Symplectic capacities from
positive S^1–equivariant symplectic homology

JEAN GUTT

MICHAEL HUTCHINGS

We use positive S^1–equivariant symplectic homology to define a sequence of symplectic capacities c_k for star-shaped domains in $\mathbb{R}^{2n}$. These capacities are conjecturally equal to the Ekeland–Hofer capacities, but they satisfy axioms which allow them to be computed in many more examples. In particular, we give combinatorial formulas for the capacities c_k of any "convex toric domain" or "concave toric domain". As an application, we determine optimal symplectic embeddings of a cube into any convex or concave toric domain. We also extend the capacities c_k to functions of Liouville domains which are almost but not quite symplectic capacities.

1 Introduction

1.1 Symplectic capacities

Let (X, ω) and (X', ω') be symplectic manifolds of the same dimension, possibly noncompact or with boundary. A *symplectic embedding* of (X, ω) into (X', ω') is a smooth embedding $\varphi\colon X \to X'$ such that $\varphi^\star \omega' = \omega$. A basic problem in symplectic

geometry is to determine for which (X, ω) and (X', ω') a symplectic embedding exists. This is already a highly nontrivial question when the symplectic manifolds in question are domains in $\mathbb{R}^{2n} = \mathbb{C}^n$, with the restriction of the standard symplectic form.

Some basic examples of interest are as follows: If $a_1, \ldots, a_n > 0$, define the *ellipsoid*

$$(1\text{-}1) \qquad E(a_1, \ldots, a_n) = \left\{ z \in \mathbb{C}^n \ \Big| \ \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \leq 1 \right\}$$

and the *polydisk*

$$(1\text{-}2) \qquad P(a_1, \ldots, a_n) = \{ z \in \mathbb{C}^n \mid \pi |z_i|^2 \leq a_i \text{ for all } i = 1, \ldots, n \}.$$

Also, define the *ball* $B(a) = E(a, \ldots, a)$. In the four-dimensional case ($n = 2$), one can compute when an ellipsoid can be symplectically embedded into an ellipsoid or polydisk, although the answers are complicated; see eg Cristofaro-Gardiner, Frenkel and Schlenk [11] and McDuff and Schlenk [26]. The question of when a four-dimensional polydisk can be symplectically embedded into a polydisk or an ellipsoid is only partially understood; for some of the latest results see Hutchings [23]. The analogous questions in higher dimensions are more complicated, and much less is understood; see eg Guth [16] and Hind and Kerman [19].

In general, when studying symplectic embedding problems, one often obstructs the existence of symplectic embeddings using various kinds of symplectic capacities. Definitions of the latter term vary; in this paper we define a *symplectic capacity* to be a function c which assigns to each symplectic manifold (X, ω), possibly in some restricted class, a number $c(X, \omega) \in [0, \infty]$, satisfying the following axioms:[1]

- **Monotonicity** If (X, ω) and (X', ω') have the same dimension, and if there exists a symplectic embedding $(X, \omega) \to (X', \omega')$, then $c(X, \omega) \leq c(X', \omega')$.

- **Conformality** If r is a positive real number then $c(X, r\omega) = rc(X, \omega)$.

For surveys on symplectic capacities, see eg Cieliebak, Hofer, Latschev and Schlenk [7] and Ostrover [29].

---

[1]One can also consider *normalized* symplectic capacities, which satisfy the additional properties $c(B(1)) = c(Z(1)) = 1$, where we define the *cylinder* $Z(a) = P(a, \infty, \ldots, \infty)$. A strong version of a conjecture of Viterbo [34] asserts that all normalized symplectic capacities agree on compact convex domains in $\mathbb{R}^{2n}$. For example, the Gromov width c_{Gr} is normalized; the Ekeland–Hofer capacity c_k^{EH} reviewed below is normalized when $k = 1$, but not normalized when $n > 1$ and $k > 1$ since then $c_k^{\mathrm{EH}}(B(1)) < c_k^{\mathrm{EH}}(Z(1))$.

One can easily define symplectic capacities in terms of symplectic embeddings to or from other symplectic manifolds. For example, the *Gromov width $c_{\mathrm{Gr}}(X, \omega)$* is defined to be the supremum over a such that the ball $B(a)$ can be symplectically embedded into (X, ω). This trivially satisfies the monotonicity and conformality axioms. A related example is the "cube capacity" studied in Section 1.3 below. However, symplectic capacities defined like this are difficult to compute, since they are just encodings of nontrivial symplectic embedding questions.

Other symplectic capacities can be defined using Floer theory or related machinery, and these tend to be more computable. For example, the Ekeland–Hofer capacities [12] are a nondecreasing sequence of capacities c_k^{EH}, indexed by a positive integer k, which are defined for compact star-shaped domains[2] in $\mathbb{R}^{2n}$. The Ekeland–Hofer capacities are defined using calculus of variations for the symplectic action functional on the loop space of $\mathbb{R}^{2n}$. Computations of these capacities are known in a few examples. To state these, if $a_1, \ldots, a_n > 0$, let $(M_k(a_1, \ldots, a_n))_{k=1,2,\ldots}$ denote the sequence of positive integer multiples of $a_1, \ldots, a_n$, arranged in nondecreasing order with repetitions. We then have:

- [12, Proposition 4] The Ekeland–Hofer capacities of an ellipsoid are given by

$$(1\text{-}3) \qquad c_k^{\mathrm{EH}}(E(a_1, \ldots, a_n)) = M_k(a_1, \ldots, a_n).$$

- [12, Proposition 5] The Ekeland–Hofer capacities of a polydisk are given by

$$(1\text{-}4) \qquad c_k^{\mathrm{EH}}(P(a_1, \ldots, a_n)) = k \cdot \min(a_1, \ldots, a_n).$$

- Generalizing (1-4), it is asserted in [7, Equation (3.8)] that if $X \subset \mathbb{R}^{2n}$ and $X' \subset \mathbb{R}^{2n'}$ are compact star-shaped domains, then for the (symplectic) Cartesian product $X \times X' \subset \mathbb{R}^{2(n+n')}$, we have

$$(1\text{-}5) \qquad c_k^{\mathrm{EH}}(X \times X') = \min_{i+j=k}\{c_i^{\mathrm{EH}}(X) + c_j^{\mathrm{EH}}(X')\},$$

  where i and j are nonnegative integers and we interpret $c_0^{\mathrm{EH}} = 0$.

More recently, embedded contact homology was used to define the *ECH capacities* of symplectic four-manifolds; see Hutchings [20]. ECH capacities can be computed in many examples, such as four-dimensional "concave toric domains" (Choi, Cristofaro-Gardiner, Frenkel, Hutchings and Ramos [4]) and "convex toric domains" (Cristofaro-Gardiner [10] and Hutchings [23]), defined in Section 1.2 below. ECH capacities give

---

[2]In this paper, a "domain" in a Euclidean space is the closure of an open set.

sharp obstructions to symplectically embedding a four-dimensional ellipsoid into an ellipsoid (McDuff [24]) or polydisk (Hutchings [21, Corollary 11]), or more generally a four-dimensional concave toric domain into a convex toric domain [10]. In some other situations, such as for some cases of symplectically embedding a four-dimensional polydisk into an ellipsoid, the ECH capacities do not give sharp obstructions, and the Ekeland–Hofer capacities are better;[3] see Hutchings [20, Remark 1.8]. The most significant weakness of ECH capacities is that they are only defined in four dimensions, and there is no known analogue of embedded contact homology in higher dimensions which might be used to define capacities.

In this paper we define a new sequence of symplectic capacities for domains in $\mathbb{R}^{2n}$ for any n. The idea is to imitate the definition of ECH capacities, but using positive S^1–equivariant symplectic homology in place of embedded contact homology. The resulting capacities conjecturally agree with the Ekeland–Hofer capacities, but they satisfy certain axioms which allow them to be computed in many more examples.

To state the axioms, define a *nice star-shaped domain* in $\mathbb{R}^{2n}$ to be a compact $2n$–dimensional submanifold X of $\mathbb{R}^{2n} = \mathbb{C}^n$ with smooth boundary Y, such that Y is transverse to the radial vector field

$$\rho = \frac{1}{2} \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right).$$

In this case, the 1–form

(1-6) $$\lambda_0 = \frac{1}{2} \sum_{i=1}^{n} (x_i \, dy_i - y_i \, dx_i)$$

on $\mathbb{C}^n$ restricts to a contact form on Y. If γ is a Reeb orbit[4] of $\lambda_0|_Y$, define its *symplectic action* by

$$\mathcal{A}(\gamma) = \int_{\gamma} \lambda_0 \in (0, \infty).$$

If we further assume that $\lambda_0|_Y$ is nondegenerate, ie each Reeb orbit of $\lambda_0|_Y$ is nondegenerate, then each Reeb orbit γ has a well-defined Conley–Zehnder index[5]

---

[3]The best currently known obstructions to symplectically embedding a four-dimensional polydisk into an ellipsoid are obtained using more refined information from embedded contact homology going beyond capacities; see Hutchings [23].

[4]By "Reeb orbit", we mean a map $\gamma \colon \mathbb{R}/T\mathbb{Z} \to Y$ for some $T > 0$, modulo translation of the domain, such that $\gamma'(t) = R(\gamma(t))$, where R is the Reeb vector field.

[5]In the special case $n = 1$, we have $Y \simeq S^1$, and we define $\mathrm{CZ}(\gamma)$ to be twice the number of times that γ covers Y.

$\mathrm{CZ}(\gamma) \in \mathbb{Z}$. In this situation, if k is a positive integer, define

$$\mathcal{A}_k^- = \min\{\mathcal{A}(\gamma) \mid \mathrm{CZ}(\gamma) = 2k + n - 1\} \in (0, \infty)$$

(we will see in a moment why this is finite), and

$$\mathcal{A}_k^+ = \sup\{\mathcal{A}(\gamma) \mid \mathrm{CZ}(\gamma) = 2k + n - 1\} \in (0, \infty].$$

In Section 4 we will define the new symplectic capacities c_k for nice star-shaped domains in $\mathbb{R}^{2n}$ for each positive integer k.

Theorem 1.1 *The capacities c_k for nice star-shaped domains in $\mathbb{R}^{2n}$ satisfy the following axioms:*

- **Conformality** *If X is a nice star-shaped domain in $\mathbb{R}^{2n}$ and r is a positive real number, then $c(rX) = r^2 c(X)$.*

- **Increasing** $c_1(X) \leq c_2(X) \leq \cdots < \infty$.

- **Monotonicity** *If X and X' are nice star-shaped domains in $\mathbb{R}^{2n}$, and if there exists a symplectic embedding $X \to X'$, then $c_k(X) \leq c_k(X')$ for all k.*

- **Reeb orbits** *If $\lambda_0|_{\partial X}$ is nondegenerate, then $c_k(X) = \mathcal{A}(\gamma)$ for some Reeb orbit γ of $\lambda_0|_{\partial X}$ with $\mathrm{CZ}(\gamma) = 2k + n - 1$. In particular,*

$$(1\text{-}7) \qquad \mathcal{A}_k^-(X) \leq c_k(X) \leq \mathcal{A}_k^+(X).$$

Remark 1.2 The numbers c_k are also discussed in Section 3.2.1 of Ginzburg and Gürel [15], with applications to multiplicity results for simple Reeb orbits.

Remark 1.3 We extend the capacities c_k to functions of star-shaped domains which are not necessarily nice (such as polydisks) as follows: if X is a star-shaped domain in $\mathbb{R}^{2n}$, then

$$c_k(X) = \sup\{c_k(X')\},$$

where the supremum is over nice star-shaped domains X' in $\mathbb{R}^{2n}$ such that there exists a symplectic embedding $X' \to X$. It follows from Theorem 1.1 that this extended definition of c_k continues to satisfy the first three axioms in Theorem 1.1, and agrees with the previous definition when X is a nice star-shaped domain.

1.2 Examples

One can compute the capacities c_k for many examples of star-shaped domains in $\mathbb{R}^{2n}$, using only the axioms in Theorem 1.1.

To describe an important family of examples, let $\mathbb{R}^n_{\geq 0}$ denote the set of $x \in \mathbb{R}^n$ such that $x_i \geq 0$ for all $i = 1, \ldots, n$. Define the moment map $\mu \colon \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ by

$$\mu(z_1, \ldots, z_n) = \pi(|z_1|^2, \ldots, |z_n|^2).$$

If Ω is a domain in $\mathbb{R}^n_{\geq 0}$, define the *toric domain*

$$X_\Omega = \mu^{-1}(\Omega) \subset \mathbb{C}^n.$$

We will study some special toric domains, defined as follows. Given $\Omega \subset \mathbb{R}^n_{\geq 0}$, define

$$\widehat{\Omega} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (|x_1|, \ldots, |x_n|) \in \Omega\}.$$

Definition 1.4 A *convex toric domain* is a toric domain X_Ω such that $\widehat{\Omega}$ is a compact convex domain in $\mathbb{R}^n$.

Example 1.5 If $n = 2$, then X_Ω is a convex toric domain if and only if

(1-8) $\Omega = \{(x_1, x_2) \mid 0 \leq x_1 \leq A, \, 0 \leq x_2 \leq g(x_1)\},$

where

$$g \colon [0, A] \to \mathbb{R}_{\geq 0}$$

is a nonincreasing concave function. Some symplectic embedding problems involving these four-dimensional domains were studied in [23]. A more general notion of "convex toric domain" in four dimensions, where Ω is convex but $\widehat{\Omega}$ is not required to be convex, is considered in [10].

We now compute the capacities c_k of a convex toric domain X_Ω in $\mathbb{R}^{2n}$. If $v \in \mathbb{R}^n_{\geq 0}$ is a vector with all components nonnegative, define[6]

(1-9) $\|v\|_\Omega^* = \max\{\langle v, w \rangle \mid w \in \Omega\}$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. Let $\mathbb{N}$ denote the set of nonnegative integers.

Theorem 1.6 *Suppose that* X_Ω *is a convex toric domain in* $\mathbb{R}^{2n}$. *Then*

(1-10) $c_k(X_\Omega) = \min\left\{\|v\|_\Omega^* \,\middle|\, v = (v_1, \ldots, v_n) \in \mathbb{N}^n, \, \sum_{i=1}^n v_i = k\right\}.$

---

[6]The reason for this notation is as follows. Let $\|\cdot\|_\Omega$ denote the norm on $\mathbb{R}^n$ whose unit ball is $\widehat{\Omega}$. Then, in (1-9), $\|\cdot\|_\Omega^*$ denotes the dual norm on $(\mathbb{R}^n)^*$, where the latter is identified with $\mathbb{R}^n$ using the Euclidean inner product.

In fact, (1-10) holds for any function c_k defined on nice star-shaped domains in $\mathbb{R}^{2n}$ and satisfying the axioms in Theorem 1.1, extended to general star-shaped domains as in Remark 1.3.

Example 1.7 The polydisk $P(a_1, \ldots, a_n)$ is a convex toric domain X_Ω, where Ω is the rectangle

$$\Omega = \{x \in \mathbb{R}^n_{\geq 0} \mid x_i \leq a_i \text{ for all } i = 1, \ldots, n\}.$$

In this case,

$$\|v\|^*_\Omega = \sum_{i=1}^n a_i v_i.$$

It then follows from (1-10) that

$$c_k(P(a_1, \ldots, a_n)) = k \cdot \min\{a_1, \ldots, a_n\}.$$

Example 1.8 The ellipsoid $E(a_1, \ldots, a_n)$ is a convex toric domain X_Ω, where Ω is the simplex

$$\Omega = \left\{ x \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i=1}^n \frac{x_i}{a_i} \leq 1 \right\}.$$

In this case,

$$\|v\|^*_\Omega = \max_{i=1,\ldots,n} a_i v_i.$$

Then (1-10) gives

$$c_k(E(a_1, \ldots, a_n)) = \min_{\sum_i v_i = k} \max_{i=1,\ldots,n} a_i v_i.$$

It is a combinatorial exercise[7] to check that

$$(1\text{-}11) \qquad \min_{\sum_i v_i = k} \max_{i=1,\ldots,n} a_i v_i = M_k(a_1, \ldots, a_n).$$

We conclude that

$$(1\text{-}12) \qquad c_k(E(a_1, \ldots, a_n)) = M_k(a_1, \ldots, a_n).$$

Comparing the above two examples with equations (1-3) and (1-4) suggests that our capacities c_k may agree with the Ekeland–Hofer capacities c_k^{EH}:

---

[7]To do the exercise, by a continuity argument we may assume that a_i/a_j is irrational when $i \neq j$, so that the positive integer multiples of the numbers a_i are distinct. If $v \in \mathbb{N}^n$ and $\sum_i v_i = k$, then the k numbers ma_i where $1 \leq i \leq n$ and $1 \leq m \leq v_i$ are distinct, which implies that the left-hand side of (1-11) is greater than or equal to the right-hand side. To prove the reverse inequality, if $L = M_k(a_1, \ldots, a_n)$, then the numbers $v_i = \lfloor L/a_i \rfloor$ satisfy $\sum_i v_i = k$ and $\max_{i=1,\ldots,n} a_i v_i = L$.

Conjecture 1.9 *Let X be a compact star-shaped domain in $\mathbb{R}^{2n}$. Then*

$$c_k(X) = c_k^{\mathrm{EH}}(X)$$

for every positive integer k.

Remark 1.10 More evidence for this conjecture: Theorem 1.6 implies that our capacities c_k satisfy the Cartesian product property (1-5) in the special case when X and X' are convex toric domains. We do not know whether the capacities c_k satisfy this property in general.

We can also compute the capacities c_k of another family of examples:

Definition 1.11 A *concave toric domain* is a toric domain X_Ω where Ω is compact and $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Example 1.12 If $n = 2$, then X_Ω is a concave toric domain if and only if Ω is given by (1-8), where $g \colon [0, A] \to \mathbb{R}_{\geq 0}$ is a convex function with $g(A) = 0$. Some symplectic embedding problems involving these four-dimensional domains were studied in [4].

Remark 1.13 A domain in $X \subset \mathbb{R}^{2n}$ is both a convex toric domain and a concave toric domain if and only if X is an ellipsoid (1-1).

Suppose that X_Ω is a concave toric domain. Let Σ denote the closure of the set $\partial\Omega \cap \mathbb{R}^n_{>0}$. Similarly to (1-9), if $v \in \mathbb{R}^n_{\geq 0}$, define[8]

$$(1\text{-}13) \qquad\qquad [v]_\Omega = \min\{\langle v, w \rangle \mid w \in \Sigma\}.$$

Theorem 1.14 *If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then*

$$(1\text{-}14) \qquad c_k(X_\Omega) = \max\left\{[v]_\Omega \mid v \in \mathbb{N}^n_{>0}, \ \sum_{i=1}^{n} v_i = k + n - 1\right\}.$$

In fact, (1-14) holds for any function c_k defined on nice star-shaped domains in $\mathbb{R}^{2n}$ and satisfying the axioms in Theorem 1.1, extended to general star-shaped domains as in Remark 1.3.

Note that in (1-14), all components of v are required to be positive, while in (1-10), we only required that all components of v be nonnegative.

---

[8]Unlike (1-9), the function $[\cdot]_\Omega$ is not a norm; instead it satisfies the reverse inequality $[v + v']_\Omega \geq [v]_\Omega + [v']_\Omega$.

Example 1.15 Let us check that (1-14) gives the correct answer when X_Ω is an ellipsoid $E(a_1, \ldots, a_n)$. Similarly to Example 1.8, we have

$$[v]_\Omega = \min_{i=1,\ldots,n} a_i v_i.$$

Thus, we need to check that

$$\text{(1-15)} \qquad \max_{\sum_i v_i = k+n-1} \min_{i=1,\ldots,n} a_i v_i = M_k(a_1, \ldots, a_n),$$

where, unlike Example 1.8, now all components of v must be positive integers. This can be proved similarly to (1-11).

A quick application of Theorem 1.14, pointed out by Schlenk [30], is to compute the Gromov width of any concave toric domain:[9]

Corollary 1.16 *If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then*

$$c_{\mathrm{Gr}}(X_\Omega) = \max\{a \mid B(a) \subset X_\Omega\}.$$

Proof Let $a_{\max}$ denote the largest real number a such that $B(a) \subset X_\Omega$. By the definition of the Gromov width c_{Gr}, we have $c_{\mathrm{Gr}}(X_\Omega) \geq a_{\max}$. To prove the reverse inequality $c_{\mathrm{Gr}}(X_\Omega) \leq a_{\max}$, suppose that there exists a symplectic embedding $B(a) \to X_\Omega$; we need to show that $a \leq a_{\max}$. By (1-12), the monotonicity property of c_1 and Theorem 1.14, we have

$$a = c_1(B(a)) \leq c_1(X_\Omega) = [(1, \ldots, 1)]_\Omega = \min\left\{ \sum_{i=1}^n w_i \mid w \in \Sigma \right\} = a_{\max}. \qquad \square$$

1.3 Application to cube capacities

We now use the above results to solve some symplectic embedding problems where the domain is a cube.

Given $\delta > 0$, define the *cube*

$$\square_n(\delta) = P(\delta, \ldots, \delta) \subset \mathbb{C}^n.$$

Equivalently,

$$\square_n(\delta) = \left\{ z \in \mathbb{C}^n \mid \max_{i=1,\ldots,n} \{\pi |z_i|^2\} \leq \delta \right\}.$$

Definition 1.17 Given a $2n$–dimensional symplectic manifold (X, ω), define the *cube capacity*

$$c_{\square}(X, \omega) = \sup\{\delta > 0 \mid \text{there exists a symplectic embedding } \square_n(\delta) \to (X, \omega)\}.$$

---

[9]The four-dimensional case of this was shown using ECH capacities in [4, Corollary 1.10].

