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A signature invariant for knotted Klein graphs

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We define some signature invariants for a class of knotted trivalent graphs using branched covers. We relate them to classical signatures of knots and links. Finally, we explain how to compute these invariants through the example of Kinoshita's knotted theta graph.

05C10, 57M12, 57M15; 57M25, 57M27

1 Introduction

The notion of knotted graph generalizes the notion of link. It has direct applications in stereo-chemistry; see Flapan [3] and Simon [22]. On the one hand, the classification of knotted graphs can be seen as an extension of the classification problem for knots. On the other hand, given a knotted graph, one can look at all its sublinks. Kinoshita [12; 13] gave an example (see Figure 1) of a nontrivial knotted theta graph such that all three subknots are trivial (see Jang, Kronaeur, Luitel, Medici, Taylor and Zupan [10] for more examples of *Brunnian* theta graphs). Hence, it is necessary to develop specific invariants for knotted graphs.

We restrict ourselves to a certain class of trivalent graphs in $\mathbb{S}^3$ with an edge-coloring called 3–Hamiltonian Klein graphs. The aim of this paper is to define some signature-like invariants for knotted such graphs. The theta graph is a 3–Hamiltonian Klein graph and these invariants are in particular suited for the study of knotted theta graphs.

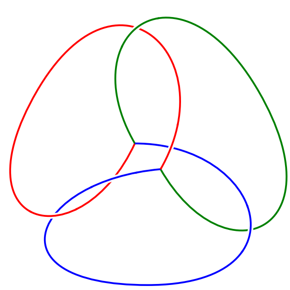

Figure 1: Kinoshita's knotted theta graph

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In [7], Gordon and Litherland explain how to compute signatures of knots from a nonorientable spanning surface F in $\mathbb{B}^4$. It is the signature of the double branched cover of $\mathbb{B}^4$ along F corrected by the normal Euler number of F (see also Kauffman and Taylor [11]). In this paper, we adopt this 4–dimensional point of view.

A Klein graph Γ is a trivalent graph endowed with a 3–coloring of its edges. For any knotted Klein graph in $\mathbb{S}^3$, one can consider its *Klein cover*; it is a branched $\mathbb{Z}_2\times\mathbb{Z}_2$–cover with Γ as branched locus. Given a spanning foam[1] F for Γ, one can construct the Klein cover W_F of $\mathbb{B}^4$ along F.

We define invariants of Γ using the signature of W_F. To do so, we define normal Euler numbers of foams. It turns out that if Γ is 3–Hamiltonian,[2] there are two ways to define the normal Euler numbers, yielding different invariants (see Theorem 3.11). The computation for the Kinoshita knotted graph shows that this graph is nontrivial (and even chiral).

Moreover, we investigate the relations between our invariants and the signatures of the different knots and links related to the knotted graph Γ. The identities we find (see Proposition 3.14) can be thought of as consistency constraints between these signatures. The concept of foam enables to interpret these constraints geometrically.

For defining our invariants, we need the notion of normal Euler numbers of immersed surfaces with boundaries. For this, we use linking numbers of rationally null-homologous curves in arbitrary 3–manifolds (see Lescop [16] for a gentle introduction on this notion). The invariance of the signatures follows from the G–signature theorem (see Atiyah and Singer [1]) in dimension 4 (see Gordon [6] for an elementary approach). Finally, in order to compute our invariants on an example, we use a result of Przytycki and Yasuhara [20], which calculates the modification by surgery of the linking matrix of a link in a rational homology spheres.

Structure of the paper

In Section 2, we introduce the notion of Klein graph, Klein foam and Klein cover. In Section 3.2 we define the invariants. For this we recall the notion of normal Euler numbers in Section 3.1. The rest of Section 3 is dedicated to the proof of invariance:

[1] Foams are natural cobordisms when working with graphs; they are surfaces with some singularities (see Section 2.1). Here we do not suppose any kind of orientability conditions on F.

[2] If the graph Γ is *not* 3–Hamiltonian, we can still define one signature invariant, which turns out to depends only on the sublinks of Γ.

Section 3.3 contains two technical lemmas about normal Euler numbers. Section 3.4 contains the proof of the Theorem 3.11. Finally in Section 4, we compute our signature invariants on Kinoshita's knotted graph.

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2 Klein covers

2.1 Graphs and foams

Throughout the paper, D_4 denotes the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (we use the multiplication convention for the group law) and D_4^* denotes the set $D_4 \setminus \{1\}$. The elements of D_4^* are denoted by a, b and c and are represented in pictures by red, blue and green, respectively.

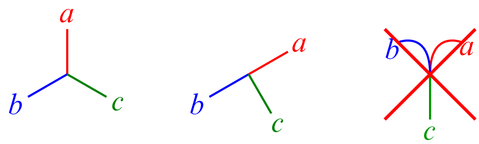
Definition 2.1 An abstract *Klein graph* is a finite trivalent multigraph Γ with an edge-coloring by D_4^* (as usual in graph theory we require that the colors of two adjacent edges are different). It is 3–*Hamiltonian* if for any two elements i and j of D_4^*, the subgraph Γ_{ij} consisting of edges colored by i or j is connected. From Section 3 on, all Klein graphs are supposed to be 3–Hamiltonian.

A *knotted Klein graph* is a Klein graph Γ together with a smooth embedding of Γ in a manifold of dimension 3. If the manifold is not given it is assumed to be $\mathbb{S}^3$.

Remarks 2.2 (1) Klein graphs are related to but slightly different from knotted trivalent graphs (KTGs) as studied in Thurston [23]. Indeed, Klein graphs are not framed and are endowed with an edge-coloring. The colorings we work with in this paper (if one forgets about the 3–Hamiltonian condition) are considered by Kronheimer and Mrowka in [15], where they are called Tait colorings.

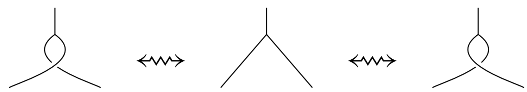
(2) We should explain what is meant by a *smooth embedding* of an abstract Klein graph. We require that each edge be smoothly embedded and that for every vertex,

none of the three tangent vectors be positively collinear with any of the two others:



(3) We consider embedded Klein graphs up to ambient isotopy. In a diagrammatic approach, this means that a graph is considered up to the classical Reidemeister moves and the following additional ones (see [17, Proposition 1.6]):

(Rv1)



(Rv2)

(4) One may wonder which trivalent graphs can be endowed with a structure of Klein graphs. It is known to be the case for planar graphs with no bridge (this follows from the 4–color theorem) and for bipartite graphs (this follows from König's theorem). However, the 3–Hamiltonian condition is more complicated to ensure. Four examples are given in Figure 2.

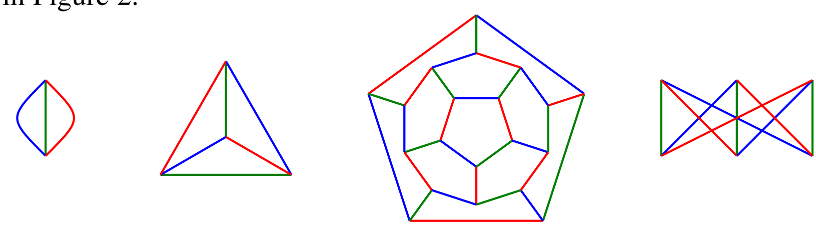


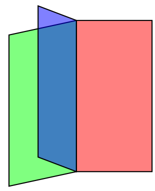
Figure 2: Examples of 3–Hamiltonian Klein graphs

(5) The 3–Hamiltonian condition is preserved by performing connected sum along a vertex. This operation is described in Figure 3. This permits us to create arbitrarily large graphs endowed with a Klein coloring.

Figure 3: Connected sum along a vertex

Definition 2.3 A *closed embedded Klein foam* E is the realization of a finite CW–complex in a manifold of dimension 4 (if the manifold is not given, it is meant to be $\mathbb{S}^4$)

and some data attached to it. For every point of the CW–complex, we require that there exists a neighborhood which is either diffeomorphic to a disk or to the following picture (the singular one-dimensional cell is called a *binding*):



.

The data attached to E is a coloring of its facets by D_4^* such that the three facets adjacent to a common binding have different colors.

The intersection F of an embedded Klein foam E with a submanifold of dimension 4 with boundary is *a Klein foam with boundary* if

- the intersection of E with ∂W is a knotted Klein graph Γ in ∂W,
- there exists a tubular neighborhood U of ∂W such that $(U, E \cap U)$ is diffeomorphic to $(\partial W \times]0,1[, \Gamma \times]0,1[)$.

In this case, Γ is the *boundary of* F and we write $\Gamma = \partial F$. We say as well that F is a *spanning foam* for Γ.

