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Braid monodromy, orderings and transverse invariants

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A closed braid β naturally gives rise to a transverse link K in the standard contact 3–space. We study the effect of the dynamical properties of the monodromy of β, such as right-veering, on the contact-topological properties of K and the values of transverse invariants in Heegaard Floer and Khovanov homologies. Using grid diagrams and the structure of Dehornoy's braid ordering, we show that $\widehat{\theta}(K) \in \widehat{\mathrm{HFK}}(m(K))$ is nonzero whenever β has fractional Dehn twist coefficient $C > 1$. (For a 3–braid, we get a sharp result: $\widehat{\theta} \neq 0$ if and only if the braid is right-veering.)

57M27, 57R17, 57R58

1 Introduction

In the standard contact 3–space $(\mathbb{R}^3, \xi_{\mathrm{std}}) = \ker(dz + r^2 d\varphi)$, links that are transverse to the contact planes can be described via braids around the z–axis (up to a stabilization equivalence). In this paper, we would like to see how dynamics of the braid monodromy affects the contact-topological properties of the corresponding transverse link. We focus on transverse link invariants that live in Heegaard Floer and Khovanov homologies of the link K: the transverse invariant $\widehat{\theta}(K) \in \widehat{\mathrm{HFK}}(m(K))$ (see Ozsváth, Szabó and Thurston [39]) as well as the invariant $\psi(K) \in \mathrm{Kh}(K)$; see Plamenevskaya [41]. A lot of previous research was aimed at using these invariants to detect *transverse nonsimplicity*, ie to show that certain transverse knots are not isotopic despite having the same smooth type and self-linking number. The Heegaard Floer homological invariant $\widehat{\theta}$ detects many such pairs, but it is unknown whether the Khovanov homological invariant ψ is effective.

The purpose of this paper is to clarify the qualitative meaning of transverse invariants of an *individual* transverse knot. It turns out that the behavior of $\widehat{\theta} \in \widehat{\mathrm{HFK}}$ is related to certain dynamical properties of the braid monodromy (acting on a disk $\mathcal{D}$ with punctures). A braid is *right-veering* if its monodromy sends *every* arc connecting $\partial\mathcal{D}$ to one of the punctures to the right of itself; see Section 2 for a detailed definition. (Otherwise, we say that the braid is non-right-veering.) The following proposition is known and has motivated the present paper.

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Proposition 1.1 (see Baldwin and Grigsby [3], Baldwin, Vela-Vick and Vértesi [5] and Plamenevskaya [41]) *Suppose K is a transverse link with a non-right-veering braid representative. Then $\widehat{\theta}(K)=0$ and $\psi(K)=0$.*

We establish a partial converse to this fact:

Theorem 1.2 *Suppose that K is a transverse 3–braid. Then K is right-veering if and only if $\widehat{\theta}(K)\neq 0$.*

Theorem 1.3 *Suppose K is a transverse braid with the fractional Dehn twist coefficient $C>1$. Then $\widehat{\theta}(K)\neq 0$.*

The fractional Dehn twist coefficient measures the amount of positive rotation the braid monodromy makes around the boundary of the disk. The condition $C>1$ means, roughly, that the braid has more positive rotation at the boundary than a full positive twist on all strands. We will give the precise definition of the FDTC in Section 5. Note, however, that the condition $C>1$ *does not* imply that the braid is positive or that it "looks like" a positive braid in any sense; indeed, such a braid can contain a lot of negative crossings and a large amount of negative rotation inside the disk (see model examples $\beta_{n,k}$ in Section 5).

It is well known that the right-veering property of diffeomorphisms of surfaces plays an important role in 3–dimensional contact topology. Namely, by Giroux's theorem contact structures can be described via compatible open books, and tight contact structures are exactly those for which every compatible open book is right-veering; see Honda, Kazez and Matić [18]. A product of positive Dehn twists is a special case of a right-veering diffeomorphism; an open book with such monodromy supports a Stein fillable contact structure. Some of the properties of the monodromy of an open book are reflected, via the compatible contact structure, by the Ozsváth–Szabó contact invariant [37]. More precisely, this invariant vanishes for open books with non-right-veering monodromy (ie for overtwisted contact structures), and is always nonzero for monodromies which are products of positive Dehn twists (ie for Stein fillable contact structures). The fractional Dehn twist coefficient of the open book monodromy also carries information on the contact structure. By Honda, Kazez and Matić [19], a contact structure supported by an open book with connected boundary and pseudo-Anosov monodromy with $C\geq 1$ is isotopic to a perturbation of a taut foliation, and therefore weakly symplectically fillable. Then, by Ozsváth and Szabó [36], the Heegaard Floer contact invariant (with *twisted* coefficients) is nonzero.

Our Theorem 1.3 for classical braids parallels the above result for contact structures, but our proof is surprisingly different. Taut foliations and fillability do not have an obvious interpretation in terms of braids; instead, we use braid structure supplied by Dehornoy's theory of braid orderings. This allows us to prove nonvanishing of $\widehat{\theta}$ via grid diagrams. Note that a generalization of $\widehat{\theta}$ for the case of braids in arbitrary open books was given in [5]; the definition is reminiscent of that of the Heegaard Floer contact invariant. It would be interesting to find a technique that could be used to prove both a general version of Theorem 1.3 and the result of [19]. An additional subtlety is that our Theorem 1.3 holds with *untwisted* coefficients (we use $\mathbb{Z}/2$ coefficients throughout); in the context of open books, a nonvanishing result with $\mathbb{Z}/2$ coefficients has not been established. (In general, the untwisted version of the Ozsváth–Szabó invariant may vanish for weakly fillable contact structures; see Ghiggini [16].)

Invariants related to braid orderings encode some of the topology of the braid closure, such as genus bounds for the underlying knot, existence of certain isotopies and some information on the geometry of the link complement; see Ito [21; 22] and Malyutin and Netsvetaev [29]. We show that braid orderings are useful in contact topology. It would be interesting to develop further connections and applications.

Right-veering seems important for certain contact-topological properties of transverse links, so we take a closer look at features of right-veering transverse braids. An important remark is in order: it is possible to start with a non-right-veering braid and stabilize it, while preserving the transverse link type, to obtain right-veering monodromy (Proposition 3.1). We therefore suggest:

Definition 1.4 Let K be a transverse link in $(S^3, \xi_{\mathrm{std}})$. We say that K is right-veering if *every* braid representative of K is a right-veering braid. Similarly, K is non-right-veering if *there exists* a non-right-veering braid representative for K.

It is interesting to ask if $\widehat{\theta}(K)$ is always nonzero whenever K is a right-veering transverse link; at the moment, we do not have any counterexamples. Another question concerns the behavior of the Khovanov homological invariant $\psi(K)$ of a right-veering link K. The analogs of Theorems 1.2 and 1.3 do not hold true for ψ, and there are examples of right-veering transverse links with vanishing ψ. We include a few observations concerning ψ.

In general, we would like to claim that right-veering transverse links have certain special properties. It turns out that right-veering is a necessary condition for tightness of the double cover $(\Sigma(K), \xi_K)$ of $(S^3, \xi_{\mathrm{std}})$ branched over K, and for equality in various versions of the slice-Bennequin inequality. Recall that the self-linking number $\mathrm{sl}(K)$ of the transverse knot K has upper bounds (see Plamenevskaya [40; 41], Rudolph [46] and Shumakovitch [47]) in terms of the slice genus $g^*(K)$, the Ozsváth–Szabó concordance invariant $\tau(K)$ and Rasmussen's concordance invariant $s(K)$:

$$\mathrm{sl}(K) \leq 2g^*(K) - 1, \quad \mathrm{sl}(K) \leq 2\tau(K) - 1, \quad \mathrm{sl}(K) \leq s(K) - 1.$$

We show that these upper bounds can only be achieved by right-veering knots.

Theorem 1.5 *Suppose that K is a non-right-veering transverse knot. Then:*

(1) $\mathrm{sl}(K) < s - 1$, $\mathrm{sl}(K) < 2\tau - 1$ *and* $\mathrm{sl}(K) < 2g^*(K) - 1$.

(2) *The branched double cover $(\Sigma(K), \xi_K)$ is overtwisted.*

These facts are straightforward observations or corollaries of known results; we collect them together to create a coherent picture. The converse statements to those of Theorem 1.5 are not true (see Proposition 3.4), although it is well known that when K is a *quasipositive* braid, the above bounds for the self-linking number become equalities, and the branched double cover is tight (even Stein fillable). We discuss this further in Section 3.

