A_∞–resolutions and the Golod property for monomial rings

ROBIN FRANKHUIZEN

Let $R = S/I$ be a monomial ring whose minimal free resolution F is rooted. We describe an A_∞–algebra structure on F. Using this structure, we show that R is Golod if and only if the product on $\operatorname{Tor}^S(R, k)$ vanishes. Furthermore, we give a necessary and sufficient combinatorial condition for R to be Golod.

1 Introduction

Let $S = k[x_1, \ldots, x_m]$ be the polynomial algebra over a field k in m variables and let $I = (m_1, \ldots, m_r)$ be an ideal generated by monomials. In that case, S/I is called a *monomial* ring. Given a monomial ring $R = S/I$, the *Poincaré series* of R is defined as

$$P(R) = \sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^R(k, k) t^j.$$

A result due to Serre states that there is an inequality of power series

$$P(R) \leq \frac{(1 + t)^m}{1 - t \left(\sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^S(R, k) t^j - 1 \right)}.$$

The ring R is said to be *Golod* if equality is obtained. The problem of when a monomial ring is Golod goes back to at least the 70s when Golod [11] showed that a monomial ring R is Golod if and only if all Massey products on the Tor–algebra $\operatorname{Tor}^S(R, k)$ vanish. In general, it is hard to directly verify the vanishing of Massey products and so in practice the Golod property is still hard to determine.

In recent decades, the Golod property has received an increasing amount of attention in topology. The Tor–algebra shows up naturally in topology as follows. Let Δ be a simplicial complex on vertex set $[m] = \{1, \ldots, m\}$ and define the *moment-angle complex* Z_Δ as follows. Let D^2 denote the 2–disc and S^1 its bounding circle. For $\sigma \in \Delta$, define

$$X_\sigma = \prod_{i=1}^{m} Y_i \subseteq (D^2)^m \quad \text{where } Y_i = \begin{cases} D^2 & \text{if } i \in \sigma, \\ S^1 & \text{if } i \notin \sigma. \end{cases}$$

Lastly, we put

$$Z_\Delta = \text{colim}_{\sigma \in \Delta} \, X_\sigma \subseteq (D^2)^m.$$

Moment-angle complexes are one of the central objects of study in toric topology. For us, the cohomology of Z_Δ is of particular interest.

Theorem 1.1 [8, Theorem 4.5.4] *Let Δ be a simplicial complex. There is an isomorphism of graded algebras*

$$H^*(Z_\Delta, k) \cong \text{Tor}^S(k[\Delta], k).$$

Here, $k[\Delta]$ denotes the *Stanley–Reisner ring*

$$k[\Delta] = S/(x_{i_1} \cdots x_{i_k} \mid \{i_1, \ldots, i_k\} \notin \Delta)$$

of the simplicial complex Δ. Note that $k[\Delta]$ is a square-free monomial ring. In general, the homotopy type of Z_Δ is not well understood, but significant progress has been made for those Z_Δ where Δ is Golod; see for example Grbić and Theriault [12; 13], Iriye and Kishimoto [15] and Beben and Grbić [4].

The preceding discussion makes clear that the Golod property is of interest in both commutative algebra and algebraic topology. Consequently, a lot of work has been done on the Golodness problem. For example, a combinatorial characterization of Golodness in terms of the homology of the lower intervals in the lattice of saturated subsets is given by Berglund in [5]. Using results from Jöllenbeck [16], it has been claimed in Berglund and Jöllenbeck [6] that R is Golod if and only if the product on $\text{Tor}^S(R, k)$ vanishes. However, recently a counterexample to this claim was found by Katthän in [19], where the error is traced back to [16]. This leads naturally to the central question this work investigates.

Question For which classes of monomial rings R is the Golod property equivalent to the vanishing of the product on $\text{Tor}^S(R, k)$?

A partial answer to this question is given by Theorem 6.5. To answer this question, we develop a new approach to the Golodness problem using A_∞–algebras. An A_∞–algebra is similar to a differential graded algebra (dga), except that associativity only holds up to coherent homotopy. By contrast with dgas, every resolution admits the structure of an A_∞–algebra (as first shown by Burke [9]) hence in particular the minimal free resolution does. The first main result of this paper characterizes vanishing of Massey

products in terms of this A_∞–structure. A monomial ring R is said to satisfy condition (B_r) if all k–Massey products are defined and contain only zero for all $k \leq r$. Denote by K_R the Koszul dga of the monomial ring R. We obtain the following result.

Theorem 4.6 *Let $R = S/I$ be a monomial ring with minimal free resolution F. Let $r \in \mathbb{N}$ and let μ_n be an A_∞–structure on F such that $F \otimes_S k$ and K_R are quasi-isomorphic as A_∞–algebras. Then R satisfies (B_r) if and only if μ_k is minimal for all $k \leq r$.*

Next, we turn our attention to the class of rooted rings. A monomial ring is said to be *rooted* if the minimal free resolution F of R is rooted in the sense of Novik [27]. Rooted resolutions include both the Taylor and Lyubeznik resolutions. Given a rooted ring with rooting map π, we give an explicit A_∞–structure in terms of π.

This A_∞–structure allows us to give a combinatorial characterization of the Golod property for rooted rings as follows. Following [16], we say that R satisfies the *gcd condition* if for all generators m_i and m_j with $\gcd(m_i, m_j) = 1$ there exists an $m_k \neq m_i, m_j$ such that m_k divides $\mathrm{lcm}(m_i, m_j)$. The second main result is then the following.

Theorem 6.5 *Let R be a rooted ring. Then the following are equivalent:*

(1) *The ring R is Golod.*

(2) *The product on $\mathrm{Tor}^S(R, k)$ vanishes.*

(3) *The ring R is gcd.*

In particular, the main result from [6] does hold when restricted to rooted rings.

2 Simplicial resolutions

Let $S = k[x_1, \ldots, x_m]$ and let I be the ideal minimally generated by monomials $m_1, \ldots, m_r$. The *Taylor resolution T* [32] of S/I is constructed as follows. Let E denote the exterior algebra on generators $u_1, \ldots, u_r$. The resolution T has underlying module $S \otimes_k E$. If $J = \{j_1 < \cdots < j_k\} \subseteq \{1, \ldots, r\}$, then we write $u_J = u_{j_1} \cdots u_{j_k}$. Furthermore, we put $m_J = \mathrm{lcm}(m_{j_1}, \ldots, m_{j_k})$. We will also write J^i to denote $\{j_1 < \cdots < \widehat{j_i} < \cdots < j_k\}$. The differential d of T is given by

$$d(u_J) = \sum_{i=1}^{|J|} (-1)^{i+1} \frac{m_J}{m_{J^i}} u_{J^i}.$$

The Taylor resolution admits a multiplication defined by

$$u_I \cdot u_J = \begin{cases} \operatorname{sgn}(I, J)\frac{m_I m_J}{m_{I \cup J}} u_{I \cup J} & \text{if } I \cap J = \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{sgn}(I, J)$ is the sign of the permutation making $I \cup J$ an increasing sequence. This multiplication induces a differential graded algebra (dga) structure on T. The *Tor–algebra* $\operatorname{Tor}^S(S/I, k)$ of S/I is

$$\operatorname{Tor}^S(S/I, k) = \bigoplus_n \operatorname{Tor}^S_n(S/I, k) = \bigoplus_n H_n(T \otimes_S k),$$

where the multiplication is induced by the multiplication on T.

