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Dynamic characterizations of quasi-isometry and applications to cohomology

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We build a bridge between geometric group theory and topological dynamical systems by establishing a dictionary between coarse equivalence and continuous orbit equivalence. As an application, we show that group homology and cohomology in a class of coefficients, including all induced and coinduced modules, are coarse invariants. We deduce that being of type FP_n (over arbitrary rings) is a coarse invariant, and that being a (Poincaré) duality group over a ring is a coarse invariant among all groups which have finite cohomological dimension over that ring. Our results also imply that every coarse self-embedding of a Poincaré duality group must be a coarse equivalence. These results were only known under suitable finiteness assumptions, and our work shows that they hold in full generality.

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1 Introduction

The philosophy of geometric group theory is to study groups not merely as algebraic objects but from a geometric point of view. There are two ways of developing a geometric perspective, by viewing groups themselves as geometric objects (for instance with the help of their Cayley graphs, which leads to the notion of quasi-isometry) or by studying groups by means of "nice" group actions on spaces which carry some topology or geometry. Once a geometric point of view is taken, an immediate question is: How much of the original algebraic structures is still visible from our new perspective? Or: Which algebraic invariants of groups are quasi-isometry invariants?

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Our goals in this paper are twofold. First, we want to connect the two geometric perspectives mentioned above by giving dynamic characterizations of quasi-isometry, or more generally, coarse equivalence. It turns out that for topological dynamical systems, the concept corresponding to coarse equivalence is given by (modified versions of) continuous orbit equivalence, as introduced in Li [32; 31]. The latter means that we can identify the orbit structure of our dynamical systems in a continuous way. The idea of developing dynamic characterizations of coarse equivalence goes back to Gromov's notion of topological couplings and has been developed further in Sauer [49] and Shalom [51]. Recently, independently from the author, a dynamic characterization of bilipschitz equivalence for finitely generated groups was obtained in Medynets, Sauer and Thom [36], which is a special case of our result.

Secondly, we want to study the behaviour of algebraic invariants of groups under coarse equivalence. More precisely, inspired by a refined, more concrete version of our dynamic characterizations, we produce many new coarse invariants of (co)homological nature. Although the proofs of the latter results — which we present here in their final form — do not rely on the first part of this paper, our dynamic characterizations played a crucial role since they provided the geometric intuition behind our arguments. We generalize the result in Gersten [21] that among groups G satisfying the finiteness condition F_n (ie there exist models for Eilenberg–Mac Lane spaces with finite n–skeleton), the cohomology groups $H^n(G, RG)$ are coarse invariants for all commutative rings R with unit. We show that for a class of coefficients (called res–invariant modules), including all induced and coinduced modules, group homology and cohomology are coarse invariants. In particular, $H^*(G, RG)$ is always a coarse invariant. This answers a question in Mosher, Sageev and Whyte [41] (see [41, Questions after Theorem 2.7]). Our results imply that being of type FP_n over R (ie the trivial RG–module R admits a projective resolution which is finitely generated up to level n) is a coarse invariant. This is a partial generalization of Shalom [51, Theorem 1.7]. A different approach is mentioned in Druțu and Kapovich [17, Theorem 9.61], and the case $R = \mathbb{Z}$ has been treated in Alonso [2]. As a consequence, we obtain that for an arbitrary commutative ring R with unit, the property of being a duality or Poincaré duality group over R is a coarse invariant among all groups which have finite cohomological dimension over R. A group G is called a duality group over R if there is a right RG–module C and an integer $n \geq 0$ with natural isomorphisms $H^k(G, A) \cong H_{n-k}(G, C \otimes_R A)$ for all $k \in \mathbb{Z}$ and all RG–modules A (see Bieri [5, Section 9.2; 4] and Brown [10, Chapter VIII, Section 10]). G is called a Poincaré duality group over R if $C \cong R$ as R–modules. C is called the dualizing module; note that we must have $C \cong H^n(G, RG)$ as right RG–modules. Our result generalizes [21, Corollary 3], as we do not need the finiteness condition F_∞ (ie F_n for all n) and can work over arbitrary rings. Examples of groups which are not duality groups over $\mathbb{Z}$ but over some other ring can be found in Davis [14], and examples of (Poincaré) duality groups which are not of type F_∞ appear in [14] and Leary [29]. We should also point out that a notion of coarse Poincaré duality group has been introduced in Mosher, Sageev and Whyte [40; 41], based on Kapovich and Kleiner [27]. However, these groups have to be finitely presented, while our results apply to arbitrary (Poincaré) duality groups (see [14; 29] for examples of Poincaré duality groups which are not finitely presented). Moreover, combined with Sauer's result [49, Theorem 1.2(ii)], we obtain that among amenable groups, being a (Poincaré) duality group over a divisible ring is a coarse invariant. This generalizes [49, Theorem 3.3.2]. We also prove a rigidity result for coarse embeddings into Poincaré duality groups. If a group G with $\mathrm{hd}_R\, G < \infty$ coarsely embeds into a Poincaré duality group H via a coarse embedding which is not a coarse equivalence, then $\mathrm{hd}_R\, G < \mathrm{cd}_R\, H$. In particular, coarse self-embeddings of Poincaré duality groups over an arbitrary ring must be coarse equivalences. Such a coarse co-Hopfian property has been studied in Kapovich and Lukyanenko [26] and Merenkov [37], but it has not been established for general Poincaré duality groups.

Let us now formulate and explain our main results in more detail. At the same time, we fix some notation. Throughout this paper, all our groups are countable and discrete. First, we recall the notion of coarse maps (see Roe [48, Definition 2.21]). Note that coarse embeddings in our sense are called uniform embeddings in [49; 51].

Definition 1.1 A map $\varphi\colon G \to H$ between two groups G and H is called a coarse map if $\varphi^{-1}(\{y\})$ is finite for all $y \in H$, and for every $S \subseteq G \times G$ with $\{st^{-1} : (s,t) \in S\}$ finite, $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\}$ is finite.

Further, $\varphi\colon G \to H$ is called a coarse embedding if for every subset $S \subseteq G \times G$, $\{st^{-1} : (s,t) \in S\}$ is finite if and only if $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\}$ is finite.

Two maps $\varphi, \phi\colon G \to H$ are called close if $\{\varphi(x)\phi(x)^{-1} : x \in G\}$ is finite. We write $\varphi \sim \phi$ in that case.

A coarse map $\varphi\colon G \to H$ is called a coarse equivalence if it is coarsely invertible, ie there is a coarse map $\psi\colon H \to G$ such that $\psi \circ \varphi \sim \mathrm{id}_G$ and $\varphi \circ \psi \sim \mathrm{id}_H$.

We say that two groups G and H are coarsely equivalent if there is a coarse equivalence $G \to H$.

Clearly, coarse embeddings are coarse maps. Examples of coarse embeddings are subgroup embeddings and quasi-isometric embeddings. For finitely generated groups, coarse equivalences coincide with quasi-isometries (see [51]). Unlike in [51; 49], in our definition, we use st^{-1} and not $s^{-1}t$ (see Remark 2.1).

Let us explain our dynamic characterizations of coarse embeddings and equivalences. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topological dynamical systems, where the groups act by homeomorphisms on locally compact Hausdorff spaces. A continuous orbit couple is a pair of continuous maps $p\colon X \to Y$ and $q\colon Y \to X$, which both preserve orbits in a continuous way, such that p and q are inverses up to orbits (ie $q(p(x))$ lies in the same G–orbit of x and similarly for $p \circ q$). "Preserving orbits in a continuous way" is made precise by continuous maps $a\colon G \times X \to H$ such that $p(g.x) = a(g,x).p(x)$ for all $g \in G$ and $x \in X$. If p and q are actual inverses (ie $q \circ p = \mathrm{id}_X$ and $p \circ q = \mathrm{id}_Y$), then our dynamical systems are called continuously orbit-equivalent.

Our first main result establishes the following dictionary: The existence of a coarse embedding $G \to H$ corresponds to the existence of a continuous orbit couple for topologically free systems $G \curvearrowright X$ and $H \curvearrowright Y$, where X is compact. The existence of a coarse equivalence $G \to H$ corresponds to the existence of a continuous orbit couple for topologically free systems $G \curvearrowright X$ and $H \curvearrowright Y$, where both X and Y are compact, and we can find a bijective coarse equivalence $G \to H$ if and only if we can find a continuous orbit equivalence for $G \curvearrowright X$ and $H \curvearrowright Y$. We refer to Theorem 2.17 for precise statements.

It turns out that for compact X, the existence of a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$ is equivalent to saying that $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent, ie there are clopen subspaces $A \subseteq X$ and $B \subseteq Y$ which are G– and H–full such that the partial actions $G \curvearrowright A$ and $H \curvearrowright B$ are continuously orbit-equivalent (in the sense of [31]). This implies that the transformation groupoids of $G \curvearrowright X$ and $H \curvearrowright Y$ are Morita equivalent. Building on this observation, we give conceptual explanations for the results in [51; 49] on coarse invariance of (co)homological dimension and Shalom's property H_{FD}.

The dynamic characterizations we described so far are abstract as the dynamical systems are not specified. It is striking that even such abstract characterizations suffice to derive the results in [51; 49]. However, to show coarse invariance of group (co)homology in particular coefficients, we need more concrete versions of our dynamic characterizations.

Inspired by Špakula and Willett [54], we first observe that in place of abstract dynamical systems, we may always take the canonical action $G \curvearrowright \beta G$ of groups G on their Stone–Čech compactifications βG. The appearance of $G \curvearrowright \beta G$ is not surprising because of its universal property. But, now, our crucial observation is that we can go even further and consider the actions $G \curvearrowright G$ of groups acting on themselves by left multiplication. By doing so, it seems that we are losing all the information as any two actions $G \curvearrowright G$ and $H \curvearrowright H$ are continuously orbit-equivalent as long as G and H have the same cardinality. The problem is that the spaces on which our groups act are no longer compact. However, we can replace compactness by asking for finiteness conditions on the maps a, which — as in the definition of continuous orbit couples — make precise that orbits are preserved in a continuous way: We require that for every $g \in G$, the map $a(g, \cdot)$ should have finite image. It is this finiteness condition which singles out "controlled" orbit equivalences which behave well in (co)homology. The point is that every coarse equivalence $G \to H$ gives rise to a "controlled" orbit equivalence between $G \curvearrowright G$ and $H \curvearrowright H$. This change of perspective, putting the emphasis on this finiteness condition, turns out to be crucial.

These ideas lead to the following results: Let R be a commutative ring with unit and W an R–module. The set $C(G, W)$ of functions $G \to W$ carries a natural RG–module structure. An RG–submodule $L \subseteq C(G, W)$ is called res–invariant if for every $f \in L$ and $A \subseteq G$, the restriction of f to A (viewed as a function on G by extending it by 0) still lies in L. Examples include $C(G, W)$, the submodule $C_f(G, W)$ of $f \in C(G, W)$ taking only finitely many values; $RG \otimes_R W$; and, for $W = R = \mathbb{R}$ or $\mathbb{C}$,

$$c_0(G, W) = \{f\colon G \to W : \lim_{x\to\infty} |f(x)| = 0\},$$

$$\ell^p(G, W) = \Big\{f\colon G \to W : \sum_{x\in G} |f(x)|^p < \infty\Big\} \quad (0 < p \leq \infty),$$

$$H^{s,p}(G, W) = \{f\colon G \to W : f \cdot (1+\ell)^s \in \ell^p(G, W)\} \quad (s \in \mathbb{R} \cup \{\infty\},\ 1 \leq p < \infty),$$

where G is finitely generated, ℓ is the word length on G and $H^{\infty,p}(G, W) = \bigcap_{s\in\mathbb{R}} H^{s,p}(G, W)$. We show that a coarse equivalence $\varphi\colon G \to H$ induces a one-to-one correspondence $L \mapsto \varphi_* L$ between res–invariant submodules of $C(G, W)$ and res–invariant submodules of $C(H, W)$, together with isomorphisms $H_*(\varphi)\colon H_*(G, L) \cong H_*(H, \varphi_* L)$ for all L. Similarly, φ induces a one-to-one correspondence between res–invariant submodules of $C(H, W)$ and res–invariant submodules of $C(G, W)$, say $M \mapsto \varphi^* M$, together with isomorphisms $H^*(\varphi)\colon H^*(H, M) \cong H^*(G, \varphi^* M)$ for all M. In particular, we obtain:

Corollary 4.41 *Among all countable discrete groups G, the following (co)homology groups are coarse invariants:*

$$H_*(G, C(G, W)), \quad H_*(G, C_f(G, W)), \quad H^*(G, C_f(G, W)), \quad H^*(G, RG \otimes_R W)$$

for every commutative ring R with unit and every R–module W;

$$H_*(G, c_0(G, R)), \quad H^*(G, c_0(G, R)), \quad \overline{H}_*(G, c_0(G, R)), \quad \overline{H}^*(G, c_0(G, R));$$
$$H_*(G, \ell^p(G, R)), \quad H^*(G, \ell^p(G, R)), \quad \overline{H}_*(G, \ell^p(G, R)), \quad \overline{H}^*(G, \ell^p(G, R))$$

for all $0 < p \leq \infty$; and, for finitely generated groups G,

$$H_*(G, H^{s,p}(G, R)), \quad H^*(G, H^{s,p}(G, R)),$$
$$\overline{H}_*(G, H^{s,p}(G, R)), \quad \overline{H}^*(G, H^{s,p}(G, R))$$

for all $s \in \mathbb{R} \cup \{\infty\}$ and $1 \leq p \leq \infty$, where $R = \mathbb{R}$ or $\mathbb{C}$.

Some of these (co)homology groups can be identified with existing (co)homology theories (for classes of groups where the latter are defined): $H^*(G, RG)$ is coarse cohomology [48, Section 5.1], $H_*(G, C_f(G, \mathbb{Z}))$ and $H_*(G, \ell^\infty(G, \mathbb{R}))$ coincide with uniformly finite homology (see Blank and Diana [6], Block and Weinberger [7] and Brodzki, Niblo and Wright [9]), and for ℓ^p coefficients, we obtain L^p–cohomology (see Elek [18], Gersten [21] and Pansu [43]). Actually, we show that every coarse map $\varphi: G \to H$ induces a map $H_*(\varphi): H_*(G, L) \to H_*(H, \varphi_* L)$ such that $H_*(\varphi) = H_*(\phi)$ if $\varphi \sim \phi$ and $H_*(\psi \circ \varphi) = H_*(\psi) \circ H_*(\varphi)$. It is then evident that coarse equivalences induce isomorphisms as they are precisely those coarse maps which are invertible modulo $\sim$. A similar remark applies to cohomology. Thus, not only these (co)homology groups, but, by functoriality, the actions of the groups of coarse equivalences (modulo $\sim$) on these (co)homology groups are coarse invariants as well. We obtain analogous results for coarse embeddings in the setting of topological res–invariant modules and reduced (co)homology.

The aforementioned results on coarse invariance of type FP_n and being a (Poincaré) duality group are immediate consequences, as is our rigidity result for coarse embeddings into Poincaré duality groups. We also deduce that vanishing of ℓ^2–Betti numbers is a coarse invariant, as observed in Mimura, Ozawa, Sako and Suzuki [38], Oguni [42] and Pansu [43], and generalized by Sauer and Schrödl [50] to all unimodular locally compact second countable groups.

As far as our methods are concerned, we use groupoid techniques as in [51; 49; 42], but instead of working with abstract dynamical systems, we base our work on concrete dynamic characterizations of coarse equivalence. The difference between our work and [21] is that we do not work with descriptions of group (co)homology in terms of Eilenberg–Mac Lane spaces, as these descriptions require finiteness conditions (like F_n or F_∞) on our groups and have to be modified whenever we change coefficients. Instead, since coarse embeddings automatically lead to "controlled" orbit equivalences satisfying the finiteness condition mentioned above, we can work directly with complexes coming from bar resolutions.

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2 Dynamical characterizations of quasi-isometry

2.1 Preliminaries

The central notions of coarse maps, embeddings and equivalences have been introduced in Section 1. We remark that it is easy to see that a coarse embedding $\varphi\colon G \to H$ is coarsely invertible if and only if H can be covered by finitely many translates of $\varphi(G)$, ie there is a finite set $F \subseteq H$ such that $H = \bigcup_{h \in F} h\varphi(G)$.

Remark 2.1 Unlike in [51], our definition of coarse maps is right-invariant, not left-invariant (ie we use st^{-1} instead of $s^{-1}t$). For finitely generated groups, this amounts to considering right-invariant word lengths and word metrics. We do so because in the following, we will consider left actions of groups, in particular the action of a group by left multiplication on itself. Of course, this is merely a matter of convention.

The following concept, due to Gromov, builds a bridge between geometric group theory and topological dynamical systems.

Definition 2.2 For two groups G and H, a (G,H) topological coupling consists of a locally compact space Ω with commuting free and proper left G– and right H–actions which admit clopen H– and G–fundamental domains $\overline{X}$ and $\overline{Y}$. Our (G,H) topological coupling is called G–cocompact if $\overline{Y}$ is compact, H–cocompact if $\overline{X}$ is compact, and cocompact if it is both G– and H–cocompact. It is called topologically free (or free) if the combined action $G \times H \curvearrowright \Omega$ is topologically free (or free).

All our spaces are Hausdorff. Also, being only concerned with the topological setting, we simply write "coupling" (without the prefix "topological"). To keep track of all the relevant data, we often write $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$.

The following result goes back to ideas of Gromov and is proven in [51; 49].

Theorem 2.3 *Let G and H be countable discrete groups.*

(i) *There exists a coarse embedding $G \to H$ if and only if there exists a H–cocompact (G,H) coupling.*
(ii) *There exists a coarse equivalence $G \to H$ if and only if there exists a cocompact (G,H) coupling.*
(iii) *There is a bijective coarse equivalence $G \to H$ if and only if there is a cocompact (G,H) coupling $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$ with $\overline{X} = \overline{Y}$.*

Proof For (i), see [49, Theorem 2.2 (i) $\Longleftrightarrow$ (ii)]. For (ii), see [49, Theorem 2.2 (iii) $\Longleftrightarrow$ (iv)]. For (iii), see [51, Remark after Theorem 2.1.2]. □

Remark 2.4 The proofs in [49; 51] show that the underlying space Ω of the (G,H) couplings can be chosen to be second countable and totally disconnected in the above statements.

Let us now isolate an idea from [36] which will be useful later on.

Lemma 2.5 *If there exists a (G,H) coupling $G \curvearrowright \Omega \curvearrowleft H$, then there exists a topologically free (G,H) coupling $G \curvearrowright \Omega' \curvearrowleft H$. If $G \curvearrowright \Omega \curvearrowleft H$ is G–cocompact, H–cocompact or cocompact, $G \curvearrowright \Omega' \curvearrowleft H$ may be chosen with the same property. If Ω is second countable and totally disconnected, we may choose Ω' with the same property.*

Proof The idea of the proof appears in the proof of [36, Theorem 3.2]. Let $G \times H \curvearrowright Z$ be a free action on the Cantor space Z. It is easy to see that $\Omega' = \Omega \times Z$ with diagonal G– and H–actions is a (G,H) coupling which is topologically free (even free). As Z is compact and totally disconnected, our additional claims follow. □

2.2 Topological couplings and continuous orbit couples

We explain the connection between topological couplings and continuous orbit couples. First of all, a topological dynamical system $G \curvearrowright X$ consists of a group G acting on a locally compact space X via homeomorphisms. We write $g.x$ for the action.

Definition 2.6 Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topological dynamical systems.

A continuous map $p\colon X \to Y$ is called a continuous orbit map if there exists a continuous map $a\colon G \times X \to H$ such that $p(g.x) = a(g,x).p(x)$ for all $g \in G$ and $x \in X$.

A continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$ consists of continuous orbit maps $p\colon X \to Y$ and $q\colon Y \to X$ such that there exist continuous maps $g\colon X \to G$ and $h\colon Y \to H$ such that $q(p(x)) = g(x).x$ and $p(q(y)) = h(y).y$ for all $x \in X$ and $y \in Y$.

Definition 2.7 A (G,H) continuous orbit couple consists of topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ and a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$. If $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free, then the (G,H) continuous orbit couple is called topologically free. We call X the G–space and Y the H–space of our (G,H) continuous orbit couple.

Remark 2.8 In this language, a continuous orbit equivalence for $G \curvearrowright X$ and $H \curvearrowright Y$ in the sense of [32] is the same as a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$ with $g \equiv e$ and $h \equiv e$, ie $p = q^{-1}$.

Definition 2.9 A (G,H) continuous orbit equivalence consists of topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ and a continuous orbit equivalence for $G \curvearrowright X$ and $H \curvearrowright Y$.

Theorem 2.10 *Let G and H be groups. There is a one-to-one correspondence between isomorphism classes of topologically free (G,H) couplings and isomorphism classes of topologically free (G,H) continuous orbit couples, with the following additional properties:*

(i) *A (G,H) coupling $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$ corresponds to a (G,H) continuous orbit couple with G–space homeomorphic to $\overline{X}$ and H–space homeomorphic to $\overline{Y}$.*

(ii) *A (G,H) coupling $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$ with $\overline{X} = \overline{Y}$ corresponds to a (G,H) continuous orbit equivalence.*

Here, the notions of isomorphisms are the obvious ones: Topological couplings $G \curvearrowright {}_{\overline{Y}_1}\Omega_{1\,\overline{X}_1} \curvearrowleft H$ and $G \curvearrowright {}_{\overline{Y}_2}\Omega_{2\,\overline{X}_2} \curvearrowleft H$ are isomorphic if there exists a $G \times H$–equivariant homeomorphism $\Omega_1 \cong \Omega_2$ sending $\overline{X}_1$ to $\overline{X}_2$ and $\overline{Y}_1$ to $\overline{Y}_2$. Continuous orbit couples (p_i, q_i) for $G \curvearrowright X_i$ and $H \curvearrowright Y_i$ for $i = 1, 2$ are isomorphic if there exist G– and H–equivariant homeomorphisms $X_1 \cong X_2$ and $Y_1 \cong Y_2$ such that we obtain commutative diagrams

$$\begin{array}{ccc} X_1 & \xrightarrow{p_1} & Y_1 \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ X_2 & \xrightarrow{p_2} & Y_2 \end{array} \qquad \begin{array}{ccc} Y_1 & \xrightarrow{q_1} & X_1 \\ \cong\Big\downarrow & & \Big\downarrow\cong \\ Y_2 & \xrightarrow{q_2} & X_2 \end{array}$$

For the proof of Theorem 2.10, we now present explicit constructions of continuous orbit couples out of topological couplings and vice versa. The constructions are really the topological analogues of those in [19, Section 3] (see also [51; 49]). In the following, we write gx (for $g \in G$ and $x \in \Omega$) and xh (for $x \in \Omega$ and $h \in H$) for the left G–and right H–actions in topological couplings, and $g.x$ and $h.y$ for the actions $G \curvearrowright X$ and $H \curvearrowright Y$ from continuous orbit couples.

2.2.1 From topological couplings to continuous orbit couples Let $G \curvearrowright_{\overline{Y}} \Omega_{\overline{X}} \curvearrowleft H$ be a (G, H) coupling. Set $X := \overline{X}$ and $Y := \overline{Y}$. Define a map $p\colon X \to Y$ by requiring $Gx \cap Y = \{p(x)\}$ for all $x \in X$. The intersection $Gx \cap Y$, taken in Ω, consists of exactly one point because Y is a G–fundamental domain. By construction, there is a map $\gamma\colon X \to G$ such that $p(x) = \gamma(x)x$. For $g \in G$, γ takes the constant value g on $X \cap g^{-1}Y$. As $X \cap g^{-1}Y$ is clopen, because X and Y are, γ is continuous. Also p is continuous as it is so on $X \cap g^{-1}Y$ for all $g \in G$.

We now define a G–action, denoted by $G \times X \to X$, $(g, x) \mapsto g.x$, as follows: For every $g \in G$ and $x \in X$, there exists a unique $\alpha(g, x) \in H$ such that $gx \in X\alpha(g, x)$. For fixed $g \in G$ and $h \in H$, we have $\alpha(g, x) = h$ for all $x \in X \cap g^{-1}Xh$. As $X \cap g^{-1}Xh$ is clopen because X is, $\alpha\colon G \times X \to H$ is continuous. Set $g.x := gx\alpha(g, x)^{-1}$. It is easy to check that α satisfies the cocycle identity $\alpha(g_1g_2, x) = \alpha(g_1, g_2.x)\alpha(g_2, x)$. Using this, it is easy to see that $G \times X \to X$, $(g, x) \mapsto g.x$, defines a (left) G–action on X by homeomorphisms.