It is immediate from the definition that $c_\square$ is a symplectic capacity.

Theorem 1.18 Let $X_\Omega \subset \mathbb{C}^n$ *be a convex toric domain or a concave toric domain.*
Then

$$c_\square(X_\Omega) = \max\{\delta \mid (\delta, \ldots, \delta) \in \Omega\}.$$

That is, $c_\square(X_\Omega)$ is the largest δ such that $\square_n(\delta)$ is a subset of X_Ω; one cannot do
better than this obvious symplectic embedding by inclusion.

Since the proof of Theorem 1.18 is short, we will give it now. We need to consider the
nondisjoint union of n symplectic cylinders,

$$L_n(\delta) = \left\{ z \in \mathbb{C}^n \mid \min_{i=1,\ldots,n} \{\pi |z_i|^2\} \le \delta \right\}.$$

Lemma 1.19 $c_k(L_n(\delta)) = \delta(k + n - 1).$

Proof Observe that $L_n(\delta) = X_{\Omega_\delta}$, where

$$\Omega_\delta = \left\{ x \in \mathbb{R}^n_{\ge 0} \mid \min_{i=1,\ldots,n} x_i \le \delta \right\}.$$

As such, Ω_δ is the union of a nested sequence of concave toric domains. By an
exhaustion argument, the statement of Theorem 1.14 is valid for X_{Ω_δ}. Similarly to
Example 1.7, we have

$$[v]_{\Omega_\delta} = \delta \sum_{i=1}^n v_i.$$

The lemma then follows from (1-14).   □

Proposition 1.20 $c_\square(L_n(\delta)) = \delta.$

Proof We have $\square_n(\delta) \subset L_n(\delta)$, so $c_\square(L_n(\delta)) \ge \delta$ follows from the definition of $c_\square$.

To prove the reverse inequality $c_\square(L_n(\delta)) \le \delta$, suppose that there exists a symplectic
embedding $\square_n(\delta') \to L_n(\delta)$; we need to show that $\delta' \le \delta$. By the monotonicity
property of the capacities c_k, we know that

$$c_k(\square_n(\delta')) \le c_k(L_n(\delta))$$

for each positive integer k. By Example 1.7 and Lemma 1.19, this means that

$$k\delta' \le \delta(k + n - 1).$$

Since this holds for arbitrarily large k, it follows that $\delta' \le \delta$, as desired.   □

Proof of Theorem 1.18 Let $\delta > 0$ be the largest real number such that $(\delta, \ldots, \delta) \in \Omega$. It follows from the definitions of convex and concave toric domains that

$$\square_n(\delta) \subset X_\Omega \subset L_n(\delta).$$

The first inclusion implies that $\delta \leq c_\square(X_\Omega)$ by the definition of $c_\square$, while the second inclusion implies that $c_\square(X_\Omega) \leq \delta$ by Proposition 1.20. Thus $c_\square(X_\Omega) = \delta$. $\qquad\square$

Remark 1.21 The proof of Theorem 1.18 shows more generally that any star-shaped domain $X \subset \mathbb{C}^n$ such that

(1-16) $$\square_n(\delta) \subset X \subset L_n(\delta)$$

satisfies $c_\square(X) = \delta$.

Remark 1.22 The proof of Theorem 1.18 also shows that if $X \subset \mathbb{C}^n$ is a star-shaped domain satisfying (1-16), then

(1-17) $$\lim_{k \to \infty} \frac{c_k(X)}{k} = c_\square(X).$$

This is related to the following question of Cieliebak and Mohnke [8].

Given a domain $X \subset \mathbb{R}^{2n}$, define the *Lagrangian capacity* $c_L(X)$ to be the supremum over $A \in \mathbb{R}$ such that there exists an embedded Lagrangian torus $T \subset X$ such that the symplectic area of every map $(D^2, \partial D^2) \to (X, T)$ is an integer multiple of A. It is asked in [8] whether if $X \subset \mathbb{R}^{2n}$ is a convex domain then

(1-18) $$\lim_{k \to \infty} \frac{c_k^{\mathrm{EH}}(X)}{k} = c_L(X).$$

It is confirmed by [8, Corollary 1.3] that (1-18) holds when X is a ball.

Observe that if X is any domain in $\mathbb{C}^n$, then the Lagrangian capacity is related to the cube capacity by

$$c_\square(X) \leq c_L(X),$$

because if $\square_n(\delta)$ symplectically embeds into X, then the restriction of this embedding maps the "corner"

$$\mu^{-1}(\delta, \ldots, \delta) \subset \square_n(\delta)$$

to a Lagrangian torus T in X such that the symplectic area of every disk with boundary on T is an integer multiple of δ. Thus, the asymptotic result (1-17) implies that if

$X \subset \mathbb{C}^n$ is a domain satisfying (1-16), then

$$\lim_{k \to \infty} \frac{c_k(X)}{k} \leq c_L(X).$$

Assuming Conjecture 1.9, this proves one inequality in (1-18) for these examples.

1.4 Liouville domains

Recall that a *Liouville domain* is a pair (X, λ) where X is a compact manifold with boundary, λ is a 1–form on X such that $d\lambda$ is symplectic, and λ restricts to a contact form on ∂X, compatibly with the boundary orientation. For example, if X is a nice star-shaped domain in $\mathbb{R}^{2n}$, then $(X, \lambda_0|_X)$ is a Liouville domain.

In Section 4 we extend the symplectic capacities c_k for nice star-shaped domains to functions of Liouville domains. These are not quite capacities, because the monotonicity property only holds under some restrictions:

Definition 1.23 Let (X, λ) and (X', λ') be Liouville domains of the same dimension. A *generalized Liouville embedding* $(X, \lambda) \to (X', \lambda')$ is a symplectic embedding $\varphi \colon (X, d\lambda) \to (X', d\lambda')$ such that

$$[(\varphi^\star \lambda' - \lambda)|_{\partial X}] = 0 \in H^1(\partial X; \mathbb{R}).$$

If X is a nice star-shaped domain in $\mathbb{R}^{2n}$, or generally if $H^1(\partial X; \mathbb{R}) = 0$, then every symplectic embedding is a generalized Liouville embedding.

Theorem 1.24 *The functions c_k of Liouville domains satisfy the following axioms:*

- **Conformality** *If (X, λ) is a Liouville domain and r is a positive real number, then $c(X, r\lambda) = rc(X, \lambda)$.*

- **Increasing** $c_1(X, \lambda) \leq c_2(X, \lambda) \leq \cdots \leq \infty$.

- **Restricted monotonicity** *If there exists a generalized Liouville embedding $(X, \lambda) \to (X', \lambda')$, then $c_k(X, \lambda) \leq c_k(X', \lambda')$.*

- **Contractible Reeb orbits** *If $c_k(X, \lambda) < \infty$, then $c_k(X, \lambda) = \mathcal{A}(\gamma)$ for some Reeb orbit γ of $\lambda|_{\partial X}$ which is contractible[10] in X.*

---

[10]Here $\mathcal{A}(\gamma)$ denotes the symplectic action of γ, which is defined by $\mathcal{A}(\gamma) = \int_\gamma \lambda$.

Remark 1.25 Monotonicity does not extend from generalized Liouville embeddings to arbitrary symplectic embeddings: in some cases there exists a symplectic embedding $(X, d\lambda) \to (X', d\lambda')$ even though $c_k(X, \lambda) > c_k(X', \lambda')$. For example, suppose that $T \subset X'$ is a Lagrangian torus. Let λ_T denote the standard Liouville form on the cotangent bundle T^*T. By the Weinstein Lagrangian tubular neighborhood theorem, there is a symplectic embedding $(X, d\lambda) \to (X', d\lambda')$, where $X \subset T^*T$ is the unit disk bundle for some flat metric on T, and $\lambda = \lambda_T|_X$. Then (X, λ) is a Liouville domain. But $\lambda|_{\partial X}$ has no Reeb orbits which are contractible in X, so by the contractible Reeb orbits axiom, $c_k(X, \lambda) = \infty$ for all k.

Note that the symplectic embedding $(X, d\lambda) \to (X', d\lambda')$ is a generalized Liouville embedding if and only if T is an *exact* Lagrangian torus in (X', λ'), that is, $\lambda'|_T$ is exact. The restricted monotonicity axiom then tells us that *if (X', λ') is a Liouville domain with $c_1(X', \lambda') < \infty$, then (X', λ') does not contain any exact Lagrangian torus*.

Remark 1.26 The functions c_k are defined for disconnected Liouville domains. However, it follows from the definition in Section 4 that

$$c_k \left(\coprod_{i=1}^{m} (X_i, \lambda_i) \right) = \max_{i=1,\dots,m} c_k(X_i, \lambda_i).$$

As a result, restricted monotonicity for embeddings of disconnected Liouville domains does not tell us anything more than it already does for their connected components.

Remark 1.27 One can ask whether, by analogy with ECH capacities [20, Proposition 1.5], the existence of a generalized Liouville embedding $\coprod_{i=1}^{m} (X_i, \lambda_i) \to (X', \lambda')$ implies that

(1-19) $$\sum_{i=1}^{m} c_{k_i}(X_i, \lambda_i) \le c_{k_1 + \dots + k_m}(X', \lambda')$$

for all positive integers $k_1, \dots, k_m$. We have heuristic reasons to expect this when the k_i are all multiples of $n - 1$. However, it is false more generally.

For example, in $2n$ dimensions, the Traynor trick [32] can be used to symplectically embed the disjoint union of n^2 copies of the ball $B(\frac{1}{2} - \varepsilon)$ into the ball $B(1)$ for any $\varepsilon > 0$. If (1-19) is true with all $k_i = 1$, then by (1-12) we obtain

$$n^2 \left(\tfrac{1}{2} - \varepsilon \right) \le n.$$

But this is false when $n > 2$ and $\varepsilon > 0$ is small enough.

The rest of the paper   In Section 2 we prove Theorems 1.6 and 1.14, computing the capacities c_k for convex and concave toric domains, using only the axioms in Theorem 1.1. In Section 3 we state the properties of positive S^1–equivariant symplectic homology and transfer morphisms that are needed to define the capacities c_k. In Section 4 we define the capacities c_k and prove that they satisfy the axioms in Theorems 1.1 and 1.24. In Section 5 we review the definition of positive S^1–equivariant symplectic homology. In Section 6, we prove the properties of positive S^1–equivariant SH that are stated in Section 3. In Section 7 we review the construction of transfer morphisms on positive S^1–equivariant SH. Finally, in Section 8 we prove the properties of transfer morphisms that are stated in Section 3.

2   Computations of the capacities c_k

We now prove Theorems 1.6 and 1.14, computing the capacities c_k for convex and concave toric domains, using only the axioms in Theorem 1.1.

2.1   Computation for an ellipsoid

To prepare for the proofs of Theorems 1.6 and 1.14, we first compute the capacities c_k for an ellipsoid (without using either of these theorems as in Examples 1.8 or 1.15).

Lemma 2.1   *The capacities c_k of an ellipsoid are given by*

$$(2\text{-}1) \qquad\qquad c_k(E(a_1, \ldots, a_n)) = M_k(a_1, \ldots, a_n).$$

Proof   By a continuity argument using the monotonicity and conformality axioms — compare [4, Section 2.2] — to prove (2-1) we may assume that a_i/a_j is irrational when $i \neq j$. In this case we can compute the capacities c_k of the ellipsoid using the Reeb orbits axiom in Theorem 1.1. For this purpose, we now review how to compute the Reeb orbits on the boundary of the ellipsoid, their actions and their Conley–Zehnder indices.

The Reeb vector field on the boundary of the ellipsoid $E(a_1, \ldots, a_n)$ is given by

$$(2\text{-}2) \qquad R = 2\pi \sum_{i=1}^{n} \frac{1}{a_i} \frac{\partial}{\partial \theta_i},$$

where θ_i denotes the angular polar coordinate on the ith summand in $\mathbb{C}^n$. Since a_i / a_j is irrational when $i \neq j$, it follows from (2-2) that there are exactly n simple Reeb orbits $\gamma_1, \ldots, \gamma_n$, where γ_i denotes the circle where $z_j = 0$ for $j \neq i$. We will also see below that $\lambda_0|_{\partial E(a_1, \ldots, a_n)}$ is nondegenerate.

It follows from (2-2) that the actions of the simple Reeb orbits are given by $\mathcal{A}(\gamma_i) = a_i$. If m is a positive integer, let γ_i^m denote the mth iterate of γ_i; then this orbit has symplectic action

$$(2\text{-}3) \qquad \mathcal{A}(\gamma_i^m) = m a_i.$$

Let S denote the set of all such symplectic actions, ie the set of real numbers $m a_i$ where m is a positive integer and $i \in \{1, \ldots, n\}$. These are all distinct, by our assumption that a_i / a_j is irrational when $i \neq j$.

We now compute the Conley–Zehnder indices of the Reeb orbits γ_i^m. Assume for the moment that $n > 1$. Recall that the Conley–Zehnder index of a contractible nondegenerate Reeb orbit γ in a contact manifold (Y, λ) with $c_1(\xi)|_{\pi_2(Y)} = 0$ can be computed by the formula

$$(2\text{-}4) \qquad \text{CZ}(\gamma) = \text{CZ}_\tau(\gamma) + 2c_1(\gamma, \tau).$$

Here τ is any (homotopy class of a) symplectic trivialization of the restriction of the contact structure $\xi = \text{Ker}(\lambda)$ to γ, $\text{CZ}_\tau(\gamma)$ denotes the Conley–Zehnder index of the path of symplectic matrices obtained by the linearized Reeb flow along γ with respect to the trivialization τ, and $c_1(\gamma, \tau)$ denotes the relative first Chern class with respect to τ of the pullback of ξ to a disk u bounded by γ; see [22, Section 3.2].

In the present case, where $Y = \partial E(a_1, \ldots, a_n)$, the contact structure ξ on γ_i^m is the sum of all of the $\mathbb{C}$ summands in $T\mathbb{C}^n = \mathbb{C}^n$ except for the ith summand. Let us use this identification to define the trivialization τ. By (2-2), the linearized Reeb flow around γ_i^m is the direct sum of rotation by angle $2\pi m a_i / a_j$ in the jth summand for each $j \neq i$. It follows that

$$\text{CZ}_\tau(\gamma_i^m) = \sum_{j \neq i} \left(2 \left\lfloor \frac{m a_i}{a_j} \right\rfloor + 1 \right).$$

On the other hand,

(2-5) $$c_1(\gamma_i^m, \tau) = m,$$

essentially because the Hopf fibration over S^2 has Euler number 1. Putting this all together, we obtain

$$\mathrm{CZ}(\gamma_i^m) = 2m + \sum_{j \neq i}\left(2\left\lfloor\frac{ma_i}{a_j}\right\rfloor + 1\right) = n - 1 + 2\sum_{j=1}^{n}\left\lfloor\frac{ma_i}{a_j}\right\rfloor.$$

Thus,

(2-6) $$\mathrm{CZ}(\gamma_i^m) = n - 1 + 2|\{L \in S \mid L \leq ma_i\}|.$$

It follows from (2-3) and (2-6) that

(2-7) $$\mathrm{CZ}(\gamma_i^m) = n - 1 + 2k \iff \mathcal{A}(\gamma_i^m) = M_k(a_1, \ldots, a_n).$$

Note that this also holds when $n = 1$, by our convention in Section 1.1.

In conclusion, it follows from (2-7) that for each positive integer k we have

$$\mathcal{A}_k^-(E(a_1, \ldots, a_n)) = M_k(a_1, \ldots, a_n) = \mathcal{A}_k^+(E(a_1, \ldots, a_n)).$$

The lemma now follows from the Reeb orbits axiom (1-7). $\square$

Remark 2.2 A useful equivalent version of (2-1) is

(2-8) $$c_k(E(a_1, \ldots, a_n)) = \min\left\{L \;\middle|\; \sum_{i=1}^{n}\left\lfloor\frac{L}{a_i}\right\rfloor \geq k\right\}.$$

Remark 2.3 Lemma 2.1, in the form (2-8), extends to the case where some of the numbers a_i are infinite, by an exhaustion argument.

2.2 Computation for convex toric domains

We now prove Theorem 1.6. We first prove that the left-hand side of (1-10) is less than or equal to the right-hand side:

Lemma 2.4 If X_Ω is a convex toric domain in $\mathbb{R}^{2n}$ then

$$c_k(X_\Omega) \leq \min\left\{\|v\|_\Omega^* \;\middle|\; v = (v_1, \ldots, v_n) \in \mathbb{N}^n, \sum_{i=1}^{n} v_i = k\right\}.$$

Proof Let $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$ with

(2-9) $$\sum_{i=1}^{n} v_i = k;$$

we need to show that $c_k(X_\Omega) \leq \|v\|_\Omega^*$. Write $L = \|v\|_\Omega^*$. By the definition (1-9) of $\|\cdot\|_\Omega^*$, we have $L = \langle v, w \rangle$, where $w \in \Omega$ is such that $\langle v, w \rangle$ is maximal. Define

$$\Omega' = \{x \in \mathbb{R}_{\geq 0}^n \mid \langle v, x \rangle \leq L\}. \tag{2-10}$$

Then by maximality of $\langle v, w \rangle$ we have $\Omega \subset \Omega'$. By the monotonicity axiom for the capacity c_k, it follows that

$$c_k(X_\Omega) \leq c_k(X_{\Omega'}).$$

Thus, it suffices to show that $c_k(X_{\Omega'}) \leq L$.

To do so, suppose first that $v_i > 0$ for all $i = 1, \ldots, n$. Then $X_{\Omega'}$ is an ellipsoid,

$$X_{\Omega'} = E\left(\frac{L}{v_1}, \ldots, \frac{L}{v_n}\right).$$

By (2-8), we have

$$c_k(X_{\Omega'}) = \min\left\{L' \mid \sum_{i=1}^n \left\lfloor \frac{L'}{L/v_i} \right\rfloor \geq k\right\}. \tag{2-11}$$

Since the v_i are integers, by (2-9) we have

$$L' = L \implies \sum_{i=1}^n \left\lfloor \frac{L'}{L/v_i} \right\rfloor = k.$$

It follows from this and (2-11) that $c_k(X_{\Omega'}) \leq L$, as desired (in fact this is an equality).

The above calculation extends to the case where some of the components v_i are zero by Remark 2.3. $\qquad\square$

We now use a different argument to prove the reverse inequality, which completes the proof of Theorem 1.6:

Lemma 2.5 *If X_Ω is a convex toric domain in $\mathbb{R}^{2n}$ then*

$$c_k(X_\Omega) \geq \min\left\{\|v\|_\Omega^* \mid v = (v_1, \ldots, v_n) \in \mathbb{N}^n, \sum_{i=1}^n v_i = k\right\}. \tag{2-12}$$

Proof If $n = 1$, then the result follows immediately from Lemma 2.1; thus we may assume that $n > 1$. Fix the positive integer k in (2-12).

To start, we perturb Ω to have some additional properties that will be useful. It follows from the conformality and monotonicity axioms that the left-hand side of (2-12) is continuous with respect to the Hausdorff metric on compact sets Ω, as in [4, Lemma 2.3].

The right-hand side is also continuous with respect to the Hausdorff metric, as in [4, Lemma 2.4]. As a result, we may assume the following, where Σ denotes the closure of the set $\partial\Omega \cap \mathbb{R}^n_{>0}$:

   (i) Σ is a smooth hypersurface in $\mathbb{R}^n$.

  (ii) The Gauss map $G\colon \Sigma \to S^{n-1}$ is a smooth embedding and ∂X_Ω is a smooth hypersurface in $\mathbb{R}^{2n}$. In particular, X_Ω is a nice star-shaped domain.

 (iii) If $w \in \Sigma$ and if $w_i = 0$ for some i, then the ith component of $G(w)$ is positive and small with respect to the positive integer k in (2-12).

We now prove (2-12) in four steps.

Step 1 We first compute the Reeb vector field on $\partial X_\Omega = \mu^{-1}(\Sigma)$.

Let $w \in \Sigma$ and let $z \in \mu^{-1}(w)$. Also, write $G(w) = (v_1, \dots, v_n)$. Observe that

$$\sum_i v_i w_i = \|G(w)\|_\Omega^*.$$

We now define local coordinates on a neighborhood of z in $\mathbb{C}^n$ as follows. For $i = 1, \dots, n$, let $\mathbb{C}_i$ denote the ith summand in $\mathbb{C}^n$. If $z_i = 0$, then we use the standard coordinates x_i and y_i on $\mathbb{C}_i$. If $z_i \neq 0$, then on $\mathbb{C}_i$ we use local coordinates μ_i and θ_i, where $\mu_i = \pi(x_i^2 + y_i^2)$ and θ_i is the angular polar coordinate.

In these coordinates, the standard Liouville form (1-6) is given by

$$\lambda_0 = \frac{1}{2} \sum_{w_i = 0} (x_i \, dy_i - y_i \, dx_i) + \frac{1}{2\pi} \sum_{w_i \neq 0} \mu_i \, d\theta_i.$$

Also, the tangent space to ∂X_Ω at z is described by

$$T_z \partial X_\Omega = \bigoplus_{w_i = 0} \mathbb{C}_i \oplus \left\{ \sum_{w_i \neq 0} \left(a_i \frac{\partial}{\partial \mu_i} + b_i \frac{\partial}{\partial \theta_i} \right) \,\Big|\, \sum_{w_i \neq 0} v_i a_i = 0 \right\}.$$

It follows from the above three equations that the Reeb vector field at z is given by

$$(2\text{-}13) \qquad R = \frac{2\pi}{\|G(w)\|_\Omega^*} \sum_{w_i \neq 0} v_i \frac{\partial}{\partial \theta_i}.$$

For future reference, we also note that the contact structure ξ at z is given by

$$(2\text{-}14) \quad \xi_z = \bigoplus_{w_i = 0} \mathbb{C}_i \oplus \left\{ \sum_{w_i \neq 0} \left(a_i \frac{\partial}{\partial \mu_i} + b_i \frac{\partial}{\partial \theta_i} \right) \,\Big|\, \sum_{w_i \neq 0} v_i a_i = 0, \ \sum_{w_i \neq 0} w_i b_i = 0 \right\}.$$

Step 2   We now compute the Reeb orbits and their basic properties.

