Algebraic & Geometric Topology

Volume 18 (2018)

Issue 7 (pages 3749–4373)

ALGEBRAIC & GEOMETRIC TOPOLOGY

msp.org/agt

EDITORS

PRINCIPAL ACADEMIC EDITORS

John Etnyre
etnyre@math.gatech.edu
Georgia Institute of Technology

Kathryn Hess
kathryn.hess@epfl.ch
École Polytechnique Fédérale de Lausanne

MANAGING EDITOR

Colin Rourke
agt@msp.warwick.ac.uk
University of Warwick

BOARD OF EDITORS

Matthew Ando	University of Illinois mando@math.uiuc.edu	Tsuyoshi Kobayashi	Nara Women's University tsuyoshi09@gmail.com
Julie Bergner	University of Virginia jeb2md@eservices.virginia.edu	Christine Lescop	Université Joseph Fourier lescop@ujf-grenoble.fr
Joan Birman	Columbia University jb@math.columbia.edu	Robert Lipshitz	University of Oregon lipshitz@uoregon.edu
Steven Boyer	Université du Québec à Montréal cohf@math.rochester.edu	Dan Margalit	Georgia Institute of Technology margalit@math.gatech.edu
Fred Cohen	University of Rochester cohf@math.rochester.edu	Norihiko Minami	Nagoya Institute of Technology nori@nitech.ac.jp
Alexander Dranishnikov	University of Florida dranish@math.ufl.edu	Andrés Navas Flores	Universidad de Santiago de Chile andres.navas@usach.cl
Tobias Ekholm	Uppsala University, Sweden tobias.ekholm@math.uu.se	Robert Oliver	Université Paris-Nord bobol@math.univ-paris13.fr
Mario Eudave-Muñoz	Univ. Nacional Autónoma de México mario@matem.unam.mx	Luis Paris	Université de Bourgogne lparis@u-bourgogne.fr
David Futer	Temple University dfuter@temple.edu	Jérôme Scherer	École Polytech. Féd. de Lausanne jerome.scherer@epfl.ch
Soren Galatius	Stanford University galatius@math.stanford.edu	Peter Scott	University of Michigan pscott@umich.edu
John Greenlees	University of Sheffield j.greenlees@sheffield.ac.uk	Zoltán Szabó	Princeton University szabo@math.princeton.edu
J. Elisenda Grigsby	Boston College grigsbyj@bc.edu	Ulrike Tillmann	Oxford University tillmann@maths.ox.ac.uk
Ian Hambleton	McMaster University ian@math.mcmaster.ca	Maggy Tomova	University of Iowa maggy-tomova@uiowa.edu
Hans-Werner Henn	Université Louis Pasteur henn@math.u-strasbg.fr	Daniel T. Wise	McGill University, Canada daniel.wise@mcgill.ca
Daniel Isaksen	Wayne State University isaksen@math.wayne.edu		

See inside back cover or msp.org/agt for submission instructions.

The subscription price for 2018 is US $465/year for the electronic version, and $730/year (+ $60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP. Algebraic & Geometric Topology is indexed by Mathematical Reviews, Zentralblatt MATH, Current Mathematical Publications and the Science Citation Index.

Algebraic & Geometric Topology (ISSN 1472-2747 printed, 1472-2739 electronic) is published 7 times per year and continuously online, by Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, c/o Department of Mathematics, University of California, 798 Evans Hall #3840, Berkeley, CA 94720-3840.

AGT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2018 Mathematical Sciences Publishers

msp

Algebraic & Geometric Topology 18 (2018) 3749–3788

Poincaré duality complexes with highly connected universal cover

BEATRICE BLEILE
IMRE BOKOR
JONATHAN A HILLMAN

Turaev conjectured that the classification, realization and splitting results for Poincaré duality complexes of dimension 3 (PD_3–*complexes*) generalize to PD_n–complexes with $(n-2)$–connected universal cover for $n \geq 3$. Baues and Bleile showed that such complexes are classified, up to oriented homotopy equivalence, by the triple consisting of their fundamental group, orientation class and the image of their fundamental class in the homology of the fundamental group, verifying Turaev's conjecture on classification.

We prove Turaev's conjectures on realization and splitting. We show that a triple (G, ω, μ), comprising a group G, a cohomology class $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and a homology class $\mu \in H_n(G; \mathbb{Z}^\omega)$, can be realized by a PD_n–complex with $(n-2)$–connected universal cover if and only if the Turaev map applied to μ yields an equivalence. We show that a PD_n–complex with $(n-2)$–connected universal cover is a nontrivial connected sum of two such complexes if and only if its fundamental group is a nontrivial free product of groups.

We then consider the indecomposable PD_n–complexes of this type. When n is odd the results are similar to those for the case $n = 3$. The indecomposables are either aspherical or have virtually free fundamental group. When n is even the indecomposables include manifolds which are neither aspherical nor have virtually free fundamental group, but if the group is virtually free and has no dihedral subgroup of order > 2 then it has two ends.

57N65, 57P10

1 Introduction

Hendricks classified Poincaré duality complexes of dimension 3 (PD_3–*complexes*) up to oriented homotopy equivalence using the "fundamental triple", comprising the fundamental group, the orientation character and the image of the fundamental class in the homology of the fundamental group.

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3749

Turaev [21] gave an alternative proof of Hendricks' result and provided necessary and sufficient conditions for a triple (G, ω, μ) — comprising a group G, a cohomology class $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and a homology class $\mu \in H_3(G; \mathbb{Z}^\omega)$, where $\mathbb{Z}^\omega$ denotes the integers $\mathbb{Z}$ regarded as a right $\mathbb{Z}[G]$–module with respect to the *twisted* structure induced by ω — to be the fundamental triple of a PD_3–complex. Central to this was that the image of μ under a specific homomorphism, which we call *the Turaev map*, be an isomorphism in the stable module category of $\mathbb{Z}[G]$, that is to say, a homotopy equivalence of $\mathbb{Z}[G]$–modules.

The results on classification and splitting allowed Turaev to show that a PD_3–complex is a nontrivial connected sum of two PD_3–complexes if and only if its fundamental group is a nontrivial free product of groups. He conjectured that these results hold for all PD_n–complexes with $(n-2)$–connected universal cover.

Baues and Bleile classified Poincaré duality complexes of dimension 4 in [2]. Their analysis showed that a PD_n–complex X, with $n \geq 3$, is classified up to oriented homotopy equivalence by the triple comprising its $(n-2)$–type $P_{n-2}(X)$, its orientation character $\omega = \omega_X \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$ and the image μ of its fundamental class in $H_n(P_{n-2}(X); \mathbb{Z}^\omega)$. (We assume that all spaces have basepoints. Thus, every map $f\colon X \to Y$ has a preferred lift to a map of universal covers. Hence, if $f^*\omega_Y = \omega_X$, there is a well-defined homomorphism $H_n(f; \mathbb{Z}^\omega)$, and it is meaningful to say that f is "oriented", ie that $f_*[X] = [Y]$. See Taylor [20] for a discussion of this issue.)

They called this the *fundamental triple of* X, as it is a generalization of Hendricks' "fundamental triple", for the $(n-2)$–type of X determines its fundamental group. Moreover, when the universal cover of the complex is $(n-2)$–connected — automatically the case when $n = 3$ — the $(n-2)$–type is an Eilenberg–Mac Lane space of type $(\pi_1(X), 1)$, so that the $(n-2)$–type and the fundamental group determine each other completely, reducing their fundamental triple to that of Hendricks.

Turaev's conjecture on classification is a direct consequence:

Theorem *There is an oriented homotopy equivalence between two* PD_n*–complexes with* $(n-2)$*–connected universal cover if and only if their fundamental triples are isomorphic.*

We prove Turaev's conjectures on realization and splitting. These are, respectively, Theorems A and B below. While our approach to Theorem A is a direct generalization of Turaev's theorem, our proof applies Baues' homotopy systems in detail.

Recall that the group G is of type FP_n if and only if the trivial $\mathbb{Z}[G]$–module $\mathbb{Z}$ has a projective resolution $\boldsymbol{P}$, with P_j finitely generated for $j \leq n$.

Theorem A *Let G be a finitely presentable group, ω a cohomology class in $H^1(G;\mathbb{Z}/2\mathbb{Z})$ and μ a homology class in $H_n(G;\mathbb{Z}^\omega)$, with $n \geq 3$.*

If G is of type FP_{n-1}, with $H^i(G;{}^\omega\mathbb{Z}[G]) = 0$ for $1 < i \leq n-1$, then (G,ω,μ) can be realized as the fundamental triple of a PD_n–complex with $(n-2)$–connected universal cover if and only if the Turaev map applied to μ yields an isomorphism in the stable module category of $\mathbb{Z}[G]$.

Theorem B *A PD_n–complex with $(n-2)$–connected universal cover decomposes as a nontrivial connected sum of two such PD_n–complexes if only if its fundamental group decomposes as a nontrivial free product of groups.*

Thus, it is enough to investigate PD_n–complexes with $(n-2)$–connected universal cover whose fundamental group is indecomposable as free product, and we turn to the analysis of such complexes. Our arguments here exploit the interaction of Poincaré duality with the *Chiswell sequence associated with a graph of groups* (see Crisp [8] and Hillman [15]).

The parity of the dimension n is significant.

When n is odd, indecomposable orientable PD_n–complexes are either aspherical or have virtually free fundamental groups, and the arguments of [15] provide similar constraints on the latter class of groups. (See Section 7.) However, implementing the realization theorem may be difficult, and we do not consider this case further.

When n is even there are indecomposable fundamental groups, G, with virtual cohomological dimension n — $\operatorname{vcd} G = n$ — and infinitely many ends. Our strongest results are for groups which are indecomposable and virtually free.

Theorem C *Let X be a PD_{2k}–complex with $(2k-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free and indecomposable as a free product. If G is finite then $X \simeq S^{2k}$ or $\mathbb{R}P^{2k}$. If G is infinite and has no dihedral subgroup of order > 2, then G has two ends and its finite subgroups have cohomological period dividing $2k$. Hence, $\widetilde{X} \simeq S^{2k-1}$. If, moreover, X is orientable, then $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$.*

In particular, Theorem C applies to closed 4–manifolds M with $\pi_2(M) = 0$ and such fundamental groups. There is no geometric connected sum decomposition theorem for 4–manifolds currently known that corresponds to Theorem B.

There is also a realization result, when $G \cong F \rtimes_\theta \mathbb{Z}$ with F finite (that is, when $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$).

Theorem D *If $G \cong F \rtimes_\theta \mathbb{Z}$, where F is finite, then $G = \pi_1(X)$ for some PD_{2k}–complex X with $\widetilde{X} \simeq S^{2k-1}$ if and only if F has cohomological period dividing $2k$ and $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$.*

Since putting this paper on arXiv in May 2016 we have learned that Theorem D is a particular case of Proposition 8 of Golasiński and Gonçalves [10]. The paper [10] also gives estimates of the number of homotopy types realizing a given fundamental group. However, we have chosen to retain our independent treatment as it is brief and is a natural complement to our more substantial results.

In general, it is not known when such a PD_n–complex is homotopy equivalent to a closed n–manifold. (This question leads to delicate issues of algebraic number theory; see Hambleton and Madsen [12].) There has been extensive research on the mixed-spherical space form problem, on the fundamental groups of manifolds with universal covering space $S^n \times \mathbb{R}^k$ for $n, k > 0$. A recurring theme is the role of finite dihedral subgroups. See Hambleton and Pedersen [13] for a survey of recent progress.

The main questions left open by our study of indecomposable, virtually free fundamental groups are

(a) what happens when G has dihedral subgroups; and

(b) are there examples with $D_4 = (\mathbb{Z}/2\mathbb{Z})^2$ as a subgroup?

As seems common in topology, there appear to be difficulties associated with 2–torsion!

Section 2 summarizes background material and fixes notation.

Section 3 contains the formulation and proof of the necessity of the condition for the realization of a fundamental triple by a PD_n–complex with $(n-2)$–connected universal cover.

Section 4 completes the proof of Theorem A, with the sufficiency of the condition in Section 3.

Section 5 contains the proof of Theorem B, showing how the fundamental triple detects connected sums.

Section 6 is an interlude outlining the notion of graphs of groups used subsequently.

Section 7 starts the discussion of indecomposable PD_n–complexes with $(n-2)$–connected universal covers, beginning with Crisp's centralizer condition.

In Section 8 we give some supporting results, and construct examples of indecomposable groups G with infinitely many ends and $\operatorname{vcd} G = n$ which are the fundamental groups of closed n–manifolds with $(n-2)$–connected universal cover.

In Section 9 we show that finite subgroups of G of odd order are metacyclic.

In Sections 10 and 11 we prove Theorems C and D.

Section 12 concludes by briefly considering possible examples with D_4 as a subgroup.

2 Background and notation

This section summarizes background material and fixes notation for the rest of the paper. Details and further references can be found in [4; 8; 15].

Let Λ be the integral group ring $\mathbb{Z}[G]$ of the group G. We write I for the *augmentation ideal*, the kernel of the augmentation map

$$\text{aug}\colon \Lambda \to \mathbb{Z}, \quad \sum_{g\in G} n_g g \mapsto \sum_{g\in G} n_g,$$

where $\mathbb{Z}$ is a Λ–bimodule with trivial Λ action. Each cohomology class $\omega \in H^1(G;\mathbb{Z}/2\mathbb{Z})$ may be viewed as a group homomorphism $\omega\colon G \to \mathbb{Z}/2\mathbb{Z} = \{0,1\}$ and yields an anti-isomorphism

$$\bar{}\colon \Lambda \to \Lambda, \quad \lambda = \sum_{g\in G} n_g g \mapsto \bar{\lambda} = \sum_{g\in G} (-1)^{\omega(g)} n_g g^{-1}. \tag{1}$$

Consequently, a right Λ–module A yields the *conjugate left Λ–module* ${}^{\omega}A$, with action given by

$$\lambda_\bullet a := a.\bar{\lambda}$$

for $\lambda \in \Lambda$ and $a \in A$. Plainly, the conjugate defines a functor from the category of left Λ–modules to the category of right Λ–modules. Similarly, a left Λ–module B yields the conjugate right Λ–module B^{ω}. If M is a Λ–bimodule, then conjugating both the left and the right Λ–module structures leads to ${}^{\omega}M^{\omega}$, with Λ–bimodule structure

$$\lambda_\bullet x_\bullet \mu := \bar{\mu}.x.\bar{\lambda}.$$

Given left Λ–modules A_j and B_i for $1 \le i \le k$ and $1 \le j \le \ell$, we sometimes write the Λ–morphism $\psi\colon \bigoplus_{j=1}^{\ell} A_j \to \bigoplus_{i=1}^{k} B_i$ in matrix form as $[\psi_{ij}]_{k\times\ell}$ for $\psi_{ij} = \mathrm{pr}_i \circ \psi \circ \mathrm{in}_j\colon A_j \to B_i$, where in_j is the j^{th} natural inclusion and pr_i the i^{th}

natural projection of the direct sum. The composition of such morphisms is given by matrix multiplication.

If B is a left Λ–module and M a Λ–bimodule, then $\mathrm{Hom}_\Lambda(B, M)$ is a right Λ–module with action given by

$$\varphi.\lambda\colon B \to M, \quad b \mapsto \varphi(b).\lambda.$$

The dual of the left Λ–module B is the left Λ–module $B^* = {}^\omega\mathrm{Hom}_\Lambda(B, \Lambda)$. The construction of the dual defines an endofunctor on the category of left Λ–modules.

Evaluation defines a natural transformation ε from the identity functor to the double dual functor, where, for the left Λ–module B,

$$\varepsilon_B\colon B \to B^{**} = {}^\omega\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(B, \Lambda), \Lambda), \quad b \mapsto \overline{\mathrm{ev}}_b,$$

with $\overline{\mathrm{ev}}_b$ defined by

$$\overline{\mathrm{ev}}_b\colon {}^\omega\mathrm{Hom}(B, \Lambda) \to \Lambda, \quad \psi \mapsto \overline{\psi(b)}.$$

The left Λ–module A defines the natural transformation η from the functor $A^\omega \otimes_\Lambda -$ to the functor $\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-, \Lambda), A)$, where, for the left Λ–module B,

$$\eta_B\colon A^\omega \otimes_\Lambda B \to \mathrm{Hom}_\Lambda(B^*, A) = \mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(B, \Lambda), A)$$

is given by

$$\eta_B(a \otimes b)\colon \psi \mapsto \overline{\psi(b)}.a$$

for $a \otimes b \in A^\omega \otimes_\Lambda B$. Both ε and η become natural equivalences when restricted to the category of finitely generated free Λ–modules.

The Λ–morphisms $f, g\colon A_1 \to A_2$ are *homotopic* if and only if the Λ–morphism $f - g\colon A_1 \to A_2$ factors through a projective Λ–module P. Associated with Λ is its *stable module category*, whose objects are all Λ–modules and whose morphisms are all homotopy classes of Λ–morphisms. Thus, an isomorphism in the stable module category of Λ is a homotopy equivalence of Λ–modules.

We work in the category of connected, well-pointed CW–complexes and pointed maps. We write $X^{[k]}$ for the k–skeleton of X, suppressing the basepoint from our notation. The inclusion of the k–skeleton into the $(k+1)$–skeleton induces a homomorphism $\pi_{k+1}(X^{[k]}) \to \pi_{k+1}(X^{[k+1]})$, whose image we denote by $\Gamma_{k+1}(X)$.

From now, we work with the fundamental group $G = \pi_1(X)$ of X and its integral group ring $\Lambda = \mathbb{Z}[G]$. We take X to be a reduced CW–complex, so that $X^{[0]} = \{*\}$, and write $u\colon \widetilde{X} \to X$ for the universal cover of X, fixing a basepoint for $\widetilde{X}$ in $u^{-1}(*)$.

We write $\boldsymbol{C}(\widetilde{X})$ for the cellular chain complex of $\widetilde{X}$ viewed as a complex of left Λ–modules. Since X is reduced, $C_0(\widetilde{X}) = \Lambda$, and the augmentation ideal coincides with the image of the boundary map $C_1(\widetilde{X}) \to C_0(\widetilde{X})$.

The homology and cohomology of X we work with are the abelian groups

$$H_q(X; A) := H_q(A \otimes_\Lambda \boldsymbol{C}(\widetilde{X})),$$
$$H^q(X; B) := H^q\big(\mathrm{Hom}_\Lambda(\boldsymbol{C}(\widetilde{X}), B)\big),$$

where A is a right Λ–module and B is a left Λ–module.

An *n–dimensional Poincaré duality complex* (PD_n–complex) comprises a reduced connected CW–complex X whose fundamental group $\pi_1(X)$ is finitely presentable, together with an *orientation character* $\omega = \omega_X \in H^1(\pi_1(X); \mathbb{Z}/2\mathbb{Z})$, viewed as a group homomorphism $\pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$, and a *fundamental class* $[X] \in H_n(X; \mathbb{Z}^\omega)$ such that, for every $r \in \mathbb{Z}$ and left $\mathbb{Z}[\pi_1(X)]$–module M, the *cap product with* $[X]$,

$$- \frown [X]\colon H^r(X; M) \to H_{n-r}(X; M^\omega), \quad \alpha \mapsto \alpha \frown [X],$$

is an isomorphism of abelian groups. We denote this by $(X, \omega, [X])$.

Wall [22; 24] showed that for $n > 3$, every PD_n–complex is *standard*, meaning that it is homotopically equivalent to an n–dimensional CW–complex with precisely one n–cell, whereas a PD_3–complex X is either standard or *weakly standard*, the latter meaning that it is homotopically equivalent to one of the form $X' \cup e^3$, where e^3 is a 3–cell and X' is a 3–dimensional CW–complex with $H^3(X'; B) = 0$ for all coefficient modules B.

In [1], Baues introduced *homotopy systems* to investigate when chain complexes and chain maps of free Λ–modules are realized by CW–complexes.

Take an integer $n > 1$. A *homotopy system of order* $(n+1)$ comprises

(a) a reduced n–dimensional CW–complex X;

(b) a chain complex $\boldsymbol{C}$ of free Λ–modules coinciding with $\boldsymbol{C}(\widetilde{X})$ in degree q for $q \le n$;

(c) a homomorphism $f_{n+1}\colon C_{n+1} \to \pi_n(X)$ with $f_{n+1} \circ d_{n+2} = 0$ such that the diagram

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{f_{n+1}} & \pi_n(X) \\ {\scriptstyle d_{n+1}}\downarrow & & \downarrow{\scriptstyle j} \\ C_n & \xleftarrow[h_n]{} & \pi_n(X, X^{[n-1]}) \end{array}$$

commutes, where j is induced by the inclusion $(X,*)\to(X,X^{[n-1]})$, and

$$h_n\colon \pi_n(X,X^{[n-1]}) \xrightarrow[\cong]{u_*^{-1}} \pi_n(\widetilde{X},\widetilde{X}^{[n-1]}) \xrightarrow[\cong]{h} H_n(\widetilde{X},\widetilde{X}^{[n-1]})$$

is the Hurewicz isomorphism h composed with u_*^{-1}, the inverse of the isomorphism induced by the universal covering map.

3 Formulation and necessity of the realization conditions

For our generalization of Tuarev's realization condition to PD_n–complexes with $n\geq 3$, we introduce a set of functors from the category of chain complexes of projective left Λ–modules to the category of left Λ–modules.

Given $\boldsymbol{f}\colon \boldsymbol{C}\to\boldsymbol{D}$, a map of chain complexes of projective left Λ–modules $\boldsymbol{C}$ and $\boldsymbol{D}$, put $T_q(\boldsymbol{C}) := \operatorname{coker}(d^{\boldsymbol{C}}_{q+1}\colon C_{q+1}\to C_q) = C_q/\operatorname{im}(d^{\boldsymbol{C}}_{q+1})$ and let $T_q(\boldsymbol{f})$ be the induced map of cokernels

$$\begin{array}{ccccc}
\operatorname{im}(d^{\boldsymbol{C}}_{q+1}) & \rightarrowtail & C_q & \twoheadrightarrow & T_q(\boldsymbol{C}) \\
\downarrow & & \downarrow{\scriptstyle f_q} & & \vdots{\scriptstyle \exists!\ T_q(\boldsymbol{f})} \\
\operatorname{im}(d^{\boldsymbol{D}}_{q+1}) & \rightarrowtail & D_q & \twoheadrightarrow & T_q(\boldsymbol{D})
\end{array}$$

Direct verification shows that each T_q is a functor from the category of chain complexes of left Λ–modules to the category of left Λ–modules.

By Lemma 4.2 in [4], chain-homotopic maps $\boldsymbol{f}\simeq\boldsymbol{g}\colon \boldsymbol{C}\to\boldsymbol{D}$ induce homotopic maps $T_q(\boldsymbol{f})\simeq T_q(\boldsymbol{g})$, that is, $T_q(\boldsymbol{f})-T_q(\boldsymbol{g})$ factors through a projective Λ–module. Hence, for each $q\in\mathbb{Z}$, T_q induces a functor from the category of chain complexes of projective left Λ–modules and chain homotopy classes of chain maps to the stable module category of Λ.

Let X be a PD_n–complex with $n\geq 3$, and let $\Lambda=\mathbb{Z}[\pi_1(X)]$. By Remark 2.3 and Lemma 3.6 in [2], we may assume that $X=X'\cup e^n$ is standard (or weakly standard if $n=3$) with

$$C_n(\widetilde{X}) = C_n(\widetilde{X}')\oplus\Lambda e,$$

where e corresponds to e^n, the element $1\otimes e\in\mathbb{Z}^\omega\otimes_\Lambda C_n(\widetilde{X})$ is a cycle representing the fundamental class $[X]$ of X, and e is a generator of $C_n(\widetilde{X})$.

Writing F^q for $T_q({}^{\omega}\mathrm{Hom}_\Lambda(-,\Lambda))$, Poincaré duality, together with Lemma 4.3 in [4], provides the homotopy equivalence of Λ–modules

$$T_{-n+1}(-\frown(1\otimes e))\colon F^{n-1}(\boldsymbol{C}(\widetilde{X}))\to T_1(\boldsymbol{C}(\widetilde{X})).$$

Construct the $(n-2)$–type $P=P_{n-2}(X)$ of X by attaching to X cells of dimension n and higher. Then the Postnikov section $p\colon X\to P$ is the identity on the $(n-1)$–skeleta and $C_i(\widetilde{X})=C_i(\widetilde{P})$ for $0\le i<n$. Composing with the isomorphism $\theta\colon T_1(\boldsymbol{C}(\widetilde{X}))\to I$, $[c]\mapsto d_1(c)$, we obtain the homotopy equivalence of left Λ–modules

$$\theta\circ T_{-n+1}(-\frown(1\otimes e))\colon F^{n-1}(\boldsymbol{C}(\widetilde{P}))\to I. \tag{2}$$

We next construct the *Turaev map*, which sends the image of the fundamental class of X in the homology of the Postnikov section to the homotopy class of the homotopy equivalence (2).

Let $\boldsymbol{C}$ be a chain complex of free left Λ–modules. We write $\overline{I}$ for the image of the augmentation ideal I under the anti-isomorphism (1). This gives rise to the short exact sequence of chain complexes $0\to\overline{I}\boldsymbol{C}\to\boldsymbol{C}\to\mathbb{Z}^\omega\otimes_\Lambda\boldsymbol{C}\to 0$, with associated connecting homomorphism $\delta_r\colon H_r(\mathbb{Z}^\omega\otimes_\Lambda\boldsymbol{C})\to H_{r-1}(\overline{I}\boldsymbol{C})$.

The set of homotopy classes of module morphisms $A\to B$, written $[A,B]$, is naturally a group and it is straightforward (see [3]) to verify that

$$\widehat{\nu}_{\boldsymbol{C},r}\colon H_r(\overline{I}\boldsymbol{C})\to[F^r(\boldsymbol{C}),I],\quad [\lambda.c]\mapsto[F^r\boldsymbol{C}\to I,\ [\varphi]\mapsto\overline{\varphi(\lambda.c)}],$$

is a homomorphism of groups. Composing $\widehat{\nu}_{\boldsymbol{C},r-1}$ with δ_r yields the Turaev map

$$\nu_{\boldsymbol{C},r}\colon H_r(\mathbb{Z}^\omega\otimes_\Lambda\boldsymbol{C})\to[F^{r-1}(\boldsymbol{C}),I].$$

Lemma 1 $$\nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))=[\theta\circ T_{-n+1}(-\frown(1\otimes e))].$$

Proof Take a diagonal

$$\Delta\colon\boldsymbol{C}(\widetilde{X})\to\boldsymbol{C}(\widetilde{X})\otimes_{\mathbb{Z}}\boldsymbol{C}(\widetilde{X})$$

and a chain homotopy $\alpha\colon\boldsymbol{C}(\widetilde{X})\to\boldsymbol{C}(\widetilde{X})$ such that $\mathrm{id}-(\mathrm{id}\otimes\mathrm{aug})\Delta=d\alpha+\alpha d$, where we have identified $\boldsymbol{C}\otimes_{\mathbb{Z}}\mathbb{Z}$ with $\boldsymbol{C}$. Let

$$\Delta e=e\otimes\lambda+\sum_\ell\sum_{0\le i<n}x_{\ell,i}\otimes y_{\ell,n-i}.$$

Direct calculation shows that $e=\mathrm{aug}(\lambda)e+\alpha de$. Since $[1\otimes e]$ generates the homology $H_n(X;\mathbb{Z}^\omega)\cong\mathbb{Z}$, this yields $[1\otimes e]=[\mathrm{aug}(\lambda)\otimes e]$, whence $\mathrm{aug}(\lambda-1)=0$. Hence,

$\lambda - 1 \in I = \mathrm{im}(d_1)$, or $\lambda = 1 + d_1(c_1)$ for some $c_1 \in C_1(\widetilde{X})$. Thus, given $\varphi \in \mathrm{Hom}_\Lambda(C_n(\widetilde{X}), \Lambda)$,

$$\varphi \frown (1 \otimes e) = \overline{\varphi(e)}(1 + d_1(c_1)).$$

By direct calculation,

$$\big(\theta \circ T_{-n+1}(- \frown (1 \otimes e))\big)([\varphi]) = \overline{\varphi(d_n(e))}(1 + d_1(c_1))$$

and

$$\nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))([\varphi]) = \hat{\nu}_{\boldsymbol{C}(\widetilde{P}),n}([d_n(e)])([\varphi]).$$

Hence, by definition, $\nu_{\boldsymbol{C}(\widetilde{P}),n}(p_*([X]))$ is represented by the Λ–morphism

$$F^{n-1}(\boldsymbol{C}(\widetilde{P})) \to I, \quad [\varphi] \mapsto \overline{\varphi(d_n(e))}.$$

To conclude the proof, note that $F^{n-1}(\boldsymbol{C}(\widetilde{P})) \to I$, $[\varphi] \mapsto \overline{\varphi(d_n(e))}.d_1(c_1)$, factors through $C_1(X)$ and is thus null-homotopic. □

As $\theta \circ T_{-n+1}(- \frown (1 \otimes e))$ is a homotopy equivalence of Λ–modules, Lemma 1 provides a necessary condition for realization.

Theorem 2 *Let P be an $(n-2)$–type. Take $\omega \in H^1(P; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P; \mathbb{Z}^\omega)$. Then (P, ω, μ) is the fundamental triple of a PD_n–complex only if $\nu_{\boldsymbol{C}(\widetilde{P}),n}(\mu)$ is a homotopy equivalence of left $\Lambda = \mathbb{Z}[\pi_1(P)]$–modules.*

Proof Let P be an $(n-2)$–type. Take $\omega \in H^1(P; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(P; \mathbb{Z}^\omega)$. Suppose that (P, ω, μ) is the fundamental triple of the PD_n–complex X. If P' is an $(n-2)$–type obtained by attaching to X cells of dimension n and higher, then there is a homotopy equivalence $f\colon P \to P'$ with $f_*(\mu) = i_*([X])$, where $i\colon X \to P'$ is the inclusion. By Lemma 1,

$$\nu_{\boldsymbol{C}(\widetilde{P}'),n}(i_*[X]) = [\theta \circ T_{-n+1}(- \frown (1 \otimes e))]$$

and hence $\nu_{\boldsymbol{C}(\widetilde{P}),n}(\mu)$ are homotopy equivalences of Λ–modules. □

Let X now be a PD_n–complex with $(n-2)$–connected universal cover. The $(n-2)$–type of X is an Eilenberg–Mac Lane space $K(\pi_1(X), 1)$, and we may identify the fundamental triple of X with $(\pi_1(X), \omega, \mu)$, where μ is the image of $[X]$ in the group homology of $\pi_1(X)$.

Lemma 3 *Let $(X, \omega, [X])$ be a PD_n–complex with $(n-2)$–connected universal cover. Then $\pi_1(X)$ is FP_{n-1}, and $H^i(\pi_1(X); {}^\omega\Lambda) = 0$ for all $1 < i \leq n-1$.*

Proof Since X is a PD_n–complex, it is finitely dominated, and so is homotopy equivalent to a complex with finite $(n-1)$–skeleton. Thus, we may assume that $X^{[n-1]}$ is finite. We construct an Eilenberg–Mac Lane space $K = K(\pi_1(X), 1)$ from X by attaching cells of dimension n and higher. As the universal cover $\widetilde{X}$ of X is $(n-2)$–connected, the cellular chain complexes of the universal covers $\widetilde{X}$ and $\widetilde{K}$ coincide in degrees below n, that is, $C_i(\widetilde{X}) = C_i(\widetilde{K})$ for $0 \le i < n$. In particular, these modules are finitely generated, and so $\pi_1(X)$ is FP_{n-1}.

Moreover, for $1 < i \le n-1$,

$$H^i(\pi_1(X); {}^{\omega}\Lambda) = H^i(X; {}^{\omega}\Lambda) \cong H_{n-i}(X; \Lambda) = 0. \qquad \square$$

We note for later reference that

$$\pi_{n-1}(\widetilde{X}) \cong H_{n-1}(\widetilde{X}; \mathbb{Z}) \cong {}^{\omega}H^1(G; \mathbb{Z}[G]),$$

by Hurewicz's theorem and Poincaré duality, respectively.

Necessary conditions for realization are a corollary to Lemma 3 and Theorem 2.

Corollary 4 (conditions for realizability) *Let G be a group. Take $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_n(G; \mathbb{Z}^{\omega})$. If (G, ω, μ) is the fundamental triple of a PD_n–complex with $(n-2)$–connected universal cover, then G is a finitely presentable group of type FP_{n-1}, $H^i(G; {}^{\omega}\Lambda) = 0$ for $1 < i \le n-1$ and $\nu_{C(\widetilde{K}),n}(\mu)$ is a homotopy equivalence of $\Lambda = \mathbb{Z}[G]$ modules.*

4 Sufficiency of the realization condition

We now establish the sufficiency of the realization conditions in Corollary 4.

Let G be a finitely presentable group of type FP_{n-1}, with $n \ge 3$. Let K' be an Eilenberg–Mac Lane space of type $(G, 1)$ with universal cover $\widetilde{K}' \to K'$. Identify the (co)homologies of G and K'. Choose $\omega \in H^1(G; \mathbb{Z}/2\mathbb{Z})$ and suppose that $H^i(G; {}^{\omega}\Lambda) = 0$ for $1 < i \le n-1$, where $\Lambda = \mathbb{Z}[G]$. Finally, take $\mu \in H_n(G; \mathbb{Z}^{\omega})$, with $\nu_{C(\widetilde{K}'),n}(\mu)$ a class of homotopy equivalences of Λ–modules.

We construct a PD_n–complex X with $(n-2)$–connected universal cover and fundamental triple (G, ω, μ).

By the hypotheses on G, we may assume that K' has been chosen with finitely many cells in each dimension below n.

Let $h\colon F^{n-1}(\boldsymbol{C}(\widetilde{K}'))\to I$ be a representative of $\nu_{\boldsymbol{C}(\widetilde{K}'),n}(\mu)$. Then h is a homotopy equivalence of Λ–modules. By Theorem 4.1 and Observation 1 in [4], h factors as

$$F^{n-1}(\boldsymbol{C}(\widetilde{K}'))\rightarrowtail F^{n-1}(\boldsymbol{C}(\widetilde{K}'))\oplus\Lambda^m\rightarrowtail I\oplus P\twoheadrightarrow I$$

for some projective Λ–module, P, and $m\in\mathbb{N}$. Let $B=(e^0\cup e^{n-1})\cup e^n$ be the n–dimensional ball and replace K' by the Eilenberg–Mac Lane space $K=K'\vee\big(\bigvee_{i=1}^m B\big)$. Then $F^{n-1}(\boldsymbol{C}(\widetilde{K}))=F^{n-1}(\boldsymbol{C}(\widetilde{K}'))\oplus\Lambda^m$ and the factorization of h becomes

$$h\colon F^{n-1}(\boldsymbol{C}(\widetilde{K}))\xrightarrow{j}I\oplus P\xrightarrow{\mathrm{pr}_I}I$$

with j surjective. Consider the Λ–morphism φ given by the composition

$$C^{n-1}(\widetilde{K})={}^{\omega}\mathrm{Hom}_\Lambda(C_{n-1}(\widetilde{K}),\Lambda)\xrightarrow{p}F^{n-1}(\boldsymbol{C}(\widetilde{K}))\xrightarrow{j}I\oplus P\xrightarrow{\left[\begin{smallmatrix}i&0\\0&\mathrm{id}\end{smallmatrix}\right]}\Lambda\oplus P,$$

where p is the projection onto the cokernel and $i\colon I\rightarrowtail\Lambda$ the inclusion. Since $F^{n-1}(\boldsymbol{C}(\widetilde{K}))=C^{n-1}(\widetilde{K})/\operatorname{im}(d_{n-1}^*)$ by definition, $\varphi\circ d_{n-1}^*=0$. As $C_{n-1}(\widetilde{K})$ is a finitely generated free Λ–module, the natural map

$${}^{\omega}\varepsilon\colon C_{n-1}(\widetilde{K})\to C_{n-1}(\widetilde{K})^{**}$$

is an isomorphism. Define

$$d_n:=({}^{\omega}\varepsilon)^{-1}\circ\varphi^*\colon(\Lambda\oplus P)^*\to C_{n-1}(\widetilde{K}).$$

It follows from the naturality of ${}^{\omega}\varepsilon$ that $d_{n-1}\circ d_n=0$.

We first consider the case when P is free, so that $P\cong\Lambda^q$ for some $q\in\mathbb{N}$ and $\Lambda\oplus P\cong\Lambda^{q+1}$.

Since $\widetilde{K}^{[n-1]}$ is $(n-2)$–connected, the Hurewicz homomorphism

$$h_q\colon\pi_q(\widetilde{K}^{[n-1]})\to H_q(\widetilde{K}^{[n-1]})$$

is an isomorphism for $q\le n-1$ and we obtain the map

$$\begin{aligned}\varphi'\colon\Lambda^{q+1}\cong(\Lambda\oplus P)^*\to\ker(d_{n-1})=H_{n-1}(\widetilde{K}^{[n-1]})&\xrightarrow{h_{n-1}^{-1}}\pi_{n-1}(\widetilde{K}^{[n-1]}),\\ x&\mapsto h_{n-1}^{-1}([d_n(x)]).\end{aligned}$$

Let $\boldsymbol{C}$ be the chain complex of Λ–modules

$$\Lambda^{q+1}\cong(\Lambda\oplus P)^*\xrightarrow{d_n}C_{n-1}(\widetilde{K}^{[n-1]})\xrightarrow{d_{n-1}}\cdots\to C_1(\widetilde{K}^{[n-1]})\to\Lambda.$$

Then $Y=(\boldsymbol{C},\varphi',K^{[n-1]})$ is a homotopy system of order n. As $C_i=0$ for $i>n$, $H^{n+2}(Y;\Gamma_nY)=0$ and, by Proposition 8.3 in [2], there is a homotopy system

$(\boldsymbol{C}, 0, X)$ of order $n+1$ realising Y, with X an n–dimensional CW–complex. By construction, $\boldsymbol{C}(\widetilde{X}) = \boldsymbol{C}$ and the universal cover of X is $(n-2)$–connected. Since $X^{[n-1]} = K^{[n-1]}$ and $\pi_q(K) = 0$ for all $q > 1$, the inclusion $i\colon K^{[n-1]} \to K$ extends to a map

$$f\colon X \to K = K(G, 1)$$

and we may take $\omega \in H^1(K; \mathbb{Z}/2\mathbb{Z})$ to be an element of $H^1(X; \mathbb{Z}/2\mathbb{Z})$.

Proposition 5 *X is a PD_n–complex with fundamental triple (G, ω, μ), that is,*

(i) $\mathbb{Z} \cong H_n(X; \mathbb{Z}^\omega) = \langle [X] \rangle$;

(ii) $f_*([X]) = \mu$;

(iii) $- \frown [X]\colon H^r(X; {}^\omega\Lambda) \to H_{n-r}(X; \Lambda)$ *is an isomorphism for every $r \in \mathbb{Z}$.*

Proof **(i)** As $\boldsymbol{C}(\widetilde{X}) = \boldsymbol{C}$ is a chain complex of finitely generated free Λ–modules, the natural map

$$\eta_{\boldsymbol{C}}\colon \mathbb{Z}^\omega \otimes_\Lambda \boldsymbol{C} \to \mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(\boldsymbol{C}, \Lambda), \mathbb{Z})$$

is an isomorphism. Hence, writing ζ^+ for the morphism $\mathrm{Hom}_\Lambda(B, \mathbb{Z}) \to \mathrm{Hom}_\Lambda(A, \mathbb{Z})$ induced by $\zeta\colon A \to B$, we obtain

$$H_n(X; \mathbb{Z}^\omega) = \ker(1 \otimes d_n) \cong \ker(\varphi^+)$$

for $\varphi\colon {}^\omega\mathrm{Hom}_\Lambda(C_{n-1}(\widetilde{K}^{[n-1]}), \Lambda) \to \Lambda \oplus \Lambda^q$ defined above.

Since both p and j are surjective, both p^+ and j^+ are injective, whence

$$\ker(\varphi^+) = \ker\left(\left(\begin{bmatrix} i & 0 \\ 0 & \mathrm{id} \end{bmatrix} \circ j \circ p\right)^+\right) = \ker\left(\begin{bmatrix} i & 0 \\ 0 & \mathrm{id} \end{bmatrix}^+\right) = \ker\left(\begin{bmatrix} i^+ & 0 \\ 0 & \mathrm{id} \end{bmatrix}\right) \cong \ker(i^+).$$

But I is generated by elements $1 - g$ for $g \in G$ and $(\psi \circ i)(1 - g) = 0$ for $\psi \in \mathrm{Hom}_\Lambda(\Lambda, \mathbb{Z})$. Hence,

$$\ker(\varphi^+) \cong \mathrm{Hom}_\Lambda(\Lambda, \mathbb{Z}) \cong \mathbb{Z},$$

generated by $\mathrm{aug} \circ \mathrm{pr}_\Lambda\colon \Lambda \oplus \Lambda^q \to \mathbb{Z}$, the projection onto the first factor followed by the augmentation map.

Let $[X] = [1 \otimes x] \in H_n(X; \mathbb{Z}^\omega)$ be the homology class corresponding to $\mathrm{aug} \circ \mathrm{pr}_\Lambda$ under the isomorphism $H_n(X; \mathbb{Z}^\omega) = \ker(1 \otimes d_n) \cong \ker(\varphi^+) \cong \mathrm{Hom}_\Lambda(\Lambda, \mathbb{Z})$. Then $x \in (\Lambda \oplus \Lambda^q)^*$ is projection onto the first factor.

(ii) By the proof of Lemma 1, $\nu_{\boldsymbol{C}(\widetilde{X}),n}([X])$ is represented by

$$F^{n-1}(\boldsymbol{C}(\widetilde{X})) \to I, \quad [\psi] \mapsto \overline{\psi(d_n(x))}.$$

Thus, given $\psi \in C_{n-1}(\widetilde{X})^* = C_{n-1}(\widetilde{K})^*$,

$$\begin{aligned}
\overline{\psi(d_n(x))} &= \overline{\psi({}^{\omega}\varepsilon^{-1}(x \circ \varphi))} \\
&= {}^{\omega}\varepsilon({}^{\omega}\varepsilon^{-1}(x \circ \varphi))(\psi) \\
&= (x \circ \varphi)(\psi) \\
&= \left(x \circ \begin{bmatrix} i & 0 \\ 0 & \mathrm{id} \end{bmatrix} \circ j \circ p\right)(\psi) \\
&= (i \circ \mathrm{pr}_{\boldsymbol{I}} \circ j)([\psi]) \\
&= h([\psi]).
\end{aligned}$$

Hence, $\nu_{\boldsymbol{C}(\widetilde{X}),n}([X])$ is the homotopy class of h, so that

$$\nu_{\boldsymbol{C}(\widetilde{K}),n}(\mu) = \nu_{\boldsymbol{C}(\widetilde{X}),n}([X]) = \nu_{\boldsymbol{C}(\widetilde{K}),n}(f_*([X])).$$

By Lemma 2.5 in [21], $\nu_{\boldsymbol{C}(\widetilde{K}),n}$ is injective, whence $\mu = f_*([X])$.

(iii) First consider $1 \le i < n-1$. Then $H_i(X;\Lambda) = H_i(K^{[n-1]};\Lambda) = 0$.