Similarly, we define a continuous map $q\colon Y \to X$ by requiring $X \cap yH = \{q(y)\}$ for all $y \in Y$, and let $\eta\colon Y \to H$ be the continuous map satisfying $q(y) = y\eta(y)$. To define an H–action on Y, let $\beta(y, h) \in G$ be such that $yh \in \beta(y, h)Y$. Again, $\beta\colon Y \times H \to G$ is continuous. Set $h.y := \beta(y, h^{-1})^{-1}yh^{-1}$. It is easy to check that β satisfies $\beta(y, h_1h_2) = \beta(y, h_1)\beta(h_1^{-1}.xh_2)$. Using this, it is again easy to see that $H \times Y \to Y$, $(h, y) \mapsto h.y$, defines an H–action on Y by homeomorphisms.

Let us check that (p,q) is a (G,H) continuous orbit couple. We need to identify $Ggx\alpha(g,x)^{-1}\cap Y$ in order to determine $p(g.x)=p(gx\alpha(g,x)^{-1})$. We have

$$Ggx\alpha(g,x)^{-1}\ni\beta(\gamma(x)x,\alpha(g,x)^{-1})^{-1}\gamma(x)x\alpha(g,x)^{-1}\in Y,$$

so

$$p(g.x)=\beta(\gamma(x)x,\alpha(g,x)^{-1})^{-1}\gamma(x)x\alpha(g,x)^{-1}=\alpha(g,x).(\gamma(x)x)=\alpha(g,x).p(x).$$

Similarly, in order to identify $q(h.y)=q(\beta(y,h^{-1})^{-1}yh^{-1})$, we need to determine $X\cap\beta(y,h^{-1})^{-1}yh^{-1}H$. As

$$X\ni\beta(y,h^{-1})^{-1}y\eta(y)\alpha(\beta(y,h^{-1})^{-1},y\eta(y))^{-1}\in\beta(y,h^{-1})^{-1}yh^{-1}H,$$

we deduce

$$\begin{aligned}q(y.h)=\beta(y,h^{-1})^{-1}y\eta(y)\alpha(\beta(y,h^{-1})^{-1},y\eta(y))^{-1}&=\beta(y,h^{-1})^{-1}.(y\eta(y))\\&=\beta(y,h^{-1})^{-1}.q(y).\end{aligned}$$

Finally, $qp(x)=q(\gamma(x)x)=\gamma(x)x\alpha(\gamma(x),x)^{-1}=\gamma(x).x$ and $pq(y)=p(y\eta(y))=\beta(y,\eta(y))^{-1}y\eta(y)=\eta(y)^{-1}.y$. All in all, we see that p and q give rise to a continuous orbit couple for $G\curvearrowright X$ and $H\curvearrowright Y$, with $g(x)=\gamma(x)$ and $h(y)=\eta(y)^{-1}$.

Note that our coupling does not need to be topologically free for this construction. However, it is clear that $G\curvearrowright\Omega\curvearrowleft H$ is topologically free (ie $G\times H\curvearrowright\Omega$ is topologically free) if and only if $G\curvearrowright X$ and $H\curvearrowright Y$ are topologically free.

Remark 2.11 Our notation differs slightly from that in [51; 49]. Our $\alpha(g,x)$ is $\alpha(g^{-1},x)^{-1}$ in [51, Section 2.2, Equation (3)] and [49, Section 2.2, Equation (2.2)]. This is closely related to Remark 2.1.

Remark 2.12 The dynamical system $G\curvearrowright X$ we constructed above can be canonically identified with $G\curvearrowright\Omega/H$. Similarly, our system $H\curvearrowright Y$ can be identified with $G\backslash\Omega\curvearrowleft H$ in a canonical way.

2.2.2 From continuous orbit couples to topological couplings Let $G\curvearrowright X$ and $H\curvearrowright Y$ be topologically free systems on locally compact spaces X and Y. Assume that we are given a continuous orbit couple for $G\curvearrowright X$ and $H\curvearrowright Y$, and let p, q, a, g and h be as in Definition 2.6, and let b: $H\times Y\to G$ be a continuous map with $q(h.y)=b(h,y).q(y)$ for all $h\in H$ and $y\in Y$. Define commuting left G– and right H–actions on $X\times H$ by $g(x,h)=(g.x,a(g,x)h)$, $(x,h)h'=(x,hh')$. Furthermore, define commuting left G– and right H–actions on $G\times Y$ by $g'(g,y)=(g'g,y)$ and $(g,y)h=(gb(h^{-1},y)^{-1},h^{-1}.y)$.

A straightforward computation, using the cocycle identities [32, Lemma 2.8] for a and b, shows that $\Theta\colon X \times H \to G \times Y,\ (x,h) \mapsto \big(g(x)^{-1} b(h^{-1}, p(x))^{-1}, h^{-1}.p(x)\big)$, is a G– and H–equivariant homeomorphism with inverse $\Theta^{-1}\colon G \times Y \to X \times H$, $(g,y) \mapsto \big(g.q(y), a(g,q(y))h(y)\big)$. Thus, the $G \times H$–space $\Omega = X \times H$, together with $\overline{X} = X \times \{e\}$ and $\overline{Y} = \Theta^{-1}(\{e\} \times Y)$, yields the desired topologically free (G,H) coupling $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$.

Note that topological freeness of $G \curvearrowright X$ and $H \curvearrowright Y$ ensures that a and b satisfy the cocycle identities (as in [32, Lemma 2.8]), which are needed in the preceding computations.

2.2.3 One-to-one correspondence

Proof of Theorem 2.10 It is straightforward to check that the constructions described in Sections 2.2.1 and 2.2.2 are inverse to each other up to isomorphism. If we start with a topologically free (G,H) coupling $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$, construct a continuous orbit couple and then again a (G,H) coupling, we end up with a (G,H) coupling of the form $G \curvearrowright {}_{\widetilde{Y}}\widetilde{\Omega}_{\widetilde{X}} \curvearrowleft H$, where $\widetilde{\Omega} = \overline{X} \times H \cong G \times \overline{Y}$, $\widetilde{X} = \overline{X} \times \{e\}$ and $\widetilde{Y} \cong \{e\} \times \overline{Y}$. It is then obvious that $\widetilde{\Omega} = \overline{X} \times H \to \Omega,\ (x,h) \mapsto xh$, is an isomorphism of the couplings $G \curvearrowright {}_{\widetilde{Y}}\widetilde{\Omega}_{\widetilde{X}} \curvearrowleft H$ and $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$. Conversely, if we start with a continuous orbit couple for topologically free systems $G \curvearrowright X$ and $H \curvearrowright Y$, construct a (G,H) coupling and then again a (G,H) continuous orbit couple, we end up with a continuous orbit couple for $G \curvearrowright \widetilde{X}$ and $H \curvearrowright \widetilde{Y}$, where $\widetilde{X} = X \times \{e\}$ and $\widetilde{Y} \cong \{e\} \times Y$. The canonical isomorphisms $X \cong X \times \{e\}$ and $Y \cong \{e\} \times Y$ yield an isomorphism between the original (G,H) continuous orbit couple and the one we obtained at the end.

Additional property (i) is clear from our constructions. For (ii), take $\overline{X} = \overline{Y}$ in the construction of Section 2.2.2. Then it is clear that our maps p and q become the identity map on $\overline{X} = \overline{Y}$, and that γ becomes the constant function with value $e \in G$ and η the constant function with value $e \in H$. Hence it is obvious that our construction yields a (G,H) continuous orbit equivalence (see also Remark 2.8). □

Remark 2.13 The maps p and q constructed in Section 2.2.1 are open. Thus the maps p and q appearing in a continuous orbit couple (Definition 2.6) are automatically open. This is also easy to see directly from the definition.

2.3 Continuous orbit couples and Kakutani equivalence

Definition 2.14 (compare also [35, Definition 4.1]) Topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent if there exist clopen subsets $A \subseteq X$ and

$B \subseteq Y$ such that $G.A = X$, $H.B = Y$ and $(X \rtimes G)|A \cong (Y \rtimes H)|B$ as topological groupoids. Here $(X \rtimes G)|A = s^{-1}(A) \cap r^{-1}(A)$ and $(Y \rtimes H)|B = s^{-1}(B) \cap r^{-1}(B)$.

Remark 2.15 $(X \rtimes G)|A$ is (isomorphic to) the transformation groupoid attached to the partial action $G \curvearrowright A$ which is obtained by restricting $G \curvearrowright X$ to A. Similarly, $(Y \rtimes H)|B$ is (isomorphic to) the transformation groupoid attached to the partial action $H \curvearrowright B$ which is obtained by restricting $H \curvearrowright Y$ to B. In view of this, two topologically free systems $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent if and only if there exist clopen subsets $A \subseteq X$ and $B \subseteq Y$ with $G.A = X$, $H.B = Y$ such that the partial actions $G \curvearrowright A$ and $H \curvearrowright B$ are continuously orbit-equivalent in the sense of [31]. This follows from [31, Theorem 2.7].

The reader may find more about partial actions in [31, Section 2].

Theorem 2.16 *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. There exists a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$ with $p(X)$ closed if and only if $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent.*

Here $p\colon X \to Y$ is as in Definition 2.6. The assumption that $p(X)$ is closed always holds if X is compact. This will be the case of interest later on.

Proof By Remark 2.15, we have to show that there exists a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$ if and only if there exist clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $X = G.A$ and $Y = H.B$ such that the partial actions $G \curvearrowright A$ and $H \curvearrowright B$ are continuously orbit-equivalent.

For "$\Longrightarrow$", suppose we are given a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$, and let p, q, a, b, g and h be as in Definition 2.6 and Section 2.2.2. For $g \in G$, let $U_g = \{x \in X : g(x) = g\}$. Then U_g is clopen, and $X = \bigsqcup_{g \in G} U_g$. For every $g \in G$, the image $V_g := p(U_g)$ is clopen, and $p\colon U_g \to V_g$ is a homeomorphism, whose inverse is given by $V_g \to U_g$, $y \mapsto g^{-1}.q(y)$. Set $B := p(X)$. By assumption, B is closed, hence clopen. We have $B = \bigcup_{g \in G} V_g$. As G is countable, this is a countable union. Hence, by inductively choosing compact open subspaces B_g of V_g, we can arrange that B is the disjoint union $B = \bigsqcup_{g \in G} B_g$. Let $A_g := U_g \cap p^{-1}(B_g)$ and $A := \bigsqcup_{g \in G} A_g$. As every A_g is clopen, $A = \bigsqcup_{g \in G} A_g$ is clopen in $X = \bigsqcup_{g \in G} U_g$. Set $\varphi := p|_A = \bigsqcup_{g \in G} p|_{A_g}$. By construction, φ is a homeomorphism with inverse $\varphi^{-1} = \bigsqcup_{g \in G} (p|_{A_g})^{-1} = \bigsqcup_{g \in G} (g^{-1}.q)|_{B_g}$.

We have $\varphi(g.x) = p(g.x) = a(g,x).p(x)$ for all $x \in A$ and $g \in G$ with $g.x \in A$. Moreover, take $y \in B_{g_1}$ and $h \in H$ with $h.y \in B_{g_2}$. Then $\varphi^{-1}(h.y) = g_2^{-1}.q(h.y) = g_2^{-1}b(h,y).q(y) = g_2^{-1}b(h,y)g_1.\varphi^{-1}(y)$. Define a map b' by setting $b'(h,y) = g_2^{-1}b(h,y)g_1$ if $y \in B_{g_1} \cap h^{-1}.B_{g_2}$. Then b' is continuous, and we have $\varphi^{-1}(h.y) = b'(h,y).\varphi^{-1}(y)$ for all $y \in B$ and $h \in H$ with $h.y \in B$. This shows that φ gives rise to a continuous orbit equivalence for $G \curvearrowright A$ and $H \curvearrowright B$. To see that $G.A = X$, take for $x' \in X$ an $x \in A$ such that $p(x) = p(x')$. Then $g(x).x = q(p(x)) = q(p(x')) = g(x').x'$, and therefore $x' \in G.x$. To see $H.B = Y$, take $y \in Y$ arbitrary. Then $p(q(y)) = h(y).y$ shows that $y = h(y)^{-1}.p(q(y)) \in H.B$. This shows "$\Longrightarrow$".

For "$\Longleftarrow$", suppose that $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent, ie there are clopen subsets $A \subseteq X$ and $B \subseteq Y$ with $X = G.A$ and $Y = H.B$ and the partial actions $G \curvearrowright A$ and $H \curvearrowright B$ are continuously orbit-equivalent via a homeomorphism $\varphi\colon A \cong B$. By definition of continuous orbit equivalence (see [31]), there exist continuous maps a' and b' satisfying $\varphi(g.x) = a'(g,x).\varphi(x)$ and $\varphi^{-1}(h.y) = b'(h,y).\varphi^{-1}(y)$ whenever this makes sense. As $X = G.A$, we can find clopen subsets $X_\gamma \subseteq \gamma.A$ for $\gamma \in G$ such that $X = \bigsqcup_{\gamma \in G} X_\gamma$ and $X_e = A$. Define $p\colon X \to Y$ by setting $p(x) := \varphi(\gamma^{-1}.x)$ for $x \in X_\gamma$. Then p is continuous, and $p(X) = B$ is clopen. Similarly, there are clopen subsets $Y_\eta \subseteq \eta.B$ such that $Y = \bigsqcup_{\eta \in H} Y_\eta$ and $Y_e = B$. We define $q\colon Y \to X$ by setting $q(y) = \varphi^{-1}(\eta^{-1}.y)$ if $y \in Y_\eta$. By construction, q is continuous.

We have

$$\begin{aligned} p(g.x) = \varphi(\gamma_2^{-1}g.x) = \varphi(\gamma_2^{-1}g\gamma_1.(\gamma_1^{-1}.x)) &= a'(\gamma_2^{-1}g\gamma_1, \gamma_1^{-1}.x).\varphi(\gamma_1^{-1}.x) \\ &= a'(\gamma_2^{-1}g\gamma_1, \gamma_1^{-1}.x).p(x) \end{aligned}$$

for $x \in X_{\gamma_1}$ and $g \in G$ with $g.x \in X_{\gamma_2}$. Set

$$a\colon G \times X \to H, \quad a(g,x) = a'(\gamma_2^{-1}g\gamma_1, \gamma_1^{-1}.x) \quad \text{for } x \in X_{\gamma_1} \cap g^{-1}.X_{\gamma_2}.$$

Then a is continuous and $\varphi(g.x) = a(g,x).\varphi(x)$ for all $g \in G$ and $x \in X$.

For $y \in Y_{\eta_1}$ and $h \in H$ such that $h.y \in Y_{\eta_2}$, we have

$$\begin{aligned} q(h.y) = \varphi^{-1}(\eta_2^{-1}h.y) = \varphi^{-1}(\eta_2^{-1}h\eta_1(\eta_1^{-1}.y)) &= b'(\eta_2^{-1}h\eta_1, \eta_1^{-1}.y).\varphi^{-1}(\eta_1^{-1}.y) \\ &= b'(\eta_2^{-1}h\eta_1, \eta_1^{-1}.y).q(y). \end{aligned}$$

Define $b\colon H \times Y \to G$ by $b(h,y) = b'(\eta_2^{-1}h\eta_1, \eta_1^{-1}.y)$ for $y \in Y_{\eta_1} \cap h^{-1}.Y_{\eta_2}$. Then b is continuous and $\varphi^{-1}(h.y) = b(h,y).\varphi^{-1}(y)$ for all $h \in H$ and $y \in Y$.

Moreover, $q(p(x)) = q(\varphi(\gamma^{-1}.x)) = \varphi^{-1}(\varphi(\gamma^{-1}.x)) = \gamma^{-1}.x$ for $x \in X_\gamma$. Define $g\colon X \to G$ by $g(x) = \gamma^{-1}$ if $x \in X_\gamma$. Then g is continuous and $q(p(x)) = g(x).x$

for all $x \in X$. For $y \in Y_\eta \cap \eta.\varphi(X_\gamma)$, we have $p(q(y)) = p(\varphi^{-1}(\eta^{-1}.y)) = \varphi(\gamma^{-1}.\varphi^{-1}(\eta^{-1}.y)) = \varphi(\gamma^{-1}b'(\eta^{-1},y).\varphi^{-1}(y)) = a'(\gamma^{-1}b'(\eta^{-1},y),\varphi^{-1}(y)).y$. Let $h\colon Y \to H$ be given by $h(y) := a'(\gamma^{-1}b'(\eta^{-1},y),\varphi^{-1}(y))$ if $y \in Y_\eta \cap \eta.\varphi(X_\gamma)$. Then h is continuous and $p(q(y)) = h(y).y$ for all $y \in Y$.

So p and q give a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$. This shows "$\Leftarrow$". □

2.4 Dynamic characterizations of coarse embeddings, equivalences and bijections

Putting together Theorem 2.3, Lemma 2.5 and Theorems 2.10 and 2.16, we obtain the following:

Theorem 2.17 *Let G and H be countable discrete groups.*

- *The following are equivalent:*
  - *There exists a coarse embedding $G \to H$.*
  - *There exist Kakutani-equivalent topologically free $G \curvearrowright X$ and $H \curvearrowright Y$, with X compact.*
  - *There is a continuous orbit couple for topologically free $G \curvearrowright X$ and $H \curvearrowright Y$, with X compact.*
- *The following are equivalent:*
  - *There is a coarse equivalence $G \to H$.*
  - *There are Kakutani-equivalent topologically free $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces X and Y.*
  - *There is a continuous orbit couple for topologically free $G \curvearrowright X$ and $H \curvearrowright Y$, with X and Y compact.*
- *There is a bijective coarse equivalence $G \to H$ if and only if there exist continuously orbit-equivalent topologically free systems $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces X and Y.*

In all these statements, the spaces X and Y can be chosen to be totally disconnected and second countable.

This is a generalization of [36, Theorem 3.2], where the authors independently prove the last item of our theorem in the special case of finitely generated groups.

Remark 2.18 The last observation in Theorem 2.17 says that we can always choose our spaces X and Y to be totally disconnected. In that case, Theorem 3.2 of [11] tells us that we can replace Kakutani equivalence in the theorem above by stable continuous orbit equivalence. Two topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ are called stably continuously orbit-equivalent if $\mathbb{Z} \times G \curvearrowright \mathbb{Z} \times X$ and $\mathbb{Z} \times H \curvearrowright \mathbb{Z} \times Y$ are continuously orbit-equivalent. Here the integers $\mathbb{Z}$ act on themselves by translation.

2.5 Dynamic characterizations of coarse embeddings, equivalences and bijections in terms of actions on Stone–Čech compactifications

Inspired by [54], we characterize coarse embeddings, equivalences and bijections in terms of Kakutani equivalence (or stable continuous orbit equivalence) and continuous orbit equivalence of actions on Stone–Čech compactifications.

Let G and H be two countable discrete groups. Let $\varphi\colon G \to H$ be a coarse embedding. Consider the Stone–Čech compactification βG of G. It is homeomorphic to the spectrum $\mathrm{Spec}(\ell^\infty(G))$, and can be identified with the space of all ultrafilters on G. We will think of elements in βG as ultrafilters on G. Given any subset $X \subseteq G$, we obviously have the identification $\{\mathcal{F} \in \beta G : X \in \mathcal{F}\} \cong \beta X$ given by $\mathcal{F} \mapsto \mathcal{F} \cap X := \{F \cap X : F \in \mathcal{F}\}$.

Now suppose that $X \subseteq G$ is a subset such that $\varphi|_X$ is injective. Setting $Y := \varphi(X) \subseteq H$, we obtain a bijection $X \cong Y$, $x \mapsto \varphi(x)$, which we again denote by φ. Let us consider the topological dynamical systems $G \curvearrowright \beta G$ and $H \curvearrowright \beta H$. We identify βX and βY with clopen subsets of βG and βH, respectively, in the way explained above. φ induces a homeomorphism $\beta\varphi\colon \beta X \cong \beta Y$, $\mathcal{F} \mapsto \varphi(\mathcal{F})$. The dynamical systems $G \curvearrowright \beta G$ and $H \curvearrowright \beta H$ restrict to partial dynamical systems $G \curvearrowright \beta X$ and $H \curvearrowright \beta Y$.

Proposition 2.19 *$\beta\varphi$ induces a continuous orbit equivalence between $G \curvearrowright \beta X$ and $H \curvearrowright \beta Y$, in the sense of* [31, Definition 2.6].

Proof For all $g \in G$, we need to find a continuous map $a\colon \{g\} \times U_{g^{-1}} \to H$ with $\beta\varphi(g.\mathcal{F}) = a(g,\mathcal{F}).\beta\varphi(\mathcal{F})$. Here $U_{g^{-1}} = \beta X \cap g^{-1}.\beta X = \{\mathcal{F} \in \beta X : g.\mathcal{F} \in \beta X\} = \{\mathcal{F} \in \beta G : X \in \mathcal{F} \text{ and } g^{-1}X \in \mathcal{F}\} \cong \beta(X \cap g^{-1}X)$. For $x \in X \cap g^{-1}X$, define the ultrafilter $\mathcal{F}_x$ by saying that $Z \in \mathcal{F}_x$ if and only if $x \in Z$. Define a map $\tilde{a}\colon \{g\} \times \{\mathcal{F}_x : x \in X \cap g^{-1}X\} \to H$ by setting $\tilde{a}(g,\mathcal{F}_x) := \varphi(gx)\varphi(x)^{-1}$. Then

$$
\begin{aligned}
(1)\quad \tilde{a}(g,\mathcal{F}_x).\beta\varphi(\mathcal{F}_x) &= \varphi(gx)\varphi(x)^{-1}.\beta\varphi(\mathcal{F}_x)\\
&= \varphi(gx)\varphi(x)^{-1}.\mathcal{F}_{\varphi(x)} = \mathcal{F}_{\varphi(gx)} = \beta\varphi(\mathcal{F}_{gx}) = \beta\varphi(g.\mathcal{F}_x)
\end{aligned}
$$

for all $g \in G$, $x \in X \cap g^{-1}X$. Let us fix $g \in G$. Set $S = \{(gx, x) : x \in G\}$. As φ is a coarse embedding and $\{st^{-1} : (s,t) \in S\} = \{g\}$ is finite, $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\} = \{\varphi(g,x)\varphi(x)^{-1} : x \in G\}$ is finite. Hence, $\mathrm{im}(\widetilde{a}) \subseteq \{\varphi(gx)\varphi(x)^{-1} : x \in G\}$ is finite, hence a compact subset of H. By the universal property of $\beta(X \cap g^{-1}X)$, there exists a continuous extension of $\widetilde{a}$ to $\{g\} \times U_{g^{-1}}$, which we denote by a. We claim that $\beta\varphi(g.\mathcal{F}) = a(g,\mathcal{F}).\beta\varphi(\mathcal{F})$ for all $\mathcal{F} \in U_{g^{-1}}$. Let $x_i \in X \cap g^{-1}X$ be a net such that $\lim_i \mathcal{F}_{x_i} = \mathcal{F}$. Then $a(g, \mathcal{F}_{x_i}) = \varphi(gx_i)\varphi(x_i)^{-1}$ converges to $a(g,\mathcal{F})$ by construction. Hence,

$$\begin{aligned} a(g,\mathcal{F}).\beta\varphi(\mathcal{F}) = \lim_i a(g,\mathcal{F}_{x_i}).\beta\varphi(\mathcal{F}_{x_i}) &\overset{(1)}{=} \lim_i \beta\varphi(g.\mathcal{F}_{x_i}) \\ &= \beta\varphi(\lim_i g.\mathcal{F}_{x_i}) = \beta\varphi(g.\mathcal{F}). \end{aligned}$$ □

The following observation will be used several times:

Lemma 2.20 *Let $\varphi\colon G \to H$ be a coarse embedding. Set $Y := \varphi(G)$. For every $y \in Y$, choose $x_y \in G$ with $\varphi(x_y) = y$. Set $X := \{x_y : y \in Y\}$.*

Then φ restricts to a bijection $X \cong Y$, and there is a finite subset $F \subseteq G$ with $G = \bigcup_{g \in F} gX$.