It might be tempting to think of the right-veering vs non-right-veering transverse knots as a dichotomy analogous to that of tight vs overtwisted contact structures, but this analogy does not go very far. Indeed, the main feature of overtwisted contact structures is that they satisfy an h–principle (see Eliashberg [12]); in particular, two overtwisted contact structures are isotopic whenever their underlying plane fields are homotopic. In the case of transverse knots, an h–principle would say that non-right-veering knots are transversely simple (ie the transverse isotopy type would be determined by the smooth knot and the self-linking number), but this is not true: some of the transversely nonsimple braids given in Birman and Menasco [7] are not right-veering.

The paper is organized as follows. In Section 2, we review the necessary definitions and background. In Section 3, we prove Theorem 1.5 along with some other easy properties of non-right-veering braids. The main results of the paper are contained in Sections 4 and 5: in Section 4, we discuss 3–braids and prove Theorem 1.2; Section 5 contains the proof of Theorem 1.3 and the related discussion of Dehornoy's floor function and the fractional Dehn twist coefficient. In Section 6, we discuss higher-order branched covers.

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2 Background

2.1 Braids and transverse links

As usual, we will write the elements of the Artin braid group B_m (braids on m strands) as braid words on standard generators, σ_1, σ_2, ..., σ_{m-1}. We draw braids from left to right, and consider braid closures as closed braids around the z–axis in $\mathbb{R}^3$. (Closed braids correspond to conjugacy classes in B_m.)

If $\mathbb{R}^3$ is endowed with the standard contact structure $\xi_{\mathrm{std}} = \ker(dz + r^2 d\phi)$, closed braids around the z–axis naturally give rise to transverse links. Moreover, two braids that produce transversely isotopic links are related by braid isotopies and positive Markov stabilization and destabilization moves [35; 48] (this fact is usually called "transverse Markov theorem"). Positive Markov stabilization of a braid $\beta \in B_m$ gives the braid $\beta\sigma_m \in B_{m+1}$; geometrically, this means adding an extra strand and a positive crossing to the given closed braid. Negative Markov stabilization of β gives the braid $\beta\sigma_m^{-1}$, which is *not* transversely isotopic to β. When we consider the transverse links corresponding to braids, negative Markov stabilization is called transverse stabilization. A basic invariant of transverse links, the self-linking number, can be computed from a braid via the formula

$$\mathrm{sl} = \#\{\text{positive crossings}\} - \#\{\text{negative crossings}\} - (\text{braid index}).$$

Transverse stabilization decreases the self-linking number by 2.

The braid group B_m is naturally identified with the mapping class group of a disk with m marked points. (We will assume that $\mathcal{D} \subset \mathbb{C}$ is the standard unit disk, and that the set $Q = \{x_1, x_2, \dots, x_m\}$ of marked points lies on the x–axis with $x_1 < x_2 < \cdots < x_m$.) The braid group acts on $(\mathcal{D}, Q)$ on the right (fixing $\partial\mathcal{D}$). Each standard generator σ_i acts by a right-handed half-twist interchanging x_i and x_{i+1}, supported in a small neighborhood of the segment $[x_i, x_{i+1}]$.

2.2 Right-veering property and braid orderings

We now recall the definition of right-veering braids. For this, we consider the action of the braid monodromy on embedded arcs that connect the boundary $\partial\mathcal{D}$ of the disk to one of the marked points in Q, while avoiding all other marked points. We will refer to such an arc as "an arc in $(\mathcal{D}, Q)$". Suppose that a_1 and a_2 are nonisotopic arcs in $(\mathcal{D}, Q)$ that start at the same point in $\partial\mathcal{D}$ and intersect transversely without nonessential intersections (that is, the two arcs form no bigons in $\mathcal{D} \setminus Q$, so that no intersections can be removed by an isotopy of the arcs). We will say that a_1 lies to the right of a_2 if the pair of tangent vectors $(\dot{a}_1, \dot{a}_2)$ at the arcs' initial point induces the original orientation on the disk $\mathcal{D}$. For arcs with nonessential intersections, the same definition can be used after the arcs are *pulled taut*, ie isotoped to eliminate all nonessential intersections. Now, let $\beta \in B_m$ be a braid, and write $\beta(a)$ for the image of an arc a in $(\mathcal{D}, Q)$ under the action of β. (The braid group acts on the right but we abuse notation.)

Definition 2.1 Let $\beta \in B_m$ be a braid. We say that β is right-veering if for any arc a in $(\mathcal{D}, Q)$, the arc $\beta(a)$ is either isotopic to a or lies to the right of a (after the arcs a and $\sigma(a)$ are pulled taut in $\mathcal{D} \setminus Q$).

If there is an arc a such that $\beta(a)$ is to the left of a (ie a is to the right of $\beta(a)$), we say that β is non-right-veering.

In the context of open book decompositions, the notion of right-veering was developed in [18] and applied to the study of contact structures. For braids, a definition where one considers arcs with both endpoints in $\partial\mathcal{D}$ was given in [5]; the present definition appears in [3]. (See also [23] for a detailed discussion and a proof that the definitions from [3] and [5] are equivalent.) The general idea of right-veering goes back to W Thurston.

Using Thurston's approach, the idea of right-veering can be used to show that the braid group B_m is orderable. (Namely, there exists a *left-invariant* linear order on B_m, so that if $\beta_1 < \beta_2$, then $\gamma\beta_1 < \gamma\beta_2$ for any $\gamma \in B_m$.) Many different orderings (known as *Nielsen–Thurston* orderings) arise from this idea; we refer the reader to [11] for a general discussion. In this paper, we will only need a specific ordering introduced by Dehornoy [10]. Historically, Dehornoy was the first one to describe a linear order on B_m; he found it from an algebraic perspective. By definition, $\beta \succ 1$ in Dehornoy's order if and only if the braid β admits a braid word that contains the generator σ_i but no σ_i^{-1} and no $\sigma_j^{\pm 1}$ for $j < i$. (Such words are called σ–positive.) For example, we have $\sigma_3\sigma_1^2\sigma_2^{-5}\sigma_3^{-1} \succ 1$ and $\sigma_2\sigma_5\sigma_3^{-2}\sigma_2^2 \succ 1$. Now, we set $\beta \succ \beta'$ if $(\beta')^{-1}\beta \succ 1$, ie $(\beta')^{-1}\beta$ admits a σ–positive word. This gives a well-defined left-invariant order [10; 11]. (From this perspective, it is nontrivial to check, for example, that β and β^{-1} can never be both σ–positive.)

We will also use some more specific terminology. We say that a braid word is σ_i–positive if it contains at least one letter σ_i but no σ_i^{-1} and no $\sigma_j^{\pm 1}$ with $j < i$. Similarly, we can define σ_i–negative words. We will say that a word is σ_i–free if it contains no $\sigma_j^{\pm 1}$ for $j \le i$.

Dehornoy's order can be given a geometric interpretation [14], by considering the action of the braid monodromy $\beta \in \mathrm{Map}(D, Q)$ on the x–axis. (Recall that we assume that the marked points are all on the x–axis, labeled from left to right.) It turns out that $\beta \succ 1$ in Dehornoy's order if and only if the image of the x–axis under β veers to the right of the x–axis, when compared at the left endpoint and pulled taut. (Note that the x–axis passes through marked points, and may coincide with its image on a few initial intervals. The image must veer right from the x–axis when they eventually diverge. Indeed, for a σ_1–positive word, the image of the x–axis will veer to the right at $\partial\mathcal{D}$, but for a word that is σ_2–free, the image will follow the x–axis up to the point x_2. For a σ_3–positive word, the image of the x–axis will veer to the right after x_2.)

It is important to note that the condition $\beta \succ 1$ does not imply that β is right-veering (since we do not consider images of arbitrary arcs). For example, $\sigma_1\sigma_2^{-1} \succ 1$ since this braid is σ_1–positive, but it is clearly non-right-veering. However, we can say that if β is non-right-veering, ie sends some arc to the left of itself, it has a conjugate $\widetilde{\beta}$ that sends the initial (leftmost) arc of the x–axis to the left of itself, so $\widetilde{\beta} \prec 1$:

Proposition 2.2 [3; 14] *Any non-right-veering braid is conjugate to a braid with a braid word that contains σ_1^{-1} but no σ_1.*

Corollary 2.3 *A non-right-veering transverse link can be represented by a braid with a word that contains σ_1^{-1} but no σ_1.*

The proposition above and its corollary will be very useful for us.

