}_{1256}\hat{R}_{1234}.$$

Set

$$\tilde{e}_a:=\gamma(e_a),\quad \tilde{e}^b:=\overline{\gamma}^*(e^b),$$

and set

$$\begin{aligned}S'&=\sum(1\otimes\tilde{e}_a)\otimes(e^a\otimes 1)\in H(A)^{\mathrm{op}}\otimes H(A),\\S''&=\sum(1\otimes e_a)\otimes(\tilde{e}^a\otimes 1)\in H(A)\otimes H(A)^{\mathrm{op}},\\\tilde{S}&=\sum(1\otimes\tilde{e}_a)\otimes(\tilde{e}^a\otimes 1)\in H(A)^{\mathrm{op}}\otimes H(A)^{\mathrm{op}}.\end{aligned}$$

Kashaev showed the following.

Theorem 3.4 [18] *We have*

$$\hat{R}=S''_{14}S_{13}\tilde{S}_{24}S'_{23}\in(H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 2}. \tag{3-9}$$

Proposition 3.5 [18] *The quantum Yang–Baxter equation* (3-8) *in* $(H(A)\otimes H(A)^{\mathrm{op}})^{\otimes 3}$ *is a consequence of the following variations of the pentagon equation for the tensors*

S, S', S'' and $\widetilde{S}$:

$$S_{23}S_{12}=S_{12}S_{13}S_{23},\quad S_{23}S'_{12}=S'_{12}S'_{13}S_{23}, \tag{3-10}$$

$$S''_{23}S_{12}=S_{12}S''_{13}S''_{23},\quad S''_{23}S'_{12}=S'_{12}S''_{13}S''_{23}, \tag{3-11}$$

and

$$S'_{23}S_{13}S''_{12}=S''_{12}S'_{23},\quad S'_{23}S'_{13}\widetilde{S}_{12}=\widetilde{S}_{12}S'_{23}, \tag{3-12}$$

$$\widetilde{S}_{23}S''_{13}S''_{12}=S''_{12}S''_{23},\quad \widetilde{S}_{23}\widetilde{S}_{13}\widetilde{S}_{12}=\widetilde{S}_{12}\widetilde{S}_{23}. \tag{3-13}$$

4 Reconstruction of the universal quantum invariant

Let $D(A)$ be the Drinfeld double of A. Recall from (2-3) the element $u=\sum\gamma(\beta)\alpha=\sum\overline{\gamma}^*(e^a)\otimes e_a$ with $R=\sum\alpha\otimes\beta=\sum(1\otimes e_a)\otimes(e^a\otimes 1)$. We have a ribbon Hopf algebra

$$D(A)^\theta=D(A)[\theta]/(\theta^2-u\gamma(u))$$

with the ribbon element θ (see eg [20]).

We also consider the algebra

$$(H(A)\otimes H(A)^{\mathrm{op}})^{\overline{\theta}}=(H(A)\otimes H(A)^{\mathrm{op}})[\overline{\theta}]/\big(\overline{\theta}^2-\phi(u\gamma(u))\big)$$

and extend ϕ to the map

$$\phi\colon D(A)^\theta\to(H(A)\otimes H(A)^{\mathrm{op}})^{\overline{\theta}}$$

by $\phi(\theta)=\overline{\theta}$.

In this section, we define tangle invariant J'' using $(H(A)\otimes H(A)^{\mathrm{op}})^{\overline{\theta}}$, which turns out to be the image of tensor power of ϕ of the universal invariant associated to $D(A)^\theta$ (Theorem 4.1).

In what follows, for simplicity, we use the notation

$$fx=f\otimes x\in A^*\otimes A$$

for $f\in A^*$ and $x\in A$. In particular we have

$$S=\sum_a e^a\otimes e_a,\quad S'=\sum_a\widetilde{e}^a\otimes e_a,\quad S''=\sum_a e^a\otimes\widetilde{e}_a,\quad \widetilde{S}=\sum_a\widetilde{e}^a\otimes\widetilde{e}_a.$$

4.1 Reconstruction of the universal quantum invariant using the Heisenberg double

Let $T = T_1 \cup \cdots \cup T_n$ be an n–component, framed, ordered tangle. Similarly to Section 2.2, set

$$N_{(H\otimes H^{\mathrm{op}})^{\bar\theta}} = \mathrm{Span}_k\{ab-ba \mid a,b \in (H(A)\otimes H(A)^{\mathrm{op}})^{\bar\theta}\} \subset (H(A)\otimes H(A)^{\mathrm{op}})^{\bar\theta}.$$

For $i=1,\dots,n$, let

$$(H(A)\otimes H(A)^{\mathrm{op}})_i^{\bar\theta} = \begin{cases} (H(A)\otimes H(A)^{\mathrm{op}})^{\bar\theta} & \text{if } \partial T_i \neq \varnothing, \\ (H(A)\otimes H(A)^{\mathrm{op}})^{\bar\theta}/N_{(H(A)\otimes H(A)^{\mathrm{op}})^{\bar\theta}} & \text{if } \partial T_i = \varnothing. \end{cases} \tag{4-1}$$

Take a diagram D of T. We define an element $J'(D) \in \bigotimes_i (H(A)\otimes H(A)^{\mathrm{op}})_i^{\bar\theta}$ modifying the definition of $J(T)$ as follows.

We duplicate D and thicken the left strands following the orientation, and denote the result by $\zeta(D)$. See Figures 6, left, and 7, right, for examples.

Then we put labels on crossings as in Figure 8, where each γ' and each $(\bar\gamma^*)'$ should be replaced with γ and $\bar\gamma^*$, respectively, if the string is oriented upwards, and with the identities otherwise.

We define the $(2i-1)^{\mathrm{st}}$ and the $2i^{\mathrm{th}}$ tensorands of $J'(D)$ as the product of the labels on the thin and the thick strands, respectively, obtained by duplicating T_i, where the labels are read off reversing the orientation, and written from left to right. Here, if T_i is a closed component, then we choose a point p on T_i and denote by p' (resp. p'') the image of p by the duplicating procedure on the thin (resp. thick) strand. We read the labels of the thin (resp. thick) strand from p' (resp. p'').

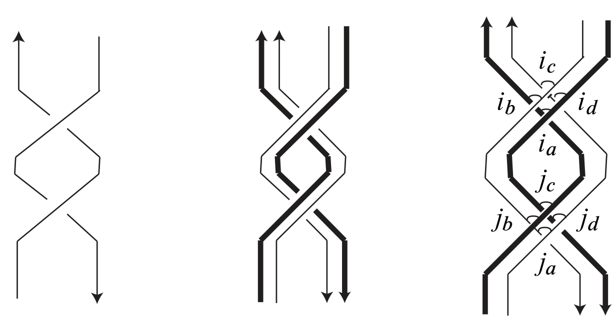

Figure 6: A tangle C (left), the diagram $\zeta(C)$ (middle) and parameters for $\zeta(C)$ (right)

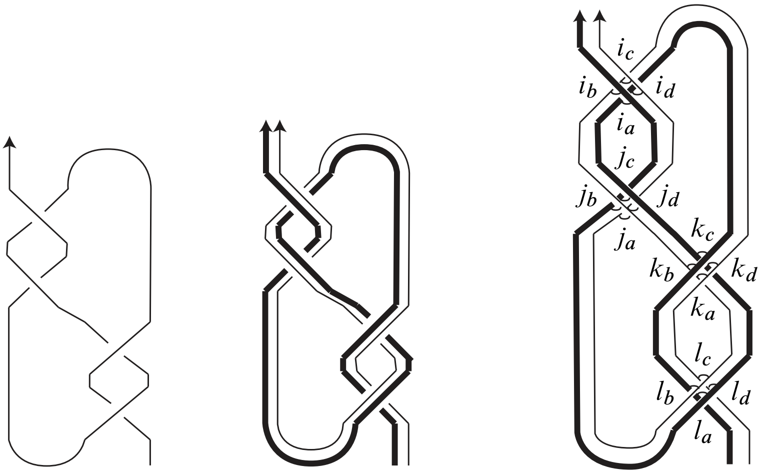

Figure 7: A tangle T_{41} (left), the diagram $\zeta(T_{41})$ (middle), and parameters for $\zeta(T_{41})$ (right)

Let

(4-2) $$(\leftarrow)\colon \{\text{tangle diagrams}\} \to \{\text{tangle diagrams}\}, \quad D \mapsto D_{(\leftarrow)},$$

where $D_{(\leftarrow)}$ is the diagram obtained from D by respectively replacing each of  and  with

 and .

For $i = 1, \dots, n$, let D_i be the subdiagram of D corresponding to T_i. We define $d(D_i)$ as the number of  minus the number of  in D_i.