Proof Clearly, the restriction of φ to X is a bijection onto Y. To prove that G can be covered by finitely many translates of X, set $S := \{(g, x_{\varphi(g)}) : g \in G\}$. Then $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\} = \{e\}$, where e is the identity in H. Since φ is a coarse embedding, $\{gx_{\varphi(g)}^{-1} : g \in G\} = \{st^{-1} : (s,t) \in S\}$ must be finite. Hence, there is a finite subset $F \subseteq G$ with $G = \bigcup_{g \in F} gX$. □

We now obtain the following characterizations of coarse embeddings, equivalences and bijections:

Corollary 2.21 *Let G and H be countable discrete groups.*

(i) *The following are equivalent:*
   - *There is a coarse embedding $G \to H$.*
   - *There is an open, dense, H–invariant subspace $\widetilde{Y} \subseteq \beta H$ such that $G \curvearrowright \beta G$ and $H \curvearrowright \widetilde{Y}$ are Kakutani-equivalent.*
   - *There is an open, dense, H–invariant subspace $\widetilde{Y} \subseteq \beta H$ such that there is a continuous orbit couple for $G \curvearrowright \beta G$ and $H \curvearrowright \widetilde{Y}$.*
   - *There is an open, dense, H–invariant subspace $\widetilde{Y} \subseteq \beta H$ such that $G \curvearrowright \beta G$ and $H \curvearrowright \widetilde{Y}$ are stably continuously orbit-equivalent.*

(ii) *There is a coarse equivalence* $G \to H$ *if and only if* $\widetilde{Y} = \beta H$ *works in the statements in* (i).

(iii) *There is a bijective coarse equivalence* $G \to H$ *if and only if* $G \curvearrowright \beta G$ *and* $H \curvearrowright \beta H$ *are continuously orbit-equivalent.*

Proof (i) Let $\varphi\colon G \to H$ be a coarse embedding. Let Y and X be as in Lemma 2.20. As the restriction of φ to X is a bijection onto Y, Proposition 2.19 yields that $G \curvearrowright \beta X$ and $H \curvearrowright \beta Y$ are continuously orbit-equivalent. As there is a finite subset $F \subseteq G$ with $G = \bigcup_{g \in F} gX$, we have $\beta G = G.\beta X$. Let $\widetilde{Y} := H.\beta Y$. Then $G \curvearrowright \beta G$ and $H \curvearrowright \widetilde{Y}$ are Kakutani-equivalent. $\widetilde{Y}$ is H–invariant by construction, and it is easy to see that $\widetilde{Y}$ is open and dense. Now (i) follows from Theorems 2.17 and 2.16 and Remark 2.18.

(ii) A coarse embedding $\varphi\colon G \to H$ is coarsely invertible if and only if there is a finite subset $F \subseteq H$ such that $H = \bigcup_{h \in F} h\varphi(G)$. This happens if and only if in the proof of (i) we get $\widetilde{Y} = \beta H$.

(iii) If $\varphi\colon G \to H$ is a bijective coarse equivalence, then we can take $X = G$ and $Y = H$ in the above proof of (i) and obtain that $G \curvearrowright \beta G$ and $H \curvearrowright \beta H$ are continuously orbit-equivalent. The reverse implication "$\Longleftarrow$" in (ii) is proven in Theorem 2.17. □

Remark 2.22 In combination with [54], Corollary 2.21 implies that nuclear Roe algebras have distinguished Cartan subalgebras, as explained in [33].

Remark 2.23 A result analogous to Corollary 2.21 is valid in the more general setting of uniformly locally finite metric spaces. In that case, transformation groupoids of Stone–Čech dynamical systems have to be replaced by coarse groupoids as constructed in [52, Section 3.2].

Remark 2.24 Corollary 2.21 shows that quasi-isometry rigidity can be interpreted as a special case of continuous orbit equivalence rigidity (in the sense of [32]), applied to actions on Stone–Čech compactifications. This points towards an interesting connection between these two types of rigidity phenomena and would be worth exploring further.

3 Applications to (co)homology, I

We now show how the results in [51; 49] on coarse invariance of (co)homological dimensions and property H_{FD} follow from Morita invariance of groupoid (co)homology. Let us first define groupoid (co)homology. We do this in a concrete and elementary way which is good enough for our purposes. We refer to [12] for a more general and more conceptual approach, and for more information about groupoids. Let $\mathcal{G}$ be an étale locally compact groupoid with unit space $X = \mathcal{G}^{(0)}$ and R a commutative ring with unit. A $\mathcal{G}$–sheaf of R–modules is a sheaf $\mathcal{A}$ of R–modules over X, ie we have a locally compact space $\mathcal{A}$ with an étale continuous surjection $\pi\colon \mathcal{A} \twoheadrightarrow X$ whose fibres are R–modules, together with the structure of a right $\mathcal{G}$–space on $\mathcal{A}$. In particular, every $\gamma \in \mathcal{G}$ induces an isomorphism of R–modules $\mathcal{A}_{r(\gamma)} \to \mathcal{A}_{s(\gamma)}$, $a \mapsto a * \gamma$. To pass from right to left actions, we write $\gamma.a := a * \gamma^{-1}$ if $\pi(a) = s(\gamma)$.

Let $\mathcal{G}^{(n)} = \{(\gamma_1, \dots, \gamma_n) \in \mathcal{G}^n : s(\gamma_i) = r(\gamma_{i+1}) \text{ for all } 1 \le i \le n-1\}$, and set $r(\gamma_1, \dots, \gamma_n) = r(\gamma_1)$. We write $\vec{\gamma}$ for elements in $\mathcal{G}^{(n)}$. Given a $\mathcal{G}$–sheaf of R–modules $\mathcal{A}$ with projection $\pi\colon \mathcal{A} \twoheadrightarrow X$, let $\Gamma_c(\mathcal{G}^{(n)}, \mathcal{A})$ be the R–module of continuous functions $f\colon \mathcal{G}^{(n)} \to \mathcal{A}$ with compact support such that $\pi(f(\vec{\gamma})) = r(\vec{\gamma})$. Now we define a chain complex

$$\cdots \xrightarrow{d_{n+1}} \Gamma_c(\mathcal{G}^{(n)}, \mathcal{A}) \xrightarrow{d_n} \Gamma_c(\mathcal{G}^{(n-1)}, \mathcal{A}) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \Gamma_c(\mathcal{G}, \mathcal{A}) \xrightarrow{d_1} \Gamma_c(X, \mathcal{A}) \to 0,$$

with $d_1(f)(x) = \sum_{\gamma \in \mathcal{G}, s(\gamma)=x} \gamma^{-1}.f(\gamma) - \sum_{\gamma \in \mathcal{G}, r(\gamma)=x} f(\gamma)$ for $f \in \Gamma_c(\mathcal{G}, \mathcal{A})$, and, for $n \ge 1$, $d_n(f) = \sum_{i=0}^{n} (-1)^i d_n^{(i)}(f)$ for $f \in \Gamma_c(\mathcal{G}^{(n)}, \mathcal{A})$, where

$$\begin{aligned}
d_n^{(0)}(f)(\gamma_1, \dots, \gamma_{n-1}) &= \sum_{\substack{\gamma_0 \in \mathcal{G} \\ s(\gamma_0) = r(\gamma_1)}} \gamma_0^{-1}.f(\gamma_0, \gamma_1, \dots, \gamma_{n-1}), \\
d_n^{(i)}(f)(\gamma_1, \dots, \gamma_{n-1}) &= \sum_{\substack{\eta, \xi \in \mathcal{G} \\ \eta\xi = \gamma_i}} f(\dots, \gamma_{i-1}, \eta, \xi, \gamma_{i+1}, \dots) \quad \text{for } 1 \le i \le n-1, \\
d_n^{(n)}(f)(\gamma_1, \dots, \gamma_{n-1}) &= \sum_{\substack{\gamma_n \in \mathcal{G} \\ r(\gamma_n) = s(\gamma_{n-1})}} f(\gamma_1, \dots, \gamma_{n-1}, \gamma_n).
\end{aligned}$$

We then define the n^{th} homology group $H_n(\mathcal{G}, \mathcal{A}) := \ker(d_n)/\operatorname{im}(d_{n+1})$. In the case $R = \mathbb{Z}$ and where $\mathcal{A}$ is a constant sheaf with trivial $\mathcal{G}$–action, we recover [35, Definition 3.1].

Let us also introduce cohomology. Let $\mathcal{G}$, R and $\mathcal{A}$ be as above, and let $\Gamma(\mathcal{G}^{(n)}, \mathcal{A})$ be the R–module of continuous functions $f\colon \mathcal{G}^{(n)} \to \mathcal{A}$ with $\pi(f(\vec{\gamma})) = r(\vec{\gamma})$. We define a cochain complex

$$0 \to \Gamma(X, \mathcal{A}) \xrightarrow{d^0} \Gamma(\mathcal{G}, \mathcal{A}) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Gamma(\mathcal{G}^{(n)}, \mathcal{A}) \xrightarrow{d^n} \Gamma(\mathcal{G}^{(n+1)}, \mathcal{A}) \xrightarrow{d^{n+1}} \cdots$$

with $d^0(f)(\gamma) = \gamma.f(s(\gamma)) - f(r(\gamma))$, and, for $n \ge 1$, $d^n(f) = \sum_{i=0}^{n+1} (-1)^i d^n_{(i)}(f)$, where

$$d^n_{(0)}(f)(\gamma_0,\dots,\gamma_n) = \gamma_0.f(\gamma_1,\dots,\gamma_n),$$
$$d^n_{(i)}(f)(\gamma_0,\dots,\gamma_n) = f(\gamma_0,\dots,\gamma_{i-1}\gamma_i,\dots,\gamma_n) \quad \text{for } 1 \le i \le n,$$
$$d^n_{(n+1)}(f)(\gamma_0,\dots,\gamma_n) = f(\gamma_0,\dots,\gamma_{n-1}).$$

We set $H^n(\mathcal{G},\mathcal{A}) := \ker(d^n)/\operatorname{im}(d^{n-1})$.

In the proof of Theorem 3.1, we will need Morita invariance of groupoid (co)homology. Morita invariance for groupoid cohomology is established in [24; 39] (see also the explanations in the introduction of [12]). For groupoid homology, Morita invariance is proven in [12, Corollary 4.6].

Now let $G \curvearrowright X$ be a topological dynamical system. For notational purposes, and to keep the conventions in the literature, let us pass to the right action $X \curvearrowleft G$, $x.g = g^{-1}.x$, and consider the corresponding transformation groupoid $X \rtimes G$ with source and range maps given by $s(x,g) = x.g$ and $r(x,g) = x$. We note that the transformation groupoid $G \ltimes X$ attached to the original action, as in [32; 31], is isomorphic to $X \rtimes G$ via $G \ltimes X \to X \rtimes G$, $(g,x) \mapsto (g.x, g)$. It is easy to see that an $(X \rtimes G)$–sheaf of R–modules is nothing but a sheaf $\mathcal{A}$ of R–modules over X, $\pi\colon \mathcal{A} \twoheadrightarrow X$, together with a left G–action on $\mathcal{A}$ via homeomorphisms (denoted by $G \times \mathcal{A} \to \mathcal{A}$, $(g,a) \mapsto g.a$) such that π becomes G–equivariant, and $\mathcal{A}_x \to \mathcal{A}_{g.x}$, $a \mapsto g.a$, is an isomorphism of R–modules. We call these G–sheaves of R–modules over X.

3.1 Isomorphisms in homology and cohomology

First of all, let us prove:

Theorem 3.1 *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems, where G and H are countable discrete groups. Suppose that $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent. Then there is an equivalence of categories between G–sheaves of R–modules over X and H–sheaves of R–modules over Y, denoted by $\mathcal{S}_X \mapsto \mathcal{S}_Y$ on the level of objects, such that $H_*(G, \Gamma_c(X,\mathcal{S}_X)) \cong H_*(H, \Gamma_c(Y,\mathcal{S}_Y))$ and $H^*(G, \Gamma(X,\mathcal{S}_X)) \cong H^*(H, \Gamma(Y,\mathcal{S}_Y))$.*

Here Γ stands for continuous sections and Γ_c for those with compact support.

Proof It is easy to see that $H_*(G,\Gamma_c(X,\mathcal{A})) \cong H_*(X \rtimes G, \mathcal{A})$ and $H^*(G,\Gamma(X,\mathcal{A})) \cong H^*(X \rtimes G, \mathcal{A})$ for topological dynamical systems $G \curvearrowright X$ and G–sheaves $\mathcal{A}$ of R–modules over X.

Now, by assumption, there are clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $X = G.A$ and $Y = H.B$ and an isomorphism of topological groupoids $\chi\colon (X \rtimes G)|A \cong (Y \rtimes H)|B$. Let $\iota_A\colon (X \rtimes G)|A \hookrightarrow X \rtimes G$ and $\iota_B\colon (Y \rtimes H)|B \hookrightarrow Y \rtimes H$ be the canonical inclusions. As A is G–full and B is H–full, ι_A and ι_B induce equivalences of categories of sheaves (see [39, Section 2.2]). So we obtain an equivalence of categories between G–sheaves of R–modules over X and H–sheaves of R–modules over Y, denoted by $\mathcal{S}_X \mapsto \mathcal{S}_Y$ on the level of objects, such that $\mathcal{S}_Y$ is uniquely determined by $\chi^*(\mathcal{S}_Y|B) = \mathcal{S}_X|A$. Our theorem now follows from Morita invariance of groupoid (co)homology. □

By the definitions of homological and cohomological dimensions, we have

$$\sup\{n : H_n(G, \Gamma_c(X, \mathcal{A})) \not\cong \{0\}\} \leq \mathrm{hd}_R(G),$$
$$\sup\{n : H^n(G, \Gamma(X, \mathcal{A})) \not\cong \{0\}\} \leq \mathrm{cd}_R(G)$$

for every topological dynamical system $G \curvearrowright X$. Here the suprema are taken over all G–sheaves $\mathcal{A}$ of R–modules over X.

Definition 3.2 A (G, H) continuous orbit couple is called $H_{*,R}G$–full if

$$\sup\{n : H_n(G, \Gamma_c(X, \mathcal{A})) \not\cong \{0\}\} = \mathrm{hd}_R(G)$$

holds for its topological dynamical system $G \curvearrowright X$. It is called $H^{*,R}G$–full if its topological dynamical system $G \curvearrowright X$ satisfies $\sup\{n : H^n(G, \Gamma(X, \mathcal{A})) \not\cong \{0\}\} = \mathrm{cd}_R(G)$.

The following is an immediate consequence of Theorem 3.1:

Corollary 3.3 *If there exists an $H_{*,R}G$–full topologically free (G, H) continuous orbit couple, then $\mathrm{hd}_R(G) \leq \mathrm{hd}_R(H)$. If there exists an $H^{*,R}G$–full topologically free (G, H) continuous orbit couple, then $\mathrm{cd}_R(G) \leq \mathrm{cd}_R(H)$.*

Remark 3.4 Together with Theorem 2.17, Corollary 3.3 can be viewed as an explanation and generalization of the results in [51; 49] concerning coarse invariance of (co)homological dimension. In our terminology, the conditions from [51; 49] that the topological dynamical system $G \curvearrowright X$ of a (G, H) continuous orbit couple admits a G–invariant probability measure and $\mathbb{Q} \subseteq R$ ensure that the (G, H) continuous orbit couple is $H_{*,R}G$–full and $H^{*,R}G$–full (see [51, Section 3.3; 49, Section 4]). Existence of a G–invariant probability measure is guaranteed if G is amenable and the G–space of our continuous orbit couple is compact. Moreover, again in our terminology, it is shown in [49, Section 4] that a (G, H) continuous orbit couple with compact G–space is $H_{*,R}G$–full if $\mathrm{hd}_R(G) < \infty$ and $H^{*,R}G$–full if $\mathrm{cd}_R(G) < \infty$. Once we know this, Theorem 1.5 of [51] and Theorem 1.2 of [49] are immediate consequences of Theorem 2.17 and Corollary 3.3. In Section 4.4, we present an alternative approach to these results.

3.2 Isomorphisms in reduced cohomology

Let $\mathcal{G}$ be an étale locally compact groupoid and $\mathfrak{L} = (\mu, \mathcal{H}, L)$ a (unitary) representation of $\mathcal{G}$ as in [47, Chapter II, Definition 1.6]. Here μ is a quasi-invariant measure on $\mathcal{G}^{(0)}$, $\mathcal{H}$ a Hilbert bundle over $(\mathcal{G}^{(0)}, \mu)$, and L a representation of $\mathcal{G}$, ie for each $\gamma \in \mathcal{G}$, $L(\gamma)$ is a unitary $\mathcal{H}_{s(\gamma)} \cong \mathcal{H}_{r(\gamma)}$, and the conditions in [47, Chapter II, Definition 1.6] are satisfied (σ in [47, Chapter II, Definition 1.6] is the trivial cocycle in our case). Let D be the modular function attached to μ, as in [47, Chapter I, Definition 3.4]. In particular, we are interested in the case $\mathcal{G} = X \rtimes G$ of a transformation groupoid attached to a topological dynamical system $G \curvearrowright X$ on a compact space X. A representation $\mathfrak{L}$ of $X \rtimes G$ gives rise — through its integrated form — to a $*$–representation of $C(X) \rtimes G$, which in turn corresponds in a one-to-one way to a covariant representation $(\pi_{\mathfrak{L}}, \sigma_{\mathfrak{L}})$ of $G \curvearrowright X$ (or rather of $(C(X), G)$).

Now let $\mathcal{G} = X \rtimes G$ and $\mathfrak{L}$ be as above. We define cohomology groups $H^n(\mathcal{G}, \mathfrak{L})$ and reduced cohomology groups $\overline{H}^n(\mathcal{G}, \mathfrak{L})$. Let us write $\mathfrak{L} = (\mu, \mathcal{H}, L)$. Let $\mathcal{G}^{(n)} = \{(\gamma_1, \dots, \gamma_n) \in \mathcal{G}^n : s(\gamma_i) = r(\gamma_{i+1}) \text{ for all } 1 \leq i \leq n-1\}$, and set $r(\gamma_1, \dots, \gamma_n) = r(\gamma_1)$. We will write $\vec{\gamma}$ for elements in $\mathcal{G}^{(n)}$. Let $\Gamma(\mathcal{G}^{(n)}, \mathcal{H})$ be the set of all Borel functions $f \colon \mathcal{G}^{(n)} \to \mathcal{H}$ with $f(\vec{\gamma}) \in \mathcal{H}_{r(\vec{\gamma})}$ such that for every compact subset $K \subseteq \mathcal{G}^{(n)}$, $\int_{\mathcal{G}^{(0)}} \sum_{\vec{\gamma} \in K, r(\vec{\gamma}) = x} \|f(\vec{\gamma})\|^2 \, d\mu(x) < \infty$, where we identify two Borel functions f_1 and f_2 if $\int_{\mathcal{G}^{(0)}} \sum_{\vec{\gamma} \in K, r(\vec{\gamma}) = x} \|f_1(\vec{\gamma}) - f_2(\vec{\gamma})\|^2 \, d\mu(x) = 0$ for every compact subset $K \subseteq \mathcal{G}^{(n)}$. The topology on $\Gamma(\mathcal{G}^{(n)}, \mathcal{H})$ is given by the following notion of convergence: a net $(f_i)_i$ converges to an element f in $\Gamma(\mathcal{G}^{(n)}, \mathcal{H})$ if $\lim_{i \to \infty} \int_{\mathcal{G}^{(0)}} \sum_{\vec{\gamma} \in K, r(\vec{\gamma}) = x} \|f(\vec{\gamma}) - f_i(\vec{\gamma})\|^2 \, d\mu(x) = 0$ for every compact subset $K \subseteq \mathcal{G}^{(0)}$. We define a cochain complex $0 \to \Gamma(\mathcal{G}^{(0)}, \mathcal{H}) \xrightarrow{d^0} \Gamma(\mathcal{G}^{(1)}, \mathcal{H}) \xrightarrow{d^1} \cdots$ with $d^0(f)(\gamma) = D^{-1/2}(\gamma) L(\gamma) f(s(\gamma)) - f(r(\gamma))$, and, for $n \geq 1$, $d^n = \sum_{i=0}^{n+1} (-1)^i d^n_{(i)}$, where

$$d^n_{(0)}(f)(\gamma_0, \dots, \gamma_n) = D^{-1/2}(\gamma_0) L(\gamma_0) f(\gamma_1, \dots, \gamma_n),$$

$$d^n_{(i)}(f)(\gamma_0, \dots, \gamma_n) = f(\gamma_0, \dots, \gamma_{i-1}\gamma_i, \dots, \gamma_n) \quad \text{for } 1 \leq i \leq n,$$

$$\delta^n_{(n+1)}(f)(\gamma_0, \dots, \gamma_n) = f(\gamma_0, \dots, \gamma_{n-1}).$$

It is easy to check that $d^n \circ d^{n-1} = 0$ for all $n \geq 1$. Thus, $\mathrm{im}(d^{n-1}) \subseteq \ker(d^n)$. Since all the d^n are continuous, we also have $\overline{\mathrm{im}}(d^{n-1}) \subseteq \ker(d^n)$. We set $H^n(\mathcal{G}, \mathfrak{L}) := \ker(d^n) / \mathrm{im}(d^{n-1})$ and $\overline{H}^n(\mathcal{G}, \mathfrak{L}) := \ker(d^n) / \overline{\mathrm{im}}(d^{n-1})$.

To see that these cohomology groups are Morita invariant (which we need in the proof of Theorem 3.5), it is straightforward to construct concrete cochain homotopies analogous to the ones in the proof of [35, Proposition 3.5], which lead to Morita invariance of groupoid cohomology as in the proof of [35, Theorem 3.6]. This yields Morita invariance of reduced groupoid cohomology as well, because the maps arising from a Morita equivalence not only induce homomorphisms in groupoid cohomology which are inverse to each other, but they also induce homomorphisms (and hence isomorphisms) in reduced groupoid cohomology.

Our goal is to prove the following:

Theorem 3.5 *Suppose there is a continuous orbit couple for topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces X and Y. Then there is a one-to-one correspondence between representations of $X \rtimes G$ and $Y \rtimes H$, denoted by $\mathfrak{L} \leftrightarrow \mathfrak{M}$, with $H^*(G, \sigma_{\mathfrak{L}}) \cong H^*(H, \sigma_{\mathfrak{M}})$ and $\overline{H}^*(G, \sigma_{\mathfrak{L}}) \cong \overline{H}^*(H, \sigma_{\mathfrak{M}})$.*

For the definition of reduced cohomology $\overline{H}^*$, we refer to [23, Chapitre III]. Also, recall that we write $(\sigma_{\mathfrak{L}}, \pi_{\mathfrak{L}})$ for the covariant representation corresponding to the integrated form of a representation $\mathfrak{L}$ of $\mathcal{G}$.

Proof Clearly, $H^*(X \rtimes G, \mathfrak{L}) \cong H^*(G, \sigma_{\mathfrak{L}})$ and $\overline{H}^*(X \rtimes G, \mathfrak{L}) \cong \overline{H}^*(G, \sigma_{\mathfrak{L}})$.

Now, if there is a continuous orbit couple for topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ on compact spaces X and Y, then by Theorem 2.16, $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent. So there exist clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $G.A = X$ and $H.B = Y$, together with an isomorphism of topological groupoids $\chi\colon (X \rtimes G)|A \cong (Y \rtimes H)|B$. As A is G–full and B is H–full, we get one-to-one correspondences $\mathfrak{L} \leftrightarrow \mathfrak{L}|A$ and $\mathfrak{M} \leftrightarrow \mathfrak{M}|B$ between representations of $X \rtimes G$ and $(X \rtimes G)|A$, and between representations of $Y \rtimes H$ and $(Y \rtimes H)|B$, respectively. Thus we obtain a one-to-one correspondence between representations of $X \rtimes G$ and $Y \rtimes H$, denoted by $\mathfrak{L} \leftrightarrow \mathfrak{M}$, where $\mathfrak{M}$ is uniquely determined by $\chi^*(\mathfrak{M}|B) = \mathfrak{L}|A$. The theorem now follows from Morita invariance of groupoid (co)homology. □

Remark 3.6 If the topological dynamical system $G \curvearrowright X$ is on a second countable space X, then every $*$–representation of $C_c(X \rtimes G)$ on a Hilbert space is the integrated form of a representation of $X \rtimes G$. Actually, $*$–representations of $C_c(X \rtimes G)$ and representations of $X \rtimes G$ are in one-to-one correspondence (see [47, Chapter II, Theorem 1.21 and Corollary 1.23]). Thus we obtain a reformulation of Theorem 3.5: Suppose there is a continuous orbit couple for topological dynamical systems $G \curvearrowright X$

and $H \curvearrowright Y$ on second countable compact spaces X and Y. By Theorem 2.16, $G \curvearrowright X$ and $H \curvearrowright Y$ are Kakutani-equivalent, so there exist clopen subspaces $A \subseteq X$ and $B \subseteq Y$ with $G.A = X$ and $H.B = Y$, together with an isomorphism of topological groupoids $\chi\colon (X \rtimes G)|A \cong (Y \rtimes H)|B$. Let $\Phi\colon C^*((X \rtimes G)|A) \cong C^*((Y \rtimes H)|B)$ be the corresponding isomorphism of groupoid C^*–algebras. Then the one-to-one correspondence $\mathfrak{L} \leftrightarrow \mathfrak{M}$ from Theorem 3.5 translates to a one-to-one correspondence $(\pi,\sigma) \leftrightarrow (\rho,\tau)$ between covariant representations of $G \curvearrowright X$ and $H \curvearrowright Y$, where (ρ,τ) is uniquely determined (up to unitary equivalence) by the requirement that $(\rho \rtimes \tau|_{C^*((Y \rtimes H)|B)}) \circ \Phi = \pi \rtimes \sigma|_{C^*((X \rtimes G)|A)}$. Here we view $C^*((Y \rtimes H)|B)$ and $C^*((X \rtimes G)|A)$ as full corners in $C(Y) \rtimes H$ and $C(X) \rtimes G$. We write $(\rho,\tau) = \mathrm{Ind}_{\Phi^{-1}}(\pi,\sigma)$ and $(\pi,\sigma) = \mathrm{Ind}_{\Phi}(\rho,\tau)$.