2.3 Transverse invariants

We will be mostly concerned with transverse link invariants in Heegaard Floer homology (although we will discuss Khovanov homological invariants as well). We now very briefly recall the constructions, referring the reader to [39; 32; 33; 25] for details.

A grid diagram is an $n \times n$ square grid, marked with X's and O's so that there is exactly one X and exactly one O in each row and each column. A grid diagram gives rise to a braid, as follows. First, consider columns of the grid. For the columns that have O above X, draw a vertical segment from X to O. For the columns that have X above O, draw two vertical segments in this column, one from the bottom of the grid to O, the other from X to the top of the grid. Now, consider the rows, and draw a horizontal segment in each column connecting X to O, so that horizontal segments cross *over* the

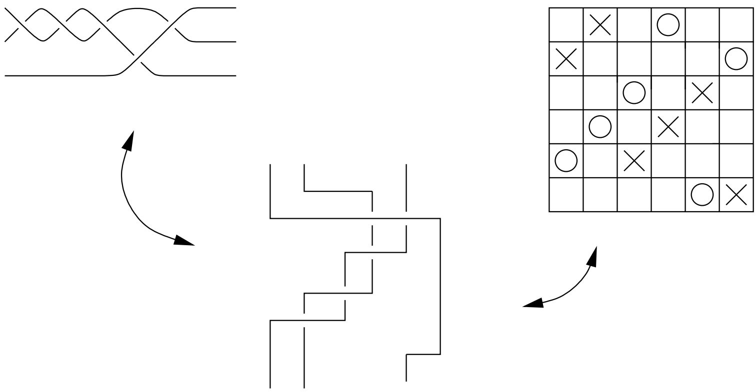

Figure 1: Braids and grid diagrams. The middle picture shows a braid running from bottom to top, constructed from the grid diagram on the right. The picture on the left shows the same braid running from left to right; the braid closure is braided around the (vertical) z–axis and represents a transverse link in $(\mathbb{R}^3, \ker(dz + r^2 d\phi))$.

vertical segments. The result is an oriented braid (running from bottom to top); as in Section 2.1, the braid gives a transverse link K. See Figure 1 for an illustration.

The grid diagram gives rise to the grid chain complex $\widehat{\mathrm{CFK}}(m(K))$ whose generators are n–tuples of intersection points of the grid (the opposite sides of the grid are identified to form a torus). Here, $m(K)$ is the mirror knot (the mirroring is needed because one associates a smooth link to a grid diagram in a different way when defining knot Floer homology). The n–tuple given by the upper-right corners of the X markings gives a distinguished cycle in the complex $\widehat{\mathrm{CFK}}(m(K))$; the corresponding element $\widehat{\theta}(K) \in \widehat{\mathrm{HFK}}(m(K))$ is an invariant of the original transverse knot K.

The definition we just gave will be most convenient for our purposes; it comes from Khandhawit and Ng's braid interpretation [25] of the original definition from [39]. The original definition [39] of the transverse invariant $\widehat{\theta}$ in Heegaard Floer homology starts by representing the transverse link K in S^3 as a push-off of a Legendrian link; the latter can be put on a grid, and the corresponding grid diagram is used to define the transverse invariant (as the class of the cycle given the upper-right corners of the X's). Note that there are other ways to define the same invariant. A more topological definition, for transverse knots in arbitrary contact 3–manifolds, was developed via open books adapted to the corresponding Legendrian knot in [27]. Finally, yet another construction, whose input is directly a transverse braid (rather than an associated Legendrian link), was given in [5], where equivalence of all the above definitions for transverse knots in S^3 was established. While we work with grid diagrams in this paper, it would be interesting to reprove and generalize our results from the perspective of [5]. Finally, we remark that there is a related transverse invariant θ^- in $\mathrm{HFK}^-(m(K))$ which potentially contains more information; however, the hat version will be sufficient for our paper.

The transverse invariant ψ in Khovanov homology was introduced in [41]. To define $\psi(K) \in \mathrm{Kh}(K)$ for a braid representative of a transverse link K, one considers the oriented braid resolution, and then takes the cycle which is the lowest quantum grading element in the component of the Khovanov chain complex $\mathrm{CKh}(K)$ corresponding to this resolution. The resulting homology class, $\psi(K)$, is independent of the choice of the braid representative of K.

We recall a few properties.

Proposition 2.4 [39; 41] *Suppose K_{stab} is a transverse stabilization of another transverse link, K. Then $\widehat{\theta}(K_{\mathrm{stab}}) = 0$ and $\psi(K_{\mathrm{stab}}) = 0$.*

Both the Heegaard Floer and the Khovanov homological invariants are functorial with respect to a positive crossing resolution in the transverse braid. More precisely:

Proposition 2.5 [2; 41] *Suppose that the braid β is obtained from β_+ by removing a positive crossing of the braid* (ie removing a generator σ_i from the braid word)*, and let K and K_+ be the corresponding transverse links. The crossing resolution cobordism gives rise to homomorphisms on Khovanov and Heegaard Floer homology, so that:*

(1) *There exists a map $F\colon \widehat{\mathrm{HFK}}(m(K_+)) \to \widehat{\mathrm{HFK}}(m(K))$ such that $F(\widehat{\theta}(K_+)) = \widehat{\theta}(K)$.*

(2) *There exists a map $G\colon \mathrm{Kh}(K_+) \to \mathrm{Kh}(K)$ such that $G(\psi(K_+)) = \psi(K)$.*

Similarly, if K_- corresponds to a transverse braid that differs from β by an additional negative crossing, the maps on Heegaard Floer and Khovanov homology send $\widehat{\theta}(K)$ to $\widehat{\theta}(K_-)$ and $\psi(K)$ to $\psi(K_-)$.

Propositions 2.2, 2.4 and 2.5 combine to give an immediate proof of Proposition 1.1: non-right-veering transverse links have $\widehat{\theta} = 0$ and $\psi = 0$. Indeed, a non-right-veering link has a braid representative where all the crossings between the first and second strands are negative; removing all of these crossings except one, we obtain a transverse braid that can be destabilized and thus has $\psi = 0$ and $\widehat{\theta} = 0$. These vanishing invariants are mapped to the invariants of the original non-right-veering link by the maps on

Khovanov and Heegaard Floer homology. (Note that the proof of vanishing of $\widehat{\theta}$ in [5] relies on a different definition of right-veering property; namely, one considers the effect of the braid monodromy on arcs with both endpoints on $\partial\mathcal{D}$. We can, however, reduce our Proposition 1.1 to [5] by "doubling" a non-right-veering arc in $(\mathcal{D}, Q)$.)

3 Non-right-veering transverse links

In this section, we discuss a few basic features of non-right-veering transverse links. (These can be turned around to say that right-veering is a *necessary* condition for certain properties.) Recall that those were defined as links that can be represented by non-right-veering braids. We first show that non-right-veering braids can become right-veering after positive Markov stabilizations. This parallels the well-known situation with open books [18]; our proof is somewhat similar in spirit to the proof of [18, Proposition 6.1].

Proposition 3.1 *Let K be an arbitrary transverse link. Then K can always be represented by a right-veering braid.*

Proof We will use the notion of stabilization along an arc. Recall that the usual Markov stabilization can be thought of as follows. For a braid $\beta \in B_m = \mathrm{Map}(\mathcal{D}, Q)$, we enlarge the disk $\mathcal{D}$ and add an extra point, x_{m+1}, to the set $Q = \{x_1, x_2, \dots, x_m\}$. The monodromy of the stabilized braid $\beta\sigma_m$ is then given by the composition of the monodromy β with a positive half-twist interchanging x_m and x_{m+1} and supported in a thin neighborhood of a standard arc connecting x_m and x_{m+1}; see Figure 2, left. More generally, we can instead consider an *arbitrary* arc connecting x_{m+1} to a point $x_j \in Q$ (and avoiding $x_i \in Q$ for $i \neq j$); see Figure 2, right. Composing β with a positive half-twist interchanging x_j and x_{m+1} and supported in a thin neighborhood of a this arc, we obtain a new stabilized braid. Clearly, stabilization along a general arc amounts to the effect of a standard stabilization composed with conjugation, so the result represents the same transverse link.