By the definition of φ,

$$H^{n-1}(X; {}^{\omega}\Lambda^{\omega}) = 0.$$

Moreover, by hypothesis,

$$H^{n-i}(X;\Lambda^{\omega}) = H^{n-i}(K^{[n-1]};\, \Lambda^{\omega}) \cong H^{n-i}(G;\Lambda^{\omega}) = 0$$

for $1 < i < n-1$. Thus,

$$- \frown (1 \otimes [X])\colon H^{n-i}(X; {}^{\omega}\Lambda) \to H_i(X;\Lambda)$$

is an isomorphism for $1 \le i < n-1$.

Next consider $i = 0$. As P and $\Lambda \oplus P$ are free, $\boldsymbol{C}(\widetilde{X})$ is a chain complex of free Λ–modules. Since the (twisted) evaluation map from a finitely generated free Λ–module to its double dual is an isomorphism,

$$\begin{aligned}
H^n(X; {}^{\omega}\Lambda) &= {}^{\omega}\mathrm{Hom}_{\Lambda}(C_n(\widetilde{X}), {}^{\omega}\Lambda)/\operatorname{im}(\varphi^*)^* = (\Lambda \oplus P)^{**}/\operatorname{im}(\varphi^*)^* \\
&\cong (\Lambda \oplus P)/\operatorname{im}(\varphi) \cong \Lambda/I \cong \mathbb{Z}.
\end{aligned}$$

The class $[\gamma]$ of the image of $(1,0)\in\Lambda\oplus P$ under the (twisted) evaluation isomorphism generates $H^n(X;{}^\omega\Lambda)$ and so, by Lemma 4.4 of [4],

$$[\gamma]\frown[X]=[\gamma]\frown[1\otimes x]=[\overline{\gamma(x)}.e_0]=[e_0],$$

where $e_0\in C_0(\widetilde{X})$ is a chain representing the basepoint. Thus,

$$-\frown[X]\colon H^n(X;{}^\omega\Lambda)\to H_0(X;\Lambda)$$

is an isomorphism.

Finally, note that by the above, $-\frown(1\otimes x)$ yields the chain homotopy equivalence

$$\begin{array}{ccccc}
\operatorname{im} d_{n-1}^* & \rightarrowtail & C_{n-1}(\widetilde{X})^* & \twoheadrightarrow & C_n(\widetilde{X})^* \\
\downarrow & & \downarrow{\scriptstyle -\frown(1\otimes x)} & & \downarrow{\scriptstyle -\frown(1\otimes x)} \\
\operatorname{im} d_2 & \rightarrowtail & C_1(\widetilde{X}) & \twoheadrightarrow & C_0(\widetilde{X})
\end{array}$$

Applying the functor ${}^\omega\mathrm{Hom}_\Lambda(-,\Lambda)$, we obtain the chain homotopy equivalence $(-\frown(1\otimes x))^*$, inducing isomorphisms

$$\begin{aligned}(-\frown[X])^*&\colon H^0(X;{}^\omega\Lambda)\to H_n(X;\Lambda),\\ (-\frown[X])^*&\colon H^1(X;{}^\omega\Lambda)\to H_{n-1}(X;\Lambda).\end{aligned}$$

By Lemma 2.1 in [4], $(-\frown(1\otimes x))^*$ induces an isomorphism in homology if and only if $-\frown(1\otimes x)$ does, whence

$$\begin{aligned}-\frown[X]&\colon H^0(X;{}^\omega\Lambda)\to H_n(X;\Lambda),\\ -\frown[X]&\colon H^1(X;{}^\omega\Lambda)\to H_{n-1}(X;\Lambda)\end{aligned}$$

are isomorphisms. □

Suppose now that P is projective, but not free.

Then there are a finitely generated Λ–module Q and a natural number q such that $P^*\oplus Q\cong\Lambda^q$. The natural isomorphisms

$$(\Lambda\oplus P)^*\oplus\Lambda^\infty\cong\Lambda^*\oplus P^*\oplus(Q\oplus P^*\oplus\cdots)\cong\Lambda\oplus(P^*\oplus Q\oplus P^*\oplus Q\oplus\cdots)\cong\Lambda^\infty$$

show that $(\Lambda\oplus P)^*\oplus\Lambda^\infty$ is a free Λ–module.

Consider the chain complex $\boldsymbol{D}$ given by

$$0\to(\Lambda\oplus P)^*\oplus\Lambda^\infty\xrightarrow{\left[\begin{smallmatrix}d_n & 0\\ 0 & \mathrm{id}\end{smallmatrix}\right]}C_{n-1}(\widetilde{K}^{[n-1]})\oplus\Lambda^\infty\xrightarrow{[\,d_{n-1}\ 0\,]}C_{n-2}(\widetilde{K}^{[n-1]})\xrightarrow{d_{n-2}}C_{n-3}(\widetilde{K}^{[n-1]})\to\cdots.$$

We attach infinitely many n–balls to $K^{[n-1]}$ to obtain a CW–complex, K', whose cellular chain complex coincides with $\boldsymbol{D}$ in dimensions below n. Then $\widetilde{K}'^{[n-1]}$ is $(n-2)$–connected, and the Hurewicz homomorphisms $h_q\colon \pi_q(\widetilde{K}'^{[n-1]}) \to H_q(\widetilde{K}'^{[n-1]})$ are isomorphisms for $q \leq n-1$. Defining the map

$$\begin{aligned}\varphi'\colon (\Lambda \oplus P)^* \oplus \Lambda^\infty \to \ker(d_{n-1}) = H_{n-1}(\widetilde{K}'^{[n-1]}) &\xrightarrow{h_{n-1}^{-1}} \pi_{n-1}(\widetilde{K}'^{[n-1]}),\\ x &\mapsto h_{n-1}^{-1}([d_n(x)]),\end{aligned}$$

we obtain the homotopy system $Y' = (\boldsymbol{D}, \varphi', K'^{[n-1]})$ of order n. As $D_i = 0$ for $i > n$, $\widehat{H}^{n+2}(Y'; \Gamma_n Y') = 0$. By Proposition 8.3 in [2], there is then a homotopy system $(\boldsymbol{C}, 0, X')$ of order $n+1$ realising Y', with X' an n–dimensional CW–complex.

Note that $\boldsymbol{D}$, the chain complex of X', is chain homotopy equivalent to the chain complex $\boldsymbol{W}$ given by

$$\begin{aligned}\cdots \to P^*\oplus Q \xrightarrow{\left[\begin{smallmatrix}\mathrm{id} & 0\\ 0 & 0\end{smallmatrix}\right]} P^*\oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 0\\ 0 & \mathrm{id}\end{smallmatrix}\right]} P^*\oplus Q \xrightarrow{\left[\begin{smallmatrix}0 & 0\\ 0 & 0\\ 0 & \mathrm{id}\end{smallmatrix}\right]} (\Lambda\oplus P)^*\oplus Q \xrightarrow{\left[\begin{smallmatrix}d_n\\ 0\end{smallmatrix}\right]} C_{n-1}(\widetilde{K}^{[n-1]})\\ \xrightarrow{d_{n-1}} C_{n-2}(\widetilde{K}^{[n-1]}) \xrightarrow{d_{n-2}} C_{n-3}(\widetilde{K}^{[n-1]}) \to \cdots.\end{aligned}$$

By Theorem 2 of [23], there is a CW–complex X, with cellular chain complex $\boldsymbol{W}$, homotopy equivalent to X'. Since $\boldsymbol{W}$ is finitely generated in each degree, the proof that X realizes (G, ω, μ) is analogous to the proof of Proposition 5.

This completes the proof of Theorem A.

5 Decomposition as connected sum

Wall constructed a new PD_n–complex from given ones using the *connected sum of* PD_n–*complexes* (see [24]). This allows PD_n–complexes to be decomposed as connected sums of other, simpler PD_n–complexes.

Take PD_n–complexes $(X_k, \omega_k, [X_k])$ for $k = 1, 2$. Then we may express X_k as the mapping cone

$$X_k = X_k' \cup_{f_k} e_k^n$$

for suitable $f_k\colon S^{n-1} \to X_k'$. Here, X_k' is an $(n-1)$–dimensional CW–complex when $n > 3$, and when $n = 3$, X_k' is 3–dimensional with $H^3(X_k'; B) = 0$ for all coefficient modules B. For $k = 1, 2$, let $\iota_k\colon X_k' \to X_1' \vee X_2'$ be the canonical inclusion of the k^{th} summand and put

$$\widehat{f_k} := \iota_k \circ f_k\colon S^{n-1} \to X_1' \vee X_2',$$

so that $\widehat{f_k}$ determines an element of $\pi_{n-1}(X_1' \vee X_2')$. Let $f_1 + f_2\colon S^{n-1} \to X_1' \vee X_2'$ represent the homotopy class $[\widehat{f_1}]+[\widehat{f_2}]$. Then the connected sum $X = X_1 \# X_2$ of X_1 and X_2 is the mapping cone of $f_1 + f_2$:

$$X_1 \# X_2 := (X_1' \vee X_2') \cup_{f_1+f_2} e^n.$$

It follows from the Seifert–van Kampen theorem that

$$\pi_1(X) = \pi_1(X_1) * \pi_1(X_2). \tag{3}$$

The canonical inclusion $\mathrm{in}_k\colon \pi_1(X_k) \to \pi_1(X)$ induces a (left or right) $\mathbb{Z}[\pi_1(X_k)]$–module structure on any (left or right) $\Lambda = \mathbb{Z}[\pi_1(X)]$–module. In particular, Λ is a $\pi_1(X_k)$–bimodule. By the universal property of the free product, the group homomorphisms $\omega_{X_k} = \mathrm{in}_k^*(\omega_X)$ uniquely determine a group homomorphism $\omega_X\colon \pi_1(X) \to \mathbb{Z}/2\mathbb{Z}$. For $k = 1, 2$, let L_k be the functor $\Lambda \otimes_{\mathbb{Z}[\pi_1(X_k)]} -$.

Let $\boldsymbol{B}$ be the subcomplex of $\boldsymbol{C}(\widetilde{X})$ containing the n–cells over the n–cell of X attached by $f_1 + f_2$. Then $\boldsymbol{B}$ is a Poincaré duality chain complex [2, page 2361] and it follows from Theorem 2.3 of [4] that $L_k(\boldsymbol{C}(\widetilde{X}_k))$ is also a Poincaré duality chain complex.

Let x denote the chain representing the n–cell attached by $f_1 + f_2$. Repeated application of Theorem 2.3 of [4] shows that $L_1(\boldsymbol{C}(\widetilde{X}_1)) + L_2(\boldsymbol{C}(\widetilde{X}_2))$ is a Poincaré duality chain complex. Hence, $(X, \omega_X, [1 \otimes x])$ is a Poincaré duality complex. This is the *connected sum of* $(X_1, \omega_{X_1}, [X_1])$ *and* $(X_2, \omega_{X_2}, [X_2])$, introduced by Wall [24].

Theorem B *A* PD_n*–complex with* $(n-2)$*–connected universal cover decomposes as a nontrivial connected sum if and only if its fundamental group decomposes as a nontrivial free product of groups.*

Proof Suppose the PD_n–complex, X, is the nontrivial connected sum of the PD_n–complexes, X_1 and X_2. Then, by (3), its fundamental group is the nontrivial free product of the fundamental groups of X_1 and X_2.

For the converse, let $(X, \omega_X, [X])$ be a PD_n–complex with $(n-2)$–connected universal cover and with $\pi_1(X) = G = G_1 * G_2$ for nontrivial groups G_1 and G_2. As $\pi_1(X)$ is finitely presentable, so are G_1 and G_2. For $j = 1, 2$, let $K_j = K(G_j; 1)$ be an Eilenberg–Mac Lane space with finite 2–skeleton. Then $K_1 \vee K_2$ is an Eilenberg–Mac Lane space $K(G_1 * G_2; 1)$, and

$$H_n(K; \mathbb{Z}^\omega) = H_n(K_1; \mathbb{Z}^{\omega_1}) \oplus H_n(K_n; \mathbb{Z}^{\omega_2}),$$

where $\omega_j \in H^1(K_j;\mathbb{Z}/2\mathbb{Z})$ for $j=1,2$ is the restriction of the orientation character $\omega \in H^1(K;\mathbb{Z}/2\mathbb{Z})$. Thus, $\mu_X = \mu_1 + \mu_2$, with $\mu_j \in H_n(K_j;\mathbb{Z}^{\omega_j})$ for $j=1,2$.

By the discussion above, if the PD_n–complex X_j, with $(n-2)$–connected universal cover realizes the fundamental triple (G_j,ω_j,μ_j), then the connected sum of X_1 and X_2 realizes the fundamental triple of X, whence, by the classification theorem in [2], X is orientedly homotopy equivalent to $X_1 \# X_2$. Hence, it is sufficient to construct realizations of (G_j,ω_j,μ_j) for $j=1,2$.

Let S_j be the functor $\Lambda \otimes_{\mathbb{Z}[G_j]} -$, so that, for $i \geq 1$,

$$C_i(\widetilde{K}) = S_1(C_i(\widetilde{K}_1)) \oplus S_2(C_i(\widetilde{K}_2)).$$

It follows that

$$F^{n-1}(\boldsymbol{C}(\widetilde{K})) = S_1\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_1))\big) \oplus S_2\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_2))\big)$$

and

$$I(\pi_1(X)) = S_1(I(G_1)) \oplus S_2(I(G_2)),$$

where the canonical inclusion is given by

$$S_j(I(G_j)) \to I(G_1 * G_2), \quad \sigma \otimes \lambda \mapsto \sigma\lambda,$$

for $\sigma \in \mathbb{Z}[\pi_1(X)]$ and $\lambda \in I(G_1 * G_2)$ viewed as an element of $I(\pi_1(X))$.

Let

$$\varphi_j\colon F^{n-1}(\boldsymbol{C}(\widetilde{K}_i)) \to I(G_j)$$

be a $\mathbb{Z}[G_j]$–morphism representing the class $\nu_{C(\widetilde{K}_j),n}(\mu_j)$. Then the class, $\nu_{C(\widetilde{K}),n}(\mu)$ of homotopy equivalences is represented by

$$\begin{array}{ccc} S_1\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_1))\big) \oplus S_2\big(F^{n-1}(\boldsymbol{C}(\widetilde{K}_2))\big) & = & F^{n-1}(\boldsymbol{C}(\widetilde{K})) \\ \downarrow{\scriptstyle S_1(\varphi_1)\oplus S_2(\varphi_2)} & & \\ S_1(I(G_1)) \oplus S_2(I(G_2)) & = & I(G_1 * G_2) \end{array}$$

and it follows from the proof of the analogous proposition for $n=3$ [21, pages 269–270] that φ_j is a homotopy equivalence of modules. By Theorem A, (G_j,ω_j,μ_j) is realized by a PD_n–complex X_j with $(n-2)$–connected universal cover. □

As in [21], Theorem B implies that when $\pi_1(X)$ is torsion-free the indecomposable summands of X are either aspherical or copies of $S^{n-1}\times S^1$ or $S^{n-1}\,\widetilde{\times}\, S^1$, where

$S^k \,\widetilde{\times}\, S^1$ is the mapping cylinder of an orientation-reversing self-homeomorphism of S^k.

In the next sections we shall consider what may happen when we allow $\pi_1(X)$ to have torsion.

6 An interlude on graphs of groups

Our arguments in the second part of this paper use the notion of graph of groups, for which our main references are [9; 18]. In particular, we rely on the fact that every finitely presentable group is accessible: it is the fundamental group of a finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. (See Theorem VI.6.3 of [9].) A *graph of groups* $(\mathcal{G}, \Gamma)$ consists of a graph Γ with origin and target functions o and t from the set of edges $E = E(\Gamma)$ to the set of vertices $V = V(\Gamma)$, and a family $\mathcal{G}$ of groups G_v for each vertex v and subgroups $G_e \leq G_{o(e)}$ for each edge e, with monomorphisms $\phi_e\colon G_e \to G_{t(e)}$. (We shall usually suppress the maps ϕ_e from our notation.) All edges are oriented, but we do not use this, and in considering paths or circuits in Γ we shall not require that the edges be compatibly oriented. The *fundamental group* of $(\mathcal{G}, \Gamma)$ is the group $\pi\mathcal{G}$ with presentation

$$\big\langle G_v, t_e \ \forall v \in V(\Gamma),\ e \in E(\Gamma) \mid \ t_e g t_e^{-1} = \phi_e(g)\ \forall g \in G_e,\ e \in E(\Gamma),\ t_f = 1\ \forall f \in E(\Upsilon) \big\rangle,$$

where Υ is some maximal tree for Γ. The generator t_e is the *stable letter* associated to the edge e. Different choices of maximal tree give isomorphic groups. We may assume that $(\mathcal{G}, \Gamma)$ is *reduced*: if an edge joins distinct vertices then the edge group is isomorphic to a proper subgroup of each of these vertex groups. If $\pi\mathcal{G}$ is indecomposable as a free product then $(\mathcal{G}, \Gamma)$ is *indecomposable*: all edge groups are nontrivial. An edge e is a *loop isomorphism* at v if $o(e) = t(e) = v$ and the inclusions induce isomorphisms $G_e \cong G_v$. It is an *MC-tie* if $o(e) \neq t(e)$ and G_e has index 2 in each of $G_{o(e)}$ and $G_{t(e)}$. (We shall give the motivation for this name later.)

In an alternative formulation, the graph of groups $(\mathcal{G}, \Gamma)$ determines a tree T on which $\pi\mathcal{G}$ acts, such that the stabilizers of edges are the conjugates of the edge groups and the stabilizers of the vertices are the conjugates of the vertex groups. A $\pi\mathcal{G}$–tree T is *terminal* if each edge stabilizer is finite and each vertex stabilizer is finite or has one end. If $(\mathcal{G}, \Gamma)$ is reduced, the corresponding $\pi\mathcal{G}$–tree T is incompressible in

the terminology of [9]. If G is a finitely generated accessible group then there is an essentially unique incompressible terminal G–tree, by Proposition IV.7.4 of [9]. Following [8], we shall say that a vertex of T is finite or infinite if its stabilizer in $\pi\mathcal{G}$ is finite or infinite, respectively. Let V_f be the subset of vertices v such that G_v is finite. Every finite subgroup of $\pi\mathcal{G}$ fixes a vertex of T, by Corollary I.4.9 of [9], and so is conjugate to a subgroup of G_v for some $v \in V$. (See Proposition I.7.11 of [9].) Thus finite vertex subgroups are *maximal* finite subgroups of $\pi\mathcal{G}$.

The two most important special cases are when Γ has a single edge e. If the vertices are distinct then $\pi\mathcal{G} \cong A *_C B$ is the generalized free product of $A = G_{o(e)}$ and $B = G_{t(e)}$ with amalgamation over $C = G_e$. If $o(e) = t(e)$ then $\pi\mathcal{G} \cong A*_\varphi$ is the HNN extension with base $A = G_{o(e)}$, associated subgroups G_e and $\phi_e(G_e)$ and characteristic isomorphism $\varphi = \phi_e$.

If σ is a subgroup of finite index in $\pi\mathcal{G}$ and T is a terminal $\pi\mathcal{G}$–tree then the stabilizers of the natural action of σ on T are finite or one-ended. Hence, σ is the fundamental group of a finite graph of groups $(\mathcal{G}_\sigma, \Gamma_\sigma)$, where $\Gamma_\sigma = \sigma\backslash T$ projects naturally onto Γ. However, if σ is a proper subgroup of $\pi\mathcal{G}$ then $(\mathcal{G}_\sigma, \Gamma_\sigma)$ need not be reduced or indecomposable.

A finitely generated group is virtually free if and only if it is the fundamental group of a finite graph of finite groups. (See Corollary IV.1.9 of [9].) It is virtually $\mathbb{Z}$ if and only if it has two ends if and only if it has a (maximal) finite normal subgroup F such that the quotient is infinite cyclic or is isomorphic to the infinite dihedral group $D_\infty = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. (See pages 129–130 of [9].)

If H is a subgroup of a group G, let $C_G(H)$ and $N_G(H)$ denote the centralizer and normalizer of H in G, respectively. If $x \in G$, let $\langle x\rangle$ be the cyclic subgroup generated by x, and let $C_G(x) = C_G(\langle x\rangle)$.

Lemma 6 *Let $(\mathcal{G}, \Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. If an edge e is a loop isomorphism or an MC-tie then $N_{\pi\mathcal{G}}(G_e)$ is infinite. If a vertex group G_v is finite then $N_{\pi\mathcal{G}}(G_v)$ is infinite if and only if there is a loop isomorphism at v.*

Proof If e is a loop isomorphism at v then the stable letter t_e associated with the edge normalizes $G_e = G_v$. If e is an MC-tie with ends u and v then G_e is normal in each of G_u and G_v. Hence, if $\alpha \in G_u \setminus G_e$ and $\beta \in G_v \setminus G_e$ then $\alpha\beta$ is an element of infinite order in $N_{\pi\mathcal{G}}(G_e)$.

Suppose that G_v is finite and $N_{\pi\mathcal{G}}(G_v)$ is infinite. The fixed-point set of the action of G_v on a terminal $\pi\mathcal{G}$–tree is a nonempty subtree which is preserved by $N_{\pi\mathcal{G}}(G_v)$. Since $N_{\pi\mathcal{G}}(G_v)$ is infinite this subtree must have a nontrivial edge, with image e in Γ having v as one vertex. Then $G_e = G_v$, since this edge is fixed by G_v, and so e must be a loop isomorphism at v, since $\mathcal{G}$ is reduced. □

Let $(\mathcal{G}, \Gamma)$ be a graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end. There is an associated "Chiswell" exact sequence of right $\mathbb{Z}[\pi\mathcal{G}]$–modules

$$0 \to \bigoplus_{v\in V_f} \mathbb{Z}[G_v\backslash\pi\mathcal{G}] \xrightarrow{\Delta} \bigoplus_{e\in E} \mathbb{Z}[G_e\backslash\pi\mathcal{G}] \to H^1(\pi\mathcal{G}; \mathbb{Z}[\pi\mathcal{G}]) \to 0,$$

in which the image of a coset $G_v g$ of G_v in $\pi\mathcal{G}$ under Δ is

$$\Delta(G_v g) = \sum_{o(e)=v} \Big(\sum_{G_e h\subset G_v} G_e hg \Big) - \sum_{t(e)=v} \Big(\sum_{G_e h\subset G_v} G_e hg \Big).$$

The outer sums are over edges e and the inner sums are over cosets of G_e in G_v. (This follows from part (1) of Theorem 2 of [7], with $i = 0$, for if $G < \pi$ then $H^0(G; \mathbb{Z}[\pi]) \cong \mathbb{Z}[G\backslash\pi]$ if G is finite and is 0 if G is infinite. The extreme terms in the sequence in the cited theorem are 0, since the vertex groups are finite or one-ended.) If C is a finite subgroup of G then the summands $\mathbb{Z}[G_v\backslash\pi\mathcal{G}]$ and $\mathbb{Z}[G_e\backslash\pi\mathcal{G}]$ are themselves direct sums of permutation modules, when considered as right $\mathbb{Z}[C]$–modules.

7 The centralizer condition of Crisp

In the remainder of this paper we shall consider indecomposable PD_n–complexes with $(n-2)$–connected universal covers. The arguments of [8] for the case $n = 3$ apply equally well in higher dimensions. When n is odd they imply that the indecomposable PD_n–complexes of this type are either aspherical or have virtually free fundamental group. Theorem 17 of [8] leads to strong constraints on the possible groups when the fundamental group is virtually free, as in [15]. The consequences are different when n is even. In particular, there may be no simple characterization of the indecomposables. However, if the PD_n–complex is indecomposable and its fundamental group is virtually free then in all known cases the fundamental group either has two ends or has order ≤ 2.

Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $G=\pi_1(X)$ and $\omega=w_1(X)$. Let $G^+=\mathrm{Ker}(\omega)$ and let X^+ be the corresponding orientable covering space. Since G is finitely presentable, it is the fundamental group of a finite graph of groups $(\mathcal{G},\Gamma)$, where all vertex groups are finite or have one end and all edge groups are finite, by Theorem VI.6.3 of [9]. Since G is indecomposable as a proper free product, by Theorem B, all edge groups are nontrivial.

The first nontrivial higher homotopy group of X is $\pi_{n-1}(X)$. As observed in Section 3, this is isomorphic to $H_{n-1}(\widetilde{X};\mathbb{Z})$ and then to ${}^{\omega}H^1(G;\mathbb{Z}[G])$, by the Hurewicz theorem and Poincaré duality, respectively. A homological argument by devissage gives isomorphisms

$$H_s(C;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong H_s(C;H_{n+1}(\widetilde{X};\mathbb{Z}))\cong H_{s+n}(C;\mathbb{Z})$$

for all subgroups $C\leq G$ and all $s\geq 1$. (See Lemma 2.10 of [14].) The work in [8] relates these homological properties of $H^1(G;\mathbb{Z}[G])$ to the presentation of $H^1(G;\mathbb{Z}[G])$ via the Chiswell exact sequence, when $n=3$. We shall see that this connection extends to all dimensions n, with due consideration of the parity of n.

Suppose first that n is odd, and that C is a finite cyclic subgroup of $G=\pi_1(X)$. Then $H_{n+1}(C;\mathbb{Z})=0$ and $H_{n+2}(C;\mathbb{Z})\cong C$. The arguments of Theorems 14 and 17 of [8] extend immediately to show that (i) if X is orientable and indecomposable then either X is aspherical or G is virtually free; and (ii) if $g\in G$ has prime order $p>1$ and $C_G(g)$ is infinite then $p=2$, $w(g)=-1$ and $C_G(g)$ has two ends. We may then apply the analysis of [15] to further constrain the possibilities. However, implementing the realization theorem may be difficult, since it involves the module $F^{n-1}(C(\widetilde{K}))=\mathrm{coker}(d^*_{n-2})$. As there is no algorithm for computing the homology of a finitely presentable group in degrees >1 [11], there may be no algorithm to provide an explicit matrix for d_{n-2} if $n>3$, in general. This may not be a problem when G is virtually free. In particular, is $S_3*_{\mathbb{Z}/2\mathbb{Z}}S_3$ the fundamental group of a PD_{2k+1}–complex with $(2k-1)$–connected universal cover for any $k>1$? (It is the group of a finite PD_3–complex [15].)

When n is even and C is finite cyclic, $H_{n+1}(C;\mathbb{Z})\cong C$ and $H_{n+2}(C;\mathbb{Z})=0$. In this case, Lemma 2.10 of [14] gives

$$H_1(C;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong H_{n+1}(C;\mathbb{Z})\cong C.$$

Let T be a terminal G–tree, with $e(T)$ ends and $\infty(T)$ vertices with infinite stabilizers, and let $\xi(T)=e(T)+\infty(T)-1$. If $g\in G$ has prime order then (since n is even)

Remark 13 of [8] gives either

$$\omega(g) = 1 \quad \text{and} \quad \xi(T^{\langle g\rangle}) = 1$$

or

$$\omega(g) = -1 \quad \text{and} \quad \xi(T^{\langle g\rangle}) = -1.$$

(The G–tree T is denoted by X in [8]). The argument of Theorem 17 of [8] then gives the following:

Theorem 7 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $G = \pi_1(X)$ and $\omega = w_1(X)$. If n is even, $x \in G$ has order $m > 1$ and $C_G(x)$ is infinite, then $C_G(x)$ is virtually $\mathbb{Z}$ and either $\omega(x) = 1$ or $4|m$. Moreover, no conjugate of x is in any infinite vertex group.*

Proof If $g \in G$ has prime order p and $\omega(g) = -1$ then $p = 2$ and $\xi(T^{\langle g\rangle}) = -1$. Hence, g does not fix any end or infinite vertex, and so $T^{\langle g\rangle}$ is a nonempty finite tree with all vertices finite. Since $C_G(g)$ leaves $T^{\langle g\rangle}$ invariant, it is finite. Hence, if $C_G(g)$ is infinite then $\omega(g) = +1$ and $\xi(T^{\langle g\rangle}) = 1$. As in [8], it follows that g fixes a ray $(\varepsilon, \varepsilon')$, but fixes no infinite vertex, and $C_G(g)$ is virtually $\mathbb{Z}$.

Suppose now that $x \in G$ has finite order m and $C_G(x)$ is infinite. If $m = 2k$ then x^k has order 2 and $C_G(x) \le C_G(x^k)$, so $C_G(x^k)$ is infinite. Hence, $\omega(x^k) = 1$, and so either $\omega(x) = 1$ or $4|m$.

If p is a prime factor of m then $x^{m/p}$ has order p, and so does not fix any infinite vertex of T. Hence, the same is true of x, and so no conjugate of x is in any infinite vertex group. □

We shall apply Theorem 7 together with the *normalizer condition — a proper subgroup of a nilpotent group is properly contained in its normalizer* [17, Proposition 5.2.4] — and the next lemma.

Lemma 8 *Let G be a group with a finite subgroup C.*

(1) *$C_G(C)$ has finite index in $N_G(C)$.*

(2) *If G has a subgroup isomorphic to $A *_C B$ and $N_G(C)$ is finite or has two ends, then either $N_A(C) = C$ or $N_B(C) = C$ or $[N_A(C) : C] = [N_B(C) : C] = 2$.*

(3) *If G has a subgroup isomorphic to $A *_C \varphi$ and $N_G(C)$ is finite or has two ends, then either $N_A(C) = C$ or $N_A(\varphi(C)) = \varphi(C)$ or*

$$[N_A(C) : C] = [N_A(\varphi(C)) : \varphi(C)] = 2.$$

Proof The first assertion is clear, since $\mathrm{Aut}(C)$ is finite.

For the second assertion, we may assume that $G = A *_C B$. The image of the subgroup generated by $N_A(C) \cup N_B(C)$ in the quotient $N_G(C)/C$ is isomorphic to $N_A(C)/C * N_B(C)/C$. Hence, if $N_G(C)$ is finite then either $N_A(C) = C$ (and $N_B(C)$ is finite) or $N_B(C) = C$ (and $N_A(C)$ is finite). If $N_G(C)$ has two ends then so does $N_G(C)/C$, and so $N_A(C)/C = N_B(C)/C = \mathbb{Z}/2\mathbb{Z}$.

If G has a subgroup isomorphic to $A *_C \varphi$ and t is the stable letter of the HNN extension, let $B = tAt^{-1}$. Then G has a subgroup isomorphic to $A *_C B$, where $C \leq A$ is identified with $\varphi(C) = tCt^{-1} \leq B$, and so (3) follows from (2). □

Lemma 9 *Let $(\mathcal{G}, \Gamma)$ be a reduced finite graph of groups. If e is an edge such that $G_{o(e)}$ and $G_{t(e)}$ are finite nilpotent groups, then either e is a loop isomorphism or it is an MC-tie. In particular, if $G_{o(e)}$ or $G_{t(e)}$ has odd order, then e must be a loop isomorphism.*

Proof This follows from the normalizer condition, Lemma 8 and Theorem 7. □

When $\pi_1(X)$ is virtually free, the following lemma complements Theorem 7.

Lemma 10 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, with n even. If $G = \pi_1(X)$ is virtually free and $g \in G^+$ has prime order $p \geq 2$, then $N_G(\langle g \rangle)$ has two ends.*

Proof We may assume that $G \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an indecomposable, reduced finite graph of finite groups. Let F be a free normal subgroup of finite index in G, and let ρ be the indecomposable factor of FC containing C. Then $N_\rho(C) = N_{FC}(C)$, and so has finite index in $N_G(C)$, since FC has finite index in G. Thus, we may assume that $G = \rho$. Since p is prime, the nontrivial edge stabilizers for the action of $C = \langle g \rangle$ on the terminal G–tree T are isomorphic to C. Hence, Γ has just one vertex and all the edges are loop isomorphisms, so G is a semidirect product $C \rtimes F(r)$, with $r \geq 0$. Clearly C is normal in this group. If $r = 0$ then G is finite and so $\widetilde{X} \simeq S^n$. But then $|G|\chi(X) = \chi(S^n) = 2$, and $G^+ = 1$, contrary to hypothesis. Therefore, $r > 0$, and so $N_G(C)$ is infinite, since it contains $C \rtimes F(r)$. Hence, $N_G(C)$ has two ends, by Theorem 7 and Lemma 8. □

8 Other consequences of the Chiswell sequence and Poincaré duality

We shall assume henceforth that n is even, and that X is a PD_n–complex with $(n-2)$–connected universal cover. If $G=\pi_1(X)$ is finite then $\widetilde{X}\simeq S^n$, and so $|G|\leq 2$. Hence $X\simeq S^n$ or $\mathbb{R}P^n$. If G has one end then $\widetilde{X}$ is contractible, and so X is aspherical. Hence, we may also assume that G has more than one end.

While our main concerns shall be with the case when all vertex groups are finite, elementary considerations give some complementary results.

Lemma 11 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $(\mathcal{G},\Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end, and such that $\pi\mathcal{G}\cong G=\pi_1(X)$. Let $g\in G$ have order q and $\omega(g)=1$, where $\omega=w_1(X)$. Then there is an exact sequence*

$$0\to\bigoplus_{v\in V_f}H_1(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_v\backslash G]))\to\bigoplus_{e\in E}H_1(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_e\backslash G]))\to\mathbb{Z}/q\mathbb{Z}\to 0.$$

Proof Since $\mathbb{Z}[G_v\backslash G]$ is a permutation $\mathbb{Z}[\langle g\rangle]$–module and $\omega(g)=1$,

$$H_0(\langle g\rangle;{}^{\omega}(\mathbb{Z}[G_v\backslash G]))$$

is a free abelian group for all $v\in V_f$. Since $H_1(\langle g\rangle;{}^{\omega}H^1(G;\mathbb{Z}[G]))\cong\mathbb{Z}/q\mathbb{Z}$ and $H_2(\langle g\rangle;{}^{\omega}H^1(G;\mathbb{Z}[G]))=0$ by Lemma 2.10 of [14], the result follows from the long exact sequence of homology for $\langle g\rangle$ associated to the short exact sequence of left $\mathbb{Z}[\pi]$–modules obtained by conjugating the Chiswell sequence. □

The result holds also if $\omega(g)=-1$ and no odd power of g is conjugate to an element of a finite vertex group. Otherwise, the righthand term of the short exact sequence may be $\mathbb{Z}/q'\mathbb{Z}$, where $q'=q$ or $\frac{1}{2}q$. However, we shall not need to consider the orientation-reversing case more closely.

Theorem 12 *Let X be an indecomposable PD_n–complex with $(n-2)$–connected universal cover, and let $(\mathcal{G},\Gamma)$ be a reduced finite graph of groups in which all edge groups are finite and all vertex groups are either finite or have one end, and such that $\pi\mathcal{G}\cong G=\pi_1(X)$. Let $\omega=w_1(X)$. Let $g\in G$ have order $q>1$. Then:*

(1) *If $q = p^r$ for some prime p and $r \geq 1$, and $\omega(g) = 1$, then g is conjugate to an element of an edge group. If g is in a finite vertex group G_v, then g is conjugate to an element of G_e for some edge e with $v \in \{o(e), t(e)\}$.*

(2) *Let $g \in G_e$, where e is an edge such that $G_{o(e)}$ and $G_{t(e)}$ each have one end, and suppose that $\omega(g) = 1$. If $xgx^{-1} \in G_{e'}$ for some $x \in G$ and edge e' such that $G_{o(e')}$ and $G_{t(e')}$ each have one end, then $x \in G_e$. Hence, $N_G(G_e) = G_e$.*

(3) *If G_v has one end for all $v \in V$ and $g \in G$ has finite order, then $\omega(g) = 1$.*

Proof If g has order p^r for some prime p and $\omega(g) = 1$, then

$$H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G])) \cong \mathbb{Z}/p^r\mathbb{Z}$$

for at least one edge e, by Lemma 11, for otherwise $\bigoplus_{e\in E} H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has exponent dividing p^{r-1}. Therefore, g must fix some coset $G_e x$, and so $xgx^{-1} \leq G_e$. If $g \in G_v$ but has no conjugate in G_e for any edge e with $v \in \{o(e), t(e)\}$, then the map from $H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_v\backslash G]))$ to $\bigoplus_{e\in E} H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has nontrivial kernel.

If $xgx^{-1} \in G'_e$ for some $x \notin G_e$ and edge e' with both adjacent vertex groups having one end, then $H^1(G; \mathbb{Z}[G])$ has more than one copy of the augmentation $\mathbb{Z}[\langle g\rangle]$–module $\mathbb{Z}$ as a direct summand. But then $H_1(\langle g\rangle; {}^{\omega}H^1(G; \mathbb{Z}[G]))$ would have at least two copies of $\mathbb{Z}/q\mathbb{Z}$ as direct summands, which would contradict Lemma 2.10 of [14].

If all vertex groups have one end, the Chiswell sequence reduces to an isomorphism $H^1(G; \mathbb{Z}[G]) \cong \bigoplus_{e\in E} \mathbb{Z}[G_e\backslash G]$. If g has order $2k$ and $\omega(g) = -1$, then $\omega(xgx^{-1}) = -1$ for all $x \in G$, and so $H_1(\langle g\rangle; {}^{\omega}(\mathbb{Z}[G_e\backslash G]))$ has exponent dividing k for all edges $e \in E$. Hence, $H_1(\langle g\rangle; {}^{\omega}H^1(G; \mathbb{Z}[G]))$ has exponent dividing k. This contradicts Lemma 2.10 of [14]. □

There are easy counterexamples to part (1) if n is odd or if $\omega(g) = -1$. Using Lemma 10, it can be shown that (1) holds if q is odd. (We do not need to know this below.) However, it does not always hold when q is even. The simplest counterexample is given by the fundamental group of the double of the nontrivial I–bundle over the lens space $L(6, 1)$, which is an amalgam of two copies of $\mathbb{Z}/6\mathbb{Z}$ over $\mathbb{Z}/3\mathbb{Z}$.

If $G \cong N \rtimes \mathbb{Z}/p\mathbb{Z}$, where N is torsion-free and p is an odd prime, then all edge groups are $\mathbb{Z}/p\mathbb{Z}$. Since $(\mathcal{G}, \Gamma)$ is reduced and indecomposable, either all vertex groups have one end or Γ has just one vertex and $G \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ (by Lemma 10).

Example Let $n \geq 4$ be even, and let M be an orientable n–manifold such that $\widetilde{M}$ is $(n-2)$–connected. Suppose that M has a self-homeomorphism g of prime order p and

with nonempty finite fixed-point set. Then g is orientation-preserving, since n is even. Let s be the number of fixed points, and let $U = M \setminus N$, where N is a $\langle g\rangle$–invariant regular neighbourhood of the fixed-point set. Then $\mu = \pi_1(U) \cong \pi_1(M)$, since $n > 2$. Let $V = U/\langle g\rangle$ and $X = D(V) = V \cup_{\partial V} V$. Then $\widetilde{X}$ is $(n-2)$–connected, by a Mayer–Vietoris argument, and $\pi_1(X) \cong (\mu * \mu * F(s-1)) \rtimes \mathbb{Z}/p\mathbb{Z}$. If μ has one end then $\pi_1(X) \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is a graph of groups, with Γ having two vertices and s edges, both vertex groups $\mu \rtimes \mathbb{Z}/p\mathbb{Z}$ and all edge groups $\mathbb{Z}/p\mathbb{Z}$.

This construction can be generalized, by starting with a finite group F which acts semifreely and with finite fixed-point set on one or several closed n–manifolds with $(n-2)$–connected universal covers. After deleting regular neighbourhoods of the fixed points, we may hope to assemble the pieces along pairs of boundary components with equivalent F–actions. Note that F must have cohomological period dividing n, since it acts freely on the boundary spheres. The analogous construction in the odd-dimensional cases gives only nonorientable examples (with F of order 2).

Explicitly: The 4–dimensional torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ has such self-maps, of orders 2, 3, 4, 5, 6 and 8. (It also has a semifree action of $Q(8)$ with finite fixed-point set.) The group $\mathbb{Z}/k\mathbb{Z}$ acts semifreely, with two fixed points, on T_k, the closed orientable surface of genus k. The corresponding diagonal action on $T_k \times T_k$ is semifree, with four fixed points. Similarly, $S^1 \times S^3$ has an orientation-preserving involution with four fixed points. (In the latter case, doubling the complement of the fixed-point set gives a virtually free group which is the free product of three two-ended factors.)

If p is prime, a locally smoothable $\mathbb{Z}/p\mathbb{Z}$–action on a closed manifold which is orientable over $\mathbb{F}_p$ cannot have exactly one fixed point. (See Corollary IV.2.3 of [5].) Thus, the above construction always leads to groups of the form $\sigma \rtimes \mathbb{Z}/p\mathbb{Z}$, where σ has a nontrivial free factor. Is there an example with $\pi_1(X)$ indecomposable and virtually a free product of PD_n–groups?

If $\pi_1(X)$ is virtually torsion-free, must the edge groups have cohomological period dividing n? In general, must $\pi_1(X)$ be virtually torsion-free? We suspect no, but have no counterexamples.

9 Virtually free fundamental group

We shall now restrict further to the class of (infinite) virtually free groups. The known indecomposable examples among manifolds with such groups are mapping tori of self-homeomorphisms of $(n-1)$–dimensional spherical space forms and unions of mapping

cylinders of double coverings of two such space forms (twisted I–bundles) with homeomorphic boundary. The fundamental groups have two ends, and graph of group structures with just one edge, which is a loop isomorphism or an MC-tie, respectively. (The examples involving mapping cylinders suggested the latter term.) There are similar constructions involving PD_{n-1}–complexes with universal cover $\simeq S^{n-1}$.

Our goal is to show that these examples are essentially all, provided that the fundamental group has no dihedral subgroup of order > 2. There are examples with dihedral subgroups and two ends, and there may still be indecomposable examples with infinitely many ends. (See Section 11 below.)

Lemma 13 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free. Let H be a nontrivial subgroup of $G_v \cap G^+$. Then there is an edge e with v as a vertex and such that $G_e \cap H \neq 1$.*

Proof Let F be a free normal subgroup of finite index in G^+. Then FH is the fundamental group of a finite orientable cover of X. If $G_e \cap H = 1$ for all edges e with v as a vertex, then the induced graph of groups structure for FH has a vertex group H with all adjacent edge groups trivial, and so H is a free factor of FH. Therefore, H is the fundamental group of an orientable PD_n–complex with $(n-2)$–connected universal cover, by Theorem B. But this is impossible, since n is even and $H \neq 1$. □

Theorem 14 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free. Then finite nilpotent subgroups of G of odd order are cyclic, and so finite subgroups of G of odd order are metacyclic.*

Proof Let F be a free normal subgroup of finite index in G and let $p\colon G \to G/F$ be the natural epimorphism. If S is a finite subgroup of G then $FS = p^{-1}p(S) \cong F \rtimes S$. On replacing FS by an indecomposable factor, if necessary, we may assume that $FS \cong \pi\mathcal{G}_S$, where $(\mathcal{G}_S, \Gamma_S)$ is an indecomposable reduced finite graph of groups, with vertex groups isomorphic to subgroups of S, and with at least one edge, since $|S| > 2$.

Suppose first that S is nilpotent, of odd order. Then Γ_S has just one vertex v and one edge, which is a loop isomorphism, by Lemma 9. Hence, $G \cong S \rtimes \mathbb{Z}$ and so has two ends. Therefore, $\widetilde{X} \simeq S^{n-1}$ and so S has periodic cohomology. Since S is nilpotent of odd order, it is cyclic.

In general, S is metacyclic, by Proposition 10.1.10 of [17], since all its Sylow subgroups are cyclic. □

This does not extend to subgroups of even order, as it stands. However, for groups of odd order "metacyclic with cyclic Sylow subgroups" is equivalent to "having periodic cohomology", and in all known examples the vertex groups have the latter property.