The following method of constructing free resolutions of monomial rings is due to Bayer, Peeva and Sturmfels [3]. Our exposition will follow that of Mermin [26]. Let $\{m_1, \dots, m_r\}$ be a set of monomials. Fix some total order $\prec$ on $\{m_1, \dots, m_r\}$. After relabelling we may assume that $m_1 \prec m_2 \prec \cdots \prec m_r$. Let Δ be a simplicial complex on the vertex set $\{1, \dots, r\}$. By abuse of notation, we will say Δ is a simplicial complex on vertex set $\{m_1, \dots, m_r\}$.

Assign a multidegree m_J to each simplex $J \in \Delta$ by defining

$$m_J = \operatorname{lcm}\{m_j \mid j \in J\}.$$

Define a chain complex F_Δ associated to Δ as follows. Let F_n be the free S–module on generators u_J with $|J| = n$. For $J = \{j_1 \prec \cdots \prec j_n\}$, write J^i to denote $\{j_1 \prec \cdots \prec \widehat{j_i} \prec \cdots \prec j_n\}$. The differential $d \colon F_n \to F_{n-1}$ is defined, for $J \in \Delta$, by

$$d(u_J) = \sum_{i=1}^{|J|} (-1)^{i+1} \frac{m_J}{m_{J^i}} u_{J^i}.$$

Example 2.1 Let Δ^r be the full r–simplex. Then F_{Δ^r} is the Taylor resolution of $R = S/I$. This also justifies the use of the same notation for both.

In general, F_Δ need not be a resolution of S/I. But we do have the following theorem.

Theorem 2.2 [3, Lemma 2.2] *Suppose Δ is a simplicial complex on vertex set $\{m_1, \dots, m_r\}$ and define, for a multidegree μ, a subcomplex*

$$\Delta_\mu = \{J \in \Delta \mid m_J \text{ divides } \mu\}.$$

Then F_Δ is a resolution of R if and only if Δ_μ is either acyclic or empty for all multidegrees μ.

A resolution F is called *simplicial* if $F = F_\Delta$ for some simplicial complex Δ.

Remark 2.3 If $\Delta' \subseteq \Delta$, then $F_{\Delta'}$ is a subcomplex of F_Δ. In particular, since each simplicial complex Δ is included in the full simplex on its vertex set, each simplicial resolution of S/I is a subcomplex of the Taylor resolution of S/I.

In the remainder of the paper we will restrict our attention to the following special type of simplicial resolution, which is due to Novik [27]. Given an monomial ideal $I = (m_1, \ldots, m_r)$ we define the *lcm–lattice* $L(I)$ to be the set of all $\mathrm{lcm}(m_{i_1}, \ldots, m_{i_k})$, where $1 \le i_1 \le \cdots \le i_k \le r$ and $k = 1, \ldots, r$. The set $L = L(I)$ admits a partial order given by divisibility. Then L forms a lattice under $a \vee b = \mathrm{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$. The lattice L has minimal element $\hat{0} = 1$ and maximal element $\hat{1} = \mathrm{lcm}(m_1, \ldots, m_r)$.

Definition 2.4 A *rooting map* on L is a map $\pi \colon L \setminus \{\hat{0}\} \to \{m_1, \ldots, m_r\}$ such that

   (1) $\pi(m)$ divides m for every $m \in L$,

   (2) $\pi(m) = \pi(n)$ whenever $\pi(m)$ divides n and n divides m.

Let π be a rooting map and $A \subseteq \{m_1, \ldots, m_r\}$ nonempty. Define $\pi(A) = \pi(\mathrm{lcm}(A))$. A set A is *unbroken* if $\pi(A) \in A$ and A is *rooted* if every nonempty $B \subseteq A$ is unbroken. Let $\mathrm{RC}(L, \pi)$ denote the set of all rooted sets with respect to L and π. Then $\mathrm{RC}(L, \pi)$ is easily seen to be a simplicial complex on vertex set $\{m_1, \ldots, m_r\}$ and we have the following result.

Theorem 2.5 [27, Theorem 1] *Let $I = (m_1, \ldots, m_r)$ be a monomial ideal and let L denote its lcm–lattice. Suppose that π is a rooting map on L. Then the chain complex $F_{\mathrm{RC}(L,\pi)}$ associated to the simplicial complex $\mathrm{RC}(L, \pi)$ is a free resolution of I.*

An important special case of this construction is the Lyubeznik resolution:

Definition 2.6 Let $I = (m_1, \ldots, m_r)$ be a monomial ideal and pick some total order $\prec$ on the m_i. After relabelling we may assume that $m_1 \prec m_2 \prec \cdots \prec m_r$. Define

$$\pi(A) = \min_\prec \{m_i \mid m_i \text{ divides } \mathrm{lcm}(A)\}.$$

Then π is easily seen to be a rooting map. The resolution associated $\mathrm{RC}(L, \pi)$ is called the *Lyubeznik resolution*.

We only consider resolutions F that are as small as possible in the sense that each F_n has the minimal number of generators. More precisely, we have the following definition.

Definition 2.7 Let S/I be a monomial ring. A free resolution $F \to S/I$ is said to be *minimal* if $d(F) \subseteq (x_1, \ldots, x_m)F$.

If the minimal free resolution of S/I is a resolution associated to $\mathrm{RC}(L, \pi)$ for some rooting map π, then I (respectively S/I) is called a *rooted ideal* (respectively a *rooted ring*). Similarly, if the Lyubeznik resolution of S/I is minimal then I (respectively S/I) is called a *Lyubeznik ideal* (respectively a *Lyubeznik ring*).