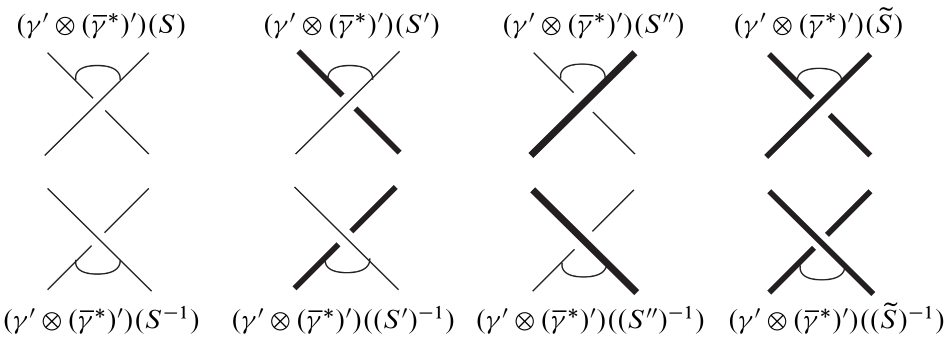

Figure 8: Labels on crossings

Set

$$J''(D) = \Big(\prod_i \overline{\theta}_i^{d(D_i)}\Big) J'(D_{(\leftarrow)}) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\overline{\theta}},$$

where $\overline{\theta}_i$ is defined following the notation (2-1).

Theorem 4.1 *We have*

$$\phi^{\otimes n} \circ J(T) = J''(D) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\overline{\theta}}.$$

If moreover T is a braid, which is a 0–framed tangle with no maxima or minima, then we have $D = D_{(\leftarrow)}$ and $\prod_i \overline{\theta}_i^{d(D_i)} = 1$. Thus we have the following.

Corollary 4.2 *If T is a braid, then we have*

$$\phi^{\otimes n} \circ J(T) = J'(D) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i,$$

where $(H(A) \otimes H(A)^{\mathrm{op}})_i$ is defined similarly to (4-1) *using $H(A) \otimes H(A)^{\mathrm{op}}$ instead of $(H(A) \otimes H(A)^{\mathrm{op}})^{\overline{\theta}}$.*

Let $f(D_i) = \#\{\text{positive self-crossings of } D_i\} - \#\{\text{negative self-crossings of } D_i\}$ be the framing of D_i. Set

$$J^0(D) = \Big(\prod_i \overline{\theta}_i^{f(D_i)}\Big) J'(D) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\overline{\theta}}.$$

We use the following lemma to prove Theorem 4.1.

Lemma 4.3 *Let T be an n–component framed tangle, and let T^0 denote T with 0–framing. Let D be a diagram of T. We have*

$$\phi^{\otimes n} \circ J(T^0) = J^0(D_{(\leftarrow)}) \in \bigotimes_i (H(A) \otimes H(A)^{\mathrm{op}})_i^{\overline{\theta}}.$$

Proof For a positive (resp. negative) crossing $c = c_1 \cup c_2$, where c_1 is the understrand, let c^0 be a tangle obtained by inserting a negative (resp. positive) kink into the bottom

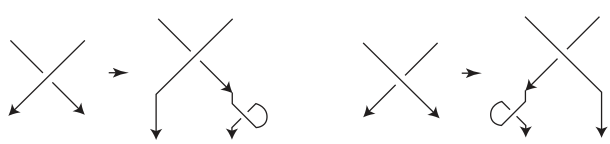

Figure 9: How to insert a kink to a crossing

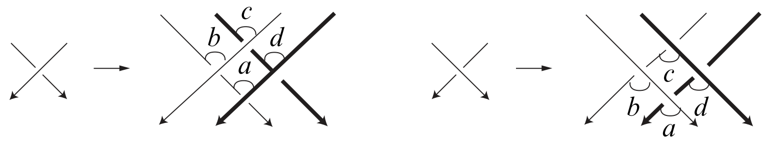

Figure 10: Labels on the colored diagrams $\zeta(c_\pm)$ associated to positive and negative crossings $c_\pm$

of c_1; see Figure 9 for examples. We take a diagram D^0 of T^0 obtained from $D_{(\leftarrow)}$ by replacing each self-crossing c by c^0 so that the framings vanish.

Since the labels on $D_{(\leftarrow)}$ to define J' are only on crossings (since there are no $\curvearrowright$ and $\smile$), in order to prove the assertion it is enough to show

(1) $\phi^{\otimes 2} \circ J(c_\pm) = J'(c_\pm)$,

(2) $\phi^{\otimes 2} \circ J(c^0_\pm) = \bar{\theta}_1^{\pm 1} J'(c_\pm)$,

for a positive (resp. negative) crossing c_+ (resp. c_-) with each strand oriented arbitrarily.

Assume that each strand of $c_\pm$ is oriented downwards. Then (1) follows from

$$\begin{aligned}
\phi^{\otimes 2} \circ J(c_+) &= \hat{R}_{1234} = S''_{14} S_{13} \widetilde{S}_{24} S'_{23} \\
&= \sum_{a,b,c,d} e_a e_b \otimes \tilde{e}_d \tilde{e}_c \otimes e^b e^c \otimes \tilde{e}^a \tilde{e}^d = J'(c_+), \\
\phi^{\otimes 2} \circ J(c_-) &= \hat{R}^{-1}_{1234} = (S'_{23})^{-1} (\widetilde{S}_{24})^{-1} (S_{13})^{-1} (S''_{14})^{-1} \\
&= \sum_{a,b,c,d} u_b u_c \otimes \tilde{u}_a \tilde{u}_d \otimes u^a u^b \otimes \tilde{u}^d \tilde{u}^c = J'(c_-),
\end{aligned}$$

where u_a, u^a, $\tilde{u}_a$, $\tilde{u}^a$ are defined by

$$\sum_a u_a \otimes u^a = S^{-1} = \sum_a \gamma(e_a) \otimes e^a, \qquad \sum_a \tilde{u}_a \otimes u^a = (S')^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes e^a,$$
$$\sum_a u_a \otimes \tilde{u}^a = (S'')^{-1} = \sum_a \gamma(e_a) \otimes \tilde{e}^a, \quad \sum_a \tilde{u}_a \otimes \tilde{u}^a = \widetilde{S}^{-1} = \sum_a \gamma(\tilde{e}_a) \otimes \tilde{e}^a;$$

see Figure 10.

Since the universal invariant of a positive (resp. negative) kink is equal to θ^{-1} (resp. θ), we have $J(c^0_\pm) = \theta_1^{\pm 1} J(c_\pm)$. Thus (2) follows from

$$\phi^{\otimes 2} \circ J(c^0_\pm) = \phi^{\otimes 2}(\theta_1^{\pm 1} J(c_\pm)) = \bar{\theta}_1^{\pm 1} (\phi^{\otimes 2} \circ J)(c_\pm) = \bar{\theta}_1^{\pm 1} J'(c_\pm),$$

where the last identity follows from (1).

For a crossing $c_\pm$ with other orientations, (1) and (2) follow similarly from

$$\phi^{\otimes 2}\circ(\gamma_{D(A)}\otimes 1)(R)=\sum_{a,b,c,d}\gamma(\tilde e_c)\gamma(\tilde e_d)\otimes\gamma(e_b)\gamma(e_a)\otimes e^b e^c\otimes \tilde e^a\tilde e^d,$$

$$\phi^{\otimes 2}\circ(1\otimes\gamma_{D(A)})(R)=\sum_{a,b,c,d}e_a e_b\otimes\tilde e_d\tilde e_c\otimes\bar\gamma^*(\tilde e^d)\bar\gamma^*(\tilde e^a)\otimes\bar\gamma^*(e^c)\bar\gamma^*(e^b),$$

$$\phi^{\otimes 2}\circ(\gamma_{D(A)}\otimes 1)(R^{-1})=\sum_{a,b,c,d}\gamma(\tilde u_d)\gamma(\tilde u_a)\otimes\gamma(u_c)\gamma(u_b)\otimes u^a u^b\otimes\tilde u^d\tilde u^c,$$

$$\phi^{\otimes 2}\circ(1\otimes\gamma_{D(A)})(R^{-1})=\sum_{a,b,c,d}u_b u_d\otimes\tilde u_a\tilde u_d\otimes\bar\gamma^*(\tilde u^c)\bar\gamma^*(\tilde u^d)\otimes\bar\gamma^*(u^b)\bar\gamma^*(u^a),$$

which completes the proof. □

Proof of Theorem 4.1 By Lemma 4.3 we have

$$\begin{aligned}\phi^{\otimes n}\circ J(T)&=\phi^{\otimes n}\Bigl(\Bigl(\prod_i\theta_i^{-f(D_i)}\Bigr)J(T^0)\Bigr)=\Bigl(\prod_i\bar\theta_i^{-f(D_i)}\Bigr)(\phi^{\otimes n}\circ J(T^0))\\&=\Bigl(\prod_i\bar\theta_i^{-f(D_i)}\Bigr)J^0(D_{(\leftarrow)})\\&=\Bigl(\prod_i\bar\theta_i^{-f(D_i)}\Bigr)\Bigl(\prod_i\bar\theta_i^{f((D_{(\leftarrow)})_i)}\Bigr)J'(D_{(\leftarrow)})\\&=\Bigl(\prod_i\bar\theta_i^{d(D_i)}\Bigr)J'(D_{(\leftarrow)}).\end{aligned}$$
□