Corollary 15 *If G has no subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ then all finite subgroups of G have periodic cohomology.*

Proof The exclusion of $(\mathbb{Z}/2\mathbb{Z})^2$ implies that finite 2–groups in G are cyclic or quaternionic. (See Proposition 5.3.6 of [17].) Since all finite p–groups of odd order in G are cyclic by Theorem 14, it follows that all finite subgroups have periodic cohomology. (See Proposition VI.9.3 of [6].) □

Finite groups with periodic cohomology fall into six families:

(I) $\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$.

(II) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times Q(2^i))$ for $i \geq 3$.

(III) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times T_k^*)$ for $k \geq 1$.

(IV) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times O_k^*)$ for $k \geq 1$.

(V) $(\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}) \times \mathrm{SL}(2, p)$ for $p \geq 5$ prime.

(VI) $\mathbb{Z}/m\mathbb{Z} \rtimes (\mathbb{Z}/q\mathbb{Z} \times \mathrm{TL}(2, p))$ for $p \geq 5$ prime.

Here m is odd, and m, q and the order of the quotient by the metacyclic subgroup $\mathbb{Z}/m\mathbb{Z} \rtimes \mathbb{Z}/q\mathbb{Z}$ are relatively prime. The first family includes cyclic groups, dihedral groups $D_{2m} = \mathbb{Z}/m\mathbb{Z} \rtimes_{-1} \mathbb{Z}/2\mathbb{Z}$ with m odd, and the groups of odd order with periodic cohomology. The group $Q(2^i)$ is the quaternionic group of order 2^i, with presentation

$$\langle x, y \mid x^{2^{i-1}} = 1,\ x^{2^{i-2}} = y^2,\ yxy^{-1} = x^{-1} \rangle,$$

and T_k^* and O_k^* are the generalized binary tetrahedral and octahedral groups, respectively. Then $T_k^* \cong O_k^{*\prime} \cong Q(8) \rtimes \mathbb{Z}/3^k\mathbb{Z}$ and has index 2 in O_k^*. If p is an odd prime then $\mathrm{TL}(2, p)$ may be defined as follows. Choose a nonsquare $\rho \in \mathbb{F}_p^\times$, and let $\mathrm{TL}(2, p) \subset \mathrm{GL}(2, p)$ be the subset of matrices with determinant 1 or ρ. The multiplication $\star$ is given by $A \star B = AB$ if A or B has determinant 1, and $A \star B = \rho^{-1}AB$ otherwise. Then $\mathrm{SL}(2, p) = \mathrm{TL}(2, p)'$ and has index 2. (Note also that $\mathrm{SL}(2, 3) \cong T_1^*$ and $\mathrm{TL}(2, 3) \cong O_1^*$.)

We shall not specify the actions in the semidirect products here, as these play no role in our arguments. We shall only need the following simple facts about such groups,

which may easily be checked by inspecting the terms of the above list. Let P be a finite group of even order with periodic cohomology. If P is not metacyclic then its Sylow 2–subgroup is quaternionic, and P has a unique element of order 2, which is central. Hence, if P has a dihedral subgroup $D_{2\ell}$ then it is metacyclic. Moreover, $D'_{2\ell}$ is then normal in P. If P is metacyclic or of type IV or VI it has a unique subgroup of index 2, while if it is of type III or V there is no subgroup. Groups of type II have three such subgroups. (See [25] for more on these groups.)

10 Virtually free groups without dihedral subgroups

Our strategy for proving Theorem 20 (the main part of Theorem C) is to use Theorem 7, the normalizer condition and Lemma 13 to show that the graph has just one edge, which is either a loop isomorphism or an MC-tie. Lemma 16 implies that if G_v is a vertex group of maximal order then there is either a loop isomorphism or an MC-tie with v as one vertex. Lemmas 17 and 19 show that if π has no dihedral subgroup then all edge groups have index ≤ 2 in adjacent vertex groups. The main result then follows fairly easily. We shall assume that X is a PD_n–complex and $G = \pi_1(X) \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is a reduced, indecomposable finite graph of finite groups, with at least one edge, and that G does not have D_4 as a subgroup. We shall not state these conditions explicitly in the lemmas.

Lemma 16 *At each vertex v there is either a loop isomorphism or an edge e with distinct vertices v and w and such that $[G_v : G_e] = 2$ and $N_G(G_e)$ has two ends.*

Proof Let F be a free normal subgroup of finite index in G. After replacing G by an indecomposable factor of FG_v, if necessary, we may assume that $G = FG_v$. Since G is infinite and indecomposable, there is at least one edge with v as a vertex. If G_v has prime order then each such edge must be a loop isomorphism. Thus, we may assume henceforth that $|G_v|$ is not prime.

If G_v is metacyclic but $|G_v|$ is not a power of 2 then G_v has a cyclic normal subgroup S of odd prime order p. If G_v has order $2^k \geq 4$ or if G is not metacyclic then it has a central element g of order 2 such that $\omega(g) = 1$, and we let $S = \langle g \rangle$. In each case, $S \leq G^+$.

By Lemma 13, there is an edge e with v as one vertex and such that $S \leq G_e$. If both vertices are v then S is normalized by G_v and by t_e, the stable letter associated to e,

since S is the unique subgroup of G_v of order p. The subgroup $\langle G_v, t_e\rangle$ has infinitely many ends unless $G_e = G_v$. Hence, $G_e = G_v$, and so e is a loop isomorphism, by Theorem 7.

If e has distinct vertices $v \neq w$ then G_w is isomorphic to its image in $FG_v/F \cong G_v$. Hence, S is also normal in G_w, and $|G_w| \leq |G_v|$. Therefore, S is normal in $G_v *_{G_e} G_w$. Theorem 7 and Lemma 8 together imply that the normalizer of any finite subgroup of G is finite or has two ends. Hence, $[G_v : G_e] \leq 2$ and $[G_w : G_e] \leq 2$. Since e is not a loop isomorphism, $[G_v : G_e] = [G_w : G_e] = 2$. Hence, G_e is normal in $G_v *_{G_e} G_w$, and so $N_G(G_e)$ has two ends. □

If G_v has no subgroup of index 2 (eg if it is metacyclic of odd order or is of type III or V) then Lemma 16 ensures that there is a loop isomorphism at v. If G_v has maximal order among finite subgroups of G and $[G_v : G_e] = 2$, then e is an MC-tie. The argument for Lemma 16 shows that if e is not a loop isomorphism then it an MC-tie for the induced graph of groups structure for FG_v. However, it is not otherwise obvious that it must be an MC-tie for the original graph of groups $(\mathcal{G}, \Gamma)$.

Lemma 17 *Let f be an edge with both vertices v. If f is not a loop isomorphism then $|G_f| = 2$ and G_v is dihedral.*

Proof Suppose that $|G_f| > 2$. Let g be an element of G_f of prime order p. Since the Sylow subgroups of G_v are cyclic or quaternionic, each Sylow p–subgroup has a unique subgroup S of order p. Therefore, if t_f is the stable letter associated to f then there is an $a \in G_v$ such that $at_f g t_f^{-1} a^{-1} = g^s$ for some $0 < s < p$. Hence, $(at_f)^{p-1}$ centralizes S. By Lemma 16, there is another edge e which is either a loop isomorphism at v or has distinct vertices u and v, and such that $[G_v : G_e] = 2$ and $N_G(G_e)$ has two ends.

If e is a loop isomorphism, then S is also centralized by some power of t_e. If $[G_v : G_e] = 2$ and $N_G(G_e)$ has two ends, we may assume that $S \leq G_e$, since $|G_f| > 2$, and then S is centralized by an element of infinite order in $N_G(G_e)$. In each case, we find that S is centralized by a nonabelian free subgroup, contradicting Theorem 7.

Therefore, we must have $G_f \cong \mathbb{Z}/2\mathbb{Z}$. If G_v is not dihedral then G_f is central in G_v. But the subgroup generated by G_v and t_f contains a nonabelian free group, and so we again contradict Theorem 7. Since $G_v \ncong D_4$, it has order at least 6. □

This lemma indicates why we could require an MC-tie to have distinct vertices:

Lemma 18 *Let e and f be distinct edges with vertices u, v and v, w, respectively. If e is a loop isomorphism at v or an MC-tie, then $w \neq u$ or v, and f is neither a loop isomorphism nor an MC-tie.*

Proof Suppose that e is a loop isomorphism at v and that f also has both vertices v. If f is a loop isomorphism then G_v is normalized by the free group generated by the stable letters t_e and t_f, which contradicts Theorem 7. Therefore, f is not a loop isomorphism, and so Lemma 17 applies.

If e is a loop isomorphism and f is an MC-tie with vertices v, w, then G_f is normalized by $G_v *_{G_f} G_w$ and by some power of t_e (since G_v has only a finite number of subgroups of index 2). This again leads to a contradiction with Theorem 7.

Finally, if $u \neq v$ and e is an MC-tie then similar arguments show that $w \neq u$ or v, and that f is not an MC-tie. □

In particular, if every edge with v as one vertex is either a loop isomorphism or an MC-tie, then there is just one edge, and so G has two ends.

Lemma 19 *Let f be an edge with vertices $v \neq w$. If $[G_w : G_f] > 2$ then G_f has order 2, and G_v or G_w is dihedral.*

Proof In order to show that G_f has order 2, we may assume without loss of generality that $G = FG_w$, where F is a free normal subgroup of finite index. Then every finite subgroup of G is isomorphic to a subgroup of G_w, and so G_w has maximal order among such subgroups. We may also assume that $o(f) = v$ and $t(f) = w$. Clearly f is neither a loop isomorphism nor an MC-tie.

There are edges e and g, with vertices u, v and w, x, respectively, which are loop isomorphisms or for which $[G_v : G_e] = 2$ or $[G_w : G_g] = 2$, by Lemma 16. In the latter case, g must be an MC-tie, by the maximality of $|G_w|$. Hence, $v \neq w$ or x, by Lemma 18, and so $g \neq f$. The subgroups G_e and G_g are centralized by elements of infinite order. Hence, G_f has a subgroup H which is the intersection of two subgroups of index ≤ 2 in G_f, and which is centralized by these elements. We shall show that we may assume that they generate a nonabelian free subgroup of $C_G(H)$.

If e and g are each loop isomorphisms then $H = G_f$ is centralized by powers of the stable letters t_e and t_f. If e is a loop isomorphism and g is an MC-tie then

$H = G_f \cap G_g$ is centralized by powers of t_e and $\alpha'\beta'$, where $\alpha' \in G_w \setminus G_g$ and $\beta' \in G_x \setminus G_g$ do not involve t_e.

If e is not a loop isomorphism then $u \neq v$, by Lemma 17, and G_e is normalized by some $\alpha\beta$, where $\alpha \in G_u \setminus G_e$ and $\beta \in G_u \setminus G_e$. Suppose that $u \neq w$ or x. If g is a loop isomorphism then $H = G_f \cap G_e$ is normalized by powers of $\alpha\beta$ and t_g, If g is an MC-tie then $H = G_f \cap G_e \cap G_g$ is normalized by powers of $\alpha\beta$ and of $\alpha'\beta'$. A similar argument applies if $u = w$ or x.

In each case, these pairs generate a nonabelian free subgroup of $C_G(H)$, which contradicts Theorem 7, unless $H = 1$. Since G_f is nontrivial and is not D_4, we then have $G_f = \mathbb{Z}/2\mathbb{Z}$.

We now return to the general case (ie we do not assume that $G = FG_w$). Since $G_f = \mathbb{Z}/2\mathbb{Z}$, it is central in the Sylow 2–subgroups of G_v and G_w, and $C_G(G_f)$ has two ends or is finite. Hence, either G_f is its own centralizer in one vertex group, in which case the vertex group is dihedral, or these Sylow subgroups both have order 4, and no element of odd order in either vertex group commutes with G_f. In the latter case, the vertex groups are metacyclic groups of the form $\mathbb{Z}/m\mathbb{Z} \rtimes_\theta \mathbb{Z}/4\mathbb{Z}$, where m is odd and $\theta\colon \mathbb{Z}/4\mathbb{Z} \to (\mathbb{Z}/m\mathbb{Z})^\times$ is injective. Such groups have a unique subgroup of index 2, and so $G_f \leq G_e$ and $G_f \leq G_g$. But then the earlier argument applies with $H = G_f$ to show that $C_G(H)$ has a nonabelian free subgroup, contradicting Theorem 7. Hence, G_v or G_w is dihedral. □

Theorem 20 *Let X be a PD_n–complex with $(n-2)$–connected universal cover, where n is even. If $G = \pi_1(X)$ is infinite, virtually free and indecomposable, and no maximal finite subgroup is dihedral, then G has two ends, and its finite subgroups have cohomological period dividing n.*

Proof Let G_v be a vertex group of maximal order. Suppose first that G_v has odd order. Then G_v has no subgroup of index 2 and none of order 2, and so every edge e with v as a vertex must be a loop isomorphism, by Lemmas 17 and 19.

Since no maximal finite subgroup is dihedral, there are no dihedral vertex groups. Therefore, if G_v has even order > 4 and f is an edge with vertices v, w, then $[G_v : G_f] \leq 2$, by Lemmas 17 and 19. Since $|G_v|$ is maximal, f is either a loop isomorphism or an MC-tie.

In each of these cases there must be just one edge, by Lemma 18, and so π has two ends.

Finally, if all vertex groups have order 4 then they are cyclic, and all proper edge groups have order 2. Hence, there is a unique subgroup S of order 2. Clearly $S \leq G^+$, and so $G = C_G(S)$ has two ends, by Lemma 10.

Since G has two ends, $\widetilde{X} \simeq S^{n-1}$, and so finite subgroups of G have cohomological period dividing n. □

In the final case there are three possibilities: $G \cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z} \rtimes_{-1} \mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/4\mathbb{Z}$.

Corollary 21 *If G has no element of even order then $G \cong S \rtimes \mathbb{Z}$, where S is a finite metacyclic group of odd order and of cohomological period dividing n.* □

In particular, if n is a power of 2 then S must be cyclic. (See Exercise VI.9.6 of [6].)

When there is 2–torsion, G need not be an extension of $\mathbb{Z}$ by a finite normal subgroup. For example, if MC is the mapping cylinder of the double cover of a lens space $L = L(2m, q)$ and $X = D(\mathrm{MC}) = \mathrm{MC} \cup_L \mathrm{MC}$ is the double, then G is an extension of the infinite dihedral group D_∞ by $\mathbb{Z}/m\mathbb{Z}$, and $\widetilde{X} \cong S^3 \times \mathbb{R}$.

Theorem 22 *Let X be an orientable PD_n–complex such that $\widetilde{X} \simeq S^{n-1}$. Then $G = \pi_1(X) \cong F \rtimes_\theta \mathbb{Z}$, where F is the maximal finite normal subgroup of G, and X is a mapping torus.*

Proof Since $\widetilde{X} \simeq S^{n-1}$, the group G has two ends. Hence, it has a maximal finite normal subgroup F and a subgroup σ of index ≤ 2 which contains F and is such that $\sigma/F \cong \mathbb{Z}$. The covering space $X_F = \widetilde{X}/F \simeq S^{n-1}/F$ associated to F is a PD_{n-1}–complex, and so the covering space X_σ associated to σ is the mapping torus of a self-homotopy-equivalence of F. Hence, $\chi(X_\sigma) = 0$, and so $\chi(X) = 0$ also. But if $\sigma \neq G$ then G/G' is finite. Since $cd_{\mathbb{Q}} G = 1$, it follows from the spectral sequence for the universal covering that $H_q(X; \mathbb{Q}) = 0$ for $0 < q < n-1$. This is also the case when $q = n-1$, by Poincaré duality. Hence, $\chi(X) = 2$ (since n is even). This is a contradiction. Therefore, $\sigma = G \cong F \rtimes \mathbb{Z}$ and X is a mapping torus. □

We shall now restate and prove Theorem C of the introduction.

Theorem C *Let X be a PD_{2k}–complex with $(2k-2)$–connected universal cover, and such that $G = \pi_1(X)$ is virtually free and indecomposable as a free product. If*

G is finite then $X \simeq S^{2k}$ or $\mathbb{R}P^{2k}$. If G is infinite and has no dihedral subgroup of order > 2 then G has two ends and its finite subgroups have cohomological period dividing $2k$. Hence, $\widetilde{X} \simeq S^{2k-1}$. If, moreover, X is orientable, then $H^1(G;\mathbb{Z}) \cong \mathbb{Z}$.

Proof If G is finite then $\widetilde{X} \simeq S^{2k}$, and so $|G|\chi(X) = \chi(S^{2k}) = 2$. Hence, either $G = 1$ and $X \simeq S^{2k}$, or $G = \mathbb{Z}/2\mathbb{Z}$ and $X \simeq \mathbb{R}P^{2k}$.

If G is infinite and has no dihedral subgroup of order > 2, then G has two ends, and its finite subgroups have cohomological period dividing $2k$, by Theorem 20.

The final assertion follows from Theorem 22. □

It remains an open question whether the conclusion of Theorem C must hold if G has a dihedral maximal finite subgroup. (There are such examples with $G = \pi_1(X)$ having two ends — see Theorem 23 below.) Lemmas 18 and 19 impose some restrictions, but leave open the possibility that, for instance, there might be a PD_{2k}–complex X with $(2k-2)$–connected universal cover and $\pi_1(X) \cong \pi\mathcal{G}$, where the underlying graph Γ is a cycle of length four, the vertex groups are dihedral and the edges are alternately MC-ties or have edge group of order 2.

11 Construction of examples

Every finite group F with cohomological period q is the fundamental group of an orientable PD_{q-1}–complex with universal cover $\simeq S^{kq-1}$ for all $k \geq 1$ [19; 24]. Since q is even, such complexes are odd-dimensional. (In fact, the only nonorientable quotients of finite group actions on spheres are the even-dimensional real projective spaces $\mathbb{R}P^{2k}$.) We may use such complexes to realize groups with two ends.

Theorem 23 *Let F be a finite group. If $G \cong F \rtimes_\theta \mathbb{Z}$ then there is a PD_{2k}–complex X with $\pi_1(X) \cong G$ and $\widetilde{X} \simeq S^{2k-1}$ if and only if F has cohomological period dividing $2k$ and $H_{2k-1}(\theta;\mathbb{Z})$ is multiplication by ± 1. If $|F| > 2$, then X is orientable if and only if $H_{2k-1}(\theta;\mathbb{Z}) = 1$.*

Proof If a PD_{2k}–complex X has fundamental group G and universal cover $\widetilde{X} \simeq S^{2k-1}$ then F has cohomological period dividing $2k$, since it acts freely on $\widetilde{X}$, and the condition $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$ follows from the Wang sequence for the projection of X onto S^1 corresponding to the epimorphism $G \to G/F \cong \mathbb{Z}$. (See Theorem 11.1 of [14] for the case $k = 2$.)

Suppose, conversely, that F has cohomological period dividing $2k$. Then there is a based orientable PD_{2k-1}–complex X_F with fundamental group F and $\widetilde{X}_F \simeq S^{2k-1}$. If $H_{2k-1}(\theta;\mathbb{Z}) = \pm 1$ then there is a self-homotopy-equivalence f of X_F which induces θ [16]. The mapping torus of f is then a PD_{2k}–complex with fundamental group G and universal cover $\simeq S^{2k-1}$.

If $|F| > 2$ then $H_{2k-1}(\theta;\mathbb{Z}) = 1$ (as an automorphism of $H_{2k-1}(F;\mathbb{Z}) \cong \mathbb{Z}/|F|\mathbb{Z}$) if and only if $H_{2k-1}(f;\mathbb{Z}) = 1$ (as an automorphism of $H_{2k-1}(X;\mathbb{Z}) \cong \mathbb{Z}$) if and only if X is orientable. □

In particular, when the dimension $2k$ is divisible by 4, there are examples X with $\pi_1(X) \cong D_{2m} \times \mathbb{Z}$ for odd $m > 1$. These do not satisfy the hypotheses of Theorem C.

Suppose now that $G \cong E *_F H$, where E and H are finite groups with periodic cohomology and $[E:F] = [H:F] = 2$. Let n be a multiple of the cohomological periods of E and H. However, there is one subtlety: We must be able to choose PD_{2k-1}–complexes X_E and X_H with fundamental groups E and H and universal covers $\simeq S^{2k-1}$ in such a way that the double covers associated to the subgroups F are homotopy equivalent. For then we may construct a PD_{2k}–complex X with fundamental group G and $\widetilde{X} \simeq S^{n-1}$ by gluing together two mapping cylinders via a homotopy equivalence of their "boundaries". See Chapter 11 of [14] for an example with $k = 2$, $E = Q(24)$, $H = \mathbb{Z}/3\mathbb{Z} \times Q(8)$ and $F = \mathbb{Z}/12\mathbb{Z}$ where this construction cannot be carried through. (The difficulty is that PD_3–complexes with fundamental group E or H have unique homotopy types: the double covers corresponding to F are lens spaces which are not homotopy equivalent. Similar examples should exist in higher dimensions.)

Theorem 23 is essentially Theorem D of the introduction, and is the case $m = 1$ of Proposition 8 of [10]. Part of the discussion of the case $G \cong E *_F H$ in the previous paragraph may also be found in the final section of [10].

12 Must finite subgroups have periodic cohomology?

There remains the key question of whether the group $(\mathbb{Z}/2\mathbb{Z})^2$ ever arises in this context. Suppose that Y is a PD_{2k}–complex with $(2k-2)$–connected universal cover and virtually free fundamental group, and that $\pi_1(Y)$ has a subgroup $C \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let F be a free normal subgroup of finite index in $\pi_1(Y)$. We may assume that

$F < \pi_1(Y)^+$. Then Y has a finite cover Y_{FC} with fundamental group $FC \cong F \rtimes C$. As in Theorem 14, some indecomposable factor X of Y_{FC} has fundamental group $G = F(r) \rtimes C$ for some $r > 1$. (Since G cannot be finite of order 4 and C does not have periodic cohomology, $r \neq 0$ or 1.) We may assume that $G \cong \pi\mathcal{G}$, where $(\mathcal{G}, \Gamma)$ is an indecomposable reduced finite graph of groups, with all vertex groups $G_v \cong C$ and all edge groups G_e of order 2. In particular, every edge group is orientable, by Theorem 7.

Since G is virtually free, it has a well-defined virtual Euler characteristic

$$\chi^{\text{virt}}(G) = \frac{\chi(F(r))}{|C|} = \frac{1-r}{4}.$$

We also have $\chi^{\text{virt}}(G) = \frac{1}{4}|V| - \frac{1}{2}|E|$, since $(\mathcal{G}, \Gamma)$ is indecomposable and reduced. Moreover, $\chi(X) = 2\chi^{\text{virt}}(G)$, by the multiplicativity of (virtual) Euler characteristic for finite covers (passage to finite-index subgroups), and so $\chi^{\text{virt}}(G) \in \frac{1}{2}\mathbb{Z}$. Hence, $|V|$ is even.

Suppose first that X is not orientable. Then $\omega|_C \neq 1$, since $\omega(F) = 1$. Hence, if e and f are two edges with $o(e) = o(f) = v$ then $G_e = G_f = \text{Ker}(\omega|_{G_v})$. If $e \neq f$ then $C_G(G_e)$ contains a nonabelian free subgroup. Hence, there is at most one edge at each vertex. Since Γ is connected, there is just one edge e, which must have distinct vertices, for otherwise $C_G(G_e)$ would have a nonabelian free subgroup. Hence, $G \cong G_u *_{G_e} G_v \cong (\mathbb{Z}/2\mathbb{Z}) \times D_\infty$ has two ends, and so $\widetilde{X} \simeq S^{2k-1}$. But then C has periodic cohomology, which is false. Therefore, X must be orientable.

Since X is finitely covered by $\#^r(S^3 \times S^1)$, $H_2(X; \mathbb{Q}) = 0$, and so $\chi(X)$ is even. Hence, $\chi^{\text{virt}}(G)$ is integral. Moreover, if there is a vertex v of valency ≤ 2 then there is an epimorphism $f\colon G \to \mathbb{Z}/2\mathbb{Z}$ which is nontrivial on G_e for all edges e with v as one vertex. But then $\text{Ker}(f) \cong \pi\widetilde{\mathcal{G}}$, where $(\widetilde{\mathcal{G}}, \Gamma)$ is a graph of groups with all vertex groups of order 2 and with trivial edge groups for the edges with v as one vertex. This is impossible if the double cover is orientable. If there is a vertex w with valency > 3 then two edges with w as one vertex have the same edge group $G_e < G_w$, and $C_G(G_e)$ contains a nonabelian free subgroup. Therefore each vertex of Γ has valency 3, so $2|E| = 3|V|$. In summary,

$$\begin{cases} X \text{ is orientable,} \\ \text{vertices of } \Gamma \text{ have valence 3,} \\ |V| \text{ is even,} \\ r = 1 + 2|V| \equiv 1 \bmod 4. \end{cases}$$

The simplest example meeting these criteria has $V = \{v, w\}$ and $E = \{a, b, c\}$, with each edge having origin v and target w. Then

$$G_v = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ c = ab \rangle$$

and

$$G_w = \langle a', b', c' \mid (a')^2 = (b')^2 = (c')^2 = 1,\ c' = a'b' \rangle.$$

The edge groups are $G_a = \langle a \rangle$, $G_b = \langle b \rangle$ and $G_c = \langle ab \rangle$, as subgroups of G_v, and for each edge x the monomorphism $\phi_x\colon G_x \to G_w$ is given by $\phi_x(x) = x'$. The edge a is a maximal tree in Γ. Let t and u be stable letters corresponding to the other edges. Then $\pi\mathcal{G}$ has the presentation

$$\langle G_v, G_w, t, u \mid a' = a,\ b' = tbt^{-1},\ a'b' = uabu^{-1} \rangle,$$

which simplifies to

$$\langle a, b, t, u \mid a^2 = b^2 = (ab)^2 = 1,\ atbt^{-1} = tbt^{-1}a = uabu^{-1} \rangle.$$

Is this the fundamental group of an orientable PD_{2k}–complex with $(2k-2)$–connected universal cover? Can the arguments of Section 2 of [8] be tweaked to rule this out?

References

[1] **H J Baues**, *Combinatorial homotopy and 4–dimensional complexes*, De Gruyter Expositions in Mathematics 2, de Gruyter, Berlin (1991) MR

[2] **H J Baues**, **B Bleile**, *Poincaré duality complexes in dimension four*, Algebr. Geom. Topol. 8 (2008) 2355–2389 MR

[3] **B Bleile**, *Poincaré duality pairs of dimension three*, PhD thesis, The University of Sydney (2004)

[4] **B Bleile**, *Poincaré duality pairs of dimension three*, Forum Math. 22 (2010) 277–301 MR

[5] **G E Bredon**, *Introduction to compact transformation groups*, Pure and Applied Mathematics 46, Academic, New York (1972) MR

[6] **K S Brown**, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer (1982) MR

[7] **I M Chiswell**, *Exact sequences associated with a graph of groups*, J. Pure Appl. Algebra 8 (1976) 63–74 MR

[8] **J Crisp**, *The decomposition of 3–dimensional Poincaré complexes*, Comment. Math. Helv. 75 (2000) 232–246 MR

[9] **W Dicks**, **M J Dunwoody**, *Groups acting on graphs*, Cambridge Studies in Advanced Mathematics 17, Cambridge Univ. Press (1989) MR

[10] **M Golasiński**, **D L Gonçalves**, *Free and properly discontinuous actions of groups $G \rtimes \mathbb{Z}^m$ and $G_1 *_{G_0} G_2$*, J. Homotopy Relat. Struct. 11 (2016) 803–824 MR

[11] **F González-Acuña**, **C M Gordon**, **J Simon**, *Unsolvable problems about higher-dimensional knots and related groups*, Enseign. Math. 56 (2010) 143–171 MR

[12] **I Hambleton**, **I Madsen**, *Actions of finite groups on $\mathbb{R}^{n+k}$ with fixed set $\mathbb{R}^k$*, Canad. J. Math. 38 (1986) 781–860 MR

[13] **I Hambleton**, **E K Pedersen**, *More examples of discrete co-compact group actions*, from "Algebraic topology: applications and new directions" (U Tillmann, S Galatius, D Sinha, editors), Contemp. Math. 620, Amer. Math. Soc., Providence, RI (2014) 133–143 MR

[14] **J A Hillman**, *Four-manifolds, geometries and knots*, Geometry & Topology Monographs 5, Geometry & Topology Publications, Coventry (2002) MR

[15] **J A Hillman**, *Indecomposable* PD_3*–complexes*, Algebr. Geom. Topol. 12 (2012) 131–153 MR

[16] **S Plotnick**, *Homotopy equivalences and free modules*, Topology 21 (1982) 91–99 MR

[17] **D J S Robinson**, *A course in the theory of groups*, Graduate Texts in Mathematics 80, Springer (1982) MR

[18] **J-P Serre**, *Trees*, Springer (2003) MR

[19] **R G Swan**, *Periodic resolutions for finite groups*, Ann. of Math. 72 (1960) 267–291 MR

[20] **L R Taylor**, *Unoriented geometric functors*, Forum Math. 20 (2008) 457–467 MR

[21] **V G Turaev**, *Three-dimensional Poincaré complexes: homotopy classification and splitting*, Mat. Sb. 180 (1989) 809–830 MR In Russian; translated in Math. USSR-Sb. 67 (1990) 261–282

[22] **C T C Wall**, *Finiteness conditions for* CW*–complexes*, Ann. of Math. 81 (1965) 56–69 MR

[23] **C T C Wall**, *Finiteness conditions for* CW *complexes, II*, Proc. Roy. Soc. Ser. A 295 (1966) 129–139 MR

[24] **C T C Wall**, *Poincaré complexes, I*, Ann. of Math. 86 (1967) 213–245 MR

[25] **C T C Wall**, *On the structure of finite groups with periodic cohomology*, from "Lie groups: structure, actions, and representations" (A Huckleberry, I Penkov, G Zuckerman, editors), Progr. Math. 306, Springer (2013) 381–413 MR

Mathematics, School of Science and Technology, University of New England
Armidale, NSW, Australia

Mathematics, School of Science and Technology, University of New England
Armidale, NSW, Australia

School of Mathematics and Statistics, University of Sydney
Sydney, NSW, Australia

bbleile@une.edu.au, ibokor@turing.une.edu.au,
jonathan.hillman@sydney.edu.au

Received: 31 October 2016 Revised: 7 July 2018

msp

Algebraic & Geometric Topology 18 (2018) 3789–3820

Cubical rigidification, the cobar construction and the based loop space

MANUEL RIVERA
MAHMOUD ZEINALIAN

We prove the following generalization of a classical result of Adams: for any pointed path-connected topological space (X, b), that is not necessarily simply connected, the cobar construction of the differential graded (dg) coalgebra of normalized singular chains in X with vertices at b is weakly equivalent as a differential graded associative algebra (dga) to the singular chains on the Moore based loop space of X at b. We deduce this statement from several more general categorical results of independent interest. We construct a functor $\mathfrak{C}_{\square_c}$ from simplicial sets to categories enriched over cubical sets with connections, which, after triangulation of their mapping spaces, coincides with Lurie's rigidification functor $\mathfrak{C}$ from simplicial sets to simplicial categories. Taking normalized chains of the mapping spaces of $\mathfrak{C}_{\square_c}$ yields a functor Λ from simplicial sets to dg categories which is the left adjoint to the dg nerve functor. For any simplicial set S with $S_0 = \{x\}$, $\Lambda(S)(x, x)$ is a dga isomorphic to $\Omega Q_\Delta(S)$, the cobar construction on the dg coalgebra $Q_\Delta(S)$ of normalized chains on S. We use these facts to show that Q_Δ sends categorical equivalences between simplicial sets to maps of connected dg coalgebras which induce quasi-isomorphisms of dgas under the cobar functor, which is strictly stronger than saying the resulting dg coalgebras are quasi-isomorphic.

18G30, 55P35, 55U10, 57T30; 18D20, 55U35, 55U40

1 Introduction

In order to compare two different models for ∞–categories, Lurie constructs in [15] a *rigidification*, or *categorification*, functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, where Set_Δ denotes the category of simplicial sets and Cat_Δ the category of simplicial categories (categories enriched over simplicial sets). For a standard n–simplex Δ^n the simplicial category $\mathfrak{C}(\Delta^n)$ has the set $[n] = \{0, 1, \dots, n\}$ as objects and for any $i, j \in [n]$ with $i \leq j$ the mapping space $\mathfrak{C}(\Delta^n)(i, j)$ is isomorphic to the simplicial cube $(\Delta^1)^{\times j-i-1}$ if $i < j$, isomorphic to Δ^0 if $i = j$, and empty if $i > j$. In particular, $\mathfrak{C}(\Delta^n)(0, n) \cong (\Delta^1)^{\times n-1}$ for $n > 0$ and we think of this simplicial $(n-1)$–cube as parametrizing a family of paths in Δ^n from 0

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3789

to n. Adams described in [1] an algebraic construction, known as the cobar construction, that when applied to a suitable differential graded coassociative coalgebra model of a simply connected space X produces a differential graded associative algebra (dga) model for the based loop space of X. Adams's construction is based on certain geometric maps $\theta_n\colon I^{n-1} \to P_{0,n}|\Delta^n|$, where $P_{0,n}|\Delta^n|$ is the space of paths in the topological n–simplex $|\Delta^n|$ from vertex 0 to vertex n, satisfying a compatibility equation that relates the cubical boundary to the simplicial face maps and the Alexander–Whitney coproduct. The definition of $\mathfrak{C}(\Delta^n)(0,n)$ resembles the construction of Adams's maps θ_n and it suggests that behind Adams's constructions there is a space-level story.

In this article we describe explicitly the relationship between Lurie's functor $\mathfrak{C}$ and Adams's cobar construction. As a consequence we obtain a generalization of the main theorem of [1] to path-connected spaces with possibly nontrivial fundamental group. To achieve this, we factor the functor $\mathfrak{C}$ through a functor $\mathfrak{C}_{\square_c}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_{\square_c}$ from the category of simplicial sets to the category of categories enriched over cubical sets with connections. If we apply the functor of normalized cubical chains (over a fixed commutative ring k) to the mapping spaces of $\mathfrak{C}_{\square_c}$, we obtain a functor $\Lambda\colon \mathrm{Set}_\Delta \to \mathrm{dgCat}_k$ from simplicial sets to dg categories satisfying the following properties. The functor Λ is the left adjoint of the dg nerve functor described by Lurie in [16]. Moreover, if S is a 0–reduced simplicial set, ie $S_0 = \{x\}$, then $\Lambda(S)(x,x)$ is a dga isomorphic to $\Omega Q_\Delta(S)$, the cobar construction on the dg coalgebra $Q_\Delta(S)$ of normalized simplicial chains with Alexander–Whitney coproduct.

From the properties of $\mathfrak{C}_{\square_c}$ described in the above paragraph we deduce that $\Lambda(S)(x,x)$ and $Q_\Delta(\mathfrak{C}(S)(x,x))$ are weakly equivalent as dgas, where $Q_\Delta(\mathfrak{C}(S)(x,x))$ is considered as a dga obtained by taking normalized simplicial chains on the simplicial monoid $\mathfrak{C}(S)(x,x)$. In fact, $Q_\Delta(\mathfrak{C}(S)(x,x))$ is a dg bialgebra (with Alexander–Whitney coproduct) but we are not concerned with the dg coalgebra structure in this article. From these results, it follows that if $f\colon S \to S'$ is a map between 0–reduced simplicial sets such that $\mathfrak{C}(f)\colon \mathfrak{C}(S) \to \mathfrak{C}(S')$ is a weak equivalence of simplicial categories (these maps are called *categorical equivalences*) then $Q_\Delta(f)\colon Q_\Delta(S) \to Q_\Delta(S')$ is a map of connected dg coalgebras which induces a quasi-isomorphism of dgas after applying the cobar functor. Maps $f\colon C \to C'$ between connected dg coalgebras which induce a quasi-isomorphism of dg algebras $\Omega f\colon \Omega C \to \Omega C'$ after applying the cobar functor Ω are called Ω–quasi-isomorphisms.

We apply the preceding discussion to the 0–reduced simplicial set $\mathrm{Sing}(X,b)$ of singular simplices on a path-connected space X with vertices at a fixed point b. From

the relationships between $\mathfrak{C}$ and $\mathfrak{C}_\square$ and between $\mathfrak{C}_\square$ and the cobar functor Ω, and from some basic homotopy-theoretic properties of $\mathfrak{C}$, we deduce that $\Omega Q_\Delta(\mathrm{Sing}(X,b))$ is weakly equivalent as a dga to the singular chains on $\Omega_b^M X$, the topological monoid of Moore loops in X based at b. In [1], Adams obtained a similar statement for a simply connected space X using different methods. Our statement does not assume X is simply connected and therefore extends Adams's classical result. The key homotopy-theoretic property of $\mathfrak{C}$ that implies our result is the following space-level statement, which lies at the heart of Section 2.2 of [15]: for any path-connected pointed space (X,b) there is a weak homotopy equivalence of simplicial monoids between $\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)$ and $\mathrm{Sing}(\Omega_b^M X)$.

We believe this extension of Adams's result has not been observed in the literature mainly because of the historical development of the cobar construction. We highlight two situations in which the simply connected hypothesis comes into play:

(1) In [1], Adams constructs a map of dg algebras from the cobar construction of the dg coalgebras of chains to the cubical singular chains on the based loop space. Comparison of spectral sequences was the main technique used at the time to measure how far a chain map is from being a quasi-isomorphism. In Adams's setup, the hypotheses in Zeeman's spectral sequence comparison theorem hold if the underlying space is simply connected and fail in general for spaces with nontrivial fundamental group.

(2) The cobar construction is not invariant under quasi-isomorphisms of dg coalgebras. Namely, there are quasi-isomorphisms of dg coalgebras $f\colon C \to C'$ for which $\Omega(f)\colon \Omega C \to \Omega C'$ is not a quasi-isomorphism of dg algebras. An explicit example is described in Proposition 2.4.3 of Loday and Vallette [14]. However, the cobar construction is invariant under quasi-isomorphisms of simply connected dg coalgebras, ie dg coalgebras C for which $C_0 \cong k$ and $C_1 = 0$, as shown in Proposition 2.2.7 of [14]. Hence, Adams's main statement regarding the relationship between the cobar construction and the based loop space also holds if we replace singular chains on a simply connected space X with the quasi-isomorphic dg coalgebra of simplicial chains associated to any simplicial set S with no nondegenerate 1–simplices whose geometric realization is weakly homotopy equivalent to X. The generalization of this statement to spaces with nontrivial fundamental group fails.

In the nonsimply connected case we go around the use of spectral sequences as described in (1) by turning the problem of showing that two dgas are quasi-isomorphic into the more fundamental problem of showing that the two simplicial monoids

$\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)$ and $\mathrm{Sing}(\Omega_b^M X)$ are weakly homotopy equivalent. Then, by looking closely at the combinatorics, we realize that the simplicial chain complex on $\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)$ is weakly equivalent as a dga to the cobar construction

$$\Omega Q_\Delta(\mathrm{Sing}(X,b)).$$

We go around (2) by using the following observation: if Set_Δ^0 denotes the category of simplicial sets with a single vertex, then the functor $Q_\Delta^K\colon \mathrm{Set}_\Delta^0 \to \mathrm{dgCoalg}_k$ defined by $Q_\Delta^K(S) = Q_\Delta(\mathrm{Sing}(|S|,x))$ sends weak homotopy equivalences of simplicial sets to Ω–quasi-isomorphisms of dg coalgebras. Notice that, in general, for any $S \in \mathrm{Set}_\Delta^0$ the connected dg coalgebra of simplicial chains $Q_\Delta(S)$ is quasi-isomorphic to $Q_\Delta^K(S)$ but not Ω–quasi-isomorphic. Hence, in order to preserve all the homological information of the based loop space, the chains functor should be always precomposed with a Kan replacement functor and the notion of weak equivalences of dg coalgebras should be taken to be Ω–quasi-isomorphisms.

We now say a few words regarding how the combinatorics in the construction of $\mathfrak{C}$ is unraveled and how its cubical version $\mathfrak{C}_{\square_c}$ is constructed. For any simplicial set S, Dugger and Spivak computed in [5] the mapping spaces $\mathfrak{C}(S)(x,y)$ in terms of necklaces. A necklace is a simplicial set of the form $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$, where in the wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$; a necklace in S from x to y is a map of simplicial sets $f\colon T \to S$, where T is a necklace and f sends the first vertex of T to x and the last vertex of T to y. For any necklace T, one may associate functorially a simplicial cube $C(T)$ and one of the main results in [5] is that $\mathfrak{C}(S)(x,y)$ is isomorphic to the colimit of the simplicial sets $C(T)$ over necklaces T in S from x to y. It is tempting to replace the simplicial cubes $C(T)$ with standard cubical sets of the same dimension to obtain a cubical version of $\mathfrak{C}$. However, there are certain maps between necklaces that are not realized by maps of cubical sets. For example the codegeneracy map $s^1\colon \Delta^3 \to \Delta^2$ which collapses the edge $[1,2]$ in Δ^3 yields a map between simplicial cubes $C(s^1)\colon C(\Delta^3) \to C(\Delta^2)$ which does not correspond to a codegeneracy map between standard cubical sets. Nonetheless, $C(s^1)$ corresponds to a *coconnection* morphism, whose definition is recalled in Section 2. Cubical sets with connections were introduced in Brown and Higgins [3] and can be thought of as cubical sets with extra degeneracies. In Section 3 we describe explicitly the morphisms in the category of necklaces and then in Section 4 we explain how cubical sets with connections arise naturally from necklaces. We use the results in Sections 3 and 4 and the description of $\mathfrak{C}(S)(x,y)$ in terms of necklaces to define $\mathfrak{C}_{\square_c}$

in Section 5. In Section 6 we show that $\mathfrak{C}_{\square_c}$ gives rise to the functor Λ which is the left adjoint of the dg nerve functor described by Lurie in [16]. Then, in Section 7 we explain how Λ relates to the cobar construction. Finally, in Section 8, we recall some homotopy-theoretic properties of the rigidification functor $\mathfrak{C}$ and use them in Section 9 to obtain algebraic models for both the based and free loop spaces on a path-connected space, extending classical results originally proven in the simply connected case.

Over a year since the results of this paper were posted on arXiv, two other preprints (Kapulkin and Voevodsky [10] and Le Grignou [12]) discussing a cubical factorization of $\mathfrak{C}$ also appeared. In [10], the authors use a cubical version of $\mathfrak{C}$ to describe a cubical approach to Lurie's theory of straightening and unstraightening. In [12], Le Grignou discusses the homotopy theory of the category of categories enriched over cubical sets with connections using the framework of model categories.

Acknowledgements We would like to thank Micah Miller, who was a part of the early stages of this project. We would also like to thank Thomas Nikolaus, Ronnie Brown, Thomas Tradler and Gabriel Drummond-Cole for conversations and their comments. Rivera acknowledges support by the ERC via the grant StG-259118–STEIN and the excellent working conditions at *Institut de Mathématiques de Jussieu-Paris Rive Gauche* (IMJ-PRG). Zeinalian was partially supported by the NSF grant DMS-1309099 and would like to thank the *Max Planck Institute for Mathematics* for their support and hospitality during his visits.