Corollary 3.7 *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topological dynamical systems on second countable compact spaces X and Y, and assume that there is a continuous orbit couple for $G \curvearrowright X$ and $H \curvearrowright Y$. Let $(\pi,\sigma) \leftrightarrow (\rho,\tau)$ be as in Remark 3.6. Then we have $H^*(G,\sigma) \cong H^*(H,\tau)$ and $\overline{H}^*(G,\sigma) \cong \overline{H}^*(H,\tau)$.*

Remark 3.8 Theorem 3.5 and Corollary 3.7 have natural analogues in homology, ie for H_* and $\overline{H}_*$.

3.3 Coarse invariance of property H_{FD}

As a consequence of Theorem 3.5, we discuss coarse invariance of Shalom's property H_{FD} from [51]. In this section (Section 3.3), we assume that our spaces are second countable. Let us start with the following:

Lemma 3.9 *Let $G \curvearrowright {}_{\overline{Y}}\Omega_{\overline{X}} \curvearrowleft H$ be a topological coupling, let α and β be as in Section 2.2.1, let $G \curvearrowright \overline{Y}$ and $H \curvearrowright \overline{X}$ be the actions given by $g.x = gx\alpha(g,x)^{-1}$ and $h.y = \beta(y,h^{-1})^{-1}xh^{-1}$, and let $\overline{X} \rtimes G$ and $\overline{Y} \rtimes H$ be the corresponding transformation groupoids. Then*

$$\overline{X} \rtimes G \to (\Omega \rtimes (G \times H))|\overline{X}, \quad (x,g) \mapsto (x,g,\alpha(g^{-1},x)^{-1}),$$
$$\overline{Y} \rtimes H \to (\Omega \rtimes (G \times H))|\overline{Y}, \quad (y,h) \mapsto (y,\beta(y,h),h),$$

are isomorphisms of topological groupoids.

Proof As

$$r(x,g) = x = r(x,g,\alpha(g^{-1},x)^{-1}),$$

$$s(x,g) = x.g = g^{-1}.x = g^{-1}x\alpha(g^{-1},x)^{-1} = s(x,g,\alpha(g^{-1},x)^{-1}),$$
$$(x,g,\alpha(g^{-1},x)^{-1})(g^{-1}x\alpha(g^{-1},x)^{-1},\overline{g},\alpha(\overline{g}^{-1},g^{-1}x\alpha(g^{-1},x)^{-1})^{-1}) \\ = (x,g\overline{g},\alpha((g\overline{g})^{-1},x)^{-1}),$$

the map $\overline{X} \rtimes G \to (\Omega \rtimes (G \times H))|\overline{X}$, $(x,g) \mapsto (x,g,\alpha(g^{-1},x)^{-1})$, is a groupoid homomorphism. It is clearly continuous, and $(\Omega \rtimes (G \times H))|\overline{X} \to \overline{X} \rtimes G$, $(x,g,h) \mapsto (x,g)$, is its continuous inverse. The proof of the second claim is analogous. □

Given a topologically free (G,H) continuous orbit couple which corresponds to the (G,H) coupling $G \curvearrowright {}_Y\Omega_X \curvearrowleft H$ with compact X and Y, the proof of Theorem 2.17 provides a concrete way to construct Kakutani-equivalent dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ together with clopen subspaces $A \subseteq X$ and $B \subseteq Y$ such that $(X \rtimes G)|A \cong (Y \rtimes H)|B$. We need the following:

Lemma 3.10 *We can modify our (G,H) continuous orbit couple above, without changing its topological dynamical system $G \curvearrowright X$, so that the described process yields a topological coupling and subspaces A and B with $A = B$ as subspaces of Ω.*

Proof In the proof of Theorem 2.16, we had constructed A and B as disjoint unions $A = \bigsqcup_g A_g$ and $B = \bigsqcup_g B_g$. Following the construction of the continuous orbit couple out of our topological coupling in Section 2.2.1, we see that these subspaces A_g and B_g were related by $gA_g = B_g$ in Ω. Set $Y' := (Y \setminus B) \sqcup A$. Then X and Y' are still fundamental domains for the H– and G–actions on Ω. So we obtain a new topologically free (G,H) coupling $G \curvearrowright {}_{Y'}\Omega_X \curvearrowleft H$. The construction in Section 2.2.1 yields a continuous orbit couple with new continuous orbit map $p'\colon X \to Y'$ satisfying $p'(X) = A$. Hence, our construction in the proof of Theorem 2.16 gives us the subspaces $A \subseteq X$ and $A \subseteq Y'$ implementing the Kakutani equivalence between $G \curvearrowright X$ and $H \curvearrowright Y'$. □

Let $G \curvearrowright {}_Y\Omega_X \curvearrowleft H$ and $G \curvearrowright X$, $H \curvearrowright Y$ be as above, with a clopen subspace $A \subseteq X \cap Y$ such that $G.A = X$, $H.A = Y$ and $(X \rtimes G)|A \cong (Y \rtimes H)|A$. Let $\Phi\colon C^*((X \rtimes G)|A) \cong C^*((Y \rtimes H)|A)$ be the induced C^*–isomorphism. Lemma 3.9 yields an isomorphism of C^*–algebras

$$C(X) \rtimes G \cong C^*(X \rtimes G) \cong C^*((\Omega \rtimes (G \times H))|X) \cong 1_X\big(C^*(\Omega \rtimes (G \times H))\big)1_X \\ \cong 1_X(C_0(\Omega) \rtimes (G \times H))1_X.$$

Here the second isomorphism is provided by Lemma 3.9. The argument that the third isomorphism is not only an identification (of dense subalgebras) on an algebraic level but also preserves C^*–norms is the same as in the proof of [30, Lemma 5.22 and Corollary 5.23]. Since 1_X is in addition a full projection, $C(X) \rtimes G$ is Morita equivalent to $C_0(\Omega) \rtimes (G \times H)$, and a $C(X) \rtimes G$–$C_0(\Omega) \rtimes (G \times H)$–imprimitivity bimodule is given by $\mathfrak{X} = 1_X(C_0(\Omega) \rtimes (G \times H))$ (with respect to the identification $C(X) \rtimes G \cong 1_X(C_0(\Omega) \rtimes (G \times H))1_X$ provided by Lemma 3.9). We obtain (up to unitary equivalence) bijections between representations of $C(X) \rtimes G$ and representations of $C_0(\Omega) \rtimes (G \times H)$ and also between covariant representations of $G \curvearrowright X$ and $G \times H \curvearrowright \Omega$. We denote both of them by $\mathrm{Ind}_{\mathfrak{X}}$. Also, let $\mathfrak{Y}$ be the $C(Y) \rtimes H$–$C_0(\Omega) \rtimes (G \times H)$–imprimitivity bimodule given by $1_Y(C_0(\Omega) \rtimes (G \times H))$ with respect to the identification $C(Y) \rtimes H \cong 1_Y(C_0(\Omega) \rtimes (G \times H))1_Y$ provided by Lemma 3.9. We define $\mathrm{Ind}_{\mathfrak{Y}}$ similarly as $\mathrm{Ind}_{\mathfrak{X}}$. Now we have two ways to go from covariant representations of $G \curvearrowright X$ to covariant representations of $H \curvearrowright Y$, namely $\mathrm{Ind}_{\Phi^{-1}}$, introduced in Remark 3.6, and $\mathrm{Ind}_{\mathfrak{Y}}^{-1} \mathrm{Ind}_{\mathfrak{X}}$. It turns out that they coincide.

Proposition 3.11 *In the situation described above,* $\mathrm{Ind}_{\mathfrak{Y}}^{-1} \mathrm{Ind}_{\mathfrak{X}}(\pi, \sigma)$ *is unitarily equivalent to* $\mathrm{Ind}_{\Phi^{-1}}(\pi, \sigma)$ *for every covariant representation* (π, σ) *of* $G \curvearrowright X$.

In the following, we write $\sim_u$ for unitary equivalence.

Proof Let $\mathrm{Ind}_{\Phi^{-1}}(\pi, \sigma) = (\rho, \tau)$, and let $\mathrm{Ind}_{\mathfrak{Y}}^{-1} \mathrm{Ind}_{\mathfrak{X}}(\pi, \sigma) = (\rho', \tau')$. Let

$$i^X \colon C^*((X \rtimes G)|A) \hookrightarrow C(X) \rtimes G,$$
$$i^Y \colon C^*((Y \rtimes H)|A) \hookrightarrow C(Y) \rtimes H$$

be the canonical embeddings. Also, let $i_X \colon C(X) \rtimes G \hookrightarrow C_0(\Omega) \rtimes (G \times H)$ and $i_Y \colon C(Y) \rtimes H \hookrightarrow C_0(\Omega) \rtimes (G \times H)$ be the embeddings obtained with the help of Lemma 3.9. Then (ρ, τ) is uniquely determined by $(\pi \rtimes \sigma) \circ i^X \circ \Phi^{-1} \sim_u (\rho \rtimes \tau) \circ i^Y$. We want to show that $\rho' \rtimes \tau'$ has the same property. (ρ', τ') is uniquely determined by the existence of a representation Π of $C_0(\Omega) \rtimes (G \times H)$ with $\Pi \circ i_X \sim_u \pi \rtimes \sigma$ and $\Pi \circ i_Y \sim_u \rho' \rtimes \tau'$. Hence, $(\rho' \rtimes \tau') \circ i^Y \sim_u \Pi \circ i_Y \circ i^Y$. On the groupoid level, $i_Y \circ i^Y$ is given by

$$Y \rtimes H|B \to Y \rtimes H \to \Omega \rtimes (G \times H), \quad (y, h) \mapsto (y, \beta(y, h), h),$$

where β is as defined in Section 2.2.1. At the same time, $i_X \circ i^X \circ \Phi^{-1}$ on the groupoid level is given by

$$(Y \rtimes H)|B \to (X \rtimes G)|A \to X \rtimes G \to \Omega \rtimes (G \times H),$$
$$(y, h) \mapsto (y, b(h^{-1}, y)^{-1}) \mapsto \big(y, b(h^{-1}, y)^{-1}, \alpha(b(h^{-1}, y), y)^{-1}\big),$$

where b comes from the groupoid isomorphism $(X \rtimes G)|A \cong (Y \rtimes H)|A$ (see Remark 2.15 and [31, Definition 2.6]) and α is as defined in Section 2.2.1. We have $\alpha(b(h^{-1}, y), y) = h^{-1}$ by [32, Lemma 2.10] (or rather its analogue for partial actions). Hence, $i_Y \circ i^Y = i_X \circ i^X \circ \Phi^{-1}$, so that $(\rho' \rtimes \tau') \circ i^Y \sim_u \Pi \circ i_Y \circ i^Y = \Pi \circ i_X \circ i^X \circ \Phi^{-1} \sim_u (\pi \rtimes \sigma) \circ i^X \circ \Phi^{-1}$. Our claim follows. □

Let $G \curvearrowright {}_Y\Omega_X \curvearrowleft H$ and $G \curvearrowright X$ and $H \curvearrowright Y$ be as above. Let $A \subseteq X \cap Y$ be a clopen subspace with $G.A = X$, $H.A = Y$ and $(X \rtimes G)|A \cong (Y \rtimes H)|A$. Let Φ: $C^*((X \rtimes G)|A) \cong C^*((Y \rtimes H)|A)$ be the induced C^*–isomorphism. Let $\Pi = (\Pi^X, \Pi^G)$ be a covariant representation of $G \curvearrowright X$ on the Hilbert space $\mathcal{H}$. Let σ be a unitary representation of G on $\mathcal{H}_\sigma$. It is clear that $(1 \otimes \Pi^X, \sigma \otimes \Pi^G)$ is a covariant representation of $G \curvearrowright X$ on $\mathcal{H}_\sigma \otimes \mathcal{H}$. Let $\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi)$ be the unitary representation of H which is part of the covariant representation $\mathrm{Ind}_{\Phi^{-1}}(1 \otimes \Pi^X, \sigma \otimes \Pi^G)$. Moreover, let τ be a unitary representation of H on $\mathcal{H}_\tau$. Let $\Theta = (\Theta^Y, \Theta^H) = \mathrm{Ind}_{\Phi^{-1}}(\Pi^X, \Pi^G)$. Denote by $\mathrm{Ind}_\Phi(\Theta, \tau)$ the unitary representation of G which is part of the covariant representation $\mathrm{Ind}_\Phi(\Theta^Y \otimes 1, \Theta^H \otimes \tau)$.

Lemma 3.12 $(1 \otimes \Pi^X \otimes 1, \sigma \otimes \mathrm{Ind}_\Phi(\Theta, \tau)) = \mathrm{Ind}_\Phi(1 \otimes \Theta^Y \otimes 1, \mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)$.

Proof We have to show that

$$(1 \otimes \Pi^X \otimes 1) \rtimes (\sigma \otimes \mathrm{Ind}_\Phi(\Theta, \tau))|_{C^*((X \rtimes G)|A)} = (1 \otimes \Theta^Y \otimes 1) \rtimes (\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)|_{C^*((Y \rtimes H)|B)} \circ \Phi.$$

Fix $g \in G$ and $h \in H$. Let f be the characteristic function of a compact subset of $(X \times \{g\}) \cap (X \rtimes G)|A$ whose image under χ lies in $(Y \times \{h\}) \cap (Y \rtimes H)|B$. It suffices to consider such f as they span a dense subset in $C^*((X \rtimes G)|A)$. We have

$$\begin{aligned}
(1 \otimes \Theta^Y \otimes 1) &\rtimes (\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau)(\Phi(f)) \\
&= ((1 \otimes \Theta^Y) \rtimes \mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi))(\Phi(f)) \otimes \tau(h) \\
&= ((1 \otimes \Pi^X) \rtimes (\sigma \otimes \Pi^G)(f)) \otimes \tau(h) \\
&= \sigma(g) \otimes \Pi(f) \otimes \tau(h) \\
&= \sigma(g) \otimes (\Theta(\Phi(f)) \otimes \tau(h)) \\
&= \sigma(g) \otimes ((\Pi^X \otimes 1) \rtimes \mathrm{Ind}_\Phi(\Theta, \tau)(f)) \\
&= (1 \otimes \Pi^X \otimes 1) \rtimes (\sigma \otimes \mathrm{Ind}_\Phi(\Theta, \tau))(f).
\end{aligned}$$

□

Let Λ be a representation of $C(X) \rtimes G$, and set $\widetilde{\Lambda} := \mathrm{Ind}_{\mathfrak{X}} \Lambda$. Let

$$\mathcal{H}_{\widetilde{\Lambda},c} := \{\eta \in \mathcal{H}_{\widetilde{\Lambda}} : \eta = \widetilde{\Lambda}(1_K)\eta \text{ for some compact } K \subseteq \Omega\},$$

and let $\mathcal{L}$ be the complex vector space of linear maps $\mathcal{H}_{\widetilde{\Lambda},c} \to \mathbb{C}$ which are bounded whenever restricted to a subspace of the form $\widetilde{\Lambda}(1_K)\mathcal{H}_{\widetilde{\Lambda}}$, with $K \subseteq \Omega$ compact. Moreover, let Λ^G be the unitary representation of G on $\mathcal{H}_\Lambda$ induced by Λ, and denote by $\widetilde{\Lambda}^G$ and $\widetilde{\Lambda}^H$ the unitary representations of G and H on $\mathcal{H}_{\widetilde{\Lambda}}$ induced by $\widetilde{\Lambda}$. As $\mathcal{H}_{\widetilde{\Lambda},c}$ is obviously invariant under the G– and H–actions, we obtain by restriction G– and H–actions on $\mathcal{H}_{\widetilde{\Lambda},c}$. Finally, by dualizing, we obtain G– and H–actions on $\mathcal{L}$.

Lemma 3.13 *There is a G–equivariant linear isomorphism $\mathcal{H}_\Lambda \cong \mathcal{L}^H$.*

Proof Up to unitary equivalence, we have $\mathcal{H}_\Lambda = \widetilde{\Lambda}(1_{\overline{X}})\mathcal{H}_{\widetilde{\Lambda}}$, and Λ^G is given by the composite

$$G \hookrightarrow C(X) \rtimes G \cong 1_{\overline{X}} C_0(\Omega) \rtimes (G \times H) 1_{\overline{X}} \xrightarrow{\widetilde{\Lambda}} \mathcal{L}(\widetilde{\Lambda}(1_{\overline{X}})\mathcal{H}_{\widetilde{\Lambda}}),$$

where the first map is given by $G \hookrightarrow C(X) \rtimes G,\ g \mapsto u_g$.

We define $L\colon \mathcal{H}_\Lambda \to \mathcal{L}$ by setting $L(\xi)(\eta) = \sum_{h \in H} \langle \widetilde{\Lambda}^H(h)\xi, \eta \rangle$. Here $\langle \cdot\,,\cdot \rangle$ is the inner product in $\mathcal{H}_{\widetilde{\Lambda}}$, and our convention is that it is linear in the second component. Note that in the definition of $L(\xi)(\eta)$, the sum is always finite since η lies in $\mathcal{H}_{\widetilde{\Lambda},c}$. It is clear that L is linear. Moreover, we have

$$L(\xi)(\widetilde{\Lambda}^H(h')\eta) = \sum_h \langle \widetilde{\Lambda}^H(h)\xi, \widetilde{\Lambda}^H(h')\eta \rangle = \sum_h \langle \widetilde{\Lambda}^H((h')^{-1}h)\xi, \eta \rangle = L(\xi)(\eta).$$

Therefore, the image of L lies in $\mathcal{L}^H$, and we obtain a linear map $\mathcal{H}_\Lambda \to \mathcal{L}^H$. We claim that the inverse is given by $R\colon \mathcal{L}^H \to \mathcal{H}_\Lambda^* \cong \mathcal{H}_\Lambda$, where the first map is given by restriction, $l \mapsto l|_{\widetilde{\Lambda}(1_{\overline{X}})\mathcal{H}_{\widetilde{\Lambda}}}$, and the second map is the canonical isomorphism, identifying $\zeta \in \mathcal{H}_\Lambda$ with the element $\langle \zeta, \cdot \rangle \in \mathcal{H}_\Lambda^*$. Note that $l|_{\widetilde{\Lambda}(1_{\overline{X}})\mathcal{H}_{\widetilde{\Lambda}}}$ is bounded because of our definition of $\mathcal{H}_{\widetilde{\Lambda},c}$. Let us show that R is the inverse of L. For $l \in \mathcal{L}^H$, we have

$$\begin{aligned} L(R(l))(\eta) &= \sum_h \langle \widetilde{\Lambda}^H(h)R(l), \eta \rangle = \sum_h \langle R(l), \widetilde{\Lambda}^H(h^{-1})\eta \rangle = \sum_h l(\widetilde{\Lambda}(1_{\overline{X}})\widetilde{\Lambda}^H(h^{-1})\eta) \\ &= \sum_h l(\widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{\overline{X}})\widetilde{\Lambda}^H(h^{-1})\eta) = \sum_h l(\widetilde{\Lambda}(1_{\overline{X}h}\eta)) = l(\eta). \end{aligned}$$

For $\xi \in \mathcal{H}_\Lambda = \widetilde{\Lambda}(1_{\overline{X}})\mathcal{H}_{\widetilde{\Lambda}}$, we have $R(L(\xi)) = \xi$ since

$$L(\xi)(\widetilde{\Lambda}(1_{\overline{X}})\eta) = \sum_h \langle \widetilde{\Lambda}^H(h)\xi, \widetilde{\Lambda}(1_{\overline{X}})\eta \rangle = \sum_h \langle \widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{\overline{X}h})\xi, \eta \rangle = \langle \xi, \eta \rangle$$

because $\widetilde{\Lambda}(1_{\overline{X}h})\xi = \xi$ if $h = e$ and $\widetilde{\Lambda}(1_{\overline{X}h})\xi = 0$ if $h \neq e$.

Finally, let us show that L is G–equivariant:

$$\begin{aligned}
L(\Lambda^G(g)\xi)(\eta) &= \sum_h \langle \widetilde{\Lambda}^H(h)(\Lambda^G(g)\xi), \eta\rangle \\
&= \sum_h \sum_j \langle \widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{g\overline{X}j^{-1}\cap\overline{X}})\widetilde{\Lambda}^G(g)\widetilde{\Lambda}^H(j)\xi, \eta\rangle \\
&= \sum_{h,j} \langle \widetilde{\Lambda}^H(h)\widetilde{\Lambda}^H(j)^{-1}\widetilde{\Lambda}(1_{g\overline{X}j^{-1}\cap\overline{X}})\widetilde{\Lambda}^H(j)\widetilde{\Lambda}^G(g)\xi, \eta\rangle \\
&= \sum_{h,j} \langle \widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{g\overline{X}\cap\overline{X}j})\widetilde{\Lambda}^G(g)\xi, \eta\rangle \\
&= \sum_h \langle \widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{g\overline{X}})\widetilde{\Lambda}^G(g)\xi, \eta\rangle \\
&= \sum_h \langle \widetilde{\Lambda}^G(g)\widetilde{\Lambda}^H(h)\widetilde{\Lambda}(1_{\overline{X}})\xi, \eta\rangle \\
&= \sum_h \langle \widetilde{\Lambda}^H(h)\xi, \widetilde{\Lambda}^G(g)^{-1}\eta\rangle \\
&= L(\xi)(\widetilde{\Lambda}^G(g)^{-1}\eta).
\end{aligned}$$
□

Corollary 3.14 *We have* $\{\Lambda^G$*–invariant vectors*$\} = \mathcal{H}_\Lambda^G \cong \mathcal{L}^{G\times H}$.

Theorem 3.15 *There exists a one-to-one correspondence between* $\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi)\otimes\tau$*–invariant vectors and* $\sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau)$*–invariant vectors.*

Proof Obviously, $(1\otimes\Pi^X\otimes 1, \sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau))$ is a covariant representation of $G \curvearrowright X$. Let $\Lambda := (1\otimes\Pi\otimes 1)\rtimes(\sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau))$. Set $\widetilde{\Lambda} := \mathrm{Ind}_{\mathfrak{X}}\Lambda$, and define $\mathcal{L}$ as in Lemma 3.13. Then Corollary 3.14 yields a one-to-one correspondence between $\sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau)$–invariant vectors and $\mathcal{L}^{G\times H}$.

Let $\mathrm{Ind}_{\Phi^{-1}}\Lambda$ be the representation of $C(Y)\rtimes H$ corresponding to

$$\mathrm{Ind}_{\Phi^{-1}}(1\otimes\Pi^X\otimes 1, \sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau)).$$

By Proposition 3.11, $\mathrm{Ind}_{\mathfrak{Y}}\mathrm{Ind}_{\Phi^{-1}}\Lambda \sim_u \widetilde{\Lambda}$. Hence, together with Lemma 3.12, Corollary 3.14 yields a one-to-one correspondence between $\mathrm{Ind}_{\Phi^{-1}}(\sigma,\Pi)\otimes\tau$–invariant vectors and $\mathcal{L}^{G\times H}$.

Thus,

$$\{\mathrm{Ind}_{\Phi^{-1}}(\sigma,\Pi)\otimes\tau\text{–invariant vectors}\} \xleftrightarrow{1-1} \mathcal{L}^{G\times H} \xleftrightarrow{1-1} \{\sigma\otimes\mathrm{Ind}_\Phi(\Theta,\tau)\text{–invariant vectors}\}.$$
□

Corollary 3.16 $\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau$ *has an invariant vector if and only if* $\sigma \otimes \mathrm{Ind}_{\Phi}(\Theta, \tau)$ *has an invariant vector.*

We now come to Shalom's property H_{FD}. Recall that a group G has H_{FD} if for every unitary representation σ of G, $\overline{H}^1(G, \sigma) \ncong \{0\}$ implies that σ contains a finite-dimensional subrepresentation.