For the example with C, with the parameters as in Figure 6, right, we have

$$\begin{aligned}J'_C=\sum_{i_a,i_b,i_c,i_d,j_a,j_b,j_c,j_d}&\gamma(e_{i_c})\gamma(e_{i_d})\bar\gamma^*(e^{j_d})\bar\gamma^*(e^{j_a})\\&\otimes\gamma(\tilde e_{i_b})\gamma(\tilde e_{i_a})\bar\gamma^*(\tilde e^{j_c})\bar\gamma^*(\tilde e^{j_b})\\&\otimes e_{j_a}e_{j_b}e^{i_b}e^{i_c}\otimes\tilde e_{j_d}\tilde e_{j_c}\tilde e^{i_a}\tilde e^{i_d}.\end{aligned}$$

For the example with T_{41}, with the parameters as in Figure 7, right, summing over all i, j, k and l, we have

$$\begin{aligned}J'_{T_{41}}=\sum&\bar\gamma^*(u^{i_c})\bar\gamma^*(u^{i_d})\gamma(u_{j_d})\gamma(u_{j_a})e^{l_b}e^{l_c}e_{k_a}e_{k_b}\\&\times u^{j_a}u^{j_b}u_{i_b}u_{i_c}\bar\gamma^*(e^{k_d})\bar\gamma^*(e^{k_a})\gamma(e_{l_c})\gamma(e_{l_d})\\&\otimes\bar\gamma^*(\tilde u^{i_b})\bar\gamma^*(\tilde u^{i_a})\gamma(\tilde u_{j_c})\gamma(\tilde u_{j_b})\tilde e^{l_a}\tilde e^{l_d}\tilde e_{k_d}\tilde e_{k_c}\\&\times\tilde u^{j_d}\tilde u^{j_c}\tilde u_{i_a}\tilde u_{i_d}\bar\gamma^*(\tilde e^{k_c})\bar\gamma^*(\tilde e^{k_b})\gamma(\tilde e_{l_b})\gamma(\tilde e_{l_a}).\end{aligned}$$

5 Extension of the universal quantum invariant to an invariant for colored diagrams

In this section we define *colored diagrams* and extend the map J' to an invariant for colored diagrams up to *colored moves*.

5.1 Colored diagrams and an extension of J'

In what follows, we consider also a virtual crossing as in Figure 11, which we call a *symmetry*. By a *crossing* we mean only a real crossing.

A *colored diagram* Z is a virtual tangle diagram consisting of *thin* strands and *thick* strands, which is obtained by pasting, horizontally and vertically, copies of fundamental tangle diagrams in Figure 2 and copies of the symmetry, where the thickness of each strand are arbitrary.

Let $\mathcal{CD}$ be the set of colored diagrams. For $\mu = (\mu_1, \dots, \mu_n), \nu = (\nu_1, \dots, \nu_n) \in \{\pm\}^n$, we denote by

$$\mathcal{CD}(\mu; \nu) \subset \mathcal{CD}$$

the set of n–component colored diagrams $Z = Z_1 \cup \dots \cup Z_n$ such that

$$Z_i \text{ is thin} \iff \mu_i = +, \quad Z_i \text{ is thick} \iff \mu_i = -,$$
$$\partial Z_i \neq \varnothing \iff \nu_i = +, \quad \partial Z_i = \varnothing \iff \nu_i = -.$$

For $i = 1, \dots, n$, set

$$H(A)_i^+ = H(A), \qquad H(A)_i^- = H(A)/N_{H(A)},$$
$$(H(A)^{\mathrm{op}})_i^+ = H(A)^{\mathrm{op}}, \quad (H(A)^{\mathrm{op}})_i^- = H(A)^{\mathrm{op}}/N_{H(A)^{\mathrm{op}}}.$$

We define the map

$$J' \colon \mathcal{CD}(\mu; \nu) \to \bigotimes_{\mu_i = +} H(A)_i^{\nu_i} \otimes \bigotimes_{\mu_j = -} (H(A)^{\mathrm{op}})_j^{\nu_j}$$

in a similar way to the definition of J' in Section 4, ie by putting the labels on the crossings as in Figure 8, not putting label for other fundamental tangle diagrams, and by taking the product of the labels.

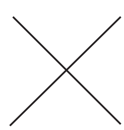

Figure 11: A symmetry, where the orientation of each strand is arbitrary

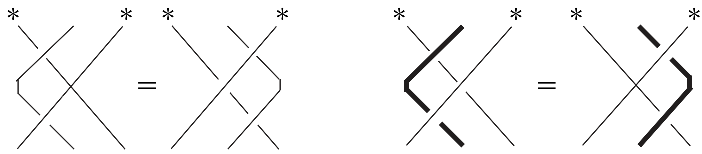

Figure 12: The colored Pachner $(2,3)$ moves, where the orientation and the thickness of each $*$–marked strand are arbitrary

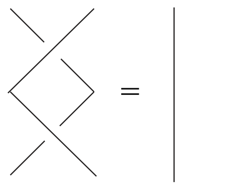

Figure 13: The colored $(0,2)$ moves, where the orientation and the thickness of each strand are arbitrary

5.2 Colored moves

We define several moves on colored diagrams as follows.

The *colored Pachner* $(2,3)$ *moves* are defined in Figure 12. Note that each colored Pachner $(2,3)$ move involves a symmetry, and thus is not the Reidemeister III move on tangle diagrams.

The *colored* $(0,2)$ *moves* are defined in Figure 13.

The *symmetry moves* are defined in Figure 14.

The *planar isotopies* are defined in Figure 15.[7]

We call each of the above move a *colored move*.

Let $\sim_{\mathrm{c}}$ be the equivalence relation on the set of colored diagrams generated by all colored moves.

Similarly, let $\sim'_{\mathrm{c}}$ be the equivalence relation on the set of colored diagrams generated by colored moves except for the moves in Figure 16.

Theorem 5.1 *The map J' is invariant under $\sim'_{\mathrm{c}}$. If $\gamma^2 = 1$, then the map J' is also invariant under $\sim_{\mathrm{c}}$.*

Proof Let Z and Z' be two colored diagrams.

[7]It is known that if two tangle diagrams D and D' are planar isotopic to each other, then D and D' are related by a sequence of the moves defined in Figure 15; see eg [17].

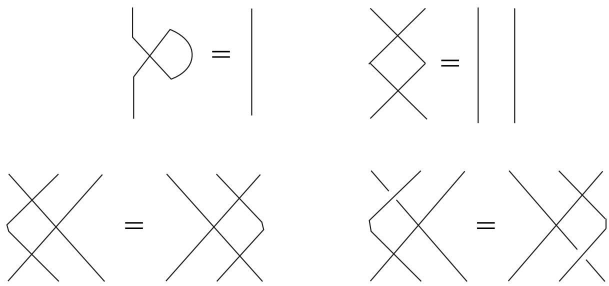

Figure 14: The symmetry moves, where the orientation and thickness of each strand are arbitrary

If Z and Z' are related by a colored Pachner $(2,3)$ move with strands oriented downwards, then $J'(Z) = J'(Z')$ follows from the pentagon relations (3-10)–(3-13). If some $*$–marked strands are upwards, then $J'(Z) = J'(Z')$ follows from the pentagon relations, after applying the antipode on each tensorand corresponding to an upward strand.

If Z and Z' are related by a colored $(0,2)$ move, then $J'(Z) = J'(Z')$ follows from the invertibility of S, S', S'' and $\widetilde{S}$.

If Z and Z' are related by a symmetry move, or by a planar isotopy which does not involve a crossing, then it is easy to see $J'(Z) = J'(Z')$.