2 Preliminaries

Denote by Set the category of sets. For any small category $\mathcal{C}$ denote by $\mathrm{Set}_{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$ with values in Set, so the objects of $\mathrm{Set}_{\mathcal{C}}$ are functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and morphisms are natural transformations between them. For example, if Δ is the category of nonempty finite ordinals with order-preserving maps then Set_{Δ} is the category of *simplicial sets*. We denote by Δ^n the *standard n-simplex*, so Δ^n is obtained by applying the Yoneda embedding to $[n]$, namely $\Delta^n\colon [m] \mapsto \mathrm{Hom}_{\Delta}([m],[n])$. Recall that morphisms in the category Δ are generated by functions of two types: cofaces $d_i\colon [n] \to [n+1]$ for $0 \le i \le n+1$ and codegeneracies $s_j\colon [n] \to [n-1]$ for $0 \le j \le n-1$. The Yoneda embedding yields simplicial set morphisms between standard simplices, $Y(d_i)\colon \Delta^n \to \Delta^{n+1}$ and $Y(s_j)\colon \Delta^n \to \Delta^{n-1}$, which we call *coface* and *codegeneracy*

(simplicial) morphisms. We say a simplicial set S is 0–reduced if the set S_0 is a singleton and we denote by Set_Δ^0 be the full subcategory of the category Set_Δ of simplicial sets whose objects are 0–reduced simplicial sets.

For any positive integer n, let $\mathbf{1}^n$ be the n–fold cartesian product of copies of the category $\mathbf{1} = \{0, 1\}$ which has two objects and one nonidentity morphism. Denote by $\mathbf{1}^0$ the category with one object and one morphism. We will consider presheaves over the category $\square_c$ which is defined as follows. The objects of $\square_c$ are the categories $\mathbf{1}^n$ for $n = 0, 1, 2, \ldots$. The morphisms in $\square_c$ are generated by functors of the following three kinds: *cubical coface functors* $\delta^\epsilon_{j,n}\colon \mathbf{1}^n \to \mathbf{1}^{n+1}$, where $j = 0, 1, \ldots, n+1$ and $\epsilon \in \{0, 1\}$, defined by

$$\delta^\epsilon_{j,n}(s_1, \ldots, s_n) = (s_1, \ldots, s_{j-1}, \epsilon, s_j, \ldots, s_n);$$

cubical codegeneracy functors $\varepsilon_{j,n}\colon \mathbf{1}^n \to \mathbf{1}^{n-1}$, where $j = 1, \ldots, n$, defined by

$$\varepsilon_{j,n}(s_1, \ldots, s_n) = (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n);$$

and *cubical coconnection functors* $\gamma_{j,n}\colon \mathbf{1}^n \to \mathbf{1}^{n-1}$, where $j = 1, \ldots, n-1$ and $n \geq 2$, defined by

$$\gamma_{j,n}(s_1, \ldots, s_n) = (s_1, \ldots, s_{j-1}, \max(s_j, s_{j+1}), s_{j+2}, \ldots, s_n).$$

Objects in the category $\mathrm{Set}_{\square_c}$ are called *cubical sets with connections* and were introduced by Brown and Higgins [3]. For any cubical set with connections K we have a collection of sets $\{K_n := K(\mathbf{1}^n)\}_{n\in\mathbb{Z}_{\geq 0}}$ together with *cubical face maps* $\partial^\epsilon_{j,n} := K(\delta^\epsilon_{j,n})\colon K_{n+1} \to K_n$, *cubical degeneracy maps* $E_{j,n} := K(\varepsilon_{j,n})\colon K_{n-1} \to K_n$ and *connections* $\Gamma_{j,n} := K(\gamma_{j,n})\colon K_{n-1} \to K_n$. For simplicity we often drop the second index in this notation and, for example, write ∂_j instead of $\partial_{j,n}$. Elements of K_n are called n–cells. The structure maps satisfy certain identities described in [3]. The *standard n-cube with connections* $\square^n_c$ is the presheaf on $\square_c$ represented by $\mathbf{1}^n$, namely, $\mathrm{Hom}_{\square_c}(_\,, \mathbf{1}^n)\colon \square_c^{\mathrm{op}} \to \mathrm{Set}$.

For a fixed commutative unital ring k denote by Ch_k the category of nonnegatively graded chain complexes over k. The tensor product over k defines on Ch_k a symmetric monoidal structure. We have normalized chains functors $Q_\Delta\colon \mathrm{Set}_\Delta \to \mathrm{Ch}_k$ and $Q_{\square_c}\colon \mathrm{Set}_{\square_c} \to \mathrm{Ch}_k$. The definition of Q_Δ is standard; we recall the definition of $Q_{\square_c}$ following [2]. First let C_*K be the chain complex such that C_nK is the free k–module generated by elements of K_n with differential $\partial\colon K_n \to K_{n-1}$ defined on $\sigma \in K_n$ by $\partial(\sigma) := \sum_{j=1}^n (-1)^j (\partial^1_{j,n-1}(\sigma) - \partial^0_{j,n-1}(\sigma))$. Let D_nK be the submodule

of C_nK which is generated by those cells in K_n which are the image of a degeneracy or of a connection map $K_{n-1} \to K_n$. The graded module D_*K forms a subcomplex of C_*K. Define $Q_{\square_c}(K)$ to be the quotient chain complex C_*K/D_*K.

The functor $Q_\Delta\colon \mathrm{Set}_\Delta \to \mathrm{Ch}_k$ lifts to a functor $Q_\Delta\colon \mathrm{Set}_\Delta \to \mathrm{dgCoalg}_k$, where $\mathrm{dgCoalg}_k$ is the category of dg coalgebras over k, via the Alexander–Whitney construction as recalled in Section 7. There is a slight abuse of notation throughout the article: depending on the context $Q_\Delta(S)$ may be considered as a chain complex or as a dg coalgebra. For example, by $\Omega Q_\Delta(S)$ we mean the cobar construction of $Q_\Delta(S)$ considered as a dg coalgebra.

The category Set_Δ has a symmetric monoidal structure given by the cartesian product of simplicial sets. We will use the following (nonsymmetric) monoidal structure on $\mathrm{Set}_{\square_c}$: for cubical sets with connections K and K' define

$$K \otimes K' := \operatorname*{colim}_{\sigma\colon \square_c^n \to K,\, \tau\colon \square_c^m \to K'} \square_c^{n+m}.$$

Using the above monoidal structures we may define Cat_Δ, the category of small categories enriched over simplicial sets; these are called *simplicial categories*. Similarly, denote by $\mathrm{Cat}_{\square_c}$ the category of small categories enriched over cubical sets with connections; these are called *cubical categories with connections*. We will also consider the category dgCat_k of small categories enriched over chain complexes over k; these are called *dg categories*.

The symbol $\cong$ will always denote isomorphism and $\simeq$ will mean that there is a zigzag of weak equivalences whenever there is a notion of weak equivalence in the underlying category.

3 The category of necklaces

We follow [5] for the next definitions and notation. A *necklace* T is a simplicial set of the form $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$, where $n_i \geq 0$ and in the wedge the final vertex of Δ^{n_i} has been glued to the initial vertex of $\Delta^{n_{i+1}}$. Each Δ^{n_i} is called a *bead* of T. Since the beads of T are ordered and the vertices of each bead Δ^{n_i} are ordered as well, there is a canonical ordering on the set V_T of vertices of any necklace T. We denote by α_T and ω_T the first and last vertices of the necklace T. A morphism $f\colon T \to T'$ of necklaces is a map of simplicial sets which preserves the first and last vertices. We

say a necklace $\Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ is of *preferred form* if $k = 0$ or each $n_i \geq 1$. Let $T = \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ be a necklace in preferred form. Denote by b_T the number of beads in T. A *joint* of T is either an initial or a final vertex in some bead. Given a necklace T write J_T for the subset of V_T consisting of all the joints of T. For any two vertices $a, b \in V_T$ we write $V_T(a,b)$ and $J_T(a,b)$ for the set of vertices and joints between a and b inclusive. Note that there is a unique subnecklace $T(a,b) \subseteq T$ with joints $J_T(a,b)$ and vertices $V_T(a,b)$. Denote by Nec the category whose objects are necklaces in preferred form and morphisms are morphisms of necklaces. Note that Nec is a full subcategory of $\mathrm{Set}_{\Delta}^{*,*} = \partial\Delta^1 \downarrow \mathrm{Set}_{\Delta}$.

Proposition 3.1 *Any nonidentity morphism in* Nec *is a composition of morphisms of the following type:*

(i) $f\colon T \to T'$ *is an injective morphism of necklaces and* $|V_{T'} - J_{T'}| - |V_T - J_T| = 1$.

(ii) $f\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \to \Delta^{m_1} \vee \cdots \vee \Delta^{m_k}$ *is a morphism of necklaces of the form* $f = f_1 \vee \cdots \vee f_k$ *such that for exactly one* p, $f_p\colon \Delta^{n_p} \to \Delta^{m_p}$ *is a codegeneracy morphism* (so $m_p = n_p - 1$) *and, for all* $i \neq p$, $f_i\colon \Delta^{n_i} \to \Delta^{m_i}$ *is the identity map of standard simplices* (so $n_i = m_i$ for $i \neq p$).

(iii) $f\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_{p-1}} \vee \Delta^1 \vee \Delta^{n_{p+1}} \vee \cdots \vee \Delta^{n_k} \to \Delta^{n_1} \vee \cdots \vee \Delta^{n_{p-1}} \vee \Delta^{n_{p+1}} \vee \cdots \vee \Delta^{n_k}$ *is a morphism of necklaces such that* f *collapses the* p^{th} *bead* Δ^1 *in the domain to the last vertex of the* $(p-1)^{st}$ *bead in the target and the restriction of* f *to all the other beads is injective.*

Proof We prove that any nonidentity morphism of necklaces $f\colon T \to T'$ is a composition of morphisms of type (i), (ii) and (iii) by induction on b_T, the number of beads of T. If $b_T = 1$, then we must have $b_{T'} = 1$ as well, so f is a morphism of simplicial sets between standard simplices which preserves first and last vertices. It follows that f is a composition of (simplicial) coface and codegeneracy morphisms. Cofaces and codegeneracies between standard simplices are morphisms of necklaces of type (i) and of type (ii) or (iii), respectively. Assume we have shown the proposition for $b_T \leq k$ and suppose $b_T = k+1$. Let $V_T = \{x_0, \ldots, x_p\}$ be the vertices of T and $x_i \preceq x_{i+1}$. Let x_{j_0} be the last vertex of the first bead of T, so $T = T(x_0, x_{j_0}) \vee T(x_{j_0}, x_p)$, where $T(x_0, x_{j_0})$ has one bead and $T(x_{j_0}, x_p)$ has k beads. Let $T_f = T'(f(x_0), f(x_{j_0})) \vee T'(f(x_{j_0}), f(x_p))$. We have an injective morphism of necklaces $t\colon T_f \to T'$ (notice that it is possible for $T_f \neq T'$ since $f(x_{j_0})$ might not be a joint of T'). It follows that $f = t \circ (g \vee h)$, where

$g\colon T(x_0, x_{j_0}) \to T'(f(x_0), f(x_{j_0}))$ and $h\colon T(x_{j_0}, x_p) \to T'(f(x_{j_0}), f(x_p))$ are the morphisms of necklaces induced by restricting f to $T(x_0, x_{j_0})$ and $T(x_{j_0}, x_p)$ respectively. By the induction hypothesis each of g and h is a composition of morphisms of type (i), (ii) and (iii) and this implies that $g \vee h$ is a composition of such morphisms as well. In fact, we have

$$g \vee h = (\mathrm{id}_{T'(f(x_0), f(x_{j_0}))} \vee h) \circ (g \vee \mathrm{id}_{T(x_{j_0}, x_p)})$$

and, clearly, the wedge of an identity morphism and a morphism which is a composition of morphisms of type (i), (ii) and (iii) is again a morphism of such form.

To conclude the proof we show that $t\colon T_f \to T'$ is of the desired form. More generally, let us prove that any nonidentity injective morphism of necklaces $t\colon R \to R'$ is a composition of morphisms of type (i) by induction on the integer $l(R, R') := |V_{R'} - J_{R'}| - |V_R - J_R|$. If $l(R, R') = 1$ then t is of type (i). Assume we have shown the claim for $l(R, R') = k$. Suppose $t\colon R \to R'$ is injective and $l(R, R') = k+1$; then we have two cases: either (a) $J_{R'} = t(J_R)$, or (b) $J_{R'} \subset t(J_R)$. In case (a), it follows that both R and R' have the same number of beads, thus $t = i \circ j$ for inclusions of necklaces $j\colon R \to S$, $i\colon S \to R'$, where S is the subnecklace of R' spanned by $t(V_R) \cup \{v\}$ and v is the smallest element of $V_{R'} - t(V_R)$. Then j is of type (i) and i is a composition of morphisms of type (i) by the induction hypothesis. For case (b), let $t(J_R) - J_{R'} = \{t(x_{i_1}), \ldots, t(x_{i_n})\}$ and consider the unique subnecklace S of R' defined by $V_S = t(V_R)$ and $J_S = t(J_R) - \{t(x_{i_1})\}$. Then we have $t = i \circ j$ for inclusions of necklaces $j\colon R \to S$, $i\colon S \to R'$ with j of type (i) and i a composition of type (i) morphisms by the induction hypothesis. □

Remark 3.2 Let us consider type (i) morphisms of the form $f\colon T \to \Delta^p$ for some integer $p \geq 1$. If $b_T = 1$ then we have an injective map of simplicial sets $f\colon \Delta^{p-1} \to \Delta^p$ which sends the first (resp. last) vertex of Δ^{p-1} to the first (resp. last) vertex of Δ^p. The morphism f determines a $(p-1)$–simplex of the simplicial set Δ^p, ie an element of $(\Delta^p)_{p-1}$. There are $p+1$ nondegenerate elements in $(\Delta^p)_{p-1}$, however only $p-1$ of these can correspond to f based on the constraint that f must preserve first and last vertices, namely, all the faces of the unique nondegenerate element in $(\Delta^p)_p$ except the first and last. If $b_T > 1$ then there is a joint $v \in J_T$ such that $f(v) \notin J_{T'}$. Moreover, since f is injective and $|V_{T'} - J_{T'}| - |V_T - J_T| = 1$, we have $f(J_T - \{v\}) = J'_T$ and $f(V_T) = V_{T'}$. It follows that $b_T = 2$ and the image of f is a subnecklace $T'_1 \vee T'_2$ of Δ^p starting and ending with the first and last vertices of Δ^p, respectively, and containing all the vertices of Δ^p. Hence, we have $T'_1 \vee T'_2 = \Delta^{p-i} \vee \Delta^i$ for some

$0 < i < p$ and each of these subnecklaces of Δ^p corresponds to a unique term in the formula for the Alexander–Whitney diagonal $Q_\Delta(\Delta^p) \to Q_\Delta(\Delta^p) \otimes Q_\Delta(\Delta^p)$ applied to the generator represented by the unique nondegenerate p–simplex in $(\Delta^p)_p$.

4 The functor $C_{\square_c}$: Nec $\to$ Set$_{\square_c}$

There is a functor $C_{\square_c}$: Nec $\to$ Set$_{\square_c}$ which associates functorially to any $\Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ in Nec a standard cube with connections of dimension $n_1 + \cdots + n_k - k$. The goal of this section is to define this functor carefully in a way which will be useful later. We start by defining a functor P: Nec $\to$ Cat, where Cat is the category of small categories. Given a necklace T and two vertices $a, b \in V_T$ we may define a small category $P_T(a,b)$ whose objects are subsets $X \subseteq V_T(a,b)$ such that $J_T(a,b) \subseteq X$ and morphisms are inclusions of sets. For any necklace $T \in$ Nec let $P(T) = P_T(\alpha, \omega)$, where $\alpha, \omega \in V_T$ are the first and last vertices of T. Let $f\colon T \to T'$ be a morphism in Nec, so f is a map of simplicial sets such that $f(\alpha) = \alpha'$ and $f(\omega) = \omega'$, where $\alpha, \omega \in V_T$ and $\alpha', \omega' \in V_{T'}$ are the first and last vertices of T and T', respectively. Notice that we have an inclusion $J_{T'} \subseteq f(J_T)$. Thus f induces a functor $P_f\colon P_T(\alpha,\omega) \to P_{T'}(\alpha',\omega')$ defined on objects by $P_f(X) = f(X)$ and on morphisms by the induced inclusion of sets. This yields a functor P: Nec $\to$ Cat. We might think of the objects of $P(T)$ as strings of 0's and 1's, as discussed below. This interpretation will yield a functor $P_{\mathbf{1}}$ which is naturally isomorphic to P. We define a total order on the vertices of a necklace by setting $a \preceq b$ if there is a directed path from a to b.

Proposition 4.1 *For any necklace T and any $a, b \in V_T$ such that $a \preceq b$, there is an isomorphism of categories $\phi_T\colon P_T(a,b) \cong \mathbf{1}^N$, where $N = |V_T(a,b) - J_T(a,b)|$.*

Proof Let $V_T(a,b) - J_T(a,b) = \{y_1, \ldots, y_N\}$ and $y_i \preceq y_{i+1}$ for $i = 1, \ldots, N-1$. Given any object X of $P_T(a,b)$ (so $J_T(a,b) \subseteq X \subseteq V_T(a,b)$) we define $\phi_T(X) := (\phi_T^1(X), \ldots, \phi_T^N(X))$ to be the object in the category $\mathbf{1}^N$, where, for $1 \le i \le N$, we have $\phi_T^i(X) = 1$ if $y_i \in X$ and $\phi_T^i(X) = 0$ if $y_i \notin X$. Given a morphism $f\colon X \to Y$ in $P_T(a,b)$ (so f is an inclusion of sets) we have an induced morphism $\phi_T(f)\colon \phi_T(X) \to \phi_T(Y)$ defined by $\phi_T(f) := (\phi_T^1(f), \ldots, \phi_T^N(f))$, where, for $1 \le i \le N$, $\phi_T^i(f)\colon \phi_T^i(X) \to \phi_T^i(Y)$ is the unique nonidentity morphism in $\mathbf{1}$ if $\phi_T^i(X) = 0$ and $\phi_T^i(Y) = 1$, and $\phi_T^i(f)$ is an identity morphism otherwise. It is clear that the functor $\phi_T\colon P_T(a,b) \to \mathbf{1}^N$ is an isomorphism of categories. □

Consider the functor $P_{\mathbf{1}}\colon \mathrm{Nec} \to \mathrm{Cat}$ defined on objects by $P_{\mathbf{1}}(T) = \mathbf{1}^{|V_T - J_T|}$ and on morphisms $f\colon T \to T'$ by $P_{\mathbf{1}}(f) = \phi_{T'} \circ P(f) \circ \phi_T^{-1}\colon \mathbf{1}^{|V_T - J_T|} \to \mathbf{1}^{|V_{T'} - J_{T'}|}$. The above proposition implies that $P_{\mathbf{1}}$ is naturally isomorphic to P. In the following proposition we describe explicitly the functor $P_{\mathbf{1}}(f)$ for morphisms $f\colon T \to T'$ of type (i), (ii) and (iii) as in Proposition 3.1.

Proposition 4.2 *Let $f\colon T \to T'$ be a morphism in* Nec *and let $N = |V_T - J_T|$.*

(1) *If f is of type* (i) *then $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^{N+1}$ is a cubical coface functor.*

(2) *If f is of type* (ii) *then $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^{N-1}$ is either a cubical coconnection functor or a cubical codegeneracy functor.*

(3) *If f is of type* (iii) *then $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^N$ is the identity functor.*

Proof For any morphism of necklaces $f\colon T \to T'$ we have $J_{T'} \subseteq f(J_T)$. For $f\colon T \to T'$ of type (i) we prove below that if $J_{T'} \subset f(J_T)$ then $P_{\mathbf{1}}(T)(f)$ is a cubical coface functor $\delta^1_{j,N}$ and if $J_{T'} = f(J_T)$ then $P_{\mathbf{1}}(T)(f)$ is a cubical coface functor $\delta^0_{j,N}$. A morphism $f\colon T \to T'$ of type (ii) collapses two vertices v and w of T into a vertex v' of T' and is injective on $V_T - \{v, w\}$. We prove below that if $v' \notin J_{T'}$ then $P_{\mathbf{1}}(T)(f)$ is a cubical coconnection functor $\gamma_{j,N}$ and if $v' \in J_{T'}$ then $P_{\mathbf{1}}(T)(f)$ is a cubical codegeneracy functor $\varepsilon_{j,N}$. The proof for the third part of the proposition will be straightforward.

(1) Let $f\colon T \to T'$ be of type (i) and write $\{y'_1, \dots, y'_{N+1}\} = V_{T'} - J_{T'}$, where $y'_i \preceq y'_{i+1}$. We have $J_{T'} \subseteq f(J_T)$ since f is a morphism of necklaces. If $J_{T'} \subset f(J_T)$ then there is $v \in J_T$ such that $f(v) = y'_j \in V_{T'} - J_{T'}$ for some $j \in \{1, \dots, N+1\}$ and $f(J_T - \{v\}) \subseteq J'_T$. Then, for any object X in $P(T)$, $v \in J_T \subseteq X$, so $y_j = f(v) \in f(X)$. Using the fact that f is injective and identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T\colon P(T) \cong \mathbf{1}^N$ we see that $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^{N+1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1, \dots, s_N) = (s_1, \dots, s_{j-1}, 1, s_j, \dots, s_N)$$

and on morphisms $\lambda = (\lambda_1, \dots, \lambda_N)\colon (s_1, \dots, s_N) \to (s'_1, \dots, s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, \dots, \lambda_{j-1}, \mathrm{id}_1, \lambda_j, \dots, \lambda_N).$$

Thus, $P_{\mathbf{1}}(f)$ is the cubical coface functor $\delta^1_{j,N}$.

If $J_{T'} = f(J_T)$ then there exists exactly one $j \in \{1, \dots, N+1\}$ such that $f^{-1}(y'_j) = \varnothing$. Then, for any object X in $P(T)$, y'_j will never be an element of $f(X)$. Using the fact

that f is injective and identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T\colon P(T)\cong \mathbf{1}^N$, we see that $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^{N+1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1,\dots,s_N) = (s_1,\dots.,s_{j-1},0,s_j,\dots,s_N)$$

and on morphisms $\lambda=(\lambda_1,\dots,\lambda_N)\colon (s_1,\dots,s_N)\to(s'_1,\dots,s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1,\dots,\lambda_{j-1},\mathrm{id}_0,\lambda_j,\dots,\lambda_N).$$

It follows that $P_{\mathbf{1}}(f)$ is the cubical coface functor $\delta^0_{j,N}$.

(2) Let $f\colon T\to T'$ be of type (ii) and write $\{y_1,\dots,y_N\}=V_T-J_T$, where $y_i\preceq y_{i+1}$ and $\{y'_1,\dots,y'_{N-1}\}=V_{T'}-J_{T'}$, where $y'_i\preceq y'_{i+1}$. There exists $v'\in V_{T'}$ such that $f^{-1}(v')=\{v,w\}$ for some $v,w\in V_T$ and $|f^{-1}(x')|=1$ for all $x'\in V_{T'}-\{v'\}$. Note that v and w are consecutive vertices in the p^{th} bead of T. We have two cases: either $v'\in V_{T'}-J_{T'}$ or $v'\in J_{T'}$.

If $v'\in V_{T'}-J_{T'}$, then $v,w\in V_T-J_T$ so we may write $v=y_j$ and $w=y_{j+1}$ for some $j\in\{1,\dots,N-1\}$. Hence, for any object X of $P(T)$ we have that if $X\cap\{y_j,y_{j+1}\}\neq\varnothing$ then $v'\in f(X)$ and if $X\cap\{y_j,y_{j+1}\}=\varnothing$ then $v'\notin f(X)$. By identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi_T\colon P(T)\cong\mathbf{1}^N$, we see that $P_{\mathbf{1}}(f)\colon \mathbf{1}^N\to\mathbf{1}^{N-1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1,\dots,s_N) = (s_1,\dots.s_{j-1},\max(s_j,s_{j+1}),s_{j+2},\dots,s_N)$$

and on morphisms $\lambda=(\lambda_1,\dots,\lambda_N)\colon(s_1,\dots,s_N)\to(s'_1,\dots,s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1,\dots,\lambda_{j-1},\sigma_{j,j+1},\lambda_{j+2},\dots,\lambda_N),$$

where $\sigma_{j,j+1}$ is the unique morphism $\max(s_j,s_{j+1})\to\max(s'_j,s'_{j+1})$ in the category $\mathbf{1}$. It follows that $P_{\mathbf{1}}(f)$ is the cubical coconnection functor $\gamma_{j,N}$.

If $v'\in J_{T'}$, we may assume without loss of generality that $w\in J_T$ and $v=y_j\in V_T-J_T$ for some $j\in\{1,\dots,N\}$. Let X be any object of $P(T)$. Every element of $X-\{y_j\}$ corresponds to a unique element in $f(X)$ via $P(f)$ (since f is of type (ii)) and if $y_j\in X$ then $P(f)$ sends y_j to the joint $v'\in f(X)$. By identifying objects X in $P(T)$ with sequences of 0's and 1's via the isomorphism $\phi\colon P(T)\cong\mathbf{1}^N$, we see that $P_{\mathbf{1}}(f)\colon\mathbf{1}^N\to\mathbf{1}^{N-1}$ is given on objects by

$$P_{\mathbf{1}}(f)(s_1,\dots,s_N) = (s_1,\dots,s_{j-1},s_{j+1},\dots,s_N)$$

and on morphisms $\lambda = (\lambda_1, \dots, \lambda_N)\colon (s_1, \dots, s_N) \to (s'_1, \dots, s'_N)$ by

$$P_{\mathbf{1}}(f)(\lambda) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_N).$$

It follows that $P_{\mathbf{1}}(f)$ is the cubical codegeneracy functor $\varepsilon_{j,N}$.

(3) If f is of type (iii) then $|V_T| = |V_{T'}| + 1$ and the injectivity of f only fails when it collapses two joints (the endpoints of the p^{th} bead Δ^1) to a joint in T'. Under the isomorphism $\phi_T\colon P(T) \cong \mathbf{1}^N$, this collapse does not have any effect since given an object X of $P(T)$ the entries in the string $\phi_T(X)$ of 0's and 1's only indicate which nonjoint vertices of T are in X. It follows that $P_{\mathbf{1}}(f)\colon \mathbf{1}^N \to \mathbf{1}^N$ is the identity functor. □

Remark 4.3 Consider two morphisms of necklaces $f\colon U \to T$ and $g\colon V \to T$. If f and g are both of type (i) and $f \neq g$, then $P_{\mathbf{1}}(f) \neq P_{\mathbf{1}}(g)$. If f and g are of both of type (ii) and $f \neq g$, we may have $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$. For example, let $U = W \vee \Delta^{m+1} \vee \Delta^n \vee W'$, $V = W \vee \Delta^m \vee \Delta^{n+1} \vee W'$ and $T = W \vee \Delta^m \vee \Delta^n \vee W'$ for any two necklaces W and W'. Consider the maps $f = \mathrm{id}_W \vee s_{m+1} \vee \mathrm{id}_{\Delta^n} \vee \mathrm{id}_{W'}$ and $g = \mathrm{id}_W \vee \mathrm{id}_{\Delta^m} \vee s_1 \vee \mathrm{id}_{W'}$, where $s_{m+1}\colon \Delta^{m+1} \to \Delta^m$ and $s_1\colon \Delta^{n+1} \to \Delta^n$ are the last and first (simplicial) codegeneracy morphisms, respectively. It follows that $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$. The identification of these two morphisms after applying $P_{\mathbf{1}}$ should be compared with the identification in the colimit defining the monoidal structure of the category of cubical sets with connections discussed in the next section. Finally, if f and g are of type (iii), then we always have $P_{\mathbf{1}}(f) = P_{\mathbf{1}}(g)$.

Corollary 4.4 *The functor* $P_{\mathbf{1}}\colon \mathrm{Nec} \to \mathrm{Cat}$ *factors as a composition* $\mathrm{Nec} \to \square_c \hookrightarrow \mathrm{Cat}$.

Proof For any object T in Nec, $P_{\mathbf{1}}(T) = \mathbf{1}^N$ is an object of $\square_c$ and, by Proposition 4.2, for any morphism f in Nec, $P_{\mathbf{1}}(f)$ is a morphism in $\square_c$. □

Hence, we may consider $P_{\mathbf{1}}$ as a functor from Nec to $\square_c$. Finally, we define a functor from the category of necklaces to the category of cubical sets as follows.

Definition 4.5 Define the functor $C_{\square_c}\colon \mathrm{Nec} \to \mathrm{Set}_{\square_c}$ to be the composition of functors $C_{\square_c} := Y \circ P_{\mathbf{1}}$, where $Y\colon \square_c \to \mathrm{Hom}_{\mathrm{Cat}}((\square_c)^{\mathrm{op}}, \mathrm{Set}) = \mathrm{Set}_{\square_c}$ is the Yoneda embedding.

Note that for any T in Nec, $C_{\square_c}(T)$ is the standard cube with connections $\square_c^N$, where $N = |V_T - J_T|$.

Remark 4.6 All nondegenerate cells of $C_{\square_c}(T)$ can be realized by injective maps of necklaces $T' \to T$. More precisely, for every nondegenerate cell $\sigma \in C_{\square_c}(T)_n$ there is a necklace T_σ, with $|V_{T_\sigma} - J_{T_\sigma}| = n$, together with an injective map of necklaces $\iota_\sigma \colon T_\sigma \to T$ such that the induced map of cubical sets with connections

$$\square_c^n \cong C_{\square_c}(T_\sigma) \xrightarrow{C_{\square_c}(\iota_\sigma)} C_{\square_c}(T)$$

corresponds to the cell σ. Notice T_σ is not unique, since any other T'_σ for which there is a map $T'_\sigma \to T_\sigma$ of type (iii) also works.

5 The cubical rigidification functor $\mathfrak{C}_{\square_c} \colon \mathbf{Set}_\Delta \to \mathbf{Cat}_{\square_c}$

The goal of this section is to show that the functor $\mathfrak{C} \colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$ defined by Lurie factors naturally through categories enriched over cubical sets with connections via a functor $\mathfrak{C}_{\square_c} \colon \mathrm{Set}_\Delta \to \mathrm{Cat}_{\square_c}$. More precisely, we construct functors $\mathfrak{C}_{\square_c} \colon \mathrm{Set}_\Delta \to \mathrm{Cat}_{\square_c}$ and $\mathfrak{T} \colon \mathrm{Cat}_{\square_c} \to \mathrm{Cat}_\Delta$ such that $\mathfrak{T} \circ \mathfrak{C}_{\square_c}$ is naturally isomorphic to $\mathfrak{C}$.

Definition 5.1 For any simplicial set S we define a category $\mathfrak{C}_{\square_c}(S)$ enriched over cubical sets with connections. Define the objects of $\mathfrak{C}_{\square_c}(S)$ to be the vertices of S, ie the elements of S_0. For any $x, y \in S_0$ define

$$\mathfrak{C}_{\square_c}(S)(x, y) := \operatorname*{colim}_{T \to S \in (\mathrm{Nec} \downarrow S)_{x,y}} C_{\square_c}(T),$$

where $(\mathrm{Nec} \downarrow S)_{x,y}$ is the category whose objects are morphisms $f \colon T \to S$ for some $T \in \mathrm{Nec}$ such that $f(\alpha_T) = x$ and $f(\omega_T) = y$. For any $x, y, z \in S_0$, the composition law

$$\mathfrak{C}_{\square_c}(S)(y, z) \otimes \mathfrak{C}_{\square_c}(S)(x, y) \to \mathfrak{C}_{\square_c}(S)(x, z)$$

is induced as follows. Note that given $T \to S \in (\mathrm{Nec} \downarrow S)_{x,y}$ and $U \to S \in (\mathrm{Nec} \downarrow S)_{y,z}$, we obtain $T \vee U \to S \in (\mathrm{Nec} \downarrow S)_{x,z}$. Then the composition

$$C_{\square_c}(U) \otimes C_{\square_c}(T) \to C_{\square_c}((T \vee U)(\alpha_U, \omega_U)) \otimes C_{\square_c}((T \vee U)(\alpha_T, \omega_T)) \to C_{\square_c}(T \vee U)$$

of morphisms of cubical sets with connections induces the desired composition law after taking colimits. Recall that $(T \vee U)(\alpha_U, \omega_U)$ denotes the unique subnecklace of $T \vee U$ with joints $J_{T \vee U}(\alpha_U, \omega_U)$ and vertices $V_{T \vee U}(\alpha_U, \omega_U)$. It follows from Remark 4.3 that the above composition passes to the colimit and yields a well-defined composition rule. Finally, it is clear that $\mathfrak{C}_{\square_c}(S)$ is functorial in S.

Remark 5.2 The set of n–cells in $\mathfrak{C}_{\square_c}(S)(x,y)$ is

$$\Big(\coprod_{(T\to S)\in(\mathrm{Nec}\downarrow S)_{x,y}} C_{\square_c}(T)_n\Big)\Big/\sim,$$

where the equivalence relation is generated by $(t\colon T\to S,\sigma)\sim(t'\colon T'\to S,\sigma')$ if there is a map of necklaces $f\colon T\to T'$ such that $t=t'\circ f$ and $C_{\square_c}(f)(\sigma)=\sigma'$. Here $t\colon T\to S$ and $t'\colon T'\to S$ are objects in $(\mathrm{Nec}\downarrow S)_{x,y}$, and σ and σ' are n–cells in $C_{\square_c}(T)$ and $C_{\square_c}(T')$, respectively. Any *nondegenerate* n–cell $[t\colon T\to S,\sigma]\in\mathfrak{C}_{\square_c}(S)(x,y)_n$ may be represented by a pair $(r\colon R\to S,\sigma_R)$, where

- R is a necklace with $|V_R-J_R|=n$ such that there are no $(u\colon U\to S)\in(\mathrm{Nec}\downarrow S)_{x,y}$ with $|V_U-J_U|=n-1$ and $f\colon R\to U$ satisfying $r=u\circ f$, and
- $\sigma_R\in C_{\square_c}(R)_n$ is the unique nondegenerate n–cell in $C_{\square_c}(R)$.

In fact, one can let $R=T_\sigma$ and $r=t\circ\iota_\sigma$ as in Remark 4.6. These representatives are not unique since we may have another representative $(r'\colon R'\to S,\sigma_{R'})$ if there is a morphism of necklaces $h\colon R\to R'$ of type (iii) such that $r'\circ h=r$. We write $[r\colon R\to S]$ for the equivalence class of the nondegenerate n–cell in $\mathfrak{C}_{\square_c}(S)(x,y)$ represented by $(r\colon R\to S,\sigma_R)$. Let v be the j^{th} vertex in V_R-J_R. The face map $\partial^1_j\colon\mathfrak{C}_{\square_c}(S)(x,y)_n\to\mathfrak{C}_{\square_c}(S)(x,y)_{n-1}$ is given by $\partial^1_j[r\colon R\to S]=[\partial^1_j r\colon R_v\to S]$, where R_v is the subnecklace of R spanned by vertices $V_R-\{v\}$ and $\partial^1_j r$ is the restriction of r to R_v. The face map $\partial^0_j\colon\mathfrak{C}_{\square_c}(S)(x,y)_n\to\mathfrak{C}_{\square_c}(S)(x,y)_{n-1}$ is given by $\partial^0_j[r\colon R\to S]=[\partial^0_j r\colon R(\alpha_R,v)\vee R(v,\omega_R)\to S]$, where $\partial^0_j r$ is the restriction of r to $R(\alpha_R,v)\vee R(v,\omega_R)$. Of course, $[\partial^1_j r\colon R_v\to S]$ and $[\partial^0_j r\colon R(\alpha_R,v)\vee R(v,\omega_R)\to S]$ may be degenerate cells in $\mathfrak{C}_{\square_c}(S)(x,y)_{n-1}$ even if $[r\colon R\to S]$ is nondegenerate.

Let us recall Lurie's construction of $\mathfrak{C}\colon\mathrm{Set}_\Delta\to\mathrm{Cat}_\Delta$. Given integers $0\le i\le j$, denote by $P_{i,j}$ the category whose objects are subsets of the set $\{i,i+1,\dots,j\}$ containing both i and j and morphisms are inclusions of sets. We have an isomorphism of categories $P_{i,j}\cong\mathbf{1}^{j-i-1}$ if $i<j$ and $P_{i,i}\cong\mathbf{1}^0$. For each integer $n\ge0$ define a simplicial category $\mathfrak{C}(\Delta^n)$ whose objects are the elements of the set $\{0,\dots,n\}$ and for any two objects i and j such that $i\le j$, $\mathfrak{C}(\Delta^n)(i,j)$ is the simplicial set $N(P_{i,j})$, where $N\colon\mathrm{Cat}\to\mathrm{Set}_\Delta$ is the nerve functor. If $j<i$, $\mathfrak{C}(\Delta^n)(i,j)$ is defined to be empty. The composition law in the simplicial category $\mathfrak{C}(\Delta^n)$ is induced by the map of categories $P_{j,k}\times P_{i,k}\to P_{i,k}$ given by union of sets. The construction of $\mathfrak{C}(\Delta^n)$ is functorial with respect to simplicial maps between standard simplices. Then the functor $\mathfrak{C}\colon\mathrm{Set}_\Delta\to\mathrm{Cat}_\Delta$ is defined by $\mathfrak{C}(S):=\mathrm{colim}_{\Delta^n\to S}\,\mathfrak{C}(\Delta^n)$.

$\mathfrak{C}$ is defined as a colimit in the category of simplicial categories. Dugger and Spivak computed in [5] the mapping spaces of $\mathfrak{C}$ explicitly via necklaces. More precisely, Proposition 4.3 of [5] states that there is an isomorphism of simplicial sets

$$\operatorname*{colim}_{T \to S \in (\mathrm{Nec}\downarrow S)_{x,y}} [\mathfrak{C}(T)(\alpha_T, \omega_T)] \cong \mathfrak{C}(S)(x, y).$$

We defined $\mathfrak{C}_{\square_c}$ having this formula in mind. We do it this way, as opposed to first defining $\mathfrak{C}_{\square_c}$ on standard simplices and then extending as a left Kan extension, to emphasize that maps of necklaces give rise to maps of cubical sets with connections and the relationship of this fact with Adams's cobar construction, as we will explain later on. The mapping spaces of the functor $\mathfrak{C}_{\square_c}$ are cubical sets with connections constructed by applying the Yoneda embedding to the category $P_{\mathbf{1}}(T)$ associated to a necklace T and then taking a colimit, while the mapping spaces in $\mathfrak{C}$ are simplicial sets obtained by applying the nerve functor to $P_{\mathbf{1}}(T)$ and then taking a colimit.

Recall we have a triangulation functor $|\cdot|\colon \mathrm{Set}_{\square_c} \to \mathrm{Set}_\Delta$ defined on a cubical set with connections K by $|K| := \mathrm{colim}_{\square_c^n \to K} N(\mathbf{1}^n) \cong \mathrm{colim}_{\square_c^n \to K} (\Delta^1)^{\times n}$. Define a functor $\mathfrak{T}\colon \mathrm{Cat}_{\square_c} \to \mathrm{Cat}_\Delta$ as follows. Given a category $\mathcal{K}$ enriched over $\mathrm{Set}_{\square_c}$ define $\mathfrak{T}(\mathcal{K})$ to be the simplicial category whose objects are the objects of $\mathcal{K}$ and whose mapping spaces are given by $|\mathcal{K}(x, y)|$ for any objects x and y in $\mathcal{K}$. We have a composition law on $\mathfrak{T}(\mathcal{K})$ induced by applying the functor $|\cdot|$ to the composition law in $\mathcal{K}$ and using the fact that for cubical sets with connections K and K' we have a natural isomorphism $|K \otimes K'| \cong |K| \times |K'|$. In fact, since colimits commute we have the isomorphisms of simplicial sets

$$\begin{aligned}
|K \otimes K'| &\cong |\operatorname*{colim}_{\square_c^n \to K,\, \square_c^m \to K'} \square_c^{n+m}| \\
&\cong \operatorname*{colim}_{\square_c^n \to K,\, \square_c^m \to K'} |\square_c^{n+m}| \\
&\cong \operatorname*{colim}_{\square_c^n \to K,\, \square_c^m \to K'} (\Delta^1)^{\times n+m} \\
&\cong \operatorname*{colim}_{\square_c^n \to K,\, \square_c^m \to K'} (\Delta^1)^{\times n} \times (\Delta^1)^{\times m} \\
&\cong \operatorname*{colim}_{\square_c^n \to K} (\Delta^1)^{\times n} \times \operatorname*{colim}_{\square_c^m \to K'} (\Delta^1)^{\times m} \\
&\cong |K| \times |K'|.
\end{aligned}$$

Proposition 5.3 *The functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$ is naturally isomorphic to the composition of functors*

$$\mathrm{Set}_\Delta \xrightarrow{\mathfrak{C}_{\square_c}} \mathrm{Cat}_{\square_c} \xrightarrow{\mathfrak{T}} \mathrm{Cat}_\Delta.$$

Proof Let $Y(\square_c)\downarrow\square_c^N$ be the category whose objects are morphisms $\square_c^n\to\square_c^N$ of cubical sets with connections and whose morphisms are given by the corresponding commutative triangles. Note $|\square_c^N|$ is the colimit in simplicial sets of the functor $Y(\square_c)\downarrow\square_c^N\to\mathrm{Set}_\Delta$ that sends an object $(\square_c^n\to\square_c^N)$ to $N(\mathbf{1}^n)\cong(\Delta^1)^{\times n}$ and a morphism in $Y(\square_c)\downarrow\square_c^N$ to the corresponding induced morphism between nerves. The identity morphism $\square_c^N\to\square_c^N$ is a terminal object in $Y(\square_c)\downarrow\square_c^N$. Therefore, $|\square_c^N|=\operatorname{colim}_{\square_c^n\to\square_c^N}N(\mathbf{1}^n)$ is given by the value of the functor on the identity morphism $\square_c^N\to\square_c^N$, so $|\square_c^N|=N(\mathbf{1}^N)$.

Let S be a simplicial set. The objects of the simplicial categories $\mathfrak{T}(\mathfrak{C}_{\square_c}(S))$ and $\mathfrak{C}(S)$ are the same, ie the elements of S_0. Since the triangulation functor $|\cdot|$ commutes with colimits, we have the natural isomorphisms

$$(\mathfrak{T}(\mathfrak{C}_{\square_c}(S)))(x,y)\cong\operatorname*{colim}_{T\to S\in(\mathrm{Nec}\downarrow S)_{x,y}}|C_{\square_c}(T)|\cong\operatorname*{colim}_{T\to S\in(\mathrm{Nec}\downarrow S)_{x,y}}N(\mathbf{1}^{|V_T-J_T|}).$$

Moreover, by Proposition 4.3 of [5] it follows that we have natural isomorphisms

$$\operatorname*{colim}_{T\to S\in(\mathrm{Nec}\downarrow S)_{x,y}}N(\mathbf{1}^{|V_T-J_T|})\cong\operatorname*{colim}_{T\to S\in(\mathrm{Nec}\downarrow S)_{x,y}}[\mathfrak{C}(T)(\alpha,\omega)]\cong\mathfrak{C}(S)(x,y).$$

Hence, we have an isomorphism of simplicial categories $\mathfrak{T}(\mathfrak{C}_{\square_c}(S))\cong\mathfrak{C}(S)$ which is functorial on S. It follows that $\mathfrak{T}\circ\mathfrak{C}_{\square_c}$ and $\mathfrak{C}$ are naturally isomorphic functors. □

6 The left adjoint $\Lambda\colon\mathbf{Set}_\Delta\to\mathbf{dgCat}_k$ of the dg nerve functor

In Section 1.3.1 of [16], Lurie defines a functor $N_{\mathrm{dg}}\colon\mathrm{dgCat}_k\to\mathrm{Set}_\Delta$, called the *dg nerve*, which is weakly equivalent to the left adjoint of the composite functor

$$\Gamma\colon\mathrm{Set}_\Delta\xrightarrow{\mathfrak{C}}\mathrm{Cat}_\Delta\xrightarrow{\mathfrak{Q}_\Delta}\mathrm{dgCat}_k,$$

where $\mathfrak{Q}_\Delta$ is the functor obtained by applying the normalized chains functor

$$Q_\Delta\colon\mathrm{Set}_\Delta\to\mathrm{Ch}_k$$

on the mapping spaces. In this section we prove that the composite functor

$$\Lambda\colon\mathrm{Set}_\Delta\xrightarrow{\mathfrak{C}_{\square_c}}\mathrm{Cat}_{\square_c}\xrightarrow{\mathfrak{Q}_{\square^c}}\mathrm{dgCat}_k,$$

where $\mathfrak{Q}_{\square^c}$ is the functor obtained by applying the normalized chains functor

$$Q_{\square_c}\colon\mathrm{Set}_{\square_c}\to\mathrm{Ch}_k$$

on the mapping spaces, is left adjoint to N_{dg}.