It is convenient here to define a (discontinuous) modification $\widetilde{G} \colon \Sigma \to \mathbb{R}^n$ of the Gauss map G by setting a component of the output to zero whenever the corresponding component of the input is zero. That is, for $i = 1, \ldots, n$ we define

$$(2\text{-}15) \qquad\qquad \widetilde{G}(w)_i = \begin{cases} G(w)_i, & w_i \neq 0, \\ 0, & w_i = 0. \end{cases}$$

Continuing the discussion from Step 1, observe from (2-13) that the Reeb vector field R is tangent to $\mu^{-1}(w)$. Let $Z(w)$ denote the number of components of w that are equal to zero; then $\mu^{-1}(w)$ is a torus of dimension $n - Z(w)$. It follows from (2-13) that if $\widetilde{G}(w)$ is a scalar multiple of an integer vector, then $\mu^{-1}(w)$ is foliated by an $(n - Z(w) - 1)$–dimensional Morse–Bott family of Reeb orbits; otherwise, $\mu^{-1}(w)$ contains no Reeb orbits.

Let V denote the set of nonnegative integer vectors v such that v is a scalar multiple of an element $\widetilde{v}$ of the image of the modified Gauss map $\widetilde{G}$. Given $v \in V$, let $d(v)$ denote the greatest common divisor of the components of v. Let $\mathcal{P}(v)$ denote the set of $d(v)$–fold covers of simple Reeb orbits in the torus $\mu^{-1}(\widetilde{G}^{-1}(\widetilde{v}))$. Then it follows from the above discussion that the set of Reeb orbits on ∂X_Ω equals $\bigsqcup_{v \in V} \mathcal{P}(v)$. Moreover, condition (iii) above implies that $v \in V$ whenever $\sum_i v_i \leq k$.

Equation (2-13) implies that each Reeb orbit $\gamma \in \mathcal{P}(v)$ has symplectic action

$$\mathcal{A}(\gamma) = \|v\|_\Omega^*.$$

Also, we can define a trivialization τ of $\xi|_\gamma$ from (2-14), identifying ξ_z for each $z \in \gamma$ with a codimension-two subspace of $\mathbb{R}^{2n}$ with coordinates x_i and y_i for each i with $w_i = 0$, and coordinates a_i and b_i for each i with $w_i \neq 0$. Then, similarly to (2-5), we have

$$(2\text{-}16) \qquad\qquad c_1(\gamma, \tau) = \sum_{i=1}^{n} v_i.$$

Step 3   We now approximate the convex toric domain X_Ω by a nice star-shaped domain X' such that $\lambda_0|_{\partial X'}$ is nondegenerate.

Given $v \in V$ with $d(v) = 1$, one can perturb ∂X_Ω in a neighborhood of the $(n - Z(v))$–dimensional torus swept out by the Reeb orbits in $\mathcal{P}(v)$, using a Morse function f on the $(n - Z(v) - 1)$–dimensional torus $\mathcal{P}(v)$, to resolve the Morse–Bott family $\mathcal{P}(v)$ into a finite set of nondegenerate Reeb orbits corresponding to the critical points of f

(possibly together with some additional Reeb orbits of much larger symplectic action). Owing to the strict convexity of Σ, each such nondegenerate Reeb orbit γ will have Conley–Zehnder index with respect to the above trivialization τ in the range

$$(2\text{-}17) \qquad\qquad Z(v) \leq \mathrm{CZ}_\tau(\gamma) \leq n - 1.$$

It then follows from (2-16) that

$$(2\text{-}18) \qquad\qquad Z(v) + 2 \sum_{i=1}^{n} v_i \leq \mathrm{CZ}(\gamma) \leq n - 1 + 2 \sum_{i=1}^{n} v_i.$$

In particular,

$$(2\text{-}19) \qquad \mathrm{CZ}(\gamma) = 2k + n - 1 \implies k \leq \sum_{i=1}^{n} v_i \leq k + \tfrac{1}{2}(n - 1 - Z(v)).$$

Moreover, even if we drop the assumption that $d(v) = 1$, then after perturbing the orbits in $\mathcal{P}(v/d(v))$ as above, the family $\mathcal{P}(v)$ will still be replaced by nondegenerate orbits, each satisfying (2-18) (possibly together with additional Reeb orbits of much larger symplectic action), as long as $d(v)$ is not too large with respect to the perturbation.

Now choose $\varepsilon > 0$ small and choose

$$R > \max\Big\{ \|v\|_\Omega^* \mid v \in \mathbb{N}^n, \ \sum_i v_i \leq k + \tfrac{1}{2}(n - 1) \Big\}.$$

We can then perturb X_Ω to a nice star-shaped domain X' with $\lambda_0|_{\partial X'}$ nondegenerate such that for each $v \in V$ with $\|v\|_\Omega^* < R$, the Morse–Bott family $\mathcal{P}(v)$ is perturbed as above, each nondegenerate orbit γ arising from each such $\mathcal{P}(v)$ has symplectic action satisfying

$$(2\text{-}20) \qquad\qquad \mathcal{A}(\gamma) \geq \|v\|_\Omega^* - \varepsilon,$$

and there are no other Reeb orbits of symplectic action less than R.

Step 4 We now put together the above inequalities to complete the proof.

It follows from (2-19) and (2-20) that

$$\mathcal{A}_k^-(X') \geq \min\Big\{ \|v\|_\Omega^* - \varepsilon \mid v \in \mathbb{N}^n, \ k \leq \sum_{i=1}^{n} v_i \leq k + \tfrac{1}{2}(n - 1 - Z(v)) \Big\}.$$

Thus, by the Reeb orbits axiom (1-7), we have

$$c_k(X') \geq \min\Big\{ \|v\|_\Omega^* - \varepsilon \mid v \in \mathbb{N}^n, \ k \leq \sum_{i=1}^{n} v_i \leq k + \tfrac{1}{2}(n - 1 - Z(v)) \Big\}.$$

Taking $\varepsilon \to 0$ and a sequence of perturbations X' converging in C^0 to X_Ω, and using conformality and monotonicity as at the beginning of the proof of this lemma, we obtain

$$c_k(X_\Omega) \ge \min\left\{\|v\|_\Omega^* \mid v \in \mathbb{N}^n,\ k \le \sum_{i=1}^n v_i \le k + \tfrac{1}{2}(n-1-Z(v))\right\}.$$

In fact, in the above minimum, we can restrict attention to v with $\sum_i v_i = k$, because if $\sum_i v_i > k$, then we can decrease some components of v to obtain a new vector $v' \in \mathbb{N}^n$ with $\sum_i v_i' = k$, and by (1-9) we will have $\|v'\|_\Omega^* \le \|v\|_\Omega^*$. This completes the proof of (2-12). $\qquad\square$

2.3 Computation for concave toric domains

We now prove Theorem 1.14. The proof is very similar to the proof of Theorem 1.6 above, but with the direction of some inequalities switched, and other slight changes.

Lemma 2.6 *If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then*

$$c_k(X_\Omega) \ge \max\left\{[v]_\Omega \mid v \in \mathbb{N}_{>0}^n,\ \sum_i v_i = k+n-1\right\}.$$

Proof Let $v \in \mathbb{N}_{>0}^n$ with

(2-21) $$\sum_i v_i = k+n-1;$$

we need to show that $c_k(X_\Omega) \ge [v]_\Omega$.

Write $L = [v]_\Omega$. By the definition (1-13), we have $L = \langle v, w \rangle$, where $w \in \Sigma$ is such that $\langle v, w \rangle$ is minimal. If we define Ω' as in (2-10), then by minimality of $\langle v, w \rangle$ we have $\Omega' \subset \Omega$. By monotonicity of the capacity c_k, we then have

$$c_k(X_{\Omega'}) \le c_k(X_\Omega).$$

So it suffices to show that $c_k(X_{\Omega'}) \ge L$. We again have (2-11), namely

$$c_k(X_{\Omega'}) = \min\left\{L' \,\Big|\, \sum_{i=1}^n \left\lfloor \frac{L'}{L/v_i} \right\rfloor \ge k\right\}.$$

Since the v_i are integers, by (2-21) we have

$$L' < L \implies \sum_{i=1}^n \left\lfloor \frac{L'}{L/v_i} \right\rfloor \le k-1.$$

It follows that $c_k(X_{\Omega'}) \ge L$, as desired (in fact this is an equality). $\qquad\square$

Lemma 2.7 If X_Ω is a concave toric domain in $\mathbb{R}^{2n}$, then

$$(2\text{-}22) \qquad c_k(X_\Omega) \le \max\left\{[v]_\Omega \mid v \in \mathbb{N}_{>0}^n, \ \sum_i v_i = k + n - 1\right\}.$$

Proof As in the proof of Lemma 2.5, we may assume that $n > 1$ and that:

(i) Σ is a smooth hypersurface in $\mathbb{R}^n$.

(ii) The Gauss map $G \colon \Sigma \to S^{n-1}$ is a smooth embedding, and ∂X_Ω is a smooth hypersurface in $\mathbb{R}^{2n}$, so that X_Ω is a nice star-shaped domain.

(iii) If $w \in \Sigma$ and $w_i = 0$ for some i, then $G(w)$ is close (with respect to k) to the set of $v \in S^{n-1}$ such that $v_j = 0$ whenever $w_j \ne 0$.

Similarly to (2-13), the Reeb vector field again preserves each torus $\mu^{-1}(w)$, on which now

$$(2\text{-}23) \qquad R = \frac{2\pi}{[G(w)]_\Omega} \sum_{w_i \ne 0} v_i \frac{\partial}{\partial \theta_i},$$

where $G(w) = (v_1, \ldots, v_n)$. Continuing to define the modified Gauss map $\widetilde{G}$ by (2-15), and defining V and $\mathcal{P}(v)$ as before, it follows that the set of Reeb orbits on ∂X_Ω is again given by $\bigsqcup_{v \in V} \mathcal{P}(v)$. Condition (iii) above implies that $v \in V$ whenever $\sum_i v_i$ is not too large, and (2-23) implies that each Reeb orbit in $\mathcal{P}(v)$ has action $[v]_\Omega$.

As in Step 3 of the proof of Lemma 2.5, we can perturb the concave toric domain X_Ω to a nice star-shaped domain X' such that the contact form $\lambda_0|_{X'}$ is nondegenerate; up to large symplectic action, the Reeb orbits come from the tori $\mathcal{P}(v)$ where $\sum_i v_i$ is not too large; and a Reeb orbit γ coming from $\mathcal{P}(v)$ has action

$$(2\text{-}24) \qquad \mathcal{A}(\gamma) \le [v]_\Omega + \varepsilon,$$

where $\varepsilon > 0$ can be chosen arbitrarily small.

If any component of v is zero, then the Conley–Zehnder index of γ will be very large, by condition (iii) above. Otherwise, to compute the Conley–Zehnder index of γ, we use a homotopy class τ of a trivialization of ξ_γ defined as in the proof of Lemma 2.5. Equation (2-16) still holds, while the inequalities (2-17) are replaced by

$$0 \le -\mathrm{CZ}_\tau(\gamma) \le n - 1.$$

(Here the sign of $\mathrm{CZ}_\tau(\gamma)$ is switched because Σ is concave instead of convex.) Thus, we obtain

$$1 - n + 2 \sum_{i=1}^n v_i \le \mathrm{CZ}(\gamma) \le 2 \sum_{i=1}^n v_i.$$

In particular, we obtain

$$(2\text{-}25) \qquad \mathrm{CZ}(\gamma) = 2k + n - 1 \implies k + \tfrac{1}{2}(n-1) \leq \sum_{i=1}^{n} v_i \leq k + n - 1.$$

It follows from (2-24) and (2-25) that

$$\mathcal{A}_k^+(X') \leq \max\Big\{[v]_\Omega + \varepsilon \mid v \in \mathbb{N}_{>0}^n,\ k + \tfrac{1}{2}(n-1) \leq \sum_{i=1}^{n} v_i \leq k + n - 1\Big\}.$$

As in Step 4 of the proof of Lemma 2.5, we deduce that

$$c_k(X_\Omega) \leq \max\Big\{[v]_\Omega \mid v \in \mathbb{N}_{>0}^n,\ k + \tfrac{1}{2}(n-1) \leq \sum_{i=1}^{n} v_i \leq k + n - 1\Big\}.$$

In the above maximum, we can restrict attention to v with $\sum_i v_i = k + n - 1$, since increasing some components of v will not decrease $[v]_\Omega$. This completes the proof of (2-22). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Input from positive S^1–equivariant symplectic homology

We now state the properties of positive S^1–equivariant symplectic homology, and transfer morphisms defined on it, that are needed to define the capacities c_k and establish their basic properties. These properties are stated in Propositions 3.1 and 3.3 below, which are proved in Sections 6 and 8, respectively.

We say that a Liouville domain (X, λ) is *nondegenerate* if the contact form $\lambda|_{\partial X}$ is nondegenerate. In this case we can define the *positive S^1–equivariant symplectic homology* $\mathrm{SH}^{S^1,+}(X, \lambda)$; see Section 5.4. This is a $\mathbb{Q}$–module.[11] To simplify notation, we often denote $\mathrm{SH}^{S^1,+}(X, \lambda)$ by $\mathrm{CH}(X, \lambda)$ below.[12]

Proposition 3.1 *The positive S^1–equivariant symplectic homology $\mathrm{CH}(X, \lambda)$ has the following properties:*

- **Free homotopy classes** $\mathrm{CH}(X, \lambda)$ *has a direct sum decomposition*

$$\mathrm{CH}(X, \lambda) = \bigoplus_\Gamma \mathrm{CH}(X, \lambda, \Gamma),$$

---

[11] It is also possible to define positive S^1–equivariant symplectic homology with integer coefficients. However, the torsion in the latter is not relevant to the construction of the capacities c_k, and it will simplify our discussion to discard it.

[12] The reason for this notation is that positive S^1–equivariant symplectic homology can be regarded as a substitute for linearized contact homology, which can be defined without transversality difficulties [3, Section 3.2].

where Γ ranges over free homotopy classes of loops in X. We let $\mathrm{CH}(X,\lambda,0)$ denote the summand corresponding to contractible loops in X.

- **Action filtration** For each $L \in \mathbb{R}$, there is a $\mathbb{Q}$–module $\mathrm{CH}^L(X,\lambda,\Gamma)$ which is an invariant of (X,λ,Γ). If $L_1 < L_2$, then there is a well-defined map

(3-1) $$\iota_{L_2,L_1}\colon \mathrm{CH}^{L_1}(X,\lambda,\Gamma) \to \mathrm{CH}^{L_2}(X,\lambda,\Gamma).$$

These maps form a directed system, and we have the direct limit

$$\lim_{L\to\infty} \mathrm{CH}^L(X,\lambda,\Gamma) = \mathrm{CH}(X,\lambda,\Gamma).$$

We denote the resulting map $\mathrm{CH}^L(X,\lambda,\Gamma) \to \mathrm{CH}(X,\lambda,\Gamma)$ by ι_L. We write $\mathrm{CH}^L(X,\lambda) = \bigoplus_\Gamma \mathrm{CH}^L(X,\lambda,\Gamma)$.

- **U map** There is a distinguished map

$$U\colon \mathrm{CH}(X,\lambda,\Gamma) \to \mathrm{CH}(X,\lambda,\Gamma),$$

which respects the action filtration in the following sense: For each $L \in \mathbb{R}$ there is a map

$$U_L\colon \mathrm{CH}^L(X,\lambda,\Gamma) \to \mathrm{CH}^L(X,\lambda,\Gamma).$$

If $L_1 < L_2$ then $U_{L_2} \circ \iota_{L_2,L_1} = \iota_{L_2,L_1} \circ U_{L_1}$. The map U is the direct limit of the maps U_L, ie

(3-2) $$\iota_L \circ U_L = U \circ \iota_L.$$

- **Reeb orbits** If $L_1 < L_2$, and if there does not exist a Reeb orbit γ of $\lambda|_{\partial X}$ in the free homotopy class Γ with action $\mathcal{A}(\gamma) \in (L_1, L_2]$, then the map (3-1) is an isomorphism.

- **δ map** There is a distinguished map

$$\delta\colon \mathrm{CH}(X,\lambda,\Gamma) \to H_*(X,\partial X;\mathbb{Q}) \otimes H_*(BS^1;\mathbb{Q})$$

which vanishes whenever $\Gamma \neq 0$.

- **Scaling** If r is a positive real number, then there are canonical isomorphisms

$$\mathrm{CH}(X,\lambda,\Gamma) \xrightarrow{\cong} \mathrm{CH}(X,r\lambda,\Gamma),$$
$$\mathrm{CH}^L(X,\lambda,\Gamma) \xrightarrow{\cong} \mathrm{CH}^{rL}(X,r\lambda,\Gamma),$$

which commute with all of the above maps.

- **Star-shaped domains** *If X is a nice star-shaped domain in $\mathbb{R}^{2n}$ and λ_0 is the restriction of the standard Liouville form (1-6), and $\lambda_0|_{\partial X}$ is nondegenerate then:*

  (i) *$\mathrm{CH}(X, \lambda_0)$ and $\mathrm{CH}^L(X, \lambda_0)$ have canonical $\mathbb{Z}$ gradings. With respect to this grading, we have*

(3-3)
$$\mathrm{CH}_*(X, \lambda_0) \simeq \begin{cases} \mathbb{Q} & \text{if } * \in n + 1 + 2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

  (ii) *The map δ sends a generator of $\mathrm{CH}_{n-1+2k}(X, \lambda_0)$ to the tensor product of a generator of $H_{2n}(X, \partial X; \mathbb{Q})$ with a generator of $H_{2k-2}(BS^1; \mathbb{Q})$.*

  (iii) *The U map has degree -2 and is an isomorphism*
  $$\mathrm{CH}_*(X, \lambda_0) \xrightarrow{\cong} \mathrm{CH}_{*-2}(X, \lambda_0),$$
  except when $* = n + 1$.

  (iv) *If $\lambda_0|_{\partial X}$ has no Reeb orbit γ with $\mathcal{A}(\gamma) \in (L_1, L_2]$ and $\mathrm{CZ}(\gamma) = n-1+2k$, then the map*
  $$\iota_{L_2, L_1} \colon \mathrm{CH}^{L_1}_{n-1+2k}(X, \lambda_0) \to \mathrm{CH}^{L_2}_{n-1+2k}(X, \lambda_0)$$
  is surjective.

Remark 3.2 One can presumably refine the "Reeb orbits" property to show that $\mathrm{CH}^L(X, \lambda, \Gamma)$ is the homology of a chain complex (with noncanonical differential) which is generated by the good Reeb orbits γ of $\lambda|_{\partial X}$ in the free homotopy class Γ with symplectic action $\mathcal{A}(\gamma) \le L$. (A Reeb orbit γ is called *bad* if it is an even multiple cover of a Reeb orbit γ' such that the Conley–Zehnder indices of γ and γ' have opposite parity; otherwise it is called *good*.) We further expect that, if $L_1 < L_2$, then one can take the differential for L_1 to be the restriction of the differential for L_2, and the map ι_{L_2, L_1} is induced by the inclusion of chain complexes. This is shown in [15, Proposition 3.3] using a different definition of equivariant symplectic homology.

Now suppose that (X', λ') is another nondegenerate Liouville domain and $\varphi \colon (X, \lambda) \to (X', \lambda')$ is a generalized Liouville embedding (see Definition 1.23) with $\varphi(X) \subset \mathrm{int}(X')$. One can then define a *transfer morphism*
$$\Phi \colon \mathrm{CH}(X', \lambda') \to \mathrm{CH}(X, \lambda);$$
see Section 7.

Proposition 3.3 *The transfer morphism Φ has the following properties:*

- **Action** Φ respects the action filtration in the following sense: for each $L \in \mathbb{R}$ there are distinguished maps

$$\Phi^L \colon \mathrm{CH}^L(X', \lambda') \to \mathrm{CH}^L(X, \lambda)$$

such that if $L_1 < L_2$ then

(3-4) $$\Phi^{L_2} \circ \iota_{L_2, L_1} = \iota_{L_2, L_1} \circ \Phi^{L_1},$$

and Φ is the direct limit of the maps Φ^L, ie

(3-5) $$\iota_L \circ \Phi^L = \Phi \circ \iota_L.$$

- **Commutativity with U**   For each $L \in \mathbb{R}$, the diagram

(3-6)
$$
\begin{array}{ccc}
\mathrm{CH}^L(X', \lambda') & \xrightarrow{\;\Phi^L\;} & \mathrm{CH}^L(X, \lambda) \\
\downarrow{\scriptstyle U^L} & & \downarrow{\scriptstyle U^L} \\
\mathrm{CH}^L(X', \lambda') & \xrightarrow{\;\Phi^L\;} & \mathrm{CH}^L(X, \lambda)
\end{array}
$$

commutes.