Proposition 2.4 (proof in the appendix) *Let Γ be a knotted Klein graph in $\mathbb{S}^3$. There exists a spanning foam for Γ in $\mathbb{B}^4$.*

2.2 Klein covers

Definition 2.5 Let M be a closed, oriented manifold. If D_4 acts on M by orientation-preserving diffeomorphisms and if for every g in D_4^*, the set M^g of fixed points of g is a submanifold of codimension 2, we say that M is a *Klein manifold*. We set $M^{\cup D_4} := \bigcup_{g \in D_4^*} M^g$.

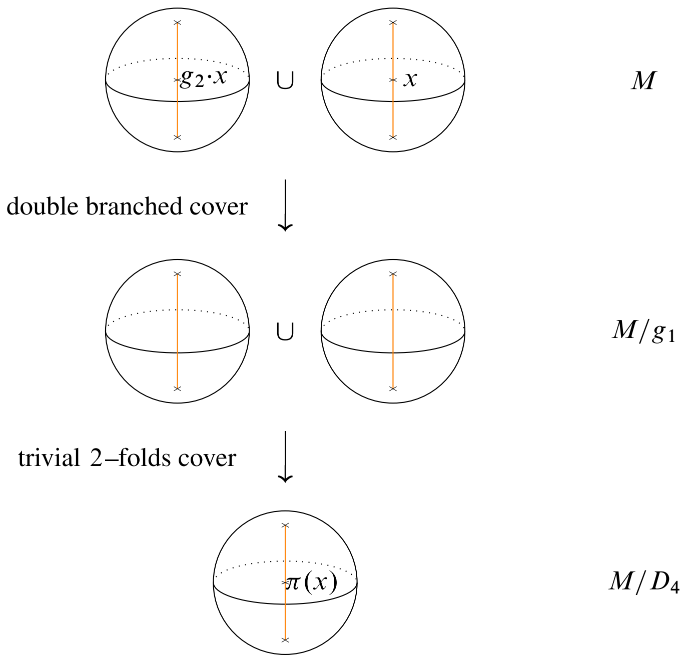
Proposition 2.6 *Let M be a Klein manifold of dimension 3 (resp. 4); then M/D_4 is a closed oriented manifold of the same dimension and $M^{\cup D_4}$ is mapped on a Klein graph (resp. Klein foam) by $\pi\colon M \twoheadrightarrow M/D_4$.*

Proof Dimensions 3 and 4 are analogous. We only treat dimension 3. Let g_1 be an element of D_4^* and x in M be a fixed point for g_1. The diffeomorphism induced by g_1 being an involution, the action of g_1 on T_xM is diagonalizable and we can find a basis of T_xM such that the matrices of the linear map induced by g_1 is equal to

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.$$

Let g_2 and g_3 be the two remaining elements of D_4. We have $g_2\cdot x = g_2g_1\cdot x = g_3\cdot x$. Hence, x is fixed by g_2 if and only if it is fixed by g_3.

Suppose that x is not fixed by g_2. Then the restriction of $\pi\colon M \to M/D_4$ to a neighborhood of $\{x, g_2\cdot x\}$ is isomorphic to a double branched cover followed by a trivial 2–folds cover:

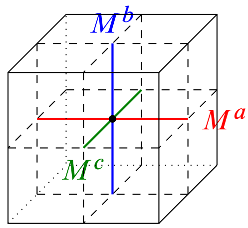


This implies that $\pi(x)$ has a neighborhood homeomorphic to a ball.

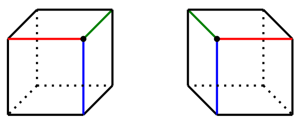
Suppose now that x is fixed by all the elements of D_4^*. We look at the action of D_4 over T_xM. This can be seen as a map from $\phi\colon D_4 \to \mathrm{GL}_3(\mathbb{R})$. The matrices $\phi(g_1)$, $\phi(g_2)$ and $\phi(g_3)$ are simultaneously diagonalizable and we can find a basis of T_xM such that these matrices are equal to

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}, \quad \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

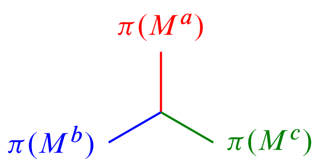
Hence, in a chart, the action of g_1, g_2 and g_3 is given by these matrices and the fixed-point loci look like



in a neighborhood of x. This implies that a neighborhood of $\pi(x)$ in M/D_4 is given by gluing the two cubes



along the three faces containing the distinguished vertex. This gives a 3–dimensional ball. This proves that M/D_4 is a manifold and that in a neighborhood of $\pi(x)$ the image of $\pi(M^{\cup D_4})$ is given in a chart by



Since M is compact, there are finitely many points which are fixed by the whole group D_4. Altogether, this implies that M/D_4 is a manifold and that the image of $M^{\cup D_4}$ is a Klein graph. For the 4–dimensional statements, all local discussions have to be multiplied by an interval. □

Definition 2.7 Let M be a closed manifold of dimension 3 (resp. 4) and Γ (resp. F) be an embedded Klein graph (resp. a Klein foam). Suppose that N is a Klein manifold of the same dimension such that $N/D_4 \simeq M$ and that $\pi(N^{\cup D_4})$ is identified with Γ (resp. F). Then we say that N is a *Klein cover of M along Γ (resp. F)*.

Proposition 2.8 (1) *For any embedded Klein graph Γ in $\mathbb{S}^3$, there exists a unique* (up to diffeomorphism) *Klein cover of $\mathbb{S}^3$ along Γ. It is denoted by M_Γ.*

(2) *For any Klein foam F embedded in $\mathbb{S}^4$, there exists a unique* (up to diffeomorphism) *Klein cover of $\mathbb{S}^4$ along F. It is denoted by V_F.*

(3) *For any Klein foam F properly embedded in $\mathbb{B}^4$, there exists a unique* (up to diffeomorphism) *Klein cover of $\mathbb{B}^4$ along F. It is denoted by W_F.* (The first point actually implies that $\partial W_F \simeq M_{\partial F}$.)

Proof We only prove the first statement; the two others are analogous. Fix an arbitrary orientation on $\Gamma = (V, E)$. The first homology group of $M := \mathbb{S}^3 \setminus \Gamma$ is generated by elements $[\gamma_e]_{e \in E}$, where γ_e is a small loop wrapping positively around the edge e. More precisely,

$$H_1(M, \mathbb{Z}) \simeq \left(\bigoplus_{z \in E} \mathbb{Z} \right) \Big/ \langle \pm[\gamma_{e_1(v)}] + \pm[\gamma_{e_2(v)}] + \pm[\gamma_{e_3(v)}] = 0 \text{ for all } v \text{ in } V \rangle,$$

where $e_1(v)$, $e_2(v)$ and $e_3(v)$ are the three edges adjacent to v and the signs ambiguity is given by the way the orientations are toward v. The coloring of Γ gives a morphism

$$\phi\colon H_1(M) \to D_4$$

which sends $[\gamma_e]$ to the color of e in Γ. Note here that the ambiguity in the orientation of Γ is not a problem since all nontrivial elements of D_4 have order 2. This is indeed a morphism, since in D_4 we have $g_1 g_2 g_3 = 1$. We can promote ϕ to a morphism

$$\phi\colon \pi_1(M) \to D_4.$$

We consider the covering M' associated with this morphism. We can complete M' in order to make it a Klein cover of $\mathbb{S}^3$. For the edges and the vertices we use the local models described in the proof of Proposition 2.6.

Uniqueness comes from the uniqueness of the covering M' and from the fact that the local models for the singular points and the edges are the only possible ones (see the proof of Proposition 2.6). The proof in dimension 4 is analogous. Note that in this case, the local models are given by the ones we described times an interval. □

3 A signature invariant

3.1 Normal Euler numbers

The aim of this part is to recall the definition of normal Euler numbers of surfaces with boundaries (Definition 3.5) and to give an additivity property that they satisfy. They have been studied by Gilmer [5] and require linking numbers of rationally null-homologous knots in arbitrary 3–manifolds. We refer to the lecture notes of Lescop

[16, Section 1.5] for details about such linking numbers. We start with the normal Euler number of a closed surface:

Definition 3.1 Let Σ be a smooth, not necessarily orientable, closed surface immersed in a smooth oriented manifold of dimension 4 and let s be a section of the normal bundle of Σ transverse to the null section s_0. Then the intersection (computed with local orientations) of s_0 with s is called the *normal Euler number of* Σ and is denoted by $e(\Sigma)$. As the name suggests the integer $e(\Sigma)$ does not depends on s.