Let us assume that Z and Z' are related by a planar isotopy which involves a crossing. Recall that $S \in H(A) \otimes H(A)$, $S' \in H(A)^{\mathrm{op}} \otimes H(A)$, $S'' \in H(A) \otimes H(A)^{\mathrm{op}}$ and $\widetilde{S} \in H(A)^{\mathrm{op}} \otimes H(A)^{\mathrm{op}}$. If the planar isotopy is not in Figure 16, then $J'(Z) = J'(Z')$ follows from

$$\begin{aligned}
&(\gamma \otimes 1)(S) = S^{-1}, && (1 \otimes \overline{\gamma}^*)(S^{-1}) = S, && (\gamma \otimes \overline{\gamma}^*)(S) = S,\\
&(\gamma \otimes 1)(S') = (S')^{-1}, && (1 \otimes \overline{\gamma}^*)((S')^{-1}) = S', && (\gamma \otimes \overline{\gamma}^*)(S') = S',\\
&(\gamma \otimes 1)(S'') = (S'')^{-1}, && (1 \otimes \overline{\gamma}^*)((S'')^{-1}) = S'', && (\gamma \otimes \overline{\gamma}^*)(S'') = S'',\\
&(\gamma \otimes 1)(\widetilde{S}) = \widetilde{S}^{-1}, && (1 \otimes \overline{\gamma}^*)(\widetilde{S}^{-1}) = \widetilde{S}, && (\gamma \otimes \overline{\gamma}^*)(\widetilde{S}) = \widetilde{S}.
\end{aligned}$$

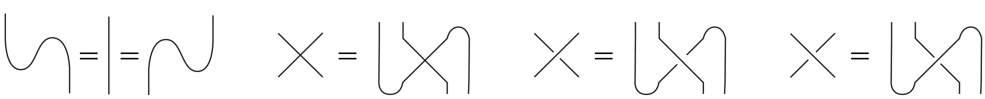

Figure 15: The planar isotopies, where the orientation and thickness of each strand are arbitrary

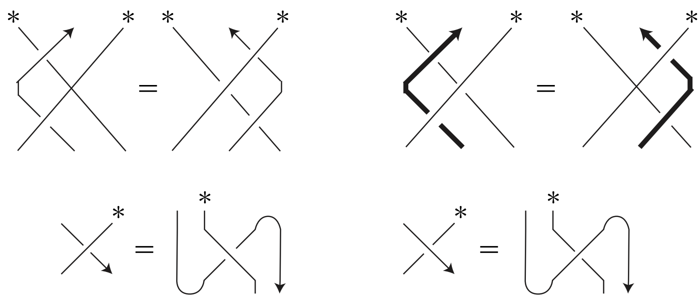

Figure 16: The colored moves which are not in generators for $\sim'_{\mathrm{c}}$, where the orientation and the thickness of each $*$–marked strand are arbitrary

If the planar isotopy is in Figure 16, then we have $J'(Z) = J'(Z')$ if $\gamma^2 = 1$, by

$$\begin{aligned}
(1 \otimes \bar{\gamma}^*)(S) &= S^{-1}, & (\gamma \otimes 1)(S^{-1}) &= S,\\
(1 \otimes \bar{\gamma}^*)(S') &= (S')^{-1}, & (\gamma \otimes 1)((S')^{-1}) &= S',\\
(1 \otimes \bar{\gamma}^*)(S'') &= (S'')^{-1}, & (\gamma \otimes 1)((S'')^{-1}) &= S'',\\
(1 \otimes \bar{\gamma}^*)(\widetilde{S}) &= \widetilde{S}^{-1}, & (\gamma \otimes 1)(\widetilde{S}^{-1}) &= \widetilde{S}.
\end{aligned}$$

If Z and Z' are related by a colored Pachner $(2,3)$ move in Figure 16, ie a colored Pachner $(2,3)$ move with middle strands oriented upwards, then Z and Z' are related by planar isotopy and the colored Pachner $(2,3)$ move with middle strands oriented downwards. Thus we have $J'(Z) = J'(Z')$ by the above argument.

Thus we have the assertion. □

5.3 Tangles and colored diagrams

Recall from Section 4.1 the diagram $\zeta(D)$ associated to a tangle diagram D. Actually $\zeta(D)$ is nothing but a colored diagram and ζ defines a map

$$\zeta\colon \{\text{tangle diagrams}\} \to \{\text{colored diagrams}\}.$$

Let $\sim_{\mathrm{RII,III}}$ be the regular isotopy, ie the equivalence relation of tangle diagrams generated by Reidemeister II, III moves and planar isotopies of tangle diagrams. We have the following.

Theorem 5.2 *Let D and D' be two diagrams such that $D \sim_{\mathrm{RII,III}} D'$. Then we have $\zeta(D) \sim_{\mathrm{c}} \zeta(D)$.*

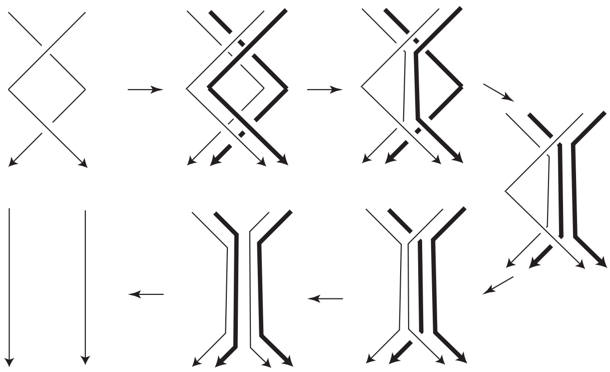

Figure 17: A realization of a Reidemeister II move using colored moves

Proof Let D and D' be two tangle diagrams related by a Reidemeister II move. We can transform $\zeta(D)$ to $\zeta(D')$ by applying colored $(0,2)$ moves four times; see Figure 17 for the case that each strand is oriented downwards.

Let D and D' be two tangle diagrams related by a Reidemeister III move. We can transform $\zeta(D)$ to $\zeta(D')$ by applying colored Pachner $(2,3)$ moves eight times; see Figure 18 for the case that each strand is oriented downwards.

Let D and D' be two tangle diagrams that are related by the planar isotopy. Then we can also transform $\zeta(D)$ to $\zeta(D')$ by the planar isotopies; see Figure 19 for examples. □

Note that the diagrammatic transformations in Figure 18 induce algebraic equations via the universal invariant J', which gives a proof of Proposition 3.5. See Table 1 for the situation.

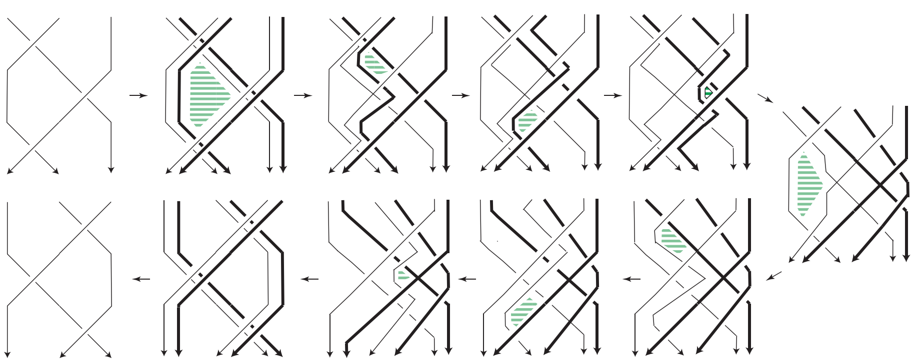

Figure 18: A realization of a Reidemeister III move using colored moves

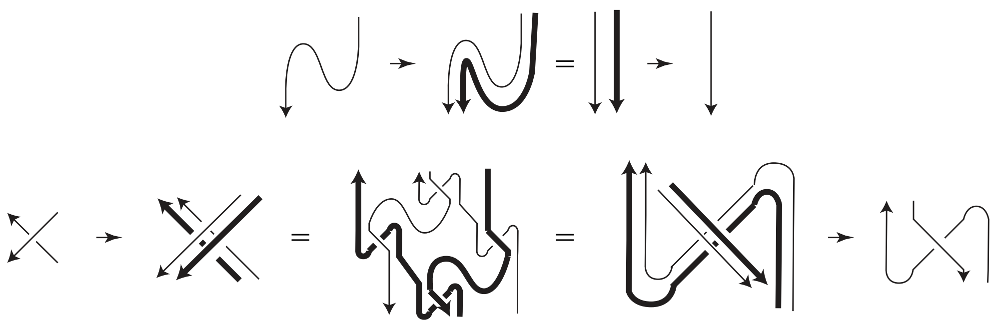

Figure 19: Realizations of planar isotopies using colored moves

topological situation	algebraic situation
Reidemeister III move	quantum Yang–Baxter equation
colored Pachner $(2,3)$ moves	pentagon relations
Figure 18 (colored $(2,3)$ move $\Rightarrow$ RIII move)	Proposition 3.5 (pentagon relation $\Rightarrow$ quantum Yang–Baxter equation)

Table 1: The correspondence between transformations and equations, induced by the universal invariant J'

6 Three-dimensional descriptions: colored diagrams and colored singular triangulations

In this section, we associate a *colored tetrahedron* to each crossing of a colored diagram Z and define a *colored cell complex* associated to Z. Using a colored cell complex we define a *colored singular triangulation* of a topological space. As a result, the universal quantum invariant J' turns out to be an invariant of colored singular triangulations, where a copy of the S–tensor is attached to each colored tetrahedron.