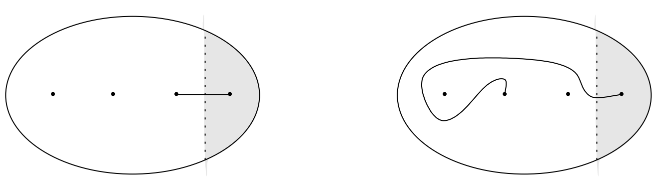

Figure 2: Markov stabilization $\beta \mapsto \beta\sigma_m$ (left) and stabilization along an arbitrary arc (right)

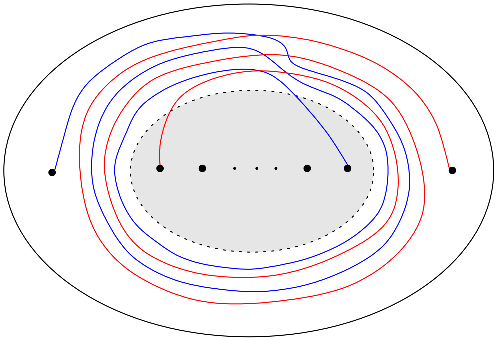

Figure 3: The monodromy of the braid β is supported inside the shaded disk. The two stabilizing arcs twist around.

Now, we start with an arbitrary (non-right-veering) braid $\beta \in B_m$, and stabilize it twice as described below, so that it becomes right-veering. First, find a positive integer N so that β composed with an N–fold positive boundary twist is right-veering. Consider a bigger disk $\widetilde{\mathcal{D}}$ such that $\mathcal{D}$ is contained inside $\widetilde{\mathcal{D}}$, and $\widetilde{\mathcal{D}} \setminus \mathcal{D}$ contains two additional marked points, x_{m+1} and x_{m+2}. We can assume that the monodromy β is supported inside $\mathcal{D}$, and β is stabilized along two arcs that connect x_{m+1} and x_{m+2} to some marked points in $\mathcal{D}$. We choose these two arcs as shown in Figure 3; both arcs make at least N full twists in $\widetilde{\mathcal{D}} \setminus \mathcal{D}$ before entering $\partial\mathcal{D}$. Moreover, one arc twists in the clockwise direction and the other counterclockwise, and they intersect so that each of them makes at least N twists around $\partial\mathcal{D}$ between their first intersection point and the endpoint inside $\mathcal{D}$.

We claim that after positively stabilizing along these two arcs (in any order), β becomes right-veering. Indeed, let a any arc in $\widetilde{\mathcal{D}}$ that starts on $\partial\widetilde{\mathcal{D}}$ and goes to one of the marked points. If a does not enter $\mathcal{D}$, then clearly the image of a must lie to the right of a. If a enters $\mathcal{D}$, then it must cross the stabilizing arcs. The monodromy of the stabilized braid is the composition of the monodromy of β with two positive half-twists supported near the stabilizing arcs. Two cases are then possible for the image of a under this new monodromy: either (1) the image of a veers right toward and around the point x_{m+1} or x_{m+2}, or (2) it follows one of the stabilizing arcs, making (at least) N right-handed full twists before entering $\mathcal{D}$. In the first case, the image of a must veer to the right and go around x_{m+1} or x_{m+2} even after it is pulled taut, which guarantees that it stays to the right of a. In the second case, the image of a will be to the right of a because β becomes right-veering when composed with N positive full twists. □

Proposition 3.2 *Suppose K is a transverse knot such that* $\mathrm{sl}(K) = s(K) - 1$ *or* $\mathrm{sl}(K) = 2\tau(K) - 1$. *Then K is right-veering.*

Proof Suppose that $\mathrm{sl}(K) = 2\tau(K) - 1$. Then right-veering of K follows from Proposition 1.1 and the fact that the equality $\mathrm{sl}(K) = 2\tau(K) - 1$ implies that $\widehat{\theta}(K) \neq 0$. This last fact seems to be absent from the literature but is known to experts; indeed, the proof given in [32, Proposition 3.4] for the case of thin knots does not actually use the thin property. The idea is that $\widehat{\theta}$ is related to another invariant, $\theta^-(K) \in \mathrm{HFK}^-(m(K))$. The latter is always nontrivial, and $\widehat{\theta}$ vanishes if and only if θ^- is in the image of the U–map. (Recall that one takes the quotient by the image of U to pass from HFK^- to $\widehat{\mathrm{HFK}}$.) If $\mathrm{sl}(K) = 2\tau(K) - 1$, the grading level of θ^- is the same as the very top of the $\mathbb{Z}[U]$–chain in HFK^-, which means θ^- can't be in the image of U.

The second statement of the proposition concerns Rasmussen's invariant $s(K)$, defined via Khovanov homology. In [44], s is only defined for knots, not for links, but an extension to the case of oriented links was given in [6]. To prove the proposition, we will need to remove a number of crossings from the given knot; as the result might be a multicomponent link, we will work with the link version of s below. (We will actually prove this part of the proposition for links.)

Recall that $\mathrm{sl}(K) \leq s(K) - 1$ when K is a knot; we claim that the same inequality holds for links. To see this, we will connect the given link to a knot by a cobordism. Under cobordisms, s changes as follows [44; 6]: if S is a smooth oriented cobordism from L_1 to L_2 such that every connected component of S has boundary in L_1, then

$$|s(L_2) - s(L_1)| \leq -\chi(S). \tag{1}$$

Given a $(k+1)$–component transverse braid L, we can insert k positive crossings to obtain a connected braid K. This gives a cobordism S from the link L to the knot K; we have $\chi(S) = -k$. Then $\mathrm{sl}(L) = \mathrm{sl}(K) - k \leq s(K) - 1 - k \leq s(L) - 1$ by (1).

We are now ready to prove the proposition. Assume that K is a non-right-veering transverse link. Then, by Corollary 2.3, K has a braid representative with only negative crossings in the σ_1–level (ie the braid word has entries σ_1^{-1} but no σ_1). Removing all of these crossings except one, we get a cobordism S from the link K to a link K' which is a negatively stabilized transverse braid. If k negative crossings were deleted, then $\mathrm{sl}(K) = \mathrm{sl}(K') - k$. On the other hand, $s(K) \geq s(K') - k$ by (1). Since K' is a stabilized transverse link, the inequality $\mathrm{sl}(K') < s(K') - 1$ is strict; therefore, $\mathrm{sl}(K) < s(K) - 1$. □

Note that in the above proposition, our proof for the case $\mathrm{sl}(K) = s(K) - 1$ could be repeated almost verbatim for the case $\mathrm{sl}(K) = 2\tau(K) - 1$, using Cavallo's extension of the tau invariant to links [9]. Indeed, if one deletes k negative crossings from the link K to obtain the link K', then $2\tau(K) \geq 2\tau(K') - k$; see [9, Section 4.3]. Since $\mathrm{sl}(L) \leq 2\tau(L) - 1$ for all links [9, Proposition 1.5], it follows that the inequality must be strict for stabilized links, and the result follows as in the case of the s–invariant. In fact, Proposition 1.5 in [9] gives a stronger bound, namely $\mathrm{sl}(L) \leq 2\tau(L) - n$, where n is the number of link components of L. One can likely show that an n–component link L is right-veering whenever $\mathrm{sl}(L) = 2\tau(L) - n$, but we will not pursue this generalization.

Proposition 3.3 *Suppose K is a transverse link such that the branched double cover $(\Sigma(K), \xi_K)$ is tight. Then K is right-veering.*

Proof Suppose K is non-right-veering, ie there is a non-right-veering braid representative σ for K. Then there is an arc a in (D, Q) whose image $\sigma(\alpha)$ can be isotoped to an arc b minimally intersecting a, so that b is to the left of a. We want to show that $(\Sigma(K), \xi_K)$ is overtwisted.

The branched double cover has an open book decomposition (S, ϕ) such that S is a double cover of D with branch set Q and the monodromy ϕ covers σ. The arc a lifts to an arc α in S, and it is clear that $\phi(\alpha)$ is isotopic to the arc β that covers b and lies to the left of α. We can conclude that ϕ is non-right-veering if we check that α and β form no bigons. Indeed, if there were a bigon B formed by α and β, then B couldn't cover a bigon in (D, Q), and so B would contain a branch point (unique for Euler characteristic reasons). But then in a neighborhood of B the covering could be modeled on $z \to z^2$, and this neighborhood admits a covering involution (modeled on $z \to -z$). The arcs α and β are both invariant under this involution, so they have to pass through the branch point, and B would in fact be the union of two bigons, each of which covers a bigon downstairs. This is a contradiction since we assumed that bigons between a and b in $(\mathcal{D}, Q)$ no longer exist. $\square$

It is natural to ask whether the converses of Propositions 1.1, 3.2 and 3.3 hold true. We have counterexamples to all statements except the one concerning $\widehat{\theta}$.