- **Commutativity with δ**   The diagram

(3-7)
$$
\begin{array}{ccc}
\mathrm{CH}(X', \lambda') & \xrightarrow{\qquad\qquad \Phi \qquad\qquad} & \mathrm{CH}(X, \lambda) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\
H_*(X', \partial X'; \mathbb{Q}) \otimes H_*(BS^1; \mathbb{Q}) & \xrightarrow{\;\rho \otimes 1\;} & H_*(X, \partial X; \mathbb{Q}) \otimes H_*(BS^1; \mathbb{Q})
\end{array}
$$

commutes. Here $\rho \colon H_*(X', \partial X'; \mathbb{Q}) \to H_*(X, \partial X; \mathbb{Q})$ denotes the composition

$$H_*(X', \partial X'; \mathbb{Q}) \to H_*\big(X', X' \setminus \varphi(\mathrm{int}(X)); \mathbb{Q}\big)$$
$$\xrightarrow{\;\cong\;} H_*(\varphi(X), \varphi(\partial X); \mathbb{Q}) = H_*(X, \partial X; \mathbb{Q}),$$

where the first map is the map on relative homology induced by the triple $\big(X', X' \setminus \varphi(\mathrm{int}(X)), \partial X'\big)$ and the second map is excision.

4 Definition of the capacities c_k

4.1 Nondegenerate Liouville domains

We first define the capacities c_k for nondegenerate Liouville domains, imitating the definition of ECH capacities in [20, Definition 4.3].

Definition 4.1 Let (X, λ) be a nondegenerate Liouville domain and let k be a positive integer. Define

$$c_k(X, \lambda) \in (0, \infty]$$

to be the infimum over L such that there exists $\alpha \in \mathrm{CH}^L(X, \lambda)$ satisfying

$$(4\text{-}1) \qquad \delta U^{k-1} \iota_L \alpha = [X] \otimes [\mathrm{pt}] \in H_*(X, \partial X) \otimes H_*(BS^1).$$

We now show that the function c_k satisfies most of the axioms of Theorem 1.24 (restricted to nondegenerate Liouville domains):

Lemma 4.2 (a) *If (X, λ) is a nondegenerate Liouville domain and $r > 0$, then*

$$c_k(X, r\lambda) = r c_k(X, \lambda).$$

(b) *If (X, λ) is a nondegenerate Liouville domain and $k > 1$, then*

$$c_{k-1}(X, \lambda) \leq c_k(X, \lambda).$$

(c) *If (X, λ) and (X', λ') are nondegenerate Liouville domains, and if there exists a generalized Liouville embedding $\varphi\colon (X, \lambda) \to (X', \lambda')$ with $\varphi(X) \subset \mathrm{int}(X')$, then*

$$c_k(X, \lambda) \leq c_k(X', \lambda').$$

(d) *If (X, λ) is a nondegenerate Liouville domain and if $c_k(X, \lambda) < \infty$, then $c_k(X, \lambda) = \mathcal{A}(\gamma)$ for some Reeb orbit γ of $\lambda|_{\partial X}$ which is contractible in X.*

Proof (a) This follows from the scaling axiom in Proposition 3.1.

(b) Suppose that $\alpha \in \mathrm{CH}^L(X, \lambda)$ satisfies (4-1). We need to show that there exists $\alpha' \in \mathrm{CH}^L(X, \lambda)$ such that

$$\delta U^{k-2} \iota_L \alpha' = [X] \otimes [\mathrm{pt}].$$

By (3-2), we can take $\alpha' = U_L \alpha$.

(c) Suppose that $\alpha' \in \mathrm{CH}^L(X', \lambda')$ satisfies

$$(4\text{-}2) \qquad \delta U^{k-1} \iota_L \alpha' = [X'] \otimes [\mathrm{pt}].$$

We need to show that there exists $\alpha \in \mathrm{CH}^L(X, \lambda)$ satisfying

$$\delta U^{k-1} \iota_L \alpha = [X] \otimes [\mathrm{pt}].$$

We claim that we can take $\alpha = \Phi^L \alpha'$, where Φ^L is the filtered transfer map from Proposition 3.3(a). To see this, we observe that

$$\delta U^{k-1} \iota_L \Phi^L \alpha' = \delta \iota_L U_L^{k-1} \Phi^L \alpha' = \delta \iota_L \Phi^L U_L^{k-1} \alpha' = \delta \Phi \iota_L U_L^{k-1} \alpha'$$
$$= (\rho \otimes 1) \delta U^{k-1} \iota_L \alpha' = (\rho \otimes 1)([X'] \otimes [\mathrm{pt}]) = [X] \otimes [\mathrm{pt}].$$

Here the first equality holds by (3-2), the second equality follows from (3-6), the third equality holds by (3-5), the fourth equality uses (3-7) and (3-2) again, and the fifth equality follows from the hypothesis (4-2).

(d) Suppose that $c_k(X, \lambda) = L < \infty$. Suppose, to get a contradiction, that there is no Reeb orbit of action L which is contractible in X. Since $\lambda|_{\partial X}$ is nondegenerate, there are only finitely many Reeb orbits of action less than $2L$. It follows that we can find $\varepsilon > 0$ such that there is no Reeb orbit which is contractible in X and has action in the interval $[L - \varepsilon, L + \varepsilon]$, and such that there exists $\alpha_+ \in \mathrm{CH}^{L+\varepsilon}(X, \lambda)$ with

$$\delta U^{k-1} \iota_{L+\varepsilon} \alpha_+ = [X] \otimes [\mathrm{pt}].$$

By the last part of the "δ map" property in Proposition 3.1, we can assume that $\alpha_+ \in \mathrm{CH}^{L+\varepsilon}(X, \lambda, 0)$. By the "Reeb orbits" property, there exists $\alpha_- \in \mathrm{CH}^{L-\varepsilon}(X, \lambda, 0)$ with $\iota_{L+\varepsilon, L-\varepsilon} \alpha_- = \alpha_+$. It follows that

$$\delta U^{k-1} \iota_{L-\varepsilon} \alpha_- = [X] \otimes [\mathrm{pt}].$$

This implies that $c_k(X, \lambda) \leq L - \varepsilon$, which is the desired contradiction.   □

4.2 Arbitrary Liouville domains

We now extend the definition of c_k to an arbitrary Liouville domain (X, λ). To do so, we use the following procedure to perturb a possibly degenerate Liouville domain to a nondegenerate one.

First recall that there is a distinguished Liouville vector field V on X characterized by $\iota_V d\lambda = \lambda$. Write $Y = \partial X$. The flow of V then defines a smooth embedding

(4-3) $$(-\infty, 0] \times Y \to X,$$

sending $\{0\} \times Y$ to Y in the obvious way, such that if ρ denotes the $(-\infty, 0]$ coordinate, then ∂_ρ is mapped to the vector field V. This embedding pulls back the Liouville form λ on X to the 1–form $e^\rho(\lambda|_Y)$ on $(-\infty, 0] \times Y$. The *completion* of (X, λ) is the pair $(\widehat{X}, \widehat{\lambda})$ defined as follows. First,

$$\widehat{X} = X \cup_Y ([0, \infty) \times Y),$$

glued using the identification (4-3). Observe that $\widehat{X}$ has a subset which is identified with $\mathbb{R} \times Y$, and we denote the $\mathbb{R}$ coordinate on this subset by ρ. The 1–form λ on X then extends to a unique 1–form $\widehat{\lambda}$ on $\widehat{X}$ which agrees with $e^\rho(\lambda|_Y)$ on $\mathbb{R} \times Y$.

Now if $f\colon Y \to \mathbb{R}$ is any smooth function, define a new Liouville domain (X_f, λ_f), where

$$X_f = \widehat{X} \setminus \{(\rho, y) \in \mathbb{R} \times Y \mid \rho > f(y)\}$$

and λ_f is the restriction of $\widehat{\lambda}$ to X_f. For example, if $f \equiv 0$, then $(X_f, \lambda_f) = (X, \lambda)$. In general, there is a canonical identification

$$Y \to \partial X_f, \quad y \mapsto (f(y), y) \in \mathbb{R} \times Y.$$

Under this identification,

$$\lambda_f|_{\partial X_f} = e^f \lambda|_Y.$$

We now consider c_k of nondegenerate perturbations of a possibly degenerate Liouville domain.

Lemma 4.3 (compare [20, Lemma 3.5])   (a) *If (X, λ) is any Liouville domain, then*

(4-4)
$$\sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}) = \inf_{f_+ > 0} c_k(X_{f_+}, \lambda_{f_+}).$$

   Here the supremum and infimum are taken over functions $f_-\colon Y \to (-\infty, 0)$ and $f_+\colon Y \to (0, \infty)$, respectively, such that the contact form $e^{f_\pm}(\lambda|_Y)$ is nondegenerate.

   (b) *If (X, λ) is nondegenerate, then the supremum and infimum in (4-4) agree with $c_k(X, \lambda)$.*

As a result of Lemma 4.3, it makes sense to extend Definition 4.1 as follows:

Definition 4.4   If (X, λ) is any Liouville domain, define $c_k(X, \lambda)$ to be the supremum and infimum in (4-4).

The proof of Lemma 4.3 will use the following simple fact:

Lemma 4.5   *If (X_{f_-}, λ_{f_-}) is nondegenerate and if $f_+ = f_- + \varepsilon$ for some $\varepsilon \in \mathbb{R}$, then (X_{f_+}, λ_{f_+}) is also nondegenerate and*

$$c_k(X_{f_+}, \lambda_{f_+}) = e^\varepsilon c_k(X_{f_-}, \lambda_{f_-}).$$

Proof We have that (X_{f_+}, λ_{f_+}) is nondegenerate because scaling the contact form on the boundary by a constant (in this case e^ε) scales the Reeb vector field and preserves nondegeneracy.

The time ε flow of the Liouville vector field V on $\hat{X}$ restricts to a diffeomorphism $X_{f_-} \to X_{f_+}$ which pulls back λ_{f_+} to $e^\varepsilon \lambda_{f_-}$. It follows that

$$c_k(X_{f_+}, \lambda_{f_+}) = c_k(X_{f_-}, e^\varepsilon \lambda_{f_-}) = e^\varepsilon c_k(X_{f_-}, \lambda_{f_-}),$$

where the second equality holds by the conformality in Lemma 4.2(a). □

Proof of Lemma 4.3 (a) If $f_-, f_+ : Y \to \mathbb{R}$ satisfy $f_- < f_+$, then inclusion defines a Liouville embedding $\varphi : (X_{f_-}, \lambda_{f_-}) \to (X_{f_+}, \lambda_{f_+})$ with $\varphi(X_{f_-}) \subset \operatorname{int}(X_{f_+})$. It then follows from the monotonicity in Lemma 4.2(c) that

$$c_k(X_{f_-}, \lambda_{f_-}) \leq c_k(X_{f_+}, \lambda_{f_+}).$$

This shows that the left-hand side of (4-4) is less than or equal to the right-hand side.

To prove the reverse inequality, for any $\varepsilon > 0$ we can find a function $f_+ : Y \to (0, \varepsilon)$ such that the contact form $e^{f_+}(\lambda|_Y)$ is nondegenerate. Now define $f_- : Y \to (-\varepsilon, 0)$ by $f_- = f_+ - \varepsilon$. By Lemma 4.5 we have

$$c_k(X_{f_+}, \lambda_{f_+}) = e^\varepsilon c_k(X_{f_-}, \lambda_{f_-}).$$

It follows that

$$\inf_{f_+ > 0} c_k(X_{f_+}, \lambda_{f_+}) \leq e^\varepsilon \sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that the right-hand side of (4-4) is less than or equal to the left-hand side.

(b) In this case, for any $\varepsilon > 0$ we can take $f_\pm = \pm \varepsilon$ in (4-4), so using Lemma 4.5 we have

$$\sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}) \geq c_k(X_{-\varepsilon}, \lambda_{-\varepsilon}) = e^{-\varepsilon} c_k(X, \lambda),$$

$$\inf_{f_+ > 0} c_k(X_{f_+}, \lambda_{f_+}) \leq c_k(X_\varepsilon, \lambda_\varepsilon) = e^\varepsilon c_k(X, \lambda).$$

Taking $\varepsilon \to 0$, we obtain

$$\sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}) \geq c_k(X, \lambda) \geq \inf_{f_+ > 0} c_k(X_{f_+}, \lambda_{f_+}).$$

The result now follows from the first half of part (a). □

Proposition 4.6 *The function c_k of Liouville domains satisfies the conformality, increasing, restricted monotonicity and contractible Reeb orbit axioms in Theorem 1.24.*

Proof The conformality and increasing axioms follow immediately from the corresponding properties in Lemma 4.2(a)–(b).

To prove the restricted monotonicity property, suppose that there exists a generalized Liouville embedding $\varphi\colon (X, \lambda) \to (X', \lambda')$. Let $f_-\colon \partial X \to (-\infty, 0)$ and $f_+\colon \partial X' \to (0, \infty)$ be smooth functions such that (X_{f_-}, λ_{f_-}) and $(X'_{f_+}, \lambda'_{f_+})$ are nondegenerate. Then we can restrict φ to X_{f_-}, and compose with the inclusion $X' \to X'_{f_+}$, to obtain a generalized Liouville embedding $\widetilde{\varphi}\colon (X_{f_-}, \lambda_{f_-}) \to (X'_{f_+}, \lambda'_{f_+})$ with $\widetilde{\varphi}(X_{f_-}) \subset \mathrm{int}(X'_{f_+})$. By the monotonicity in Lemma 4.2(c), we have

$$c_k(X_{f_-}, \lambda_{f_-}) \leq c_k(X'_{f_+}, \lambda'_{f_+}).$$

It follows that

$$\sup_{f_- < 0} c_k(X_{f_-}, \lambda_{f_-}) \leq \inf_{f_+ > 0} c_k(X'_{f_+}, \lambda'_{f_+}),$$

which means that $c_k(X, \lambda) \leq c_k(X', \lambda')$.

The contractible Reeb orbit axiom follows from the corresponding property in Lemma 4.2(d) and a compactness argument. $\square$

4.3 Nice star-shaped domains

We now study c_k of nice star-shaped domains and complete the proofs of Theorems 1.1 and 1.24.

Proof of Theorems 1.1 and 1.24 By Proposition 4.6, it is enough to show that the functions c_k, restricted to nice star-shaped domains, satisfy the axioms in Theorem 1.1. The conformality and increasing axioms follow immediately from the corresponding properties in Proposition 4.6. The monotonicity axiom in Theorem 1.1 follows from the restricted monotonicity axiom in Proposition 4.6, because if X and X' are nice star-shaped domains in $\mathbb{R}^{2n}$, then any symplectic embedding $X \to X'$ is automatically a generalized Liouville embedding. Finally, the Reeb orbits axiom follows from Lemma 4.7(b) below. $\square$

Lemma 4.7 *Let X be a nice star-shaped domain in $\mathbb{R}^{2n}$. Suppose that $\lambda_0|_{\partial X}$ is nondegenerate. Then:*

(a) $c_k(X, \lambda_0)$ is the infimum over L such that the degree $n - 1 + 2k$ summand in $\mathrm{CH}(X, \lambda_0)$ is in the image of the map $\iota_L \colon \mathrm{CH}^L(X, \lambda_0) \to \mathrm{CH}(X, \lambda_0)$.

(b) $c_k(X, \lambda_0) = \mathcal{A}(\gamma)$ for some Reeb orbit γ of $\lambda_0|_{\partial X}$ with $\mathrm{CZ}(\gamma) = n - 1 + 2k$.

Proof   (a)   This follows immediately from the definition of c_k and the star-shaped domains property in Proposition 3.1.

(b) This follows from (a), similarly to the proof of Lemma 4.2(d).   □

Remark 4.8   If one is only interested in nice star-shaped domains, then one can take the characterization of c_k in Lemma 4.7(a) as the definition of c_k.

5   Definition of positive S^1–equivariant SH

Our remaining goal is to prove Propositions 3.1 and 3.3. We now review what we need to know about positive S^1–equivariant symplectic homology for this purpose.

(Positive) symplectic homology was developed by Viterbo [33], using work of Cieliebak, Floer and Hofer [14; 5]. The S^1–equivariant version of (positive) symplectic homology was originally defined by Viterbo [33], and an alternative definition using family Floer homology was given by Bourgeois and Oancea [3, Section 2.2], following a suggestion of Seidel [31]. We will use the family Floer homology definition here, because it is more amenable to computations. We follow the treatment in [18], with some minor tweaks which do not affect the results.

We will only consider (positive, S^1–equivariant) symplectic homology for Liouville domains, even though it can be defined for more general compact symplectic manifolds with contact-type boundary. We restrict to Liouville domains in order to be able to define transfer morphisms.

5.1   Symplectic homology

Let (X, λ) be a Liouville domain with boundary Y. Let R_λ denote the Reeb vector field associated to λ on Y. Below, let $\mathrm{Spec}(Y, \lambda)$ denote the set of periods of Reeb orbits, and let $\varepsilon = \frac{1}{2} \min \mathrm{Spec}(Y, \lambda)$.

Recall from Section 4.2 that the completion $(\widehat{X}, \widehat{\lambda})$ of (X, λ) is defined by

$$\widehat{X} := X \cup ([0, \infty) \times Y) \qquad \text{and} \qquad \widehat{\lambda} := \begin{cases} \lambda & \text{on } X, \\ e^\rho \lambda|_Y & \text{on } [0, \infty) \times Y, \end{cases}$$

where ρ denotes the $[0, \infty)$ coordinate. Write $\hat{\omega} = d\hat{\lambda}$. Consider a 1–periodic Hamiltonian on $\hat{X}$, ie a smooth function

$$H \colon S^1 \times \hat{X} \to \mathbb{R},$$

where $S^1 = \mathbb{R}/\mathbb{Z}$. Such a function H determines a vector field X_H^θ on $\hat{X}$ for each $\theta \in S^1$, defined by $\hat{\omega}(X_H^\theta, \cdot) = dH(\theta, \cdot)$. Let $\mathcal{P}(H)$ denote the set of 1–periodic orbits of X_H, ie smooth maps $\gamma \colon S^1 \to \hat{X}$ satisfying the equation $\gamma'(\theta) = X_H^\theta(\gamma(\theta))$.

Definition 5.1 An *admissible Hamiltonian* is a smooth function $H \colon S^1 \times \hat{X} \to \mathbb{R}$ satisfying the following conditions:

(1) The restriction of H to $S^1 \times X$ is negative, autonomous (ie S^1–independent), and C^2–small (so that there are no nonconstant 1–periodic orbits). Furthermore,

(5-1) $$H > -\varepsilon$$

on $S^1 \times X$.

(2) There exists $\rho_0 \geq 0$ such that on $S^1 \times [\rho_0, \infty) \times Y$ we have

(5-2) $$H(\theta, \rho, y) = \beta e^\rho + \beta'$$

with $0 < \beta \notin \mathrm{Spec}(Y, \lambda)$ and $\beta' \in \mathbb{R}$. The constant β is called the *limiting slope* of H.

(3) There exists a small, strictly convex, increasing function $h \colon [1, e^{\rho_0}] \to \mathbb{R}$ such that on $S^1 \times [0, \rho_0] \times Y$, the function H is C^2–close to the function sending $(\theta, \rho, x) \mapsto h(e^\rho)$. The precise sense of "small" and "close" that we need here is explained in Remarks 5.2 and 5.6.

(4) The Hamiltonian H is nondegenerate, ie all 1–periodic orbits of X_H are non-degenerate.

We denote the set of admissible Hamiltonians by $\mathcal{H}_{\mathrm{adm}}$.

Remark 5.2 Condition (1) implies that the only 1–periodic orbits of X_H in X are constants; they correspond to critical points of H.

The significance of condition (2) is as follows. On $S^1 \times [0, \infty) \times Y$, for a Hamiltonian of the form $H_1(\theta, \rho, y) = h_1(e^\rho)$, we have

$$X_{H_1}^\theta(\rho, y) = -h_1'(e^\rho) R_\lambda(y).$$

Hence, for such a Hamiltonian H_1 with h_1 increasing, a 1–periodic orbit of X_{H_1} maps to a level $\{\rho\} \times Y$, and the image of its projection to Y is the image of a (not necessarily simple) periodic Reeb orbit of period $h_1'(e^\rho)$. In particular, condition (2) implies that there is no 1–periodic orbit of X_H in $[\rho_0, \infty) \times Y$.

Condition (3) ensures that for any nonconstant 1–periodic orbit γ_H of X_H, there exists a (not necessarily simple) periodic Reeb orbit γ of period $T < \beta$ such that the image of γ_H is close to the image of γ in $\{\rho\} \times Y$, where $T = h'(e^\rho)$.

Definition 5.3 An S^1–family of almost complex structures $J: S^1 \to \text{End}(T\widehat{X})$ is *admissible* if it satisfies the following conditions:

- J^θ is $\widehat{\omega}$–compatible for each $\theta \in S^1$.

- There exists $\rho_1 \geq 0$ such that on $[\rho_1, \infty) \times Y$, the almost complex structure J^θ does not depend on θ, is invariant under translation of ρ, sends ξ to itself compatibly with $d\lambda$, and satisfies

$$(5\text{-}3) \qquad\qquad J^\theta(\partial_\rho) = R_\lambda.$$

We denote the set of all admissible J by $\mathcal{J}$.