Example 2.8 Let $S = k[x, y, z]$ and let I be the ideal generated by $m_1 = xy$, $m_2 = yz$ and $m_3 = xz$. Order the generators as $m_1 \prec m_2 \prec m_3$. Let π be the rooting map of the Lyubeznik resolution as in [Definition 2.6](#). Then the rooted sets are m_1, m_2, m_3, m_1m_2 and m_1m_3. So the Lyubeznik resolution is

$$S^2 \xrightarrow{d_2} S^3 \xrightarrow{d_1} S,$$

where the differential is given by

$$d_1 = \begin{bmatrix} xy & yz & xz \end{bmatrix} \quad \text{and} \quad d_2 = \begin{bmatrix} -z & -z \\ x & 0 \\ 0 & y \end{bmatrix}.$$

In particular, the resolution is minimal and so I is a Lyubeznik ideal.

We point out that the class of rooted rings is fairly general. It includes for example monomial ideals whose lcm–lattice is a geometric lattice as well as matroid ideals of modular matroids [27]. The inclusion of Lyubeznik rings in rooted rings is strict since not every rooting map arises from a total order on the monomial generators as Example 4.1 of [7] shows. Finally, not every monomial ring is rooted. Let I be the ideal with monomial generators

$$x_1x_4x_5x_6, \quad x_2x_4x_5x_6, \quad x_3x_4x_5x_6, \quad x_2x_4x_5x_7, \quad x_3x_4x_5x_7$$

$$x_1x_3x_5x_7, \quad x_1x_2x_4x_7, \quad x_1x_4x_6x_7, \quad x_1x_5x_6x_7, \quad x_3x_4x_6x_7$$

$$x_2x_5x_6x_7, \quad x_2x_3x_6x_7, \quad x_1x_2x_3x_7,$$

and let F denote the minimal free resolution. As is shown in [28], the matrices of the differential of F cannot be chosen in $\{0, \pm 1\}$ and consequently F cannot be supported on *any* simplicial complex and hence, in particular, not on a complex $\mathrm{RC}(L, \pi)$ coming from a rooting map π.

3 A_∞–algebras

In this section we will discuss some basic aspects of the theory of A_∞–algebras. The notion was first introduced by Stasheff [29; 30] in the context of algebraic topology. Since their introduction A_∞–algebras have found applications in various branches of mathematics such as geometry [10], algebra [31] and mathematical physics [21; 24]. Though the following section aims to be self-contained, a more extensive introduction can be found in [20]. The exposition below follows that of [22].

In what follows all signs are determined by the *Koszul sign convention*

(1) $$(f \otimes g)(x \otimes y) = (-1)^{|g| \cdot |x|} fx \otimes gy.$$

Definition 3.1 Let R be a commutative ring and $A = \bigoplus A_n$ a $\mathbb{Z}$–graded free R–module. An A_∞–algebra structure on A consists of maps $\mu_n \colon A^{\otimes n} \to A$ for each $n \geq 1$ of degree $n - 2$ satisfying the *Stasheff identities*

(2) $$\sum (-1)^{r+st} \mu_u (1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = 0,$$

where the sum runs over all decompositions $n = r + s + t$ with $r, t \geq 0$, $s \geq 1$ and $u = r + t + 1$.

Observe that when applying (2) to an element additional signs appear because of the Koszul sign convention (1). In the special case when $\mu_3 = 0$, it follows that μ_2 is strictly associative and so A is a differential graded algebra with differential μ_1 and multiplication μ_2. An A_∞–algebra A is called *strictly unital* if there exists an element $1 \in A$ that is a unit for μ_2 and such that for all $n \neq 2$

$$\mu_n(a_1 \otimes \cdots \otimes a_n) = 0$$

whenever $a_i = 1$ for some i.

The notion of a morphism between A_∞–algebras will also be needed.

Definition 3.2 Let (A, μ_n) and $(B, \bar{\mu}_n)$ be A_∞–algebras. A *morphism* of A_∞–algebras (or an A_∞–*morphism*) $f \colon A \to B$ is a family of linear maps

$$f_n \colon A^{\otimes n} \to B$$

of degree $n - 1$ satisfying the *Stasheff morphism identities*

(3) $$\sum (-1)^{r+st} f_u (1^{\otimes r} \otimes \mu_s \otimes 1^{\otimes t}) = \sum (-1)^w \bar{\mu}_q (f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q})$$

for every $n \geq 1$. The first sum runs over all decompositions $n = r + s + t$ with $s \geq 1$ and $r, t \geq 0$, where $u = r + t + 1$. The second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + i_2 + \cdots + i_q$ with all $i_s \geq 1$. The sign on the right-hand side of (3) is given by

$$w = \sum_{p=1}^{q-1}(q-p)(i_p - 1).$$

If A and B are strictly unital, an A_∞–morphism is also required to satisfy $f_1(1) = 1$ and

$$f_n(a_1 \otimes \cdots \otimes a_n) = 0$$

if $n \geq 2$ and $a_i = 1$ for some i.

A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism in the usual sense.

Let A be an A_∞–algebra. Then its homology HA is an associative algebra. A crucial result relating the A_∞–algebra A and its homology algebra HA is the *homotopy transfer theorem*.

Theorem 3.3 (homotopy transfer theorem [17]; see also [25]) *Let (A, μ_n) be an A_∞–algebra over a field R and let HA be its homology algebra. There exists an A_∞–algebra structure μ_n' on HA such that*

(1) *$\mu_1' = 0$ and $\mu_2' = H(\mu_2)$ and the higher μ_n' are determined by μ_n,*

(2) *there exists an A_∞–quasi-isomorphism $HA \to A$ lifting the identity morphism of HA.*

Moreover, this A_∞–structure is unique up to isomorphism of A_∞–algebras.

An explicit way of constructing A_∞–structures on the homology of a dga is due to Merkulov [25] and will be discussed next.

Definition 3.4 Let A be a chain complex and $B \subseteq A$ a subcomplex. A *transfer diagram* is a diagram of the form

(4)
$$B \underset{p}{\overset{i}{\rightleftarrows}} A \circlearrowright \phi$$

where $pi = 1_B$ and $ip - 1 = d\phi + \phi d$.

Some authors use the term strong deformation retract for what we call a transfer diagram.

Let (A, d) be a dga and let B be a subcomplex of A such that there exists a transfer diagram as in (4). Let $\cdot$ denote the product on A. Define linear maps $\lambda_n \colon A^{\otimes n} \to A$ as follows. First, put $\lambda_2(a_1, a_2) = a_1 \cdot a_2$ and we set

$$
(5) \qquad \lambda_n = \sum_{\substack{s+t=n \\ s,t \geq 1}} (-1)^{s+1} \lambda_2(\phi \lambda_s, \phi \lambda_t).
$$

Now, define a second series of maps $\mu_n \colon B^{\otimes n} \to B$ by setting $\mu_1 = d$ and, for $n \geq 2$,

$$
(6) \qquad \mu_n = p \circ \lambda_n \circ i^{\otimes n}.
$$

The following theorem will be crucial in the remainder of the paper.