Remark 3.2 If we choose a local system of orientations $O(\Sigma)$ of Σ, we can define the Euler class $e(N_\Sigma)$ of the normal bundle N_Σ. This is an element of $H^2(\Sigma, O(\Sigma))$. We consider $[\Sigma] \in H_2(\Sigma, O(\Sigma))$, the fundamental class of Σ in this local system of orientations. The normal Euler number of Σ is then equal to $e(N_\Sigma)([\Sigma])$.

We now extend this definition to the case when the 4–manifold and surface have boundary.

Definition 3.3 (relative normal Euler number) Let W be an oriented 4–manifold with boundary and Σ be a smooth, properly immersed surface with boundary in W. Let $L = l_1 \cup \cdots \cup l_k$ be the boundary of Σ. Let us choose some parallels $\tilde{l}_1, \dots, \tilde{l}_k$ of $l_1, \dots, l_k$ in ∂W. The normal Euler number of Σ relatively to $\tilde{l}_1, \dots, \tilde{l}_k$ is the intersection number of Σ with a section s of the normal bundle of Σ (transverse to the null section s_0) such that $\partial s = \tilde{l}_1 \cup \cdots \cup \tilde{l}_k$. We denote it by $e(\Sigma; \tilde{l}_1, \dots, \tilde{l}_k)$.

Proposition 3.4 [5, page 311] *Let W be an oriented 4–manifold with boundary and Σ be a smooth, properly immersed surface with boundary. Let $L = l_1 \cup \cdots \cup l_k$ be the boundary of Σ. Suppose that each l_i is rationally null-homologous in ∂W. Let us choose $\tilde{l}_1^1, \dots, \tilde{l}_k^1$ and $\tilde{l}_1^2, \dots, \tilde{l}_k^2$, two sets of parallels of $l_1, \dots, l_k$. For each i pick an orientation of l_i and orient $\tilde{l}_i^1$ and $\tilde{l}_i^2$ accordingly. Then, for every i in $[1, l]$, $\mathrm{lk}(l_i, \tilde{l}_i^1) - \mathrm{lk}(l_i, \tilde{l}_i^2)$ is an integer, and we have*

$$e(\Sigma; \tilde{l}_1^2, \dots, \tilde{l}_k^2) - e(\Sigma; \tilde{l}_1^1, \dots, \tilde{l}_k^1) = \sum_{i=1}^{k} (\mathrm{lk}(l_i, \tilde{l}_i^2) - \mathrm{lk}(l_i, \tilde{l}_i^1)).$$

Proof Let T_i be a tubular neighborhood of l_i in ∂W. An homotopy from $\tilde{l}_i^1$ to $\tilde{l}_i^2$ can be thought of as a section s_{h_i} of the normal bundle of $l_i \times [-\epsilon, \epsilon]$ in $T_i \times [-\epsilon, \epsilon]$. The intersection of $s_{h_i}(l_i \times [-\epsilon, \epsilon])$ with $l_i \times [-\epsilon, \epsilon]$ is equal to $\mathrm{lk}(l_i, \tilde{l}_i^1) - \mathrm{lk}(l_i, \tilde{l}_i^2)$. Let

s_1 be a section of the normal bundle of Σ used to compute $e(\Sigma; \tilde{l}_1^1, \dots, \tilde{l}_k^1)$. Gluing the s_{h_i} to s_1, one obtains a section s_2 which bounds $\tilde{l}_1^1 \cup \dots \cup \tilde{l}_k^1$. Hence,

$$\Sigma \cap s_2(\Sigma) - \Sigma \cap s_1(\Sigma) = \sum_{i=1}^{k} (\mathrm{lk}(l_i, \tilde{l}_i^2) - \mathrm{lk}(l_i, \tilde{l}_i^1)). \qquad \square$$

This yields the following definition:

Definition 3.5 (normal Euler numbers, boundary case) Let W be an oriented 4–manifold with boundary and Σ be a smooth surface with boundary properly immersed. Suppose that every component $l_1, \dots, l_k$ of the boundary of Σ is rationally null-homologous in ∂W, and let s be a section of the normal bundle of Σ (transverse to s_0) and denote $l'_1, \dots, l'_k$ the parallels of $l_1, \dots, l_k$ induced by s. We define $e(\Sigma) = \Sigma \cap s(\Sigma) - \sum_{i=1}^{k} \mathrm{lk}(l_i, l'_i)$.

Remarks 3.6 (1) Suppose that every component of Σ has a nonempty boundary. Then it is possible to find a nowhere vanishing section s of the normal bundle of Σ. Let us denote by $l'_1, \dots, l'_k$ the parallels of $l_1, \dots, l_k$ induced by s. Then we have

$$e(\Sigma) = -\sum_{i=1}^{k} \mathrm{lk}(l_i, l'_i).$$

(2) If ∂W is a rational homology sphere, then the conditions on the l_i are automatically satisfied.

Proposition 3.7 *Let W be an oriented 4–manifold with boundary and Σ be a smooth surface with boundary properly immersed in W. Suppose that each connected component of $\partial\Sigma$ is rationally null-homologous in ∂W.*

- *The normal Euler number $e(\Sigma, -W)$ of Σ in the manifold W endowed with the opposite orientation is equal to $-e(\Sigma, W)$.*
- *Let W' be an oriented 4–manifold with boundary and Σ' be a smooth surface with boundary immersed in W' such that $\partial\Sigma' \subseteq \partial W'$. Suppose that ϕ is an orientation-reversing diffeomorphism from ∂W to $\partial W'$ which maps $\partial\Sigma$ on $\partial\Sigma'$. Then $e(\Sigma \cup_\phi \Sigma', W \cup_\phi W') = e(\Sigma, W) + e(\Sigma', W')$. Note that $\Sigma \cup_\phi \Sigma'$ is a closed surface in a closed oriented manifold.*

Proof The first assertion directly follows from the definition. For the second one, we choose a set of parallels $\tilde{l}_1, \dots, \tilde{l}_k$ for $\partial\Sigma$ and a section s of the normal bundle of Σ bounding them. We choose a section s' of the normal bundle of Σ' bounding $\phi(\tilde{l}_1), \dots, \phi(\tilde{l}_k)$. We can glue s and s' along ϕ, this gives a section s'' of the normal bundle of $\Sigma' \cup_\phi \Sigma$. Hence, we have

$$\begin{aligned} e(\Sigma' \cup_\phi \Sigma) &= (\Sigma' \cup_\phi \Sigma) \cap s''(\Sigma' \cup_\phi \Sigma) \\ &= \Sigma \cap s(\Sigma) + \Sigma' \cap s'(\Sigma') \\ &= \Sigma \cap s(\Sigma) - \sum_{i=1}^{k} \mathrm{lk}_{\partial W}(l_i, \tilde{l}_i) + \sum_{i=1}^{k} \mathrm{lk}_{\partial W}(l_i, \tilde{l}_i) + \Sigma' \cap s(\Sigma') \\ &= \Sigma \cap s(\Sigma) - \sum_{i=1}^{k} \mathrm{lk}_{\partial W}(l_i, \tilde{l}_i) - \sum_{i=1}^{k} \mathrm{lk}_{\partial W'}(\phi(l_i), \phi(\tilde{l}_i)) + \Sigma' \cap s(\Sigma') \\ &= e(\Sigma, W) + e(\Sigma', W'). \end{aligned}$$

□

3.2 The invariants

We start by introducing some notations which will be used throughout the rest of the paper.

Notation 3.8 Let F be a Klein foam with boundary properly embedded in $\mathbb{B}^4$. Recall that we have a Klein cover $\pi\colon W_F \to W_F/D_4 \simeq \mathbb{B}^4$. For every element i in D_4^*, we denote by

- $\widehat{F}_i$ the fixed-points surface of the diffeomorphism i in W_F,
- W_F/i the manifold $W_F/\langle i \rangle$ (where $\langle i \rangle$ is the subgroup of order two generated by i),
- $\widetilde{F}_i^i$ the image of $\widehat{F}_i$ in W_F/i,
- $\widetilde{F}_{jk}^i$ the image of $\widehat{F}_j \cup \widehat{F}_k$ in W_F/i, where j and k denote the two other elements of D_4^*,
- F_{jk} the image of $\widehat{F}_j \cup \widehat{F}_k$ in $\mathbb{B}^4$.

These notations are summarized in Figure 4.

Note that $\widetilde{F}_i^i$ and $\widetilde{F}_{jk}^i$ are properly embedded surfaces in W_F/i and that F_{jk} is a properly embedded surface in $\mathbb{B}^4$: in fact, this is the union of the facets of F colored by j and k. Moreover, the Klein cover π is the composition of two double branched covers $W_F \to W_F/i \to \mathbb{B}^4$. The last one can be seen as the double branched cover of $\mathbb{B}^4$ along F_{jk} and the first one as a double branched cover of W_F/i along $\widetilde{F}_i^i$.