6.1 Colored tetrahedra

Consider a tetrahedron Γ in the oriented space $\mathbb{R}^3$ with an ordering of its 2–faces f_1, f_2, f_3, f_4. We stick Γ by two strands going into Γ at f_1 (resp. f_3) and out of Γ at f_2 (resp. f_4). Note that there are two types of such tetrahedra up to rotation as in Figure 20, where such a tetrahedra is presented by a crossing so that the strand piercing f_1 and f_2 is over. We consider two types of strands, depicted by thick and thin strands, and then there are eight types of such tetrahedra, which we call *colored tetrahedra*, presented by eight types of crossings as in Figure 21.

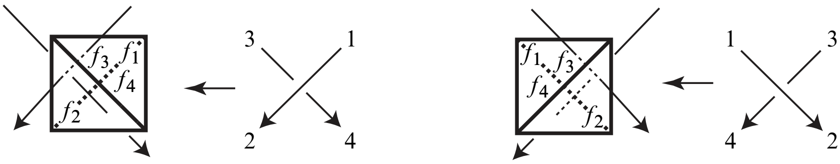

Figure 20: Two types of tetrahedra which are attached to two ordered strands

6.2 Colored diagrams and colored cell complexes

We define a *colored cell complex* $\mathcal{C}(Z)$ associated to a colored diagram Z as follows.

Recall that Z consists of fundamental tangles and symmetries. Let $\{c_1, \dots, c_k\}$ be the set of crossings in Z. To each crossing c_i, associate a colored tetrahedron Γ_i as in Section 6.1. See Figure 22 for an example.

We define $\mathcal{C}(Z)$ to be the cell complex obtained from colored tetrahedra $\Gamma_1, \dots, \Gamma_k$ by gluing them along their 2–faces as follows:

(1) 2–faces F and F' of $\Gamma_1, \dots, \Gamma_k$ are glued if and only if F and F' are adjacent along Z.

(2) We mark by $*$ the vertex of each 2–face of $\Gamma_1, \dots, \Gamma_k$ as in Figure 23 depending on the thickness of strands and the order of the faces in a tetrahedron, and glue adjacent faces F and F' so that the $*$–marked vertices are attached.

6.3 Colored singular triangulations and colored ideal triangulations

For a space X, a *singular triangulation* (see eg [32; 2]) of X consists of a finite-index set I, a function $d\colon I \to \mathbb{N}$, and continuous maps $f_i\colon \Delta^{d(i)} \to X$ for $i \in I$, where Δ^n is the standard n simplex, such that $(I, d, \{f_i\}_{i \in I})$ is a finite cell decomposition of X,

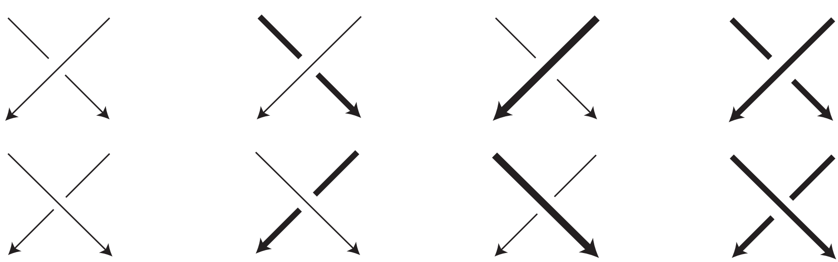

Figure 21: Colored tetrahedra

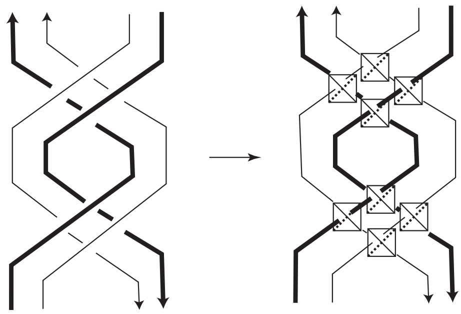

Figure 22: How to associate tetrahedra on a colored diagram

and for each $i \in I$ and a face F in $\Delta^{d(i)}$, the restriction $f_i|_F$ is the composition $f_j \circ g$ of an affine isomorphism $g\colon F \to \Delta^{d(j)}$ and f_j for some $j \in I$.

Let $\mathcal{C}(Z)$ be the colored cell complex of a colored diagram Z, which we can naturally regard a singular triangulation. Consider

$$X = \mathcal{C}(Z)/(e_1 = e_1', \dots, e_k = e_k',\ v_1 = v_1', \dots, v_l = v_l')$$

to be a singular triangulation obtained from $\mathcal{C}(Z)$ by identifying some pairs of edges $(e_1, e_1'), \dots, (e_k, e_k')$, where $k \geq 0$, and some pairs of vertices $(v_1, v_1'), \dots, (v_l, v_l')$, where $l \geq 0$, in $\mathcal{C}(Z)$. We call X a *colored singular triangulation* (coloring) of type Z. In particular, if X is an ideal triangulation of some topological space $\widetilde{X}$, then we call it a *colored ideal triangulation* of $\widetilde{X}$.

Let $\mathcal{CT}(Z)$ be the set of colored singular triangulations of type Z and set

$$\mathcal{CT} = \bigcup_{Z \in \mathcal{CD}} \mathcal{CT}(Z).$$

Remark 6.1 In this remark, we assume 3–manifolds are connected, compact, oriented, and with nonempty boundary.

In [9; 6; 7], Benedetti–Petronio and Baseilhac–Benedetti used so-called $\mathcal{N}$*–graphs* to represent *branched* ideal triangulations of a 3–manifolds and dual oriented *standard*

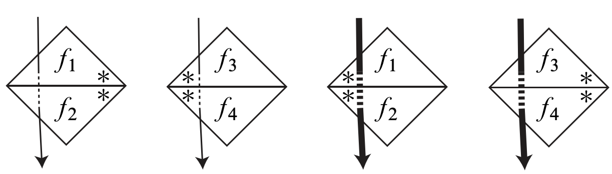

Figure 23: How to mark a vertex of each triangle by $*$

branched spines of them. In this remark we consider abstract $\mathcal{N}$–graphs, ie we do not take planar immersions of them.

Let $\mathcal{BTR}$ the set of branched ideal triangulations of 3–manifolds, $\mathcal{BSP}$ the set of oriented standard branched spines of 3–manifolds, and $\mathcal{NG}$ the set of $\mathcal{N}$–graphs with the color $0 \in \mathbb{Z}/3\mathbb{Z}$ on every edge.

We have the bijections

$$\mathcal{NG} \to \mathcal{BSP}, \quad G \mapsto BSP(G),$$
$$\mathcal{NG} \to \mathcal{BTR}, \quad G \mapsto BTR(G),$$

where $BSP(G)$ is obtained from G [6] so that a 4–valent vertex encodes a branched tetrahedron, and $BTR(G)$ is the branched ideal triangulation, which is the dual of the oriented standard branched spine $BSP(G)$.

Let $\mathcal{CCD}$ be the set of equivalence classes of *closed* colored diagrams up to planar isotopies and symmetry moves. We have the surjective map

$$p\colon \mathcal{CCD} \to \mathcal{NG}, \quad Z \mapsto p(Z),$$

where $p(Z)$ is the $\mathcal{N}$–graph obtained from Z by reversing the orientation of thick strands.

It is not difficult to check that the branched ideal triangulation $BTR(p(Z))$ is the colored singular triangulation of type Z obtained from $\mathcal{C}(Z)$ by identifying some edges and vertices so that $BSP(p(Z))$ becomes a standard spine, ie so that the complement of the vertices in the singular set of $BSP(p(Z))$ is a union of segments, and that the complement of the singular set in $BSP(p(Z))$ is a union of disks. For an example with link complements, see the proof of Proposition 7.1.

6.4 Colored moves and colored singular triangulations

We can translate colored moves on the set $\mathcal{CD}$ of colored diagrams defined in Section 5.2 to moves on the set $\mathcal{CT}$ of colored singular triangulations as follows.

For colored diagrams Z and Z', let X and X' be colored singular triangulations of types Z and Z', respectively. Let

$$\psi\colon \mathcal{C}(Z) \to X, \quad \psi'\colon \mathcal{C}(Z') \to X'$$

be the projections. We say that X and X' are related by a colored Pachner $(2,3)$ move if

(1) the colored diagrams Z and Z' are related by a colored Pachner $(2,3)$ move, and

(2) $\psi = \psi'$ on the exteriors $\mathcal{C}(Z) \setminus W = \mathcal{C}(Z') \setminus W'$, where W (resp. W') is the subcomplex of $\mathcal{C}(Z)$ (resp. $\mathcal{C}(Z')$) consisting of the three (resp. two) tetrahedra corresponding to the three (resp. two) crossings of Z (resp. Z') involved in the colored Pachner $(2,3)$ move.