Given $J \in \mathcal{J}$, and $\gamma_-, \gamma_+ \in \mathcal{P}(H)$, let $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ denote the set of maps

$$u: \mathbb{R} \times S^1 \to \widehat{X}$$

satisfying Floer's equation

$$(5\text{-}4) \qquad \frac{\partial u}{\partial s}(s, \theta) + J^\theta(u(s, \theta))\left(\frac{\partial u}{\partial \theta}(s, \theta) - X_H^\theta(u(s, \theta))\right) = 0$$

as well as the asymptotic conditions

$$\lim_{s \to \pm\infty} u(s, \cdot) = \gamma_\pm.$$

If J is generic and $u \in \widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$, then $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ is a manifold near u whose dimension is the Fredholm index of u, defined by

$$\text{ind}(u) = \text{CZ}_\tau(\gamma_+) - \text{CZ}_\tau(\gamma_-).$$

Here CZ_τ denotes the Conley–Zehnder index computed using trivializations τ of $\gamma_\pm^\star T\widehat{X}$ that extend to a trivialization of $u^\star T\widehat{X}$. Note that $\mathbb{R}$ acts on $\widehat{\mathcal{M}}(\gamma_-, \gamma_+; J)$ by translation of the domain; we denote the quotient by $\mathcal{M}(\gamma_-, \gamma_+; J)$.

Definition 5.4 Let $H \in \mathcal{H}_{\mathrm{adm}}$, and let $J \in \mathcal{J}$ be generic. Define the Floer chain complex $(\mathrm{CF}(H, J), \partial)$ as follows. The chain module $\mathrm{CF}(H, J)$ is the free $\mathbb{Q}$–module[13] generated by the set of 1–periodic orbits $\mathcal{P}(H)$. If $\gamma_-, \gamma_+ \in \mathcal{P}(H)$, then the coefficient of γ_+ in $\partial \gamma_-$ is obtained by counting Fredholm index 1 elements of $\mathcal{M}(\gamma_-, \gamma_+; J)$ with signs determined by a system of coherent orientations as in [13]. (The chain complexes for different choices of coherent orientations are canonically isomorphic.)

Let $\mathrm{HF}(H, J)$ denote the homology of the chain complex $(\mathrm{CF}(H, J), \partial)$. Given H, the homologies for different choices of generic J are canonically isomorphic to each other, so we can denote this homology simply by $\mathrm{HF}(H)$.

The construction of the above canonical isomorphisms is a special case of the following more general construction. Given two admissible Hamiltonians $H_1, H_2 \in \mathcal{H}_{\mathrm{adm}}$, write $H_1 \leq H_2$ if $H_1(\theta, x) \leq H_2(\theta, x)$ for all $(\theta, x) \in S^1 \times \widehat{X}$. In this situation, one defines a *continuation morphism* $\mathrm{HF}(H_1) \to \mathrm{HF}(H_2)$ as follows; see [18, Theorem 4.5]. Choose generic $J_1, J_2 \in \mathcal{J}$ so that the chain complexes $\mathrm{CF}(H_i, J_i)$ are defined for $i = 1, 2$. Choose a generic homotopy $\{(H_s, J_s)\}_{s \in \mathbb{R}}$ such that H_s satisfies (5-2) for some β and β' depending on s; $J_s \in \mathcal{J}$ for each $s \in \mathbb{R}$; $\partial_s H_s \geq 0$; $(H_s, J_s) = (H_1, J_1)$ for $s \ll 0$; and $(H_s, J_s) = (H_2, J_2)$ for $s \gg 0$. One then defines a chain map $\mathrm{CF}(H_1, J_1) \to \mathrm{CF}(H_2, J_2)$ as a signed count of Fredholm index 0 maps $u \colon \mathbb{R} \times S^1 \to \widehat{X}$ satisfying the equation

$$(5\text{-}5) \qquad \frac{\partial u}{\partial s} + J_s^\theta \circ u \left(\frac{\partial u}{\partial \theta} - X_{H_s}^\theta \circ u \right) = 0$$

and the asymptotic conditions $\lim_{s \to -\infty} u(s, \cdot) = \gamma_1$ and $\lim_{s \to \infty} u(s, \cdot) = \gamma_2$. The induced map on homology gives a well-defined map $\mathrm{HF}(H_1) \to \mathrm{HF}(H_2)$. If $H_2 \leq H_3$, then the continuation map $\mathrm{HF}(H_1) \to \mathrm{HF}(H_3)$ is the composition of the continuation maps $\mathrm{HF}(H_1) \to \mathrm{HF}(H_2)$ and $\mathrm{HF}(H_2) \to \mathrm{HF}(H_3)$.

Definition 5.5 We define the *symplectic homology* of (X, λ) to be the direct limit

$$\mathrm{SH}(X, \lambda) := \varinjlim{}_{H \in \mathcal{H}_{\mathrm{adm}}} \mathrm{HF}(H)$$

with respect to the partial order $\leq$ and continuation maps defined above.

---

[13]It is also possible to use $\mathbb{Z}$ coefficients here, but we will use $\mathbb{Q}$ coefficients in order to later establish the Reeb orbits property in Proposition 3.1, which leads to the Reeb orbits property of the capacities c_k. In special cases when the Conley–Zehnder index of a 1–periodic orbit is unambiguously defined, for example when all 1–periodic orbits are contractible and $c_1(TX)|_{\pi_2(X)} = 0$, the chain complex is graded by minus the Conley–Zehnder index.

5.2 Positive symplectic homology

Positive symplectic homology is a modification of symplectic homology in which constant 1–periodic orbits are discarded.

To explain this, let $H\colon S^1 \times \widehat{X} \to \mathbb{R}$ be a Hamiltonian in $\mathcal{H}_{\mathrm{adm}}$. The *Hamiltonian action* functional $\mathcal{A}_H\colon C^\infty(S^1, \widehat{X}) \to \mathbb{R}$ is defined by

$$\mathcal{A}_H(\gamma) := -\int_{S^1} \gamma^\star \widehat{\lambda} - \int_{S^1} H(\theta, \gamma(\theta))\, d\theta.$$

If $J \in \mathcal{J}$, then the differential on the chain complex $(\mathrm{CF}(H, J), \partial)$ decreases the Hamiltonian action $\mathcal{A}_H$. As a result, for any $L \in \mathbb{R}$, we have a subcomplex $\mathrm{CF}^{\leq L}(H, J)$ of $\mathrm{CF}(H, J)$, generated by the 1–periodic orbits with Hamiltonian action less than or equal to L.

To see what this subcomplex can look like, note that the 1–periodic orbits of $H \in \mathcal{H}_{\mathrm{adm}}$ fall into two classes: (i) constant orbits corresponding to critical points in X, and (ii) nonconstant orbits contained in $[0, \rho_0] \times Y$.

If x is a critical point of H on X, then the action of the corresponding constant orbit is equal to $-H(x)$. By (5-1), this is less than ε.

By Remark 5.2, a nonconstant 1–periodic orbit of X_H is close to a 1–periodic orbit of $-h'(e^\rho)R_\lambda$ located in $\{\rho\} \times Y$ for $\rho \in [0, \rho_0]$ with $h'(e^\rho) \in \mathrm{Spec}(Y, \lambda)$. The Hamiltonian action of the latter loop is given by

$$(5\text{-}6) \qquad -\int_{S^1} e^\rho \lambda(-h'(e^\rho)R_\lambda)\, d\theta - \int_{S^1} h(e^\rho)\, d\theta = e^\rho h'(e^\rho) - h(e^\rho).$$

Since h is strictly convex, the right-hand side is a strictly increasing function of ρ.

Remark 5.6 In Definition 5.1, we assume that h is sufficiently small that the right-hand side of (5-6) is close to the period $h'(e^\rho)$, and in particular greater than ε. We also assume that H is sufficiently close to $h(e^\rho)$ on $S^1 \times [0, \rho_0] \times Y$ that the Hamiltonian actions of the 1–periodic orbits are well approximated by the right-hand side of (5-6), so that:

(i) The Hamiltonian action of every 1–periodic orbit of X_H corresponding to a critical point on X is less than ε, and the Hamiltonian action of every other 1–periodic orbit is greater than ε.

(ii) If γ is a Reeb orbit of period $T < \beta$ and if γ' is a 1–periodic orbit of X_H in $[0, \rho_0] \times Y$ associated to γ, then

$$|\mathcal{A}_H(\gamma') - T| < \min\{\beta^{-1}, \tfrac{1}{3} \operatorname{gap}(\beta)\}.$$

Here $\operatorname{gap}(\beta)$ is the minimum difference between two elements of $\operatorname{Spec}(Y, \lambda)$ that are less than β.

We can now define positive symplectic homology.

Definition 5.7 Let (X, λ) be a Liouville domain, let H be a Hamiltonian in $\mathcal{H}_{\mathrm{adm}}$ and let $J \in \mathcal{J}$.

Consider the quotient complex

$$\mathrm{CF}^+(H, J) := \frac{\mathrm{CF}(H, J)}{\mathrm{CF}^{\leq \varepsilon}(H, J)}.$$

The homology of the quotient complex is independent of J, so we can denote this homology by $\mathrm{HF}^+(H)$. More generally, if $H_1 \leq H_2$, then the chain map used to define the continuation map $\mathrm{HF}(H_1) \to \mathrm{HF}(H_2)$ descends to the quotient, since the Hamiltonian action is nonincreasing along a solution of (5-5) when the homotopy is nondecreasing. Thus we obtain a well-defined continuation map $\mathrm{HF}^+(H_1) \to \mathrm{HF}^+(H_2)$ satisfying the same properties as before.

We now define the *positive symplectic homology* of (X, λ) to be the direct limit

$$\mathrm{SH}^+(X, \lambda) := \varinjlim_{H \in \mathcal{H}_{\mathrm{adm}}} \mathrm{HF}^+(H).$$

Positive symplectic homology can sometimes be better understood using certain special admissible Hamiltonians, obtained as follows.

Definition 5.8 [1] Let (X, λ) be a Liouville domain. An *admissible Morse–Bott Hamiltonian* is an autonomous Hamiltonian $H \colon \widehat{X} \to \mathbb{R}$ such that:

(1) The restriction of H to X is a Morse function which is negative and C^2–small (so that the Hamiltonian vector field has no nonconstant 1–periodic orbits).

(2) There exists $\rho_0 \geq 0$ such that on $[\rho_0, \infty) \times Y$ we have

$$H(\rho, x) = \beta e^\rho + \beta'$$

with $0 < \beta \notin \operatorname{Spec}(Y, \lambda)$ and $\beta' \in \mathbb{R}$.

(3)   On $[0, \rho_0) \times Y$ we have

$$H(\rho, x) = h(e^\rho),$$

where h is as in Definition 5.1, and moreover $h'' - h' > 0$.

We denote the set of admissible Morse–Bott Hamiltonians by $\mathcal{H}_{\mathrm{MB}}$.

Given $H \in \mathcal{H}_{\mathrm{MB}}$, each 1–periodic orbit of X_H is either (i) a constant orbit correspond-
ing to a critical point of H in X, or (ii) a nonconstant 1–periodic orbit, with image in
$\{\rho\} \times Y$ for $\rho \in (0, \rho_0)$, whose projection to Y has the same image as a Reeb orbit of
period $e^\rho h'(\rho)$. Since H is autonomous, every Reeb orbit γ with period less than β
gives rise to an S^1 family of 1–periodic orbits of X_H, which we denote by S_γ.

An admissible Morse–Bott Hamiltonian as in Definition 5.8 can be deformed into an
admissible Hamiltonian as in Definition 5.1, which will be time-dependent and have
nondegenerate 1–periodic orbits:

Lemma 5.9 [6, Proposition 2.2; 1, Lemma 3.4] *An admissible Morse–Bott Hamil-
tonian H can be perturbed to an admissible Hamiltonian H' whose 1–periodic orbits
consist of the following:*

   (i)   *Constant orbits at the critical points of H.*

   (ii) *For each Reeb orbit γ with period less than β, two nondegenerate orbits $\hat{\gamma}$
         and $\check{\gamma}$. Given a trivialization τ of $\xi|\gamma$, their Conley–Zehnder indices are given
         by $-\mathrm{CZ}_\tau(\hat{\gamma}) = \mathrm{CZ}_\tau(\gamma) + 1$ and $-\mathrm{CZ}_\tau(\check{\gamma}) = \mathrm{CZ}_\tau(\gamma)$.*

Remark 5.10 The references [6; 1] use the notation $\gamma_{\min}$ instead of $\hat{\gamma}$, and γ_{Max}
instead of $\check{\gamma}$. The motivation is that these orbits are distinguished in their S^1–family
as critical points of a perfect Morse function on S^1.

5.3 S^1–equivariant symplectic homology

Let (X, λ) be a Liouville domain with boundary Y. We now review how to define the
S^1–equivariant symplectic homology $\mathrm{SH}^{S^1}(X, \lambda)$, and the positive S^1–equivariant
symplectic homology $\mathrm{SH}^{S^1,+}(X, \lambda)$.

The S^1–equivariant symplectic homology $\mathrm{SH}^{S^1}(X, \lambda)$ is defined as a limit as $N \to \infty$
of homologies $\mathrm{SH}^{S^1,N}(X, \lambda)$, where N is a nonnegative integer. To define the latter,

fix the perfect Morse function $f_N \colon \mathbb{C}P^N \to \mathbb{R}$ defined by

$$f_N([w^0 : \cdots : w^n]) = \frac{\sum_{j=0}^{N} j\,|w^j|^2}{\sum_{j=0}^{N} |w^j|^2}.$$

Let $\tilde{f}_N \colon S^{2N+1} \to \mathbb{R}$ denote the pullback of f_N to S^{2N+1}. We will consider gradient flow lines of $\tilde{f}_N$ and f_N with respect to the standard metric on S^{2N+1} and the metric that this induces on $\mathbb{C}P^N$.

Remark 5.11 The family of functions f_N has the following two properties, which are needed below. We have two isometric inclusions $i_0, i_1 \colon \mathbb{C}P^N \to \mathbb{C}P^{N+1}$ defined by $i_0([z_0 : \cdots : z_N]) = [z_0 : \cdots : z_N : 0]$ and $i_1([z_0 : \cdots : z_N]) = [0 : z_0 : \cdots : z_N]$. Then:

(1) The images of i_0 and i_1 are invariant under the gradient flow of f_{N+1}.

(2) We have $f_N = f_{N+1} \circ i_0 = f_{N+1} \circ i_1 + \text{constant}$, so that the gradient flow of f_{N+1} pulls back via i_0 or i_1 to the gradient flow of f_N.

Now choose a "parametrized Hamiltonian"

(5-7) $$H \colon S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$$

which is S^1–invariant in the sense that

$$H(\theta + \varphi, x, \varphi z) = H(\theta, x, z) \quad \text{for all } \theta, \varphi \in S^1 = \mathbb{R}/\mathbb{Z}, \ x \in \widehat{X}, \ z \in S^{2N+1}.$$

Here the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on $S^{2N+1} \subset \mathbb{C}^{N+1}$ is defined by $\varphi \cdot z = e^{2\pi i \varphi} z$.

Definition 5.12 A parametrized Hamiltonian H as above is *admissible* if:

(i) For each $z \in S^{2N+1}$, the Hamiltonian

$$H_z = H(\,\cdot\,,\,\cdot\,, z) \colon S^1 \times \widehat{X} \to \mathbb{R}$$

satisfies conditions (1), (2) and (3) in Definition 5.1, with β and β' independent of z.

(ii) If z is a critical point of $\tilde{f}_N$, then the 1–periodic orbits of H_z are nondegenerate.

(iii) H is nondecreasing along downward gradient flow lines of $\tilde{f}_N$.

Let $\mathcal{P}^{S^1}(\tilde{f}_N, H)$ denote the set of pairs (z, γ), where $z \in S^{2N+1}$ is a critical point of $\tilde{f}_N$ and γ is a 1–periodic orbit of the Hamiltonian H_z. Note that S^1 acts freely on the set $\mathcal{P}^{S^1}(\tilde{f}_N, H)$ by

$$\varphi \cdot (z, \gamma) = (\varphi \cdot z, \gamma(\,\cdot - \varphi)).$$

If $p = (z, \gamma) \in \mathcal{P}^{S^1}(\tilde{f}_N, H)$, let S_p denote the orbit of (z, γ) under this S^1 action.

Next, choose a generic map

(5-8) $$J \colon S^1 \times S^{2N+1} \to \mathcal{J}, \quad (\theta, z) \mapsto J_z^\theta,$$

which is S^1–invariant in the sense that

$$J_{\varphi \cdot z}^{\theta + \varphi} = J_z^\theta$$

for all $\varphi, \theta \in S^1$ and $z \in S^{2N+1}$.

Let $p^- = (z^-, \gamma^-)$ and $p^+ = (z^+, \gamma^+)$ be distinct elements of $\mathcal{P}^{S^1}(\tilde{f}_N, H)$. Let $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ be the set of pairs (η, u), where $\eta \colon \mathbb{R} \to S^{2N+1}$ and $u \colon \mathbb{R} \times S^1 \to \widehat{X}$, satisfying the equations

$$\dot{\eta} + \vec{\nabla} \tilde{f}_N(\eta) = 0,$$

(5-9) $$\partial_s u + J_{\eta(s)}^\theta \circ u(\partial_\theta u - X_{H_{\eta(s)}^\theta} \circ u) = 0,$$

$$\lim_{s \to \pm\infty} (\eta(s), u(s, \cdot)) \in S_{p^\pm}.$$

Here the middle equation is a modification of Floer's equation (5-4) which is "parametrized by η". Note that $\mathbb{R}$ acts on the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ by reparametrization: if $\sigma \in \mathbb{R}$, then

$$\sigma \cdot (\eta, u) = (\eta(\,\cdot - \sigma), u(\,\cdot - \sigma, \cdot)).$$

In addition, S^1 acts on the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ as follows: if $\tau \in S^1$, then

$$\tau \cdot (\eta, u) := (\tau \cdot \eta, u(\cdot, \cdot - \tau)).$$

Let $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ denote the quotient of the set $\widehat{\mathcal{M}}(S_{p^-}, S_{p^+}; J)$ by these actions of $\mathbb{R}$ and S^1.

If J is generic, then $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ is a manifold near (η, u) of dimension

$$\mathrm{ind}(\eta, u) = (\mathrm{ind}(f_N, z^-) - \mathrm{CZ}_\tau(\gamma^-)) - (\mathrm{ind}(f_N, z^+) - \mathrm{CZ}_\tau(\gamma^+)) - 1.$$

Here $\mathrm{ind}(f_N, z^\pm)$ denotes the Morse index of the critical point $z^\pm$ of f_N and CZ_τ denotes the Conley–Zehnder index with respect to a trivialization τ of $(\gamma^\pm)^\star T\widehat{X}$ that extends over $u^\star T\widehat{X}$.

Definition 5.13 [3, Section 2.2] Define a chain complex $(\mathrm{CF}^{S^1, N}(H, J), \partial^{S^1})$ as follows. The chain module $\mathrm{CF}^{S^1, N}(H, J)$ is the free $\mathbb{Q}$–module[14] generated by the orbits S_p. If S_{p^-} and S_{p^+} are two such orbits, then the coefficient of S_{p^+} in $\partial^{S^1} S_{p^-}$ is a signed count of elements (η, u) of $\mathcal{M}^{S^1}(S_{p^-}, S_{p^+}; J)$ with $\mathrm{ind}(\eta, u) = 1$.

---

[14]It is also possible to define $\mathrm{SH}^{S^1, +}$, using $\mathbb{Z}$ coefficients, as with SH.

We denote the homology of this chain complex by $\mathrm{HF}^{S^1,N}(H)$. This does not depend on the choice of J, by the usual continuation argument; one defines continuation chain maps using a modification of (5-9) in which the second line is replaced by an "η–parametrized" version of Floer's continuation equation (5-5).

We now define a partial order on the set of pairs (N, H), where N is a nonnegative integer and H is an admissible parametrized Hamiltonian (5-7), as follows. Let $\tilde{\imath}_0 \colon S^{2N+1} \to S^{2N+3}$ denote the inclusion sending $z \mapsto (z, 0)$. (This lifts the inclusion i_0 defined in Remark 5.11.) Then $(N_1, H_1) \le (N_2, H_2)$ if and only if

- $N_1 \le N_2$, and
- $H_1 \le (\tilde{\imath}_0^\star)^{N_2-N_1} H_2$ pointwise on $S^1 \times \hat{X} \times S^{2N_1+1}$.

In this case we can define a continuation map $\mathrm{HF}^{S^1,N_1}(H_1) \to \mathrm{HF}^{S^1,N_2}(H_2)$ using an increasing homotopy from H_1 to $(\tilde{\imath}_0^\star)^{N_2-N_1} H_2$ on $S^1 \times \hat{X} \times S^{2N_1+1}$.

Definition 5.14 Define the S^1–*equivariant symplectic homology*

$$\mathrm{SH}_*^{S^1}(X, \lambda) := \varinjlim_{N,H} \mathrm{HF}_*^{S^1,N}(H).$$

It is sometimes useful to describe S^1–equivariant symplectic homology in terms of individual Hamiltonians on $S^1 \times \hat{X}$, rather than S^{2N+1}–families of them, by the following procedure.

Remark 5.15 [17, Section 2.1.1] Fix an admissible Hamiltonian $H' \colon S^1 \times \hat{X} \to \mathbb{R}$ and a nonnegative integer N. Consider a sequence of admissible parametrized Hamiltonians $\{H_k\}_{k=0,\ldots,N}$ as in (5-7), where H_k is defined on $S^1 \times \hat{X} \times S^{2k+1}$, with the following properties:

- For each $k = 0, \ldots, N-1$, the pullbacks $\tilde{\imath}_0^\star H_{k+1}$ and $\tilde{\imath}_1^\star H_{k+1}$ agree with H_k up to a constant. Here $\tilde{\imath}_1 \colon S^{2k+1} \to S^{2k+3}$ denotes the lift of i_1 sending $z \mapsto (0, z)$.
- For each $k = 0, \ldots, N$ and each $z \in \mathrm{Crit}(\tilde{f}_k)$, we have

$$(5\text{-}10) \qquad\qquad H_k(\theta, x, z) = H'(\theta - \phi(z), x) + c.$$

  Here c is a constant depending on k and z, and the map $\phi \colon \mathrm{Crit}(\tilde{f}_k) \to S^1$ sends a critical point $(0, \ldots, 0, e^{2\pi i \psi}, 0, \ldots, 0) \mapsto \psi$.