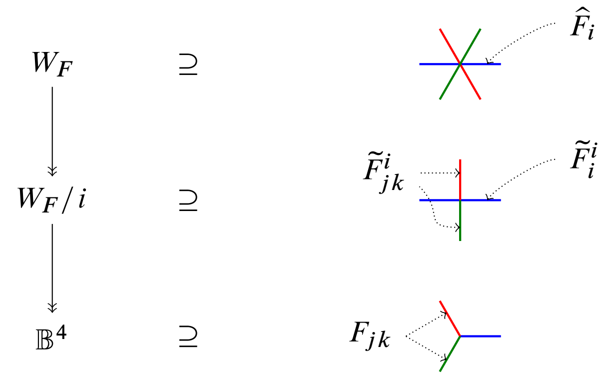


Figure 4: Decomposition of a Klein cover into two double branched covers

Denoting by a, b and c the three elements of D_4^*, the following diagram explains the three decompositions of the Klein cover of $\mathbb{B}^4$ on F in double branched covers:

$$\begin{array}{ccccc} & & W_F & & \\ & \swarrow & \downarrow & \searrow & \\ W_F/a & & W_F/b & & W_F/c \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{B}^4 & & \end{array}$$

Moreover, we will use the same system of notations

- for a Klein foam without boundary E embedded in $\mathbb{S}^4$,
- for a Klein graph Γ embedded in $\mathbb{S}^3$.

In this last case, the Klein cover M_Γ has the following decomposition in double branched covers:

$$\begin{array}{ccccc} & & M_\Gamma & & \\ & \swarrow & \downarrow & \searrow & \\ M_\Gamma/a & & M_\Gamma/b & & M_\Gamma/c \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{S}^3 & & \end{array}$$

Remarks 3.9 (1) The subgraphs Γ_{ab}, Γ_{bc} and Γ_{ca} are links in $\mathbb{S}^3$. For $\{i,j,k\} = \{a,b,c\}$, $\widetilde{\Gamma}^i_{jk}$ and $\widetilde{\Gamma}^i_i$ are links in M_Γ/i and $\widehat{\Gamma}_a$, $\widehat{\Gamma}_b$ and $\widehat{\Gamma}_c$ are links in M_Γ.

(2) If Γ is 3–Hamiltonian, Γ_{ab}, Γ_{bc} and Γ_{ca} are knots. In this case M_Γ/a, M_Γ/b and M_Γ/c are rational homology spheres whose first homology group has no 2–torsion (see for instance [21, page 213]).

Finally, we define normal Euler numbers for Klein foams as follows:

Definition 3.10 Let F be a Klein foam with boundary in $(\mathbb{B}^4, \mathbb{S}^3)$. The *weak normal Euler number* $e(F)$ of F is given by the formula

$$e(F) = e(F_{ab}) + e(F_{bc}) + e(F_{ca}).$$

If ∂F is 3–Hamiltonian, we define the *strong normal Euler number* $\widetilde{e}(F)$ of F by

$$\widetilde{e}(F) = e(\widetilde{F}_a^a) + e(\widetilde{F}_b^b) + e(\widetilde{F}_c^c).$$

We are now able to state the main theorem and to define our invariants. Recall that for a knotted Klein graph Γ in $\mathbb{S}^3$ there exists a spanning foam for Γ in $\mathbb{B}^4$ (Proposition 2.4).

Theorem 3.11 *Let Γ be a knotted 3–Hamiltonian Klein graph in $\mathbb{S}^3$. Let F be a spanning foam for Γ in $\mathbb{B}^4$. We denote the signature of the 4–manifold W_F by $\sigma(W_F)$. Then*

- *the integer $\sigma(\Gamma) := \sigma(W_F) + \frac{1}{2}e(F)$,*
- *the rational $\widetilde{\sigma}(\Gamma) := \sigma(W_F) + \frac{1}{2}\widetilde{e}(F)$,*
- *the rational $\delta(\Gamma) := \frac{1}{2}\widetilde{e}(F) - \frac{1}{2}e(F)$*
- *and the rationals*

$$\delta_{ab}(\Gamma) := \tfrac{1}{4}e(\widetilde{F}_a^a) + \tfrac{1}{4}e(\widetilde{F}_b^b) - \tfrac{1}{2}e(F_{ab}),$$
$$\delta_{bc}(\Gamma) := \tfrac{1}{4}e(\widetilde{F}_b^b) + \tfrac{1}{4}e(\widetilde{F}_c^c) - \tfrac{1}{2}e(F_{bc}),$$
$$\delta_{ca}(\Gamma) := \tfrac{1}{4}e(\widetilde{F}_c^c) + \tfrac{1}{4}e(\widetilde{F}_a^a) - \tfrac{1}{2}e(F_{ca})$$

only depend on the knotted Klein graph Γ. (These quantities are called *signature invariants of* Γ.)

Of course, we have the following relations between these invariants:

$$\delta(\Gamma) = \widetilde{\sigma}(\Gamma) - \sigma(\Gamma) \quad \text{and} \quad \delta(\Gamma) = \delta_{ab}(\Gamma) + \delta_{bc}(\Gamma) + \delta_{ca}(\Gamma).$$

Before relating our invariants with signatures of knots and links, let us recall the following definition, due to Gilmer [4; 5]:

Definition 3.12 Let M be a rational homology sphere with a first homology group of odd order and L an (unoriented) link in M. We can find an oriented four-dimensional manifold W and a surface F such that the pair (W, F) bounds r copies of (M, L) for a positive integer r. Then the *signature* $\xi(L)$ of L is defined by the formula

$$\xi(L) = \frac{1}{r}\Big(\sigma(W_F) - 2\sigma(W) + \frac{1}{2}e(F)\Big),$$

where W_F denotes the double branched cover of W along F.

Remark 3.13 This definition is given in [4] for an oriented link and in a more general setting, where ξ depends on a choice of a branched covering of M along L. With our assumptions, such a choice is unique and ξ does not depend on the orientation of L (see [5, page 295]). If M is $\mathbb{S}^3$ this definition agrees with the signature of an unoriented link given by Murasugi [19].

Proposition 3.14 *Let Γ be a knotted 3–Hamiltonian Klein graph in $\mathbb{S}^3$. The following relations hold:*

$$\begin{aligned} \sigma(\Gamma) &= \sigma(\Gamma_{ab}) + \sigma(\Gamma_{bc}) + \sigma(\Gamma_{ca}), \\ \tilde{\sigma}(\Gamma) &= \xi(\tilde{\Gamma}_a^a) + \xi(\tilde{\Gamma}_b^b) + \xi(\tilde{\Gamma}_c^c), \\ \delta_{ij}(\Gamma) &= \tfrac{1}{2}\xi(\tilde{\Gamma}_i^i) + \tfrac{1}{2}\xi(\tilde{\Gamma}_j^j) - \sigma(\Gamma_{ij}) \quad \text{for all } i \neq j \text{ in } D_4^* \end{aligned}$$

Note that $\tilde{\Gamma}_a^a$, $\tilde{\Gamma}_b^b$ and $\tilde{\Gamma}_c^c$ are links *in rational homology spheres. In these formulas $\sigma(K)$ denotes the signature of a knot K in $\mathbb{S}^3$.*

The proofs of Theorem 3.11 and Proposition 3.14 are postponed to Section 3.4.

Remarks 3.15 (1) In some cases, this proposition enables us to compute the signature of the knots $\tilde{\Gamma}_i^i$ by taking advantage of some symmetries. See Section 4 for an example.

(2) The invariants σ, $\tilde{\sigma}$ and δ_{ij} are additive with respect to connected sum along a vertex (see Remark 2.2(5)). For σ, it follows from the additivity of the signature of knots with respect to connected sum. For the invariants δ_{ij}, it follows from the additivity of the normal Euler numbers of surfaces. From this, one easily deduces the additivity of $\tilde{\sigma}$.

The proofs of the results of this section require some properties on normal Euler numbers and coverings, which are described in the next section.

3.3 Normal Euler numbers and double branched covers

In this part, we explain how normal Euler numbers behave when considering double branched coverings. We consider three different situations. The first two appear in Lemma 3.16 and are classical. The third one is quite specific to our situation. It is given in Lemma 3.18.