We define other colored moves on colored singular triangulations similarly.

Then the colored Pachner $(2,3)$ move on $\mathcal{CT}$ turns out to be the Pachner $(2,3)$ move on singular triangulations, defined in Figure 24, left, replacing two tetrahedra sharing one face with three tetrahedra, or its inverse. See Figure 25 for an example, where we color overstrands red and understrands blue so that we can distinguish them in 3–spaces in the lowest picture.

The colored $(0,2)$ move on $\mathcal{CT}$ turns out to be the $(0,2)$ move on singular triangulations, defined in Figure 24, right, replacing two adjacent 2–faces with two tetrahedra, or its inverse.

Correspondingly to the equivalence relations $\sim_{\mathrm{c}}$ and $\sim'_{\mathrm{c}}$ on $\mathcal{CD}$, we define the equivalence relations $\sim_{\mathrm{ct}}$ and $\sim'_{\mathrm{ct}}$ on $\mathcal{CT}$; ie $\sim_{\mathrm{ct}}$ is generated by all colored moves, and $\sim'_{\mathrm{ct}}$ is generated by colored moves except for the moves in Figure 16.

Let $\pi\colon \mathcal{CT} \to \mathcal{CD}$ be the map such that $\pi(X) = Z$ for $X \in \mathcal{CT}(Z)$.

For $\mu = (\mu_1, \dots, \mu_m)$ and $\nu = (\nu_1, \dots, \nu_m)$ in $\{\pm\}^m$ for $m \geq 0$, recall from Section 5.1 the subset $\mathcal{CD}(\mu;\nu) \subset \mathcal{CD}$. Let $\mathcal{CT}(\mu;\nu)$ be the set of colored singular triangulations of types in $\mathcal{CD}(\mu;\nu)$. Note that $\mathcal{CT} = \bigcup_{\mu,\nu \in \{\pm\}^m, m \geq 0} \mathcal{CT}(\mu;\nu)$.

Proposition 6.2 *The composition*

$$J' \circ \pi\colon \mathcal{CT}(\mu;\nu) \to \bigotimes_{i \in I_+} H(A)_i^{\nu_i} \otimes \bigotimes_{j \in I_-} (H(A)^{\mathrm{op}})_j^{\nu_j}$$

of the restriction of π to $\mathcal{CT}(\mu;\nu)$ and the universal quantum invariant J' is invariant under $\sim'_{\mathrm{ct}}$. If $\gamma^2 = 1$, then $J' \circ \pi$ is also invariant under $\sim_{\mathrm{ct}}$.

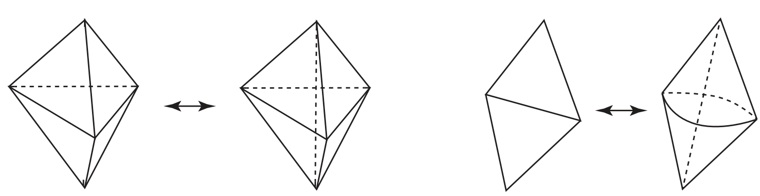

Figure 24: The Pachner $(2,3)$ move (left) and the $(0,2)$ move (right)

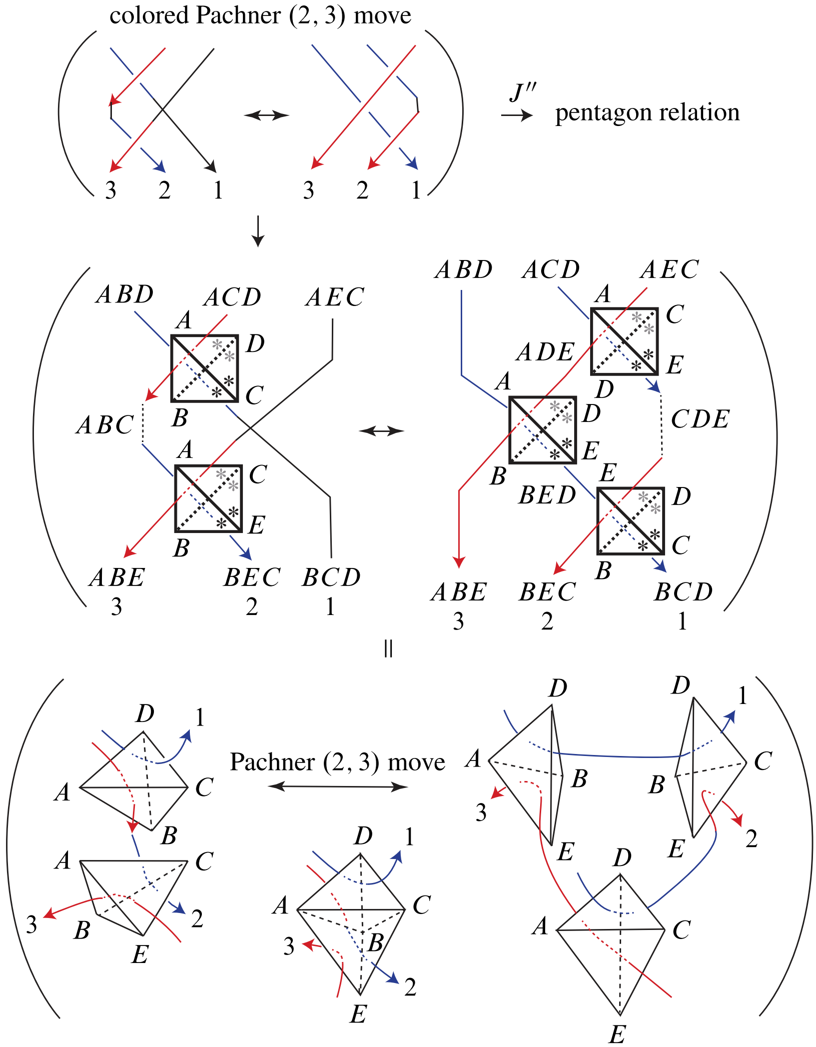

Figure 25: A colored Pachner $(2,3)$ move of colored diagrams and a Pachner $(2,3)$ move of colored cell complexes, whose image under J' turns out to be the pentagon relation $S_{23}S_{12} = S_{12}S_{13}S_{23}$

Proof Note that the projection map π induces the map

$$\mathcal{CT}/\sim'_{\mathrm{ct}} \to \mathcal{CD}/\sim'_{\mathrm{c}} \quad (\text{resp. } \mathcal{CT}/\sim_{\mathrm{ct}} \to \mathcal{CD}/\sim_{\mathrm{c}}),$$

which shows the invariance of $J' \circ \pi$ under $\sim'_{\mathrm{ct}}$ (resp. $\sim_{\mathrm{ct}}$ if $\gamma^2 = 1$). □

We call $J' \circ \pi$ the *universal quantum invariant* of colored singular triangulations. The invariance of $J' \circ \pi$ under colored Pachner $(2,3)$ moves is shown by pentagon relations.

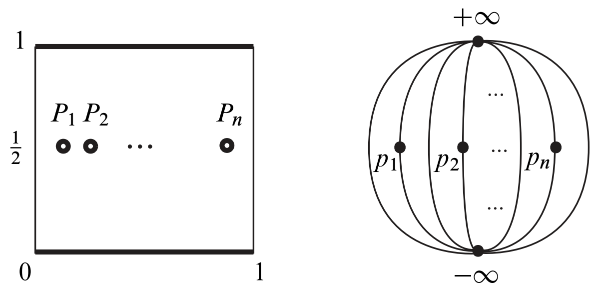

Figure 26: A punctured disk (left) and its leaves-ideal triangulation (right)

7 Octahedral triangulation of tangle complements

In this section we define *ideal triangulations of tangle complements*, and construct examples called the *octahedral triangulations*. We will show that the octahedral triangulation associated to a tangle diagram D naturally admits a structure of a colored ideal triangulation of type $\zeta(D)$.

7.1 Ideal triangulations of tangle complements

Let M be a compact manifold of dimension $n \leq 3$, possibly with nonempty boundary. Let F be an $(n-1)$–submanifold of ∂M. Let $F_1, \dots, F_k$ be the connected components of F. Let $M /\!\!/ F$ denote the topological space obtained from M by collapsing each F_i into a point. An *ideal triangulation* of the pair (M, F) is defined to be a singular triangulation of $M /\!\!/ F$ such that each vertex of the singular triangulation is on a point arising from F.