Definition 3.17 A (G, H) continuous orbit couple is called $\overline{H}^1 G$–faithful if its G– and H–spaces are second countable compact, and its topological dynamical system $G \curvearrowright X$ has the property that for every unitary representation σ of G with $\overline{H}^1(G, \sigma) \ncong \{0\}$, there exists a covariant representation (Π^X, Π^G) of $G \curvearrowright X$ such that $\overline{H}^1(G, \sigma \otimes \Pi^G) \ncong \{0\}$.

Theorem 3.18 *Let G and H be countable discrete groups. Suppose there exists an $\overline{H}^1 G$–faithful topologically free (G, H) continuous orbit couple. If H has property H_{FD}, then G has property H_{FD}.*

Proof of Theorem 3.18 By Lemma 3.10, we may assume that our $\overline{H}^1 G$–faithful topologically free (G, H) continuous orbit couple corresponds to a topologically free (G, H) coupling $G \curvearrowright {}_Y\Omega_X \curvearrowleft H$ with second countable compact spaces X and Y, which leads to topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ together with a clopen subspace $A \subseteq X \cap Y$ with $G.A = X$, $H.A = Y$ and $(X \rtimes G)|A \cong (Y \rtimes H)|A$. Now let σ be a unitary representation of G with $\overline{H}^1(G, \sigma) \ncong \{0\}$. By $\overline{H}^1 G$–faithfulness, there exists a covariant representation (Π^X, Π^G) of $G \curvearrowright X$ with $\overline{H}^1(G, \sigma \otimes \Pi^G) \ncong \{0\}$. By Corollary 3.7, $\overline{H}^1(H, \mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi)) \cong \overline{H}^1(G, \sigma \otimes \Pi^G)$, so that $\overline{H}^1(H, \mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi)) \ncong \{0\}$. As H has property H_{FD}, $\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi)$ must have a finite-dimensional subrepresentation. Thus, Proposition A.1.12 of [3] implies that there is a unitary representation τ of H such that $\mathrm{Ind}_{\Phi^{-1}}(\sigma, \Pi) \otimes \tau$ has an invariant vector. By Corollary 3.16, $\sigma \otimes \mathrm{Ind}_{\Phi}(\Theta, \tau)$ must have an invariant vector. Again by [3, Proposition A.1.12], this implies that σ has a finite-dimensional subrepresentation. Hence, G has property H_{FD}. □

Remark 3.19 A (G, H) continuous orbit couple with second countable compact G– and H–spaces is $\overline{H}^1 G$–faithful if its topological dynamical system $G \curvearrowright X$ admits a G–invariant probability measure. To see this, let μ be such a measure. Let (Π^X, Π^G) be the canonical covariant representation of $G \curvearrowright X$ on $L^2(\mu)$. Then Π^G contains the trivial representation, so that $\sigma \otimes \Pi^G$ contains σ. This shows $\overline{H}^1 G$–faithfulness.

In particular, this is the case when G is amenable. Therefore, Theorems 2.17 and 3.18 imply [51, Theorem 4.3.3]. The case of amenable groups is not the only situation where invariant probability measures exist. It follows easily from [13] and Theorem 2.17 that for residually finite groups G and H with coarsely equivalent box spaces, there exists a (G, H) continuous orbit couple with second countable compact G– and H–spaces such that its topological dynamical system $G \curvearrowright X$ admits a G–invariant probability measure. A similar statement applies to sofic groups with coarsely equivalent spaces of graphs (see [1]). Note, however, that having coarsely equivalent box spaces is a strong assumption, as this implies commensurability for finitely presented, residually finite groups by [15].

4 Applications to (co)homology, II

We now turn to coarse invariants of (co)homological nature.

4.1 Coarse maps and res–invariant modules

Let G be a group, R a commutative ring with unit and W an R–module. Let $C(G, W)$ be the set of functions from G to W. The G–action on itself by left multiplication induces a canonical left RG–module structure on $C(G, W)$. Explicitly, given $g \in G$ and $f \in C(G, W)$, $g.f$ is the element in $C(G, W)$ given by $(g.f)(x) = f(g^{-1}x)$ for all $x \in G$. We are interested in the following class of RG–submodules of $C(G, W)$. Given a subset A of G, let 1_A be its indicator function, ie $1_A \in C(G, R)$ is given by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Here 1 is the unit of R. Given $f \in C(G, W)$ and $A \subseteq G$, we form the pointwise product $1_A \cdot f \in C(G, W)$. This is nothing but the restriction of f to A, extended by 0 outside of A to give a function $G \to W$.

Definition 4.1 An RG–submodule $L \subseteq C(G, W)$ is called res–invariant if $1_A \cdot f$ lies in L for all $f \in L$ and $A \subseteq G$.

Examples 4.2 For arbitrary R and W, the submodules $C(G, W)$,

$$C_f(G, W) = \{f \in C(G, W) : f \text{ takes finitely many values}\},$$
$$RG \otimes_R W \cong \{f \in C(G, W) : \operatorname{supp}(f) \text{ is finite}\}$$

are res–invariant.

If $R = \mathbb{R}$ or $R = \mathbb{C}$ and $W = R$, then $c_0(G, W) = \{f \in C(G, W) : \lim_{x \to \infty} |f(x)| = 0\}$ is res–invariant, and $\ell^p(G, W) = \left\{f \in C(G, W) : \sum_{x \in G} |f(x)|^p < \infty\right\}$ is res–invariant for all $0 < p \leq \infty$.

Let G be a finitely generated discrete group and ℓ the right-invariant word length coming from a finite symmetric set of generators. Let $R = \mathbb{R}$ or $R = \mathbb{C}$ and $W = R$. As in [25], we define for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ the Sobolev space $H^{s,p}(G, W) := \{f\colon G \to W : f \cdot (1+\ell)^s \in \ell^p(G, W)\}$, and $H^{\infty,p}(G, W) := \bigcap_{s \in \mathbb{R}} H^{s,p}(G, W)$. All these Sobolev spaces are res–invariant.

In the last examples (ℓ^p, c_0 and $H^{s,p}$), we can also replace W by any normed space over R.

We are also interested in the following topological setting; let R be a topological field and W an R–module.

Definition 4.3 A topological res–invariant RG–submodule L of $C(G, W)$ is a res–invariant RG–submodule of $C(G, W)$ together with the structure of a topological R–vector space on L such that

$$L \to L, \quad f \mapsto 1_A \cdot f, \quad \text{is continuous for every } A \subseteq G, \tag{2}$$

$$L \to L, \quad f \mapsto g.f, \quad \text{is continuous for every } g \in G. \tag{3}$$

When we consider topological res–invariant modules, R will always be a topological field, though we might not mention this explicitly. For instance, in Examples 4.2, $\ell^p(G, W)$ and $c_0(G, W)$ are topological res–invariant modules. Also, $H^{s,p}(G, W)$ becomes a topological res–invariant module with respect to the topology induced by the norm $\|f\|_{s,p} = \|f \cdot (1+\ell)^s\|_{\ell^p(G,W)}$ for $s \in \mathbb{R}$, and with respect to the projective limit topology for $s = \infty$.

In the following, we explain how coarse maps interact with res–invariant modules. Recall that all our groups are countable and discrete, and that a map $\varphi\colon G \to H$ between groups G and H is a coarse map if $\varphi^{-1}(\{y\})$ is finite for every $y \in H$, and, for all $S \subseteq G \times G$, $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\}$ must be finite if $\{st^{-1} : (s,t) \in S\}$ is finite (Definition 1.1).

Remark 4.4 Let $\varphi\colon G \to H$ be a coarse map. Given $g \in G$, define the set $S = \{(g^{-1}x, x) \in G \times G : x \in G\}$. Then $\{st^{-1} : (s,t) \in S\} = \{g^{-1}\}$ is finite, so that $\{\varphi(g^{-1}x)\varphi(x)^{-1} : x \in G\}$ is finite. In other words, we can find a finite decomposition $G = \bigsqcup_{i \in I} X_i$, where I is a finite-index set, and a finite subset $\{h_i : i \in I\} \subseteq H$ such that $\varphi(g^{-1}x) = h_i^{-1}\varphi(x)$ for all $x \in X_i$ and $i \in I$.

Recall that two maps $\varphi, \phi\colon G \to H$ are close (written $\varphi \sim \phi$) if $\{\varphi(x)\phi(x)^{-1} : x \in G\}$ is finite (Definition 1.1).

Remark 4.5 If $\varphi, \phi\colon G \to H$ are close, then there is a finite decomposition $G = \bigsqcup_{i \in I} X_i$, where I is a finite-index set, and a finite subset $\{h_i : i \in I\} \subseteq H$ such that we have $\phi(x) = h_i \varphi(x)$ for all $x \in X_i$ and $i \in I$.

Let R and W be as above, and $\varphi\colon G \to H$ a coarse map. Given $f \in C(G, W)$, define $\varphi_*(f) \in C(H, W)$ by setting $\varphi_*(f)(y) = \sum_{x \in G, \varphi(x) = y} f(x)$. Moreover, given $f \in C(H, W)$, define $\varphi^*(f) = f \circ \varphi \in C(G, W)$.

Definition 4.6 Given a res–invariant RG–submodule L of $C(G, W)$, let $\varphi_* L$ be the smallest res–invariant RH–submodule of $C(H, W)$ containing $\{\varphi_*(f) : f \in L\}$. Given a res–invariant RH–submodule M of $C(G, W)$, let $\varphi^* M$ be the smallest res–invariant RG–submodule of $C(H, W)$ containing $\{\varphi^*(f) : f \in M\}$.

Lemma 4.7 *We have*

$$\varphi_* L = \langle \{h.\varphi_*(f) : h \in H,\ f \in L\} \rangle_R, \tag{4}$$

$$\varphi^* M = \langle \{1_A \cdot \varphi^*(f) : f \in M,\ A \subseteq G\} \rangle_R. \tag{5}$$

Proof We obviously have "$\supseteq$" in (4). To show "$\subseteq$", it suffices to show that the right-hand side is res–invariant as it is obviously an RH–submodule. Given $B \subseteq H$, we have for all $h \in H$ and $f \in L$ that

$$1_B \cdot (h.\varphi_*(f)) = h.(1_{h^{-1}B} \cdot \varphi_*(f)) = h.(\varphi_*(1_{\varphi^{-1}(h^{-1}B)} \cdot f)),$$

which lies in the right-hand side as L is res–invariant.

For (5), we again have "$\supseteq$" by construction. As the right-hand side is res–invariant, it suffices to show that it is an RG–submodule in order to prove "$\subseteq$". Given $g \in G$, by Remark 4.4 we can find a finite decomposition $G = \bigsqcup_{i \in I} X_i$ and a finite subset $\{h_i : i \in I\} \subseteq H$ such that $\varphi(g^{-1}x) = h_i^{-1} \varphi(x)$ for all $x \in X_i$ and $i \in I$. Then, for all $A \subseteq G$, $g.(1_A \cdot \varphi^*(f)) = 1_{gA} \cdot (g.\varphi^*(f)) = \sum_{i \in I} 1_{X_i} \cdot 1_{gA} \cdot (g.\varphi^*(f)) = \sum_{i \in I} 1_{X_i} \cdot 1_{gA} \cdot (\varphi^*(h_i.f))$ lies in the right-hand side of (5) as M is an RH–submodule. □

Note that in general, $\varphi_* L$ is not equal to $\{\varphi_*(f) : f \in L\}$, and $\varphi^* M$ is not equal to $\{\varphi^*(f) : f \in M\}$.

Lemma 4.8 (i) *If $\varphi, \phi\colon G \to H$ are coarse maps with $\varphi \sim \phi$, then $\varphi_* L = \phi_* L$ and $\varphi^* M = \phi^* M$ for all L and M.*

(ii) *$\psi_* \varphi_* L = (\psi \circ \varphi)_* L$ and $\varphi^* \psi^* N = (\psi \circ \varphi)^* N$ for all L, N and coarse maps $\varphi\colon G \to H$ and $\psi\colon H \to K$.*

Proof (i) Let us show $\varphi_* L = \phi_* L$. By Remark 4.5, there is a finite decomposition $G = \bigsqcup_{i \in I} X_i$ and a finite subset $\{h_i : i \in I\} \subseteq H$ such that $\phi(x) = h_i \varphi(x)$ for all $x \in X_i$ and $i \in I$. Then

$$\phi_*(f) = \sum_{i \in I} \phi_*(1_{X_i} \cdot f) = \sum_{i \in I} h_i . \varphi_*(1_{X_i} \cdot f) \in \varphi_* L$$

for all $f \in L$. Hence, $\phi_* L \subseteq \varphi_* L$. By symmetry, we have $\phi_* L = \varphi_* L$.

Let us show $\varphi^* M = \phi^* M$. Let I, $\{X_i : i \in I\}$ and $\{h_i : i \in I\}$ be as above. We have that

$$\varphi^*(f) = \sum_{i \in I} 1_{X_i} \cdot \varphi^*(f) = \sum_{i \in I} 1_{X_i} \cdot \phi^*(h_i . f) \in \phi^* M$$

for all $f \in M$. Hence, $\varphi^* M \subseteq \phi^* M$. By symmetry, we have $\varphi^* M = \phi^* M$.

(ii) Let us show $\psi_* \varphi_* L = (\psi \circ \varphi)_* L$. Obviously, "$\supseteq$" holds as $\psi_* \varphi_* L \ni \psi_*(\varphi_*(f)) = (\psi \circ \varphi)_*(f)$ for all $f \in L$. Let us show "$\subseteq$". By (4), it suffices to show that $\psi_*(h . \varphi_*(f)) \in (\psi \circ \phi)_* L$ for all $h \in H$ and $f \in L$. By Remark 4.4, we can find a finite decomposition $H = \bigsqcup_{i \in I} Y_i$ and a finite subset $\{k_i : i \in I\} \subseteq K$ such that $\psi(h^{-1} y) = k_i^{-1} \psi(y)$ for all $y \in Y_i$ and $i \in I$. Then

$$\begin{aligned}\psi_*(h . \varphi_*(f)) = \sum_{i \in I} \psi_*(1_{Y_i} \cdot (h . \varphi_*(f))) &= \sum_{i \in I} k_i . \psi_*\big(1_{h^{-1} Y_i} \cdot (\varphi_*(f))\big) \\ &= \sum_{i \in I} k_i . (\psi \circ \varphi)_*(1_{\varphi^{-1}(h^{-1} Y_i)} \cdot f)\end{aligned}$$

lies in $(\psi \circ \varphi)_* L$ for all $f \in L$ as L is res–invariant. This shows "$\subseteq$".

Let us show $\varphi^* \psi^* N = (\psi \circ \varphi)^* N$. The inclusion "$\supseteq$" holds as $\varphi^* \psi^* N \ni \varphi^*(\psi^*(f))$ for all $f \in N$. Let us prove "$\subseteq$". By (5), it suffices to prove that $\varphi^*(1_B \cdot \psi^*(f)) \in (\psi \circ \varphi)^* N$ for all $B \subseteq H$ and $f \in N$. We have

$$\varphi^*(1_B \cdot \psi^*(f)) = 1_{\varphi^{-1}(B)} \cdot \varphi^*(\psi^*(f)) = 1_{\varphi^{-1}(B)} \cdot (\psi \circ \varphi)^*(f),$$

which lies in $(\psi \circ \varphi)^* N$ as the latter is res–invariant. This shows "$\subseteq$". □

4.2 Coarse embeddings and res–invariant modules

Recall that a map $\varphi \colon G \to H$ between groups G and H is a coarse embedding if for every $S \subseteq G \times G$, $\{s t^{-1} : (s,t) \in S\}$ is finite if and only if $\{\varphi(s) \varphi(t)^{-1} : (s,t) \in S\}$ is finite (Definition 1.1).

Lemma 4.9 *Let $\varphi\colon G \to H$ be a coarse embedding, and let $Y := \varphi(G)$. Then we can find $X \subseteq G$ such that $X \to Y$, $x \mapsto \varphi(x)$, is a bijection. In addition, we can find a finite decomposition $G = \bigsqcup_{i=1}^{I} X_i$ with $g(i) \in G$ for $1 \le i \le I$ and $h(i) \in H$ for $1 \le i \le I$, such that $X_i = g(i)^{-1} X(i)$ for some $X(i) \subseteq X$, with $g(1) = e$* (identity in G), *$h(1) = e$* (identity in H), *$X_1 = X(1) = X$ and $\varphi(x) = h(i)\varphi(g(i)x)$ for all $x \in X_i$ and $1 \le i \le I$.*

Proof By Lemma 2.20, we can find X such that the restriction of φ to X is bijective onto its image and that there are finitely many $g(i) \in G$, $1 \le i \le I$, such that $G = \bigcup_{i=1}^{I} g(i)^{-1} X$, where we can certainly arrange $g(1) = e$. Now define recursively $X_1 := X$ and $X(i) = X \setminus g(i)(g(1)^{-1} X_1 \cup \cdots \cup g(i-1)^{-1} X_{i-1})$. Then $G = \bigsqcup_{i=1}^{I} g(i)^{-1} X(i)$. Using Remark 4.4, we can further decompose each $X(i)$ to guarantee that there exist $h(i) \in H$ for $1 \le i \le I$ such that $\varphi(x) = h(i)\varphi(g(i)x)$ for all $x \in g(i)^{-1} X(i)$ and $1 \le i \le I$. Setting $X_i := g(i)^{-1} X(i)$, we are done. □

Lemma 4.10 *Let $\varphi\colon G \to H$ be a coarse embedding, and fix $h \in H$. There exists a finite subset $F \subseteq G$ such that for all $x, \tilde{x} \in G$ with $\varphi(\tilde{x}) = h^{-1}\varphi(x)$, we must have $\tilde{x} \in Fx$.*

Proof Let $S = \{(s,t) \in G : \varphi(s) = h^{-1}\varphi(t)\}$. Then $\{\varphi(s)\varphi(t)^{-1} : (s,t) \in S\} = \{h^{-1}\}$ is finite, so that $F = \{st^{-1} : (s,t) \in S\}$ is finite since φ is a coarse embedding. □

Let $\varphi\colon G \to H$ be a coarse embedding, and set $Y := \varphi(G)$. Lemma 2.20 yields a subset $X \subseteq G$ such that the restriction of φ to X is a bijection $\tilde{\varphi}\colon X \cong Y$, $x \mapsto \varphi(x)$. It is clear that $H = \bigcup_{h \in H} hY$. Enumerate H, say $H = \{h_1, h_2, \dots\}$, where $h_1 = e$ is the identity. Define recursively $Y_1 := Y$ and $Y_j := Y \setminus h_j^{-1}(h_1 Y_1 \cup \cdots \cup h_{j-1} Y_{j-1})$. By construction, we have a decomposition as a disjoint union $H = \bigsqcup_{j=1}^{\infty} h_j Y_j$. Clearly, for all $h \in H$,

$$hY \cap h_j Y_j = \varnothing \quad \text{for all but finitely many } j. \tag{6}$$

Definition 4.11 Define $\omega\colon H \to G$ by setting $\omega(y) = \tilde{\varphi}^{-1}(h_j^{-1} y)$ for $y \in h_j Y_j$.

By construction,

$$(\varphi \circ \omega)(y) = h_j^{-1} y \quad \text{for } y \in h_j Y_j. \tag{7}$$

Take F as in Lemma 4.10 for $h = e$. Then $(\omega \circ \varphi)(x) \in Fx$ for all $x \in G$, so $\{(\omega \circ \varphi)(x)x^{-1} : x \in G\}$ is finite, ie

$$\omega \circ \varphi \sim \mathrm{id}_G. \tag{8}$$

In general, preimages under ω can be infinite, so that for an arbitrary $f \in C(H,W)$, $\omega_*(f)$ may not be defined. However, we can define $\omega_*(f)$ for $f \in \varphi_* L$, where $L \subseteq C(G,W)$ is a res–invariant RG–submodule. We need some preparation. The following is an immediate consequence of (4) and (6):

Lemma 4.12 *We have* $\varphi_* L = \bigoplus_{j=1}^{\infty} 1_{h_j Y_j} \cdot (\varphi_* L)$ *as* R*–modules.*

Let F be as in Lemma 4.10 for $h = e$. For every $x \in G$, define $F_x \subseteq F$ by $\{\tilde{x} \in G : \varphi(\tilde{x}) = \varphi(x)\} = F_x x$. For every subset $F_i \subseteq F$, define $X_i = \{x \in G : F_x = F_i\}$. Then $G = \bigsqcup_{F_i \subseteq F} X_i$, and by construction, we have the following:

Lemma 4.13 $\varphi^*(\varphi_*(f)) = \sum_{F_i \subseteq F} 1_{X_i} \cdot \bigl(\sum_{g \in F_i} g^{-1}.f\bigr)$.

Similarly, let F be as in Lemma 4.10 for some fixed $h \in H$. Let $X \subseteq G$ be as above. For all $x \in X$, define $F_x \subseteq F$ by setting $\{\tilde{x} \in G : \varphi(\tilde{x}) = h^{-1}\varphi(x)\} = F_x x$. For a subset $F_i \subseteq F$, let $X_i = \{x \in X : F_x = F_i\}$. We have $X = \bigsqcup_{F_i \subseteq F} X_i$ and, by construction:

Lemma 4.14 $1_Y \cdot (h.\varphi_*(f)) = \varphi_*\bigl(\sum_{F_i \subseteq F} 1_{X_i} \cdot \bigl(\sum_{g \in F_i} g^{-1}.f\bigr)\bigr)$.

Now we are ready for the following:

Lemma 4.15 *Let* $L \subseteq C(G,W)$ *be a* res*–invariant* RG*–submodule. Then* $\varphi_* L \to L$, $f \mapsto \omega_*(f)$*, is well defined, where* $\omega_*(f)(x) = \sum_{y \in H,\, \omega(y) = x} f(y)$.

Proof By Lemma 4.12, it suffices to show that for every j and $f \in 1_{h_j Y_j} \cdot \varphi_* L$, $\omega_*(f)$ lies in L. For such f, we know that $\omega_*(f) = 1_X \cdot \varphi^*(h_j^{-1}.f)$. As f lies in $1_{h_j Y_j} \cdot \varphi_* L$, the function $h_j^{-1}.f$ lies in $1_{Y_j} \cdot \varphi_* L \subseteq 1_Y \cdot \varphi_* L$. Hence, it suffices to show that $1_X \cdot \varphi^*(\varphi_* L) \subseteq L$. By (4), it is enough to show that $\varphi^*(h.\varphi_*(f)) \in L$ for all $f \in L$. This follows immediately from Lemmas 4.13 and 4.14. □

Definition 4.16 For φ and L as above, set

$$\varphi^{*-1} L := \{f \in C(H,W) : \varphi^*(h.f) \in L \text{ for all } h \in H\}.$$

Note that the construction of $\varphi^{*-1} L$ makes sense for general coarse maps φ, but we will only be interested in the case where φ is a coarse embedding. We collect a few properties of $\varphi^{*-1} L$:

Lemma 4.17 (a) $\varphi^{*-1}L$ *is a* res*–invariant* RH*–submodule of* $C(H,W)$*.*

(b) *For* $f\in C(H,W)$*,* $f\in\varphi^{*-1}L$ *if and only if* $1_{hY}\cdot f\in\varphi^{*-1}L$ *for all* $h\in H$*.*

(c) $\varphi^{*-1}L$ *is the biggest* res*–invariant* RH*–submodule* M *of* $C(G,W)$ *such that* $\varphi^*(f)\in L$ *for all* $f\in M$*.*

(d) *Let* ω *be as in Definition 4.11. Then* $\omega^*(f)\in\varphi^{*-1}L$ *for all* $f\in L$*.*

(e) $\varphi^*\varphi^{*-1}L=L$*.*

Proof (a) $\varphi^{*-1}L$ is H–invariant by definition. To see that $\varphi^{*-1}L$ is res–invariant, take $B\subseteq H$ and $f\in\varphi^{*-1}L$. Then, for all $h\in H$, $\varphi^*(h.(1_B\cdot f))=\varphi^*(1_{hB}\cdot(h.f))=1_{\varphi^{-1}(hB)}\varphi^*(h.f)\in L$, so $1_B\cdot f\in\varphi^{*-1}L$.

(b) This follows from $\varphi^*(h.f)=\varphi^*(1_Y\cdot(h.f))=\varphi^*(h.(1_{h^{-1}Y}\cdot f))$ for all $f\in C(H,W)$.