Lemma 3.16 (1) *Let Σ_1 be a surface embedded in a 4–manifold W and $\widetilde{W}$ be the double branched cover of W along Σ_1. We define $\widetilde{\Sigma}_1$ to be the preimage of Σ_1 in $\widetilde{W}$. We have*

$$e(\Sigma_1) = 2e(\widetilde{\Sigma}_1).$$

(2) *Suppose that another surface Σ_2 intersects Σ_1 transversely. We denote by $\widetilde{\Sigma}_2$ the preimage of Σ_2 in $\widetilde{W}$. We have*

$$e(\Sigma_2) = \tfrac{1}{2}e(\widetilde{\Sigma}_2).$$

Proof (1) We denote by $\pi\colon \widetilde{W} \to W$ the canonical projection. We choose a section $\widetilde{s}$ of the normal bundle of $\widetilde{\Sigma}_1$ transverse to the trivial section $\widetilde{s}_0$. The section $\widetilde{s}$ induces via π a section s of the normal bundle of Σ. This section s is not transverse to the normal section s_0 of Σ. However, there are only finitely many intersection points. In a neighborhood of every intersection point, the double branched cover can be written in an appropriate chart as

$$\pi\colon \mathbb{B}^4 := \{(z_1,z_2) \in \mathbb{C}^2 \mid |z_1| \le 1,\ |z_2| \le 1\} \to \mathbb{B}^4, \quad (z_1,z_2) \mapsto (z_1, z_2^2),$$

with $\widetilde{\Sigma}_1 = \{(z_1,0) \mid |z_1| \le 1\}$ and[3] $\widetilde{s}(\widetilde{\Sigma}_1) = \{(z_1,z_1) \mid |z_1| \le 1\}$. Locally, the intersection number of $\widetilde{\Sigma}_1$ with $\widetilde{s}$ is ± 1, the sign depending on the orientation of $\mathbb{B}^4$. With this setting, $\Sigma_1 = \{(z_1,0) \mid |z_1| \le 1\}$ and $s = \{(z_1,z_1^2) \mid |z_1| \le 1\}$. The local intersection number of Σ_1 with s is ± 2, with the same sign. We obtain the result by summing over all intersection points.

(2) We consider a section s of the normal bundle of Σ_2 transverse to the null section. We may assume that $\Sigma_2 \cap s(\Sigma_2)$ and Σ_1 are disjoint. The section s induces a section $\widetilde{s}$ of the normal bundle of $\widetilde{\Sigma}_2$. The section $\widetilde{s}$ is transverse to the null section. Moreover, the map π induces a two-to-one correspondence between $\widetilde{\Sigma}_2 \cap \widetilde{s(\Sigma_2)}$ and $\Sigma_2 \cap s(\Sigma_2)$ which respects the intersection signs. □

[3] When working in a chart, we identify a section of the normal bundle of a surface with its image in the ambient 4–manifold by an appropriate exponential map. In particular, the null section is identified with the surface itself.

Definition 3.17 Let Σ and Σ' be two surfaces embedded in a 4–manifold W. We say that Σ and Σ' intersect *quasitransversely* if for all x in $\Sigma \cap \Sigma'$, there exists a neighborhood U of x in W such that

$$(U, \Sigma \cap U, \Sigma' \cap U, x) \simeq \big(\mathbb{R}^4, \mathbb{R}^2 \times \{(0,0)\}, \mathbb{R} \times \{0\} \times \mathbb{R} \times \{0\}, \{(0,0,0,0)\}\big).$$

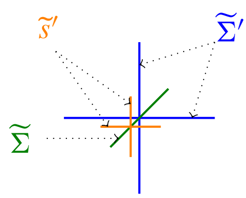
Lemma 3.18 *Let Σ be a surface embedded in an oriented 4–manifold W. Let $\widetilde{W}$ a double branched cover of W along Σ. Suppose that a surface Σ' in W intersects Σ quasitransversely. Let $\widetilde{\Sigma}'$ be the preimage of Σ' in $\widetilde{W}$. We have*

$$e(\Sigma') = \tfrac{1}{2} e(\widetilde{\Sigma}').$$

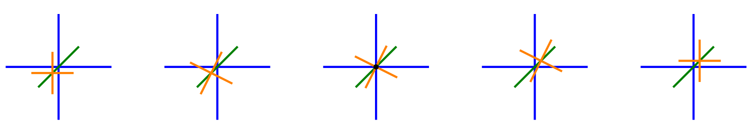
(Note that the surface $\widetilde{\Sigma}'$ is not embedded.)

Proof As a double branched cover, the manifold $\widetilde{W}$ is naturally endowed with an involution. We denote it by τ. The intersection of Σ and Σ' is a finite collection $\mathcal{C}$ of circles. Its preimage in W is the intersection of $\widetilde{\Sigma}'$ and $\widetilde{\Sigma}$, the preimage of Σ in W.

We construct a τ–invariant section $\widetilde{s}'$ of the normal bundle of $\widetilde{\Sigma}'$ in a tubular neighborhood of the preimage of $\mathcal{C}$ in $\widetilde{W}$. In order to do this, we fix a basepoint x_C on each circle C and define the section locally. This can be extended along the circle:

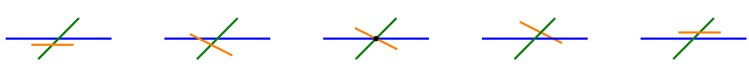


When one returns to the basepoint x_C, we may face a gluing problem. In that case we complete the section as described in the picture:



This yields a pair of transverse intersection points of $\widetilde{s}'$ with the null section of the normal bundle.

The image s' of this (incomplete) section gives an (incomplete) transverse section of the normal bundle of Σ' transverse to the trivial section. The image of the neighborhoods of the intersection points is given by:



Hence, to each pair of intersection points of $\widetilde{\Sigma}'$ and $\widetilde{s}'$ in $\widetilde{W}$ corresponds a single intersection point of Σ' and s' in W with the same sign. We extend the section s' to a complete section of the normal bundle of Σ'. Its preimage extends $\widetilde{s}'$ into a complete section of the normal bundle of $\widetilde{\Sigma}'$. What happens far from the circles is given by Lemma 3.16(2). □

3.4 Proof of the main theorem

The principle of the proof is inspired by [7; 8], which use the 4–dimensional point of view of Kauffman and Taylor [11] on signature of knots. This uses the G–signature theorem of Atiyah and Singer [1].

Following [6], let us recall the definition of signature in the context of 4–dimensional manifolds. Let W be a compact oriented 4–manifold (with or without boundary). The intersection form of W induces a hermitian form φ on $H_2(W,\mathbb{C})$. Take a finite group G which acts on W by orientation-preserving diffeomorphisms. Then, for every g in G, g_* preserves the form φ on $H_2(W,\mathbb{C})$ and we may choose a G–invariant, φ–orthogonal decomposition $H_2(W,\mathbb{C}) = H^+ \oplus H^- \oplus H^0$ such that φ is positive definite on H^+, negative definite on H^- and zero on H^0. The *signature of* g is defined by

$$\sigma(W,g) := \sigma(H_2(W), g_*) = \mathrm{tr}(g_{*|H^+}) - \mathrm{tr}(g_{*|H^-}),$$

where tr denotes the trace of linear endomorphisms.

Remark 3.19 The signature $\sigma(W)$ of a manifold W is the signature of the identity of W.

The following formula connects the sum of the signature of the elements of G and the signature of the quotient manifold W/G and is known in the literature as a standard transfer argument. See [9, Equation (5)] when G is a cyclic group.

Proposition 3.20 *Let W be a compact oriented 4–manifold endowed with an action of a finite group G by orientation-preserving diffeomorphisms. Then we have*

$$\sum_{g\in G} \sigma(W,g) = |G|\sigma(W/G).$$

Sketch of the proof Let π be the projection $\pi\colon W \to W/G$. Then there exist cellular decompositions of W and W/G compatible with the action of G such that π is cellular. Moreover, at the level of chains one can define a map which assigns to each 2–chain of W/G the sum of its preimages by π_*. This induces a transfer map $t\colon H_2(W/G) \to H_2(W)^G \hookrightarrow H_2(W)$ respecting intersections forms in the sense that $\varphi_W(tx, ty) = |G|\varphi_{W/G}(x, y)$. Moreover, we have $\pi_* \circ t = |G|\mathrm{id}_{H_2(W/G)}$ and $t \circ \pi_* = \sum_{g\in G} g_*$ on $H_2(W)$. It follows that

$$\begin{aligned}\sum_{g\in G} \sigma(W, g) &= \sum_{g\in G} \mathrm{tr}(g_{*|H^+}) - \mathrm{tr}(g_{*|H^-})\\ &= \mathrm{tr}(t \circ \pi_{*|H^+}) - \mathrm{tr}(t \circ \pi_{*|H^-})\\ &= \mathrm{tr}(t \circ \pi_{*|(H^+)^G}) - \mathrm{tr}(t \circ \pi_{*|(H^-)^G})\\ &= |G|(\dim(H^+)^G - \dim(H^-)^G)\\ &= |G|\big(\dim(H_2(W/G))^+ - \dim(H_2(W/G)^-)\big)\\ &= |G|\sigma(W/G). \end{aligned}$$ □

The G–signature theorem for 4–manifolds [6, Theorem 2] takes an especially simple form for involutions:

Theorem 3.21 *Let W be an oriented closed 4–manifold and τ be an orientation-preserving smooth involution. Then the set of fixed points of τ consists of a surface F_τ and some isolated points, and we have*

$$\sigma(W, \tau) = e(F_\tau).$$

Remark 3.22 If τ is not an involution (but still has finite order), the statement is a bit more complicated and there is some contribution coming from the isolated fixed points.