Theorem 3.5 [25, Theorem 3.4] *Let (A, d) be a dga and B a subcomplex of A such that there exists a transfer diagram of the form (4). Then the maps μ_n defined in (6) give the structure of an A_∞–algebra on B.*

4 A_∞–resolutions and the Golod property

Let R be a monomial ring. Recall R is *Golod* if there is an equality of power series

$$
(7) \qquad P(R) = \frac{(1+t)^m}{1 - t\left(\sum_{j=0}^{\infty} \dim \operatorname{Tor}_j^S(R, k) t^j - 1\right)}.
$$

The Golod property admits an equivalent description in terms of Massey products, which will be defined next.

Definition 4.1 Let (A, d) be a differential graded algebra. If $a \in A$, we write $\bar{a}$ to denote $(-1)^{\deg(a)+1} a$.

Let $\alpha_1, \alpha_2 \in HA$. The length-2 *Massey product* $\langle \alpha_1, \alpha_2 \rangle$ is the product $\alpha_1 \alpha_2$ in the homology algebra HA.

Let $\alpha_1, \ldots, \alpha_n \in HA$ be homology classes with the property that each length-$(j-i+1)$ Massey product $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains zero for $i < j$ and $j - i < n - 1$. A *defining system* $\{a_{ij}\}$ consists of

(1) representing cycles $a_{i-1,i}$ of the homology classes α_i, for $i = 1, \ldots, n$,

(2) elements a_{ij}, for $j > i + 1$, such that

$$
da_{ij} = \sum_{i < k < j} \bar{a}_{ik} a_{kj}.
$$

Note that the existence is guaranteed by the condition that $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains zero for $i < j$ and $j - i < n - 1$. The length-n *Massey product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined as the set

$$\langle \alpha_1, \ldots, \alpha_n \rangle = \left\{ \left[\sum_{0 < i < n} \bar{a}_{0i} a_{in} \right] \,\middle|\, \{a_{ij}\} \text{ is a defining system} \right\} \subseteq H^{s+2-n},$$

where $s = \sum_{i=1}^{n} \deg \alpha_i$.

A Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is said to be *trivial* if it contains zero. The *Koszul homology* of a monomial ring R is $H(R) = \mathrm{Tor}^S(R, k)$. The Golod property and Massey products are related by the following theorem.

Theorem 4.2 ([11]; see also [14, Section 4.2]) *Let R be a monomial ring. Then R is Golod if and only if all Massey products on the Koszul homology $\mathrm{Tor}^S(R, k)$ are trivial.*

Following [18], we will say that a dga A satisfies *condition* (B_r) if all k–ary Massey products are defined and contain only zero for all $k \leq r$. Recall the following lemma.

Lemma 4.3 [23, Proposition 2.3] *Let A be a dga satisfying (B_{r-1}). Then the k–ary Massey product $\langle a_1, \ldots, a_r \rangle$ is defined and contains only one element for any choice $a_1, \ldots, a_r \in H(A)$.*

Let R be a monomial ring and let K_S be the Koszul resolution of the base field k over S. The *Koszul dga* K_R of R is defined as $K_R = R \otimes_S K_S$. The Koszul dga and the Taylor resolution are related by a zigzag of dga quasi-isomorphisms

$$T \otimes_S k \longleftarrow T \otimes_S K_S \longrightarrow R \otimes_S K_S = K_R.$$

Consequently, Massey products on $\mathrm{Tor}^S(R, k)$ can be computed using either K_R or $T \otimes_S k$. Again following [18], we say that a monomial ring R satisfies (B_r) if the dga K_R of R satisfies (B_r).

Lemma 4.4 *Let R be a monomial ring. Then R is Golod if and only if R satisfies condition (B_r) for all $r \in \mathbb{N}$.*

Proof It is clear that if R satisfies condition (B_r) for every r then R is Golod. Conversely, suppose that R is Golod. We proceed by induction on r. The case $r = 2$ is trivial. So assume R satisfies (B_{r-1}). By Lemma 4.3, the Massey product $\langle a_1, \ldots, a_r \rangle$

is defined and contains only one element for any choice $a_1, \ldots, a_r \in \mathrm{Tor}^S(R, k)$. Since R is Golod, it follows by Theorem 4.2 that this element must be zero and so R satisfies (B_r). □

In general it is very hard to study Massey products directly. However, A_∞–algebras provide a systematic way of studying Massey products in view of the following theorem.

Theorem 4.5 [22, Theorem 3.1] *Let A be a differential graded algebra. Up to a sign, the higher A_∞–multiplications μ_n' on HA from Theorem 3.3 give Massey products. That is to say, if $\alpha_1, \ldots, \alpha_n \in HA$ are homology classes such that the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$ is defined then*

$$\pm \mu_n'(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \ldots, \alpha_n \rangle.$$

A map of S–modules $f \colon M \to N$ is said to be minimal if $f \otimes 1 \colon M \otimes_S k \to N \otimes_S k$ is zero. It is readily verified that f is minimal if and only if f maps into $(x_1, \ldots, x_m)N$. Using Theorem 4.5, we can describe under what conditions the Massey products on $\mathrm{Tor}^S(R, k)$ vanish.

Theorem 4.6 *Let $R = S/I$ be a monomial ring with minimal free resolution F. Let $r \in \mathbb{N}$ and let μ_n be an A_∞–structure on F such that $F \otimes_S k$ and K_R are quasi-isomorphic as A_∞–algebras. Then R satisfies (B_r) if and only if μ_k is minimal for all $k \leq r$.*

Proof Since μ_n is an A_∞–structure on F, it follows that $\mu_n \otimes 1$ is an A_∞–structure on $F \otimes_S k$. Now, assume μ_n is minimal for all $k \leq r$. Since $\mathrm{Tor}^S(R, k)$ is the homology of the A_∞–algebra $F \otimes_S k$ the homotopy transfer theorem (Theorem 3.3) implies that $\mathrm{Tor}^S(R, k)$ inherits an A_∞–structure μ_n'. Since F is minimal, $\mathrm{Tor}^S(R, k)$ is isomorphic to $F \otimes_S k$ and we can take $\mu_n' = \mu_n \otimes 1$. Let $k \leq r$ and let $\alpha_1, \ldots, \alpha_k \in \mathrm{Tor}^S(R, k)$ be such that the Massey product $\langle \alpha_1, \ldots, \alpha_k \rangle$ is defined. By Theorem 4.5 we have

$$\pm(\mu_k \otimes 1)(\alpha_1, \ldots, \alpha_k) \in \langle \alpha_1, \ldots, \alpha_k \rangle.$$

Since μ_k is minimal, we have $(\mu_k \otimes 1)(\alpha_1, \ldots, \alpha_k) = 0$. Therefore, $\langle \alpha_1, \ldots, \alpha_k \rangle$ is trivial and so R satisfies (B_r).