Proposition 3.4 *There exist right-veering transverse knots whose self-linking number is not maximal in the corresponding smooth knot type, whose branched double cover is overtwisted, and for which $\psi = 0$.*

Proof Consider the transverse knots K_1 and K_2 that are the $(3,2)$–cables of the trefoil discovered by Etnyre and Honda [13]; see also [32, Proposition 3.3]. These examples show that this smooth knot type is not transversely simple: the knots K_1 and K_2 have $\mathrm{sl}(K_1)=\mathrm{sl}(K_2)=3$ but are not transversely isotopic. Indeed, K_1 can be transversely destabilized but K_2 cannot. By [32], $\widehat{\theta}(K_2)\neq 0$; thus, by Proposition 1.1, K_2 is right-veering. Since $\mathrm{sl}(K_2)=\mathrm{sl}(K_1)$ is nonmaximal, the inequalities $\mathrm{sl}<s-1$ and $\mathrm{sl}<2\tau-1$ must be strict. The knots K_1 and K_2 can be represented as transverse push-offs of Legendrian knots L_1 and L_2 that differ by an "SZ move" [32, Figure 6]; by [26, Proposition 2.8] this implies that K_1 and K_2 can be related via "negative flypes" (see [26] for definitions of these modifications), and then by [17, Theorem 1.2] it follows that the branched double covers $(\Sigma(K_1),\xi_{K_1})$ and $(\Sigma(K_2),\xi_{K_2})$ are contactomorphic. Because K_1 is a transversely stabilized knot, $(\Sigma(K_1),\xi_{K_1})$ is overtwisted by [42, Proposition 1.3]; then $(\Sigma(K_2),\xi_{K_2})$ is also overtwisted. Finally, $\psi(K_2)=0$ (as stated in [32], the Khovanov homology component of the relevant bidegree is trivial). □

Note, however, that the converses of Propositions 1.1, 3.2 and 3.3 hold if the braid is *quasipositive* (rather than simply right-veering). Recall:

Definition 3.5 A braid σ is called quasipositive if it is a product of conjugates of standard generators, ie $\sigma=\prod w\sigma_i w^{-1}$.

As an automorphism of the punctured disk, σ is represented as a product of (nonstandard) positive half-twists.

Definition 3.6 We say that a transverse link K is quasipositive if K can be represented by a quasipositive braid.

Orevkov [34] proved that a quasipositive braid remains quasipositive after a positive Markov destabilization. Thus, in contrast to Proposition 3.1, *every* braid representing a quasipositive transverse link will be quasipositive.

The following properties of quasipositive braids are well known. Fillability of branched double cover is clear because positive half-twists in the punctured disk lift to positive twists in the covering open book; see eg [42]. The statements about transverse invariants follow from Proposition 2.5, as one can resolve a number of positive crossings in a quasipositive braid to obtain the trivial braid. Calculations of the τ– and s–invariants for quasipositive braids are known from standard theory [36; 44], and sl is easily computed.

Proposition 3.7 *Suppose K is a quasipositive transverse knot. Then:*

(1) $\mathrm{sl}(K) = 2g^*(K) - 1 = 2\tau(K) - 1 = s(K) - 1$.

(2) *The branched double cover $(\Sigma(K), \xi_K)$ is Stein fillable.*

(3) *The transverse invariants $\psi(K)$ and $\widehat{\theta}(K)$ are both nonzero.*

In a sense, quasipositivity is a very strong condition. It would be interesting to define a weaker notion of "strong right-veering" that would still imply the analog of Proposition 3.7.

4 Right-veering 3–braids

In this section, we prove Theorem 1.2, namely show that $\widehat{\theta}$ is nonzero for all right-veering 3–braids.

Proof of Theorem 1.2 If β is non-right-veering, then $\widehat{\theta}(K) = 0$ by Proposition 1.1. We need to prove that $\widehat{\theta} \neq 0$ for right-veering 3–braids.

According to Murasugi's classification [31], 3–braids come in the following types (up to conjugation):

(a) $h^d \sigma_1 \sigma_2^{-a_1} \sigma_1 \sigma_2^{-a_2} \cdots \sigma_1 \sigma_2^{-a_n}$, where $a_i \geq 0$ with some $a_i > 0$.

(b) $h^d \sigma_2^m$, where $m \in \mathbb{Z}$.

(c) $h^d \sigma_1^m \sigma_2^{-1}$, where $m = -1, -2, -3$.

Here, $h = (\sigma_1\sigma_2)^3$ is a full positive twist of the 3–braid, and the exponent d is an arbitrary integer.

We notice that some of the cases above clearly give non-right-veering braids. Indeed, if $d \leq 0$ in (a) or (c), or $d < 0$ in (b), or $d = 0$ and $m < 0$ in (b), we get braid words where σ_2 or σ_1 has only negative exponents. To prove the theorem, it suffices to show that $\widehat{\theta}$ is nonzero in all of the remaining cases, namely:

(a′) $h^d \sigma_1 \sigma_2^{-a_1} \sigma_1 \sigma_2^{-a_2} \cdots \sigma_1 \sigma_2^{-a_n}$, where $d > 0$ and $a_i \geq 0$ with some $a_i > 0$.

(b′) $h^d \sigma_2^m$, where $d > 0$ and $m \in \mathbb{Z}$, or $d = 0$ and $m \geq 0$.

(c′) $h^d \sigma_1^m \sigma_2^{-1}$, where $d > 0$ and $m = -1, -2, -3$.

Next, all braids in (c′) are clearly quasipositive, so $\widehat{\theta}$ is nonzero by [2]; see also Proposition 3.7 above. Nonvanishing is also obvious for the braids in (b′) with $m \geq 0$, since these are all positive.

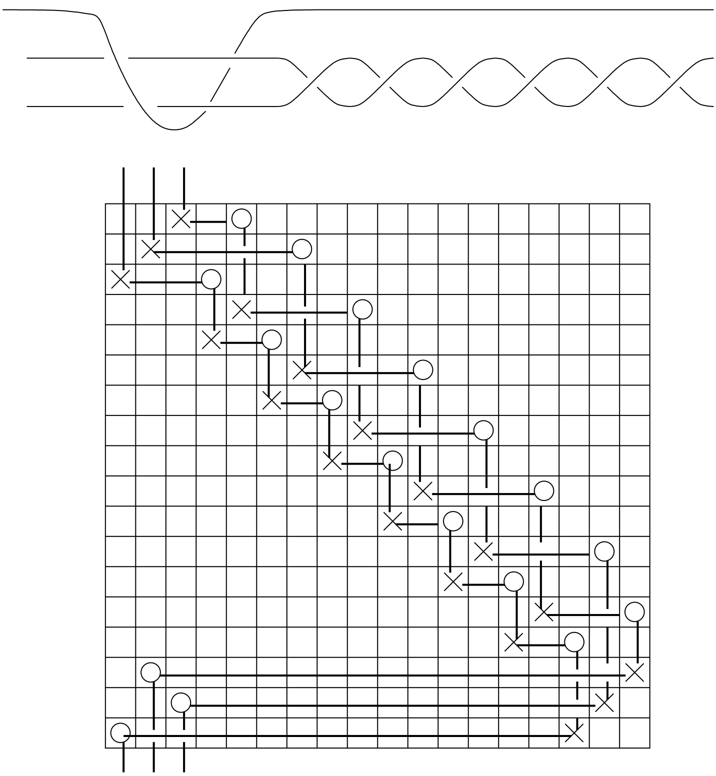

Figure 4: A grid diagram for $\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2^{-k}$

Next, we will show that

$$\widehat{\theta}(h\cdot\sigma_2^{-k}) \neq 0 \quad \text{for all } k>0. \tag{2}$$

These are braids from (b′) with $d=1$. Once this case is established, the rest of (b′) and (a′) follows from functoriality (Proposition 2.5); indeed, all the other braids in (b′) and (a′) can be obtained from the model braid $h\cdot\sigma_2^{-k}$ by insertion of additional positive crossings.

Observe that $h\cdot\sigma_2^{-k}=(\sigma_1\sigma_2)^3\sigma_2^{-k}=\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2^{-k+2}$.