Recall Lurie's definition of N_{dg}. Let $\mathcal{C}$ be a dg category. For each $n \geq 0$, define $N_{\mathrm{dg}}(\mathcal{C})_n$ to be the set of all ordered pairs of sets $(\{X_i\}_{0\leq i\leq n}, \{f_I\})$, such that:

(1) $X_0, X_1, \ldots, X_n$ are objects of the dg category $\mathcal{C}$.

(2) I is a subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subseteq [n]$ with $m \geq 0$ and f_I is an element of $\mathcal{C}(X_{i_-}, X_{i_+})_m$ satisfying

$$df_I = \sum_{1\leq j\leq m} (-1)^j (f_{I-\{i_j\}} - f_{i_j<\cdots<i_1<i_+} \circ f_{i_-<i_m<\cdots<i_j}).$$

The structure maps in $N_{\mathrm{dg}}(C)$ are defined as follows. If $\alpha\colon [m]\to[n]$ is a nondecreasing function, then the induced map $N_{\mathrm{dg}}(\mathcal{C})_n \to N_{\mathrm{dg}}(\mathcal{C})_m$ is given by

$$(\{X_i\}_{0\leq 1\leq n}, \{f_I\}) \mapsto (\{X_{\alpha(j)}\}_{0\leq j\leq m}, \{g_J\}),$$

where $g_J = f_{\alpha(J)}$ if $\alpha|_J$ is injective, $g_J = \mathrm{id}_{X_i}$ if $J = \{j, j'\}$ with $\alpha(j) = i = \alpha(j')$, and $g_J = 0$ otherwise.

Theorem 6.1 *The functor $\Lambda\colon \mathrm{Set}_\Delta \to \mathrm{dgCat}_k$ is left adjoint to $N_{\mathrm{dg}}\colon \mathrm{dgCat}_k \to \mathrm{Set}_\Delta$.*

Proof First, we show that for any standard simplex Δ^n and any dg category $\mathcal{C}$ there is a bijection

$$\theta_{n,\mathcal{C}}\colon \mathrm{dgCat}_k(\Lambda(\Delta^n), \mathcal{C}) \cong \mathrm{Set}_\Delta(\Delta^n, N_{\mathrm{dg}}(\mathcal{C}))$$

which is functorial with respect to morphisms in the category Δ. Given a dg functor $F\colon \Lambda(\Delta^n)\to\mathcal{C}$ we construct an n–simplex

$$\theta_{n,\mathcal{C}}(F) = (\{X_0, \ldots, X_n\}, \{f_I\})$$

in $N_{\mathrm{dg}}(\mathcal{C})_n$. The objects of $\Lambda(\Delta^n)$ are the integers $0, 1, \ldots, n$ so we let $X_i = F(i)$ for $i = 0, 1, \ldots, n$. For every subset $I = \{i_- < i_1 < \cdots < i_m < i_+\} \subseteq [n]$ define σ_I to be the generator of the chain complex $\Lambda(\Delta^n)(i_-, i_+) = Q_{\square_c}(\mathfrak{C}_{\square_c}(\Delta^n)(i_-, i_+))$ represented by the nondegenerate element of $(\mathfrak{C}_{\square_c}(\Delta^n)(i_-, i_+))_m$ which is the one-bead subnecklace inside Δ^n consisting of the $(m+1)$–simplex with i_- as first vertex, i_+ as last vertex, and $i_1, \ldots, i_m$ as nonjoint vertices; in other words, σ_I is represented by the $(m+1)$–simplex inside Δ^n spanned by vertices $i_-, i_1, \ldots, i_m, i_+$. It follows from Remark 3.2 that

$$d\sigma_I = \sum_{j=1}^{m} (-1)^j (\partial^1_j \sigma_I - \partial^0_j \sigma_I) = \sum_{j=1}^{m} (-1)^j (\sigma_{I-\{i_j\}} - \sigma_{i_j<\cdots<i_1<i_+} \circ \sigma_{i_-<i_m<\cdots<i_j}).$$

Define $f_I = F(\sigma_I)\colon X_{i_-} \to X_{i_+}$. Since the dg functor F commutes with differentials at the level of mapping spaces, f_I satisfies property (2) in the definition of the dg nerve functor. The functoriality of $\theta_{n,\mathcal{C}}$ with respect to simplicial maps between standard simplices follows from Proposition 4.2. Finally, since the functor Λ preserves colimits, $\theta_{n,\mathcal{C}}$ induces a functorial bijection

$$\mathrm{dgCat}_k(\Lambda(S), \mathcal{C}) \cong \mathrm{Set}_\Delta(S, N_{\mathrm{dg}}(\mathcal{C}))$$

for any simplicial set S and dg category $\mathcal{C}$. □

Remark 6.2 Let S be a simplicial set and $x, y \in S_0$. A generator ξ of degree n in the chain complex $\Lambda(S)(x,y)$ is an equivalence class which may be represented by a nondegenerate n–cell σ in the cubical set with connections $\mathfrak{C}_{\square_c}(S)(x,y)$. Since $\mathfrak{C}_{\square_c}(S)(x,y)$ is defined as a colimit, the nondegenerate n–cell σ is itself an equivalence class $[r\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \to S]$, where $(r\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \to S) \in (\mathrm{Nec}\downarrow S)_{x,y}$ and $n_1 + \cdots + n_k - k = n$ and such that there is no $(u\colon \Delta^{m_1} \vee \cdots \vee \Delta^{m_l} \to S) \in (\mathrm{Nec}\downarrow S)_{x,y}$ with $m_1 + \cdots + m_l - l < n$ together with a map of necklaces

$$f\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_k} \to \Delta^{m_1} \vee \cdots \vee \Delta^{m_l}$$

satisfying $r = u \circ f$. Moreover, any

$$s\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_i} \vee \Delta^1 \vee \Delta^{n_i+1} \vee \cdots \vee \Delta^{n_k} \to S$$

satisfying $r \circ \pi = s$, where $\pi\colon \Delta^{n_1} \vee \cdots \vee \Delta^{n_i} \vee \Delta^1 \vee \Delta^{n_i+1} \vee \cdots \vee \Delta^{n_k} \to \Delta^{n_1} \vee \cdots \vee \Delta^{n_k}$ is the map of simplicial sets which collapses the $(i+1)^{\mathrm{st}}$ bead in the domain necklace to a point, also represents the equivalence class σ. This follows essentially from Proposition 4.2(3).

7 Rigidification and the cobar construction

In this section, we relate the functor $\mathfrak{C}_{\square_c}\colon \mathrm{Set}_\Delta^0 \to \mathrm{Cat}_{\square_c}$ to the cobar functor

$$\Omega\colon \mathrm{dgCoalg}_k^0 \to \mathrm{dgAlg}_k.$$

More precisely, we prove that $\Omega Q_\Delta(S)$, the cobar construction on the dg coalgebra of normalized chains on a simplicial set S with one vertex x, is isomorphic as a dga to $\Lambda(S)(x,x)$, where Λ is the functor obtained by applying the normalized cubical chains functor on the mapping spaces of $\mathfrak{C}_{\square_c}$, or, naturally isomorphically, the left adjoint to the dg nerve functor, as described in the previous section. Then we deduce a relationship

between $\mathfrak{C}\colon \mathrm{Set}_\Delta^0 \to \mathrm{Cat}_\Delta$ and $\Omega\colon \mathrm{dgCoalg}_k^0 \to \mathrm{dgAlg}_k$: we show $\Omega Q_\Delta(S)$ is naturally weakly equivalent (quasi-isomorphic) as a dga to $\Gamma(S)(x,x)$, where $\Gamma\colon \mathrm{Set}_\Delta \to \mathrm{dgCat}$ is the functor obtained by applying normalized chains to the mapping spaces of $\mathfrak{C}$.

Let k be a fixed commutative ring. We may consider k as a graded k–module concentrated on degree 0. A graded coassociative coalgebra (C,Δ) over k is *counital* if it is equipped with a degree 0 map $\epsilon\colon C \to k$, called the *counit*, such that $(\epsilon\otimes\mathrm{id})\circ\Delta = \mathrm{id} = (\mathrm{id}\otimes\epsilon)\circ\Delta$.

We say a differential graded coassociative coalgebra (dg coalgebra, for short) (C,∂,Δ) over a commutative ring k is *connected* if $C_0 \cong k$. Given a connected dg coalgebra (C,∂,Δ) which is free as a k–module in each degree, the *cobar construction* of C is the differential graded associative algebra $(\Omega C, D)$ defined as follows. Consider the graded k–module $s\overline{C}$, where $\overline{C}_i = C_i$ for $i>0$ and $\overline{C}_0 = 0$ and s is the shift by -1, ie $(s\overline{C})_i = \overline{C}_{i+1}$. Let $\Delta = \mathrm{id}\otimes 1 + 1\otimes\mathrm{id} + \Delta'$ and for any $c\in\overline{C}$ write $\Delta'(c) = \sum c'\otimes c''$. The underlying algebra of the cobar construction is the tensor algebra

$$\Omega C = T s\overline{C} = k \oplus s\overline{C} \oplus (s\overline{C}\otimes s\overline{C}) \oplus (s\overline{C}\otimes s\overline{C}\otimes s\overline{C}) \oplus \cdots$$

and the differential D is defined by extending $D(sc) = -s\partial c + \sum(-1)^{\deg c'} sc'\otimes sc''$ as a derivation to all of ΩC. This construction yields a functor $\Omega\colon \mathrm{dgCoalg}_k^0 \to \mathrm{dgAlg}_k$, where dgAlg_k is the category of augmented dg algebras over k.

For any simplicial set S, the chain complex $Q'_\Delta(S)$ of *unnormalized* chains over k has a natural coproduct $\Delta\colon Q'_\Delta(S) \to Q'_\Delta(S)\otimes Q'_\Delta(S)$ given by

$$\Delta(x) = \bigoplus_{p+q=n} f^p(x)\otimes l^q(x)$$

for any $x\in Q_\Delta(S)_n$, where f^p denotes the *front p–face map* (induced by the map $[p]\to[p+q]$, $i\mapsto i$) and l^q is the *last q–face map* (induced by the map $[q]\to[p+q]$, $i\mapsto i+p$). This coproduct is known as the Alexander–Whitney diagonal map. Moreover, this dg coalgebra structure passes to the *normalized* chain complex $Q_\Delta(S)$. Thus, we may consider Q_Δ as a functor $Q_\Delta\colon \mathrm{Set}_\Delta \to \mathrm{dgCoalg}_k$. In particular, $Q_\Delta(S)$ is a dg coalgebra which is free as a k–module in each degree. If S is 0–reduced, ie $S_0 = \{x\}$, then $Q_\Delta(S)$ is counital and connected with counit map given by the composition $Q_\Delta(S) \twoheadrightarrow Q_\Delta(S)_0 = k[x] \xrightarrow{\cong} k$. From now on all of the coalgebras in this article will be assumed to be counital.

Theorem 7.1 *Let S be a 0–reduced simplicial set with $S_0 = \{x\}$. There is an isomorphism of differential graded algebras $\Lambda(S)(x,x) \cong \Omega Q_\Delta(S)$.*

Proof For each integer $n \geq 0$ the boundary map $\partial\colon Q'_\Delta(S)_n \to Q'_\Delta(S)_{n-1}$ and the coproduct $\Delta\colon Q'_\Delta(S)_n \to \bigoplus_{p+q=n} Q'_\Delta(S)_p \otimes Q'_\Delta(S)_q$ can be written as sums $\partial = \sum_{i=0}^{n}(-1)^i \partial_i$ and $\Delta = \sum_{i=0}^{n} \Delta_i$ as usual. In particular, for $\sigma \in S_n$, $\Delta_0(\sigma) = \min\sigma \otimes \sigma$ and $\Delta_n\sigma = \sigma \otimes \max\sigma$, where $\min\sigma$ and $\max\sigma$ denote the first and last vertices of σ, respectively. The truncated maps $\partial' = \sum_{i=1}^{n-1}(-1)^i \partial_i$ and $\Delta' = \sum_{i=1}^{n-1}(-1)^i \Delta_i$ also define a differential graded coassociative coalgebra structure on $Q'_\Delta(S)$. Consider the dga $\Omega Q'_\Delta(S) = \Omega(Q'_\Delta(S), \partial', \Delta')$. First, we show

$$\Lambda(S)(x,x) = Q_{\square_c}(\mathfrak{C}_{\square_c}(S)(x,x)) \cong \Omega Q'_\Delta(S)/\sim$$

for some equivalence relation $\sim$ and then we construct an isomorphism

$$\Omega Q'_\Delta(S)/\sim \cong \Omega Q_\Delta(S).$$

The dga $\Omega Q'_\Delta(S)$ has as underlying complex the tensor algebra $Ts\overline{Q'_\Delta(S)}$ together with differential $D'_\Omega = \partial' + \Delta'$ extended as a derivation to all of $Ts\overline{Q'_\Delta(S)}$. We denote a monomial $s\sigma_1 \otimes \cdots \otimes s\sigma_k \in Ts\overline{Q'_\Delta(S)}$ by $[\sigma_1|\cdots|\sigma_k]$. Let $s_0(x) \in Q'_\Delta(S)_1$ be the generator corresponding to the degenerate 1–simplex at x. We take a quotient of $Ts\overline{Q'_\Delta(S)}$ by the equivalence relation generated by

$$[\sigma_1|\cdots|\sigma_k] \sim [\sigma_1|\cdots|\sigma_{i-1}|\sigma_{i+1}|\cdots|\sigma_k]$$

if for some $1 \leq i \leq k$ we have $\sigma_i = s_0(x)$ (in particular, $[\sigma_1] \sim 1_k$ if $\sigma_1 = s_0(x)$); and

$$[\sigma_1|\cdots|\sigma_k] \sim 0$$

if $\sigma_i \in Q'_\Delta(S)_{n_i}$ is a degenerate simplex with $n_i > 1$ for some $1 \leq i \leq k$. The first relation corresponds to the identification in the colimit defining $\mathfrak{C}_{\square_c}(S)(x,x)$ arising from Remark 4.6; the second relation corresponds to modding out by degenerate chains in the definition of the normalized chain complex $Q_{\square_c}(\mathfrak{C}_{\square_c}(S)(x,x))$. Both the differential D'_Ω and the algebra structure of $Ts\overline{Q'_\Delta(S)}$ pass to the quotient

$$Ts\overline{Q'_\Delta(S)}/\sim.$$

It is clear that we have an isomorphism of dgas

$$Q_{\square_c}(\mathfrak{C}_{\square_c}(S)(x,x)) \cong \Omega Q'_\Delta(S)/\sim$$

since necklaces in S correspond to monomials of generators in $Q'_\Delta(S)$.

We define an isomorphism of dgas

$$\widetilde{\varphi}\colon \Omega Q'_\Delta(S)/\sim \to \Omega Q_\Delta(S).$$

Given $\sigma \in Q'_\Delta(S)$ denote by $\overline{\sigma}$ the equivalence class of σ in $Q_\Delta(S)$. First define $\varphi[\sigma] = [\overline{\sigma}]$ if $\deg \sigma > 1$, $\varphi[\sigma] = \overline{\sigma} + 1_k$ if $\deg \sigma = 1$, and $\varphi(1_k) = 1_k$. Extend φ as an algebra map to obtain a map $\varphi\colon \Omega Q'_\Delta(S) \to \Omega Q_\Delta(S)$. It follows by a short computation that the map φ is a chain map. Moreover, φ induces a map of dgas $\widetilde{\varphi}\colon \Omega Q'_\Delta(S)/\sim \to \Omega Q_\Delta(S)$. The map $\widetilde{\varphi}$ is an isomorphism of dgas, in fact, the inverse map $\psi\colon \Omega Q_\Delta(S) \to \Omega Q'_\Delta(S)/\sim$ is given by defining $\psi[\overline{\sigma}] = [[\sigma]]$ if $\deg \sigma > 1$, $\psi[\overline{\sigma}] = [[\sigma]] - [1_k]$ if $\deg \sigma = 1$, and $\psi(1_k) = [1_k]$ and then extending ψ as an algebra map, where $[[\sigma]]$ denotes the equivalence class of $[\sigma] \in \Omega Q'_\Delta$ in the quotient $\Omega Q'_\Delta(S)/\sim$. □

We now relate the dgas $\Omega Q_\Delta(S)$ and $\Gamma(S)(x,x)$. We will use the following lemma, which follows from an acyclic models argument.

Lemma 7.2 *For any cubical set with connections K the chain complex $Q_\Delta(|K|)$ is naturally weakly equivalent to $Q_{\square_c}(K)$, where $|\cdot|\colon \mathrm{Set}_{\square_c} \to \mathrm{Set}_\Delta$ is the triangulation functor.*

Proof This proposition follows from the acyclic models theorem applied to the two functors

$$Q_\Delta \circ |\cdot|,\ Q_{\square_c}\colon \mathrm{Set}_{\square_c} \to \mathrm{Ch}_k.$$

Define the collection of models in $\mathrm{Set}_{\square_c}$ to be $\mathcal{M} = \{\square_c^0, \square_c^1, \dots\}$, where $\square_c^j$ is the standard j–cube with connections. It is clear that both $Q_\Delta \circ |\cdot|$ and $Q_{\square_c}$ are acyclic on these models. Recall a functor $F\colon \mathcal{C} \to \mathrm{Ch}_k$ is *free* on $\mathcal{M}$ if there exist a collection $\{M_j\}_{j\in\mathcal{J}}$, where each M_j is an object in $\mathcal{M}$ (possibly with repetitions, possibly not including all of the objects in $\mathcal{M}$) together with elements $m_j \in F(M_j)$ such that for any object X of $\mathcal{C}$ we have that

$$\{F(f)(m_j) \in F(X) \mid j \in \mathcal{J},\ (f\colon M_j \to X) \in \mathcal{C}(M_i, X)\}$$

forms a basis for $F(X)$. Clearly $Q_{\square_c}$ is free on $\mathcal{M}$ since we can take $M_j = \square_c^j$ and $\mathcal{J} = \{0,1,2,\dots\}$, and define $m_j \in Q_{\square_c}(M_j) = Q_{\square_c}(\square_c^j)$ to be the generator corresponding to the unique nondegenerate element in $(\square_c^j)_j$ (ie m_j is the top nondegenerate cell of $\square_c^j$). Note that the simplicial set $|\square_c^j| \cong (\Delta^1)^{\times j}$ has $j!$ nondegenerate j–simplices $\sigma_1^j, \dots, \sigma_{j!}^j \in |\square_c^j|_j$. Hence, $Q_\Delta \circ |\cdot|$ is also free on $\mathcal{M}$ since we

can take $\{M_1^0, M_1^1, M_1^2, M_2^2, \dots, M_1^j, \dots, M_{j!}^j, M_1^{j+1}, \dots\}_{j\in\mathcal{J}}$, where $M_k^j = \square_c^j$, $\mathcal{J} = \{0, 1, 2, \dots\}$ and $m_k^j \in Q_\Delta(|M_k^j|)$ is the generator corresponding to the j–simplex $\sigma_k^j \in |\square_c^j|_j$.

We have a natural isomorphism of functors $H_0(Q_\Delta \circ |\cdot|) \cong H_0(Q_{\square_c})$; in fact, for any $K \in \mathrm{Set}_{\square_c}$ there is a natural bijection between $|K|_0$ and K_0 and any two vertices x and y are connected by a sequence of 1–simplices in $|K|_1$ if and only if they are connected by a sequence of 1–cubes in K_1. By the acyclic models theorem there exist natural transformations $\phi\colon Q_\Delta \circ |\cdot| \to Q_{\square_c}$ and $\psi\colon Q_{\square_c} \to Q_\Delta \circ |\cdot|$ such that each composition $\phi\circ\psi$ and $\psi\circ\phi$ is chain homotopic to the identity map. □

We use the above lemma to relate $\Omega Q_\Delta(S)$ and $\Gamma(S)(x,x)$.

Proposition 7.3 *Let S be a 0–reduced simplicial set with $S_0 = \{x\}$. The differential graded associative algebras $\Omega Q_\Delta(S)$ and $\Gamma(S)(x,x)$ are naturally weakly equivalent.*

Proof By Theorem 7.1 we have an isomorphism

$$\Omega Q_\Delta(S) \cong \Lambda(S)(x,x) = Q_{\square_c}(\mathfrak{C}_{\square_c}(S)(x,x)).$$

By Lemma 7.2 and the fact that the triangulation functor and chains functor preserve the monoidal structures, the dgas $Q_{\square_c}(\mathfrak{C}_{\square_c}(S)(x,x))$ and $Q_\Delta|\mathfrak{C}_{\square_c}(S)(x,x)|$ are naturally weakly equivalent. Finally, note that we have isomorphisms

$$Q_\Delta|C_{\square_c}(S)(x,x)| = Q_\Delta((\mathfrak{T}\circ\mathfrak{C}_{\square_c})(S)(x,x)) \cong Q_\Delta(\mathfrak{C}(S)(x,x)) = \Gamma(S)(x,x). \quad \square$$

8 Properties of $\mathfrak{C}$: $\mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$

We recall several homotopy-theoretic properties of the rigidification functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$, in particular, its behavior with respect to Kan weak equivalences and its relationship with path spaces. These will be used in the final section of the article.

A map of simplicial sets $f\colon S \to S'$ is called a *Kan weak equivalence* if it is a weak equivalence in the Quillen model structure, namely, if f induces a weak homotopy equivalence of spaces $|f|\colon |S| \to |S'|$. A map of simplicial sets $f\colon S \to S'$ is called a *categorical equivalence* if f induces a weak equivalence $\mathfrak{C}(f)\colon \mathfrak{C}(S) \to \mathfrak{C}(S')$ of simplicial categories in the Bergner model structure. Recall that a functor of simplicial categories $F\colon \mathcal{C} \to \mathcal{C}'$ is called a *weak equivalence of simplicial categories* if

- F induces an essentially surjective functor at the level of homotopy categories, and
- for all $x, y \in \mathcal{C}$, $F\colon \mathfrak{C}(S)(x,y) \to \mathcal{C}'(F(x), F(y))$ is a Kan weak equivalence of simplicial sets.

The Quillen model structure on Set_Δ has Kan equivalences as weak equivalences and Kan complexes as fibrant objects. There is a different model structure on Set_Δ, the Joyal model structure, which has categorical equivalences as weak equivalences and quasicategories as fibrant objects. Moreover, the Quillen model structure is a left Bousfield localization of the Joyal model structure. In particular, a categorical equivalence is always a Kan weak equivalence. The converse is not true in general, but a Kan weak equivalence between Kan complexes is always a categorical equivalence. This is Proposition 17.2.8 in [17], which we record below.

Proposition 8.1 *If $f\colon S \to S'$ is a Kan weak equivalence between Kan complexes S and S' then $\mathfrak{C}(f)\colon \mathfrak{C}(S) \to \mathfrak{C}(S')$ is a weak equivalence of simplicial categories.*

A map $f\colon C \to C'$ of connected dg coalgebras is called a *quasi-isomorphism* if f induces an isomorphism of coalgebras after passing to homology. On the other hand, a map $f\colon C \to C'$ of connected dg coalgebras is called an *Ω–quasi-isomorphism* if f induces a quasi-isomorphism of dgas $\Omega f\colon \Omega C \to \Omega C'$. An Ω–quasi-isomorphism between connected dg coalgebras is always a quasi-isomorphism. The converse is not true in general, namely, a quasi-isomorphism between connected dg coalgebras might not be an Ω–quasi-isomorphism. However, if C and C' are connected dg coalgebras which are *simply connected* (ie $C_1 = 0 = C'_1$) then a quasi-isomorphism $f\colon C \to C'$ is an Ω–quasi-isomorphism. This follows by comparing Eilenberg–Moore spectral sequences. There are model structures of the category of connected dg coalgebras having each of these two notions as the weak equivalences, but we do not need these for the purposes of this paper.

Let Set^0_Δ be the full subcategory of the category Set_Δ of simplicial sets whose objects are 0–reduced simplicial sets. Let $\mathrm{dgCoalg}^0_k$ be the full subcategory of the category $\mathrm{dgCoalg}_k$ of dg coalgebras whose objects are connected dg coalgebras. The normalized chains functor restricts to a functor $Q_\Delta\colon \mathrm{Set}^0_\Delta \to \mathrm{dgCoalg}^0_k$.

Proposition 8.2 *The functor $Q_\Delta\colon \mathrm{Set}^0_\Delta \to \mathrm{dgCoalg}^0_k$ sends Kan weak equivalences to quasi-isomorphisms and categorical equivalences to Ω–quasi-isomorphisms.*

Proof The proof of the first part of the proposition is well known. For the second, suppose $f\colon S \to S'$ is a categorical equivalence and $S_0 = \{x\}$, $S'_0 = \{x'\}$. Then we have an induced Kan weak equivalence of simplicial sets $\mathfrak{C}(f)\colon \mathfrak{C}(S)(x,x) \to \mathfrak{C}(S)(x',x')$. This induces a dga quasi-isomorphism

$$Q_\Delta \mathfrak{C}(f)\colon Q_\Delta(\mathfrak{C}(S)(x,x)) \to Q_\Delta(\mathfrak{C}(S)(x',x')).$$

The result follows since the dgas $Q_\Delta(\mathfrak{C}(S)(x,x))$ and $Q_\Delta(\mathfrak{C}(S)(x',x'))$ are naturally weakly equivalent to the dgas $\Omega Q_\Delta(S)$ and $\Omega Q_\Delta(S')$, respectively, by Proposition 7.3. □

For any pointed topological space (X,b), denote by $\mathrm{Sing}(X,b)$ the subsimplicial set of $\mathrm{Sing}(X)$ whose n–simplices are the continuous maps $|\Delta^n| \to X$ that take all vertices of $|\Delta^n|$ to b. Define a new functor $Q^K_\Delta\colon \mathrm{Set}^0_\Delta \to \mathrm{dgCoalg}^0_k$ by $Q^K_\Delta(S) := Q_\Delta(\mathrm{Sing}(|S|,x))$, where $S_0 = \{x\}$ and $\mathrm{Sing}(|S|,x)$ is the Kan complex of singular simplices $|\Delta^n| \to |S|$ sending all vertices of $|\Delta^n|$ to $x \in |S|$. In general, the functor Q_Δ does not send Kan weak equivalences of simplicial sets to Ω–quasi-isomorphisms, but Q^K_Δ does.

Proposition 8.3 *The functor $Q^K_\Delta\colon \mathrm{Set}^0_\Delta \to \mathrm{dgCoalg}^0_k$ sends Kan weak equivalences of simplicial sets to Ω–quasi-isomorphisms of dg coalgebras.*

Proof Let $S, S' \in \mathrm{Set}^0_\Delta$ with $S_0 = \{x\}$ and $S'_0 = \{x'\}$. If $f\colon S \to S'$ is a Kan weak equivalence then $|f|\colon (|S|,x) \to (|S'|,x')$ is a homotopy equivalence of pointed spaces. The functor $(X,b) \mapsto \mathrm{Sing}(X,b)$ from the category of pointed spaces to Set^0_Δ sends homotopy equivalences of pointed spaces to Kan weak equivalences of 0–reduced Kan complexes. Thus, $\mathrm{Sing}(|f|)\colon \mathrm{Sing}(|S|,x) \to \mathrm{Sing}(|S'|,x')$ is a Kan weak equivalence. It follows from Propositions 8.1 and 8.2 that $Q_\Delta(\mathrm{Sing}(|f|))\colon Q_\Delta(\mathrm{Sing}(|S|,x)) \to Q_\Delta(\mathrm{Sing}(|S'|,x'))$ is an Ω–quasi-isomorphism. □

We now explain the relationship between mapping spaces of $\mathfrak{C}$ and different kinds of spaces of paths in a path-connected topological space. This relationship is deduced from the homotopy-theoretic properties of $\mathfrak{C}$ as studied in Section 2.2 of [15] and in [4] using different methods.

For any simplicial category $\mathcal{C}$ define the *simplicial nerve* $N_\Delta(\mathcal{C})$ to be the simplicial set whose set of n–simplices is given by

$$(N_\Delta(\mathcal{C}))_n = \mathrm{Hom}_{\mathrm{Cat}_\Delta}(\mathfrak{C}(\Delta^n), \mathcal{C}).$$

It follows that $N_\Delta\colon \mathrm{Cat}_\Delta \to \mathrm{Set}_\Delta$ is the right adjoint of $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$. If $\mathcal{C}$ is a topological category, then the *topological nerve* $N_{\mathrm{Top}}(\mathcal{C})$ is defined to be the simplicial nerve of the simplicial category $\mathrm{Sing}(\mathcal{C})$ obtained by applying Sing to each morphism space of $\mathcal{C}$. As is well known, for any topological monoid G, $|N_{\mathrm{Top}}(G)|$ is a model for the classifying space BG.

In Section 2.2 of [15], Lurie shows that the pair of adjoint functors $(\mathfrak{C}, N_\Delta)$ defines a Quillen equivalence between model categories Set_Δ with the Joyal model structure and Cat_Δ with the Bergner model structure. In particular, for any fibrant simplicial category $\mathcal{C}$ (a simplicial category whose mapping spaces are Kan complexes) the counit map $\mathfrak{C}(N_\Delta(\mathcal{C})) \to \mathcal{C}$ is a weak equivalence of simplicial categories. This also follows from Theorem 1.5 of [4].

Let X be a path-connected topological space and let $x, y \in X$. Define the space of Moore paths in X between x and y to be

$$P^{\boldsymbol{M}}_{x,y}X = \{(\gamma, r) \mid \gamma\colon [0,\infty) \to X,\ \gamma(0) = x,\ \gamma(s) = y \text{ for } r \leq s,\ r \in [0,\infty)\}$$

topologized as a subset of $\mathrm{Map}([0,\infty), X)\times[0,\infty)$, where $\mathrm{Map}([0,\infty), X)$ is equipped with the compact–open topology. Define a functor

$$\mathcal{P}\colon \mathrm{Top} \to \mathrm{Cat}_{\mathrm{Top}}$$

from the category of topological spaces to the category of topological categories as follows. For any $X \in \mathrm{Top}$ the objects of $\mathcal{P}(X)$ are the points of X. For any $x, y \in X$, define the space of morphisms $\mathcal{P}(X)(x, y) := P^{\boldsymbol{M}}_{x,y}X$ with composition rule induced by concatenation of paths. We call $\mathcal{P}\colon \mathrm{Top} \to \mathrm{Cat}_{\mathrm{Top}}$ the *path category functor*.

The functor $\mathfrak{C}\colon \mathrm{Set}_\Delta \to \mathrm{Cat}_\Delta$ is a simplicial model for the path category functor as shown in Proposition 8.4 below. Denote by $\mathrm{Sing}(\mathcal{P}X)$ the simplicial category obtained by applying Sing to the morphism spaces of the topological category $\mathcal{P}X$.

Proposition 8.4 *Let X be a path-connected topological space. The simplicial categories $\mathfrak{C}(\mathrm{Sing}(X))$ and $\mathrm{Sing}(\mathcal{P}X)$ are weakly equivalent.*

Proof Choose $b \in X$. The topological category $\mathcal{P}X$ is weakly equivalent to ΩX, the topological category with a single object b and as morphism space $\Omega X(b,b) = \Omega^{\boldsymbol{M}}_b X$ the space of based Moore loops at b with composition law given by concatenation of loops. A weak equivalence $\mathcal{P}X \to \Omega X$ of topological categories is given by fixing a collection of paths $\mathcal{O} = \{\gamma_x\}_{x\in X}$, where γ_x is a path from b to x. More

precisely, we have a functor $F_{\mathcal{O}}\colon \mathcal{P}X \to \Omega X$ given on objects by sending all objects of $\mathcal{P}X$ to the single object of ΩX and on morphisms $F_{\mathcal{O}}\colon \mathcal{P}X(x,y) \to \Omega X(b,b)$ is the continuous map $F_{\mathcal{O}}(\gamma) = \gamma_y^{-1} * \gamma * \gamma_x$, where $*$ denotes concatenation. The functor $F_{\mathcal{O}}$ is clearly a weak equivalence of topological categories. The topological nerve $N_{\mathrm{Top}} = N_\Delta \circ \mathrm{Sing}\colon \mathrm{Cat}_{\mathrm{Top}} \to \mathrm{Set}_\Delta$ sends weak homotopy equivalences of topological categories to Kan weak equivalence of simplicial sets. Thus, the simplicial sets $N_{\mathrm{Top}}(\mathcal{P}X)$ and $N_{\mathrm{Top}}(\Omega X)$ are Kan weakly equivalent. Moreover, the geometric realization $|N_{\mathrm{Top}}(\Omega X)|$ is a model for $B(\Omega X)$, the classifying space of the topological monoid of based loops. It follows from $B(\Omega X) \simeq X$ that the simplicial sets $N_{\mathrm{Top}}(\mathcal{P}X)$ and $\mathrm{Sing}(X)$ are Kan weakly equivalent. On the other hand, since the homotopy category of $N_{\mathrm{Top}}(\mathcal{P}X)$ is a groupoid it follows that $N_{\mathrm{Top}}(\mathcal{P}X)$ is a Kan complex [9]. By Proposition 8.1 we have that $\mathfrak{C}(N_{\mathrm{Top}}(\mathcal{P}X))$ and $\mathfrak{C}(\mathrm{Sing}(X))$ are weakly equivalent as simplicial categories. Since $\mathfrak{C} \circ N_\Delta(\mathcal{C}) \simeq \mathcal{C}$ for any $\mathcal{C} \in \mathrm{Set}_\Delta$ whose mapping spaces are Kan complexes, it follows that $\mathfrak{C}(N_{\mathrm{Top}}(\mathcal{P}X)) = \mathfrak{C}(N_\Delta(\mathrm{Sing}(\mathcal{P}X))) \simeq \mathrm{Sing}(\mathcal{P}X)$. Hence, the simplicial categories $\mathfrak{C}(\mathrm{Sing}(X))$ and $\mathrm{Sing}(\mathcal{P}X)$ are weakly equivalent. □

We have the following corollary.

Corollary 8.5 *Let X be a path-connected topological space and $b \in X$. The simplicial categories with one object $\mathfrak{C}(\mathrm{Sing}(X,b))$ and $\mathrm{Sing}(\Omega X)$ are weakly equivalent.*

Proof For path-connected X the inclusion $\mathrm{Sing}(X,b) \hookrightarrow \mathrm{Sing}(X)$ is a Kan weak equivalence of Kan complexes, so $\mathfrak{C}(\mathrm{Sing}(X))(b,b) \simeq \mathfrak{C}(\mathrm{Sing}(X,b))(b,b)$. Hence, by Proposition 8.4, $\mathfrak{C}(\mathrm{Sing}(X,b)) \simeq \mathrm{Sing}(\Omega X)$. □

We finish this section by describing more explicitly the weak equivalence of simplicial sets between $\mathfrak{C}(\mathrm{Sing}(X))(x,y)$ and $\mathrm{Sing}(\mathcal{P}X)(x,y)$ given by Proposition 8.4. We review this for completeness but it is not strictly necessary to follow Section 9. We follow Chapter 2 of [15].

Define a cosimplicial object $J^\bullet\colon \Delta \to (\partial\Delta^1 \downarrow \mathrm{Set}_\Delta)$ by letting J^n be the quotient of the standard simplex Δ^{n+1} by collapsing the last face (ie the face spanned by vertices $[0,\dots,n]$) to a vertex. The quotient simplicial set J^n has exactly two vertices, which we denote by the integers 0 and $n+1$. For any $S \in \mathrm{Set}_\Delta$ and $x, y \in S_0$, there is a simplicial set $\mathrm{Hom}^R_S(x,y)$ called the *right mapping space* defined by letting $\mathrm{Hom}^R_S(x,y)_n$ be the set of all morphisms of simplicial sets $\varphi\colon J^n \to S$ such that $\varphi(0) = x$ and $\varphi(n+1) = y$, together with structure face and degeneracy maps defined

to coincide with the corresponding structure maps of on S_{n+1}. Define a cosimplicial simplicial set $Q^\bullet$ by letting $Q^n := \mathfrak{C}(J^n)(0, n+1)$ and denote by $|-|_{Q^\bullet}\colon \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ the realization functor associated to $Q^\bullet$. Recall Proposition 2.2.4.1 of [15]:

Proposition 8.6 *Let S be an quasicategory containing a pair of objects x and y. There is a natural Kan weak equivalence of simplicial sets*

$$f\colon |\mathrm{Hom}^R_S(x,y)|_{Q^\bullet} \to \mathfrak{C}(S)(x,y).$$

In Proposition 2.2.2.7 of [15], Lurie shows there is a Kan weak equivalence of simplicial sets

$$g\colon |S|_{Q^\bullet} \cong \operatorname*{colim}_{\Delta^n \to S} \mathfrak{C}(J^n)(0, n+1) \to \operatorname*{colim}_{\Delta^n \to S} \Delta^n \cong S$$

for any simplicial set S. Hence, for a quasicategory S and $x, y \in S_0$ we have a zigzag of Kan weak equivalences

$$\mathrm{Hom}^R_S(x,y) \xleftarrow{g} |\mathrm{Hom}^R_S(x,y)|_{Q^\bullet} \xrightarrow{f} \mathfrak{C}(S)(x,y).$$

Now consider the above zigzag of Kan weak equivalences in the case $S = \mathrm{Sing}(X)$ for a topological space X. There is a Kan weak equivalence of simplicial sets

$$\theta\colon \mathrm{Hom}^R_{\mathrm{Sing}(X)}(x,y) \to \mathrm{Sing}(P^M_{x,y}X)$$

given as follows. A simplex $\varphi\colon J^n \to \mathrm{Sing}(X) \in \mathrm{Hom}^R_{\mathrm{Sing}(X)}(x,y)$ corresponds to a continuous map $\sigma_\varphi\colon |\Delta^{n+1}| \to X$ which collapses the last face of $|\Delta^{n+1}|$ to x and sends the last vertex of $|\Delta^{n+1}|$ to y. For each point p in the last face of $|\Delta^{n+1}|$ there is a straight line segment from p to the last vertex of $|\Delta^{n+1}|$. These straight line segments give a family of disjoint paths inside $|\Delta^{n+1}|$ which start in the last face and end in the last vertex and such a family is parametrized by $|\Delta^n|$. The continuous map σ_φ induces a continuous map $|\Delta^n| \to P^M_{x,y}X$ which corresponds to a simplex $\theta(\varphi)\colon \Delta^n \to \mathrm{Sing}(P^M_{x,y}X)$. The map θ is clearly a Kan weak equivalence of simplicial sets. It follows from the above zigzag formed by Kan weak equivalences f and g that $\mathfrak{C}(\mathrm{Sing}(X))(x,y) \simeq \mathrm{Sing}(P^M_{x,y}X)$.

9 Algebraic models for loop spaces

In this section we deduce an extension of a classical theorem of Adams from our previous results and discuss a few consequences. We start by showing that for a path-connected pointed space (X,b), $\Lambda(\mathrm{Sing}(X,b))(b,b)$ and $S_*(\Omega^M_b X;k)$ are weakly equivalent as dgas.

Proposition 9.1 *Let (X,b) be a pointed path-connected topological space. The differential graded associative algebras $\Lambda(\mathrm{Sing}(X,b))(b,b)$ and $S_*(\Omega^M_b X;k)$ are weakly equivalent.*

Proof By definition, we have $\Lambda(\mathrm{Sing}(X,b))(b,b) = Q_{\square_c}\big(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)\big)$. By Lemma 7.2 we have a quasi-isomorphism of chain complexes

$$Q_{\square_c}\big(\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)\big) \simeq Q_\Delta\big(|\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)|\big).$$

Moreover, this quasi-isomorphism is a weak equivalence of dgas since the monoidal structures are preserved under the triangulation functor. By Proposition 5.3, we have an isomorphism

$$Q_\Delta\big(|\mathfrak{C}_{\square_c}(\mathrm{Sing}(X,b))(b,b)|\big) \cong Q_\Delta\big(\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)\big).$$

Finally, by Corollary 8.5, we have

$$Q_\Delta\big(\mathfrak{C}(\mathrm{Sing}(X,b))(b,b)\big) \simeq S_*(\Omega^M_b X;k)$$

as dgas. □

In [1], Adams introduced the cobar construction and constructed a chain map of dgas $\varphi\colon \Omega Q_\Delta(\mathrm{Sing}(X,b)) \to C^\square_*(\Omega^M_b X;k)$, where $C^\square_*(\Omega^M_b X;k)$ denotes the normalized singular cubical chains on $\Omega^M_b X$. Moreover, Adams showed that if X is simply connected then φ is a quasi-isomorphism. The proof of this fact relied on associating a spectral sequence to $\Omega Q_\Delta(\mathrm{Sing}(X,b))$ and then comparing it to the Serre spectral sequence for the fibration $\Omega^M_b X \to PX \to X$. The simple connectivity assumption was used in order for the hypotheses of the Zeeman comparison theorem for spectral sequences to be satisfied.

We now deduce an extension of Adams's classical theorem (Corollary 9.2 below) to the case when X is a path-connected space with possibly nontrivial fundamental group. Note that we have not relied on spectral sequence arguments but rather on categorical and space-level arguments as discussed in the previous section.

Corollary 9.2 *For any pointed path-connected space (X,b), the differential graded algebras $\Omega(Q_\Delta(\mathrm{Sing}(X,b)))$ and $S_*(\Omega^M_b X;k)$ are weakly equivalent.*

Proof This follows directly from Theorem 7.1 and Proposition 9.1. □

We conclude with two remarks and an application to model the free loop space.

Remark 9.3 It follows from the above discussion that we may recover the homology of the based loop space of $|S|$ by taking the cobar construction on any connected dg coalgebra Ω–quasi-isomorphic to $Q^K_\Delta(S)$. In general, $Q_\Delta(S)$ and $Q^K_\Delta(S)$ are quasi-isomorphic but not necessarily Ω–quasi-isomorphic. However, if $S_0 = \{x\}$ and $S_1 = \{s_0(x)\}$, where $s_0(x)$ denotes the degenerate 1–simplex at x, then $Q_\Delta(S)$ and $Q^K_\Delta(S)$ are simply connected dg coalgebras and the natural map of dg coalgebras $\iota\colon Q_\Delta(S) \to Q^K_\Delta(S)$ is a quasi-isomorphism. Thus, by Proposition 2.2.7 in [14], ι is an Ω–quasi-isomorphism. Consequently, $\Omega Q_\Delta(S)$ is weakly equivalent as a dg algebra (ie quasi-isomorphic) to $S_*(\Omega^M_x|S|;k)$.