Proof of Theorem 3.11 We first prove the invariance of σ and $\widetilde{\sigma}$. This implies the invariance of δ. The invariance of the δ_{ij} is postponed to the proof of Proposition 3.14.

Let us consider two spanning foams $(\mathbb{B}^4_1, F_1)$ and $(\mathbb{B}^4_2, F_2)$ for $(\mathbb{S}^3, \Gamma)$ (the indices of the $\mathbb{B}^4$ are simply labels with no geometrical meaning). Then we can consider the gluing $(\mathbb{S}^4, E) = (\mathbb{B}^4_1, F_1) \cup_{(\mathbb{S}^3,\Gamma)} (-\mathbb{B}^4_2, F_2)$. Lifting to Klein covers, we get the gluing $V_E = W_{F_1} \cup_{M_\Gamma} (-W_{F_2})$. Novikov additivity [14, Theorem 5.3] gives

$$\sigma(V_E) = \sigma(W_{F_1}) - \sigma(W_{F_2}). \tag{1}$$

The action of the Klein group D_4 on V_E defines a collection of signatures which, thanks to Proposition 3.20, satisfies the relation

$$\sigma(V_E) + \sum_{g\in D_4^*} \sigma(V_E, g) = 4\sigma(\mathbb{S}^4) = 0.$$

Together with Theorem 3.21 this gives

$$-\sigma(V_E)=\sigma(V_E,a)+\sigma(V_E,b)+\sigma(V_E,c)=e(\widehat{E}_a)+e(\widehat{E}_b)+e(\widehat{E}_c). \tag{2}$$

In the diagram

$$\begin{array}{ccccc} & & V_E & & \\ & \swarrow & \downarrow & \searrow & \\ V_E/a & & V_E/b & & V_E/c \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{S}^4 & & \end{array}$$

the three top arrows are double branched covers along $\widetilde{E}^a_a$, $\widetilde{E}^b_b$ and $\widetilde{E}^c_c$, respectively. So, applying Lemma 3.16(1), we get

$$-\sigma(V_E)=\tfrac{1}{2}e(\widetilde{E}^a_a)+\tfrac{1}{2}e(\widetilde{E}^b_b)+\tfrac{1}{2}e(\widetilde{E}^c_c).$$

Note that in the gluing $V_E/a=(W_{F_1}/a)\cup_{M_\Gamma/a}(-W_{F_2}/a)$ the 3–manifold M_Γ/a is a rational homology sphere (see Remarks 3.15) so that, thanks to Proposition 3.7 and Definition 3.10, we obtain

$$\begin{aligned}-\sigma(V_E)&=\tfrac{1}{2}\big(e((\widetilde{F_1})^a_a)-e((\widetilde{F_2})^a_a)+e((\widetilde{F_1})^b_b)-e((\widetilde{F_2})^b_b)+e((\widetilde{F_1})^c_c)-e((\widetilde{F_2})^c_c)\big)\\ &=\tfrac{1}{2}\widetilde{e}(F_1)-\tfrac{1}{2}\widetilde{e}(F_2).\end{aligned}$$

From this and relation (1) we deduce

$$\sigma(W_{F_1})+\tfrac{1}{2}\widetilde{e}(F_1)=\sigma(W_{F_2})+\tfrac{1}{2}\widetilde{e}(F_2).$$

This proves the invariance of $\widetilde{\sigma}$.

On the other hand, starting again from relation (2) we get

$$\begin{aligned}-\sigma(V_E)&=\tfrac{1}{2}(e(\widehat{E}_a)+e(\widehat{E}_b))+\tfrac{1}{2}(e(\widehat{E}_b)+e(\widehat{E}_c))+\tfrac{1}{2}(e(\widehat{E}_c)+e(\widehat{E}_a))\\ &=\tfrac{1}{2}e(\widehat{E}_a\cup\widehat{E}_b)+\tfrac{1}{2}e(\widehat{E}_b\cup\widehat{E}_c)+\tfrac{1}{2}e(\widehat{E}_c\cup\widehat{E}_a)\\ &=e(\widetilde{E}^c_{a,b})+e(\widetilde{E}^a_{b,c})+e(\widetilde{E}^b_{c,a})\\ &=\tfrac{1}{2}(e(E_{ab})+e(E_{bc})+e(E_{ca}))\\ &=\tfrac{1}{2}e(F_1)-\tfrac{1}{2}e(F_2).\end{aligned}$$

The third and fourth equalities follow from Lemmas 3.18 and 3.16, while the last one holds because of Proposition 3.7 and Definition 3.10. Again from this and relation (1) we deduce

$$\sigma(W_{F_1})+\tfrac{1}{2}e(F_1)=\sigma(W_{F_2})+\tfrac{1}{2}e(F_1).$$

This proves the invariance of σ. □

Before proving Proposition 3.14, we need the following lemma:

Lemma 3.23 *Let F be a foam with boundary in $\mathbb{B}^4$; then we have*

$$\sigma(W_F) = \sigma(W_F/a) + \sigma(W_F/b) + \sigma(W_F/c).$$

Proof Using Proposition 3.20 for the manifold W_F endowed with the action of the Klein group D_4, we get

$$\sigma(W_F) + \sigma(W_F, a) + \sigma(W_F, b) + \sigma(W_F, c) = 4\sigma(\mathbb{B}^4) = 0.$$

Now the same proposition applied to the action of the involution a on W_F gives

$$\sigma(W_F) + \sigma(W_F, a) = 2\sigma(W_F/a)$$

and analogous formulas for the involutions b and c. The result follows easily. □

Proof of Proposition 3.14 Let F be an arbitrary spanning foam for Γ in $\mathbb{B}^4$. The surface F_{ab} is a spanning surface for the knot Γ_{ab}. Thanks to [7, Theorem 2 and Corollary 5], we have

$$\sigma(\Gamma_{ab}) = \sigma(W_F/c) + \tfrac{1}{2}e(F_{ab}). \tag{3}$$

We obviously have analogous formulas for $\sigma(\Gamma_{bc})$ and $\sigma(\Gamma_{ca})$. This gives

$$\begin{aligned}\sigma(\Gamma_{ab}) + \sigma(\Gamma_{bc}) + \sigma(\Gamma_{ca})& \\ &= \sigma(W_F/c) + \sigma(W_F/a) + \sigma(W_F/b) + \tfrac{1}{2}(e(F_{ab}) + e(F_{bc}) + e(F_{ca})) \\ &= \sigma(W_F) + \tfrac{1}{2}e(F) \\ &= \sigma(\Gamma),\end{aligned}$$

where the second equality comes from Lemma 3.23. This is exactly the first part of Proposition 3.14.

The second part is in some sense a generalization. Here $\widetilde{\Gamma}_a^a$ is a link in M_Γ/a and $\widetilde{F}_a^a$ is a spanning surface for this link (living in W_F/a). Recall that M_Γ/a is a rational homology sphere whose first homology group has odd order. Hence, Definition 3.12 gives

$$\xi(\widetilde{\Gamma}_a^a) = \sigma(W_F) - 2\sigma(W_F/a) + \tfrac{1}{2}e(\widetilde{F}_a^a). \tag{4}$$

Doing the sum with the analogous formulas for $\xi(\widetilde{\Gamma}_b^b)$ and $\xi(\widetilde{\Gamma}_c^c)$ we get

$$\begin{aligned}
&\xi(\widetilde{\Gamma}_a^a)+\xi(\widetilde{\Gamma}_b^b)+\xi(\widetilde{\Gamma}_c^c)\\
&\quad=3\sigma(W_F)-2(\sigma(W_F/a)+\sigma(W_F/b)+\sigma(W_F/c))+\tfrac{1}{2}(e(\widetilde{F}_a^a)+e(\widetilde{F}_b^b)+e(\widetilde{F}_c^c))\\
&\quad=\sigma(W_F)+\tfrac{1}{2}\widetilde{e}(F)\\
&\quad=\widetilde{\sigma}(\Gamma),
\end{aligned}$$

where the second equality follows from Lemma 3.23.

It remains to show that

$$\delta_{ij}(\Gamma)=\tfrac{1}{2}\xi(\widetilde{\Gamma}_i^i)+\tfrac{1}{2}\xi(\widetilde{\Gamma}_j^j)-\sigma(\Gamma_{ij})\quad\text{for all } i\neq j \text{ in } D_4^*.$$

Note that this formula implies the invariance of the δ_{ij} and therefore completes the proof of Theorem 3.11.