We establish (2) for braids of the form $\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2^{-k}$ with $k>0$ by a direct examination of a grid diagram representing the corresponding transverse link. An example of the corresponding grid diagram is shown in Figure 4; in our picture $k=6$, but the general pattern should be clear to the reader. The element $\widehat{\theta}(K)\in\widehat{\mathrm{HFK}}(m(K))$ is given by the cycle made of "upper-right corners" of the X–markings in the figure. If this cycle is

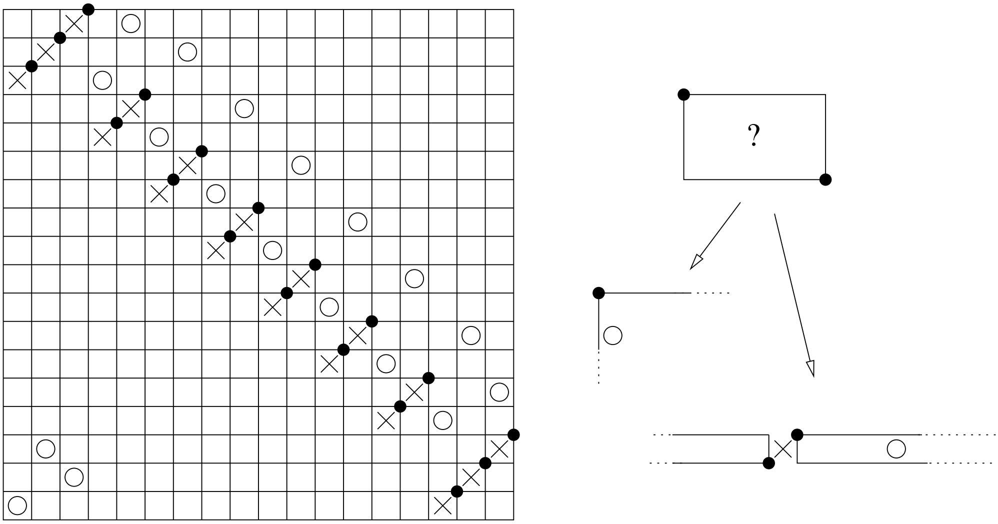

Figure 5: The cycle $\widehat{\theta}$ (left) and looking for empty rectangles (right)

null-homologous, it is the boundary of another element in $\widehat{\mathrm{CFK}}(m(K))$. We now need to recall the definition of the boundary map in grid homology [30]. In fact, instead of $\widehat{\mathrm{CFK}}$, we can work with another chain complex, $\widetilde{\mathrm{CFK}}$ [39], that has the same chain group but a slightly different boundary map. The homology $\widetilde{\mathrm{HFK}}$ of $\widetilde{\mathrm{CFK}}$ is not quite a knot invariant — it depends on the size of the grid — but nonvanishing of $\widehat{\theta}$ in $\widehat{\mathrm{HFK}}$ is equivalent to nonvanishing of the corresponding element $\widetilde{\theta}$ in $\widetilde{\mathrm{HFK}}$. (The class $\widetilde{\theta}$ is given by the "same" generator as $\widehat{\theta}$, the upper-right corners of X's.) The advantage of working with $\widetilde{\mathrm{CFK}}$ over $\widehat{\mathrm{CFK}}$ is that the differential has a simpler form and will be easier to analyze. For a generator $\boldsymbol{x}$ of $\widetilde{\mathrm{CFK}}$, given by an n–tuple of intersection points $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$, the boundary $\partial \boldsymbol{x}$ is obtained by summing over all n–tuples of the form $\boldsymbol{y} = (x_1, \dots, y_i, \dots, y_j, \dots x_n)$ that differ from $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ by exactly two entries and such that there is an empty rectangle in the grid diagram whose top-right and bottom-left corners are x_i and x_j, while the top-left and bottom-right corners are y_i and y_j. (A rectangle is "empty" if it contains no X and O markings and none of the entries of $\boldsymbol{x}$ and $\boldsymbol{y}$.)

Examining the diagram in Figure 5, we immediately see that there are no empty rectangles that can contribute to a differential killing the θ–cycle for our link. Indeed, notice that we have to check for two types of rectangles, those contained inside the square and those "wrapping around" the toric grid; the picture shows that no empty rectangles (of any type) can be found in the middle of the grid, where the X's and O's fall into the given pattern. Empty rectangles involving the intersections near the corners of the square grid (outside of the pattern) do not exist either, as inspection of the diagram shows. □

It should be noted that 3–braids are quite special and satisfy many properties that might not hold in general. In particular, we have the following:

Proposition 4.1 *A transverse 3–braid is right-veering if and only if its branched double cover is tight.*

Proof The "if" part is always true by Proposition 3.3; we now establish the "only if" part, which is specific to 3–braids. This is an easy corollary of known results. Indeed, the branched double cover of a 3–braid has an open book decomposition whose page is a punctured torus (or, rather, a genus 1 surface S with one boundary component). The page arises as a double cover of the disk branched over 3 points, and the braid monodromy lifts to a diffeomorphism ϕ of the page S, ie the monodromy of the open book. Every arc α with endpoints on ∂S covers an arc a in $(\mathcal{D}, \{x_1, x_2, x_3\})$. (Note that this statement is specific to the genus 1 case.) Indeed, α can be represented as an image $\rho(\alpha_0)$ of a standard arc α_0 in S covering an arc a_0 in $(\mathcal{D}, \{x_1, x_2, x_3\})$, under some diffeomorphism $\rho \in \mathrm{Map}(S)$. Since S has genus 1 and one boundary component, any such ρ is a lift of an element $r \in \mathrm{Map}(\mathcal{D}, \{x_1, x_2, x_3\})$, so that α covers $r(a_0)$.

Now suppose that the given transverse 3–braid is right-veering. Then the corresponding open book for the double cover must be right-veering as well, since a non-right-veering arc "upstairs" would cover a non-right-veering arc "downstairs". Indeed, if the image $\phi(\alpha)$ is to the left of α for some arc α in S and α covers an arc a in $(\mathcal{D}, \{x_1, x_2, x_3\})$, then the image of a under the braid monodromy must lie to the left of a (one should be careful with nonessential intersections, but the arcs can be pulled taut both upstairs and downstairs as in the proof of Proposition 3.3). Once we know that (S, ϕ) is right-veering, we can use a result of Honda, Kazez and Matić [20]: right-veering open books with a punctured torus page support tight contact structures. □

While we chose to discuss liftings of arcs explicitly in the above proof, one could instead use the results of [1], where it is shown that the right-veering monodromies (a′), (b′) and (c′) listed in our proof of Theorem 1.2 are exactly the monodromies that correspond to tight contact structures (and give nonvanishing Ozsváth–Szabó contact invariant). This is also relevant for the corollary below:

Corollary 4.2 *Let K be a transverse 3–braid. Then the following conditions are equivalent:*

(1) $\widehat{\theta}(K) \neq 0$.

(2) *The branched double cover $(\Sigma(K), \xi_K)$ is tight.*

(3) *The Heegaard Floer contact invariant $c(\xi_K)$ is nonzero.*

Additionally, if $\psi(K) \neq 0$, then conditions (1)–(3) hold, but the converse is not true.

Proof Theorem 1.2 and Proposition 4.1 give (1) $\iff$ (2). The equivalence (2) $\iff$ (3) for open books with punctured torus page is known from [20, Theorem 1.2; 1]. The Khovanov homological statement follows from the above together with Proposition 1.1. The analog of Theorem 1.2 does not hold for the transverse invariant ψ in Khovanov homology. Indeed, while right-veering braids from (a′) are all quasialternating and have $\psi \neq 0$, the braid $h \cdot \sigma_2^{-k}$ from (b′) considered in the proof above has $\psi(h \cdot \sigma_2^{-k}) = 0$, even though its branched double cover is tight; see [4, Example 7.8]. □

It is interesting to compare the above corollary with the results of [4], where in certain cases nonvanishing of $c(\xi_K)$ is derived from nonvanishing of $\psi(K)$ via the Ozsváth–Szabó spectral sequence (see [38; 45]).

5 Fractional Dehn twist coefficient

It is natural to ask whether Theorem 1.2 extends to braids of higher index. It would perhaps be too optimistic to expect that the transverse invariant $\widehat{\theta}$ is nonzero for *all* right-veering braids. Our Theorem 1.3 establishes that $\widehat{\theta}$ does not vanish as long as the braid has enough positive twisting.

We will obtain Theorem 1.3 as a corollary of a stronger statement, Theorem 5.2 below. Theorem 5.2 uses braid orderings and is stated in terms of Dehornoy's floor of a braid. We first recall the notion of Dehornoy floor and prove Theorem 5.2. Then we will discuss the fractional Dehn twist coefficient and its relation to Dehornoy's floor and complete the proof of Theorem 1.3.