Conversely, assume that R satisfies (B_r). We need to show that μ_k is minimal for all $k \leq r$. For $k = 2$, we have $(\mu_2 \otimes 1)(a_1, a_2) = a_1 a_2$ but the product on $\mathrm{Tor}^S(R, k)$ is zero because R satisfies (B_r). Now, let $3 \leq k \leq r$. Since R satisfies (B_k),

for all $a_1, \ldots, a_k$ the Massey product $\langle a_1, \ldots, a_k \rangle$ is defined and contains only zero. Since $(\mu_k \otimes 1)(a_1, \ldots, a_k) \in \langle a_1, \ldots, a_k \rangle$ we have $(\mu_k \otimes 1)(a_1, \ldots, a_k) = 0$ for all $a_1, \ldots, a_k$. Consequently, μ_k is minimal as required. □

Corollary 4.7 *Let $R = S/I$ be a monomial ring with minimal free resolution F. Let μ_n be an A_∞–structure on F such that $F \otimes_S k$ and K_R are quasi-isomorphic as A_∞–algebras. Then R is Golod if and only if μ_n is minimal for all $n \geq 1$.*

Corollary 4.7 was first proved in [9] using different methods.

The following immediate corollary to Corollary 4.7 is well known; see for example Proposition 5.2.4(4) of [2], where it is proved using different methods.

Corollary 4.8 [2, Proposition 5.2.4(4)] *Let $R = S/I$ be a monomial ring with minimal free resolution F. If F admits the structure of a dga, then R is Golod if and only if the product on $\mathrm{Tor}^S(R, k)$ vanishes.*

5 Homotopy transfer on the Taylor resolution

Corollary 4.7 implies that monomial rings with minimal dga resolution are Golod if and only if the product on $\mathrm{Tor}^S(S/I, k)$ vanishes. However, there exists monomial rings whose minimal resolution does not admit the structure of a dga [1]. On the other hand, every free resolution of a monomial ring S/I admits an A_∞–structure [9].

In general, it is not clear how to obtain an explicit description of such an A_∞–structure. Instead of considering general A_∞–structures on resolutions, we will consider only those that arise as a deformation of the dga structure on the Taylor resolution. To make this idea precise we will use rooting maps to construct transfer diagrams on the Taylor resolution. In that case Theorem 3.5 tells us how to construct an A_∞–structure to which we may apply Corollary 4.7.

Let π be a rooting map and let F be the free resolution of S/I associated to $\mathrm{RC}(L, \pi)$. Recall that F_n is the free S–module on u_J, where $J \in \mathrm{RC}(L, \pi)$ with $|J| = n$. The remainder of this section is devoted to computing an explicit A_∞–algebra structure on F. Let T will denote the Taylor resolution of S/I. We will write d for the differential of F whereas ∂ will be reserved for the "simplicial" differential, ie

$$\partial u_J = \sum_{i=1}^{|J|} (-1)^{i+1} u_{J^i}$$

on a basis set u_J of F. If u_J is a basis set of F we define $[u_J] = (1/m_J)u_J$.

Let $u_{J_1}, \ldots, u_{J_n}$ be rooted sets and $\alpha_1, \ldots, \alpha_n \in S$. Then for $u = \sum \alpha_k u_{J_k}$, we set $[u] = \sum (\alpha_k / m_{J_k}) u_{J_k}$. The following lemma will be used extensively.

Lemma 5.1 *For any basis set u_J of F we have $d[u_J] = [\partial u_J]$.*

Proof We have

$$d[u_J] = \frac{1}{m_J} du_J = \frac{1}{m_J} \sum_{i=1}^{|J|} (-1)^{i+1} \frac{m_J}{m_{J^i}} u_{J^i}$$

$$= \sum_{i=1}^{|J|} (-1)^{i+1} \frac{1}{m_{J^i}} u_{J^i} = \sum_{i=1}^{|J|} (-1)^{i+1} [u_{J^i}]$$

$$= [\partial u_J]. \qquad \square$$

Let π be a rooting map. For $u_J \in T$, define $\pi(u_J) = u_i$ if $\pi(m_J) = m_i$. Define a map $p': T \to F$ as follows. Let $u \in T$ and write $u = u_{i_1} \cdots u_{i_k}$. For $q = 1, \ldots, k$ define $I_q = \{i_1, \ldots, i_q\}$. For a permutation $\sigma \in S_k$, put $\sigma I_q = \{i_{\sigma(1)}, \ldots, i_{\sigma(q)}\}$. We define

$$(8) \qquad p'(u) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \pi(u_{\sigma I_1}) \pi(u_{\sigma I_2}) \cdots \pi(u_{\sigma I_k}).$$

Geometrically, the map p' can be thought of as similar to the barycentric subdivision of a simplex. For example, if $u_{i_1,i_2} \in T$ and we think of $\pi(u_{i_1,i_2})$ as its barycenter then p' replaces u_{i_1,i_2} by its barycentric subdivision

$$p'(u_{i_1,i_2}) = u_{i_2} \pi(u_{i_1,i_2}) - u_{i_1} \pi(u_{i_1,i_2}).$$

In the same way, given $u_{i_1,i_2,i_3} \in T$ the right-hand terms in

$$p(u_{i_1,i_2,i_3}) = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) \pi(u_{i_{\sigma(1)}}) \pi(u_{i_{\sigma(1)},i_{\sigma(2)}}) \pi(u_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}})$$

are precisely the six constituent triangles in the barycentric subdivision of a 2–simplex. Before proceeding, we need to verify that $\mathrm{im}(p') \subseteq F$. For $\sigma \in S_k$, we need to show that

$$\pi(u_{\sigma I_1}) \pi(u_{\sigma I_2}) \cdots \pi(u_{\sigma I_k})$$

is rooted. Since $u_{\sigma I_1} \subseteq u_{\sigma I_2} \subseteq \cdots \subseteq u_{\sigma I_k}$, it follows that for all $j_1, \ldots, j_k$ we have

$$\pi(\pi(u_{I_{j_1}}), \pi(u_{I_{j_2}}), \ldots, \pi(u_{I_{j_k}})) = \pi(u_{I_{j_k}}).$$

Therefore, $\pi(u_{\sigma I_1}) \pi(u_{\sigma I_2}) \cdots \pi(u_{\sigma I_k})$ is rooted and so $\mathrm{im}(p') \subseteq F$.