Next, choose a sequence of families of almost complex structures $J_k \colon S^1 \times S^{2k+1} \to \mathcal{J}(\widehat{X})$ for $k = 0, \ldots, N$ such that:

- J_k is generic, so that the chain complex $(\mathrm{CF}^{S^1,k}(H_k, J_k), \partial^{S^1})$ is defined.
- $\tilde{\imath}_0^\star J_{k+1} = \tilde{\imath}_1^\star J_{k+1} = J_k$.

The chain complex $(\mathrm{CF}^{S^1,N}(H_N, J_N), \partial^{S^1})$ can now be described as follows. By (5-10), we can identify the chain module as

$$\text{(5-11)} \qquad \mathrm{CF}^{S^1,N}(H_N, J_N) = \mathbb{Q}\{1, u, \ldots, u^N\} \otimes_{\mathbb{Q}} \mathrm{CF}(H', J_0).$$

This identification sends a pair (z, γ), where $z \in \mathrm{Crit}(\tilde{f}_N)$ is a lift of an index $2k$ critical point of f_N and γ is a reparametrization of a 1–periodic orbit γ' of H', to $u^k \otimes \gamma'$.

Since the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\tilde{\imath}_1$, the differential has the form

$$\text{(5-12)} \qquad \partial^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_i(\gamma),$$

where the operator φ_i on $\mathrm{CF}(H', J_0)$ does not depend on k. In particular, φ_0 is the differential on $\mathrm{CF}(H', J_0)$. We can also formally write

$$\partial^{S^1} = \sum_{i=0}^{N} u^{-i} \otimes \varphi_i,$$

where it is understood that u^{-i} annihilates terms of the form $u^j \otimes \gamma$ with $i > j$.

The usual continuation arguments show that the homology of this chain complex does not depend on the choice of sequences $\{H_k\}$ and $\{J_k\}$ satisfying the above assumptions. We denote this homology by $\mathrm{HF}^{S^1,N}(H')$.

Since in the above construction we assume that the sequences $\{H_k\}$ and $\{J_k\}$ respect the inclusions $\tilde{\imath}_0$, it follows that when $N_1 \leq N_2$ we have a well-defined map $\mathrm{HF}^{S^1,N_1}(H') \to \mathrm{HF}^{S^1,N_2}(H')$ induced by inclusion of chain complexes.

As before, if $H_1' \leq H_2'$, then there is a continuation map $\mathrm{HF}^{S^1,N}(H_1') \to \mathrm{HF}^{S^1,N}(H_2')$ satisfying the usual properties.

As in [3, Section 2.3], we now have:

Proposition 5.16 *The S^1–equivariant homology of (X, λ) is given by*

$$\mathrm{SH}_*^{S^1}(X, \lambda) = \lim_{N \in \mathbb{N},\, H' \in \mathcal{H}_{\mathrm{adm}}} \mathrm{HF}^{S^1,N}(H').$$

5.4 Positive S^1–equivariant symplectic homology

Like symplectic homology, S^1–equivariant symplectic homology also has a positive version in which constant 1–periodic orbits are discarded.

Definition 5.17 Let $H\colon S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ be an admissible parametrized Hamiltonian. The *parametrized action functional* $\mathcal{A}_H\colon C^\infty(S^1, \widehat{X}) \times S^{2N+1} \to \mathbb{R}$ is defined by

$$(5\text{-}13) \qquad \mathcal{A}_H(\gamma, z) := -\int_\gamma \widehat{\lambda} - \int_{S^1} H(\theta, \gamma(\theta), z)\, d\theta.$$

Lemma 5.18 *If H is an admissible parametrized Hamiltonian and if J is a generic S^1–invariant family of almost complex structures as in (5-8), then the differential ∂^{S^1} on $\mathrm{CF}^{S^1,N}(H, J)$ does not increase the parametrized action (5-13).*

Proof Given a solution (η, u) to the equations (5-9), one can think of η as fixed and regard u as a solution to an instance of (5-5), where J_s and H_s in (5-5) are determined by η. By Definition 5.12(iii), this instance of (5-5) corresponds to a nondecreasing homotopy of Hamiltonians. Consequently, the action is nonincreasing along this solution of (5-5), as before. $\qquad\square$

It follows from Lemma 5.18 that for any $L \in \mathbb{R}$ we have a subcomplex $\mathrm{CF}^{S^1,N,\leq L}(H, J)$ of $\mathrm{CF}^{S^1,N}(H, J)$, spanned by S^1–orbits of pairs (z, γ) where $z \in \mathrm{Crit}(\tilde{f}_N)$ and γ is a 1–periodic orbit of H_z with $\mathcal{A}_H(z, \gamma) \leq L$.

As in Section 5.2, if the S^1–orbit of (z, γ) is a generator of $\mathrm{CF}^{S^1,N}(H, J)$, then there are two possibilities: (i) γ is a constant orbit corresponding to a critical point of H_z on X and $\mathcal{A}_H(z, \gamma) < \varepsilon$, or (ii) γ is close to a Reeb orbit in $\{\rho\} \times Y$ with period $-h'(e^\rho)$ and $\mathcal{A}_H(z, \gamma)$ is close to this period, in particular $\mathcal{A}_H(z, \gamma) > \varepsilon$.

Definition 5.19 Consider the quotient complex

$$(5\text{-}14) \qquad \mathrm{CF}^{S^1,N,+}(H, J) := \frac{\mathrm{CF}^{S^1,N}(H, J)}{\mathrm{CF}^{S^1,N,\leq\varepsilon}(H, J)}.$$

As in Definition 5.7, the homology of the quotient complex is independent of J, so we can denote this homology by $\mathrm{HF}^{S^1,N,+}(H)$, and we have continuation maps $\mathrm{HF}^{S^1,N_1,+}(H_1) \to \mathrm{HF}^{S^1,N_2,+}(H_2)$ when $(N_1, H_1) \leq (N_2, H_2)$. We now define the *positive S^1–equivariant symplectic homology* by

$$(5\text{-}15) \qquad \mathrm{SH}^{S^1,+}(X, \lambda) := \varinjlim_{N,H} \mathrm{HF}^{S^1,N,+}(H).$$

Returning to the situation of Remark 5.15, define $\mathrm{HF}^{S^1,N,+}(H')$ to be the homology of the quotient of the chain complex (5-11) by the subcomplex spanned by $u^k \otimes \gamma$, where γ is a critical point of H' in X. We then have the following analogue of Proposition 5.16:

Proposition 5.20 *The positive S^1–equivariant homology of (X, λ) is given by*

$$\mathrm{SH}^{S^1,+}(X, \lambda) = \varinjlim_{N \in \mathbb{N},\, H' \in \mathcal{H}_{\mathrm{adm}}} \mathrm{HF}^{S^1,N,+}(H').$$

6 Properties of positive S^1–equivariant SH

Let (X, λ) be a Liouville domain. We now show that the positive S^1–equivariant homology $\mathrm{SH}^{S^1,+}(X, \lambda)$ defined in Section 5, which we denote by $\mathrm{CH}(X, \lambda)$ for short, satisfies all of the properties in Proposition 3.1.

6.1 Free homotopy classes

Given an admissible Hamiltonian H, we can decompose the complex $\mathrm{CF}^{S^1,N,\leq L}(H, J)$ into a direct sum

$$\mathrm{CF}^{S^1,N,\leq L}(H, J) = \bigoplus_{\Gamma} \mathrm{CF}^{S^1,N,\leq L}(H, J, \Gamma).$$

Here Γ ranges over free homotopy classes of loops in X and $\mathrm{CF}^{S^1,N,\leq L}(H, J, \Gamma)$ denotes the subset of $\mathrm{CF}^{S^1,N,\leq L}(H, J)$ generated by S^1–orbits of pairs (z, γ), where γ represents the free homotopy class Γ.

The differentials and continuation maps defined in Section 5 all count certain cylinders, and thus respect the above direct sum decomposition. As a result, we obtain a corresponding direct sum decomposition in (5-14) and (5-15), so that we can decompose

$$\mathrm{CH}(X, \lambda) = \bigoplus_{\Gamma} \mathrm{CH}(X, \lambda, \Gamma),$$

where $\mathrm{CH}(X, \lambda, \Gamma)$ is defined like $\mathrm{CH}(X, \lambda)$ but only using loops in the free homotopy class Γ.

Similar remarks apply to all of the constructions to follow; we will omit the free homotopy class Γ below to simplify notation.

6.2   Action filtration

Given $L \in \mathbb{R}$, we now define a version of positive S^1–equivariant symplectic homology "filtered up to action L", which we denote by $\mathrm{CH}^L(X, \lambda)$. This will only depend on the largest element of $\mathrm{Spec}(Y, \lambda)$ which is less than or equal to L. Thus, we can assume without loss of generality that $L \notin \mathrm{Spec}(Y, \lambda)$.

As in Definition 5.19, we can consider the quotient complex

$$\mathrm{CF}^{S^1, N, +, \leq L}(H, J) := \frac{\mathrm{CF}^{S^1, N, \leq L}(H, J)}{\mathrm{CF}^{S^1, N, \leq \varepsilon}(H, J)}.$$

As in Definition 5.7, the homology of the quotient complex is independent of J, so we can denote this homology by $\mathrm{HF}^{S^1, N, +, \leq L}(H)$. If $(N_1, H_1) \leq (N_2, H_2)$, then the continuation chain map induces a well-defined map $\mathrm{HF}^{S^1, N_1, +, \leq L}(H_1) \to \mathrm{HF}^{S^1, N_2, +, \leq L}(H_2)$.

Definition 6.1   We define the positive S^1–equivariant symplectic homology filtered up to action L to be

$$\mathrm{CH}^L(X, \lambda) := \mathrm{SH}^{S^1, +, \leq L}(X, \lambda) := \varinjlim_{N, H} \mathrm{HF}^{S^1, N, +, \leq L}(H).$$

It follows from Remark 5.6(ii) that if $L \notin \mathrm{Spec}(Y, \lambda)$, then $\mathrm{CH}^L(X, \lambda)$ depends only on the largest element of $\mathrm{Spec}(Y, \lambda)$ that is less than L.

Given an admissible parametrized Hamiltonian H, a nonnegative integer N, a generic parametrized almost complex structure J as in (5-8) and real numbers $L_1 < L_2$, we have an inclusion of chain complexes

$$(6\text{-}1) \qquad \mathrm{CF}^{S^1, N, +, \leq L_1}(H, J) \to \mathrm{CF}^{S^1, N, +, \leq L_2}(H, J).$$

The usual continuation map argument shows that the induced map on homology

$$(6\text{-}2) \qquad \mathrm{HF}^{S^1, N, +, \leq L_1}(H) \to \mathrm{HF}^{S^1, N, +, \leq L_2}(H)$$

does not depend on the choice of J, and commutes with the continuation map for $(N_1, H_1) \leq (N_2, J_2)$.

Definition 6.2   We define the map

$$(6\text{-}3) \qquad \iota_{L_2, L_1} \colon \mathrm{CH}^{L_1}(X, \lambda) \to \mathrm{CH}^{L_2}(X, \lambda)$$

to be the direct limit over pairs (N, H) of the maps (6-2).

We then have the required property

$$\text{(6-4)} \qquad\qquad \lim_{L\to\infty} \text{CH}^L(X,\lambda) = \text{CH}(X,\lambda),$$

because we can compute the direct limit

$$\varinjlim_{N,H,L} \text{HF}^{S^1,N,+,\leq L}(H)$$

either by first taking the limit over pairs (N, H), which gives the left-hand side of (6-4), or by first taking the limit over L, which gives the right-hand side of (6-4).

Remark 6.3 One can equivalently define $\text{CH}^L(X,\lambda)$ by repeating the definition of $\text{CH}(X,\lambda)$, but using appropriate admissible Hamiltonians where the limiting slope is equal to L.

6.3 U map

We now define the U map on $\text{CH}(Y,\lambda)$, similarly to [3, Section 2.4].

Recall from Remark 5.15 that given an admissible Hamiltonian $H': S^1 \times \hat{X} \to \mathbb{R}$ and a nonnegative integer N, we can choose a pair (H_N, J_N) so that the chain complex $(\text{CH}^{S^1,N}(H_N, J_N), \partial^{S^1})$ has the nice form given by (5-11) and (5-12).

It follows from (5-12) that the operation of "multiplication by u^{-1}", sending a chain complex generator $u^i \otimes \gamma$ to $u^{i-1} \otimes \gamma$ when $i > 0$ and to 0 when $i = 0$, is a chain map. This induces a map on the homology $\text{HF}^{S^1,N}(H')$, which we denote by $U_{N,H'}$. A priori this map also depends on the choice of pair (H_N, J_N), but the usual continuation map argument shows that it does not. In addition, if $(N_1, H_1') \leq (N_2, H_2')$, then the continuation map $\text{HF}^{S^1,N_1}(H_1') \to \text{HF}^{S^1,N_2}(H_2')$ fits into a commutative diagram

$$
\begin{array}{ccc}
\text{HF}^{S^1,N_1}(H_1') & \longrightarrow & \text{HF}^{S^1,N_2}(H_2') \\
\downarrow{\scriptstyle U_{N_1,H_1'}} & & \downarrow{\scriptstyle U_{N_2,H_2'}} \\
\text{HF}^{S^1,N_1}(H_1') & \longrightarrow & \text{HF}^{S^1,N_2}(H_2')
\end{array}
$$

It then follows from Proposition 5.16 that we obtain a well-defined map

$$U = \varinjlim_{N,H'} U_{N,H'}$$

on $\text{SH}_*^{S^1}(X,\lambda)$.

Since the U map is induced by chain maps which respect (in fact preserve) the symplectic action filtration, it also follows from Proposition 5.20 that we obtain a well-defined U map on $\mathrm{CH}(Y, \lambda)$. Similarly, we obtain a well-defined U map on $\mathrm{CH}^L(Y, \lambda)$. This completes the proof of the "U map" property.

For use in Section 6.7 below, we also note that there is the following Gysin-type exact sequence:

Proposition 6.4 *If (X, λ) is a Liouville domain, then there is a long exact sequence*

$$(6\text{-}5) \qquad \cdots \to \mathrm{SH}^+(X, \lambda) \to \mathrm{CH}(X, \lambda) \xrightarrow{U} \mathrm{CH}(X, \lambda) \to \mathrm{SH}^+(X, \lambda) \to \cdots .$$

Proof With the above definition of U, this follows as in [3, Proposition 2.9]. This was also shown earlier in [2] using a slightly different definition of positive S^1–equivariant symplectic homology. $\qquad \square$

6.4 Reeb orbits

Let $L_1 < L_2$ be such that there does not exist a Reeb orbit γ of $\lambda|_{\partial X}$ having action $\mathcal{A}(\gamma)$ in the interval $(L_1, L_2]$. As in Section 6.2, we can also assume without loss of generality that $L_1 \notin \mathrm{Spec}(Y, \lambda)$. Then, for every triple (N, H, J), if the limiting slope of H is sufficiently large, then the inclusion of chain complexes (6-1) is the identity map. It follows that the map (6-2) is an isomorphism, and consequently the direct limit map (6-3) is an isomorphism, as desired.

6.5 δ map

To define the delta map, we have the following:

Proposition 6.5 *Let (X, λ) be a Liouville domain. Then there is a canonical long exact sequence*

$$(6\text{-}6)$$

$$H_*(X, \partial X) \otimes H_*(BS^1) \longrightarrow \mathrm{SH}^{S^1}(X, \lambda)$$

$$\overset{\delta}{\nwarrow} \qquad \swarrow$$

$$\mathrm{SH}^{S^1,+}(X, \lambda)$$

Proof For any triple (N, H, J) as in Definition 5.19, by definition we have a short exact sequence of chain complexes

$$(6\text{-}7) \qquad 0 \to \mathrm{CF}^{S^1,N,\leq\varepsilon}(H, J) \to \mathrm{CF}^{S^1,N}(H, J) \to \mathrm{CF}^{S^1,N,+}(H, J) \to 0.$$

Since continuation maps respect symplectic action, we can take the direct limit of the resulting long exact sequences on homology to obtain a canonical long exact sequence

$$(6\text{-}8) \qquad \cdots \to \mathrm{SH}^{S^1,\leq\varepsilon}(X,\lambda) \to \mathrm{SH}^{S^1}(X,\lambda) \to \mathrm{SH}^{S^1,+}(X,\lambda) \to \cdots,$$

where we define

$$(6\text{-}9) \qquad \mathrm{SH}^{S^1,\leq\varepsilon}(X,\lambda) = \varinjlim_{N,H} \mathrm{HF}^{S^1,N,\leq\varepsilon}(H,J).$$

To compute (6-9), note that we have a canonical isomorphism

$$(6\text{-}10) \qquad \mathrm{HF}^{S^1,N,\leq\varepsilon}(H,J) = H_*(X,\partial X) \otimes \mathbb{Q}\{1,u,\ldots,u^N\}.$$

For proofs of counterparts of this isomorphism for different definitions of S^1–equivariant symplectic homology, see [33, Proposition 1.3; 2, Lemma 4.8]. In our context, the isomorphism (6-10) holds because if we compute the left-hand side as in Remark 5.15, then the chain complex comes from the critical points of H' on X, so that we have

$$(6\text{-}11) \qquad \mathrm{CF}^{S^1,N,\leq\varepsilon}(H,J) = C_{\mathrm{Morse}}(X,H') \otimes \mathbb{Q}\{1,u,\ldots,u^N\}.$$

Here $C_{\mathrm{Morse}}(X,H')$ denotes the chain complex for the Morse cohomology of H', whose differential counts upward gradient flow lines, and u^i represents the index $2i$ critical point of f_N. The differential on left side of (6-11) agrees on the right side with the tensor product of the Morse differential and the identity on $\mathbb{Q}\{1,u,\ldots,u^N\}$. Since the gradient of H' points out of X along ∂X, the Morse cohomology agrees with the relative homology $H_*(X,\partial X)$. This proves (6-10), and taking the direct limit over pairs (N,H) gives a canonical isomorphism

$$(6\text{-}12) \qquad \mathrm{SH}^{S^1,\leq\varepsilon}(X,\lambda) = H_*(X,\partial X) \otimes H_*(BS^1).$$

Putting this into (6-8) proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The map δ vanishes on $\mathrm{CH}(X,\lambda,\Gamma)$ for every free homotopy class $\Gamma \neq 0$, because the maps in the long exact sequence (6-8) preserve the free homotopy class, and the homology (6-12) is entirely supported in the summand corresponding to $\Gamma = 0$.

6.6 Scaling

If $(\widehat{X},\widehat{\lambda})$ is the completion of (X,λ), then the completion of $(X,r\lambda)$ is naturally identified with the same manifold $\widehat{X}$, with the 1–form $r\widehat{\lambda}$.

If $H\colon S^1 \times \hat{X} \to \mathbb{R}$ is an S^1–dependent Hamiltonian and if X_H denotes the (S^1–dependent) Hamiltonian vector field for H defined using $\hat{\omega}$, then the Hamiltonian vector field for H defined using $r\hat{\omega}$ is $r^{-1}X_H$. It follows that if H is an admissible Hamiltonian for (X, λ), then rH is an admissible Hamiltonian for $(X, r\lambda)$, with the same 1–periodic orbits. Note here that $\mathrm{Spec}(Y, r\lambda) = r\,\mathrm{Spec}(Y, \lambda)$, so the conditions involving the action spectrum are preserved. In particular, if $\varepsilon = \frac{1}{2}\min\mathrm{Spec}(Y, \lambda)$ as usual, then

$$r\varepsilon = \tfrac{1}{2}\min\mathrm{Spec}(Y, r\lambda).$$

Likewise, if $H\colon S^1 \times \hat{X} \times S^{2N+1} \to \mathbb{R}$ is an admissible parametrized Hamiltonian for (X, λ), then rH is an admissible parametrized Hamiltonian for $(X, r\lambda)$.

If J is an admissible parametrized almost complex structure (5-8) as needed to define the (positive) S^1–equivariant symplectic homology of (X, λ), then J is not quite admissible for $(X, r\lambda)$, because the condition (5-3) only holds up to a constant. However, one can still define (positive) S^1–equivariant symplectic homology using parametrized almost complex structures that satisfy this weaker version of admissibility — see [28, Section 1.3.2] — and a continuation argument shows that the resulting (positive) S^1–equivariant symplectic homology will be canonically isomorphic.

Putting this together, we have a canonical isomorphism of chain complexes

$$\mathrm{CF}^{S^1,N,\leq L}(H, J) = \mathrm{CF}^{S^1,N,\leq rL}(rH, J).$$

We then have a canonical isomorphism of quotient chain complexes

$$\frac{\mathrm{CF}^{S^1,N,\leq L}(H, J)}{\mathrm{CF}^{S^1,N,\leq \varepsilon}(H, J)} = \frac{\mathrm{CF}^{S^1,N,\leq rL}(rH, J)}{\mathrm{CF}^{S^1,N,\leq r\varepsilon}(rH, J)}.$$

Taking the direct limit over pairs (N, H) gives the desired canonical isomorphism

$$\mathrm{CH}^L(X, \lambda) = \mathrm{CH}^{rL}(X, r\lambda).$$

We can also take $L = +\infty$, giving the desired canonical isomorphism

$$\mathrm{CH}(X, \lambda) = \mathrm{CH}(X, r\lambda).$$

These scaling isomorphisms preserve the U and δ maps, since the U map is purely formal, and the holomorphic curves counted by the δ maps are the same.