Remark 9.4 In the case of a simplicial complex, an explicit and smaller model for the based loop space can be given using a Kan fibrant replacement functor. Let K be a simplicial complex with an ordering of its vertices and let v be a vertex of K. Let fK be the simplicial set obtained by defining the face maps in accordance with the ordering of the vertices and adding degeneracies freely to K. The cobar construction on $Q_\Delta(fK)$ might not yield the homology of the based loop space of $|fK|$. However, we may consider the Kan fibrant replacement $\mathrm{Ex}^\infty(fK)$ of fK. $\mathrm{Ex}^\infty(fK)$ is a Kan complex weakly equivalent to fK, so it follows that the Kan complexes $\mathrm{Ex}^\infty(fK)$ and $\mathrm{Sing}(|fK|)$ are weakly equivalent. Thus, $\mathfrak{C}(\mathrm{Ex}^\infty(fK))$, $\mathfrak{C}(\mathrm{Sing}(|fK|))$ and $\mathrm{Sing}(\mathcal{P}|fK|)$ are weakly equivalent simplicial categories. Therefore, $\Lambda(\mathrm{Ex}^\infty(fK))(v,v)$ is a dga model for the based loop space of $|fK|$ at v. This remark explains an example of Kontsevich outlined in [11]. In [8], a similar construction was also described for any simplicial set, which was then compared to Kan's loop group construction.

Finally, a chain complex model for the free loop space of a path-connected topological space may be obtained as follows. For any dga A denote by $\mathrm{CH}_*(A)$ the Hochschild chain complex of A. For the definition we refer the reader to any standard reference, such as [13].

Corollary 9.5 *For any pointed path-connected space (X,b), the Hochschild chain complex $\mathrm{CH}_*(\Omega(Q_\Delta(\mathrm{Sing}(X,b))))$ is quasi-isomorphic to $S_*(LX;k)$, the singular chains on the free loop space of X.*

Proof This is a direct consequence of the fact that the Hochschild chain complex of the dga $S_*(\Omega^M X;k)$ is quasi-isomorphic to $S_*(LX;k)$ (a theorem usually attributed

to Goodwillie [6]), Corollary 9.2 and the invariance of Hochschild chains under weak equivalences of dgas. □

As explained in Remark 2.23 of [7], for any connected dg coalgebra C there is a quasi-isomorphism of chain complexes

$$\mathrm{coCH}_*(C) \simeq \mathrm{CH}_*(\Omega C),$$

where $\mathrm{coCH}_*(C)$ denotes the co-Hochschild chain complex of C; we refer to [7] for definitions and further details. As a consequence, we obtain a model for the free loop space LX of a path-connected space X that does not require passing to the based loop space, which we expect to be convenient in studying string topology.

Corollary 9.6 *For any pointed path-connected space (X, b), the co-Hochschild complex $\mathrm{coCH}_*(Q_\Delta(\mathrm{Sing}(X, b)))$ is quasi-isomorphic to $S_*(LX; k)$.*

Proof This follows directly from Lemma 7.2 and the fact that $\mathrm{coCH}_*(C) \simeq \mathrm{CH}_*(\Omega C)$ for any connected dg coalgebra C. □

References

[1] **J F Adams**, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 409–412 MR

[2] **R Antolini**, *Geometric realisations of cubical sets with connections, and classifying spaces of categories*, Appl. Categ. Structures 10 (2002) 481–494 MR

[3] **R Brown**, **P J Higgins**, *On the algebra of cubes*, J. Pure Appl. Algebra 21 (1981) 233–260 MR

[4] **D Dugger**, **D I Spivak**, *Mapping spaces in quasi-categories*, Algebr. Geom. Topol. 11 (2011) 263–325 MR

[5] **D Dugger**, **D I Spivak**, *Rigidification of quasi-categories*, Algebr. Geom. Topol. 11 (2011) 225–261 MR

[6] **T G Goodwillie**, *Cyclic homology, derivations, and the free loopspace*, Topology 24 (1985) 187–215 MR

[7] **K Hess**, *The Hochschild complex of a twisting cochain*, J. Algebra 451 (2016) 302–356 MR

[8] **K Hess**, **A Tonks**, *The loop group and the cobar construction*, Proc. Amer. Math. Soc. 138 (2010) 1861–1876 MR

[9] **A Joyal**, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra 175 (2002) 207–222 MR

[10] **C Kapulkin**, **V Voevodsky**, *Cubical approach to straightening*, preprint (2018) Available at `https://tinyurl.com/Kapulkin-Voevodsky-2018`

[11] **M Kontsevich**, *Symplectic geometry of homological algebra*, preprint (2009) Available at `https://www.ihes.fr/~maxim/TEXTS/Symplectic_AT2009.pdf`

[12] **B Le Grignou**, *Composition + homotopy = cubes*, preprint (2018) arXiv

[13] **J-L Loday**, *Cyclic homology*, 2nd edition, Grundl. Math. Wissen. 301, Springer (1998) MR

[14] **J-L Loday**, **B Vallette**, *Algebraic operads*, Grundl. Math. Wissen. 346, Springer (2012) MR

[15] **J Lurie**, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton Univ. Press (2009) MR

[16] **J Lurie**, *Higher algebra*, book project (2017) Available at `https://tinyurl.com/Lurie-Higher-alg`

[17] **E Riehl**, *Categorical homotopy theory*, New Mathematical Monographs 24, Cambridge Univ. Press (2014) MR

Department of Mathematics, University of Miami
Coral Gables, FL, United States

Department of Mathematics, City University of New York, Lehman College
Bronx, NY, United States

`manuelr@math.miami.edu`, `mahmoud.zeinalian@lehman.cuny.edu`

Received: 25 February 2017 Revised: 26 July 2018

msp

Algebraic & Geometric Topology 18 (2018) 3821–3838

Noncrossing partitions and Milnor fibers

THOMAS BRADY
MICHAEL J FALK
COLUM WATT

For a finite real reflection group W we use noncrossing partitions of type W to construct finite cell complexes with the homotopy type of the Milnor fiber of the associated W–discriminant Δ_W and that of the Milnor fiber of the defining polynomial of the associated reflection arrangement. These complexes support natural cyclic group actions realizing the geometric monodromy. Using the shellability of the noncrossing partition lattice, this cell complex yields a chain complex of homology groups computing the integral homology of the Milnor fiber of Δ_W.

20F55; 52C35, 05E99

1 Introduction

Suppose $g \in \mathbb{C}[z_1, \dots, z_n]$ is a quasihomogeneous polynomial, defining the hypersurface $V = g^{-1}(0)$ in $\mathbb{C}^n$. Then g restricts to a locally trivial fibration $g\colon \mathbb{C}^n - V \to \mathbb{C}^*$, the global Milnor fibration, with fiber $g^{-1}(1)$, the Milnor fiber of g; see Milnor [25]. The topology of $g^{-1}(1)$ and the monodromy of the bundle are invariants of the singularity type of g at the origin. Of special interest is the case where $g = Q_W$ is a product of complex linear forms defining the arrangement $\mathcal{A} = \mathcal{A}_W$ of reflecting hyperplanes in $\mathbb{C}^n$ of a finite real or complex reflection group W; see Dimca and Lehrer [19], Settepanella [30; 31] and Măcinic and Papadima [27].

In this setting W acts on $\mathbb{C}^n$, preserving $V = \bigcup_{H \in \mathcal{A}} H$, the quotient $W\backslash\mathbb{C}^n$ is homeomorphic to $\mathbb{C}^n$, and under this homeomorphism $W\backslash V$ is carried to a hypersurface Δ_W in $\mathbb{C}^n$. This hypersurface is the zero locus of a quasihomogeneous polynomial P_W, well-defined up to polynomial automorphism of $\mathbb{C}^n$, called the discriminant associated with W (see Section 3).

The fundamental group of $\mathbb{C}^n - \Delta_W$ is the generalized braid group (or Artin group) $B(W)$ associated with W. If W has type A_{d-1} then P_W is the classical discriminant for polynomials of degree d and $B(W)$ is isomorphic to the classical braid group

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3821

on d strands. In this paper, we construct a noncrossing partition (NCP) model for the Milnor fiber $F_P = P_W^{-1}(1)$ and study its structure, including the monodromy action, in the case where W is a real reflection group. We also construct an NCP model for the Milnor fiber $F_Q = Q_W^{-1}(1)$ of the reflection arrangement $\mathcal{A}_W$. Both models arise as subcomplexes of appropriate covering spaces of a finite $K(B(W), 1)$ which is defined in terms of noncrossing partitions; see Bessis [4], Brady [7] and Brady and Watt [8].

The NCP model for F_P has a natural filtration by subcomplexes, which are seen to be homotopy equivalent to bouquets of spheres using the lexicographic shellability of the noncrossing partition lattice. This yields a chain complex computing $H_*(F_P, \mathbb{Z})$ whose terms are homology groups of truncations of this lattice.

The standard approach to calculating the (co)homology of F_Q (resp. F_P) is via the co(homology) of $\mathbb{C}^n - V$ (resp. $\mathbb{C}^n - \Delta_W$) with twisted coefficients in $R[t, t^{-1}]$ for some ring R. In the case of F_P, a spectral sequence is often used to compute the cohomology with coefficients in $R[t, t^{-1}]$ of the Salvetti complex, a small cellular model for $\mathbb{C}^n - \Delta_W$ (see De Concini and Salvetti [15], De Concini, Salvetti and Stumbo [16], Callegaro [12], Callegaro and Salvetti [13], Frenkel [20] and Salvetti [29]). Our approach is to construct a small model $\widehat{F}_P$ for F_P and use shellability of the NCP lattice to simplify the direct calculation of homology with integer coefficients. The disadvantage with this approach compared to using the Salvetti complex is the large ranks encountered in our chain complex. For example, the rank of the top-dimensional group in this complex is equal to the number of NCPs which are not contained in proper standard parabolic subgroups (see Athanasiadis, Brady and Watt [3]). The corresponding ranks for the model $\widehat{F}_Q$ of F_Q increase by the factor $|W|/2$.

2 NCP models for subgroups of $B(W)$

2.1 Background

Let W be a finite, irreducible, real reflection group of rank n and let T be the set of all reflections in W. For background on finite reflection groups, see [6; 22]. Equip W with the total reflection length function $w \mapsto |w|$ (with respect to the generating set T) and with the partial order $\leq$ given by $u \leq w$ whenever $|u| + |u^{-1}w| = |w|$ (see [2]). We will use the notation $u \lessdot w$ for the case where w covers u. Fix a specific Coxeter element γ in W and define the noncrossing partitions to be the elements in the interval $[e, \gamma]$ in the poset $(W, \leq)$. The poset of W–noncrossing partitions is a lattice, L (see [9]),

whose order complex is denoted $|L|$. If W is of type A_n, then W is isomorphic to the group of permutations of $\{1,\dots,n+1\}$ and the noncrossing partitions are those elements whose cycle structure gives a classical noncrossing partition; see [7].

We define $B(W)$ to be the group with generating set

$$\{[w] : w \in L, w \neq e\}$$

subject to the relations

$$[w_1][w_1^{-1}w_2] = [w_2] \quad \text{whenever } w_1 \leq w_2.$$

It is shown in [4; 7; 8] that $B(W)$ is isomorphic to the generalized braid group of type W.

We recall from [4; 7; 8] the contractible, n–dimensional, simplicial complex X whose k–simplices are ordered $(k+1)$–tuples from $B(W)$ of the form $(g_0, g_1, \dots, g_k)$ with $g_i = g_0[w_i]$ for some chain $e < w_1 < w_2 < \cdots < w_k$ in L. It is convenient to use the notation $(g_0, e < w_1 < \cdots < w_k)$ for such a simplex. Thus the simplices of X are identified with pairs (g, σ), where $g \in B(W)$ and σ is an *initialized chain* in L, that is, σ is a chain of the form $e < w_1 < \cdots < w_k$. As $B(W)$ acts freely on X, the quotient $K := B(W)\backslash X$ is a $K(B(W), 1)$ and X is its universal cover. The action of $B(W)$ on X is given by

$$g \cdot (g_0, g_1, \dots, g_k) = (gg_0, gg_1, \dots, gg_k),$$

or, in terms of the pair notation,

$$g \cdot (g_0, e < w_1 < \cdots < w_k) = (gg_0, e < w_1 < \cdots < w_k). \tag{2-1}$$

It is immediate that the simplex $(g_0, e < w_1 < \cdots < w_k)$ has k faces of the form $(g_0, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k)$, for $1 \leq i \leq k$, each obtained by deleting one of $w_1, w_2, \dots, w_k$ from σ. The remaining face is obtained by deleting e from σ and hence is given by the ordered set

$$(g_0[w_1], g_0[w_2], \dots, g_0[w_k]) = (g_0[w_1]) \cdot (e, [w_1]^{-1}[w_2], \dots, [w_1]^{-1}[w_k]).$$

In pair notation, this is denoted $(g_0[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$.

2.2 Quotients of X

If H is a normal subgroup of $B(W)$ then we can form a CW–complex X_H whose cells are of the form (Hg, σ), where σ is an initialized chain in L and the first component

is a right H coset. If $\sigma = e < w_1 < \cdots < w_k$, then this cell has k boundary faces of the form

$$(Hg, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k) \quad \text{for } 1 \le i \le k,$$

with the remaining face given by $(Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$. It will be convenient to refer to $(Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$ as the *top* face of (Hg, σ) and to $(Hg, e < w_1 < \cdots < w_{k-1})$ as the *bottom* face of (Hg, σ). Since X is contractible, X_H is a $K(H,1)$. The action of the quotient group $H\backslash B(W)$ on X_H is given by $(Hg_1)(Hg_2, \sigma) = (Hg_1g_2, \sigma)$.

We now highlight a particular feature of these complexes X_H.

Lemma 2.1 *In X_H, each k–cell of the form*

$$c_k = (Hg, e < w_1 < w_2 < \cdots < w_k) \quad \textit{with } |w_k| < n$$

is incident on precisely two $(k+1)$–cells of the form

$$c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).$$

Proof Suppose that the cell

$$c_k = (Hg, e < w_1 < w_2 < \cdots < w_k),$$

with $|w_k| < n$, is incident on a $(k+1)$–cell of the form

$$c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).$$

Since the chain of c_k does not contain γ, the cell c_k must be obtained by deleting either e or γ from the chain of c_{k+1}. In the latter case, c_k is the bottom face of c_{k+1}, forcing $Hg' = Hg$ and $u_i = w_i$ for $i = 1, \ldots, k$. In the former case, c_k is the top face

$$(Hg'[u_1], e < u_1^{-1}u_2 < \cdots < u_1^{-1}u_k < u_1^{-1}\gamma),$$

so that $Hg' = Hg[w_1]^{-1}$ and $u_1 = \gamma w_k^{-1}$, $u_2 = \gamma w_k^{-1}w_1$, $\ldots$, $u_k = \gamma w_k^{-1}w_{k-1}$. □

In our examples, H will arise as the kernel of a specific homomorphism ϕ with domain $B(W)$. In this case we will denote $X_{\ker(\phi)}$ by X_ϕ. We can then identify the coset Hg with the element $\phi(g)$ and denote the cells of X_H as pairs $(\phi(g), \sigma)$, where σ is an initialized chain in L. If $\sigma = e < w_1 < \cdots < w_k$, then the cell $(\phi(g), \sigma)$ has top face given by $\big(\phi(g)\phi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\big)$.

Example 2.2 If ϕ is the trivial homomorphism, then $H = B(W)$ and $X_\phi = K$ is the $K(B(W), 1)$ introduced earlier. The cells of K are of the form (e, σ), where σ is an initialized chain in L. If $\sigma = e < w_1 < \cdots < w_k$, then this cell has top face given by

$$\big(e\phi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\big) = (e, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k).$$

Thus $X = X_\phi$ can be identified with the quotient of $|L|$ under the equivalence relation generated by identifying $w_1 < w_2 < \cdots < w_k$ with $e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k$.

Example 2.3 The standard projection $s\colon B(W) \to W$, $[w] \mapsto w$, which takes each NCP generator of $B(W)$ to the corresponding NCP in W, is a homomorphism by our presentation of $B(W)$. The kernel of s is the pure braid group $PB(W)$ associated to W, and X_s is a $K(\pi, 1)$ for $\pi = PB(W)$. (By [11; 18], X_s is homotopy equivalent to the complement $M = \mathbb{C}^n - \bigcup_{H\in\mathcal{A}} H$, where $\mathcal{A}$ is the complexification of the associated real reflection arrangement.) The cells of X_s can be identified with pairs (w, σ), for $w \in W$ and σ an initialized chain in L. The top face of the cell $(w, e < w_1 < \cdots < w_k)$ is $(ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$.

We will consider two further examples of this construction in Sections 4 and 5. These will give the models for the fibers mentioned in the introduction. The next section establishes the homotopy types of these fibers.

3 Discriminants and Milnor fibers

In this section we recall some definitions and basic facts about discriminants and Milnor fibers.

3.1 The W–discriminant

Recall that W is a real reflection group whose action on $\mathbb{R}^n$ has been complexified to an action on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$. Let $\{f_1, \ldots, f_n\}$ be a set of basic invariants for the ring of W–invariant, complex polynomials [14]. The function $f = (f_1, \ldots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ induces a homeomorphism of $W\backslash\mathbb{C}^n$ with $\mathbb{C}^n$. Recall that T denotes the set of reflections in W, and for each $t \in T$, let $H_t \subset \mathbb{C}^n$ denote its fixed complex hyperplane and $\lambda_t\colon \mathbb{C}^n \to \mathbb{C}$ be a complex linear form with kernel H_t. The polynomial $Q = \prod_{t\in T} \lambda_t\colon \mathbb{C}^n \to \mathbb{C}$ has the property that $Q(wx) = \det(w)Q(x)$ for all $w \in W$ and $x \in \mathbb{C}^n$ and hence Q^2 is invariant under the action of W. It follows that $Q^2 = P(f_1, \ldots, f_n)$ for some quasihomogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ whose

weights are equal to the degrees of the f_i. The polynomial P is called the discriminant of W. It is unique up to polynomial automorphism of $\mathbb{C}^n$. The action of W on $\mathbb{C}^n$ leaves the affine algebraic hypersurface $V = Q^{-1}(0) = \bigcup_{t\in T} H_t$ invariant and its quotient $\Delta_W = W\backslash V$ is identified with the affine algebraic hypersurface $\Delta := P^{-1}(0)$. The space $M = \mathbb{C}^n - V$ is a $K(PB(W), 1)$, and W acts freely on M, so the space $f(M) = W\backslash M \cong \mathbb{C}^n - \Delta$ is a $K(B(W), 1)$ [10; 18]. In what follows, we identify $PB(W)$ with $\pi_1(M, z_0)$ and $B(W)$ with $\pi_1(f(M), f(z_0))$.

3.2 Milnor fibers of P and Q

The restriction to $\mathbb{C}^n - g^{-1}(0)$ of any quasihomogeneous polynomial $g\colon \mathbb{C}^n \to \mathbb{C}$ is a locally trivial fibration whose fiber $g^{-1}(1)$ is called the Milnor fiber of g [25]. The Milnor fibers of P and Q will be denoted by F_P and F_Q. These spaces are determined up to polynomial diffeomorphism by W. Then $(Q^2)^{-1}(1) = Q^{-1}(1) \cup Q^{-1}(-1)$ is invariant under the action of W and $F_P = W\backslash((Q^2)^{-1}(1)) \cong W^+\backslash F_Q$, where $W^+ = \{w \in W : \det(w) = 1\}$. The space F_Q is a connected, regular, W^+–cover of F_P since the action of W^+ on F_Q is free.

We will show that each of F_P and F_Q is homotopy equivalent to a complex of the form X_ϕ. Our proofs will make use of the following general result.

Proposition 3.1 *Suppose $g\colon E \to \mathbb{C}^*$ is a fibration with fiber F, where each of E and F has the homotopy type of a connected CW–complex. Then F is homotopy equivalent to the cover of E corresponding to the kernel of $g_*\colon \pi_1(E) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$.*

Proof Let i denote the inclusion of F into E. The exact sequence of the fibration implies that $i_*\colon \pi_1(F) \to \pi_1(E)$ is an injection whose image is $\ker(g_*)$. Let $p\colon F' \to E$ denote the connected cover of E corresponding to $\ker(g_*)$. Then the inclusion i lifts to a map $h\colon F \to F'$ with $p \circ h = i$. We show that h is a homotopy equivalence. Since p_* and i_* are injections, it follows that $h_*\colon \pi_1(F) \to \pi_1(F')$ is an injection and hence is an isomorphism. Since $p_*\colon \pi_k(F') \to \pi_k(E)$ and $i_*\colon \pi_k(F) \to \pi_k(E)$ are isomorphisms for all $k \geq 2$ (the latter by the exact sequence of the fibration), $h_*\colon \pi_k(F) \to \pi_k(F')$ is also. As each of F and F' has the homotopy type of a CW–complex, h is a homotopy equivalence. □

Since algebraic sets are homotopy equivalent to CW–complexes, Proposition 3.1 yields the following corollary.

Corollary 3.2 (i) *The Milnor fiber F_P of P is homotopy equivalent to the cover of $f(M) = W\backslash M$ corresponding to the kernel of the map $P_*\colon B(W) \to \mathbb{Z}$.*

(ii) *The Milnor fiber F_Q of Q is homotopy equivalent to the cover of $f(M) = W\backslash M$ corresponding to the group $f_*(\ker(Q_*))$.*

Proof (ii) By Proposition 3.1, F_Q is homotopy equivalent to the connected domain Y of a covering map $\rho\colon Y \to M$ for which $\rho_*(\pi_1(Y, y_0)) = \ker(Q_*)$. Since the map $f\colon M \to W\backslash M$ is a finite cover, the map $f \circ \rho$ is a covering map with $(f \circ \rho)_*(\pi_1(Y, y_0)) = f_*(\ker(Q_*))$, as required. □

3.3 The characteristic homomorphism

It remains to identify the homomorphisms

$$Q_*\colon PB(W) = \pi_1(M) \to \mathbb{Z} \quad \text{and} \quad P_*\colon B(W) = \pi_1(W\backslash M) \to \mathbb{Z}.$$

First we describe convenient generating sets for their respective domains. Fix $\epsilon > 0$. For each reflection $t \in T$, choose a point $z_t \in H_t - \bigcup_{t' \in T, t' \neq t} H_{t'}$ and let D_t be the closed disc of radius ϵ centered at z_t in the complex line L_t which passes through z_t and is orthogonal to H_t. The complex structure induces a natural orientation on each L_t. By shrinking ϵ, if necessary, we may assume that $D_t \cap \bigcup_{t' \in T, t' \neq t} (H_{t'} \cup D_{t'}) = \varnothing$ for all $t \in T$. Now choose a basepoint z_0 in M and for each $t \in T$ choose a path σ_t in M which starts at z_0 and ends on the boundary of D_t. Let γ_t be the loop which travels along σ_t, then around the boundary of D_t in a positive orientation and finally back to z_0 along σ_t. The set of homotopy classes $\overline{\gamma}_t$ for $t \in T$ generates $\pi_1(M, z_0)$ [10]; see [1] for a recent generalization.

Since W acts freely on M, the restriction of f to M is a regular covering map onto $f(M) = W\backslash M$. We specify a set of elements of $\pi_1(W\backslash M, f(z_0))$ which corresponds to the set of reflections in $W \cong \pi_1(W\backslash M, f(z_0))/f_*\pi_1(M, z_0)$. Under this isomorphism, if δ is any path in M which starts at z_0 and ends at $w(z_0)$ (or which starts at $w(z_0)$ and ends at $w(w(z_0))$), then the homotopy class of $f \circ \delta$ corresponds to $w \in W$. If $t \in T$ is a reflection, let δ_t be the path in M which travels first along σ_t, then half way around the boundary of D_t in the positive sense and finally along the reverse of $t\sigma_t$ to $t(z_0)$. Similarly, let δ_t' be the path from $t(z_0)$ to z_0 in M which travels first along $t\sigma_t$, then around the other half of the boundary of D_t in the positive sense and finally along the reverse of σ_t to $t(t(z_0)) = z_0$. Note that $f \circ \delta_t = f \circ \delta_t'$,

let Γ_t be the corresponding homotopy class and note that the composition Γ_t followed by Γ_t is represented by the path $f \circ \gamma_t$. From the short exact sequence

$$\pi_1(M, z_0) \to \pi_1(W\backslash M, f(z_0)) \to W$$

the group $\pi_1(W\backslash M, f(z_0))$ is generated by $\{\overline{f \circ \gamma_t}, \Gamma_t : t \in T\}$ and, in fact, by the smaller set $\{\Gamma_t : t \in T\}$, because $\overline{f \circ \gamma_t} = \Gamma_t^2$. For $t \in T$, the generator $\Gamma_t \in \pi_1(W\backslash M, f(z_0))$ corresponds to the generator $[t]$ in the presentation for $B(W)$ from Section 2.1.

Proposition 3.3 (i) *The map* $Q_*\colon \pi_1(M, z_0) \to \pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ *is given by*

$$Q_*(\overline{\gamma}_t) = 1 \quad \text{for each } t \in T.$$

(ii) *The map* $P_*\colon \pi_1(W\backslash M, f(z_0)) \to \pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ *is given by*

$$P_*(\Gamma_t) = 1 \quad \text{for each } t \in T.$$

Proof (i) By scaling each λ_t if necessary, we may assume that $\lambda_t(z_0) = 1$. Then $Q_* = \left(\prod_{t\in T} \lambda_t\right)_* = \sum_{t\in T}(\lambda_{t*})$ (since $\mathbb{C}^*$ is a topological group). The result now follows since our choices ensure that the winding number of $(\lambda_t)_*(\overline{\gamma}_{t'})$ about the origin is one if $t = t'$ and zero otherwise.

(ii) First note that

$$P_*(\Gamma_t^2) = P_*(\overline{f \circ \gamma_t}) = (Q^2)_*(\overline{\gamma}_t) = Q_*(\overline{\gamma}_t) + Q_*(\overline{\gamma}_t) = 2.$$

Since P_* is a group homomorphism, it follows that $P_*(\Gamma_t) = 1$. □

Remark 3.4 Our calculation of P_* from Q_* can be modified to apply when W is a Shephard group, the symmetry group of a complex polytope. The construction of a model for the Milnor fiber of the W–discriminant (in the next section) will also be valid in this case.

4 NCP model for the Milnor fiber of the discriminant

Consider the homomorphism $P_*\colon B(W) \to \mathbb{Z}$ and the cover of K given by $X_{P_*} := \ker(P_*)\backslash X$.

Proposition 4.1 *X_{P_*} is homotopy equivalent to the Milnor fiber of the discriminant P.*

Proof This follows from Corollary 3.2(i) and Proposition 3.3, since corresponding covers of K and $W\backslash M$ are homotopy equivalent. □

Remark 4.2 By Section 2.2, X_{P_*} can be identified with the CW–complex whose cells are pairs (m,σ) for $m\in\mathbb{Z}$ and σ an initialized chain in L. Since $P_*([t])=1$ for each reflection generator $[t]$ of $B(W)$, it follows that $P_*([w])=|w|$, the reflection length of w, for each NCP $w\in W$. Hence, the top face of the cell $(m, e<w_1<\cdots<w_k)$ is the cell

$$(m+|w_1|, e<w_1^{-1}w_2<\cdots<w_1^{-1}w_k).$$

Definition 4.3 We define the *small* NCP *model* $\widehat{F}_{\boldsymbol{P}}$ of the Milnor fiber of P_W to be the finite subcomplex of X_{P_*} consisting of the cells of the form

$$(m, e<w_1<w_2<\cdots<w_k) \quad \text{with } 0\le m<n-|w_k|.$$

Remark 4.4 We observe that $\widehat{F}_{\boldsymbol{P}}$ is the union of cells of the form

$$(0, e\lessdot w_1\lessdot w_2\lessdot\cdots\lessdot w_{n-1})$$

together with their faces. In particular, $\widehat{F}_{\boldsymbol{P}}$ is $(n-1)$–dimensional.

Theorem 4.5 *The subcomplex $\widehat{F}_{\boldsymbol{P}}$ is a strong deformation retract of X_{P_*}.*

Proof We construct an acyclic matching (see Chapter 11 of [24]) which pairs cells of $X_{P_*}-\widehat{F}_{\boldsymbol{P}}$. Suppose $c_{k+1}=(m,\sigma)$ and that the chain σ ends in γ. Then this matching pairs c_{k+1} with its top (resp. bottom) face if $m\ge 0$ (resp. $m<0$). In particular, the matching pairs cells whose chains end in γ with cells whose chains do not end in γ.

To show that this matching is acyclic, consider an alternating path

$$(l_1,\sigma_1)\succ_m(l_2,\sigma_2)\prec(l_3,\sigma_3)\succ_m(l_4,\sigma_4)\prec\cdots$$

in this matching, where $(l_i,\sigma_i)\prec(l_{i+1},\sigma_{i+1})$ means (l_i,σ_i) is a facet of (l_{i+1},σ_{i+1}) and $(l_j,\sigma_j)\succ_m(l_{j+1},\sigma_{j+1})$ means (l_j,σ_j) is matched with its facet (l_{j+1},σ_{j+1}). By definition of the matching, if i is odd, the chain σ_i ends in γ and σ_{i+1} does not end in γ. By Lemma 2.1, each such (l_{i+1},σ_{i+1}) is incident on precisely two $(k+1)$–cells with chains ending in γ. These must be (l_i,σ_i) and (l_{i+2},σ_{i+2}). If $l_i<0$ it follows that (l_{i+1},σ_{i+1}) is the bottom face of (l_i,σ_i) and hence $l_{i+2}>l_i$ for all i. Similarly, if $l_i\ge 0$ it follows that (l_{i+1},σ_{i+1}) is the top face of (l_{i+2},σ_{i+2}) and hence $l_{i+2}<l_i$ for all i. In particular, the path cannot form a cycle and the matching is acyclic.

It remains to show that the set of critical cells is precisely the set of cells of $\widehat{F}_{\boldsymbol{P}}$. Let $c_k=(m,e<w_1<\cdots<w_k)$ be a cell of X_{P_*}. When $w_k=\gamma$, the cell c_k is matched

with its top or bottom face according to whether $m \geq 0$ or $m < 0$ and, hence, is not critical. When $w_k \neq \gamma$, two cases arise. If $m < 0$ then c_k is matched as the bottom face of $(m, e < w_1 < \cdots < w_k < \gamma)$ and is not critical. On the other hand, if $m \geq 0$, then c_k is matched as the top face of

$$(m-n+|w_k|, e < \gamma w_k^{-1} < \gamma w_k^{-1} w_1 < \cdots < \gamma w_k^{-1} w_{k-1} < \gamma)$$

if and only if $m-n+|w_k| \geq 0$. □

Corollary 4.6 *The finite complex $\widehat{F}_P$ is homotopy equivalent to the Milnor fiber F_P and is a $K(\pi, 1)$ for $\pi = \ker(P_*)$.*

Example 4.7 Let W be a dihedral group acting on $\mathbb{R}^2$ with t reflections, denoted by $R_1, R_2, \ldots, R_t$. Thus the NCPs are $\{e, R_1, R_2, \ldots, R_t, \gamma\}$, where γ can be taken to be a rotation through twice the angle between adjacent lines of symmetry. Here $n = 2$ and $\widehat{F}_P$ is 1–dimensional with two vertices, namely $(0, e)$ and $(1, e)$. The complex $\widehat{F}_P$ has a 1–cell $(0, e < R_i)$ for each chain $e < R_i$ and the endpoints of this 1–cell are $(0, e)$ and $(1, e)$. Thus $\widehat{F}_P$ has the homotopy type of the suspension of a 0–dimensional subcomplex on t points and $\ker(P_*)$ is free of rank $t-1$. This agrees with [26, Theorem 1]; P has weights 2 and t.

Example 4.8 Let W be the group $A_3 \cong \Sigma_4$, the symmetric group; the polynomial P is the classical discriminant for univariate quartics. Choose γ to be the four-cycle (1234). Here $n = 3$ and $\widehat{F}_P$ is 2–dimensional. (See Figure 1.) $\widehat{F}_P$ has three vertices, namely $(0, e)$, $(1, e)$ and $(2, e)$. Each transposition R contributes a 1–cell $(0, e < R)$ with endpoints $(0, e)$ and $(1, e)$ together with a 1–cell $(1, e < R)$ with endpoints $(1, e)$ and $(2, e)$. Each length-two NCP $\alpha \in \{(123), (124), (134), (234), (12)(34), (14)(23)\}$ contributes a single 1–cell $(0, e < \alpha)$ with endpoints $(0, e)$ and $(2, e)$. Finally, for each of the 16 chains of the form $e < R < \alpha$ corresponding to factorizations of γ by transpositions, we have a 2–cell $(0, e < R < \alpha)$ whose boundary is glued along the three 1–cells $(0, e < R)$, $(0, e < \alpha)$ and $(1, e < R^{-1}\alpha)$.

Remark 4.9 The cells of $\widehat{F}_P$ are simplices, but $\widehat{F}_P$ is not a simplicial complex; rather it is a Δ–complex in the sense of [21]. It can be realized as the nerve of the germ (in the sense of category theory; see [17]) with objects $\{0, 1, \ldots, n-1\}$ and morphisms $i \xrightarrow{\alpha} j$, where α is an NCP of length $j - i$. The composition, when defined, is determined by multiplication of NCPs.

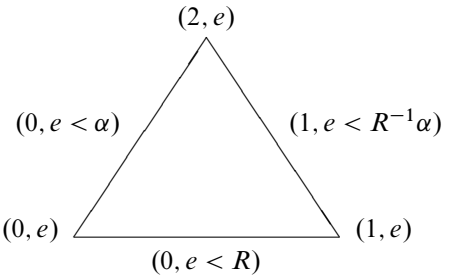


Figure 1: Typical 2–cell $(0, e < R < \alpha < \gamma)$ of $\widehat{F}_P$ for $W = A_3$

Remark 4.10 The monodromy action on $\widehat{F}_P$ is obtained by composing the monodromy action on X_{P_*} with the retraction defined by the acyclic matching. The monodromy action is determined by the action of either generator of $\mathbb{Z}$. Since one of the two cells identified in Lemma 2.1 has a simpler formula than the other we will describe the action of -1. Explicitly a cell of the form

$$(m, e < w_1 < w_2 < \cdots < w_k)$$

for $0 < m < n - |w_k|$ is taken to

$$(m-1, e < w_1 < w_2 < \cdots < w_k),$$

while

$$(0, e < w_1 < w_2 < \cdots < w_k)$$

for $|w_k| < n$ is taken to

$$(|w_1| - 1, e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k < w_1^{-1}\gamma),$$

since, in the second case, the action of -1 on the k–cell $(0, e < w_1 < w_2 < \cdots < w_k)$ as a cell in X_{P_*} takes it to the k–cell $(-1, e < w_1 < w_2 < \cdots < w_k)$, which is the bottom face of the $(k+1)$–cell $(-1, e < w_1 < w_2 < \cdots < w_k < \gamma)$, which in turn has top face $(|w_1| - 1, e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k < w_1^{-1}\gamma)$.

5 NCP model for the Milnor fiber of the arrangement

The homomorphisms s and P_* from Example 2.3 and Section 4 combine to give the homomorphism

$$\psi = (P_*, s)\colon B(W) \to \mathbb{Z} \times W, \quad x \mapsto (P_*(x), s(x)).$$

The space X_ψ (constructed as in Section 2.2) is then a $K(\pi, 1)$ for $\pi = \ker(\psi)$.

Proposition 5.1 *The complex X_ψ is homotopy equivalent to the Milnor fiber F_Q of Q.*

Proof The map $f\colon M \to W\backslash M \cong \mathbb{C}^n - \Delta$ induces an injection $f_*\colon \pi_1(M) \to B(W)$ whose image is the kernel of $s\colon B(W) \to W$, by [10]. Now

$$\begin{aligned}\ker(\psi) &= \ker(P_*) \cap \ker(s)\\ &= \ker(P_*) \cap \operatorname{im}(f_*)\\ &= f_*(\ker(P_* \circ f_*))\\ &= f_*\big(\ker((Q^2)_*)\big) \quad \text{since } P \circ f = Q^2\\ &= f_*(\ker(Q_*)) \quad \text{since } (Q^2)_* = 2(Q_*).\end{aligned}$$

It follows that X_ψ is homotopy equivalent to the cover of K corresponding to $f_*(\ker(Q_*))$ and hence to the cover of $W\backslash M$ corresponding to $f_*(\ker(Q_*))$. However, this latter cover is homotopy equivalent to F_Q by Corollary 3.2(ii). □

Remark 5.2 The cover X_ψ is a simplicial complex. The vertices of the k–cell $c_k = \big((P_*(x), s(x)), e < w_1 < \cdots < w_k\big)$ are

$$\big((P_*(x), s(x)), e\big) \qquad \text{and} \qquad \big((P_*(x) + |w_i|, s(xw_i)), e\big) \quad \text{for } 1 \le i \le k.$$

This set of $k+1$ vertices uniquely determines c_k.

Remark 5.3 The map $\psi = (P_*, s)$ is not onto. For each $x \in B(W)$, the integer $P_*(x)$ is even if and only if $s(x)$ belongs to the subgroup $W^+ < W$ of Section 3. Thus X_ψ can be identified with the CW–complex whose cells are triples (m, w, σ), where σ is an initialized chain in L and the parity of $m \in \mathbb{Z}$ is the same as that of $w \in W$. The top face of $(m, w, e < w_1 < \cdots < w_k)$ is $(m + |w_1|, ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$, while the other faces are given by $(m, w, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k)$ for $1 \le i \le k$.

Definition 5.4 We define the *small* NCP *model* $\widehat{F}_Q$ of the Milnor fiber of Q to be the finite subcomplex of X_ψ which is the preimage of $\widehat{F}_P$ under the covering $X_\psi \to X_{P_*}$ determined by the subgroup inclusion $\ker(\psi) \subseteq \ker(P_*)$.

Remark 5.5 The complex $\widehat{F}_Q$ is the union of all cells of the form

$$(0, w, e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1}), \quad \text{where } w \in W^+,$$

together with their faces.

Theorem 5.6 *The subcomplex $\widehat{F}_Q$ is a strong deformation retract of the cover X_ψ.*

Proof The simplicial complex X_ψ is the cover of X_{P_*} determined by the subgroup inclusion $\ker(P_*, s) < \ker(P_*)$. By the homotopy lifting property, the strong deformation retraction of X_{P_*} onto $\widehat{F}_P$ (Theorem 4.5) is covered by a strong deformation retraction of X_ψ onto $\widehat{F}_Q$. □

Corollary 5.7 *The finite simplicial complex $\widehat{F}_Q$ is homotopy equivalent to the Milnor fiber F_Q and is a $K(\pi, 1)$ for the group $\ker(P_*, s)$.*

Example 5.8 When W is the dihedral group acting on $\mathbb{R}^2$ with t reflections, the subcomplex $\widehat{F}_Q$ has the following description. Each vertex $(0, w, e)$ has $w \in W^+$, the rotation subgroup of W, while each vertex $(1, w, e)$ has $w \in W - W^+$, the set of reflections of W. Furthermore, for each rotation $w \in W^+$ and each reflection R there is another reflection S with $w = RS$ giving an edge in $\widehat{F}_Q$ labeled $(0, w, e < S)$, starting at $(0, w, e)$ and ending at $(1, R, e)$. Thus $\widehat{F}_Q$ is the complete bipartite graph $K_{t,t}$, which is homotopy equivalent to a bouquet of $(t-1)^2$ circles. This is consistent with the calculations in [26, Theorem 1].

Remark 5.9 The monodromy action on $\widehat{F}_Q$ is obtained as in Remark 4.10 by composing the monodromy action on X_ψ with the retraction defined by the acyclic matching. In this case, the action on X_ψ is by shifting the height of cells by multiples of 2. However, since $\widehat{F}_Q$ is a simplicial complex, it is sufficient to compute the action on 0–cells. We describe the action of $-2 \in \mathbb{Z}$.

Explicitly, we define the action of -2 on 0–cells by

$$((m, w), e) \mapsto \begin{cases} ((m-2, w), e) & \text{if } 2 \le m \le n-1, \\ ((n-1, w\gamma), e) & \text{if } m = 1, \\ ((n-2, w\gamma), e) & \text{if } m = 0. \end{cases}$$

The second case is explained as follows. The action of -2 on the 0–cell $((1, w), e)$ as a cell in X_ψ takes it to the 0–cell $((-1, w), e)$. This last 0–cell is the bottom face of the 1–cell $((-1, w), e < \gamma)$, which in turn has top face $((n-1, w\gamma), e)$. The third case is similar.

6 Structure and homology of $\widehat{F}_P$

Although our model $\widehat{F}_P$ of the Milnor fiber of P is not a simplicial complex, it does have a combinatorial description as a sequence of mapping cones. The domains of the mappings in question are order complexes of truncations of the noncrossing

partition lattice L and the lexicographic shellability of L yields a chain complex which computes $H_*(\widehat{F}_P, \mathbb{Z})$.

Let $L_{[i,j]} = \{w \in L : i \le |w| \le j\}$ and let $A_i = \{(m,\sigma) \in \widehat{F}_P : m \ge n-i-1\}$, where i and j are integers with $0 \le i \le j \le n$. Note that A_i has dimension i and that $A_0 \subset A_1 \subset \cdots \subset A_{n-1} = \widehat{F}_P$. Define $g_i\colon |L_{[1,n-i-1]}| \to A_{n-i-2}$ by

$$w_1 < w_2 < \cdots < w_k \mapsto (i + |w_1|, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k).$$

Proposition 6.1 *The mapping cone of g_i is cellularly isomorphic to A_{n-i-1}.*

Proof $|L_{[0,k]}|$ is simplicially isomorphic to the cone on $|L_{[1,k]}|$ since L has a unique minimal element e. Under this identification, the subcomplex $|L_{[1,k]}|$ of $|L_{[0,k]}|$ corresponds to $|L_{[1,k]}| \times \{1\} \subset |L_{[1,k]}| \times [0,1]$. The map $|L_{[0,n-i-1]}| \to A_{n-i-1}$ given by

$$e < w_1 < w_2 < \cdots < w_k \mapsto (i, e < w_1 < w_2 < \cdots < w_k)$$

combines with the inclusion of A_{n-i-2} into A_{n-i-1} to give a map

$$\widehat{g}_i\colon (|L_{[1,n-i-1]}| \times [0,1]) \amalg A_{n-i-2} \to A_{n-i-1}.$$

One can show that $\widehat{g}_i$ is an identification map which identifies precisely the pair $(x,0)$ with $(x',0)$ for each $x, x' \in |L_{[1,n-i-1]}|$ and $(x,1)$ with $g_i(x)$. □

Lemma 6.2 *$H_q(A_p, A_{p-1}) \cong \widetilde{H}_{q-1}(|L_{[1,p]}|)$ for all $q \ge 1$.*

Proof The filtration $A_0 \subset A_1 \subset \cdots \subset A_{n-1} = \widehat{F}_P$ yields

$$H_q(A_p, A_{p-1}) \cong \widetilde{H}_q(A_p/A_{p-1}) \cong \widetilde{H}_q(\Sigma(|L_{[1,p]}|)) \cong \widetilde{H}_{q-1}(|L_{[1,p]}|),$$

where Σ denotes suspension. The second isomorphism follows from Proposition 6.1. □

Definition 6.3 For each i, define the i^{th} face map $d_i\colon C_{p-1}(L_{[1,p]}) \to C_{p-2}(L_{[1,p]})$ by

$$d_i(w_1 \lessdot w_2 \lessdot \cdots \lessdot w_p) = (-1)^{i-1}(w_1 \lessdot w_2 \lessdot \cdots \lessdot \widehat{w_i} \lessdot \cdots \lessdot w_p)$$

and the top face map $\Omega\colon C_{p-1}(L_{[1,p]}) \to C_{p-2}(L_{[1,p-1]})$ by

$$\Omega(w_1 \lessdot w_2 \lessdot \cdots \lessdot w_p) = (w_1^{-1}w_2 \lessdot w_1^{-1}w_3 \lessdot \cdots \lessdot w_1^{-1}w_p).$$

Theorem 6.4 *The homology of $\widehat{F}_P$ is isomorphic to the homology of the chain complex whose p^{th} group is $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ and whose boundary homomorphism is given, at the level of chains, by $\sum a_\sigma \sigma \mapsto \sum a_\sigma \Omega(\sigma)$.*

Proof First note that

$$H_q(A_p, A_{p-1}) \cong \widetilde{H}_{q-1}(|L_{[1,p]}|) \cong \begin{cases} \mathbb{Z}^{n_p} & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

where the last equality uses the lexicographical shellability of the NCP lattices [3; 5]. By Theorem 39.4 of [28], the homology of $\widehat{F}_{\boldsymbol{P}}$ is isomorphic to the homology of the chain complex with p^{th} group given by $H_p(A_p, A_{p-1})$ and boundary homomorphism given by the connecting homomorphism of the exact sequence of the triple (A_p, A_{p-1}, A_{p-2}).