By symmetry it is enough to consider $i=a$ and $j=b$. We have

$$\begin{aligned}
&\tfrac{1}{2}\xi(\widetilde{\Gamma}_a^a)+\tfrac{1}{2}\xi(\widetilde{\Gamma}_b^b)-\sigma(\Gamma_{ab})\\
&\quad=\tfrac{1}{2}\big(\sigma(W_F)-2\sigma(W_F/a)+\tfrac{1}{2}e(\widetilde{F}_a^a)+\sigma(W_F)-2\sigma(W_F/b)+\tfrac{1}{2}e(\widetilde{F}_b^b)\big)\\
&\qquad\qquad-\big(\sigma(W_F/c)+\tfrac{1}{2}e(F_{ab})\big)\\
&\quad=\sigma(W_F)-(\sigma(W_F/a)+\sigma(W_F/b)+\sigma(W_F/c))+\tfrac{1}{4}e(\widetilde{F}_a^a)+\tfrac{1}{4}e(\widetilde{F}_b^b)-\tfrac{1}{2}e(F_{ab})\\
&\quad=\delta_{ab}(\Gamma).
\end{aligned}$$

The last equality follows from Lemma 3.23. □

4 An example

In this section we compute our signature invariants on Γ, the Kinoshita knotted graph. We describe a spanning foam F for Γ by a movie given in Figure 5. In between the successive frames of the movie, one has a canonical foamy cobordism. The spanning foam F of Γ is obtained by composing all these cobordisms together and finally gluing this foam with a trivial half-theta foam. This gives a foam whose boundary is Γ.

Since all sublinks of Γ are trivial, we have $\sigma(\Gamma)=0$. In order to determine the other signature invariants it is enough to compute $e(F_{ab})$, $e(F_{bc})$, $e(F_{ac})$, $e(\widetilde{F}_a^a)$, $e(\widetilde{F}_b^b)$ and $e(\widetilde{F}_c^c)$.

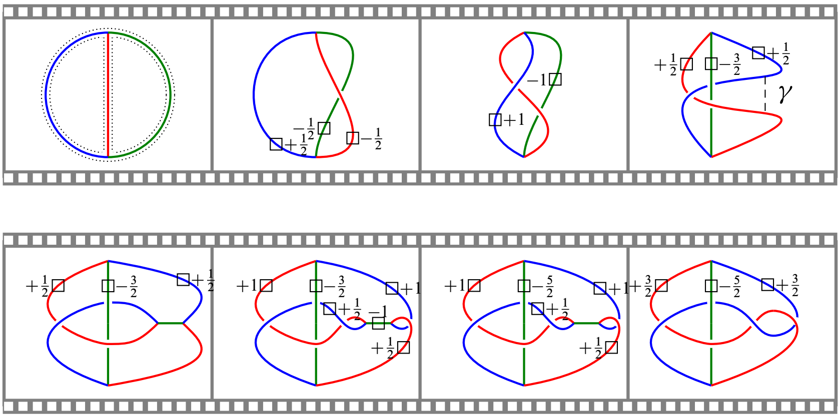


Figure 5: Movie describing a spanning foam for the Kinoshita knotted graph

As explained in Section 3.1, normal Euler numbers of surfaces with boundary can be computed via some linking numbers. We first inspect the surfaces F_{ij}.

The standard theta diagram can be seen as a framed graph equipped with three specific parallels (see the first frame in Figure 5). Following these parallels during the various steps of the movie gives for each pair $\{i, j\}$ a *nonvanishing* section of the normal bundle of the surface (with boundary) F_{ij}. The boxes with integers or half-integers drawn in the movie encode twists or half-twists between an edge and its parallel.

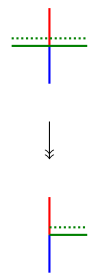
The normal Euler number $e(F_{ij})$ is equal to $-l_{ij}$, where l_{ij} is the linking number in $\mathbb{S}^3$ of the knot $\Gamma_{ij} = \partial F_{ij}$ with its parallel (see Definition 3.5 and Remark 3.6(3)), that is, the sum of all values of the boxes in the last frame plus the number of crossings counted algebraically.

Note that between frames 6 and 7, a box on a green (c) edge travels to the other green edge (because we need to unzip the first edge). This does not impact the validity of the computation.

We have

- $e(F_{ab}) = -\left(1 + \frac{3}{2} + \frac{3}{2}\right) = -4$,
- $e(F_{bc}) = e(F_{ac}) = -\left(1 + \frac{3}{2} - \frac{5}{2}\right) = 0$.

We now explain how to compute $e(\widetilde{F}_i^i)$ for i in $\{a, b, c\}$. As we will see, it is possible to deduce $e(\widetilde{F}_a^a)$ and $e(\widetilde{F}_b^b)$ from $e(\widetilde{F}_c^c)$ by using symmetries. For computing $e(\widetilde{F}_c^c)$, we consider a section of the normal bundle of $\widetilde{F}_c^c$ (in W_F/c) transverse to the trivial section. Such a section can be read on the movie. Indeed, consider each step of the movie as a framed graph where every edge colored by c comes with a parallel attached to its adjacent edges.[4] The preimage of these parallels in the double cover along the knot consisting of edges colored by a and b is a genuine parallel of the preimage of the edge, as shown in the following picture:



Following these parallels during the movie gives an appropriate section of $\widetilde{F}_c^c$ in W_F/c. The surface $\widetilde{F}_c^c$ has two connected components: a knotted sphere S coming from the small horizontal green edge on frames 5, 6 and 7 and another component Σ whose boundary is the knot $\widetilde{\Gamma}_c^c$.

Let us first deal with the knotted sphere S. We isolate the interesting part of the movie and depict it in Figure 6. This shows that we have $e(S) = +2$.

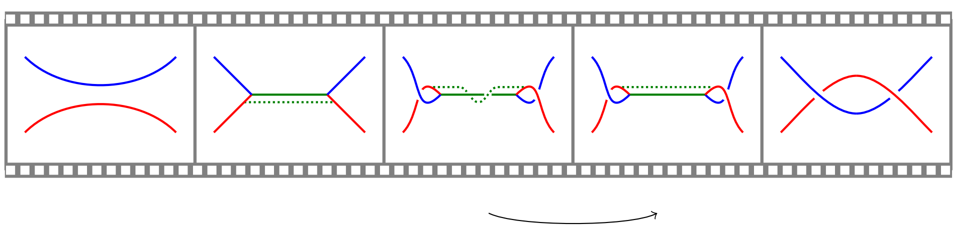


Figure 6: Movie for a positive clasp. The indicated move introduces two positive singular points in the double branched cover.

In order to compute $e(\Sigma)$, we consider the section of Σ given by the preimage of the parallels. This section does not intersect Σ, hence $e(\Sigma)$ is equal to $-\ell_c$, where ℓ_c is equal to the linking number of the knot $\widetilde{\Gamma}_c^c$ with the preimage in $\partial W_F/c$ of its preferred parallel of the edge of Γ colored by c.

For computing ℓ_c, we use the following theorem:

[4] Here the apparent choice of side has no effect.

Theorem 4.1 (Theorem 1.1 of [20], restricted to our case) *Let K_1 and K_2 be two knots in $\mathbb{S}^3$, $J = (J_1, \dots J_l)$ a link in $\mathbb{S}^3$ disjoint from K_1 and K_2, and $r_1, \dots r_l$ some rational numbers. Let M be the manifold obtained by Dehn surgery along J with coefficients $r_1, \dots r_l$. Suppose that M is a rational homology sphere; then*

$$\begin{aligned}\mathrm{lk}_M(K_1,K_2) - \mathrm{lk}_{\mathbb{S}^3}(K_1,K_2) \\ = -(\mathrm{lk}_{\mathbb{S}^3}(K_1,J_1),\dots,\mathrm{lk}_{\mathbb{S}^3}(K_1,J_l))G^{-1}(\mathrm{lk}_{\mathbb{S}^3}(K_2,J_1),\dots,\mathrm{lk}_{\mathbb{S}^3}(K_2,J_l))^t,\end{aligned}$$

where $G = (g_{ij})_{1\le i\le l, 1\le j\le l}$ is the $l \times l$ matrix defined by

$$g_{ij} = \begin{cases} \mathrm{lk}_{\mathbb{S}^3}(J_i, J_j) & \text{if } i \neq j, \\ r_i & \text{if } i = j. \end{cases}$$

Indeed, Montesinos' trick [18, Section 2] tells us that if one changes a knot by the local moves

positive clasp move negative clasp move

the double branched cover of the new knot is obtained by a surgery along $\widetilde{\gamma}$ with coefficient $-\frac{1}{2}$ for a positive clasp move and $+\frac{1}{2}$ for a negative clasp move, where $\widetilde{\gamma}$ is the preimage of γ in the double branched cover. These coefficients are given in the canonical[5] basis longitude/meridian of $\widetilde{\gamma}$.