6.7 Star-shaped domains

When X is a nice star-shaped domain, the chain complex $CF^{S^1,N}(H, J)$ has a canonical $\mathbb{Z}$ grading, in which the grading of a pair (z, γ) is $\mathrm{ind}(z) - CZ(\gamma)$. Here $\mathrm{ind}(z)$ denotes the Morse index of the corresponding critical point of f_N, while $CZ(\gamma)$ denotes the Conley–Zehnder index of γ, computed using a global trivialization of TX.

With respect to this grading, the long exact sequence (6-6) has the form

(6-13)

$$H_{*+n}(X, \partial X) \otimes H_*(BS^1) \xrightarrow{\hspace{4cm}} SH_*^{S^1}(X)$$

with maps δ ([−1]) and into $SH_*^{S^1,+}(X)$.

For a nice star-shaped domain X, we have $SH_*^{S^1}(X) = 0$; see [17, Section 1.3.2]. Assertions (i) and (ii) in the star-shaped domains property follow. (The computation (3-3) also follows from [18, Theorem 1.1] together with the description of the Reeb orbits on the boundary of an ellipsoid in the proof of Lemma 2.1.)

To prove assertion (iii), note that for a nice star-shaped domain, the Gysin-type sequence (6-5) with gradings has the form

$$\cdots \to SH_k^+(X) \to CH_k(X) \xrightarrow{U} CH_{k-2}(X) \to SH_{k-1}^+(X) \to \cdots .$$

On the other hand, if X is a nice star-shaped domain then

$$SH_*^+(X) = \begin{cases} \mathbb{Q} & \text{if } * = n+1, \\ 0 & \text{otherwise;} \end{cases}$$

see [17, Section 1.2.4]. Plugging this with (3-3) into the above Gysin-type sequence, we immediately see that the U map $CH_*(X, \lambda) \to CH_{*-2}(X, \lambda)$ is an isomorphism except when $* = n+1$.

Finally, we need to prove assertion (iv). Suppose that $\lambda_0|_{\partial X}$ is nondegenerate and has no Reeb orbit γ with action $\mathcal{A}(\gamma) \in (L_1, L_2]$ and Conley–Zehnder index $CZ(\gamma) = n - 1 + 2k$. We need to show that the map

(6-14) $\iota_{L_2,L_1} \colon CH_{n-1+2k}^{L_1}(X, \lambda_0) \to CH_{n-1+2k}^{L_2}(X, \lambda_0)$

is surjective. As in Section 6.2, we can assume without loss of generality that $L_1, L_2 \notin \mathrm{Spec}(Y, \lambda)$.

To prove that (6-14) is surjective, we compute positive S^1–equivariant symplectic homology using an admissible Hamiltonian $H'\colon S^1 \times \widehat{X} \to \mathbb{R}$ as in Remark 5.15. Furthermore, we assume that H' is perturbed from an admissible Morse–Bott Hamiltonian as in Lemma 5.9, with boundary slope $\beta > L_2$. As a result, if $L < \beta$ is not close to the action of a Reeb orbit, then the chain complex $\mathrm{CF}^{S^1,N,+,\leq L}(H_N, J_N)$ is generated by symbols $u^k \otimes \check{\gamma}$ and $u^k \otimes \widehat{\gamma}$, where $0 \leq k \leq N$ and γ is a Reeb orbit with action $\mathcal{A}(\gamma) \leq L$. Furthermore, the grading of a generator is given by

$$|u^k \otimes \check{\gamma}| = \mathrm{CZ}(\gamma) + 2k,$$
$$|u^k \otimes \widehat{\gamma}| = \mathrm{CZ}(\gamma) + 2k + 1.$$

Now fix N, H_N and J_N. The differential on the chain complex $\mathrm{CF}^{S^1,N,+,\leq L}(H_N, J_N)$ does not increase the symplectic action of Reeb orbits. This means that we can define an integer-valued filtration $\mathcal{F}$ on the chain complex as follows: Denote the real numbers in the action spectrum $\mathrm{Spec}(Y, \lambda)$ by

$$a_1 < a_2 < \cdots.$$

If γ is a Reeb orbit with action $\mathcal{A}(\gamma) = a_j$, then we define the filtration by

$$\mathcal{F}(u^i \otimes \check{\gamma}) = \mathcal{F}(u^i \otimes \widehat{\gamma}) = j.$$

Let $\mathcal{F}_j \mathrm{CF}^{\leq L}$ denote the subcomplex of $\mathrm{CF}^{S^1,N,+,\leq L}(H_N, J_N)$ spanned by generators with filtration $\leq j$. Let

$$\mathcal{G}_j \mathrm{CF}^{\leq L} = \mathcal{F}_j \mathrm{CF}^{\leq L} / \mathcal{F}_{j-1} \mathrm{CF}^{\leq L}$$

denote the associated graded complex.

It is shown in [18, Section 3.2] that the homology of $\bigoplus_j \mathcal{G}_j \mathrm{CF}^{\leq L}$ is generated by $u^0 \otimes \check{\gamma}$ and $u^N \otimes \widehat{\gamma}$, where γ ranges over the good Reeb orbits with action less than L. It follows that if N is sufficiently large with respect to k and L, then the degree $n - 1 + 2k$ part of $\bigoplus_j \mathcal{G}_j \mathrm{CF}^{\leq L}$ is generated by $u^0 \otimes \check{\gamma}$, where γ is a good Reeb orbit with action less than L and Conley–Zehnder index equal to $n - 1 + 2k$. Therefore, the inclusion of chain complexes

$$(6\text{-}15) \qquad \mathrm{CF}^{S^1,N,+,\leq L_1}(H_N, J_N) \to \mathrm{CF}^{S^1,N,+,\leq L_2}(H_N, J_N)$$

induces an injection

$$\mathcal{G}_j \mathrm{CF}^{\leq L_1} \to \mathcal{G}_j \mathrm{CF}^{\leq L_2}$$

for each j. Furthermore, under our assumption on k, L_1 and L_2, if N is sufficiently large, then the above injection in grading $n - 1 + 2k$ is an isomorphism

$$\mathcal{G}_j \mathrm{CF}_{n-1+2k}^{\leq L_1} \xrightarrow{\cong} \mathcal{G}_j \mathrm{CF}_{n-1+2k}^{\leq L_2}$$

for each j. It now follows from the algebraic Lemma 6.6 below that the inclusion (6-15) induces a surjection on the degree $n - 1 + 2k$ homology

$$(6\text{-}16) \qquad \mathrm{HF}_{n-1+2k}^{S^1,N,+,\leq L_1}(H_N, J_N) \to \mathrm{HF}_{n-1+2k}^{S^1,N,+,\leq L_2}(H_N, J_N).$$

Lemma 6.6 *Let*

$$0 = \mathcal{F}_0 C_* \subset \mathcal{F}_1 C_* \subset \cdots \subset \mathcal{F}_J C_* = C_*,$$
$$0 = \mathcal{F}_0 C_*' \subset \mathcal{F}_1 C_*' \subset \cdots \subset \mathcal{F}_J C_*' = C_*'$$

be filtered chain complexes. Denote the associated graded chain complexes by $\mathcal{G}_j C_* = \mathcal{F}_j C_* / \mathcal{F}_{j-1} C_*$ and $\mathcal{G}_j C_*' = \mathcal{F}_j C_*' / \mathcal{F}_{j-1} C_*'$. Let $\phi \colon C_* \to C_*'$ be a map of filtered chain complexes. For a given grading k, suppose that for each j, the map ϕ induces a surjection $H_k(\mathcal{G}_j C_*) \to H_k(\mathcal{G}_j C_*')$ and an injection $H_{k-1}(\mathcal{G}_j C_*) \to H_{k-1}(\mathcal{G}_j C_*')$. Then ϕ induces a surjection $H_k C_* \to H_k C_*'$ and an injection $H_{k-1} C_* \to H_{k-1} C_*'$.

Proof Since the filtrations are bounded, it is enough to prove by induction on j that ϕ induces a surjection $H_k(\mathcal{F}_j C_*) \to H_k(\mathcal{F}_j C_*')$ and an injection $H_{k-1}(\mathcal{F}_j C_*) \to H_{k-1}(\mathcal{F}_j C_*')$. Assume that the claim holds for $j - 1$. We then have a commutative diagram with exact rows

$$
\begin{array}{ccccccc}
H_k(\mathcal{F}_{j-1}C_*) & \longrightarrow & H_k(\mathcal{F}_j C_*) & \longrightarrow & H_k(\mathcal{G}_j C_*) & \longrightarrow & H_{k-1}(\mathcal{F}_{j-1}C_*) \\
\downarrow{\scriptstyle\text{surj}} & & \downarrow & & \downarrow{\scriptstyle\text{surj}} & & \downarrow{\scriptstyle\text{inj}} \\
H_k(\mathcal{F}_{j-1}C_*') & \longrightarrow & H_k(\mathcal{F}_j C_*') & \longrightarrow & H_k(\mathcal{G}_j C_*') & \longrightarrow & H_{k-1}(\mathcal{F}_{j-1}C_*')
\end{array}
$$

where the vertical arrows are induced by ϕ. Surjectivity of the second vertical arrow then follows from chasing this diagram. (This is one of the two "four lemmas" that imply the "five lemma".) Likewise, the injectivity claim for j follows by chasing the commutative diagram with exact rows

$$
\begin{array}{ccccccc}
H_k(\mathcal{G}_j C_*) & \longrightarrow & H_{k-1}(\mathcal{F}_{j-1}C_*) & \longrightarrow & H_{k-1}(\mathcal{F}_j C_*) & \longrightarrow & H_{k-1}(\mathcal{G}_j C_*) \\
\downarrow{\scriptstyle\text{surj}} & & \downarrow{\scriptstyle\text{inj}} & & \downarrow & & \downarrow{\scriptstyle\text{inj}} \\
H_k(\mathcal{G}_j C_*') & \longrightarrow & H_{k-1}(\mathcal{F}_{j-1}C_*') & \longrightarrow & H_{k-1}(\mathcal{F}_j C_*') & \longrightarrow & H_{k-1}(\mathcal{G}_j C_*')
\end{array}
$$

This concludes the proof of Lemma 6.6. $\qquad\square$

Since (6-16) is a surjection, by taking the direct limit over N and H', and using an action-filtered version of Proposition 5.20, we conclude that the map (6-14) is surjective, as desired.

7 Definition of transfer morphisms

Let (V, λ_V) and (W, λ_W) be Liouville domains. Let $\varphi: V \to W$ be a Liouville embedding, ie a smooth embedding such that $\varphi^\star \lambda_W = \lambda_V$. Assume also as in Section 3 that $\varphi(V) \subset \text{int}(W)$. In this situation one can define a "transfer morphism"

(7-1) $$\phi_{V,W}^{(S^1,+)}: \text{SH}^{(S^1,+)}(W, \lambda_W) \to \text{SH}^{(S^1,+)}(V, \lambda_V).$$

Here the superscript $(S^1, +)$ means that the superscripts S^1 and $+$ are optional (but the same in all three places).

A transfer morphism for symplectic homology was defined by Viterbo [33], and extended by the first author in his PhD thesis [18] for (positive) equivariant symplectic homology. We now review what we need to know about the definition of the transfer morphisms (7-1), and then explain how to extend the construction to generalized Liouville embeddings as in Definition 1.23.

7.1 Transfer morphisms for (positive) symplectic homology

To construct transfer morphisms, we introduce a special class $\mathcal{H}_{\text{stair}}(V, W)$ of Hamiltonians on $S^1 \times \widehat{W}$ called "admissible stair Hamiltonians". The transfer morphism is defined as a direct limit of continuation morphisms between an admissible Hamiltonian $H_1 \in \mathcal{H}_{\text{adm}}(W)$ and an admissible stair Hamiltonian $H_2 \in \mathcal{H}_{\text{stair}}(V, W)$.

Below, identify V with its image under the Liouville embedding φ. Given $\delta > 0$ small, there is a unique neighborhood U of ∂V in $W \setminus \text{int}(V)$, together with a symplectomorphism

$$(U, \omega_W) \simeq ([0, \delta] \times \partial V, d(e^\rho \lambda_V)),$$

such that the Liouville vector field for λ_W on the left-hand side corresponds to ∂_ρ on the right-hand side. Here ρ denotes the $[0, \delta]$ coordinate.

Definition 7.1 A Hamiltonian $H_2: S^1 \times \widehat{W} \to \mathbb{R}$ is in $\mathcal{H}_{\text{stair}}(V, W)$ if and only if:

(1) The restriction of H_2 to $S^1 \times V$ is negative, autonomous (ie S^1–independent), and C^2–small (so that there are no nonconstant 1–periodic orbits). Furthermore,

(7-2)
$$H > -\varepsilon$$

on $S^1 \times V$, where $\varepsilon = \frac{1}{2} \min\{\mathrm{Spec}(\partial V, \lambda_V) \cup \mathrm{Spec}(\partial W, \lambda_W)\}$.

(2) On $S^1 \times U \cong S^1 \times [0, \delta] \times \partial V$, with ρ denoting the $[0, \delta]$ coordinate, we have:

- There exists $0 < \rho_0 < \frac{\delta}{4}$ such that for $\rho_0 \le \rho \le \delta - \rho_0$ we have

(7-3)
$$H_2(\theta, \rho, y) = \beta e^\rho + \beta',$$

where $y \in \partial V$, $0 < \beta \notin \mathrm{Spec}(\partial V, \lambda_V) \cup \mathrm{Spec}(\partial W, \lambda_W)$ and $\beta' \in \mathbb{R}$.

- There exists a strictly convex increasing function $h_1 \colon [1, e^{\rho_0}] \to \mathbb{R}$ such that on $S^1 \times [0, \rho_0] \times \partial V$, the function H_2 is C^2–close to the function sending $(\theta, \rho, y) \mapsto h_1(e^\rho)$. Here and in the rest of this definition, the meanings of "close" and "small" are as in Remarks 5.2 and 5.6.

- There exists a small, strictly concave, increasing function $h_2 \colon [e^{\delta - \rho_0}, e^\delta] \to \mathbb{R}$ such that on $S^1 \times [\delta - \rho_0, \delta] \times \partial V$, the function H_2 is C^2–close to the function sending $(\theta, \rho, y) \mapsto h_2(e^\rho)$.

(3) On $S^1 \times W \setminus (V \cup U)$, the function H_2 is C^2–close to a constant.

(4) On $S^1 \times [0, +\infty) \times \partial W$, with ρ' denoting the $[0, \infty)$ coordinate, we have:

- There exists $\rho'_1 > 0$ such that for $\rho' \ge \rho'_1$ we have

$$H_2(\theta, \rho', p) = \mu e^{\rho'} + \mu',$$

with $0 < \mu \notin \mathrm{Spec}(\partial V, \lambda_V) \cup \mathrm{Spec}(\partial W, \lambda_W)$, $\mu < \beta(e^\delta - 1)/e^\delta$ and $\mu' \in \mathbb{R}$.

- There exists a strictly convex, increasing function $h_3 \colon [1, e^{\rho'_1}] \to \mathbb{R}$ such that $h_3 - h_3(1)$ is small and, on $S^1 \times [0, \rho'_1] \times \partial W$, the function H_2 is C^2–close to the function sending $(\theta, \rho', p) \mapsto h_3(e^{\rho'})$.

(5) The Hamiltonian H_2 is nondegenerate, ie all 1–periodic orbits of X_{H_2} are nondegenerate.

We denote the set of admissible stair Hamiltonians by $\mathcal{H}_{\mathrm{stair}}(V, W)$.

The graph of an admissible stair Hamiltonian H_2 is shown schematically in Figure 1.

The 1–periodic orbits of H_2 lie either in the interior of V (which we call region I), in $[0, \rho_0] \times \partial V$ (region II), in $[\delta - \rho_0, \delta] \times \partial V$ (region III), in $W \setminus (V \cup U)$ (region IV) or in $[0, \rho'_1] \times \partial W$ (region V).
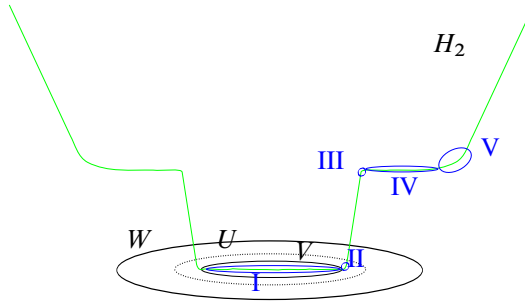
Figure 1: Graph of an admissible stair Hamiltonian H_2 on $S^1 \times \widehat{W}$ [18]

I The 1–periodic orbits in region I correspond to critical points of H_2 on V.

II In region II, the 1–periodic orbits are associated to Reeb orbits of λ_V on ∂V as in Remark 5.6.

III In region III, the 1–periodic orbits are likewise associated to Reeb orbits of λ_V on ∂V.

IV The 1–periodic orbits in region IV correspond to critical points of H_2 on $W \setminus (V \cup U)$.

V In region V, the 1–periodic orbits are associated to Reeb orbits of λ_W on ∂W.

The Hamiltonian actions of the 1–periodic orbits are ordered as follows:

$$\mathcal{A}(\text{IV}) < \mathcal{A}(\text{V}) < 0 < \mathcal{A}(\text{I}) < \mathcal{A}(\text{II}).$$

This means that every 1–periodic orbit in region IV has Hamiltonian action less than every 1–periodic orbit in region V, and so forth.

We now consider the Floer chain complex $\text{CF}(H_2, J_2)$, where $J_2 \colon S^1 \to \text{End}(T\widehat{W})$ is an S^1–family of almost complex structures on $\widehat{W}$. As in Definition 5.3, we assume that J_2^θ is $\widehat{\omega}_W$–compatible for each $\theta \in S^1$, and that

$$J_2^\theta(\partial_{\rho'}) = R_{\lambda_W}$$

on $[\rho_1', \infty) \times \partial W$. This is enough to give a well-defined chain complex $\text{CF}(H_2, J_2)$; see [27, Section 1.2.3]. We also assume that

(7-4) $$J_2^\theta(\partial_\rho) = R_{\lambda_V}$$

on $[\rho_0, \delta - \rho_0] \times \partial V$.

Let $C^{\mathrm{I,III,IV,V}}(H_2, J_2)$ denote the subcomplex of $\mathrm{CF}(H_2, J_2)$ generated by 1–periodic orbits lying in regions I, III, IV and V. Let $C^{\mathrm{III,IV,V}}(H_2, J_2)$ denote the subcomplex of $\mathrm{CF}(H_2, J_2)$ generated by 1–periodic orbits lying in regions III, IV and V. These are subcomplexes because the action decreases along Floer trajectories, and [9, Lemma 2.3] shows that there does not exist any Floer trajectory from region III to region I or II. We then have quotient chain complexes

$$C^{\mathrm{I,II}}(H_2, J_2) = C^{\mathrm{I,II,III,IV,V}}(H_2, J_2)/C^{\mathrm{III,IV,V}}(H_2, J_2),$$
$$C^{\mathrm{II}}(H_2, J_2) = C^{\mathrm{I,II,III,IV,V}}(H_2, J_2)/C^{\mathrm{I,III,IV,V}}(H_2, J_2).$$

Given H_2 and J_2 as above, let $H_2^V \in \mathcal{H}_{\mathrm{adm}}(V)$ denote the admissible Hamiltonian for V which agrees with H_2 on $V \cup ([0, \delta - \rho_0] \times \partial V)$, and which agrees with the right-hand side of (7-3) on $[\rho_0, \infty) \times \partial V$. Let J_2^V denote the admissible S^1–family of almost complex structures on $\widehat{V}$ which agrees with J_2 on $V \cup ([0, \delta - \rho_0] \times \partial V)$, and which satisfies (7-4) on $[\rho_0, \infty) \times \partial V$. Observe that we have canonical identifications of chain modules

(7-5)
$$C^{\mathrm{I,II}}(H_2, J_2) = \mathrm{CF}(H_2^V, J_2^V),$$
$$C^{\mathrm{II}}(H_2, J_2) = \mathrm{CF}^+(H_2^V, J_2^V),$$

because the generators on both sides correspond to the same 1–periodic orbits in $V \cup ([0, \delta - \rho_0] \times \partial V)$.

Proposition 7.2 [18, Proposition 4.4] *The canonical identifications* (7-5) *induce isomorphisms on homology*

$$H(C^{\mathrm{I,II}}(H_2, J_2), \partial) = \mathrm{HF}(H_2^V, J_2^V),$$
$$H(C^{\mathrm{II}}(H_2, J_2), \partial) = \mathrm{HF}^+(H_2^V, J_2^V).$$

Given H_2 and J_2 as above, suppose that $H_1 \in \mathcal{H}_{\mathrm{adm}}(W)$ satisfies $H_1 \leq H_2$ pointwise. Let J_1 be an admissible S^1–family of almost complex structures on $\widehat{W}$. We then have a well-defined continuation map

(7-6) $\mathrm{HF}(H_1, J_1) \to \mathrm{HF}(H_2, J_2),$

defined as in (5-5).

Definition 7.3 We define the transfer morphism on Floer homology to be the composition

$$\phi_{H_2^V, H_1} \colon \mathrm{HF}(H_1, J_1) \to \mathrm{HF}(H_2, J_2) \to H(C^{\mathrm{I,II}}(H_2, J_2)) = \mathrm{HF}(H_2^V, J_2^V).$$

Here the first arrow is the continuation map (7-6), the second map is induced by projection onto the quotient chain complex and the equality sign on the right is the canonical isomorphism from Proposition 7.2. Concretely, this map counts solutions of (5-5) going from a 1–periodic orbit of X_{H_1} to a 1–periodic orbit of X_{H_2} lying in region I or II.