(c) If M is an res–invariant RH–submodule of $C(H,W)$, then $f\in M$ implies $h.f\in M$ for all $h\in H$, and hence, by (b), we conclude that $f\in\varphi^{*-1}L$.

(d) By (b), it suffices to prove $1_{hY}\cdot\omega^*(f)\in\varphi^{*-1}L$ for all $h\in H$. By (6), it suffices to prove $1_{h_jY_j}\cdot\omega^*(f)\in\varphi^{*-1}L$ for all j. For all $y\in h_jY_j$, $1_{h_jY_j}\cdot\omega^*(f)(y)=f(\omega(y))=f(\tilde\varphi^{-1}(h_j^{-1}y))=\varphi_*(1_X\cdot f)(h_j^{-1}y)$, hence $1_{h_jY_j}\cdot\omega^*(f)=h_j.\varphi_*(1_X\cdot f)$. Let $h\in H$ be arbitrary. Lemmas 4.14 and 4.13 imply that $\varphi^*(hh_j.\varphi_*(1_X\cdot f))$ lies in L. Hence, $\omega^*(f)$ lies in $\varphi^{*-1}L$.

(e) We have $\varphi^*(f)\subseteq L$ for all $f\in\varphi^{*-1}L$ by construction (see also (c)). Hence $\varphi^*\varphi^{*-1}L\subseteq L$ by minimality of $\varphi^*\varphi^{*-1}L$. To show $L\subseteq\varphi^*\varphi^{*-1}L$, it suffices to show that $1_X\cdot L\subseteq\varphi^*\varphi^{*-1}L$ as $L=\sum_j g(i)^{-1}.(1_X\cdot L)$ by Lemma 4.9. Let $f\in 1_X\cdot L$. Then $\omega^*(f)\in\varphi^{*-1}L$ by (d), and $1_X\cdot\varphi^*(\omega^*(f))\in\varphi^*\varphi^{*-1}L$. But we have $1_X\cdot\varphi^*(\omega^*(f))=1_X\cdot(\omega\circ\varphi)^*(f)=1_X\cdot f=f$ as $\omega\circ\varphi=\mathrm{id}$ on X. □

Lemma 4.18 *If* $\varphi,\phi\colon G\to H$ *are coarse embeddings with* $\varphi\sim\phi$*, then* $\varphi^{*-1}L=\phi^{*-1}L$*.*

If $\varphi\colon G\to H$*,* $\psi\colon H\to K$ *are coarse embeddings, then* $\psi^{*-1}\varphi^{*-1}L=(\psi\circ\varphi)^{*-1}L$*.*

Proof By Remark 4.5, we have, for $f\in C(H,W)$, $\varphi^*(f)=\sum_i 1_{X_i}\cdot\phi^*(h_i.f)$. Hence, $\phi^{*-1}L\subseteq\varphi^{*-1}L$. By symmetry, $\phi^{*-1}L=\varphi^{*-1}L$.

If $f\in\psi^{*-1}\varphi^{*-1}L$, then $\psi^*(f)\in\varphi^{*-1}L$, and thus $(\psi\circ\varphi)^*(f)=\varphi^*(\psi^*(f))\in L$. Lemma 4.17(c) implies $f\in(\psi\circ\varphi)^{*-1}L$. To show $(\psi\circ\varphi)^{*-1}L\subseteq\psi^{*-1}\varphi^{*-1}L$, take $f \in (\psi \circ \varphi)^{*-1} L$. To show $f \in \psi^{*-1} \varphi^{*-1} L$, it suffices to show for all $k \in K$ and $h \in H$ that $\varphi^*(h.\psi^*(k.f)) \in L$. By Remark 4.4, we have $\psi(h^{-1} y) = k_j^{-1} \psi(y)$ for all $y \in Y_j$ and $j \in J$, for suitable J, Y_j and k_j, so that $\varphi^*(h.\psi^*(k.f)) = \varphi^*\big(\sum_j 1_{Y_j} \cdot \psi^*(k_j k.f)\big) = \sum_j 1_{\varphi^{-1}(Y_j)} (\psi \circ \varphi)^*(k_j k.f)$, which lies in L as f lies in $(\psi \circ \varphi)^{*-1} L$. □

Our next goal is to define a suitable topology on $\varphi_* L$ in the case that L is a topological res–invariant RG–submodule of $C(G, W)$ and φ is a coarse embedding. We start with some preparations.

Lemma 4.19 *Let $\widetilde{Y} \subset Y$ and $\widetilde{X} = X \cap \varphi^{-1}(\widetilde{Y})$. Then $1_{\widetilde{X}} \cdot L \to 1_{\widetilde{Y}} \cdot (\varphi_* L)$, $f \mapsto \varphi_*(f)$, is bijective.*

Proof Injectivity holds as we can recover f from $\varphi_*(f)$ using

$$\varphi^*(\varphi_*(f))(\tilde{x}) = \varphi_*(f)(\varphi(\tilde{x})) = \sum_{\substack{x \in G \\ \varphi(x) = \varphi(\tilde{x})}} f(x) = f(\tilde{x})$$

for $f \in 1_{\widetilde{X}} \cdot L$ and $\tilde{x} \in \widetilde{X}$. For surjectivity, (4) implies that it suffices to show that for all $h \in H$ and $f \in L$, $1_{\widetilde{Y}} \cdot (h.\varphi_*(f))$ lies in the image of our map. This follows immediately from Lemma 4.14. □

For $j \in \mathbb{Z}$, $j \geq 1$, set $X_j := X \cap \varphi^{-1}(Y_j)$. Obviously, for all $j \geq 1$, we have $1_{h_j Y_j} \cdot (\varphi_* L) = h_j.(1_{Y_j} \cdot (\varphi_* L))$. Thus, $1_{X_j} \cdot L \to 1_{h_j Y_j} \cdot (\varphi_* L)$, $f \mapsto h_j.\varphi_*(f)$, is an isomorphism. For $J \in \mathbb{Z}$, $J \geq 1$, define

$$\Phi^J \colon \bigoplus_{j=1}^{J} 1_{X_j} \cdot L \to \varphi_* L, \quad (f_j)_j \mapsto \sum_{j=1}^{J} h_j.\varphi_*(f_j).$$

Definition 4.20 Let L be a topological res–invariant RG–submodule of $C(G, W)$. Let τ be the finest topology on $\varphi_* L$ such that for all $J \in \mathbb{Z}$, $J \geq 1$, Φ^J is continuous. Here $1_{X_j} \cdot L$ is given the subspace topology from L, and $\bigoplus_{j=1}^{J} 1_{X_j} \cdot L$ is given the product topology.

The proof of the following lemma is straightforward:

Lemma 4.21 *τ is the finest topology on $\varphi_* L$ satisfying the following properties:*

(T_1) *$(\varphi_* L, \tau)$ is a topological res–invariant RH–submodule of $C(H, W)$.*

(T_2) *$L \to (\varphi_* L, \tau)$, $f \mapsto \varphi_*(f)$, is continuous.*

Lemma 4.22 *Let ω be as in Definition 4.11. Then $\omega_*\colon \varphi_* L \to L$ is continuous.*

Proof By definition of the topology of $\varphi_* L$, it suffices to show that for every j, $1_{X_j} \cdot L \to 1_{h_j Y_j} \cdot (\varphi_* L)$, $f \mapsto \omega_*(h_j.\varphi_*(f))$, is continuous. But it is easy to see that for $f \in 1_{X_j} \cdot L$, $\omega_*(h_j.\varphi_*(f)) = \varphi^*(\varphi_*(f))$. Continuity now follows from Lemma 4.13. □

Now let us define a suitable topology on $\varphi^* M$ in the case that M is a topological res–invariant RH–submodule of $C(H,W)$ and φ is a coarse embedding. Again, some preparations are necessary. Let $\varphi\colon G \to H$ be a coarse embedding and M a res–invariant RH–submodule of $C(H,W)$.

Lemma 4.23 *Let $\widetilde{X} \subseteq G$ be such that the restriction of φ to $\widetilde{X}$ is injective. Let $\widetilde{Y} := \varphi(\widetilde{X})$. Then $1_{\widetilde{Y}} \cdot M \to 1_{\widetilde{X}} \cdot (\varphi^* M)$, $f \mapsto 1_{\widetilde{X}} \cdot \varphi^*(f)$ is a bijection.*

Proof For every $f \in 1_{\widetilde{Y}} \cdot M$ and $y \in H$, we have

$$\varphi_*(1_{\widetilde{X}} \cdot \varphi^*(f))(y) = \sum_{\substack{x \in \widetilde{X} \\ \varphi(x) = y}} \varphi^*(f)(x) = \sum_{\substack{x \in \widetilde{X} \\ \varphi(x) = y}} (f)(\varphi(x)) = f(y).$$

Hence, $\varphi_*(1_{\widetilde{X}} \cdot \varphi^*(f)) = f$, and our map is injective. To show surjectivity, it suffices by (5) to show that for every $f \in M$ and $A \subseteq G$, $1_{\widetilde{X}} \cdot (1_A \cdot \varphi^*(f))$ lies in the image of our map. This follows from $1_{\widetilde{X}} \cdot (1_A \cdot \varphi^*(f)) = 1_{A \cap \widetilde{X}} \cdot \varphi^*(f) = 1_{\widetilde{X}} \cdot \varphi^*(1_{\varphi(A \cap \widetilde{X})} \cdot f)$. □

Now let $Y = \varphi(G)$. Lemma 2.20 gives us $X \subseteq G$ such that $\varphi|_X$ is a bijection $X \cong Y$, $x \mapsto \varphi(x)$. By Lemma 4.9, we can find a finite decomposition $G = \bigsqcup_{i=1}^{I} X_i$ and finite subsets $\{g(i) : 1 \leq i \leq I\} \subseteq G$ and $\{h(i) : i \leq i \leq I\} \subseteq H$ such that $X_i = g(i)^{-1} X(i)$ for some $X(i) \subseteq X$ and $\varphi(x) = h(i)\varphi(g(i)x)$ for all $x \in X_i$ and $1 \leq i \leq I$. Let $Y_i := \varphi(X_i)$ and $\Phi\colon \bigoplus_{i=1}^{I} 1_{Y_i} \cdot M \to \varphi^* M$, $(f_i)_i \mapsto \sum_{i=1}^{I} 1_{X_i} \cdot \varphi^*(f_i)$. As we obviously have $\varphi^* M = \bigoplus_{i=1}^{I} 1_{X_i} \cdot (\varphi^* M)$, Φ is surjective. And by Lemma 4.23, Φ is injective. Thus, Φ is an isomorphism of R–modules.

Definition 4.24 Let M be a topological res–invariant RH–submodule of $C(H,W)$. Define the topology τ on $\varphi^* M$ so that Φ becomes a homeomorphism. Here $1_{Y_i} \cdot M$ is given the subspace topology from M, and $\bigoplus_{i=1}^{I} 1_{Y_i} \cdot M$ is given the product topology.

The following lemma is straightforward to prove:

Lemma 4.25 *τ is the finest topology on φ^*M satisfying the following properties:*

(T^1) *(φ^*M, τ) is a topological* res*–invariant RG–submodule of $C(G,W)$.*

(T^2) *$M \to (\varphi^*M, \tau)$, $f \mapsto \varphi^*(f)$, is continuous.*

Now we define a suitable topology on $\varphi^{*-1}L$ for a topological res–invariant RG–submodule L of $C(G,W)$ and a coarse embedding φ. Lemma 4.17(b) implies that $\varphi^{*-1}L = \prod_j 1_{h_j Y_j} \cdot (\varphi^{*-1}L)$. The following is easy to verify:

Lemma 4.26 *For every j, $\Phi^{(j)}\colon 1_{X_j} \cdot L \to 1_{h_j Y_j} \cdot (\varphi^{*-1}L)$, $f \mapsto h_j.\varphi_*(f)$, is a bijection whose inverse is given by $1_{h_j Y_j} \cdot (\varphi^{*-1}L) \to 1_{X_j} \cdot L$, $f \mapsto 1_{X_j} \cdot \varphi^*(h_j^{-1}.f)$.*

Definition 4.27 Let L be a topological res–invariant RG–submodule of $C(G,W)$. Define the topology τ on $\varphi^{*-1}L$ so that $\prod_j \Phi^{(j)}\colon \prod_j 1_{X_j} \cdot L \to \prod_j 1_{h_j Y_j} \cdot (\varphi^{*-1}L) = \varphi^{*-1}L$ becomes a homeomorphism. Here $1_{X_j} \cdot L$ is given the subspace topology coming from L, and $\prod_j 1_{X_j} \cdot L$ is given the product topology.

The following is straightforward to prove:

Lemma 4.28 *τ is the coarsest topology on $\varphi^{*-1}L$ satisfying the following properties:*

(T^{-1}) *$(\varphi^{*-1}L, \tau)$ is a topological* res*–invariant RH–submodule of $C(H,W)$.*

(T^{-2}) *$(\varphi^{*-1}L, \tau) \to L$, $f \mapsto \varphi^*(f)$, is continuous.*

Lemma 4.29 *Let L, φ, ω and $\varphi^{*-1}L$ be as above. Then $\omega^*\colon L \to \varphi^{*-1}L$ is continuous.*

Proof It suffices to show continuity of $L \to 1_{X_j} \cdot L$, $f \mapsto 1_{X_j} \cdot \varphi^*(h_j^{-1}.\omega^*(f))$, for all j. And we have $1_{X_j} \cdot \varphi^*(h_j^{-1}.\omega^*(f)) = 1_{X_j} \cdot \varphi^*\big(h_j^{-1}.(1_{h_j Y_j} \cdot \omega^*(f))\big) = 1_{X_j} \cdot (\varphi^*\varphi_*(1_{X_j} \cdot f)) = 1_{X_j} \cdot f$, which clearly depends continuously on f. □

Lemma 4.30 *Let L, φ and $\varphi^{*-1}L$ be as above. We have $\varphi^*\varphi^{*-1}L = L$ as topological* res*–invariant modules.*

Proof Let τ be the topology of L and $\tilde{\tau}$ the topology of $\varphi^*\varphi^{*-1}L$. As $\varphi^*\colon \varphi^{*-1}L \to (L,\tau)$ is continuous by (T^{-2}), we must have $\tau \subseteq \tilde{\tau}$ by Lemma 4.25. To prove $\tilde{\tau} \subseteq \tau$, we show that id: $(L,\tau) \to (\varphi^*\varphi^{*-1}L, \tilde{\tau})$ is continuous. By construction of $\tilde{\tau}$ it suffices to show that $L \to 1_{Y_i} \cdot \varphi^{*-1}L$, $f \mapsto \varphi_*(1_{X_i} \cdot f)$, is continuous for all i. By construction of the topology on $\varphi^{*-1}L$, it is enough to show that $L \to L$, $f \mapsto 1_{X_j} \cdot \varphi^*\big(h_j^{-1}.(\varphi_*(1_{X_i} \cdot f))\big)$, is continuous. This now follows from Lemmas 4.14 and 4.13. □

We have the following topological analogue of Lemma 4.8, which is straightforward to prove.

Lemma 4.31 (i) *If $\varphi, \phi\colon G \to H$ are coarse embeddings with $\varphi \sim \phi$, then $\varphi_* L = \phi_* L$, $\varphi^* M = \phi^* M$ and $\varphi^{*-1} L = \phi^{*-1} L$ as topological* res*–invariant modules for all topological* res*–invariant RG–submodules L of $C(G,W)$ and all topological* res*–invariant RH–submodules M of $C(H,W)$.*

(ii) *If $\varphi\colon G \to H$ and $\psi\colon H \to K$ are coarse embeddings, then $\psi_*\varphi_* L = (\psi\circ\varphi)_* L$, $\varphi^*\psi^* N = (\psi\circ\varphi)^* N$ and $\psi^{*-1}\varphi^{*-1} L = (\psi\circ\varphi)^{*-1} L$ as topological* res*–invariant modules for all topological* res*–invariant RG–submodules L of $C(G,W)$ and all topological* res*–invariant RK–submodules N of $C(K,W)$.*

4.3 Coarse maps and (co)homology

Let us explain how coarse maps induce maps in group (co)homology. We first need to write group (co)homology in terms of groupoids.

Let G be a group, R a commutative ring with unit and L an RG–module. We write $g.f$ for the action of $g \in G$ on $f \in L$. We recall the chain and cochain complexes coming from the bar resolution (see [10, Chapter III, Section 1]): Let $(C_*(L), \partial_*)$ be the chain complex $\cdots \xrightarrow{\partial_3} C_2(L) \xrightarrow{\partial_2} C_1(L) \xrightarrow{\partial_1} C_0(L)$ with $C_0(L) = L$ and $C_n(L) = C_f(G^n, L) \cong R[G^n] \otimes_R L$, where C_f stands for maps with finite support, and $\partial_n = \sum_{i=0}^n (-1)^i \partial_n^{(i)}$, where

$$\partial_n^{(0)}(f)(g_1,\dots,g_{n-1}) = \sum_{g_0\in G} g_0^{-1}.f(g_0,g_1,\dots,g_{n-1}),$$

$$\partial_n^{(i)}(f)(g_1,\dots,g_{n-1}) = \sum_{\substack{g,\bar g\in G\\ g\bar g = g_i}} f(g_1,\dots,g_{i-1},g,\bar g,g_{i+1},\dots,g_{n-1}) \quad \text{for } 1\le i\le n-1,$$

$$\partial_n^{(n)}(f)(g_1,\dots,g_{n-1}) = \sum_{g_n\in G} f(g_1,\dots,g_{n-1},g_n).$$

Let $(C^*(L), \partial^*)$ be the cochain complex $C^0(L) \xrightarrow{\partial^0} C^1(L) \xrightarrow{\partial^1} C^2(L) \xrightarrow{\partial^2} \cdots$, where $C^0(L) = L$, $C^n(L) = C(G^n, L)$ for $n \ge 1$ and $\partial^n = \sum_{i=0}^{n+1} (-1)^i \partial^n_{(i)}$, with

$$\partial^n_{(0)}(f)(g_0,\dots,g_n) = g_0.f(g_1,\dots,g_n),$$

$$\partial^n_{(i)}(f)(g_0,\dots,g_n) = f(g_0,\dots,g_{i-1}g_i,\dots,g_n) \quad \text{for } 1\le i\le n,$$

$$\partial^n_{(n+1)}(f)(g_0,\dots,g_n) = f(g_0,\dots,g_{n-1}).$$

Now let W be an R–module and $L \subseteq C(G, W)$ be an RG–submodule. Consider the transformation groupoid $\mathcal{G} := G \rtimes G$ attached to the left multiplication action of G on G. By definition, $\mathcal{G} = \{(x, g) : x \in G,\ g \in G\}$, and the range and source maps are given by $r(x, g) = x$ and $s(x, g) = g^{-1}x$, whereas the multiplication is given by $(x, g_1)(g_1^{-1}x, g_2) = (x, g_1 g_2)$. Define $\sigma\colon \mathcal{G} \to G,\ (x, g) \mapsto g$. Let $\mathcal{G}^{(0)} = G$ and, for $n \geq 1$, set

$$\mathcal{G}^{(n)} := \{(\gamma_1, \dots, \gamma_n) \in \mathcal{G}^n : s(\gamma_i) = r(\gamma_{i+1}) \text{ for all } 1 \leq i \leq n-1\},$$

and, for $n \geq 1$, define $\sigma\colon \mathcal{G}^{(n)} \to G^n$ as the restriction of $\sigma^n\colon \mathcal{G}^n \to G^n$ to $\mathcal{G}^{(n)}$.

Note that $\mathcal{G}^{(n)} = \{((x_1, g_1), \dots, (x_n, g_n)) \in \mathcal{G}^n : g_i^{-1}x_i = x_{i+1} \text{ for all } 1 \leq i \leq n-1\}$, so that we have a bijection

$$\mathcal{G}^{(n)} \cong G \times G^n, \quad ((x_1, g_1), \dots, (x_n, g_n)) \mapsto (x_1, g_1, \dots, g_n). \tag{9}$$

This is because, for $2 \leq i \leq n$, x_i is determined by the equation $x_i = g_{i-1}^{-1} \cdots g_1^{-1} x_1$. We will often use this identification of $\mathcal{G}^{(n)}$ with $G \times G^n$ without explicitly mentioning it.

Now, given $f \in C(\mathcal{G}^{(n)}, W)$ and $\vec{g} \in G^n$, we view $f|_{\sigma^{-1}(\vec{g})}$ as the map in $C(G, W)$ given by $x \mapsto f(x, \vec{g})$. Set $\operatorname{supp}(f) := \{\vec{g} \in G^n : f|_{\sigma^{-1}(\vec{g})} \neq 0\}$.

Let us define a chain complex $(D_*(L), d_*)$ as follows: For $n = 0, 1, 2, \dots$, set

$$D_n(L) := \{f \in C(\mathcal{G}^{(n)}, W) : \operatorname{supp}(f) \text{ is finite, } f|_{\sigma^{-1}(\vec{g})} \in L \text{ for all } \vec{g} \in G^n\}.$$

Moreover, for all $n \geq 1$, define maps $d_n\colon D_n(L) \to D_{n-1}(L)$ by setting $d_n = \sum_{i=0}^{n} (-1)^i d_n^{(i)}$ with $d_n^{(i)} = (\delta_n^{(i)})_*$, where $\delta_1^{(0)} = s$, $\delta_1^{(1)} = r$ and, for $n \geq 2$,

$$\begin{aligned}
\delta_n^{(0)}(\gamma_1, \dots, \gamma_n) &= (\gamma_2, \dots, \gamma_n), \\
\delta_n^{(i)}(\gamma_1, \dots, \gamma_n) &= (\gamma_1, \dots, \gamma_i \gamma_{i+1}, \dots, \gamma_n) \quad \text{for } 1 \leq i \leq n-1, \\
\delta_n^{(n)}(\gamma_1, \dots, \gamma_n) &= (\gamma_1, \dots, \gamma_{n-1}).
\end{aligned}$$

Here, we use the same notation as in Section 4.1, ie

$$(\delta_n^{(i)})_*(f)(\vec{\eta}) = \sum_{\substack{\vec{\gamma} \in \mathcal{G}^{(n)} \\ \delta_n^{(i)}(\vec{\gamma}) = \vec{\eta}}} f(\vec{\gamma}).$$

Let us define a cochain complex $(D^*(L), d^*)$ by setting, for all $n = 0, 1, 2, \dots$,

$$D^n(L) := \{f \in C(\mathcal{G}^{(n)}, W) : f|_{\sigma^{-1}(\vec{g})} \in L \text{ for all } \vec{g} \in G^n\}.$$

Moreover, for all n, define maps $d^n\colon D^n(L) \to D^{n+1}(L)$ by $d^n = \sum_{i=0}^{n+1}(-1)^i d^n_{(i)}$, with $d^n_{(i)} = (\delta^n_{(i)})^*$ (as in Section 4.1, $(\delta^n_{(i)})^*(f) = f \circ \delta^n_{(i)}$), where $\delta^0_{(0)} = s$, $\delta^0_{(1)} = r$ and, for all $n \geq 1$,

$$\begin{aligned} \delta^n_{(0)}(\gamma_0, \dots, \gamma_n) &= (\gamma_1, \dots, \gamma_n), \\ \delta^n_{(i)}(\gamma_0, \dots, \gamma_n) &= (\gamma_0, \dots, \gamma_{i-1}\gamma_i, \dots, \gamma_n) \quad \text{for } 1 \leq i \leq n, \\ \delta^n_{(n+1)}(f)(\gamma_0, \dots, \gamma_n) &= (\gamma_0, \dots, \gamma_{n-1}). \end{aligned}$$

We are also interested in the topological setting, where we assume that R is a topological field and $L \subseteq C(G, W)$ a RG–submodule together with the structure of a topological R–vector space such that the G–action $G \curvearrowright L$ is by homeomorphisms. Equip the above chain and cochain complexes $C_*(L)$ and $C^*(L)$ with the topologies of pointwise convergence. We also equip $D_*(L)$ and $D^*(L)$ with the topologies of pointwise convergence, ie $f_i \in C(\mathcal{G}^{(n)}, W)$ converges to $f \in C(\mathcal{G}^{(n)}, W)$ if and only if $\lim_i f_i|_{\sigma^{-1}(\vec{g})} = f|_{\sigma^{-1}(\vec{g})}$ in L for all $\vec{g} \in G^n$.