Let $D_n = [0,1]^2 \setminus (P_1 \cup \dots \cup P_n)$ be a punctured disk, where $P_1, \dots, P_n$ are small disks with the centers arranged on the line $[0,1] \times \{\frac{1}{2}\}$ as in Figure 26, left. We define the *leaves-ideal triangulation* l_n of D_n to be the ideal triangulation of the pair $\big(D_n, ([0,1] \times \{0,1\}) \cup \partial P_1 \cup \dots \cup \partial P_n\big)$ as in Figure 26, right, where $-\infty$, $+\infty$, $p_1, \dots, p_n$ denote the vertices corresponding to $[0,1] \times \{0\}$, $[0,1] \times \{1\}$, $\partial P_1, \dots, \partial P_n$, respectively. Here we formally define l_0 as a segment having $\{\pm\infty\}$ as its vertices. In particular we call l_1 a *leaf*.

Let $T = T_1 \cup \dots \cup T_n$ be an n–component tangle. Let $E = \overline{[0,1]^3 \setminus N(T)}$ be the complement of T in the cube, where $N(T)$ is a tubular neighborhood of T in the cube. Let F_T be the intersection $\partial E \cap N(T)$, which consists of annuli and tori. Then an *ideal triangulation of the tangle complement* E of T in the cube is defined to be an ideal triangulation of $(E, F_T \cup F_{z=0} \cup F_{z=1})$, where $F_{z=0} = [0,1]\times[0,1]\times\{0\}$ and $F_{z=1} = [0,1]\times[0,1]\times\{1\}$, such that its restriction to each boundary component $[0,1]\times\{0,1\}\times[0,1]$ is a leaves-ideal triangulation. The vertices corresponding to $F_{z=0}$, and $F_{z=1}$ are denoted by $-\infty$ and $+\infty$, respectively.

7.2 Colored ideal triangulations for octahedral triangulations of tangle complements

A tangle diagram D is called *nonsplitting* if

(1) the 4–regular plane graph giving the diagram D is connected, and

(2) there is not a component of D such that crossings along the path of the component are only overpassing or only underpassing.

Let T be a tangle and D its nonsplitting diagram which has at least one crossing. We define a cell complex $\mathcal{O}(D)$, which we call the *octahedral triangulation* associated to D, which is an ideal triangulation of the tangle complement E. If in addition D is a link diagram, then $\mathcal{O}(D)$ is nothing but the octahedral triangulation studied in eg [10; 34] in the context of the hyperbolic geometry.

Step 1 Take a colored diagram. Recall from Section 4 the colored diagram $\zeta(D)$ obtained from D by duplicating and thickening the left strands following the orientation.

Step 2 (preparing and placing octahedra) Let $\{c_1, \dots c_k\}$ be the set of crossings of the diagram D. In a neighborhood of $\zeta(c_i)$, there are four crossings t_1^i, t_2^i, t_3^i, t_4^i as in Figure 27, where t_1^i is the right crossing when we see strands oriented downwards, and t_2^i, t_3^i, t_4^i are defined one by one in a counterclockwise order. As in Figure 27, for $j = 1,2,3,4$, we associate a tetrahedron $\Lambda_j^i = n_j^i \tilde{e}_j^i \tilde{e}_j'^i s_j^i$ to each t_j^i. Then we glue the four tetrahedra Λ_1^i, Λ_2^i, Λ_3^i, Λ_4^i together to obtain an octahedron $o_i = n^i e_{12}^i e_{23}^i e_{34}^i e_{41}^i s^i$ so that n_j^i, $\tilde{e}_j^i$, $\tilde{e}_j'^i$ and s_j^i are going to n^i, $e_{j-1,j}^i$, $e_{j,j+1}^i$ and s^i, respectively, where the index j should be considered modulo 4. We place o_i between the two original strands of c_i so that n^i and s^i are placed on the overstrand and the understrand, respectively.

Step 3 (gluing octahedra) We glue the octahedra $o_1, \dots, o_k$ as follows.

For each positive (resp. negative) crossing c_i, we pull the vertices e_{23}^i and e_{41}^i (resp. e_{12}^i and e_{34}^i) upwards, put them on $+\infty$, and glue the two edges $n^i – e_{23}^i$ and $n^i – e_{41}$ (resp. $n^i – e_{12}^i$ and $n^i – e_{34}$). Similarly, pull the vertices e_{12}^i and e_{34}^i (resp. e_{23}^i and e_{41}^i) downwards, put them on $-\infty$, and glue the two edges $s^i – e_{12}^i$ and $s^i – e_{34}$ (resp. $s^i – e_{23}^i$

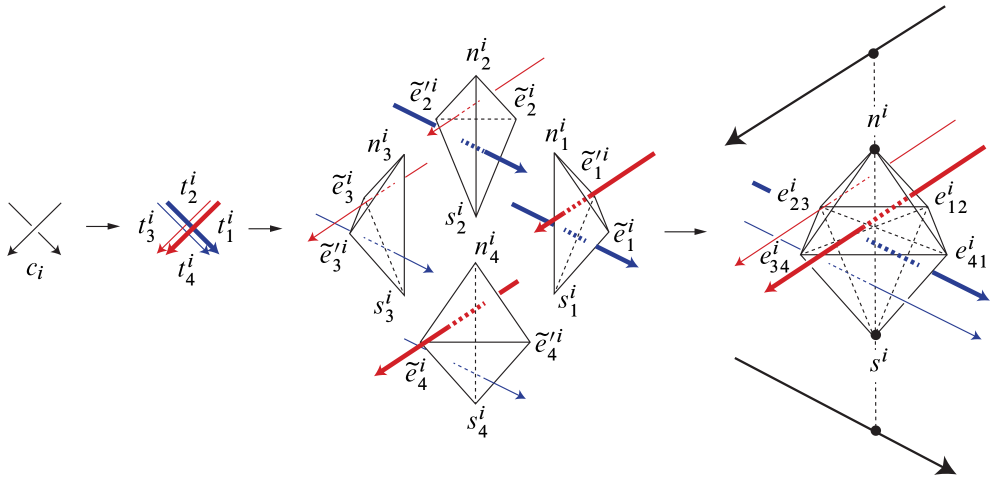

Figure 27: Octahedral triangulation around a crossing

and s^i–e_{41}); see Figure 28. Note that the boundary of the octahedron o_i consists of four leaves corresponding to the four edges of c_i; see Figure 29. We glue the octahedra $o_1, \dots, o_k$ along the pairs of leaves which are adjacent on D so that $\pm\infty$ are attached compatibly. We call the result the *octahedral triangulation* of the complement of T associated to a diagram D, and denote it by $\mathcal{O}(D)$.

It is not difficult to check that $\mathcal{O}(D)$ is an ideal triangulation of the complement of T. Moreover, we have the following.

Proposition 7.1 *The octahedral triangulation $\mathcal{O}(D)$ associated to a tangle diagram D admits a colored ideal triangulation of type $\zeta(D)$.*

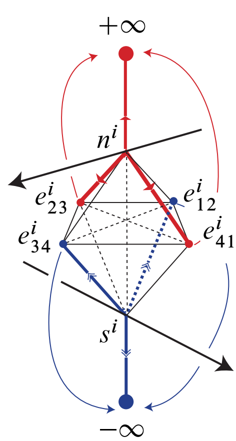

Figure 28: How to glue the edges in an octahedron

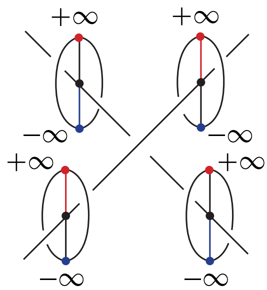

Figure 29: Leaves corresponding to the four edges of a crossing

Proof Recall that in Step 2 of the definition of $\mathcal{O}(D)$, we associate an octahedron o_i to each crossing c_i, where the octahedron is obtained from four tetrahedra as in Figure 27. Actually we can obtain o_i also as the colored cell complex $\mathcal{C}(\zeta(c_i))$ as depicted in Figure 30. In Step 3, we glued the octahedra and triangles as in Figure 31, which follows the gluing rule of the colored tetrahedra and triangles defined in Section 6.2. As the result we have $\mathcal{C}(\zeta(D))$, and finally we identify the edges of each octahedron as in Figure 28, which gives $\mathcal{O}(D)$, a colored ideal triangulation of type $\zeta(D)$. This completes the proof. □

Remark 7.2 A tangle complement could admit more than one colored ideal triangulations up to the equivalence relation $\sim'_{\mathrm{ct}}$, and the universal quantum invariant J' could give different values on them. We expect that *the universal quantum invariant is an*