It remains for us to compute the boundary homomorphism from $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ to $\widetilde{H}_{p-2}(|L_{[1,p-1]}|)$. The isomorphism from $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ to $H_p(A_p, A_{p-1})$ of Lemma 6.2 is induced by $b_p\colon |L_{[1,p]}| \to A_p$, $\sigma \mapsto (n-p-1, e*\sigma)$, where $e*\sigma$ means the simplex represented by $e < w_1 < \cdots < w_l$ when σ is the simplex represented by $w_1 < \cdots < w_l$. Let $\sum a_\sigma \sigma$ in $C_{p-1}(|L_{[1,p]}|)$ be a cycle, so that

$$0 = \partial\Big(\sum a_\sigma \sigma\Big) = \sum a_\sigma \partial(\sigma) = \sum a_\sigma \sum_i d_i(\sigma).$$

Since each σ is maximal, the $(p-1)$–chains $d_i\sigma$ and $d_j\sigma$ have different length distributions whenever $i \neq j$. (The length distribution of $w_1 < w_2 < \cdots < w_p$ is $(|w_1|, |w_2|, \ldots, |w_p|)$.) It follows that

$$\sum a_\sigma d_i(\sigma) = 0 \quad \text{for each } 1 \leq i \leq p. \tag{6-1}$$

Using the fact that the connecting homomorphism is defined on the level of chains by the boundary map in $\widehat{F}_{\boldsymbol{P}}$, we get

$$\begin{aligned} \partial\Big(b_p\Big(\sum a_\sigma \sigma\Big)\Big) &= \partial\Big(\sum a_\sigma (n-1-p, e*\sigma)\Big) \\ &= \sum a_\sigma \partial(n-1-p, e*\sigma) \\ &= \sum a_\sigma \Big((n-p, e*\Omega(\sigma)) + \sum_i (n-1-p, e*d_i\sigma)\Big) \\ &= \sum a_\sigma (n-p, e*\Omega(\sigma)) \quad \text{by (6-1)} \\ &= b_{p-1}\Big(\sum a_\sigma \Omega(\sigma)\Big). \end{aligned}$$

□

Remark 6.5 By Proposition 6.1 and Theorem 6.4, the total rank of the cellular chain complex of $\widehat{F}_{\boldsymbol{P}}$ is the sum of the ranks of the chain groups of $|L_{[1,p]}|$, while the

complex introduced in this section has total rank equal to the sum of the ranks of the reduced homology groups of the $|L_{[1,p]}|$, which vanish except in the top dimension.

Example 6.6 In the case $W = C_3$, the symmetry group of the cube, there are nine reflections, which we label $1, \dots, 9$, and nine length-two elements given by the length-two products

$$12,\ 14,\ 18,\ 19,\ 27,\ 29,\ 45,\ 47,\ 78.$$

The complex $|L_{[1,1]}|$ is a set of nine discrete points while $|L_{[1,2]}|$ has the homotopy type of the graph shown in Figure 2; see [23].

The upper face map applied to each chain $R \lessdot \alpha$ yields the reflection $R^{-1}\alpha$. These values are given by the labels along the edges of the graph. Therefore the map $\widetilde{H}_1(|L_{[1,2]}|) \to \widetilde{H}_0(|L_{[1,1]}|)$ is given by reading, for each loop in the graph, the signed sum of the corresponding labels. The resulting 8×10 matrix has rank 7 and hence $H_2(\widehat{F}_P) \cong \mathbb{Z}^3$ and $H_1(\widehat{F}_P) \cong \mathbb{Z}$.

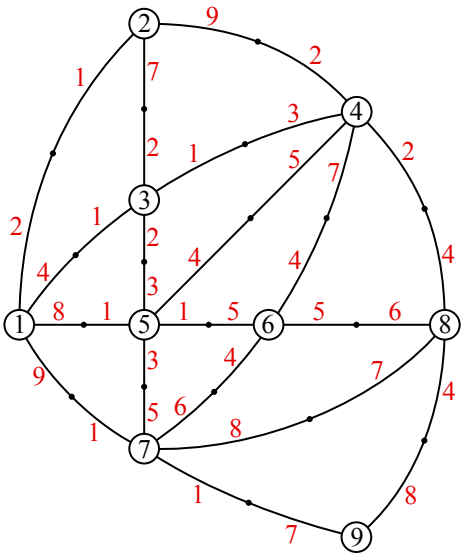


Figure 2: $W = C_3$

Acknowledgements This project began while Falk was in residence at Dublin City University in Spring, 2012, under a Fulbright US Scholars grant. He expresses gratitude to DCU and the Fulbright Commission in Ireland for support and hospitality.

References

[1] **D Allcock**, **T Basak**, *Geometric generators for braid-like groups*, Geom. Topol. 20 (2016) 747–778 MR

[2] **D Armstrong**, *Generalized noncrossing partitions and combinatorics of Coxeter groups*, Mem. Amer. Math. Soc. 949, Amer. Math. Soc., Providence, RI (2009) MR

[3] **C A Athanasiadis**, **T Brady**, **C Watt**, *Shellability of noncrossing partition lattices*, Proc. Amer. Math. Soc. 135 (2007) 939–949 MR

[4] **D Bessis**, *The dual braid monoid*, Ann. Sci. École Norm. Sup. 36 (2003) 647–683 MR

[5] **A Björner**, *Shellable and Cohen–Macaulay partially ordered sets*, Trans. Amer. Math. Soc. 260 (1980) 159–183 MR

[6] **N Bourbaki**, *Lie groups and Lie algebras, Chapters* 7–9, Springer (2005) MR

[7] **T Brady**, *A partial order on the symmetric group and new* $K(\pi,1)$*'s for the braid groups*, Adv. Math. 161 (2001) 20–40 MR

[8] **T Brady**, **C Watt**, $K(\pi,1)$*'s for Artin groups of finite type*, Geom. Dedicata 94 (2002) 225–250 MR

[9] **T Brady**, **C Watt**, *Non-crossing partition lattices in finite real reflection groups*, Trans. Amer. Math. Soc. 360 (2008) 1983–2005 MR

[10] **E Brieskorn**, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. 12 (1971) 57–61 MR

[11] **E Brieskorn**, *Sur les groupes de tresses (d'après V I Arnol'd)*, from "Séminaire Bourbaki, 1971/1972", Lecture Notes in Math. 317, Springer (1973) Exposé 401, 21–44 MR

[12] **F Callegaro**, *The homology of the Milnor fiber for classical braid groups*, Algebr. Geom. Topol. 6 (2006) 1903–1923 MR

[13] **F Callegaro**, **M Salvetti**, *Integral cohomology of the Milnor fibre of the discriminant bundle associated with a finite Coxeter group*, C. R. Math. Acad. Sci. Paris 339 (2004) 573–578 MR

[14] **C Chevalley**, *Invariants of finite groups generated by reflections*, Amer. J. Math. 77 (1955) 778–782 MR

[15] **C De Concini**, **M Salvetti**, *Cohomology of Artin groups*, Math. Res. Lett. 3 (1996) 293–297 MR

[16] **C De Concini**, **M Salvetti**, **F Stumbo**, *The top-cohomology of Artin groups with coefficients in rank-*1 *local systems over* $\mathbb{Z}$, Topology Appl. 78 (1997) 5–20 MR

[17] **P Dehornoy**, **F Digne**, **E Godelle**, **D Krammer**, **J Michel**, *Foundations of Garside theory*, EMS Tracts in Mathematics 22, Eur. Math. Soc., Zürich (2015) MR

[18] **P Deligne**, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972) 273–302 MR

[19] **A Dimca**, **G Lehrer**, *Cohomology of the Milnor fibre of a hyperplane arrangement with symmetry*, from "Configuration spaces" (F Callegaro, F Cohen, C De Concini, E M Feichtner, G Gaiffi, M Salvetti, editors), Springer INdAM Ser. 14, Springer (2016) 233–274 MR

[20] **È V Frenkel'**, *Cohomology of the commutator subgroup of the braid group*, Funktsional. Anal. i Prilozhen. 22 (1988) 91–92 MR In Russian; translated in Funct. Anal. Appl. 22 (1988) 248–250

[21] **A Hatcher**, *Algebraic topology*, Cambridge Univ. Press (2002) MR

[22] **J E Humphreys**, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics 29, Cambridge Univ. Press (1990) MR

[23] **A Kenny**, *Geometrically constructed bases for homology of non-crossing partition lattices*, Electron. J. Combin. 16 (2009) art. id. 48, 8 pages MR

[24] **D Kozlov**, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics 21, Springer (2008) MR

[25] **J Milnor**, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies 61, Princeton Univ. Press (1968) MR

[26] **J Milnor**, **P Orlik**, *Isolated singularities defined by weighted homogeneous polynomials*, Topology 9 (1970) 385–393 MR

[27] **A D Măcinic**, **Ş Papadima**, *On the monodromy action on Milnor fibers of graphic arrangements*, Topology Appl. 156 (2009) 761–774 MR

[28] **J R Munkres**, *Elements of algebraic topology*, Addison-Wesley (1984) MR

[29] **M Salvetti**, *The homotopy type of Artin groups*, Math. Res. Lett. 1 (1994) 565–577 MR

[30] **S Settepanella**, *A stability-like theorem for cohomology of pure braid groups of the series A, B and D*, Topology Appl. 139 (2004) 37–47 MR

[31] **S Settepanella**, *Cohomology of pure braid groups of exceptional cases*, Topology Appl. 156 (2009) 1008–1012 MR

School of Mathematical Sciences, Dublin City University
Dublin, Ireland

Department Mathematics and Statistics, Northern Arizona University
Flagstaff, AZ, United States

School of Mathematical Sciences, Dublin Institute of Technology
Dublin, Ireland

tom.brady@dcu.ie, michael.falk@nau.edu, colum.watt@dit.ie

Received: 16 June 2017 Revised: 12 June 2018

msp

Algebraic & Geometric Topology 18 (2018) 3839–3885

Knot Floer homology and Khovanov–Rozansky homology for singular links

NATHAN DOWLIN

The (untwisted) oriented cube of resolutions for knot Floer homology assigns a complex $C_F(S)$ to a singular resolution S of a knot K. Manolescu conjectured that when S is in braid position, the homology $H_*(C_F(S))$ is isomorphic to the HOMFLY-PT homology of S. Together with a naturality condition on the induced edge maps, this conjecture would prove the existence of a spectral sequence from HOMFLY-PT homology to knot Floer homology. Using a basepoint filtration on $C_F(S)$, a recursion formula for HOMFLY-PT homology and additional $\mathfrak{sl}_n$–like differentials on $C_F(S)$, we prove Manolescu's conjecture. The naturality condition remains open.

57M27

1 Introduction

The last few decades have seen tremendous growth within the field of knot theory. Many new knot invariants have been constructed, including the categorifications of several classical knot polynomials. These categorifications typically take the form of a multigraded homology theory, whose graded Euler characteristic returns the polynomial in question.

Some of the most notable categorifications include HOMFLY-PT homology, $\mathfrak{sl}_n$ homology and knot Floer homology, whose graded Euler characteristics return the HOMFLY-PT polynomial, the $\mathfrak{sl}_n$ polynomial and the Alexander polynomial, respectively. The HOMFLY-PT polynomial of a link L is a two variable polynomial $P_H(a,q)(L)$, and it is determined by the skein relation

$$aP_H(a,q)(L_+) - a^{-1}P_H(a,q)(L_-) = (q-q^{-1})P_H(a,q)(L_0),$$

where L_+, L_- and L_0 are identical except at one crossing, where L_+ has a positive crossing, L_- has a negative crossing and L_0 has the oriented smoothing (see Freyd, Yetter, Hoste, Lickorish, Millett and Ocneanu [4] and Przytycki and Traczyk [16]). Together with the normalization $P_H(a,q)(\text{unknot}) = 1$, this relation uniquely determines the HOMFLY-PT polynomial.

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3839

We can obtain a single-variable polynomial invariant $P_n(q)(L)$ by setting $a = q^n$ in the HOMFLY-PT polynomial. For $n \geq 1$, $P_n(q)(L)$ gives the $\mathfrak{sl}_n$ polynomial, and setting $n = 0$ gives the Alexander polynomial. The most popular of the $\mathfrak{sl}_n$ polynomials is the Jones polynomial, which is obtained by setting $n = 2$, and $\mathfrak{sl}_2$ homology is isomorphic to Khovanov's original categorification of the Jones polynomial, known as Khovanov homology [8]. An explicit description of this isomorphism is described by Hughes [6].

The fact that the $\mathfrak{sl}_n$ and Alexander polynomials are specializations of the HOMFLY-PT polynomial led to the following conjecture of Dunfield, Gukov and Rasmussen:

Conjecture 1.1 [3] *For all $n \geq 1$, there is a spectral sequence from* HOMFLY-PT *homology to $\mathfrak{sl}_n$ homology, and there is a spectral sequence from* HOMFLY-PT *homology to knot Floer homology.*

Rasmussen was able to use similarities in the constructions of HOMFLY-PT homology and $\mathfrak{sl}_n$ homology to prove the first part of this conjecture. In particular, he showed [18] that there are a family of spectral sequences $E_k(n)$ for $n \geq 1$ such that the E_2 page is HOMFLY-PT homology and the E_∞ page is $\mathfrak{sl}_n$ homology.

Unfortunately, due to the fundamental differences between Khovanov–Rozansky homology and knot Floer homology, the second part of Conjecture 1.1 has remained unsolved for the last decade. This conjectured spectral sequence from HOMFLY-PT homology to knot Floer homology will be the focus of this paper.

It turns out that all three of these homology theories can be constructed via an oriented cube of resolutions. Let $C_H(D, d_0^H + d_1^H)$ denote the cube of resolutions for HOMFLY-PT homology, where d_i^H denotes the component of the differential which increases the cube grading by i. (For both HOMFLY-PT and $\mathfrak{sl}_n$ homology, the higher face maps d_i with $i \geq 2$ are zero.) The HOMFLY-PT homology $H_H(K)$ is defined to be

$$H_H(K) = H_*\big(H_*(C_H(D), d_0^H), (d_1^H)^*\big),$$

where D is a braid diagram of the knot K and $(d_1^H)^*$ is the induced map on homology. HOMFLY-PT homology is a knot invariant, which means that the homology does not depend on the choice of braid diagram for K. Note that this homology can be equivalently viewed as the E_2 page of the spectral sequence induced by the cube filtration on the HOMFLY-PT complex.

The $\mathfrak{sl}_n$ homology $H_n(K)$ is defined in the same way. If $C_n(D, d_0^{(n)} + d_1^{(n)})$ denotes the cube of resolutions for $\mathfrak{sl}_n$ homology, then $H_n(K)$ is given by

$$H_n(K) = H_*\big(H_*(C_n(D), d_0^{(n)}), (d_1^{(n)})^*\big),$$

where, again, D is a diagram for K (not necessarily a braid), and the homology is independent of the choice of D.

Knot Floer homology is a completely different story. There is not a standard way to define the complex $CFK^-(K)$, as it depends on a choice of Heegaard diagram for K, and there are many different ways to make this choice for a knot K. However, the chain homotopy type of $CFK^-(K)$ does not depend on the choice of diagram.

Using a particular choice of Heegaard diagram together with a bit of algebra, Oszváth and Szabó developed an oriented cube of resolutions for knot Floer homology with twisted coefficients [15]. This construction was modified by Manolescu [12] to give an untwisted cube of resolutions for knot Floer homology, which is chain homotopy equivalent to $CFK^-(K)$. We will denote this complex by $(C_F(D), d_0^F + d_1^F + \cdots + d_k^F)$. Unlike the HOMFLY-PT and $\mathfrak{sl}_n$ homology, the knot Floer homology $HFK^-(K)$ is the *total* homology of this complex:

$$HFK^-(K) \cong H_*(C_F(D), d_0^F + d_1^F + \cdots + d_k^F).$$

For all of these complexes, each vertex in the cube of resolutions can be viewed as a complex corresponding to the complete resolution S at that vertex. We will denote these complexes by $C_H(S)$, $C_n(S)$ and $C_F(S)$, with the corresponding homologies given by $H_H(S)$, $H_n(S)$ and $H_F(S)$.

Manolescu made the following conjecture:

Conjecture 1.2 [12] *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

An immediate consequence of this conjecture is an isomorphism

$$H_*(C_H(D), d_0^H) \cong H_*(C_F(D), d_0^F),$$

which maps each vertex in the HOMFLY-PT cube of resolutions to the same vertex in the knot Floer cube of resolutions. As discussed in [12], one would obtain a spectral sequence from HOMFLY-PT homology to knot Floer homology if the induced edge maps agree. In other words, if f is the isomorphism between them, then the existence

of such a spectral sequence would follow from the square below being commutative:

$$\begin{array}{ccc} H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \\ \Big\downarrow{\scriptstyle (d_1^H)^*} & & \Big\downarrow{\scriptstyle (d_1^F)^*} \\ H_*(C_H(D), d_0^H) & \xrightarrow{f} & H_*(C_F(D), d_0^F) \end{array}$$

This idea can also be explained in terms of the spectral sequences induced by the cube filtrations on $C_H(D)$ and $C_F(D)$. Letting $E_k^H(D)$ and $E_k^F(D)$ denote the two spectral sequences, we see that the HOMFLY-PT homology is given by $E_2^H(D)$ and the knot Floer homology is given by $E_\infty^F(D)$. Manolescu's conjecture would imply an isomorphism $E_1^H(D) \cong E_1^F(D)$, and the induced edge maps commuting with this isomorphism would imply that $E_2^H(D) \cong E_2^F(D)$. This would give a spectral sequence whose E_2 page is isomorphic to HOMFLY-PT homology and whose E_∞ page is $HFK^-(K)$.

Manolescu showed that for a connected singular braid S, both $H_H(S)$ and $H_F(S)$ have a purely algebraic formulation in terms of Tor groups. Letting R denote the polynomial ring $\mathbb{Q}[U_1, \dots, U_k]$, where k is the number of edges in the singular braid S, he showed that there are ideals L, Q and N in R such that

$$H_H(S) \cong \mathrm{Tor}_R(R/L, R/Q) \quad \text{and} \quad H_F(S) \cong \mathrm{Tor}_R(R/L, R/N).$$

These Tor groups can naturally be viewed as bigraded vector spaces, where the dimension in each bigrading is finite. Thus, for S connected, Conjecture 1.2 is equivalent to an isomorphism of bigraded vector spaces

$$\mathrm{Tor}_R(R/L, R/Q) \cong \mathrm{Tor}_R(R/L, R/N).$$

Unfortunately, these Tor groups turned out to be difficult to compare due to the nonlocal nature of the ideal N.

In this paper, we will prove Conjecture 1.2 using a very different approach. First, we define an additional family of differentials on $C_F(S)$ for all $n \geq 1$. We will denote this complex by $C_{F(n)}$.

Theorem 1.3 *For all $n \geq 1$, there is an isomorphism $H_*(C_{F(n)}(S)) \cong H_{n+1}(S)$.*

The differential on $C_{F(n)}$ can be filtered by the Alexander grading, and when we only consider those differentials which preserve the Alexander grading, we get back the complex $C_F(S)$. Thus, using the Alexander filtration on $C_{F(n)}$, we get the following:

Corollary 1.4 *For all $n \geq 2$, there is a spectral sequence which starts at $H_F(S)$ and converges to $H_n(S)$.*

For all $n \geq 1$, there is also a known spectral sequence which starts at $H_H(S)$ and converges to $H_n(S)$; see Rasmussen [18]. Thus, $H_H(S)$ and $H_F(S)$ are both "limits" of $\mathfrak{sl}_n$ homology (in a suitable sense).

We are able to use these additional differentials together with a basepoint filtration to prove Conjecture 1.2.

Theorem 1.5 *Let S denote a complete resolution of a diagram D in braid position. Then $H_H(S) \cong H_F(S)$ as bigraded vector spaces.*

Corollary 1.6 *Let D be a braid diagram and $E_2^F(D)$ the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. Then the graded Euler characteristic of $E_2^F(D)$ is the* HOMFLY-PT *polynomial.*

This corollary provides evidence for the conjecture that $E_2^F(D)$ is in fact isomorphic to HOMFLY-PT homology.

Remark 1.7 The spectral sequences from $H_F(S)$ to $H_n(S)$ may seem strange in the context of the conjectured spectral sequence from Khovanov homology to knot Floer homology, since, assuming this conjecture is true, we would have rank inequalities among the reduced homologies

$$\mathrm{rk}(\overline{H}_H(K)) \geq \mathrm{rk}(\overline{H}_n(K)) \geq \mathrm{rk}(\overline{H}_{n-1}(K)) \geq \cdots \geq \mathrm{rk}(\overline{H}_2(K)) \geq \mathrm{rk}(\widehat{HFK}(K))$$

for any knot K. The intuition behind this is that knot Floer homology has large homology at each vertex in the cube of resolutions, but unlike HOMFLY-PT and $\mathfrak{sl}_n$ homology, it has higher face maps which allow the total homology of a knot to still be smaller.

Outline of the paper

In Section 2 we give background on HOMFLY-PT and $\mathfrak{sl}_n$ homology for singular braids. In Section 3 we describe a recursion formula for $\mathfrak{sl}_n$ homology known as the

composition product — see Wagner [21] — and prove a generalization to HOMFLY-PT homology. In Section 4 we describe the knot Floer homology of singular braids and define a filtration on the associated complex. The main theorems are proved in this section using the filtration together with the composition product formulas from Section 3.

Acknowledgements

The author would like to thank Akram Alishahi, Andrew Manion, Ciprian Manolescu, Peter Ozsváth, Sucharit Sarkar and Zoltán Szabó for their suggestions, comments and edits of this paper. The author would also like to thank the referee for their valuable comments and additions. The author was supported through NSF grant DMS-1606421.

2 The Khovanov–Rozansky homology of singular links

2.1 Singular resolutions and the ground ring

A complete resolution S of a knot K in braid position can be viewed as an oriented planar graph with the following properties:

(1) All vertices are either 2–valent or 4–valent.

(2) The number of incoming edges is equal to the number of outgoing edges at each vertex.

(3) If Z is an oriented cycle in S, then the unique disc $D \subset \mathbb{R}^2$ with boundary Z intersects the center of the braid.

Let $e_1, \dots, e_k$ denote the edges of S. To each edge e_i, we assign an indeterminate U_i. All three homology theories will be defined over the ground ring $R = \mathbb{Q}[U_1, \dots, U_k]$.

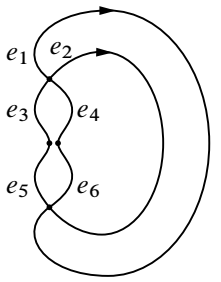


Figure 1: An example of a singular braid diagram

2.2 HOMFLY-PT homology and $\mathfrak{sl}_n$ homology

This section will give a brief description of the HOMFLY-PT and $\mathfrak{sl}_n$ homologies as defined in [9; 10]. We will use the grading conventions from [18], though we will leave out the overall grading shifts coming from the braid number. The reader can refer to these resources for further background. The HOMFLY-PT and $\mathfrak{sl}_n$ complexes have the same generators, with the $\mathfrak{sl}_n$ complex having strictly more differentials than the HOMFLY-PT complex (see Remark 2.1). For this reason, we will start by defining the HOMFLY-PT complex, then we will describe the additional differentials to make the $\mathfrak{sl}_n$ complex.

Remark 2.1 In all of our homology theories, the differential is a map $d\colon C \to C$ with $d^2 = 0$, where C is free R–module. However, we will also commonly refer to a particular component of d as a "differential" as well. For example, if the coefficient of y in the basis expansion of $d(x)$ is $a \in R$, then we will say that there is a differential from x to y with coefficient a.

The HOMFLY-PT complex for links comes equipped with a triple-grading, and the $\mathfrak{sl}_n$ complex with a bigrading. One of the gradings in both theories, however, comes from the height in the cube of resolutions, so it will be fixed for a single resolution. The HOMFLY-PT complex will therefore come with a bigrading, and the $\mathfrak{sl}_n$ complex with a single grading. For the HOMFLY-PT complex, these gradings are called the quantum grading, denoted gr_q, and the horizontal grading, denoted gr_h. Multiplication by the U_i increases the quantum grading by 2 and preserves the horizontal grading.

Let $V_2(S)$ denote the 2–valent vertices in S and $V_4(S)$ the 4–valent vertices of S. For vertices v in $V_2(S)$, there is a unique outgoing edge e_i and a unique incoming edge e_j. Define $L(v)$ to be the linear term $U_i - U_j$. Similarly, for vertices v in $V_4(S)$ there are two outgoing edges e_i and e_j and two incoming edges e_k and e_l. We define $L(v)$ to be the linear term $U_i + U_j - U_k - U_l$ and $Q(v)$ to be the quadratic term $U_iU_j - U_kU_l$.

The HOMFLY-PT complex is a tensor product of complexes $C_H(v)$ for each vertex v. For v in $V_2(S)$, $C_H(v)$ is defined as

$$R\{0,-2\} \xrightarrow{L(v)} R\{0,0\},$$

where $R\{i,j\}$ refers to the free R–module of rank 1 shifted by i in gr_q and by j in gr_h, so that the generator lies in bigrading $\{i,j\}$. For v in $V_4(S)$, $C_H(v)$ is defined as in Figure 2.

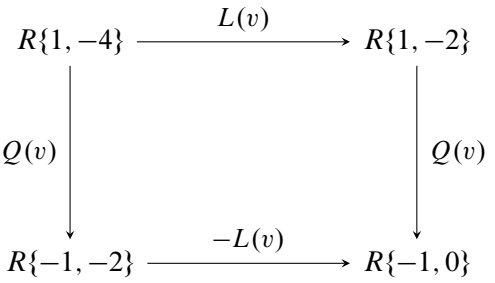


Figure 2: The HOMFLY-PT complex at a 4–valent vertex v

Note that the differential is homogeneous of degree $\{2,2\}$. The HOMFLY-PT complex for the singular diagram S is given by

$$C_H(S) = \bigotimes_{v \in S} C_H(v),$$

where the tensor product is taken over R and the HOMFLY-PT homology $H_H(S)$ is the homology of $C_H(S)$.

We will now define the additional differentials which give $\mathfrak{sl}_n$ homology. For a vertex v in S with outgoing edges E_{out} and incoming edges E_{in}, let the potential w_n be given by

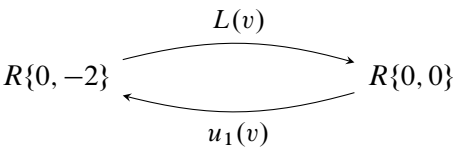
$$w_n(v) = \sum_{e_i \in E_{\text{out}}} U_i^{n+1} - \sum_{e_j \in E_{\text{in}}} U_j^{n+1}.$$

For v in $V_2(S)$, let $u_1(v)$ be the unique element in R such that $u_1(v)L(v) = w_n(v)$. For v in $V_4(S)$, we can choose $u_1(v)$ and $u_2(v)$ such that

$$u_1(v)L(v) + u_2(v)Q(v) = w_n(v).$$

Unlike the 2–valent case, the choice is not unique, but the reader can refer to [9, page 5] for the precise choice. (It is not relevant for our discussion.)

For each vertex v, we will add new differentials to $C_H(v)$ to make a new complex $C_n(V)$. For v in $V_2(S)$, $C_n(v)$ is given by



and for v in $V_4(S)$, $C_n(v)$ is given by Figure 3.

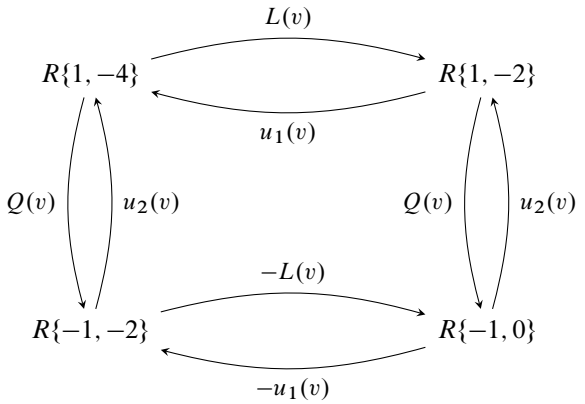


Figure 3: The $\mathfrak{sl}_n$ complex at a 4–valent vertex v

Observe that for both types of vertices, the differential on $C_n(v)$ satisfies $d^2 = w_n(v)I$. Such a complex is called a matrix factorization with potential w_n. Since d^2 is nonzero, its homology is not well defined. However, we are interested in the tensor product of $C_n(v)$ over all vertices v in S. Define the $\mathfrak{sl}_n$ complex $C_n(S)$ by

$$C_n(S) = \bigotimes_{v \in S} C_n(v),$$

where again the tensor product is taken over R.

As mentioned above, the HOMFLY-PT differentials are homogeneous of degree $\{2,2\}$. These differentials are denoted by d_+. The new differentials, those with coefficients $u_1(v)$ and $u_2(v)$, are homogeneous of degree $\{2n,-2\}$. These are denoted by d_-. The total differential $d_{\text{tot}} = d_+ + d_-$ is not homogeneous in this bigrading. However, if we look at the grading $\text{gr}_n = \text{gr}_q + \frac{1}{2}(n-1)\,\text{gr}_h$, then d_{tot} is homogeneous of degree $n+1$.

Additionally, $d_{\text{tot}}^2 = 0$. This can be seen from the fact that the potential is additive under tensor product, so $d_{\text{tot}}^2 = \sum_{v \in S} w_n(v)$. The sum must be zero because each edge e_i is an outgoing edge for one vertex, which will contribute U_i^{n+1}, and an incoming edge for another vertex, which will contribute $-U_i^{n+1}$.

This shows that $C_n(S)$ is a well-defined chain complex which is homogeneous with respect to the grading gr_n. We define the $\mathfrak{sl}_n$ homology $H_n(S)$ to be the homology of this complex.

Remark 2.2 The definitions given here correspond to the unreduced theories in [18] as opposed to the middle or reduced homologies. To translate between our definition

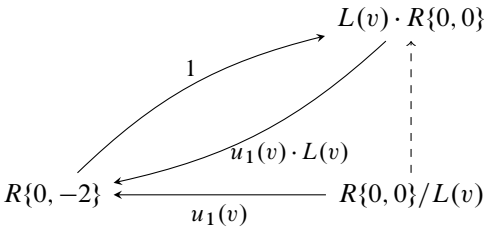


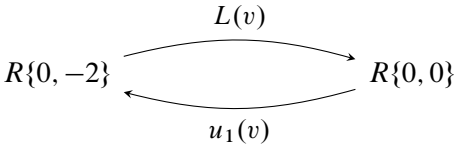
Figure 4

and Rasmussen's, it suffices to show they are the same for connected diagrams S, as in both theories disjoint union corresponds to tensor product over $\mathbb{Q}$.

Let v_0 be a bivalent vertex in S (insert one if necessary). The linear relations coming from all of the vertices besides v form a regular sequence in R — this is shown in Rasmussen's proof of Lemma 3.11 in [18]. We can write our complex $C_n(S)$ as

$$C_n(S) = C_n^Q(S) \otimes_R (R\{0,-2\} \underset{u_1(v_0)}{\overset{L(v_0)}{\rightleftarrows}} R\{0,0\}) \otimes_R C_n^L(S - v_0),$$

where $C_n^Q(S)$ is the Koszul complex on the terms $(Q(v), u_2(v))$ for all 4–valent vertices v, and $C_n^L(S - v_0)$ is the Koszul complex on the terms $(L(v), u_1(v))$ for all vertices except v_0 (see [18, Definition 3.7] for the definition of a Koszul complex in the context of $\mathbb{Z}$–graded matrix factorizations). Each Koszul factor in $C_n^L(S - v_0)$ is given by



for some vertex v. This complex can be rewritten as in Figure 4, where the dashed line refers to the fact that multiplication by $L(v)$ in $R\{0,0\}/L(v)$ maps into $L(v)\cdot R\{0,0\}$ in the obvious way (ie it is recording the R–module structure, not the differential).

After tensoring this matrix factorization with the remaining terms to obtain $C_n(S)$, the map by 1 in this diagram gives a contractible subcomplex, so our complex $C_n(S)$ is chain homotopy equivalent to the complex in which the linear Koszul factor is replaced with the quotient complex $R\{0,0\}/L(v)$. Since all of the elements $L(v)$ in $C_n^L(S - v_0)$ form a regular sequence, we can do this recursively until $C_n^L(S - v_0)$ has been replaced with the complex $R/L = R\{0,0\}/\{L(v) = 0 \text{ for } v \neq v_0\}$. (The fact that the elements form a regular sequence guarantees that the maps labeled by 1 in the

diagram are always isomorphisms rather than just surjections, making the subcomplex contractible at each stage.)

Since $L(v_0)$ is the negative of the sum of the $L(v)$ for $v \neq v_0$, $L(v_0)$ is in the ideal L as well. Thus, we have shown that

$$C_n(S) \cong C_n^{\mathcal{Q}}(S) \otimes_R (R\{0,-2\} \underset{u_1(v_0)}{\overset{0}{\rightleftarrows}} R\{0,0\}) \otimes_R R/L.$$

This corresponds to the definition of the unreduced theory in [18, Definition 2.14]. The middle theory can be obtained by dropping the Koszul complex $R\{0,-2\} \underset{u_1(v_0)}{\overset{0}{\rightleftarrows}} R\{0,0\}$.

Remark 2.3 This argument that Koszul complexes on regular sequences can be simplified to quotients R/I where I is the ideal generated by the elements of the regular sequence will be repeated throughout the paper. When we are using this argument, we will simply say that we are canceling the corresponding differentials. The above discussion describes why no new differentials arise as a result of this cancelation.

2.3 Rasmussen's spectral sequences

Rasmussen [18] showed that there is a family of spectral sequences $E_k(n)$ which start at HOMFLY-PT homology and converge to $\mathfrak{sl}_n$ homology. The existence of these spectral sequences is somewhat difficult to prove for the case of knots and links, but the argument is much simpler for fully singular diagrams.

With respect to the horizontal grading, d_+ has grading 2 and d_- has grading -2. If we take homology with respect to d_+, all induced differentials decrease the horizontal grading. Thus, there is a well-defined spectral sequence $(E_k(n), d_k(n))$, where d_k is defined to be the part of d_{tot} which increases the horizontal grading by $2-4k$.

The E_1 page of this spectral sequence is $H_*(C_n(S), d_+)$, which is exactly the definition of HOMFLY-PT homology. Since $C_n(S)$ is bounded in horizontal grading, the E_∞ page is the homology with respect to d_{tot}, or $\mathfrak{sl}_n$ homology. It turns out (Corollary 2.7) that this spectral sequence collapses at the E_2 page, given by

$$H_*(H_*(C_n(S), d_+), d_-^*).$$

We will denote this page by $H^{\pm}(C_n(S))$. Note that $H^{\pm}(C_n(S))$ is bigraded, as both d_+ and d_- are homogeneous. In order to see why this spectral sequence collapses, we must first introduce the rotation number.

Definition 2.4 Given a (possibly singular) diagram D, let D' denote the diagram obtained by replacing each crossing or singularization with the oriented smoothing. The resulting diagram is a collection of oriented circles. These circles are called the *Seifert circles* of D.

Definition 2.5 The *rotation number* of a (possibly singular) diagram D is the sum of the signs of the Seifert circles, with a circle contributing a $+1$ if it is oriented counterclockwise and a -1 if it is oriented clockwise.

The fact that all higher differentials are trivial follows from the following lemma:

Lemma 2.6 [18] *The homology $H^{\pm}(C_n(S))$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$, where $r(S)$ is the rotation number of S. Since S is a singular braid oriented clockwise, $r(S)$ is the negative of the number of strands in S.*

Since none of the higher differentials preserve the horizontal grading, they must all be trivial, causing the spectral sequence to collapse.

Corollary 2.7 *Viewing $H^{\pm}(C_n(S))$ as singly graded with grading gr_n, there is a graded isomorphism $H^{\pm}(C_n(S)) \cong H_n(S)$.*

Remark 2.8 The reader familiar with [18] may note that our homology lies in $\mathrm{gr}_h = 2r(S)$, while Rasmussen's lies in $\mathrm{gr}_h = 1 + r(S)$. This difference comes from the fact that our homology is unreduced, which decreases the grading by 1, and because we are leaving out the overall grading shift of $-r(S)$ in [18, Definition 2.14].

3 A recursion formula for the Khovanov–Rozansky homology of singular links

3.1 The composition product

3.1.1 Jaeger's formula The first composition product formula was defined by Jaeger in [7]. In order to discuss the composition product, we must first define labelings of a diagram. Let K be a knot with corresponding diagram D. Viewing D as an oriented 4–valent graph, we say that a subset S of the edges of D is a homological cycle if at each vertex in D the number of incoming edges in S is equal to the number of outgoing edges in S. A *labeling* f of the diagram D is a function from the set of edges in D to the set $\{1, 2\}$ such that $f^{-1}(1)$ is a homological cycle. (Note that $f^{-1}(1)$ is a homological cycle if and only if $f^{-1}(2)$ is a homological cycle.)

Figure 5: Nonadmissible labelings

We will place a restriction on which homological cycles are allowed. A homological cycle is said to make a turn at a crossing c if it has one incoming edge at c and one outgoing edge at c and those edges are not diagonal from one another. Let $T(f)$ denote the number of turns of the labeling f. A labeling f is *admissible* if $f^{-1}(1)$ doesn't make any left turns at positive crossings or right turns at negative crossings.

Since the homological cycle $f^{-1}(1)$ uniquely determines the labeling f, we will say that a homological cycle Z is admissible if the unique labeling f with $f^{-1}(1) = Z$ is admissible. The two cycles $f^{-1}(1)$ and $f^{-1}(2)$ can both be viewed as diagrams of links if we retain the crossing information whenever one of them contains all four edges at a crossing, and forget it otherwise. We will refer to these diagrams as $D_{f,1}$ and $D_{f,2}$, respectively. Note that $r(D_{f,1}) + r(D_{f,2}) = r(D)$.

With the HOMFLY-PT polynomial P_H as defined in the introduction, define

$$P'_H(a, q, D) = \left(\frac{a - a^{-1}}{q - q^{-1}}\right)(a^{w(D)})\, P_H(a, q, D).$$

Note that P'_H is invariant under Reidemeister II and III moves, but performing a Reidemeister I move changes the writhe, so one picks up a factor of a or a^{-1} depending on the sign of the crossing. With this normalization, $P'_H(\mathcal{O}) = (a - a^{-1})/(q - q^{-1})$, and $P'_H(\varnothing) = 1$, where $\varnothing$ denotes the empty diagram and $\mathcal{O}$ denotes the crossingless diagram for the unknot. Jaeger's composition product can be stated as

$$\sum_{f \text{ admissible}} (q - q^{-1})^{T(f)} a_1^{r(D_{f,2})} a_2^{-r(D_{f,1})} P'_H(a_1, q, D_{f,1}) P'_H(a_2, q, D_{f,2}) = P'_H(a_1 a_2, q, D). \tag{1}$$

The proof of this formula is combinatorial in nature — one can show that it behaves properly under Reidemeister moves and that it satisfies the necessary skein relation via local computations. In fact, Jaeger [7] showed that this formula is invariant under all Reidemeister moves, so the formula holds for arbitrary diagrams D instead of just braid diagrams.

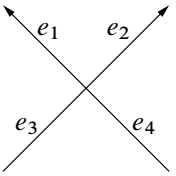


Figure 6: A labeled 4–valent vertex

3.1.2 The composition product for singular graphs Defining

$$P'_n(q, D) = P'_H(q^n, q, D),$$

the composition product formula can be specialized to the $\mathfrak{sl}_n$ polynomials to give

$$\sum_{f \text{ admissible}} (q - q^{-1})^{T(f)} q^{mr(D_{f,1}) - nr(D_{f,2})} P'_n(q, D_{f,1}) P'_m(q, D_{f,2}) = P'_{m+n}(q, D). \tag{2}$$

This formula was extended by Wagner to singular braids in the following way. If S is a singular braid, we can define labelings of S in the same way as labelings for knots. We will drop the admissibility condition at 4–valent vertices since they no longer correspond to positive or negative crossings. Note that as with the nonsingular diagrams, we have $r(S_{f,1}) + r(S_{f,2}) = r(S)$. Given a labeling f of S, let $T_1(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_1 and e_3 in Figure 6. Similarly, let $T_2(S_{f,1})$ denote the number of vertices $v \in V_4(S)$ at which $f^{-1}(1)$ contains the edges e_2 and e_4. With this terminology, the composition product for singular braids can be stated as

$$P'_{m+n}(S) = \sum_{f \in L(S)} q^{\sigma_{m,n}(f)} P'_n(S_{f,1}) P'_m(S_{f,2}),$$

where $P'_k(S)$ is the unreduced $\mathfrak{sl}_k$ polynomial of S,

$$\sigma_{m,n}(f) = T_2(S_{f,1}) - T_1(S_{f,1}) + mr(S_{f,1}) - nr(S_{f,2})$$

and $r(S_{f,i})$ is the negative of the number of strands in the singular braid $S_{f,i}$.

Remark 3.1 The $\mathfrak{sl}_k$ polynomials for singular braids are defined by the skein relations

$$P_k(D_x) = qP_k(D_0) - q^k P_k(D_+) = q^{-1} P_k(D_0) - q^{-k} P_k(D_-)$$

and

$$P'_k(D_x) = qP'_k(D_0) - P'_k(D_+) = q^{-1} P'_k(D_0) - P'_k(D_-),$$

where D_+, D_-, D_x and D_0 are diagrams which are the same away from a crossing, but locally D_+ has a positive crossing, D_- has a negative crossing, D_x has the singularization and D_0 has the oriented smoothing. For a fully singular graph S, the writhe is zero, so $P'_k(S) = (q^k - q^{-k})/(q - q^{-1})P_k(S)$. Thus, $P'_k(S)$ is categorified by the unreduced $\mathfrak{sl}_k$ homology of S, and $P_k(S)$ is categorified by the reduced $\mathfrak{sl}_k$ homology of S.