As usual, let $\Delta \in B_n$ denote the Garside element,

$$\Delta = (\sigma_1\sigma_2 \ldots \sigma_{n-1})(\sigma_1\sigma_2 \cdots \sigma_{n-2}) \cdots (\sigma_1\sigma_2)\sigma_1.$$

Recall that Δ generates the center of the group B_n. Geometrically, Δ is the positive half-twist on all strands, so

$$\Delta^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n$$

is the full twist.

Definition 5.1 [11] Let $\beta \in B_n$ be an arbitrary braid. Let m be an integer such that $\Delta^{2m} \preceq \beta \prec \Delta^{2m+2}$, where $\prec$ is Dehornoy's ordering on B_n. Then $m = \lfloor\beta\rfloor_D$ is called Dehornoy's floor of β.

Note that several slightly different definitions of $\lfloor\beta\rfloor_D$ exist in the literature [29; 21; 11]; the present one best fits our purposes. Intuitively, Dehornoy's floor tells us how many positive full twists can be extracted from a given braid, so that the "leftover" braid is still nonnegative in Dehornoy's ordering. (Since $\prec$ is a total ordering, and for any given braid β we have that $\Delta^{-N} \prec \beta \prec \Delta^N$ when N is very large, the floor function is well defined.) Note that Dehornoy's floor is *not* conjugacy-invariant (and thus not an invariant of *closed braids*); for example, $\lfloor\Delta^2\sigma_1\sigma_2^{-1}\rfloor_D = 1$ but $\lfloor\Delta^2\sigma_2\sigma_1^{-1}\rfloor_D = 0$, despite the fact that these two 3–braids are conjugate.

We are now ready to state our result.

Theorem 5.2 *If the transverse link K can be represented by a braid β with $\lfloor\beta\rfloor_D \geq 1$, then $\hat{\theta}(K) \neq 0$.*

Proof We will use the same idea as in the proof of Theorem 1.2: the question can be reduced to a family of model braids, which can then be attacked directly via grid diagrams.

Indeed, suppose that $\lfloor\beta\rfloor_D \geq 1$. This means that $\beta = \Delta^2\gamma$, where $\gamma \succeq 1$ with respect to Dehornoy's order. Then γ has a braid word which is either σ_1–positive or σ_1–free. Both cases guarantee that the braid word for γ has no entries of σ_1 with negative exponent.

For any $k > 0$, consider a braid $\beta_{n,k} \in B_n$ of the form

$$\beta_{n,k} = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_{n-1}\cdots\sigma_2\sigma_1)(\sigma_2\sigma_3\cdots\sigma_{n-1})^{-k}.$$

In the n–braid $\beta_{n,k}$, all the strands except the first are twisted together k times in the negative direction, while the first strand twists positively once around all the other strands.

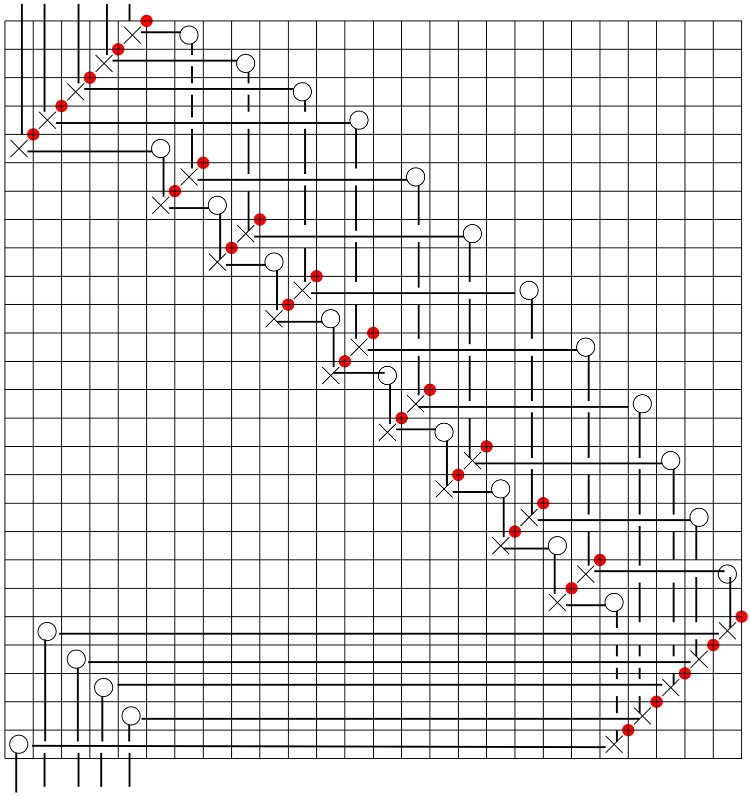

Figure 6: The cycle $\widehat{\theta}$ for a braid $\beta'_{n,k} = \beta_{n,k}$ with $n = 5$ and $k = 2$

The family of braids $\beta_{n,k}$ is universal in the following sense: any given braid $\beta \in B_n$ with $\lfloor \beta \rfloor_D \geq 1$ can be obtained from a braid $\beta_{n,k}$ with k large enough by insertion of a number of positive crossings. Indeed, suppose that $\beta = \Delta^2\gamma$ is an n–braid, and the braid γ has no negative crossings in the σ_1 level (ie no entries of σ_1 with negative exponent). Suppose that γ has k negative crossings. Then γ can be obtained from the braid $\gamma' = (\sigma_2\sigma_3\cdots\sigma_{n-1})^{-k}$ by insertion of a number of positive crossings: we get each negative crossing σ_i^{-1} of γ by inserting positive crossings to cancel all the other negative crossings in the braid $(\sigma_2\sigma_3\cdots\sigma_{n-1})^{-1}$ (the latter expression repeats k times in the braid γ', so we can get k arbitrary negative crossings by this procedure). Then we can insert additional positive crossings to create positive crossings of γ. Applying this procedure to $\beta_{n,k} = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_{n-1}\cdots\sigma_2\sigma_1)\gamma'$, we will get the braid $(\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_{n-1}\cdots\sigma_2\sigma_1)\gamma$, which in turn can be converted into $\Delta^2\gamma = \beta$ by inserting extra positive crossings to obtain the full twist Δ^2 from the braid $(\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_{n-1}\cdots\sigma_2\sigma_1)$.

For a braid β on n strands with k negative crossings and $\lfloor \beta \rfloor_D \geq 1$, we have shown that β can be obtained from $\beta_{n,k}$ by insertion of positive crossings. By functoriality of $\widehat{\theta}$ [2] (see Proposition 2.5 above), it will follow that $\widehat{\theta}(\beta) \neq 0$ if we check that $\widehat{\theta}(\beta_{n,k}) \neq 0$ for all n and k.

We will examine a grid diagram for a braid $\beta_{n,k}$; more precisely, we draw a diagram for an equivalent braid

$$\beta'_{n,k} = (\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_{n-1}\cdots\sigma_2\sigma_1)\Delta'(\sigma_2\sigma_3\cdots\sigma_{n-1})^{-k}(\Delta')^{-1},$$

where Δ' is a positive half-twist on strands $2, 3, \dots, n$. Note that $\beta'_{n,k} = \beta_{n,k}$ in B_n, since Δ' commutes with $\sigma_2, \sigma_3, \dots, \sigma_n$. Consider a grid diagram for $\beta'_{n,k}$ as shown in Figure 6 (the example in the figure has $n = 5$ and $k = 2$). This diagram has a similar pattern and the same features as the diagram for a 3–braid we considered in Figure 4. As before, we can see that there are no empty rectangles whose top-left and bottom-right corners are given by the distinguished intersection points forming the $\widehat{\theta}$–cycle. Therefore, $\widehat{\theta} \neq 0$. $\square$

We now discuss the definition of the fractional Dehn twist coefficient; its relation to Dehornoy's floor will give an immediate proof of Theorem 1.3. Most of the discussion below is only needed to state Lemma 5.4 and not strictly necessary otherwise. However, we choose to include a fairly detailed definition of the FDTC since the connection between transverse invariants and the twisting of the monodromy, as measured by the FDTC, is the central theme of this paper. The idea of the fractional Dehn twist coefficient first appeared in [15] and was developed in the context of open books and contact topology in [18]. For classical braids, a similar notion (via a somewhat different approach) was studied in [28]. A generalization of FDTC to the case of braids in arbitrary open books and a detailed proof that different definitions are equivalent is given in [24].