The following is now immediate:

Lemma 4.32 (i) *We have isomorphisms χ_* of chain complexes and χ^* of cochain complexes given by*

$$\begin{aligned} \chi_n\colon C_n(L) \to D_n(L), &\quad \chi_n(f)(x, \vec{g}) = f(\vec{g})(x), \\ \chi^n\colon C^n(L) \to D^n(L), &\quad \chi^n(f)(x, \vec{g}) = f(\vec{g})(x). \end{aligned}$$

(ii) *In the topological setting, χ_* and χ^* from* (i) *are topological isomorphisms.*

By definition of group (co)homology, $H_n(G, L) = H_n(C_*(L))$ and $H^n(G, L) = H^n(C^*(L))$. By definition of reduced group (co)homology, we have $\overline{H}_n(G, L) = \overline{H}_n(C_*(L))$ and $\overline{H}^n(G, L) = \overline{H}^n(C^*(L))$ in the topological setting (recall that $\overline{H}_n(C_*(L)) = \ker(\partial_n)/\overline{\operatorname{im}(\partial_{n+1})}$ and $\overline{H}^n(C_*(L)) = \ker(\partial^n)/\overline{\operatorname{im}(\partial^{n-1})}$). Hence, we obtain:

Corollary 4.33 (i) *The maps χ_* and χ^* from Lemma 4.32 induce isomorphisms $H_n(\chi_*)\colon H_n(G, L) \cong H_n(D_*(L))$ and $H^n(\chi^*)\colon H^n(G, L) \cong H^n(D^*(L))$ for all n.*

(ii) *In the topological setting, χ_* and χ^* from Lemma 4.32 induce isomorphisms $\overline{H}_n(\chi_*)\colon \overline{H}_n(G, L) \cong \overline{H}_n(D_*(L))$ and $\overline{H}^n(\chi^*)\colon \overline{H}^n(G, L) \cong \overline{H}^n(D^*(L))$ for all n.*

In this groupoid picture of group (co)homology, let us now explain how coarse maps induce chain and cochain maps. Let $\varphi\colon G \to H$ be a coarse map. Let $\mathcal{G} = G \rtimes G$ and $\mathcal{H} = H \rtimes H$. Define $\varphi^1\colon \mathcal{G} \to \mathcal{H}$, $(x,g) \mapsto (\varphi(x), \varphi(x)\varphi(g^{-1}x)^{-1})$. It is easy to see that φ^1 is a groupoid homomorphism. This means that if γ_1 and γ_2 are composable, then so are $\varphi^1(\gamma_1)$ and $\varphi^1(\gamma_2)$, and we have $\varphi^1(\gamma_1\gamma_2) = \varphi^1(\gamma_1)\varphi^1(\gamma_2)$. For all $n \geq 1$, define $\varphi^n\colon \mathcal{G}^{(n)} \to \mathcal{H}^{(n)}$, $(\gamma_1, \dots, \gamma_n) \mapsto (\varphi^1(\gamma_1), \dots, \varphi^1(\gamma_n))$. Moreover, if $\varphi\colon G \to H$ is a coarse embedding, let $\omega\colon H \to G$ be as above, and define $\omega^1\colon \mathcal{H} \to \mathcal{G}$, $(y,h) \mapsto (\omega(y), \omega(y)\omega(h^{-1}y)^{-1})$, and for all $n \geq 1$, define $\omega^n\colon \mathcal{H}^{(n)} \to \mathcal{G}^{(n)}$, $(\eta_1, \dots, \eta_n) \mapsto (\omega^1(\eta_1), \dots, \omega^1(\eta_n))$. Now let L be a res–invariant RG–submodule of $C(G,W)$. For $f \in D_n(L)$, consider $(\varphi^n)_*(f)(\vec{\eta}) = \sum_{\vec{\gamma} \in \mathcal{G}^{(n)},\, \varphi^n(\vec{\gamma}) = \vec{\eta}} f(\vec{\gamma})$. If φ is a coarse embedding and ω is as above, set $(\omega^n)_*(f)(\vec{\gamma}) = \sum_{\vec{\eta} \in \mathcal{H}^{(n)},\, \omega^n(\vec{\eta}) = \vec{\gamma}} f(\vec{\eta})$ for $f \in D_n(\varphi_* L)$.

Lemma 4.34 (i) *Let $\varphi\colon G \to H$ be a coarse map. For all n, $D_n(\varphi)\colon D_n(L) \to D_n(\varphi_* L)$, $f \mapsto (\varphi^n)_*(f)$, is well defined and gives rise to a chain map $D_*(\varphi)\colon D_*(L) \to D_*(\varphi_* L)$. If $\psi\colon H \to K$ is another coarse map, then we have*

$$D_*(\psi \circ \varphi) = D_*(\psi) \circ D_*(\varphi). \tag{10}$$

If L is a topological res–invariant RG–submodule of $C(G,W)$ and φ is a coarse embedding, then for all n, $D_n(\varphi)$ is continuous.

(ii) *If φ is a coarse embedding, then $D_n(\omega)\colon D_n(\varphi_* L) \to D_n(L)$, $f \mapsto (\omega^n)_*(f)$, is well defined and gives rise to a chain map $D_*(\omega)\colon D_*(\varphi_* L) \to D_*(L)$. If L is a topological res–invariant module, then $D_n(\omega)$ is continuous for all n.*

Note that for (10) to make sense, we implicitly use Lemma 4.8(ii).

Proof (i) To show that $D_n(\varphi)$ is well defined, we have to show that $(\varphi^n)_*(f) \in D_n(\varphi_* L)$ for all $f \in D_n(L)$. It suffices to treat the case that $\mathrm{supp}(f) = \{\vec{g}\}$ for a single $\vec{g} = (g_1, \dots, g_n) \in G^n$, as a general element in $D_n(L)$ is a finite sum of such f. Let us first show that $(\varphi^n)_*(f)$ has finite support. As φ is a coarse map,

$$F := \{\varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G,\ 1 \leq i \leq n\} \quad \text{is finite.} \tag{11}$$

Clearly, $\mathrm{supp}((\varphi^n)_*(f)) \subseteq F^n$. To show that for every $\vec{h} = (h_1, \dots, h_n) \in H^n$, $(\varphi^n)_*(f)|_{\sigma^{-1}(\vec{h})}$ lies in $\varphi_* L$, define

$$A := \{x \in G : \varphi(g_{i-1}^{-1} \cdots g_1^{-1}x)\varphi(g_i^{-1} \cdots g_1^{-1}x)^{-1} = h_i \text{ for all } 1 \leq i \leq n\}.$$

Then $\varphi^n(x,\vec{g}) \in \sigma^{-1}(\vec{h})$ if and only if $x \in A$. Hence,

$$(\varphi^n)_*(f)(y,\vec{h}) = \sum_{\substack{x\in A\\ \varphi^n(x,\vec{g})=(y,\vec{h})}} f(x,\vec{g}) = \sum_{\substack{x\in A\\ \varphi(x)=y}} f(x,\vec{g}) = \varphi_*(1_A\cdot(f|_{\sigma^{-1}(\vec{g})}))(y),$$

so that

$$(\varphi^n)_*(f)|_{\sigma^{-1}(\vec{h})} = \varphi_*(1_A\cdot(f|_{\sigma^{-1}(\vec{g})})). \tag{12}$$

As $f|_{\sigma^{-1}(\vec{g})}$ lies in L, L is res–invariant and $\varphi_*(\tilde{f}) \in \varphi_*L$ for all $\tilde{f} \in L$, this shows that $(\varphi^n)_*(f)|_{\sigma^{-1}(\vec{h})} \in \varphi_*L$. Hence, $D_n(\varphi)$ is well defined for all n. Now $(D_n(\varphi))_n$ is a chain map because φ^n is a groupoid homomorphism for all n. Equation (10) holds because we have $(\psi^n)_* \circ (\varphi^n)_* = ((\psi\circ\varphi)^n)_*$ for all n. Equation (12) shows continuity of $D_n(\varphi)$ for all n as the right-hand side depends continuously on f. This is because L satisfies (2) and the topology on φ_*L satisfies (T_2).

(ii) To show $D_n(\omega)$ is well defined, take $f \in D_n(\varphi_*L)$. We may assume $\mathrm{supp}(f) = \{\vec{h}\}$ for $\vec{h} = (\bar{h}_1,\dots,\bar{h}_n)$ and $f|_{\sigma^{-1}(\vec{h})} \in 1_{hY}\cdot(\varphi_*L)$. By (6),

$$hY \cup \bar{h}_1^{-1}hY \cup \cdots \cup (\bar{h}_n^{-1}\cdots\bar{h}_1^{-1}hY) \subseteq \bigcup_{j=1}^{J} h_jY_j$$

for some J. Thus, for all $y \in hY$ and $1 \le i \le n$, $\omega(\bar{h}_i^{-1}\cdots\bar{h}_1^{-1}y) = \tilde{\varphi}^{-1}(h_j^{-1}y)$ for some $1 \le j \le J$. Let $S = \{(\bar{h}_{i-1}^{-1}\cdots\bar{h}_1^{-1}y, \bar{h}_i^{-1}\cdots\bar{h}_1^{-1}y) : y \in hY,\ 1 \le i \le n\}$. The set $\{\varphi(\omega(s))\varphi(\omega(t))^{-1} : (s,t) \in S\} \subseteq \{h_j^{-1}h_k : 1 \le j,k \le J\}$ is finite, so that $F := \{\omega(s)\omega(t)^{-1} : (s,t) \in S\}$ is finite as φ is a coarse embedding. Hence, we have $\mathrm{supp}((\omega^n)_*(f)) \subseteq F^n$. A similar formula as (12) shows that $(\omega^n)_*$ is well defined, and continuous in the topological setting. □

Now let M be a res–invariant RH–submodule of $C(H,W)$. For $f \in D^n(M)$, consider $(\varphi^n)^*(f) = f\circ\varphi^n$. If φ is a coarse embedding and L a res–invariant RG–submodule of $C(G,W)$, set, for $f \in D^n(L)$, $(\omega^n)^*(f) = f\circ\omega^n$.

Lemma 4.35 (i) *Let φ be a coarse map. For all n,*

$$D^n(\varphi)\colon D^n(M) \to D^n(\varphi^*M), \quad f \mapsto (\varphi^n)^*(f),$$

is well defined and gives rise to a cochain map $D^*(\varphi)\colon D^*(M) \to D^*(\varphi^*M)$. If $\psi\colon H \to K$ is another coarse map, we have

$$D^*(\psi\circ\varphi) = D^*(\varphi)\circ D^*(\psi). \tag{13}$$

If M is a topological res*–invariant RH–submodule of $C(H,W)$ and φ is a coarse embedding, then $D^n(\varphi)$ is continuous for all n.*

(ii) *If φ is a coarse embedding, then*

$$D^n(\omega)\colon D^n(L) \to D^n(\varphi^{*-1}L), \quad f \mapsto (\omega^n)^*(f),$$

is well defined and gives rise to a cochain map $D^*(\omega)\colon D^*(L) \to D^*(\varphi^{*-1}L)$. If L is a topological res*–invariant module, then $D^n(\omega)$ is continuous for all n.*

For (13) to make sense, we implicitly use Lemma 4.8(ii).

Proof (i) To show $D^n(\varphi)$ is well defined, we have to show $(\varphi^n)^*(f) \in D^n(\varphi^*M)$ for all $f \in D^n(M)$, ie $(\varphi^n)^*(f)|_{\sigma^{-1}(\vec{g})} \in \varphi^*M$ for all $\vec{g} = (g_1,\dots,g_n) \in G^n$. $F = \{\varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G,\ 1 \le i \le n\}$ is finite by (11). We also know that $\varphi^n(x,\vec{g}) \in \sigma^{-1}(F^n)$ for all $x \in G$. For $\vec{h} = (h_1,\dots,h_n) \in F^n$, let

$$A_{\vec{h}} := \{x \in G : \varphi(g_{i-1}^{-1}\cdots g_1^{-1}x)\varphi(g_i^{-1}\cdots g_1^{-1}x)^{-1} = h_i \text{ for all } 1 \le i \le n\}.$$

Then $G = \bigsqcup_{\vec{h}\in F^n} A_{\vec{h}}$ and, for $x \in A_{\vec{h}}$, we have $\varphi^n(x,\vec{g}) = (\varphi(x),\vec{h})$. Hence,

$$(\varphi^n)^*(f)|_{\sigma^{-1}(\vec{g})}(x) = f(\varphi^n(x,\vec{g})) = \sum_{\vec{h}\in F^n} 1_{A_{\vec{h}}}(x)\cdot(f|_{\sigma^{-1}(\vec{h})})(\varphi(x)),$$

and thus

$$(\varphi^n)^*(f)|_{\sigma^{-1}(\vec{g})} = \sum_{\vec{h}\in F^n} 1_{A_{\vec{h}}}\cdot\varphi^*(f|_{\sigma^{-1}(\vec{h})}). \tag{14}$$

As $f|_{\sigma^{-1}(\vec{h})} \in M$, $\varphi^*(\tilde{f}) \in \varphi^*M$ for all $\tilde{f} \in M$ and φ^*M is res–invariant, this shows that $(\varphi^n)^*(f)|_{\sigma^{-1}(\vec{g})} \in \varphi^*M$. Hence, $D^n(\varphi)$ is well defined for all n. Then $(D^n(\varphi))_n$ is a cochain map because φ^n is a groupoid homomorphism for all n. Equation (13) holds because we have $(\varphi^n)^* \circ (\psi^n)^* = ((\psi\circ\varphi)^n)^*$ for all n. Equation (14) shows that $D^n(\varphi)$ is continuous for all n as the right-hand side depends continuously on f because the topology on φ^*M satisfies (T^1) and (T^2).

(ii) Given $f \in D^n(L)$ and $\vec{h} = (\bar{h}_1,\dots,\bar{h}_n) \in H^n$, we show $(\omega^n)^*(f)|_{\sigma^{-1}(\vec{h})} \in \varphi^{*-1}L$. By Lemma 4.17(b), it suffices to show $1_{hY}\cdot((\omega^n)^*(f)|_{\sigma^{-1}(\vec{h})}) \in \varphi^{*-1}L$ for all $h \in H$. As we saw in the proof of Lemma 4.34(ii),

$$F = \{\omega(\bar{h}_{i-1}^{-1}\cdots\bar{h}_1^{-1}y)\omega(\bar{h}_i^{-1}\cdots\bar{h}_1^{-1}y)^{-1} : y \in hY,\ 1 \le i \le n\}$$

is finite. Thus, $\omega^n(y,\vec{h}) \in \sigma^{-1}(F^n)$ for all $y \in hY$. For $\vec{g} \in F^n$, let $B_{\vec{g}} = \{y \in hY : \omega(\bar{h}_{i-1}^{-1}\cdots\bar{h}_1^{-1}y)\omega(\bar{h}_i^{-1}\cdots\bar{h}_1^{-1}y)^{-1} = g_i \text{ for all } 1 \le i \le n\}$. We then have

$hY = \bigsqcup_{\vec{g} \in F^n} B_{\vec{g}}$, and $\omega^n(y, \vec{h}) = (\omega(y), \vec{g})$ for $y \in B_{\vec{g}}$, so $1_{hY} \cdot ((\omega^n)^*(f)|_{\sigma^{-1}(\vec{h})}) = \sum_{\vec{g} \in F^n} 1_{B_{\vec{g}}} \cdot \omega^*(f|_{\sigma^{-1}(\vec{g})})$, which lies in $\varphi^{*-1}L$ by Lemma 4.17(d). This formula also shows continuity in the topological setting. □

Our next goal is to show that coarse maps which are close induce the same chain and cochain maps up to homotopy. Let $\varphi, \phi\colon G \to H$ be two coarse embeddings with $\varphi \sim \phi$. Let L be a res–invariant RG–submodule of $C(G, W)$ and M a res–invariant RH–submodule of $C(H, W)$. Let $\mathcal{G} = G \rtimes G$ and $\mathcal{H} = H \rtimes H$. Define $\theta\colon G \to \mathcal{H}$, $x \mapsto (\varphi(x), \varphi(x)\phi(x)^{-1})$. For $n \geq 0$ and $1 \leq h \leq n+1$, let $\kappa_n^{(h)}\colon \mathcal{G}^{(n)} \to \mathcal{H}^{(n+1)}$ be given by $\kappa_0^{(1)} = \theta$ and, for $n \geq 1$,

$$\kappa_n^{(h)}(\gamma_1, \dots, \gamma_n) = (\varphi^1(\gamma_1), \dots, \varphi^1(\gamma_{h-1}), \theta(r(\gamma_h)), \phi^1(\gamma_h), \dots, \phi^1(\gamma_n)) \quad \text{for } 1 \leq h \leq n,$$

$$\kappa_n^{(n+1)}(\gamma_1, \dots, \gamma_n) = \big(\phi^1(\gamma_1), \dots, \phi^1(\gamma_n), \theta(s(\gamma_n))\big).$$

Moreover, for $n \geq 1$ and $1 \leq h \leq n$, let $\kappa^n_{(h)}\colon \mathcal{G}^{(n-1)} \to \mathcal{H}^{(n)}$ be given by $\kappa^1_{(1)} = \theta$ and, for $n \geq 2$,

$$\kappa^n_{(h)}(\gamma_1, \dots, \gamma_{n-1}) = (\varphi^1(\gamma_1), \dots, \varphi^1(\gamma_{h-1}), \theta(r(\gamma_h)), \phi^1(\gamma_h), \dots, \phi^1(\gamma_{n-1})) \quad \text{for } 1 \leq h \leq n-1,$$

$$\kappa^n_{(n)}(\gamma_1, \dots, \gamma_{n-1}) = \big(\phi^1(\gamma_1), \dots, \phi^1(\gamma_{n-1}), \theta(s(\gamma_{n-1}))\big).$$

Lemma 4.36 (i) *$k_n^{(h)} = (\kappa_n^{(h)})_*\colon D_n(L) \to D_{n+1}(\varphi_* L) = D_{n+1}(\phi_* L)$ is well defined for all n and h. The sum $k_n := \sum_{h=1}^{n+1} (-1)^{h+1} k_n^{(h)}$ gives a chain homotopy $D_*(\varphi) \sim_h D_*(\phi)$.*

(ii) *$k^n_{(h)} = (\kappa^n_{(h)})^*\colon D^n(M) \to D_{n-1}(\varphi^* M) = D_{n-1}(\phi^* M)$ is well defined for all n and h. The sum $k^n := \sum_{h=1}^{n} (-1)^{h+1} k^n_{(h)}$ gives a cochain homotopy $D^*(\varphi) \sim_h D^*(\phi)$.*

Proof (i) Let us show that $k_n^{(h)}$ is well defined, ie $(\kappa_n^{(h)})_*(f) \in D_{n+1}(\varphi_* L)$ for all $f \in D_n(L)$. We may assume $\mathrm{supp}(f) = \{\vec{g}\}$ for a single $\vec{g} = (g_1, \dots, g_n) \in G^n$, as a general element in $D_n(L)$ is a finite sum of such f. We first show $\mathrm{supp}((\kappa_n^{(h)})_*(f))$ is finite. By (11) and because $\varphi \sim \phi$, we know that

$$F := \{\varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G,\ 1 \leq i \leq n\} \cup \{\varphi(x)\phi(x)^{-1} : x \in G\} \cup \{\phi(x)\phi(g_i^{-1}x)^{-1} : x \in G,\ 1 \leq i \leq n\}$$

is finite. As $\kappa_n^{(h)}(x,\vec{g})$ lies in $\sigma^{-1}(F^{n+1})$ for all $x \in G$, we conclude $\mathrm{supp}((\kappa_n^{(h)})_*(f))$ is contained in F^{n+1}, which is finite. Let us show that $(\kappa_n^{(h)})_*(f)|_{\sigma^{-1}(\vec{h})}$ lies in $\varphi_* M$ for every $\vec{h} = (h_1, \dots, h_{n+1}) \in H^{n+1}$. Define

$$\begin{aligned} A := \bigl\{x \in G\colon\ & \varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)\varphi(g_i^{-1} \cdots g_1^{-1} x)^{-1} = h_i \text{ for all } 1 \le i \le h{-}1, \\ & \varphi(g_{h-1}^{-1} \cdots g_1^{-1} x)\phi(g_{h-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_h, \\ & \phi(g_{i-1}^{-1} \cdots g_1^{-1} x)\phi(g_i^{-1} \cdots g_1^{-1} x)^{-1} = h_{i+1} \text{ for all } h \le i \le n\bigr\}. \end{aligned}$$

Then we have $\kappa_n^{(h)}(x,\vec{g}) \in \sigma^{-1}(\vec{h})$ if and only if $x \in A$. Hence, $(\kappa_n^{(h)})_*(f)|_{\sigma^{-1}(\vec{h})} = \varphi_*(1_A \cdot (f|_{\sigma^{-1}(\vec{g})}))$. Since $f|_{\sigma^{-1}(\vec{g})}$ lies in L, which is res–invariant, and $\varphi_*(\tilde{f}) \in \varphi_* L$ for all $\tilde{f} \in L$, we see that $(\kappa_n^{(h)})_*(f)|_{\sigma^{-1}(\vec{h})} \in \varphi_* L$. Hence, $k_n^{(h)}$ is well defined for all n and h. A straightforward computation shows that k_n indeed gives us the desired chain homotopy.

(ii) Let us show that $k_{(h)}^n$ is well defined, ie $(\kappa_{(h)}^n)^*(f)|_{\sigma^{-1}(\vec{g})} \in \varphi^* M$ for all $\vec{g} = (g_1, \dots, g_{n-1}) \in G^{n-1}$ and $f \in D^n(M)$. As in the proof of (i), note that

$$\begin{aligned} F := {} & \{\varphi(x)\varphi(g_i^{-1}x)^{-1} : x \in G,\ 1 \le i \le n-1\} \\ & \cup \{\varphi(x)\phi(x)^{-1} : x \in G\} \cup \{\phi(x)\phi(g_i^{-1}x)^{-1} : x \in G,\ 1 \le i \le n-1\} \end{aligned}$$

is finite, and that $\kappa_{(h)}^n(x,\vec{g}) \in \sigma^{-1}(F^n)$. For $\vec{h} = (h_1, \dots, h_n) \in F^n$, set

$$\begin{aligned} A_{\vec{h}} := \bigl\{x \in G\colon\ & \varphi(g_{i-1}^{-1} \cdots g_1^{-1} x)\varphi(g_i^{-1} \cdots g_1^{-1} x)^{-1} = h_i \text{ for all } 1 \le i \le h-1, \\ & \varphi(g_{h-1}^{-1} \cdots g_1^{-1} x)\phi(g_{h-1}^{-1} \cdots g_1^{-1} x)^{-1} = h_h, \\ & \phi(g_{i-1}^{-1} \cdots g_1^{-1} x)\phi(g_i^{-1} \cdots g_1^{-1} x)^{-1} = h_{i+1} \text{ for all } h \le i \le n-1\bigr\}. \end{aligned}$$

Then $G = \bigsqcup_{\vec{h} \in F^n} A_{\vec{h}}$, and, for $x \in A_{\vec{h}}$, $\kappa_{(h)}^n(x,\vec{g}) = (\varphi(x), \vec{h})$. Hence,

$$(\kappa_{(h)}^n)^*(f)|_{\sigma^{-1}(\vec{g})}(x) = f(\kappa_{(h)}^n(x,\vec{g})) = \sum_{\vec{h} \in F^n} 1_{A_{\vec{h}}}(x) \cdot (f|_{\sigma^{-1}(\vec{h})})(\varphi(x))$$

and thus

$$(\kappa_{(h)}^n)^*(f)|_{\sigma^{-1}(\vec{g})} = \sum_{\vec{h} \in F^n} 1_{A_{\vec{h}}} \cdot (\varphi^*(f|_{\sigma^{-1}(\vec{h})})).$$

Since $f|_{\sigma^{-1}(\vec{h})} \in M$, $\varphi^*(\tilde{f}) \in \varphi^* M$ for all $\tilde{f} \in M$ and $\varphi^* M$ is res–invariant, this shows that $(\kappa_{(h)}^n)^*(f)|_{\sigma^{-1}(\vec{g})} \in \varphi^* M$. Hence, $k_{(h)}^n$ is well defined. It is straightforward to check that k^n indeed gives us the desired cochain homotopy. □

Now let $\varphi\colon G \to H$ be a coarse embedding, $\omega\colon H \to G$ as above and L an res–invariant RG–submodule of $C(G,W)$. Define $\vartheta\colon H \to \mathcal{H}$, $y \mapsto (y, y(\varphi\circ\omega)(y)^{-1})$. For $n \geq 0$ and $1 \leq h \leq n+1$, let $\lambda_n^{(h)}\colon \mathcal{H}^{(n)} \to \mathcal{H}^{(n+1)}$ be given by $\lambda_0^{(1)} = \vartheta$ and, for $n \geq 1$,

$$\lambda_n^{(h)}(\eta_1,\dots,\eta_n) = (\eta_1,\dots,\eta_{h-1},\vartheta(r(\eta_h)),(\varphi\circ\omega)^1(\eta_h),\dots,(\varphi\circ\omega)^1(\eta_n)) \quad \text{for } 1 \leq h \leq n,$$

$$\lambda_n^{(n+1)}(\eta_1,\dots,\eta_n) = \bigl(\eta_1,\dots,\eta_n,\vartheta(s(\eta_n))\bigr).$$

Moreover, for $n \geq 1$ and $1 \leq h \leq n$, let $\lambda^n_{(h)}\colon \mathcal{H}^{(n-1)} \to \mathcal{H}^{(n)}$ be given by $\lambda^1_{(1)} = \vartheta$ and, for $n \geq 2$,

$$\lambda^n_{(h)}(\eta_1,\dots,\eta_{n-1}) = (\eta_1,\dots,\eta_{h-1},\vartheta(r(\eta_h)),(\varphi\circ\omega)^1(\eta_h),\dots,(\varphi\circ\omega)^1(\eta_{n-1})) \quad \text{for } 1 \leq h \leq n-1,$$

$$\lambda^n_{(n)}(\eta_1,\dots,\eta_{n-1}) = \bigl(\eta_1,\dots,\eta_{n-1},\vartheta(s(\eta_{n-1}))\bigr).$$

Lemma 4.37 (i) *We have $D_*(\omega\circ\varphi) \sim_h \mathrm{id}$. The map $l_n^{(h)} = (\lambda_n^{(h)})_*\colon D_n(\varphi_* L) \to D_n(\varphi_* L)$ is well defined for all n and h. The sum $l_n := \sum_{h=1}^{n+1}(-1)^{h+1} l_n^{(h)}$ gives a chain homotopy $D_*(\varphi\circ\omega) \sim_h \mathrm{id}$.*

(ii) *We have $D^*(\omega\circ\varphi) \sim_h \mathrm{id}$. The map $l^n_{(h)} = (\lambda^n_{(h)})_*\colon D^n(\varphi^{*-1}L) \to D_n(\varphi^{*-1}L)$ is well defined for all n and h. The sum $l^n := \sum_{h=1}^{n}(-1)^{h+1} l^n_{(h)}$ gives a cochain homotopy $D^*(\varphi\circ\omega) \sim_h \mathrm{id}$.*

Proof (i) $D_*(\omega\circ\varphi) \sim_h \mathrm{id}$ follows from Lemma 4.36(i) and (8). That $l_n^{(h)}$ is well defined can be proven as Lemma 4.34(ii). It is straightforward to check that l_n gives the desired chain homotopy.