Lemma 5.2 *The map p' is a chain map with respect to the differential ∂.*

Proof It is sufficient to prove the result for basis elements $u_I \in T$. Write $I = \{i_1, \dots, i_k\}$. We first show that

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

We have

$$\partial p'(u_I) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \partial(\pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_k}))$$

$$= \sum_{\sigma \in S_k} \sum_{j=1}^{k} (-1)^{j+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \widehat{\pi(u_{\sigma I_j})} \cdots \pi(u_{\sigma I_k}).$$

Now, fix some $j < k$ and let τ_j be the transposition $(\sigma(j), \sigma(j+1))$. Then the summands indexed by σ and $\tau_j \sigma$ cancel. Indeed, if $q < j$ then τ_j acts as the identity on σI_q and so $u_{\sigma I_q} = u_{\tau_j \sigma I_q}$. On the other hand, if $q \geq j+1$ then the underlying sets of σI_q and $\tau_j \sigma I_q$ are the same. Since $\pi(u_J)$ depends only on the set J and not on the ordering we have

$$\pi(u_{\sigma I_q}) = \pi(u_{\tau_j \sigma I_q}),$$

and so the summands indexed by σ and $\tau_j \sigma$ cancel. Note that since the map $\sigma \to \tau_j \sigma$ is an involution these permutations cancel in pairs. Therefore, we obtain

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

For $\sigma \in S_k$, write

$$G_\sigma = \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}),$$

and so

(9) $$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) G_\sigma.$$

We now compute $p' \partial(u_I)$. For $j \in \{1, \dots, k\}$ and $\sigma \in S_{k-1}$, set $I_q(j) = I_q \setminus \{j\}$ and

$$F_{\sigma,j} = \pi(u_{\sigma I_1(j)}) \cdots \pi(u_{\sigma I_{j-1}(j)}) \pi(u_{\sigma I_{j+1}(j)}) \cdots \pi(u_{\sigma I_k(j)}).$$

Then

(10) $$p' \partial u = \sum_{j=1}^{k} (-1)^{j+1} p'(u_{I_k(j)}) = \sum_{j=1}^{k} \sum_{\sigma \in S_{k-1}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}.$$

Given $j \in \{1, \ldots, k\}$, we can embed S_{k-1} into S_k by fixing j. Therefore, we have

$$p'\partial u = \sum_{j=1}^{k} \sum_{\sigma \in S_{k-1}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j} = \sum_{j=1}^{k} \sum_{\substack{\sigma \in S_k \\ \sigma(j)=j}} (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}.$$

Now, fix $j \in \{1, \ldots, k\}$ and fix $\sigma \in S_k$ such that $\sigma(j) = j$. Define ρ to be the cycle $(j \cdots k)$ and let $\tau = \sigma\rho$. Then we have $G_\tau = F_{\sigma,j}$ and

$$(-1)^{k+1} \operatorname{sgn}(\tau) G_\tau = (-1)^{2k+j+1} G_{\sigma\rho} = (-1)^{j+1} \operatorname{sgn}(\sigma) F_{\sigma,j}.$$

Since the sums in both (9) and (10) have $k!$ terms, it follows that they are equal. □

Let $i\colon F \to T$ denote the inclusion.

Lemma 5.3 *For all $u \in T$, we have*

$$\pi(u) i p' \partial u = i p' u.$$

Proof It is sufficient to prove the result for basis elements $u_I \in T$. Write $I = \{i_1, \ldots, i_k\}$. As in the proof of Lemma 5.2, we have

$$\partial p'(u_I) = \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}).$$

Since p' is a chain map by Lemma 5.2, we have

$$
\begin{aligned}
\pi(u_I) i p' \partial u_I &= \pi(u_I) \partial i p'(u_I) \\
&= \pi(u_I) \sum_{\sigma \in S_k} (-1)^{k+1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \\
&= \sum_{\sigma \in S_k} (-1)^{k+1+k-1} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \pi(u_I) \\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_{k-1}}) \pi(u_{\sigma I_k}) \\
&= i p'(u_I),
\end{aligned}
$$

where we have used that $\pi(u_I) = \pi(u_{I_k}) = \pi(u_{\sigma I_k})$. □

Lemma 5.4 *The composition $i p'$ and the identity 1_T are chain homotopic as chain maps $(T, \partial) \to (T, \partial)$.*

Proof Define $\phi'\colon T \to T$ by induction as follows. Set $\phi'_0 = \phi'_1 = 0$ and

$$\phi'_2(u_{i_1} u_{i_2}) = \pi(u_{i_1, i_2}) u_{i_1} u_{i_2}.$$

For $k > 2$, write $u = u_{i_1} \cdots u_{i_k}$ and define

$$\phi'_k(u) = \pi(u)(u - \phi'_{k-1}(\partial u)).$$

We need to show that $1_T - i\, p' = \partial \phi' + \phi' \partial$. We proceed by induction on k. If $k = 1$, there is nothing to prove. If $k = 2$, we have

$$\begin{aligned}
\partial \phi_2(u_{i_1} u_{i_2}) &= \partial(\pi(u_{i_1,i_2})u_{i_1} u_{i_2}) \\
&= u_{i_1} u_{i_2} - \pi(u_{i_1,i_2})u_{i_2} + \pi(u_{i_1,i_2})u_{i_1} \\
&= (1_F - i\, p')(u_{i_1} u_{i_2}).
\end{aligned}$$

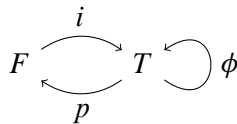Now, let $k > 2$. Using Lemma 5.3, we finish the proof:

$$\begin{aligned}
\partial \phi'_k(u) &= u - \phi'_{k-1}\partial u - \pi(u)(\partial u - \partial \phi'_{k-1}\partial u) \\
&= u - \phi'_{k-1}\partial u - \pi(u)(\partial u - \partial u + i\, p'\partial u + \phi_{k-2}\partial^2 u) \\
&= u - \phi_{k-1}\partial u - \pi(u)i\, p'\partial u \\
&= u - i\, p'u - \phi_{k-1}\partial u. \qquad \square
\end{aligned}$$

Define a map $p\colon T \to F$ as follows. For $u_J \in T$, let

$$(11) \qquad\qquad p(u_J) = m_J[p'(u_J)],$$

where p' is the map from (8). Then we have the following theorem.