Since the graph (denoted by Γ') in the 4th frame is still a trivial theta graph, all double branched cover are diffeomorphic to $\mathbb{S}^3$. Figure 7 describes the double branched cover of $\mathbb{S}^3$ along Γ'_{ab}.

The knot $\widetilde{\gamma}$ is a framed trivial knot with framing +1. In Montesinos' trick, the surgery coefficient is $-\frac{1}{2}$ for the basis given by the framing. For the standard basis it is therefore $\frac{-1+2}{2} = \frac{1}{2}$. Moreover, we have $\mathrm{lk}_{\mathbb{S}^3}(\widetilde{\Gamma}'^c_c, \widetilde{\gamma}) = \pm 3$ (the sign depends on which orientation we take for $\widetilde{\gamma}$). Thanks to Theorem 4.1, we have $\ell_c = -(\pm 3)^2\left(\frac{1}{2}\right)^{-1} = -18$ and finally $e(\widetilde{F}^c_c) = 2 + 18 = 20$.

The remaining normal Euler numbers are $e(\widetilde{F}^a_a)$ and $e(\widetilde{F}^b_b)$.

First of all, the three bicolored knots of Γ are trivial, so that $\sigma(\Gamma_{ab}) = \sigma(\Gamma_{bc}) = \sigma(\Gamma_{ca}) = 0$. Moreover, because of the symmetry of Γ, the knots $\widetilde{\Gamma}^a_a$, $\widetilde{\Gamma}^b_b$ and $\widetilde{\Gamma}^c_c$ are

[5] The longitude is required to be the preimage of the dotted arc.

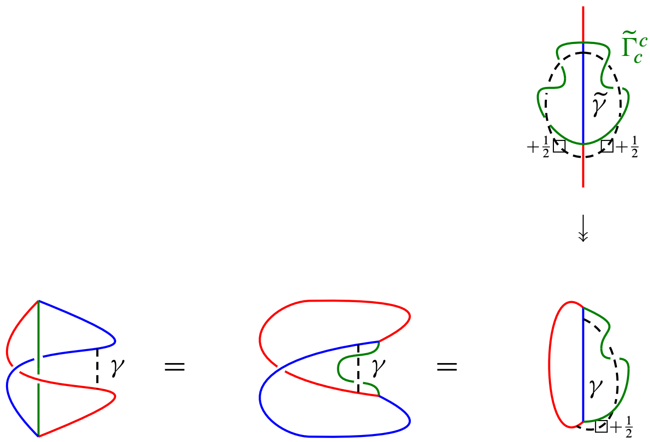


Figure 7

the same. Denote by s the signature of this knot. By symmetry of the movie relative to the colors a and b we have $e(\widetilde{F}_a^a) = e(\widetilde{F}_b^b)$. Denote by e this value. We have

$$\delta_{ab}(\Gamma) = \delta_{bc}(\Gamma) = \delta_{ca}(\Gamma) = \tfrac{1}{2}s + \tfrac{1}{2}s - 0 = s,$$
$$\delta_{ab}(\Gamma) = \tfrac{1}{4}e(\widetilde{F}_a^a) + \tfrac{1}{4}e(\widetilde{F}_b^b) - \tfrac{1}{2}e(F_{ab}) = \tfrac{1}{2}e + 2,$$
$$\delta_{bc}(\Gamma) = \tfrac{1}{4}e(\widetilde{F}_b^b) + \tfrac{1}{4}e(\widetilde{F}_c^c) - \tfrac{1}{2}e(F_{bc}) = \tfrac{1}{4}e + 5,$$

which implies that $e = 12$ and $s = 8$. In conclusion we have

$$\delta_{ab}(\Gamma) = \delta_{bc}(\Gamma) = \delta_{ac}(\Gamma) = 8, \quad \sigma(\Gamma) = 0, \quad \widetilde{\sigma}(\Gamma) = 24.$$

In particular, the Kinoshita graph is not trivial.

Remark 4.2 As an intermediate result, our computations give the signature of the knot $\widetilde{\Gamma}_a^a$ in $\mathbb{S}^3$ (which is 8). Note that we did not need to determine this knot, which happens to be the mirror image of 10_{124} (see for example [2]).

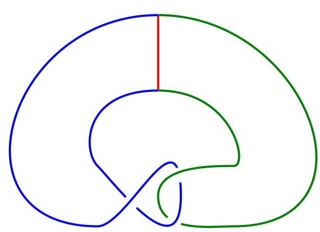


Figure 8: The trefoil theta graph

The same kind of computations can be done for other knotted graphs. For the trefoil theta graph (Figure 8), we find

$$\delta_{ab}(\Gamma) = \delta_{ca}(\Gamma) = \tfrac{4}{3}, \quad \sigma(\Gamma) = -2,$$
$$\delta_{bc}(\Gamma) = 4, \quad \widetilde{\sigma}(\Gamma) = \tfrac{14}{3}.$$

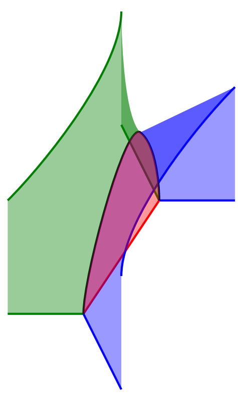
As for the Kinoshita graph, the symmetries give an intermediate result: the signature of $\sigma(\widetilde{\Gamma}_a^a)$, which is a knot in the lens space $L(3,1)$, is equal to $\frac{2}{3}$.

Appendix: Spanning foams

The aim of this section is to prove the following statement:

Proposition 2.4 *Let Γ be a knotted Klein graph in $\mathbb{S}^3$. There exists a spanning foam for Γ in $\mathbb{B}^4$.*

We need to introduce some elementary foamy cobordisms. A *zip* (resp. *unzip*) is a cobordism between Klein graphs which is locally given by the following picture read from top to bottom (resp. bottom to top):



A zip adds two vertices to the Klein graph, while an unzip removes two vertices from the Klein graph.

A *clasp* is a cobordism which locally performs a bicolor crossing change. It is described by the movie in Figure 6. Note that a clasp starts with a zip and ends with an unzip.

Lemma A.1 *Let Γ be a knotted Klein graph in $\mathbb{S}^3$. There exits a cobordism F in $\mathbb{S}^3 \times I$ from Γ to a Klein graph with no vertices* (ie a link).

Proof Let D be a diagram for Γ; we consider all edges[6] of D colored by a. With a sequence of Reidemeister moves, we can shrink these edges until they are not involved in any crossing. Then we unzip these edges. Before performing the unzip, we may have to twist one end of these edges with the (Rv1) Reidemeister move. □

Lemma A.2 *Let Γ be a knotted Klein graph in $\mathbb{S}^3$ with no vertices. This can be seen as a link L colored by D_4^*, which means that $L = L_a \cup L_b \cup L_c$. There exists a (foamy) cobordism from L to a link $L' = L'_a \cup L'_b \cup L'_c$, where the components L'_a, L'_b and L'_c are in three disjoint balls.*

Proof In order to unlink the components L_a, L_b and L_c of L, it is enough to perform crossing changes on bicolored crossings. These crossing changes can be achieved by clasps. □

Proof of Proposition 2.4 The construction is sketched in Figure 9. Thanks to Lemmas A.1 and A.2, we can construct a cobordism F in $\mathbb{S}^3 \times I$ from Γ to a link $L' = L'_a \cup L'_b \cup L'_c$, where the components L'_a, L'_b and L'_c are in disjoint balls. We now pick Seifert surfaces for L'_a, L'_b and L'_c, push them in $\mathbb{B}^4$ and concatenate them with F. This gives a spanning foam for Γ. □

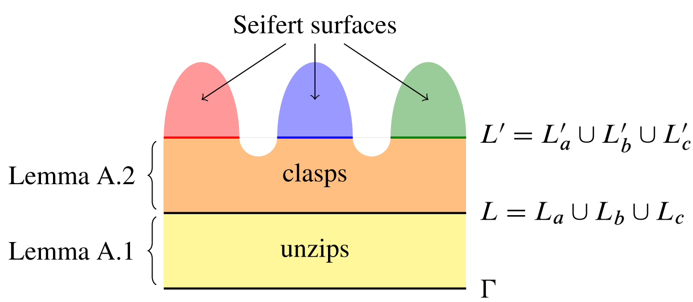


Figure 9: The construction of a spanning foam for a knotted Klein graph Γ

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[6] Here, we mean *real* edges, ie not circles.

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