(ii) $D^*(\omega\circ\varphi) \sim_h \mathrm{id}$ follows from Lemma 4.36(ii) and (8). The same proof as for Lemma 4.35(ii) shows that $l^n_{(h)}$ is well defined. It is straightforward to check that l^n gives the desired cochain homotopy. □

Combining Corollary 4.33 and Lemmas 4.34, 4.35, 4.36 and 4.37, we obtain:

Theorem 4.38 *Let $\varphi\colon G \to H$ be a coarse map, L a res–invariant RG–submodule of $C(G,W)$ and M a res–invariant RH–submodule of $C(G,W)$.*

(i) *$D_*(\varphi)$ induces homomorphisms $H_*(\varphi)\colon H_*(G,L) \to H_*(H,\varphi_* L)$. If φ is a coarse embedding, $H_*(\varphi)$ is an isomorphism with inverse $H_*(\omega)$. If in addition L*

is a topological res*–invariant module,* $D_*(\varphi)$ *also induces topological isomorphisms* $\overline{H}_*(\varphi)\colon \overline{H}_*(G,L) \cong \overline{H}_*(H,\varphi_* L)$.

If $\phi\colon G \to H$ *is a coarse map with* $\varphi \sim \phi$, *then* $H_*(\varphi) = H_*(\phi)$, *and* $\overline{H}_*(\varphi) = \overline{H}_*(\phi)$ *in the topological case.*

If $\psi\colon H \to K$ *is another coarse map, then* $H_*(\psi \circ \varphi) = H_*(\psi) \circ H_*(\varphi)$, *and* $\overline{H}_*(\psi \circ \varphi) = \overline{H}_*(\psi) \circ \overline{H}_*(\varphi)$ *in the topological case.*

(ii) $D^*(\varphi)$ *induces homomorphisms* $H^*(\varphi)\colon H^*(H,M) \to H^*(G,\varphi^* M)$. *If* φ *a coarse embedding, then* $H^*(\varphi)\colon H^*(H,\varphi^{*-1} L) \to H^*(G,L)$ *is an isomorphism with inverse* $H^*(\omega)$. *If in addition* L *and* M *are topological* res*–invariant modules, then* $D^*(\varphi)$ *also induces continuous homomorphisms* $\overline{H}^*(\varphi)\colon \overline{H}^*(H,M) \to \overline{H}^*(G,\varphi^* M)$ *and topological isomorphisms* $\overline{H}^*(\varphi)\colon \overline{H}^*(H,\varphi^{*-1} L) \to \overline{H}^*(G,L)$.

If $\phi\colon G \to H$ *is a coarse map with* $\varphi \sim \phi$, *then* $H^*(\varphi) = H^*(\phi)$, *and* $\overline{H}^*(\varphi) = \overline{H}^*(\phi)$ *in the topological case.*

If $\psi\colon H \to K$ *is another coarse map, then* $H^*(\psi \circ \varphi) = H^*(\varphi) \circ H^*(\psi)$, *and* $\overline{H}^*(\psi \circ \varphi) = \overline{H}^*(\varphi) \circ \overline{H}^*(\psi)$ *in the topological case.*

In particular, for coarse equivalences, that is, coarse embeddings which are invertible modulo $\sim$, we get:

Corollary 4.39 *If* $\varphi\colon G \to H$ *is a coarse equivalence, then we obtain isomorphisms*

$$H_*(\varphi)\colon H_*(G,L) \cong H_*(H,\varphi_* L), \quad H^*(\varphi)\colon H^*(H,M) \cong H^*(G,\varphi^* M),$$

and, in the topological case,

$$\overline{H}_*(\varphi)\colon \overline{H}_*(G,L) \cong \overline{H}_*(H,\varphi_* L), \quad \overline{H}^*(\varphi)\colon \overline{H}^*(H,M) \cong \overline{H}^*(G,\varphi^* M).$$

Remark 4.40 Our constructions are functorial in W: Let $L_1 \subseteq C(G,W_1)$ and $L_2 \subseteq C(G,W_2)$ be res–invariant RG–submodules, and assume that an R–linear map $\omega\colon W_1 \to W_2$ induces an RG–linear map $\lambda\colon L_1 \to L_2$. Then we also obtain an induced map $\varphi_*\lambda\colon \varphi_* L_1 \to \varphi_* L_2$, and we get commutative diagrams

$$\begin{array}{ccc} D_*(L_1) & \xrightarrow{D_*(\varphi)} & D_*(\varphi_* L_1) \\ \downarrow{\scriptstyle D_*(\lambda)} & & \downarrow{\scriptstyle D_*(\varphi_*\lambda)} \\ D_*(L_2) & \xrightarrow{D_*(\varphi)} & D_*(\varphi_* L_2) \end{array} \qquad \begin{array}{ccc} H_*(G,L_1) & \xrightarrow{H_*(\varphi)} & H_*(H,\varphi_* L_1) \\ \downarrow{\scriptstyle H_*(\lambda)} & & \downarrow{\scriptstyle H_*(\varphi_*\lambda)} \\ H_*(G,L_2) & \xrightarrow{H_*(\varphi)} & H_*(H,\varphi_* L_2) \end{array}$$

A similar statement applies to reduced homology in the topological setting, and to (reduced) cohomology.

4.4 Consequences

Let us apply our results to Examples 4.2. Corollary 4.41(i)(c) below generalizes the result in [21] that $H^n(G, RG)$ is a coarse invariant for groups with property F_n. The reader may also consult [48, Example 5.21]. Corollary 4.41(ii)(1) was known in special cases. For instance, in [18], group cohomology with ℓ^p coefficients has been identified with nonreduced L^p–cohomology, and in [46; 8; 34], reduced group cohomology in degree 1 (ie $\overline{H}^1$) with ℓ^p coefficients has been identified with L^p–cohomology, as studied in [22; 43]. Since L^p–cohomology is known to be a coarse invariant, this gives the special case of (ii)(1) where $p \in [1, \infty[$ and our groups are finitely generated. Also, the case $p = \infty$ in (ii)(1) was known, since $H_*(G, \ell^\infty G)$ can be identified with uniformly finite homology (see [6; 9]).

Corollary 4.41 *Let G and H be countable discrete groups and $\varphi\colon G \to H$ a coarse equivalence.*

(i) *For every commutative ring R with unit and every R–module W, φ induces isomorphisms*

   (a) $H_*(G, C(G, W)) \cong H_*(H, C(H, W))$,

   (b) $H_*(G, C_f(G, W)) \cong H_*(H, C_f(H, W))$,
   $H^*(G, C_f(G, W)) \cong H^*(H, C_f(H, W))$,

   (c) $H^*(H, RH \otimes_R W) \cong H^*(G, RG \otimes_R W)$.

(ii) *Let $R = \mathbb{R}$ or $R = \mathbb{C}$ and $W = R$.*

   (1) *For all $0 < p \leq \infty$, φ induces isomorphisms*

$$\begin{aligned} H_*(G, \ell^p(G, W)) &\cong H_*(H, \ell^p(H, W)), \\ H^*(H, \ell^p(H, W)) &\cong H^*(G, \ell^p(G, W)), \\ \overline{H}_*(G, \ell^p(G, W)) &\cong \overline{H}_*(H, \ell^p(H, W)), \\ \overline{H}^*(H, \ell^p(H, W)) &\cong \overline{H}^*(G, \ell^p(G, W)), \end{aligned}$$

   (2) *φ induces isomorphisms*

$$\begin{aligned} H_*(G, c_0(G, W)) &\cong H_*(H, c_0(H, W)), \\ H^*(H, c_0(H, W)) &\cong H^*(G, c_0(G, W)), \\ \overline{H}_*(G, c_0(G, W)) &\cong \overline{H}_*(H, c_0(H, W)), \\ \overline{H}^*(H, c_0(H, W)) &\cong \overline{H}^*(G, c_0(G, W)). \end{aligned}$$

(3) *Let G and H be a finitely generated discrete groups. Then, for all $s \in \mathbb{R} \cup \{\infty\}$ and $1 \le p \le \infty$, φ induces isomorphisms*

$$H_*(G, H^{s,p}(G, W)) \cong H_*(H, H^{s,p}(H, W)),$$
$$H^*(H, H^{s,p}(H, W)) \cong H^*(G, H^{s,p}(G, W)),$$
$$\overline{H}_*(G, H^{s,p}(G, W)) \cong \overline{H}_*(H, H^{s,p}(H, W)),$$
$$\overline{H}^*(H, H^{s,p}(H, W)) \cong \overline{H}^*(G, H^{s,p}(G, W)).$$

Proof The point is that $L(G) = C(G, W)$, $C_f(G, W)$, $RG \otimes_R W$, $\ell^p(G, W)$, $c_0(G, W)$ or $H^{s,p}(G, W)$ has the property that for every coarse equivalence $\varphi\colon G \to H$, we have $\varphi_* L(G) = L(H)$ (and also topologically in the topological setting). Our claim now follows from Corollary 4.39. □

As an immediate consequence, we obtain a new proof of the result in [49] that homological and cohomological dimensions over R are preserved by coarse embeddings as long as these dimensions are finite.

Corollary 4.42 *Let R be a commutative ring with unit. Let G and H be countable discrete groups, and assume that there is a coarse embedding $\varphi\colon G \to H$.*

If G has finite homological dimension over R, ie $\mathrm{hd}_R\, G < \infty$, then $\mathrm{hd}_R\, G \le \mathrm{hd}_R\, H$.

If G has finite cohomological dimension over R, ie $\mathrm{cd}_R\, G < \infty$, then $\mathrm{cd}_R\, G \le \mathrm{cd}_R\, H$.

Proof Assume that $\mathrm{hd}_R\, G = n < \infty$. Let W be an RG–module with $H_n(G, W) \ncong \{0\}$. Define $W \hookrightarrow C(G, W)$, $w \mapsto f_w$, where $f_w(x) = x^{-1}.w$. It is easy to see that this is an embedding of RG–modules when we view W as an R–module to construct $C(G, W)$ (ie we define the RG–module structure by setting $(g.f)(x) = f(g^{-1}.x)$ for $f \in C(G, W)$). The long exact sequence in homology gives us $0 \to H_n(G, W) \to H_n(G, C(G, W)) \to \cdots$ because the $(n+1)^{\mathrm{st}}$ group homology of G vanishes for all coefficients by assumption. Hence, $H_n(G, C(G, W)) \ncong \{0\}$. By Theorem 4.38(i), we have $H_n(H, \varphi_* C(G, W)) \cong H_n(G, C(G, W)) \ncong \{0\}$. Thus, $\mathrm{hd}_R\, H \ge n$.

Now assume that $\mathrm{cd}_R\, G = n < \infty$. By [10, Proposition (2.3)], we know that $H^n(G, RG \otimes_R W) \ncong \{0\}$ for some R–module W. By Theorem 4.38(ii),

$$H^n(H, \varphi^{*-1}(RG \otimes_R W)) \cong H^n(G, RG \otimes_R W) \ncong \{0\}.$$

Thus, $\mathrm{cd}_R\, H \ge n$. □

We also obtain a new proof for the following result, first proven in [49]:

Corollary 4.43 *Let R, G and H be as above. Assume that $\varphi\colon G \to H$ is a coarse embedding. If G is amenable and $\mathbb{Q} \subseteq R$, then $\operatorname{hd}_R G \le \operatorname{hd}_R H$ and $\operatorname{cd}_R G \le \operatorname{cd}_R H$.*

Proof As explained in [49, Section 4], it was observed in [51] that our assumptions on G and R imply the existence of an RG–linear split $C_f(G,R) \to R$ for the canonical homomorphism $R \to C_f(G,R)$ embedding R as constant functions. Hence, given an arbitrary RG–module V, we obtain by tensoring with V over R that the canonical homomorphism $V \to C_f(G,V)$ splits. Note that G acts on $C_f(G,V)$ diagonally, so $C_f(G,V)$ is not a res–invariant module in our sense. But $C_f(G,V) \cong C_f(G,V_{\mathrm{triv}})$, where V_{triv} is the R–module V viewed as a RG–module with trivial G–action. Hence, $\operatorname{hd}_R G = \sup_n\{n : H_n(G, C_f(G,W)) \ncong \{0\}$ for some R-module $W\}$. As $H_n(H, \varphi_* C_f(G,W)) \cong H_n(G,W)$ by Theorem 4.38(i), we conclude that $\operatorname{hd}_R G \le \operatorname{hd}_R H$. The proof for cd_R is analogous. □

At this point, the following interesting question arises naturally:

Question 4.44 Let R be a commutative ring with unit and G and H countable discrete groups with no R–torsion. If G and H are coarsely equivalent, do we always have $\operatorname{hd}_R G = \operatorname{hd}_R H$ and $\operatorname{cd}_R G = \operatorname{cd}_R H$?

Having no R–torsion means that orders of finite subgroups must be invertible in R, and this is certainly a hypothesis we have to include. For instance, Theorem 1.4 of [44] implies that the answer to Question 4.44 is affirmative if our groups lie in the class $H\mathcal{F}$. This class $H\mathcal{F}$ has been introduced by Kropholler [28] and is defined as the smallest class of groups containing all finite groups and every group G which acts cellularly on a finite-dimensional contractible CW–complex with all isotropy subgroups already in $H\mathcal{F}$. All countable elementary amenable groups and all countable linear groups lie in $H\mathcal{F}$, and it is closed under subgroups, extensions, and countable direct unions.

Corollary 4.45 (to Theorem 1.4 in [45]) *If G and H are in $H\mathcal{F}$, then the answer to Question 4.44 is affirmative.*

Proof Theorem 1.4 of [45] implies that

$$\operatorname{cd}_R G = \sup\{\operatorname{cd}_R G' : G' \text{ coarsely embeds into } G \text{ and } \operatorname{cd}_R G' < \infty\}, \tag{15}$$

and similarly for H. Now Corollary 4.42 implies $\operatorname{cd}_R G = \operatorname{cd}_R H$. Equality for hd_R follows because for countable groups, cd_R is infinite if and only if hd_R is infinite by [5, Theorem 4.6]. □

Remark 4.46 The proof of Corollary 4.45 shows that Question 4.44 has an affirmative answer among all groups satisfying (15). In particular, for groups satisfying [45, Conjecture 1.6], Question 4.44 has an affirmative answer. While counterexamples to [45, Conjecture 1.6] are presented in [20], these examples still satisfy (15), as becomes clear in [20]. Hence, also for them, Question 4.44 has an affirmative answer.

Let us now show that being of type FP_n over a ring R is a coarse invariant. An alternative approach, based on [27], has been sketched in [17, Theorem 9.61]. The case $R = \mathbb{Z}$ is treated in [2]. Recall that for a commutative ring R with unit, a group G is of type FP_n over R if the trivial RG–module R has a projective resolution $\cdots \to P_1 \to P_0 \to R \to 0$, where P_i is finitely generated for all $0 \le i \le n$.

Corollary 4.47 *Let G and H be two countable discrete groups. Assume G and H are coarsely equivalent. Then G is of type FP_n over R if and only if H is of type FP_n over R.*

Proof By [5, Proposition 2.3], G is of type FP_n over R if and only if G is finitely generated and $H_k\big(G, \prod_I RG\big) \cong \{0\}$ for all $1 \le k \le n$, where I is an index set with $|I| = \max(\aleph_0, |R|)$. The map $\prod_I RG \to C\big(G, \prod_I R\big)$, $(f_i)_i \mapsto f$, where $(f(x))_i = f_i(x)$, identifies $\prod_I RG$ with the RG–submodule $L(G)$ of $C\big(G, \prod_I R\big)$ consisting of those functions f with the property that for every $i \in I$, $(f(x))_i = 0$ for all but finitely many $x \in G$. Clearly, $L(G)$ is res–invariant. Denote the analogous res–invariant RH–submodule of $C\big(H, \prod_I R\big)$ by $L(H)$. It is easy to see that given a coarse equivalence $\varphi\colon G \to H$, we have $\varphi_* L(G) = L(H)$. Hence, by Theorem 4.38(i), we have $H_k\big(G, \prod_I RG\big) \cong H_k(G, L(G)) \cong H_k(H, L(H)) \cong H_k\big(H, \prod_I RH\big)$. □

As another consequence, we generalize the result in [21] that for groups of type F_∞, being a (Poincaré) duality group over $\mathbb{Z}$ is a coarse invariant. We obtain an improvement since we can work over arbitrary rings R and do not need the F_∞ assumption. We only need our groups to have finite cohomological dimension over R. Recall that a group G is called a duality group over R if there is a right RG–module C and an integer $n \ge 0$ with natural isomorphisms $H^k(G, A) \cong H_{n-k}(G, C \otimes_R A)$ for all $k \in \mathbb{Z}$ and all RG–modules A (see [5, Section 9.2; 4; 10, Chapter VIII, Section 10]). G is called a Poincaré duality group over R if $C \cong R$ as R–modules. The class of duality groups is closed under extensions and under taking graphs of groups, with certain hypotheses (see [5; 16]). Examples of groups which are not duality groups over $\mathbb{Z}$ but over some other ring can be found in [14], and examples of (Poincaré) duality groups which are not of type F_∞ appear in [14; 29]. The second part of the following corollary generalizes [51, Theorem 3.3.2].

Corollary 4.48 *Let R be a commutative ring with unit. Let G and H be countable discrete groups with finite cohomological dimension over R. If G and H are coarsely equivalent, then G is a (Poincaré) duality group over R if and only if H is a (Poincaré) duality group over R.*

If G and H are amenable and $\mathbb{Q} \subseteq R$, then G is a (Poincaré) duality group over R if and only if H is a (Poincaré) duality group over R.

Proof By [4, Theorem 5.5.1 and Remark 5.5.2], we know that a group G is a duality group if and only if it has finite cohomological dimension, there is n such that $H^k(G, A) \cong \{0\}$ for all $k \neq n$ and all induced RG–modules A, and G is of type FP_n over R. The second property is a coarse invariant by Corollary 4.41(i)(c). The third property is a coarse invariant by Corollary 4.47. Hence, being a duality group is a coarse invariant. Being a Poincaré duality group means being a duality group and having dualizing module isomorphic to R. By Corollary 4.41(i)(c), the dualizing module is a coarse invariant. Thus being a Poincaré duality group is also a coarse invariant. The second part follows from the first part of the corollary and Corollary 4.43. □

If Question 4.44 has an affirmative answer, then we can replace the assumption of finite cohomological dimension by having no R–torsion in the first part of Corollary 4.48.

As another consequence, we obtain the following rigidity result for coarse embeddings into Poincaré duality groups. The proof follows that of [5, Proposition 9.22].

Corollary 4.49 *Let G and H be countable discrete groups. Let H be a Poincaré duality group over a commutative ring R with unit. Assume that there is a coarse embedding $\varphi\colon G \to H$ which is not a coarse equivalence. If $\mathrm{hd}_R\, G < \infty$, then $\mathrm{hd}_R\, G < \mathrm{cd}_R\, H$. If, in addition, G is of type FP_∞ (ie FP_n for all n), then $\mathrm{cd}_R\, G < \mathrm{cd}_R\, H$.*

In particular, every coarse self-embedding of a Poincaré duality group over R must be a coarse equivalence.

Proof Let $n = \mathrm{cd}_R\, H$. Let $D = H^n(R, RH)$. As H is a Poincaré duality group over R, $D \cong R$ as R–modules, and the RH–module structure of D is given by a group homomorphism $H \to R^*$, $h \mapsto u_h$. We know that $\mathrm{hd}_R\, G \le \mathrm{cd}_R\, G \le n$ by [5, Theorem 4.6] and Corollary 4.42. Now let L be a res–invariant RG–submodule of $C(G, W)$. Then, by Theorem 4.38(i),

$$H_n(G, L) \cong H_n(H, \varphi_* L) \cong H^0(H, \mathrm{Hom}_R(D, \varphi_* L)) \cong (\mathrm{Hom}_R(D, \varphi_* L))^H,$$

where we used that H is a Poincaré duality group over R. Clearly, $\mathrm{Hom}_R(D, \varphi_* L) \cong \varphi_* L$ as R–modules, and the H–action of $\mathrm{Hom}_R(D, \varphi_* L)$ becomes $h_\bullet f = u_h \cdot (h.f)$ for $f \in \varphi_* L$. Now take $f \in (\varphi_* L)^H$. If $f \neq 0$, then $f(y) \neq 0$ for some $y \in H$, and it follows from $h_\bullet f = f$ for all $h \in H$ that $f(y) \neq 0$ for all $y \in H$. This, however, contradicts Lemma 4.12 as H cannot be contained in a finite union of the $h_j Y_j$ if φ is not a coarse equivalence. Hence, $H_n(G, L) \cong (\varphi_* L)^H \cong \{0\}$. This implies $\mathrm{hd}_R\, G < n$ (compare also the proof of Corollary 4.42). The rest follows from [5, Theorem 4.6(c)] and that Poincaré duality groups are of type FP_∞. □

Corollary 4.49 implies that for a Poincaré duality group H and an arbitrary group G, if G coarsely embeds into H and H coarsely embeds into G, then G and H must be coarsely equivalent (ie H is "UE rigid" in Shalom's terminology [51, Section 6.2]).

Question 4.50 In Corollary 4.49, do we always get $\mathrm{cd}_R\, G < \mathrm{cd}_R\, H$, even without the FP_∞ assumption? In other words, is the analogue of the main theorem in [53] true for coarse embeddings?

We present one more application: vanishing of ℓ^2–Betti numbers is a coarse invariant. This was shown in [43] for groups of type F_∞, for more general groups in [42] (as explained in [50]), and for all countable discrete groups in [38, Corollary 6.3]. Recently, Sauer and Schrödl were even able to cover all unimodular locally compact second countable groups [50]. As vanishing of the n^{th} ℓ^2–Betti number is equivalent to $\overline{H}^n(G, \ell^2 G) \cong \{0\}$ by [44, Proposition 3.8], Corollary 4.41(ii)(1) gives another approach to the aforementioned result.

Corollary 4.51 *Let G and H be countable discrete groups which are coarsely equivalent. Then, for all n, the n^{th} ℓ^2–Betti number of G vanishes if and only if the n^{th} ℓ^2–Betti number of H vanishes.*

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