Since the continuation map decreases action, it follows that in the above composition, we can start with the homology of the quotient by $\mathrm{CF}^{\leq \varepsilon}(H_1, J_1)$, to obtain a transfer map on positive Floer homology,

$$\phi^+_{H_2^V, H_1} : \mathrm{HF}^+(H_1, J_1) \to H\left(\frac{\mathrm{CF}(H_2, J_2)}{\mathrm{CF}^{\leq \varepsilon}(H_2, J_2)} \right) \to H(C^{\mathrm{II}}(H_2, J_2)) = \mathrm{HF}^+(H_2^V, J_2^V).$$

The above transfer maps $\phi_{H_2^V, H_1}$ and $\phi^+_{H_2^V, H_1}$ depend only on H_1 and H_2^V, and more generally commute with continuation maps for increasing H_1 and H_2^V; see [18, Proposition 4.7]. Consequently, we can define a transfer morphism on (positive) symplectic homology by taking direct limits:

$$\phi_{V,W} = \varinjlim_{H_1, H_2^V} \phi_{H_2^V, H_1} : \mathrm{SH}(W, \lambda_W) \to \mathrm{SH}(V, \lambda_V),$$

$$\phi^+_{V,W} = \varinjlim_{H_1, H_2^V} \phi^+_{H_2^V, H_1} : \mathrm{SH}^+(W, \lambda_W) \to \mathrm{SH}^+(V, \lambda_V).$$

7.2 Transfer morphisms for (positive) S^1–equivariant symplectic homology

Recall that to define (positive) S^1–equivariant symplectic homology, we modify the definition of (positive) symplectic homology, by replacing the notion of admissible Hamiltonians $H : S^1 \times \widehat{X} \to \mathbb{R}$ in Definition 5.1 by the notion of admissible parametrized Hamiltonians $H : S^1 \times \widehat{X} \times S^{2N+1} \to \mathbb{R}$ in Definition 5.12. In an analogous way, one can modify the definition of admissible stair Hamiltonians $H_2 : S^1 \times \widehat{W} \to \mathbb{R}$ in Definition 7.1, to define a notion of "admissible parametrized stair Hamiltonians" $H_2 : S^1 \times \widehat{W} \times S^{2N+1} \to \mathbb{R}$. We can then repeat the constructions in Section 7.1 to obtain transfer maps

$$\phi^{S^1}_{H_2^V, H_1} : \mathrm{HF}^{S^1, N}(H_1) \to \mathrm{HF}^{S^1, N}(H_2^V),$$

(7-7) $$\phi^{S^1, +}_{H_2^V, H_1} : \mathrm{HF}^{S^1, N, +}(H_1) \to \mathrm{HF}^{S^1, N, +}(H_2^V).$$

We can then take the direct limit over H_1, H_2^V, and N to define transfer morphisms

$$\phi_{V,W}^{S^1}\colon \mathrm{SH}^{S^1}(W,\lambda_W) \to \mathrm{SH}^{S^1}(V,\lambda_V),$$

$$\phi_{V,W}^{S^1,+}\colon \mathrm{SH}^{S^1,+}(W,\lambda_W) \to \mathrm{SH}^{S^1,+}(V,\lambda_V).$$

Remark 7.4 One can also describe the transfer morphism (7-7) for fixed N in the context of Remark 5.15 and Proposition 5.20. Here one starts with an admissible stair Hamiltonian $H_2'\colon S^1 \times \widehat{W} \to \mathbb{R}$ and an admissible Hamiltonian $H_1'\colon S^1 \times \widehat{V} \to \mathbb{R}$ with $H_1' \le (H_2')^V$. Recall that the homology $\mathrm{HF}^{S^1,N,+}(H_1')$ appearing in Proposition 5.20 is the homology of a chain complex generated by symbols $u^k \otimes \gamma$, where $k \in \{0,\dots,N\}$ and γ is a nonconstant 1–periodic orbit of $X_{H_1'}$. The differential has the form

$$\partial_1^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_{1,i}(\gamma).$$

Likewise, the homology $\mathrm{HF}^{S^1,N,+}((H_2')^V)$ is the homology of a chain complex generated by symbols $u^k \otimes \gamma$, where $k \in \{0,\dots,N\}$ and γ is a nonconstant 1–periodic orbit of $X_{(H_2')^V}$. The differential has the form

$$\partial_2^{S^1}(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \varphi_{2,i}(\gamma).$$

We now construct the transfer map (7-7) using continuation maps for homotopies which respect the inclusions $\tilde{\imath}_0$ and $\tilde{\imath}_1$ as in Remark 5.15. This transfer map will then be induced by a chain map having the form

$$(7\text{-}8) \qquad\qquad \psi(u^k \otimes \gamma) = \sum_{i=0}^{k} u^{k-i} \otimes \psi_i(\gamma).$$

7.3 Transfer morphisms for generalized Liouville embeddings

We now extend the definition of transfer morphisms for a generalized Liouville embedding $\varphi\colon (V,\lambda_V) \to (W,\lambda_W)$ with $\varphi(V) \subset \mathrm{int}(W)$.

Lemma 7.5 Let $\varphi\colon (V,\lambda_V) \hookrightarrow (W,\lambda_W)$ be a generalized Liouville embedding with $\varphi(V) \subset \mathrm{int}(W)$. Then there exists a 1–form λ_W' on W such that

(1) $d\lambda_W' = d\lambda_W$,

(2) $\lambda_W' = \lambda_W$ near ∂W,

(3) $\varphi^\star \lambda_W' = \lambda_V$.

Proof Given $\delta > 0$, define

$$V_\delta = V \cup ([0, \delta] \times \partial V) \subset \hat{V}.$$

As in [25, Theorem 3.3.1], if δ is sufficiently small then we can extend φ to a symplectic embedding

$$\varphi_\delta \colon (V_{\delta, V}) \to (W, \omega_W).$$

Now use the map φ_δ to identify V_δ with its image in W. Then the 1–form $\lambda_W - V$ is closed on V_δ.

By hypothesis, the de Rham cohomology class of this 1–form restricted to $[0, \delta] \times \partial V$ is zero. Thus, there is a function $g \colon [0, \delta] \times \partial V$ such that

$$dg = (\lambda_W - V)|_{[0,\delta] \times \partial V}.$$

Let $\beta \colon [0, \delta] \to \mathbb{R}$ be a smooth function with $\beta(\rho) \equiv 0$ for ρ close to 0 and $\beta(\rho) \equiv 1$ for ρ close to δ. We can then take

$$\lambda'_W = \begin{cases} \lambda_V & \text{on } V, \\ V + d(\beta g) & \text{on } [0, \delta] \times \partial V, \\ \lambda_W & \text{on } W \setminus V_\delta. \end{cases}$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now, given a generalized Liouville embedding as above, let λ'_W be a 1–form on W provided by Lemma 7.5. We then have an honest Liouville embedding

$$\varphi \colon (V, \lambda_V) \to (W, \lambda'_W).$$

As explained in Sections 7.1 and 7.2, this induces transfer maps

$$(7\text{-}9) \qquad\qquad \mathrm{SH}^{(S^1,+)}(W, \lambda'_W) \to \mathrm{SH}^{(S^1,+)}(V, \lambda_V).$$

The construction in Section 5 of (positive, S^1–equivariant) symplectic homology of (W, λ_W) depends only on the contact form $\lambda_W|_{\partial W}$ on the boundary and the symplectic form $\omega_W = d\lambda_W$ on the interior. Indeed, replacing the Liouville form λ_W by another Liouville form λ'_W with the same exterior derivative and restriction to the boundary does not change any of the chain complexes or maps in the definition of (positive, S^1–equivariant) symplectic homology,[15] since the classes of admissible Hamiltonians

---

[15]One might worry that the Hamiltonian action of a noncontractible loop can change if $\lambda_W - \lambda'_W$ is not exact. However, for the Hamiltonians that we are using, the only noncontractible 1–periodic orbits are associated to Reeb orbits and their action does not change.

used are determined by the restriction to the boundary and the Hamiltonian vector fields are determined by the symplectic form. (For stronger results on invariance of symplectic homology, see [18, Section 4.3].) Thus, we have a canonical isomorphism

$$\text{(7-10)} \qquad \text{SH}^{(S^1,+)}(W, \lambda_W) = \text{SH}^{(S^1,+)}(W, \lambda'_W).$$

We can now finally make the following definition:

Definition 7.6 Suppose $\varphi \colon (V, \lambda_V) \to (W, \lambda_W)$ is a generalized Liouville embedding with $\varphi(V) \subset \text{int}(W)$. Let λ'_W be a 1–form provided by Lemma 7.5. Define the *transfer morphism*

$$\text{(7-11)} \qquad \phi_{V,W}^{(S^1,+)} \colon \text{SH}^{(S^1,+)}(W, \lambda_W) \to \text{SH}^{(S^1,+)}(V, \lambda_V)$$

to be the composition of the canonical isomorphism (7-10) with the map (7-9).

The transfer morphism (7-11) does not depend on the choice of λ'_W, because the admissible Hamiltonians, chain complexes and chain maps in the definition of the transfer morphism depend only on the symplectic form on each Liouville domain and the contact form on the boundary of each Liouville domain.

8 Properties of transfer morphisms

Let $\varphi \colon (X, \lambda) \to (X', \lambda')$ be a generalized Liouville embedding with $\varphi(X) \subset \text{int}(X')$. Let

$$\Phi \colon \text{CH}(X', \lambda') \to \text{CH}(X, \lambda)$$

denote the transfer map $\phi_{X,X'}^{S^1,+}$ defined in Section 7. We now prove that this map satisfies the properties in Proposition 3.3.

8.1 Action

The transfer map Φ is a direct limit over H_1, H_2^X and N of continuation maps

$$\text{(8-1)} \qquad \text{HF}^{S^1,N,+}(H_1) \to \text{HF}^{S^1,N,+}(H_2^X),$$

where H_1 and H_2^X are appropriate parametrized Hamiltonians for X' and X, respectively. Since the continuation map (8-1) is induced by a chain map which decreases symplectic action, it is the direct limit over L of maps

$$\text{(8-2)} \qquad \text{HF}^{S^1,N,+,\leq L}(H_1) \to \text{HF}^{S^1,N,+,\leq L}(H_2^X).$$

We now define

$$\Phi^L \colon \mathrm{CH}^L(X', \lambda') \to \mathrm{CH}^L(X, \lambda)$$

to be the direct limit over H_1, H_2^X and N of the maps (8-2). Here, as in Section 6.2, we assume without loss of generality that $L \notin \mathrm{Spec}(\partial X', \lambda') \cup \mathrm{Spec}(\partial X, \lambda)$. The required properties (3-4) and (3-5) follow from Definition 6.2.

8.2 Commutativity with U

We now show that the transfer map Φ commutes with the U map defined in Section 6.3.

Recall that the map Φ can be computed as a direct limit of maps (7-8) from Remark 7.4. And recall from Section 6.3 that in this setup, the U map is the direct limit of chain maps given by "multiplication by u^{-1}". So it is enough to prove that for each nonnegative integer N, we have a commutative diagram of chain maps

$$
\begin{array}{ccc}
\mathrm{CF}^{S^1,N,+}(H_1') & \xrightarrow{\ \psi\ } & \mathrm{CF}^{S^1,N,+}((H_2')^V) \\[2pt]
\downarrow{\scriptstyle u^{-1}} & & \downarrow{\scriptstyle u^{-1}} \\[2pt]
\mathrm{CF}^{S^1,N,+}(H_1') & \xrightarrow{\ \psi\ } & \mathrm{CF}^{S^1,N,+}((H_2')^V)
\end{array}
$$

Here the chain complexes depend on S^{2N+1}–families of Hamiltonians and almost complex structures as in Remark 5.15, which we are omitting from the notation.

It is enough to check this commutativity on a generator $u^k \otimes \gamma$. If $k = 0$, then both compositions are zero, since ψ does not increase the exponent of u. If $k > 0$, then the lower-left composition is given by

$$\psi(u^{-1}(u^k \otimes \gamma)) = \psi(u^{k-1} \otimes \gamma) = \sum_{i=0}^{k-1} u^{k-1-i} \otimes \psi_i(\gamma),$$

while the upper-right composition is given by

$$u^{-1}\psi(u^k \otimes \gamma) = u^{-1} \sum_{i=0}^{k} u^{k-i} \otimes \psi_i(\gamma) = \sum_{i=0}^{k-1} u^{k-i-1} \otimes \psi_i(\gamma).$$

These are equal, and this completes the proof that $\Phi U = U \Phi$.

To prove that $\Phi^L U^L = U^L \Phi^L$, as before we can assume without loss of generality that $L \notin \mathrm{Spec}(\partial X', \lambda') \cup \mathrm{Spec}(\partial X, \lambda)$. We then repeat the above argument, restricted to orbits with action less than L.

8.3 Commutativity with δ

To conclude, we now prove the commutativity with δ in Proposition 3.3. Note that a closely related result was proved in [33, Theorem 5.2], and our proof will use some of the same ideas.

Recall that the δ map is defined starting from the short exact sequence of chain complexes (6-7). If H_1 and H_2^X are Hamiltonians as in the definition of the transfer map in Section 7.2, then we have a commutative diagram

$$
\begin{array}{ccccc}
C_{\mathrm{Morse}}(X', H_1) \otimes \mathbb{Q}\{1, u, \ldots, u^N\} & \longrightarrow & \mathrm{CF}^{S^1, N}(H_1, J_1) & \longrightarrow & \mathrm{CF}^{S^1, N, +}(H_1, J_1) \\
\downarrow & & \downarrow & & \downarrow \\
C_{\mathrm{Morse}}(X, H_2^X) \otimes \mathbb{Q}\{1, u, \ldots, u^N\} & \longrightarrow & \mathrm{CF}^{S^1, N}(H_2^X, J_2^X) & \longrightarrow & \mathrm{CF}^{S^1, N, +}(H_2^X, J_2^X)
\end{array}
$$

Here the rows are from the short exact sequences of chain complexes (6-7) for X' and X. The center vertical arrow is the continuation chain map which, in the direct limit, gives the transfer morphism $\phi_{X,X'}^{S^1}$. The right vertical arrow is the continuation chain map which, in the direct limit, gives the transfer morphism $\Phi = \phi_{X,X'}^{S^1;+}$. The left vertical arrow is the restriction of the center vertical arrow. As in the proof of [33, Theorem 5.2], this left arrow simply discards critical points in $X' \setminus X$ (here we are identifying X with its image in X' under the symplectic embedding), and is the Morse continuation map from $H_1|_X$ to $H_2|_X$.

The above commutative diagram gives rise to a morphism of long exact sequences on homology. One square of this is the commutative diagram

$$
\begin{array}{ccc}
\mathrm{HF}^{S^1, N, +}(H_1, J_1) & \longrightarrow & H_*(X', \partial X') \otimes \mathbb{Q}\{1, u, \ldots, u^N\} \\
\downarrow{\scriptstyle \phi_{H_2^X, H_1}^{S^1;+}} & & \downarrow{\scriptstyle \rho \otimes 1} \\
\mathrm{HF}^{S^1, N, +}(H_2^X, J_2^X) & \longrightarrow & H_*(X, \partial X) \otimes \mathbb{Q}\{1, u, \ldots, u^N\}
\end{array}
$$

Here the horizontal arrows are the connecting homomorphisms which, in the direct limit, give the δ maps for X' and X. Thus, taking the direct limit over N, H_1 and H_2^X, we obtain the desired commutative diagram (3-7).

References

[1] **F Bourgeois**, **A Oancea**, *Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces*, Duke Math. J. 146 (2009) 71–174 MR

[2] **F Bourgeois**, **A Oancea**, *The Gysin exact sequence for S^1–equivariant symplectic homology*, J. Topol. Anal. 5 (2013) 361–407 MR

[3] **F Bourgeois**, **A Oancea**, *S^1–equivariant symplectic homology and linearized contact homology*, Int. Math. Res. Not. 2017 (2017) 3849–3937 MR

[4] **K Choi**, **D Cristofaro-Gardiner**, **D Frenkel**, **M Hutchings**, **V G B Ramos**, *Symplectic embeddings into four-dimensional concave toric domains*, J. Topol. 7 (2014) 1054–1076 MR

[5] **K Cieliebak**, **A Floer**, **H Hofer**, *Symplectic homology, II: A general construction*, Math. Z. 218 (1995) 103–122 MR

[6] **K Cieliebak**, **A Floer**, **H Hofer**, **K Wysocki**, *Applications of symplectic homology, II: Stability of the action spectrum*, Math. Z. 223 (1996) 27–45 MR

[7] **K Cieliebak**, **H Hofer**, **J Latschev**, **F Schlenk**, *Quantitative symplectic geometry*, from "Dynamics, ergodic theory, and geometry" (B Hasselblatt, editor), Math. Sci. Res. Inst. Publ. 54, Cambridge Univ. Press (2007) 1–44 MR

[8] **K Cieliebak**, **K Mohnke**, *Punctured holomorphic curves and Lagrangian embeddings*, Invent. Math. 212 (2018) 213–295 MR

[9] **K Cieliebak**, **A Oancea**, *Symplectic homology and the Eilenberg–Steenrod axioms*, Algebr. Geom. Topol. 18 (2018) 1953–2130 MR

[10] **D Cristofaro-Gardiner**, *Symplectic embeddings from concave toric domains into convex ones*, preprint (2014) arXiv To appear in J. Differential Geom.

[11] **D Cristofaro-Gardiner**, **D Frenkel**, **F Schlenk**, *Symplectic embeddings of four-dimensional ellipsoids into integral polydiscs*, Algebr. Geom. Topol. 17 (2017) 1189–1260 MR

[12] **I Ekeland**, **H Hofer**, *Symplectic topology and Hamiltonian dynamics, II*, Math. Z. 203 (1990) 553–567 MR

[13] **A Floer**, **H Hofer**, *Coherent orientations for periodic orbit problems in symplectic geometry*, Math. Z. 212 (1993) 13–38 MR

[14] **A Floer**, **H Hofer**, *Symplectic homology, I: Open sets in $\mathbb{C}^n$*, Math. Z. 215 (1994) 37–88 MR

[15] **V L Ginzburg**, **B Z Gürel**, *Lusternik–Schnirelmann theory and closed Reeb orbits*, preprint (2016) arXiv

[16] **L Guth**, *Symplectic embeddings of polydisks*, Invent. Math. 172 (2008) 477–489 MR

[17] **J Gutt**, *On the minimal number of periodic Reeb orbits on a contact manifold*, PhD thesis, Université de Strasbourg and Université libre de Bruxelles (2014) Available at https://tel.archives-ouvertes.fr/tel-01016954v2

[18] **J Gutt**, *The positive equivariant symplectic homology as an invariant for some contact manifolds*, J. Symplectic Geom. 15 (2017) 1019–1069 MR

[19] **R Hind**, **E Kerman**, *New obstructions to symplectic embeddings*, Invent. Math. 196 (2014) 383–452 MR

[20] **M Hutchings**, *Quantitative embedded contact homology*, J. Differential Geom. 88 (2011) 231–266 MR

[21] **M Hutchings**, *Recent progress on symplectic embedding problems in four dimensions*, Proc. Natl. Acad. Sci. USA 108 (2011) 8093–8099 MR

[22] **M Hutchings**, *Lecture notes on embedded contact homology*, from "Contact and symplectic topology" (F Bourgeois, V Colin, A Stipsicz, editors), Bolyai Soc. Math. Stud. 26, János Bolyai Math. Soc., Budapest (2014) 389–484 MR

[23] **M Hutchings**, *Beyond ECH capacities*, Geom. Topol. 20 (2016) 1085–1126 MR

[24] **D McDuff**, *The Hofer conjecture on embedding symplectic ellipsoids*, J. Differential Geom. 88 (2011) 519–532 MR

[25] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, 3rd edition, Oxford Univ. Press (2017) MR

[26] **D McDuff**, **F Schlenk**, *The embedding capacity of 4–dimensional symplectic ellipsoids*, Ann. of Math. 175 (2012) 1191–1282 MR

[27] **A Oancea**, *La suite spectrale de Leray–Serre en cohomologie de Floer pour variétés symplectiques compactes à bord de type contact*, PhD thesis, Université Paris-Sud (2003) Available at `https://tel.archives-ouvertes.fr/tel-00005504`

[28] **A Oancea**, *A survey of Floer homology for manifolds with contact type boundary or symplectic homology*, Ensaios Mat. 7, Soc. Brasil. Mat., Rio de Janeiro (2004) 51–91 MR

[29] **Y Ostrover**, *When symplectic topology meets Banach space geometry*, from "Proceedings of the International Congress of Mathematicians, II" (S Y Jang, Y R Kim, D-W Lee, I Ye, editors), Kyung Moon Sa, Seoul (2014) 959–981 MR

[30] **F Schlenk**, *Symplectic embedding problems, a survey*, in preparation

[31] **P Seidel**, *A biased view of symplectic cohomology*, from "Current developments in mathematics, 2006" (B Mazur, T Mrowka, W Schmid, R Stanley, S-T Yau, editors), Int., Somerville, MA (2008) 211–253 MR

[32] **L Traynor**, *Symplectic packing constructions*, J. Differential Geom. 42 (1995) 411–429 MR

[33] **C Viterbo**, *Functors and computations in Floer homology with applications, I*, Geom. Funct. Anal. 9 (1999) 985–1033 MR

[34] **C Viterbo**, *Metric and isoperimetric problems in symplectic geometry*, J. Amer. Math. Soc. 13 (2000) 411–431 MR

Mathematical Institute, Universität zu Köln
Köln, Germany

Mathematics Department, University of California, Berkeley
Berkeley, CA, United States

gutt@math.uni-koeln.de,   hutching@math.berkeley.edu