Theorem 5.5 *Let π be a rooting map for a monomial ideal I and let F be the resolution of S/I associated to π. Then there exists a transfer diagram*

where $i\colon F \to T$ is the inclusion and $p\colon T \to F$ is the map from (11).

Proof Let $u_J \in T$ and define ϕ by $\phi(u_J) = m_J[\phi'(u_J)]$. Then, using Lemma 5.4,

$$\begin{aligned}
d\phi(u_J) &= m_J d[\phi'(u_J)] = m_J[\partial \phi'(u_J)] = m_J[u_J - i\, p'u_J - \phi'\partial u_J] \\
&= u_J - i\, pu_J - \phi d u_J,
\end{aligned}$$

and so 1_T and $i\, p$ are homotopic. On the other hand, we clearly have $pi = 1_F$, which finishes the proof. $\qquad \square$

6 The Golod property for rooted rings

Let $R = S/I$ be a rooted ring with rooting map π and minimal free resolution F. The purpose of this section is to provide necessary and sufficient conditions for R being Golod. Following [16], we have the following definition.

Definition 6.1 Let $R = S/I$ be a monomial ring and write $I = (m_1, \ldots, m_r)$. We say that R satisfies the *gcd condition* if for all generators m_i and m_j with $\gcd(m_i, m_j) = 1$ there exists an $m_k \neq m_i, m_j$ such that m_k divides $\operatorname{lcm}(m_i, m_j)$.

We have the following lemma, where we write $\pi(m_i, m_j)$ for $\pi(\{m_i, m_j\})$.

Lemma 6.2 Let $R = S/I$ be a rooted ring with rooting map π and write $I = (m_1, \ldots, m_r)$. Then R satisfies the gcd condition if and only if $\pi(m_i, m_j) \neq m_i, m_j$ whenever $\gcd(m_i, m_j) = 1$.

Proof First, assume that $\pi(m_i, m_j) \neq m_i, m_j$ whenever $\gcd(m_i, m_j) = 1$. Since $\pi(m_i, m_j)$ divides $\operatorname{lcm}(m_i, m_j)$, we can take $m_k = \pi(m_i, m_j)$ and so R satisfies the gcd condition.

Conversely, suppose that R satisfies the gcd condition and take m_i and m_j with $\gcd(m_i, m_j) = 1$. For contradiction, assume that $\pi(m_i, m_j) = m_i$. By the gcd condition, there exists some $m_k \neq m_i, m_j$ such that m_k divides $\operatorname{lcm}(m_i, m_j)$. We claim that the set $\{m_i, m_j, \pi(m_j, m_k)\}$ is rooted. To prove this, we need to verify that every subset is unbroken. Since $\pi(m_i, m_j) = m_i$, it follows immediately that $\{m_i, m_j\}$ is unbroken. For $\{m_j, \pi(m_j, m_k)\}$, note that

$$\pi(m_j, m_k) \mid \operatorname{lcm}(m_j, \pi(m_j, m_k)) \mid \operatorname{lcm}(m_j, m_k),$$

and so $\pi(m_j, \pi(m_j, m_k)) = \pi(m_j, m_k)$ as π is a rooting map. Thus, $\{m_j, \pi(m_j, m_k)\}$ is unbroken. Next, consider $\{m_i, \pi(m_j, m_k)\}$. Since $\pi(m_i, m_j) = m_i$, we have

$$\pi(m_i, m_j) \mid \operatorname{lcm}(m_i, \pi(m_j, m_k)) \mid \operatorname{lcm}(m_i, m_j),$$

and so $\pi(m_i, \pi(m_j, m_k)) = \pi(m_i, m_j) = m_i$. Consequently, $\{m_i, \pi(m_j, m_k)\}$ is unbroken. Similarly, we have that $\{m_i, m_j, \pi(m_j, m_k)\}$ is unbroken as

$$\pi(m_i, m_j) \mid \operatorname{lcm}(m_i, m_j, \pi(m_j, m_k)) \mid \operatorname{lcm}(m_i, m_j)$$

and thus $\pi(m_i, m_j, \pi(m_j, m_k)) = \pi(m_i, m_j) = m_i$. Therefore, $\{m_i, m_j, \pi(m_j, m_k)\}$ is rooted as claimed.

Let $u = u_i u_j \pi(u_j, u_k)$. Since $\pi(m_j, m_k)$ divides $\mathrm{lcm}(m_i, m_j)$, we have

$$du = \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(m_j, \pi(m_j, m_k))} u_j \pi(u_j, u_k) - \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(m_i, \pi(m_j, m_k))} u_i \pi(u_j, u_k) + u_i u_j.$$

Hence, $du \notin (x_1, \ldots, x_m)F$, which is a contradiction as R is rooted. Therefore, $\pi(m_i, m_j) \neq m_i$. Swapping the roles of i and j, we see that $\pi(m_i, m_j) \neq m_j$, which finishes the proof. $\square$

The following lemma is straightforward but included for completeness.

Lemma 6.3 *If u_I and u_J are basis elements of T such that $\gcd(m_I, m_J) \neq 1$, then*
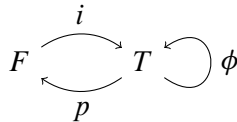
$$p\lambda_2(u_I, u_J) \in (x_1, \ldots, x_m)F.$$

Proof Indeed, we have

$$p\lambda_2(u_I \otimes u_J) = p\left(\frac{m_I m_J}{m_{I \cup J}} u_{I \cup J}\right) = \frac{m_I m_J}{m_{I \cup J}} p(u_{I \cup J}).$$

By assumption $(m_I m_J)/(m_{I \cup J}) \neq 1$ and so the result follows. $\square$

Lemma 6.4 *Let R be a rooted ring. If R is gcd then R is Golod.*

Proof Let F be the minimal free resolution of R. Then by Theorem 5.5 there is a transfer diagram

$$F \underset{p}{\overset{i}{\rightleftarrows}} T \circlearrowright \phi$$

where $i\colon F \to T$ is the inclusion and $p\colon T \to F$ is the map from (11). By Theorem 3.5, we obtain an A_∞–structure μ_n on F. From Corollary 4.7 it follows that it is sufficient to show that each μ_n is minimal. Recall that $\mu_n = p\lambda_n$, where

$$\lambda_n = \sum_{\substack{s+t=n \\ s,t \geq 1}} (-1)^{s+1} \lambda_2(\phi\lambda_s \otimes \phi\lambda_t).$$