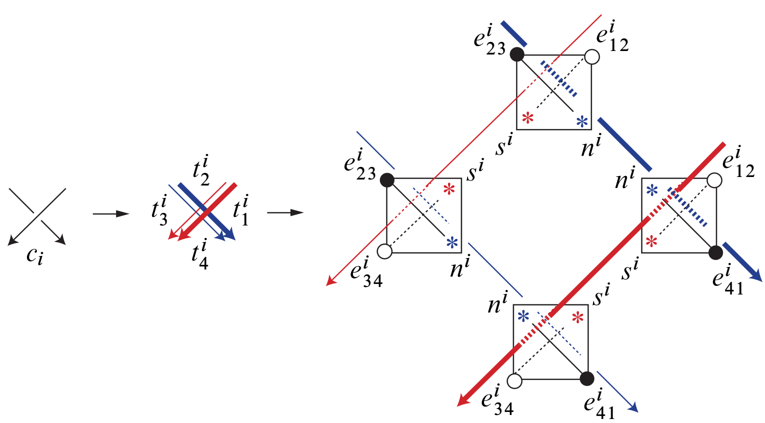

Figure 30: The colored ideal triangulation and the octahedron at a crossing of a tangle

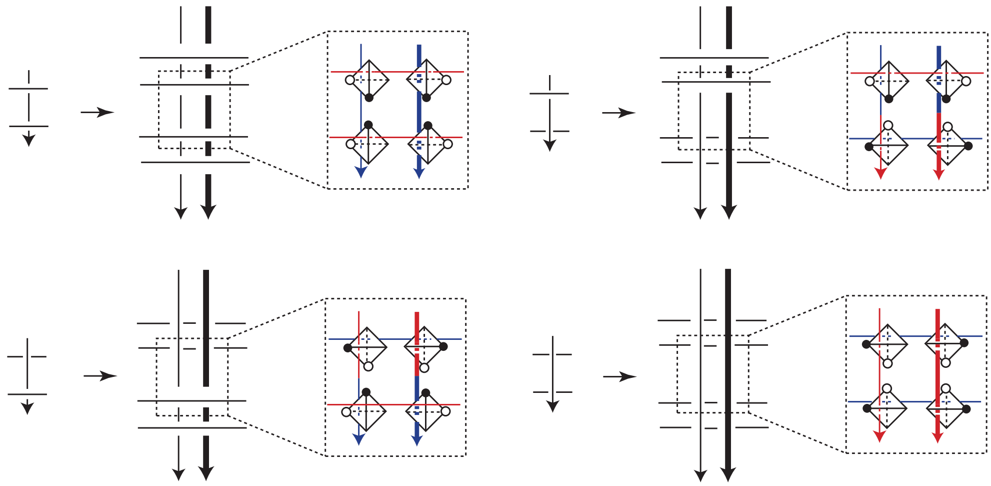

Figure 31: How we glued the octahedra in the octahedral triangulations, where the black dots are attached to $+\infty$ and the white dots are attached to $-\infty$

invariant of pairs of 3*–manifolds and some geometrical inputs obtained from the color*, which we will study in future work.

References

[1] **S Baaj**, **G Skandalis**, *Unitaires multiplicatifs et dualité pour les produits croisés de* C^**–algèbres*, Ann. Sci. École Norm. Sup. 26 (1993) 425–488 MR

[2] **S Baseilhac**, **R Benedetti**, *Quantum hyperbolic invariants of* 3*–manifolds with* $\mathrm{PSL}(2,\mathbb{C})$*–characters*, Topology 43 (2004) 1373–1423 MR

[3] **S Baseilhac**, **R Benedetti**, *Classical and quantum dilogarithmic invariants of flat* $\mathrm{PSL}(2,\mathbb{C})$*–bundles over* 3*–manifolds*, Geom. Topol. 9 (2005) 493–569 MR

[4] **S Baseilhac**, **R Benedetti**, *Quantum hyperbolic geometry*, Algebr. Geom. Topol. 7 (2007) 845–917 MR

[5] **S Baseilhac**, **R Benedetti**, *The Kashaev and quantum hyperbolic link invariants*, J. Gökova Geom. Topol. 5 (2011) 31–85 MR

[6] **S Baseilhac**, **R Benedetti**, *Analytic families of quantum hyperbolic invariants*, Algebr. Geom. Topol. 15 (2015) 1983–2063 MR

[7] **S Baseilhac**, **R Benedetti**, *Non ambiguous structures on* 3*–manifolds and quantum symmetry defects*, Quantum Topol. 8 (2017) 749–846 MR

[8] **R Benedetti**, **C Petronio**, *Branched standard spines of* 3*–manifolds*, Lecture Notes in Math. 1653, Springer (1997) MR

[9] **R Benedetti**, **C Petronio**, *Spin structures on 3–manifolds via arbitrary triangulations*, Algebr. Geom. Topol. 14 (2014) 1005–1054 MR

[10] **J Cho**, **H Kim**, **S Kim**, *Optimistic limits of Kashaev invariants and complex volumes of hyperbolic links*, J. Knot Theory Ramifications 23 (2014) art. id. 1450049 MR

[11] **V G Drinfel'd**, *Quantum groups*, from "Proceedings of the International Congress of Mathematicians, I" (A M Gleason, editor), Amer. Math. Soc., Providence, RI (1987) 798–820 MR

[12] **L D Faddeev**, **R M Kashaev**, *Quantum dilogarithm*, Modern Phys. Lett. A 9 (1994) 427–434 MR

[13] **K Hikami**, **R Inoue**, *Braiding operator via quantum cluster algebra*, J. Phys. A 47 (2014) art. id. 474006 MR

[14] **K Hikami**, **R Inoue**, *Braids, complex volume and cluster algebras*, Algebr. Geom. Topol. 15 (2015) 2175–2194 MR

[15] **M Kapranov**, *Heisenberg doubles and derived categories*, J. Algebra 202 (1998) 712–744 MR

[16] **R M Kashaev**, *Quantum dilogarithm as a $6j$–symbol*, Modern Phys. Lett. A 9 (1994) 3757–3768 MR

[17] **R M Kashaev**, *A link invariant from quantum dilogarithm*, Modern Phys. Lett. A 10 (1995) 1409–1418 MR

[18] **R M Kashaev**, *The Heisenberg double and the pentagon relation*, Algebra i Analiz 8 (1996) 63–74 MR In Russian; translated in St. Petersburg Math. J. 8 (1997) 585–592

[19] **R M Kashaev**, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997) 269–275 MR

[20] **C Kassel**, *Quantum groups*, Graduate Texts in Math. 155, Springer (1995) MR

[21] **R J Lawrence**, *A universal link invariant using quantum groups*, from "Differential geometric methods in theoretical physics" (A I Solomon, editor), World Scientific, Teaneck, NJ (1989) 55–63 MR

[22] **R J Lawrence**, *A universal link invariant*, from "The interface of mathematics and particle physics" (D G Quillen, G B Segal, S T Tsou, editors), Inst. Math. Appl. Conf. Ser. New Ser. 24, Oxford Univ. Press (1990) 151–156 MR

[23] **J-H Lu**, *On the Drinfel'd double and the Heisenberg double of a Hopf algebra*, Duke Math. J. 74 (1994) 763–776 MR

[24] **S Majid**, *Quantum double for quasi-Hopf algebras*, Lett. Math. Phys. 45 (1998) 1–9 MR

[25] **S Majid**, *Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$*, Math. Proc. Cambridge Philos. Soc. 125 (1999) 151–192 MR

[26] **H Murakami**, **J Murakami**, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. 186 (2001) 85–104 MR

[27] **A Ocneanu**, *Chirality for operator algebras*, from "Subfactors" (H Araki, Y Kawahigashi, H Kosaki, editors), World Scientific, River Edge, NJ (1994) 39–63 MR

[28] **T Ohtsuki**, *Colored ribbon Hopf algebras and universal invariants of framed links*, J. Knot Theory Ramifications 2 (1993) 211–232 MR

[29] **N Y Reshetikhin**, **V G Turaev**, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990) 1–26 MR

[30] **M A Semenov-Tian-Shansky**, *Poisson Lie groups, quantum duality principle, and the quantum double*, from "Mathematical aspects of conformal and topological field theories and quantum groups" (P J Sally, Jr, M Flato, J Lepowsky, N Reshetikhin, G J Zuckerman, editors), Contemp. Math. 175, Amer. Math. Soc., Providence, RI (1994) 219–248 MR

[31] **S Suzuki**, *On the universal* sl_2 *invariant of boundary bottom tangles*, Algebr. Geom. Topol. 12 (2012) 997–1057 MR

[32] **V G Turaev**, **O Y Viro**, *State sum invariants of* 3*–manifolds and quantum* $6j$*–symbols*, Topology 31 (1992) 865–902 MR

[33] **J Weeks**, *Computation of hyperbolic structures in knot theory*, from "Handbook of knot theory" (W Menasco, M Thistlethwaite, editors), Elsevier, Amsterdam (2005) 461–480 MR

[34] **Y Yokota**, *On the complex volume of hyperbolic knots*, J. Knot Theory Ramifications 20 (2011) 955–976 MR

Department of Mathematical and Computing Science, Tokyo Institute of Technology
Tokyo, Japan

sakie@c.titech.ac.jp

Received: 24 August 2017 Revised: 17 April 2018