To define the FDTC of a given braid $\beta \in B_n$, equip the disk $\mathcal{D} \setminus \{x_1, x_2, \dots, x_n\}$ with a complete hyperbolic metric of finite volume such that $\partial\mathcal{D}$ is geodesic. Let $\widetilde{\mathcal{D}}$ be the universal cover of the punctured disk; $\widetilde{\mathcal{D}}$ admits an isometric embedding into the Poincaré disk $\mathbb{H}^2$. We can compactify $\widetilde{\mathcal{D}}$ by adding points at infinity; the result is a closed disk $\overline{\mathcal{D}}$. Choose a basepoint $* \in \partial\mathcal{D}$ and its lift $\widetilde{*} \in \partial\overline{\mathcal{D}}$. Let $\beta \in B_n = \mathrm{Map}(\mathcal{D}, \{x_1, \dots, x_n\})$ be a braid. Consider the lift $\widetilde{\beta}\colon \widetilde{\mathcal{D}} \to \widetilde{\mathcal{D}}$ with $\widetilde{\beta}(\widetilde{*}) = \widetilde{*}$. It extends uniquely to a homeomorphism $\overline{\beta}\colon \overline{\mathcal{D}} \to \overline{\mathcal{D}}$ that fixes the component $\widetilde{C}$ of the preimage $\pi^{-1}(\partial\mathcal{D})$ of $\partial\mathcal{D}$ under the covering map π such that $\widetilde{C}$ contains $\widetilde{*}$. Notice that $\partial\overline{\mathcal{D}}\setminus\widetilde{C}$ can be identified with $\mathbb{R}$, so $\overline{\beta}$ induces a homeomorphism of $\mathbb{R}$. Moreover, the identification $\partial\overline{\mathcal{D}}\setminus\widetilde{C}\equiv\mathbb{R}$ can be chosen so that the full twist around $\partial\mathcal{D}$ (ie the braid $\Delta^2\in B_n$) induces the homeomorphism $x\mapsto x+1$. Since Δ is in the center of B_n, any braid $\beta\in B_n$ will induce a homeomorphism of $\mathbb{R}$ that commutes with the translation $x\mapsto x+1$. This means that we have a map

$$\Theta\colon B_n\to\widetilde{\mathrm{Homeo}}^+(S^1),$$

where $\widetilde{\mathrm{Homeo}}^+(S^1)$ is the group of homeomorphisms of $\mathbb{R}$ that are lifts of orientation-preserving homeomorphisms of S^1. The map Θ is called the *Nielsen–Thurston* map.

Recall that a basic invariant of a map $h\in\widetilde{\mathrm{Homeo}}^+(S^1)$ is given by its *translation number*, defined as

$$T(h)=\lim_{N\to\infty}\frac{h^N(x)-x}{N}$$

for any $x\in\mathbb{R}$. Finally:

Definition 5.3 The fractional Dehn twist coefficient of a braid $\beta\in B_n$ is defined as the translation number of $\Theta(\beta)$,

$$C=T(\Theta(\beta)).$$

The FDTC is well defined, ie independent of the choice of the hyperbolic metric on the punctured disk and other choices [28].

Another definition, emphasizing the meaning of FDTC as the amount of rotation about the boundary of $\mathcal{D}$, can be given in the spirit of [18]. One uses the Nielsen–Thurston classification to find a free isotopy connecting (an iterate of) ϕ to its pseudo-Anosov, periodic or reducible representative, and considers the winding number of an arc traced by a basepoint in $\partial\mathcal{D}$ under this isotopy. We refer the reader to [24] for a detailed definition and discussion.

The fractional Dehn twist coefficient is related to Dehornoy's floor as follows. (Note that unlike Dehornoy's floor, FDTC is invariant under conjugation and thus gives an invariant of a *closed* braid.)

Lemma 5.4 [28] *Let $\beta\in B_m$ be a braid with fractional Dehn twist coefficient C. Then*

$$\lfloor\beta\rfloor_D+1\geq C\geq\lfloor\beta\rfloor_D\quad\textit{and}\quad C=\lim_{n\to\infty}\frac{\lfloor\beta^n\rfloor_D}{n}.\qquad\square$$

Proof of Theorem 1.3 This follows immediately from Theorem 5.2 and Lemma 5.4, since any $\beta \in B_m$ representing a closed braid with fractional Dehn twist coefficient $C > 1$ has $\lfloor \beta \rfloor_D \geq 1$. (Note that $\lfloor \beta \rfloor_D$ is an integer, and if $\lfloor \beta \rfloor_D \leq 0$ then, by Lemma 5.4, we get $C \leq 1$.) □

6 Higher-order simple covers

We have seen that the properties of contact manifolds arising as double covers of $(S^3, \xi_{\mathrm{std}})$ branched over transverse braids often reflect the properties of the branch locus. Indeed, quasipositive braids give rise to Stein fillable covers, while non-right-veering braids have overtwisted covers. In this section we turn our attention to higher-order covers. If one considers cyclic covers, the same properties hold:

Theorem 6.1 *Let $(\Sigma_n(K), \xi_n)$ be the n–fold cyclic cover of $(S^3, \xi_{\mathrm{std}})$ branched over a transverse link K. Then:*

(1) *If K is represented by a quasipositive braid, $(\Sigma_n(K), \xi_n)$ is Stein fillable.*

(2) *If K is a non-right-veering transverse link, $(\Sigma_n(K), \xi_n)$ is overtwisted.*

Proof The statements above are essentially contained in [17, Theorem 1.3 and Proposition 4.2] — while [17, Proposition 4.2] is stated in terms of braids "with a row of negative crossings"; by Corollary 2.3 this is equivalent to non-right-veering. □

The situation changes dramatically if we consider *simple* covers instead of *cyclic* ones. It is known that any contact 3–manifold can be represented as a 3–fold simple cover of $(S^3, \xi_{\mathrm{std}})$ branched over some transverse link K; thus, 3–fold simple covers are of particular interest. We show that in this setting, positivity and right-veering properties of the branch locus have no bearing on the properties of the cover:

Theorem 6.2 *Any contact 3–manifold can be represented both as a 3–fold simple cover of a positive braid and a 3–fold simple cover of a negative braid.*

Proof If we ignore the contact structures, the counterpart of this statement is known in low-dimensional topology: any 3–manifold can be represented as a 3–fold simple cover of a positive braid (or of a negative braid). The proof (see eg [43, Theorem 25.2, Step 2]) is based on the fact that the branch locus can be modified in certain ways without affecting the 3–fold simple branched cover. One of the transformations [43, Figure 24.3]

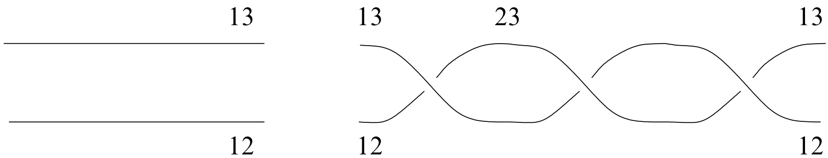

Figure 7: Modification of the branch locus for a 3–fold simple cover

replaces two adjacent parallel strands in the braid by three positive crossings provided that the branching pattern is as shown in Figure 7. (As usual, the labels indicate how the sheets of the cover meet the branch locus; for example, one passes from sheet 1 to sheet 2 after going along a small loop around a given strand.) This allows to kill a negative crossing (replacing it by two positive ones) whenever there arcs with different labels meeting at the negative crossing. It is easy to achieve that all crossings have this property by first isotoping the link as in [43, Theorem 25.2, Step 2, and Figure 25.1]. We now observe that the same proof goes through in presence of contact structures, because the branched double cover corresponding to each of the transverse tangles in Figure 7 is a Darboux ball. (Namely, we close up these tangles to obtain the trivial braid on two strands and the braid σ_1^3; both 3–fold simple branched covers are the standard contact 3–spheres. Indeed, following the discussion of [43, Figure 23.7], one sees that the cover corresponding to the braid σ_1^3 in Figure 7 is given by an open book with a disk page. This discussion is compatible with contact structures, so the cover is $(S^3, \xi_{\mathrm{std}})$.) The same move in the contact category is also discussed in [8, Section 3.2.2.2]; thanks to John Etnyre for pointing this out to us.

Inserting triples of positive crossings into a given transverse braid, we can obviously convert it into a positive braid. The branched double cover will remain the same contact manifold during this procedure, so we have shown that any contact manifold is a 3–fold simple cover of a positive braid. Performing the same modification backwards, we can insert a triple of negative crossings into any branch locus (without changing the cover); this shows that any contact 3–manifold is a cover of $(S^3, \xi_{\mathrm{std}})$ branched over a negative braid. □

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