Thus, it is sufficient to prove that $p\lambda_2$ maps into the maximal ideal. Let u_I and u_J be basis elements of T. We may assume that $\gcd(m_I, m_J) = 1$ since otherwise $p\lambda_2(u_I \otimes u_J) \in (x_1, \ldots, x_m)F$ by Lemma 6.3. Write $I = \{i_1, \ldots, i_k\}$ and $J = \{i_{k+1}, \ldots, i_n\}$, where $n = k + l$. By definition of p we have

$$p(u_{i_1} \cdots u_{i_n}) = m \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_n}),$$

where $m = \mathrm{lcm}(m_I, m_J) = m_I m_J$ and $u_{\sigma I_p} = u_{i_{\sigma(1)}} \cdots u_{i_{\sigma(p)}}$. Write

$$\alpha_\sigma = \frac{m}{\mathrm{lcm}(\pi(m_{\sigma I_1}), \ldots, \pi(m_{\sigma I_n}))};$$

then

$$p(u_{i_1} \cdots u_{i_n}) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\alpha_\sigma \pi(u_{\sigma I_1}) \cdots \pi(u_{\sigma I_n}).$$

We need to show that $\alpha_\sigma \in (x_1, \ldots, x_m)$ for all $\sigma \in S_n$. Suppose that $\alpha_\sigma = 1$ for some $\sigma \in S_n$. Without loss of generality, we may assume $i_{\sigma(1)} \in I$. Set

$$q = \min\{q' \mid i_{\sigma(q')} \in J\}.$$

By assumption, $\mathrm{lcm}(\pi(m_{\sigma I_1}), \ldots, \pi(m_{\sigma I_n}))$ is divisible by $m_{i_{\sigma(q)}}$. Therefore, since $\gcd(m_{i_{\sigma(q)}}, m_I) = 1$, we have $\gcd(m_{i_{\sigma(q)}}, \pi(m_{\sigma I_k})) = 1$ for all $k < q$. Therefore, $\mathrm{lcm}(\pi(m_{\sigma I_q}), \ldots, \pi(m_{\sigma I_n}))$ is still divisible by $m_{i_{\sigma(q)}}$.

We claim that

$$m_{i_{\sigma(q)}} \notin \{\pi(m_{\sigma I_q}), \ldots, \pi(m_{\sigma I_n})\}.$$

Indeed, assume that $m_{i_{\sigma(q)}} = \pi(m_{\sigma I_s})$ for some $s \geq q$. We have that $\pi(m_{\sigma I_s}) = \pi(m_{i_{\sigma(1)}}, \ldots, m_{i_{\sigma(s)}})$. Then

$$m_{i_{\sigma(q)}} \mid \mathrm{lcm}(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}}) \mid \mathrm{lcm}(m_{i_{\sigma(1)}}, \ldots, m_{i_{\sigma(s)}}),$$

and so $m_{i_{\sigma(q)}} = \pi(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}})$ since π is a rooting map. But by definition of q we have $\gcd(m_{i_{\sigma(1)}}, m_{i_{\sigma(q)}}) = 1$ so this contradicts I being gcd by [Lemma 6.2](#). Therefore

$$m_{i_{\sigma(q)}} \notin \{\pi(m_{\sigma I_q}), \ldots, \pi(m_{\sigma I_n})\}.$$

Define

$$u = u_{i_{\sigma(q)}} \pi(u_{\sigma I_q}) \cdots \pi(u_{\sigma I_n});$$

we claim that u is in F. To see that u is rooted, let $v \subseteq \{u_{i_{\sigma(q)}}, \pi(u_{\sigma I_q}), \ldots, \pi(u_{\sigma I_n})\}$. If $u_{i_{\sigma(q)}} \notin v$ then there is nothing to prove as $\{\pi(u_{\sigma I_q}), \ldots, \pi(u_{\sigma I_n})\}$ is rooted. So, assume $u_{i_{\sigma(q)}} \in v$. We can write

$$v = u_{i_{\sigma(q)}} \pi(u_{\sigma I_{q_1}}) \cdots \pi(u_{\sigma I_{q_k}})$$

for some $q_i \geq q$. We have

$$\pi(u_{\sigma I_{q_k}}) \mid m_v \mid m_{\sigma I_{q_k}},$$

and so $\pi(v) = \pi(u_{\sigma I_{q_k}}) \in v$. Hence, u is rooted as claimed. But $du \notin (x_1, \dots, x_m)F$ since $m_{i_{\sigma(q)}}$ divides $\mathrm{lcm}(\pi(m_{\sigma I_q}), \dots, \pi(m_{\sigma I_n}))$, which contradicts minimality of F. □

We now come to the main theorem of this section.

Theorem 6.5 *Let R be a rooted ring. Then the following are equivalent:*

(1) *The ring R is Golod.*

(2) *The product on $\mathrm{Tor}^S(R, k)$ vanishes.*

(3) *The ring R is gcd.*

Proof The implication $(1) \Rightarrow (2)$ is immediate from the definition and $(3) \Rightarrow (1)$ follows by Lemma 6.4. We prove $(2) \Rightarrow (3)$. Since the product on $\mathrm{Tor}^S(R, k)$ is just $\mu_2 \otimes 1$, the product vanishes if and only if μ_2 is minimal. Let m_i and m_j be generators such that $\gcd(m_i, m_j) = 1$. Then

$$\mu_2(u_i, u_j) = \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(\pi(m_i, m_j)m_i)} \pi(u_i, u_j)u_i - \frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(\pi(m_i, m_j)m_j)} \pi(u_i, u_j)u_j.$$

If $\pi(m_i, m_j) = m_j$ then

$$\frac{\mathrm{lcm}(m_i, m_j)}{\mathrm{lcm}(\pi(m_i, m_j)m_j)} = 1,$$

which contradicts minimality of μ_2 and so $\pi(m_i, m_j) \neq m_j$. By the same argument, $\pi(m_i, m_j) \neq m_i$ and thus R is gcd by Lemma 6.2. □

Remark 6.6 The equivalence between the second and third statements of Theorem 6.5 is known. See for example Lemma 2.4 of [19]

Example 6.7 Let $S = k[x_1, \dots, x_9]$ and let I be the ideal

$$(x_2 x_5 x_8, \quad x_2 x_3 x_8 x_9, \quad x_5 x_6 x_7 x_8, \quad x_1 x_2 x_4 x_5, \quad x_1 x_2 x_3, \quad x_4 x_5 x_6, \quad x_7 x_8 x_9).$$

Label the generators by $u_1, \dots, u_9$ and order them by $u_1 \prec u_2 \prec \dots \prec u_9$. Let L be the Lyubeznik resolution with respect to the ordering $\prec$. Then L is easily seen to be minimal. Plainly, I satisfies the gcd condition and so S/I is Golod.

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School of Mathematics, University of Southampton
Southampton, United Kingdom

robinfrankhuizen@gmail.com