3.2 The categorification for $\mathfrak{sl}_n$ homology

Wagner further showed that the composition product formula is true on the level of categorifications:

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{\sigma_{m,n}(f)\} \quad \text{(with polynomial grading)},$$

where the $\mathfrak{sl}_n$ homology groups are singly graded, using the polynomial grading gr_n. In order for us to generalize this theorem, it will be useful to have a bigraded version of it using our $(\mathrm{gr}_q, \mathrm{gr}_h)$ gradings. Since the bigraded $H_n(S)$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$, and $\mathrm{gr}_n = \mathrm{gr}_q + \frac{1}{2}(n-1)\,\mathrm{gr}_h$, the q–grading is uniquely determined by gr_n:

$$\mathrm{gr}_n = \mathrm{gr}_q + (n-1)r(S).$$

Thus, if we view each $H_n(S)$ as singly graded, where the grading is gr_q instead, the formula becomes

$$\begin{aligned} &H_{m+n}(S)\{(n+m-1)r(S)\} \\ &\quad = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{\sigma_{m,n}(f) + (n-1)r(S_{f,1}) + (m-1)r(S_{f,2})\}. \end{aligned}$$

This can be simplified to

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2nr(S_{f,2})\}.$$

Finally, to make this bigraded, we need to add in the horizontal grading. Since $H_{m+n}(S)$ lies in horizontal grading $2r(S)$, $H_n(S_{f,1})$ lies in horizontal grading $2r(S_{f,1})$, and $H_m(S_{f,2})$ lies in horizontal grading $2r(S_{f,2})$, we see that the tensor product

$$H_n(S_{f,1}) \otimes H_m(S_{f,2})$$

always lies in horizontal grading $2r(S_{f,1}) + 2r(S_{f,2})$. Since $r(S_{f,1}) + r(S_{f,2}) = r(S)$, it follows that the tensor product lies in grading $2r(S)$ — the same grading as $H_{m+n}(S)$.

Thus, we can add the horizontal grading to the composition product formula, with no grading shift required for each labeling f. The bigraded formula is then

$$H_{m+n}(S) = \bigoplus_{f \in L(S)} H_n(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2n\,r(S_{f,2}), 0\},$$

where the bigrading is given by $(\mathrm{gr}_q, \mathrm{gr}_h)$.

3.3 A categorification for HOMFLY-PT homology

In relating these formulas to knot Floer homology, we will be most interested in the case when $n = 1$. In this case, the previous formula becomes

$$H_{m+1}(S) = \bigoplus_{f \in L(S)} H_1(S_{f,1}) \otimes H_m(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}. \tag{3}$$

In this section we will prove a generalization of this formula to HOMFLY-PT homology. Letting $H_H(S)$ denote the HOMFLY-PT homology with the standard bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$, define $H_H(S)\langle k\rangle$ to be HOMFLY-PT homology with a new grading $(\mathrm{gr}_q + k\,\mathrm{gr}_h, \mathrm{gr}_h)$.

Theorem 3.2 *There is an isomorphism of bigraded vector spaces*

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\} \cong H_H(S)\langle 1\rangle.$$

The proof of this theorem will rely heavily the fact that for all $n \geq 1$, there is a differential $d_-(n)$ on $H_H(S)$ which is homogeneous with bigrading $\{2n, -2\}$, and

$$H_*(H_H(S), d_-(n)) \cong H_n(S),$$

where we are viewing $H_n(S)$ as a bigraded vector space. (This is the homology $H_n^{\pm}(S)$ from Section 2.3.) Recall that as a bigraded vector space, $H_n(S)$ lies in a single horizontal grading, namely $\mathrm{gr}_h = 2r(S)$.

The theorem will be proved in two parts. First we will show that any bigraded vector space with certain algebraic properties must be isomorphic to $H_H(S)\langle k\rangle$, and then we will show that our construction satisfies those properties for $k = 1$.

Lemma 3.3 *Let $H(S)$ denote a bigraded vector space over $\mathbb{Q}$ with the following properties:*

(1) *$H(S)$ is bounded above and below in horizontal grading.*

(2) *$H(S)$ is bounded below in q–grading.*

(3) *$H(S)$ is finite-dimensional in each bigrading.*

(4) *For each $n \geq n_0$, there is a differential d_n on $H(S)$ which is homogeneous with bigrading $\{2n, -2\}$ such that $H_*(H(S), d_n) \cong H_n(S)$, where the isomorphism is as bigraded vector spaces.*

Then $H(S) \cong H_H(S)$.

Note that HOMFLY-PT homology itself satisfies these conditions, so they are not vacuous.

Proof Let $H^{i,j}$ denote the homology in bigrading $\{i, j\}$, and similarly for $H_H^{i,j}(S)$. The lemma states that for any integers i and j, $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$.

Suppose for some i and j, the dimensions do not agree. Since $H(S)$ is bounded in horizontal grading, we can take choose i_0 and j_0 so that j_0 is minimized subject to the constraint that $\dim(H^{i_0,j_0}(S)) \neq \dim(H_H^{i_0,j_0}(S))$. Note that the choice for i_0 may not be unique.

Since both $H(S)$ and $H_H(S)$ are bounded below in q–grading, there exists a constant $a(S)$ such that for all $i \leq a(S)$, $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S)) = 0$.

Choose $n > \max(i_0 - a, n_0)$. We will now use the fact that $H_*(H(S), d_n) \cong H_n(S)$ as bigraded vector spaces. We can rewrite $\mathfrak{sl}_n$ homology as the homology of $H_H(S)$ with respect to $d_-(n)$:

$$H_*(H(S), d_n) \cong H_*(H_H(S), d_-(n)).$$

The differentials on both of these complexes have bigrading $\{2n, -2\}$. We can put an equivalence relation on $\mathbb{Z}^2$, where $(i, j) \sim (i', j')$ if $(i - i', j - j') = k(2n, -2)$, and both complexes must split according to this equivalence relation. In other words, for a fixed equivalence class A, the sum

$$\bigoplus_{(i,j)\in A} H^{i,j}(S)$$

gives a subcomplex of $(H(S), d_n)$, and similarly for $(H_H(S), d_-(n))$.

Consider the summand corresponding to the equivalence class of (i_0, j_0). For the complex $(H(S), d_n)$, this summand looks like

$$\cdots \xrightarrow{d_n} H^{i_0-2n, j_0+2}(S) \xrightarrow{d_n} H^{i_0,j_0}(S) \xrightarrow{d_n} H^{i_0+2n, j_0-2}(S) \xrightarrow{d_n} \cdots$$

and, for $(H_H(S), d_-(n))$,

$$\cdots \xrightarrow{d_-(n)} H_H^{i_0-2n,j_0+2}(S) \xrightarrow{d_-(n)} H_H^{i_0,j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n,j_0-2}(S) \xrightarrow{d_-(n)} \cdots .$$

Now, since the complex is bounded below in q–grading and we've chosen n sufficiently large, all of the chain groups before the (i_0, j_0) summand are trivial, so the complexes become

$$\cdots 0 \xrightarrow{d_n} 0 \xrightarrow{d_n} H^{i_0,j_0}(S) \xrightarrow{d_n} H^{i_0+2n,j_0-2}(S) \xrightarrow{d_n} \cdots$$

and

$$\cdots 0 \xrightarrow{d_-(n)} 0 \xrightarrow{d_-(n)} H_H^{i_0,j_0}(S) \xrightarrow{d_-(n)} H_H^{i_0+2n,j_0-2}(S) \xrightarrow{d_-(n)} \cdots .$$

Since both complexes are bounded in horizontal grading, the two chain complexes are both finitely generated. They therefore have a well-defined Euler characteristic, and since the homologies are isomorphic, the Euler characteristics must be the same. Since the alternating sum of the dimension of homology is the same as the alternating sum of the dimension of the chain groups themselves, we have

$$\sum_{k=0}^{N}(-1)^k \dim(H^{i_0+2nk,j_0-2k}(S)) = \sum_{k=0}^{N}(-1)^k \dim(H_H^{i_0+2nk,j_0-2k}(S)).$$

Furthermore, we know that there is an equality $\dim(H^{i,j}(S)) = \dim(H_H^{i,j}(S))$ for $j < j_0$. Thus, the two subcomplexes above have the same dimension in all of the bigradings except bigrading (i_0, j_0). But this means that for $k \geq 1$, the terms in the two sums are equal, which forces the $k = 0$ terms to be equal. This contradicts our assumption that $\dim(H^{i_0,j_0}(S)) \neq \dim(H_H^{i_0,j_0}(S))$, proving the isomorphism. □

Corollary 3.4 *Let $H(S)$ be a bigraded vector space over $\mathbb{Q}$ that satisfies the conditions in Lemma 3.3 with one difference — instead of d_n having bigrading $\{2n, -2\}$, it has bigrading $\{2n-2k, -2\}$. Then $H(S) \cong H_H(S)\langle k\rangle$.*

Proof After the change in grading $(\mathrm{gr}_q, \mathrm{gr}_h) \mapsto (\mathrm{gr}_q + k\,\mathrm{gr}_h, \mathrm{gr}_h)$, the conditions of Lemma 3.3 are satisfied, so this homology is isomorphic to $H_H(S)$. Shifting back to the original gradings proves the corollary. □

Proof of Theorem 3.2 To prove Theorem 3.2, we need to show that our homology

$$\bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}$$

satisfies the conditions of this corollary for $k = 1$. It satisfies Lemma 3.3(1)–(3) since each of the summands do, so we just need to define the d_n differentials.

We will define d_n for $n \geq 2$ as follows. It will preserve the direct sum decomposition, and it will act on each $H_1(S_{f,1}) \otimes H_H(S_{f,2})$ summand by $1 \otimes d_-(n-1)$, where $d_-(n-1)$ is the standard $\mathfrak{sl}_{n-1}$ differential on the HOMFLY-PT homology of $S_{f,2}$.

Since $H_*(H_H(S_{f,2}), d_-(n-1)) \cong H_{n-1}(S_{f,2})$, the homology with respect to d_n is

$$\bigoplus_f H_1(S_{f,1}) \otimes H_{n-1}(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}.$$

From Section 3.2, we know that this sum is isomorphic as a bigraded vector space to $H_n(S)$. The differential d_n has bigrading $\{2n-2, -2\}$, so applying Corollary 3.4, this proves Theorem 3.2. □

4 The knot Floer complex at a vertex in the cube of resolutions

4.1 Definition of the complex

We will assume that the reader is familiar with Heegaard diagrams and knot Floer homology. For background on the subject, refer to [13; 14; 17]. The oriented cube of resolutions for HFK was originally defined with twisted coefficients by Ozsváth and Szabó, and they noted some similarities between their complex and HOMFLY-PT homology [15]. The complex was further studied by Gilmore, who reframed the relationships in terms of framed trivalent graphs [5].

However, we will be dealing with the untwisted version defined by Manolescu [12]. This is in some ways the most natural version, as it doesn't involve twisted coefficients, and the total homology of the complex is the usual knot Floer homology. The oriented cube of resolutions uses the Heegaard diagram $\mathcal{H}(S)$ shown in Figure 7.

Note that in [12], there is a marked bivalent vertex on the leftmost strand in the braid at which an α and a β circle are removed. Our picture is slightly different — instead of having a marked edge, we place an additional X and O outside of our braid; we will denote them by X_{new} and O_{new}. Since discs are not allowed to pass through X_{new}, this can be viewed as puncturing the sphere, making our diagram a truly planar diagram. We will also set U_{new} equal to zero to avoid increasing the ground ring.

The knot Floer complex corresponding to this Heegaard diagram is denoted by $CFK^-(S)$. There are several versions of knot Floer homology, and the minus sign

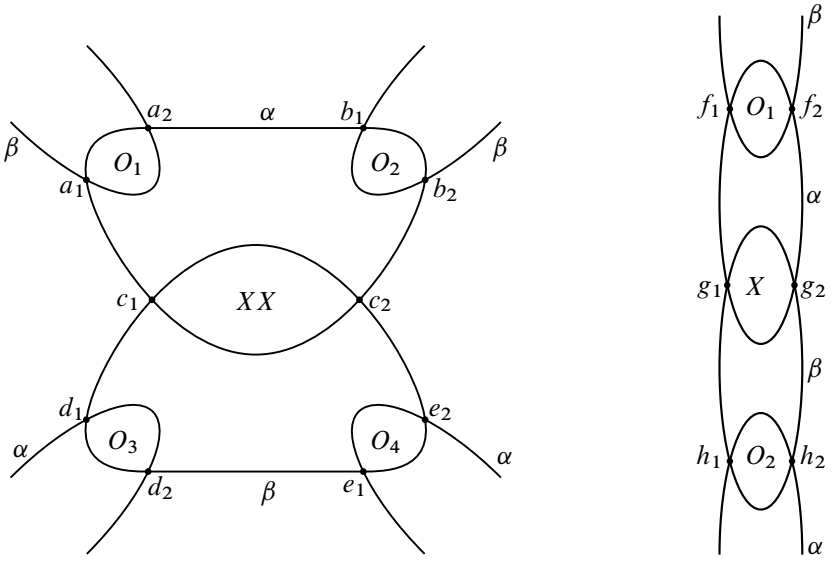


Figure 7: The local Heegaard diagram for a singular link at a 4–valent vertex (left) and a 2–valent vertex (right)

refers to the fact that none of the U_i in the ground ring will be set to 0 (except for U_{new}). For consistency with the HOMFLY-PT and $\mathfrak{sl}_n$ definitions, we will be working with $\mathbb{Q}$ coefficients.

We can relate the complex $CFK^-(S)$ to the complex from Manolescu's Heegaard diagram as follows. Place a bivalent vertex on the leftmost strand of the braid, and let α_1 and β_1 be the corresponding α and β circles. We can handleslide α_1 over all the other α circles and β_1 over all of the other β circles, so that we are left with Manolescu's diagram together with the diagram shown in Figure 8. The complex for this diagram is homotopy equivalent to the tensor product of Manolescu's complex with $H_*(S^1)$, so its homology has twice the rank of Manolescu's. (This can also be seen by Manolescu's formula for disjoint union of singular braids [12, page 198].)

Remark 4.1 Strictly speaking, this complex also depends on some auxiliary information, including a choice of a path of almost complex structures and orientations on the moduli spaces of the holomorphic disks, called a *system of orientations*. There are many choices for orienting these moduli spaces, but it was shown by Alishahi and Eftekhary that there is always a system of orientations such that Maslov index 2 α–degenerations come with positive sign, and the β–degenerations come with negative sign (see [1, Section 5.1]). Note that CFK^- is a special case of their sutured HF^-

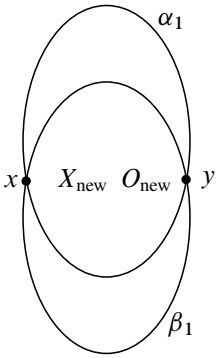


Figure 8: The extra unknot outside the braid diagram

construction. Sarkar [20] showed that any two systems of orientations satisfying this property give chain homotopy equivalent complexes. We will leave out any discussion about complex structure, since none of the discs counted in our computations depend on the complex structure.

The complex ascribed to a vertex in the cube of resolutions, which we will denote by $C_F(S)$, is the tensor product of $CFK^-(S)$ with a certain Koszul complex. Using the terminology from the previous section, we can define $C_F(S)$ as

$$C_F(S) = CFK^-(S) \otimes \bigotimes_{v \in V_4(S)} (R \xrightarrow{L(v)} R).$$

We denote the Koszul complex by $K(S)$. The bigrading on $C_F(S)$ will be described in Section 4.2.3.

4.1.1 The generators of $CFK^-(S)$ In order to understand the homology of $C_F(S)$, we are going to need some tools for understanding $CFK^-(S)$. Let $E(S)$ denote the set of edges of S, and let x be a generator of the complex $CFK^-(S)$ (ie an n–tuple of intersection points of the α and β curves). We ascribe a subset Z of E to the generator x as follows.

Each O_i in the Heegaard diagram is contained in a unique minimal bigon. The boundary of this bigon contains two intersection points — if either of these intersection points are in the n–tuple x, then e_i is in Z. For example, in Figure 7, left, there are five types of generators: (a,d), (a,e), (b,d), (b,e) and (c). The underlying sets of edges of these generators are e_1e_3, e_1e_4, e_2e_3, e_2e_4 and $\varnothing$, respectively.

As observed in [15], Z must satisfy two conditions. First, for any vertex v in S, the number of incoming edges in Z must equal the number of outgoing edges in Z, and second, Z cannot contain all four edges at any 4–valent vertex in S. In other words, Z must be a disjoint union of oriented circles contained in S. We call such a set of edges a *multicycle*. Note that multicycles differ from the homological cycles in Section 3.1 in that multicycles cannot contain all four edges at a vertex.

Let $CFK^-(Z)$ denote the R–module spanned by generators x where the multicycle underlying x is Z, and let $C_F(Z)$ be the tensor product of $CFK^-(Z)$ with the Kozsul complex:

$$C_F(Z) = CFK^-(Z) \otimes \bigotimes_{v \in V_4(S)} (R \xrightarrow{L(v)} R).$$

4.2 The filtered complex and the spectral sequence from HOMFLY-PT homology to *HFK*

4.2.1 A filtration on $CFK^-(S)$ It turns out that there is a filtration on $CFK^-(S)$ that divides generators according to their underlying cycles. In other words, if there is a filtration-preserving differential from x to y, then x and y have the same underlying cycle.

In [15, Section 3], Ozsváth and Szabó choose a distinguished generator x_0 of $CFK^-(S)$ corresponding to the empty cycle. Using Figure 9, this generator is defined to contain the intersection point c_1 at each 4–valent vertex and the intersection point g_1 at each bivalent vertex.

The filtration is induced by placing additional basepoints p_i in our Heegaard diagram, as shown in Figure 9. The markings p_i are in canonical bijection with regions in $\mathbb{R}^2 - S$. Let N denote the number of such regions.

Definition 4.2 The *basepoint filtration* is a map

$$\mathcal{F}\colon \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathbb{Z}^N.$$

The i^{th} component of $\mathcal{F}(x)$ is defined to be the multiplicity of ϕ at p_i, where ϕ is a homotopy class in $\pi_2(x, x_0)$ with multiplicity 0 at X_{new}. The map $\mathcal{F}$ extends to a (multi)filtration on $CFK^-(S)$ with multiplication by U_j preserving the filtration level.

Lemma 4.3 *The filtration $\mathcal{F}$ is well defined, ie it does not depend on the choice of ϕ.*

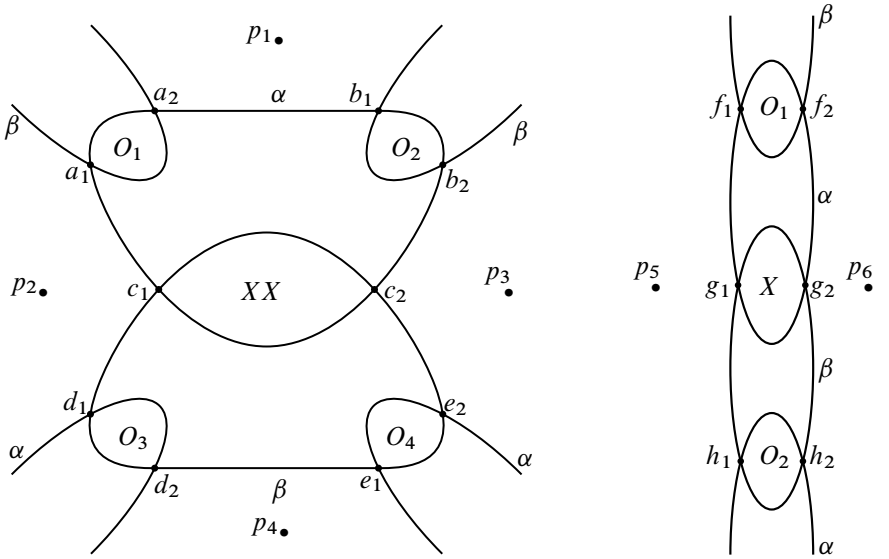


Figure 9: Local diagrams with additional markings

Proof Since any two such ϕ differ by a periodic domain, it is sufficient to show that any periodic domain with multiplicity 0 at X_{new} has multiplicity 0 at these markings. This follows from that fact that for any α or β circle, the markings and X_{new} lie on the same side. So, for any periodic domain, the multiplicity at any of these points is the same as that of X_{new}, which is required to be 0. □

Let $x(Z)$ denote the generator which, at each v not in Z, has the same intersection point as x_0 and for each e_i in Z has the leftmost generator on the bigon containing O_i.

Lemma 4.4 *If x is $CFK^-(S)$ with underlying cycle Z, then $\mathcal{F}(x) = \mathcal{F}(x(Z))$.*

Proof We will build a homotopy class $\phi \in \pi_2(x, x(Z))$ such that ϕ has multiplicity 0 at every p_i. Start with ϕ being constant on x. For each bivalent vertex v not in Z, if y has the left intersection point, we do nothing, and if x has the right intersection point, we add one of the two bigons which goes from the right intersection point to the left intersection point. In Figure 9, these are the bigons from g_2 to g_1 which pass through O_1 and O_2, respectively.

Similarly, for each 4–valent vertex v not in Z and each edge e in Z where x has a different intersection point from $x(Z)$, we can find a bigon from the x intersection point to the $x(Z)$ intersection point. Defining ϕ_1 to be the union of these bigons, we get a homotopy class from x to $x(Z)$ which does not pass through any of the basepoints. □

Define F: {multicycles} $\to \mathbb{Z}^N$ as follows. Let Z be a multicycle, and let C be a 2–chain in S^2 with boundary Z. If we require that C has multiplicity 0 on the outer region (the one corresponding to X_{new}), it is clear that this 2–chain is unique. Define the i^{th} component of $F(Z)$ to be the multiplicity of C at the region corresponding to p_i.

Lemma 4.5 *If x is a generator of $CFK^-(S)$ with underling cycle Z, then $\mathcal{F}(x) = F(Z)$.*

Proof We have already shown that $\mathcal{F}(x) = \mathcal{F}(x(Z))$, so it suffices to show that $\mathcal{F}(x(Z)) = F(Z)$.

Write Z in terms of its components $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_k$. For each i, Ozsváth and Szabó identify two Maslov index 1 homotopy classes $\phi_1^i(Z_i)$ and $\phi_2^i(Z_i)$ in $\pi_2(x(Z), x(Z - Z_i))$ (see [15, Section 3]). Let $\phi_1(Z)$ be the homotopy class in $\pi_2(x(Z), x_0)$ obtained by composing the $\phi_1^i(Z_i)$ homotopy classes. The multiplicities of $\phi_1(Z)$ at each basepoint are equal to the multiplicity of C in that region, so $\mathcal{F}(x(Z)) = F(Z)$. □

Remark 4.6 Although there were two choices of homotopy $\phi_1^i(Z_i)$ and $\phi_2^i(Z_i)$ for each component of the cycle Z_i, the two classes have the same multiplicity at all of the basepoints, so it doesn't matter which one we choose.

Corollary 4.7 *Two generators x and y of $CFK^-(Z)$ have the same filtration level if and only if they have the same underlying cycle.*

Proof If x and y have the same underlying cycle Z, then $\mathcal{F}(x) = F(Z) = \mathcal{F}(y)$. For the other direction, suppose $\mathcal{F}(x) = \mathcal{F}(y)$, and let Z_x and Z_y be their underlying cycles. Then $F(Z_x) = \mathcal{F}(x) = \mathcal{F}(y) = F(Z_y)$. But F is injective, so this implies $Z_x = Z_y$. □

We extend this filtration to $C_F(S)$ by placing the whole Koszul complex in filtration level $\{0, 0, \ldots, 0\}$. The $\mathbb{Z}^N$–filtration can be turned into a $\mathbb{Z}$–filtration by summing over the components of $\mathcal{F}$. Let d_k denote the component of the differential on $C_F(S)$ which decreases the $\mathbb{Z}$–filtration by k.

Corollary 4.8 *The differential d_0 preserves $C_F(Z)$, ie it does not change the underlying cycle of a generator.*

This corollary tells us that the filtered complex $(C_F(S), d_0)$ splits as a direct sum

$$(C_F(S), d_0) = \bigoplus_Z (C_F(Z), d_0).$$

4.2.2 Homology of a cycle Before computing the homology $H(C_F(Z), d_0)$, we will need a definition. If S is a singular braid and Z is a multicycle in S, let $S - Z$ denote the diagram obtained by removing all edges in Z from S. Note that $S - Z$ is still a singular braid because Z is an oriented cycle in the graph.

Given a cycle Z, the complex $CFK^-(Z)$ is easy to compute. Each intersection point in the Heegaard diagram lies on a unique convex bigon (convex in the traditional planar geometry sense), and this bigon either contains an X, an XX or a O_i. There are canonical bijections between the O_i bigons and the edges e_i, between the X bigons and $V_2(S)$, and between the XX bigons and $V_4(S)$.

Given a generator x, let $W_2(x)$ denote the set of vertices at which x has an intersection point on one of the X bigons, and let $W_4(x)$ denote the set of vertices at which x has an intersection point on one of the XX bigons. $W_2(x)$ and $W_4(x)$ are uniquely determined by the underlying cycle Z of x. In particular, $W_2(x)$ and $W_4(x)$ are those vertices which are not endpoints of any edges in Z. We can therefore define $W_2(Z)$ and $W_4(Z)$ accordingly. Note that x is uniquely determined by a choice of one of the two corresponding intersection points at each edge e in $\mathbb{Z}$, each vertex v in $W_2(Z)$ and each vertex v in $W_4(Z)$.

The complex for a cycle Z can now be described as follows. Each edge e_i in Z corresponds to two intersection points, which are connected by a bigon containing O_i. These are the only filtered differentials involving these two intersection points, so $CFK^-(Z)$ is going to come with a tensor factor of the Koszul complex

$$\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R.$$

Remark 4.9 The fact that bigons are the only discs that contribute can be seen by a Maslov index argument — in fact, bigons are the only Maslov index one 2–chains in $\pi_2(x, y)$ for any two generators x and y in $CFK^-(Z)$. We show this in detail (and in greater generality) in Lemma 4.18.

Each vertex v in $W_2(Z)$ also corresponds to two intersection points. They are connected by two bigons, one which passes through O_i (where e_i is the outgoing edge from v)

and one which passes through O_j (where e_j is the incoming edge at v). These two bigons will give a coefficient of $\pm(U_i - U_j)$. Thus, we also get a tensor factor of the Koszul complex

$$\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R.$$

Proving that the signs are correct requires slightly more advanced machinery, and will be discussed at the end of the section.

Finally, the vertices v in $W_4(Z)$ correspond to two intersection points, also connected by two bigons. One passes through O_i and O_j, where e_i and e_j are the outgoing edges of v, and the other passes through O_k and O_l, where e_k and e_l are the incoming edges at v. These two bigons will contribute a coefficient of $\pm(U_iU_j - U_kU_l)$, giving us the last Koszul complex

$$\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R.$$

These are all the generators and all the differentials, so the total complex is given by

$$CFK^-(Z) = \left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R\right] \tag{4}$$

and so the total complex for $C_F(Z)$ is given by

$$C_F(Z) = \left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R\right] \otimes \left[\bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R\right].$$

Lemma 4.10 *The filtered homology $H_*(C_F(Z), d_0)$ is isomorphic to $H_H(S - Z)$.*

Proof The U_i in the first tensor product form a regular sequence in R, so we can cancel all of these differentials. The resulting complex is chain homotopy equivalent to the original complex over $\mathbb{Q}$. This has the effect of setting U_i equal to zero for all e_i in Z. Let R_Z be the quotient $R/\{U_i = 0 \text{ for } e_i \in Z\}$. Note that this is precisely the ground ring for the singular braid $S - Z$.

We are left with the complex

$$\left[\bigotimes_{v \in W_2(Z)} R_Z \xrightarrow{L(v)} R_Z\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R_Z \xrightarrow{Q(v)} R_Z\right] \otimes \left[\bigotimes_{v \in V_4(S)} R_Z \xrightarrow{L(v)} R_Z\right].$$

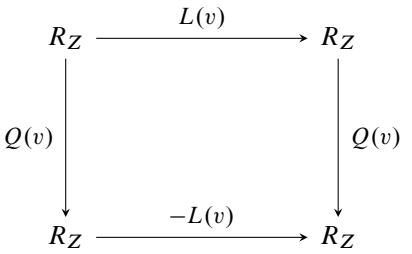


Figure 10: The complex $C_F(Z)$ at a vertex $v \in V_4(S-Z)$

For each 4–valent vertex v in $S-Z$, we have tensor factors $R_Z \xrightarrow{L(v)} R_Z$ and $R_Z \xrightarrow{Q(v)} R_Z$, which together give a factor of the complex shown in Figure 10, which is precisely the HOMFLY-PT complex $C_H(v)$.

For 2–valent vertices v in $S-Z$, there are two possibilities to consider — v is 2–valent in S ($v \in W_2(Z)$), and v is 4–valent in S ($v \in V_4(S), v \notin W_4(Z)$). When v is 2–valent in S, we get the factor

$$R_Z \xrightarrow{L(v)} R_Z,$$

which is again the HOMFLY-PT complex $C_H(v)$ for $S-Z$. For v 4–valent in S, let e_i and e_j be the outgoing edges at v, and e_k and e_l the incoming edges at v. Since $S-Z$ is 2–valent at v, we know that Z must include one outgoing edge and one incoming edge. Without loss of generality, assume they are e_i and e_k. To avoid confusion, we will write out the terms of the linear elements, as $L(v)$ refers to $U_i+U_j-U_k-U_l$ in S, while $L(v)$ refers to U_j-U_l in $S-Z$.

In $C_F(Z)$, we have the factor

$$R_Z \xrightarrow{U_i+U_j-U_k-U_l} R_Z.$$

In the HOMFLY-PT complex for $S-Z$, on the other hand, we have the factor

$$R_Z \xrightarrow{U_j-U_l} R_Z.$$

Fortunately, since e_i and e_k are in Z, U_i and U_k are zero in R_Z, so $U_i+U_j-U_k-U_l = U_j-U_l$, making the above complexes isomorphic.

Thus, after canceling the Koszul complex on the edges in Z, we get exactly the HOMFLY-PT complex for $S-Z$. It follows that $H_*(C_F(Z), d_0) \cong H_H(S-Z)$. □

Corollary 4.11 *The filtered homology decomposes as the direct sum*

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S-Z).$$

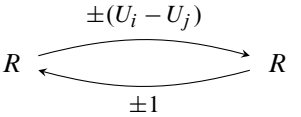


Figure 11: Local complex when allowing discs to pass through the X basepoint

Remark 4.12 (signs) Since we are discussing Koszul complexes, the $\pm$ in the terms $\pm(U_i - U_j)$ and $\pm(U_iU_j - U_kU_l)$ are not relevant. Some will have to come with positive signs and some with negative to make $d^2 = 0$, but where they are doesn't impact the chain homotopy type. What we need to show is that the two bigons in each case come with *different* signs.

Each two-valent vertex corresponds to a specific X marking in the diagram. This X lies within the same α circle as O_i and the same β circle as O_j. In CFK^-, we do not allow discs to pass through the X basepoints. However, if we do allow them to pass through only this X, we get a new complex. In this complex, d^2 is nonzero — instead, it is a multiple of the identity. This multiple is determined by the α and β degenerations, which will correspond to the α and β circles containing X. Since the α circle contains O_i, it gives a coefficient of U_i, and similarly, the β circle gives a coefficient of U_j. Since we chose a system of orientations such that the α and β degenerations come with opposite signs, this gives

$$d^2 = \pm(U_i - U_j)I.$$

Moreover, the additional differentials are also subject to the basepoint filtration, so we get

$$d_0^2 = \pm(U_i - U_j)I.$$

This X basepoint lies inside a minimal bigon, and this bigon now contributes to the differential with a coefficient of ± 1. The local contribution therefore must be the complex in Figure 11, so U_i and U_j must come with opposite sign.

The argument for the quadratic term is the same, only instead of allowing discs to pass through an X, we are allowing them to pass through an XX. The α degeneration is U_iU_j and the β degeneration is U_kU_l, and they must come with opposite sign, so we get the complex in Figure 12, which proves that U_iU_j and U_kU_l come with opposite sign.

4.2.3 Gradings The knot Floer complex comes equipped with two gradings: the Maslov grading M and the Alexander grading A. The differential decreases the Maslov

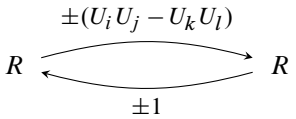


Figure 12: Local complex when allowing discs to pass through the XX basepoint

grading by 1 and preserves the Alexander grading. Multiplication by U_i decreases the Maslov grading by 2 and decreases the Alexander grading by 1.

Certain linear combinations of the Maslov and Alexander gradings return analogs of the quantum and horizontal gradings from the Khovanov–Rozansky complex. Let gr_q be given by $-2M+2A$ and gr_h by $-2M+4A$. Note that the knot Floer differential has bigrading $\{2,2\}$ with respect to this differential and multiplication by U_i changes the bigrading by $\{2,0\}$, the same as the Khovanov–Rozansky complex. Instead of the Maslov and Alexander gradings, we will henceforth use the quantum and horizontal gradings.

Before computing gradings, we need to introduce some terminology. For a multicycle Z, let $T_1(Z)$ denote the number of vertices $v \in V_4(S)$ at which Z contains the edges e_1 and e_3 in Figure 13. Similarly, let $D_1(Z)$ denote the number of vertices at which Z contains the edges e_1 and e_4, $D_2(Z)$ the number of vertices at which Z contains the edges e_2 and e_3, and $T_2(Z)$ the number of vertices at which Z contains the edges e_2 and e_4.

We will now compute the bigrading on the knot Floer complex to get a graded version of Corollary 4.11. Since until now we have only defined our complex up to an overall grading shift, the following definition pins down the absolute bigrading on $C_F(S)$. Recall that the subcomplex corresponding to the empty cycle $C_F(Z_\varnothing)$ is canonically isomorphic to the HOMFLY-PT complex $C_H(S)$, and with the new gradings $(\mathrm{gr}_q, \mathrm{gr}_h)$, they are isomorphic as bigraded complexes up to an overall grading shift.

Definition 4.13 We define the bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$ on $C_F(S)$ so that the subcomplex $C_F(Z_\varnothing)$ is isomorphic to $C_H(S)\{-2r(S), 0\}$ as bigraded chain complexes, where $Z_\varnothing$ is the empty cycle.

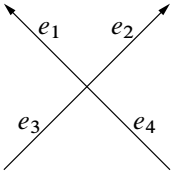


Figure 13: A labeled 4–valent vertex

Let Z denote a k–component multicycle in S. Viewing Z as a braid diagram for the k–component unlink, we can define $r(Z)$ to be the rotation number of Z, so $r(Z) = -k$. As before, let $x(Z)$ denote the generator corresponding to Z at the bottom of the Koszul complex (ie with the largest horizontal grading). Similarly, let x_0 denote the generator corresponding to the empty cycle with the largest horizontal grading. In the proof of Lemma 4.5 we utilize a set of k Maslov index one discs whose composition takes $x(Z)$ to x_0 defined by Ozsváth and Szabó [15]. This composition has coefficient which is a polynomial in R of degree $T_2(Z) + \frac{1}{2}(D_1(Z) + D_2(Z))$. Using the fact that differentials have bigrading $\{2, 2\}$ and that this is a composition of k differentials, we can see that $x(Z)$ and x_0 differ in grading by

$$\{2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}.$$

The bottom generator of the HOMFLY-PT complex has bigrading $\{-|V_4(S)|, 0\}$, so x_0 has bigrading $\{-2r(S) - |V_4(S)|, 0\}$. Thus, $x(Z)$ has bigrading

$$\begin{aligned}\{-2r(S) - |V_4(S)| + 2r(Z) + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}& \\ = \{-2r(S - Z) - |V_4(S)| + 2T_2(Z) + D_1(Z) + D_2(Z), 2r(Z)\}.&\end{aligned}$$

The bottom generator of the HOMFLY-PT complex for $S - Z$ has bigrading equal to $\{-|V_4(S - Z)|, 0\}$, so we get

$$\begin{aligned}&H(C_F(Z), d_0) \cong \\ &H_H(S-Z)\{-2r(S-Z)+|V_4(S-Z)|-|V_4(S)|+2T_2(Z)+D_1(Z)+D_2(Z), 2r(Z)\}.\end{aligned}$$

The grading shift in this formula can be simplified somewhat. The quantity $|V_4(S-Z)|-|V_4(S)|$ is the negative of the number of 4–valent vertices in S at which Z contains two edges:

$$|V_4(S - Z)| - |V_4(S)| = -T_1(Z) - T_2(Z) - D_1(Z) - D_2(Z).$$

Thus, the formula becomes

$$H(C_F(Z), d_0) \cong H_H(S - Z)\{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}$$

and we get a graded version of Corollary 4.11:

$$H_*(C_F(S), d_0) \cong \bigoplus_Z H_H(S - Z)\{-2r(S - Z) + T_2(Z) - T_1(Z), 2r(Z)\}.$$

We will be able to connect this formula to the composition product with the following lemma:

Lemma 4.14 *Let f be a labeling of S. The $\mathfrak{sl}_1$ homology of $S_{f,1}$ is given by*

$$H_1(S_{f,1}) = \begin{cases} \mathbb{Q}\{0, 2r(S_{f,1})\} & \text{if } S_{f,1} \text{ is a multicycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Applying this lemma, the formula becomes

$$H_*(C_F(S), d_0) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_H(S_{f,2})\{T_2(S_{f,1}) - T_1(S_{f,1}) - 2r(S_{f,2}), 0\}.$$

But by Theorem 3.2, this is isomorphic to $H_H(S)\langle 1\rangle$. Thus, we have proved the following theorem:

Theorem 4.15 *There is an isomorphism of bigraded groups*

$$H_*(C_F(S), d_0) \cong H_H(S)\langle 1\rangle.$$

Corollary 4.16 *There is a spectral sequence whose E_1 page is $H_H(S)\langle 1\rangle$ and which converges to $H_F(S)$.*

Proof This is just the spectral sequence induced by the basepoint filtration on $C_F(S)$. □

Manolescu's conjecture is thus equivalent to this spectral sequence collapsing at the E_1 page.

4.3 Additional differentials and the spectral sequences from *HFK* to $\mathfrak{sl}_n$

We are going to add differentials to the complex $C_F(S)$ so that the total homology is isomorphic to $H_{n+1}(S)$ for any $n \geq 1$. These new differentials do not preserve the Alexander grading, so using the Alexander grading as a filtration, this induces a spectral sequence from $H_F(S)$ to $H_{n+1}(S)$.

The complex $C_F(S)$ is constructed as a tensor product of complexes $CFK^-(S)$ and a Koszul complex $K(S)$ on linear elements:

$$C_F(S) = CFK^-(S) \otimes K(S).$$

The complex $CFK^-(S)$ does not count discs which pass through the X or XX markings. For the new differential, we are going to count these discs with certain polynomial coefficients.

Each X marking in the Heegaard diagram corresponds to a 2–valent vertex v in S. Whenever a holomorphic disc passes through this X with multiplicity k, it picks up a coefficient of $u_1(v)^k$. The only exception is the special marking X_{new}, at which we still require discs to have multiplicity 0. Similarly, each XX corresponds to a 4–valent vertex v in S. If a holomorphic disc passes through this XX with multiplicity k, it picks up a coefficient of $u_2(v)^k$. We will call this new complex $CFK_n^-(S)$.

Note that there is no guarantee that the differential on this complex squares to zero — in fact, it doesn't. To fix this, we will also modify the differential on the Koszul complex. Originally, it was given by

$$K(S) = \bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R.$$

We are going to add in differentials to make it a matrix factorization:

$$K_n(S) = \bigotimes_{v \in V_4(S)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R.$$

The total complex $C_{F(n)}(S)$ is defined to be the tensor product of $CFK_n^-(S)$ and $K_n(S)$,

$$C_{F(n)}(S) = CFK_n^-(S) \otimes K_n(S).$$

Lemma 4.17 *The differential on $C_{F(n)}(S)$ satisfies $d^2 = 0$.*

Proof At each vertex v in S, we will show that d^2 has a contribution of $w_n(v)I$, with

$$w_n(v) = \sum_{e_i \in E_{\text{out}}} U_i^{n+1} - \sum_{e_j \in E_{\text{in}}} U_j^{n+1}.$$

The lemma will then follow from the fact that $\sum_{v \in S} w_n(v)I = 0$.

The quantity d^2 has two contributions, one from $CFK_n^-(S)$ and one from $K_n(S)$. The contribution from $K_n(S)$ can be computed directly to be

$$\sum_{v \in V_4(S)} L(v)u_1(v).$$

The contribution from $CFK_n^-(S)$ can be computed via the α and β degenerations. We orient the moduli spaces so that the α and β degenerations come with opposite signs, with the α degenerations being positive and the β degenerations negative (see Remark 4.1 for an explanation of the signs).

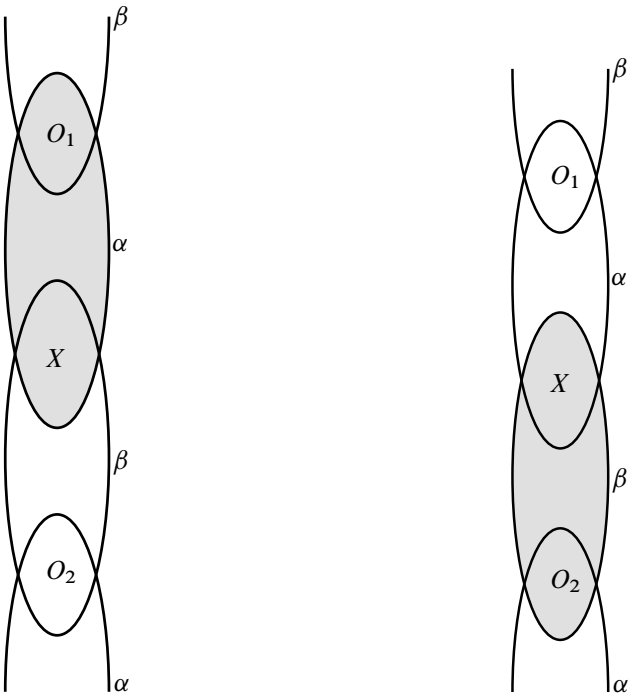


Figure 14: α–degenerations (left) and β–degenerations (right) at a bivalent vertex

At each 2–valent vertex v in S, we have one α circle and one β circle, shown in Figure 14 . The α circle contains U_i and X, and the X contributes coefficient $u_1(v)$, so the α degeneration contributes $U_i u_1(v)$. Similarly, the β circle contains U_j and X, so its contribution is $-U_j u_1(v)$. Thus, the net contribution at v is $(U_i - U_j)u_1(v)$. This can be simplified to $L(v)u_1(v) = w_n(v)$.

At each 4–valent vertex v in S, we also have one α circle and one β circle, shown in Figure 15. The α circle contains U_i, U_j and XX, and the XX contributes coefficient $u_2(v)$, so its contribution is $U_i U_j u_2(v)$. The β circle contains U_k, U_l and XX, so its contribution is $-U_k U_l u_2(v)$. Thus, the net contribution at v is $(U_i U_j - U_k U_l)u_2(v)$. This can be simplified to $Q(v)u_2(v)$.

Thus, counting the contribution from $K_n(S)$, we see that d^2 is given by

$$\begin{aligned} d^2 &= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} Q(v)u_2(v) + \sum_{v \in V_4(S)} L(v)u_1(v) \\ &= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} L(v)u_1(v) + Q(v)u_2(v) \end{aligned}$$

$$= \sum_{v \in V_2(S)} w_n(v) + \sum_{v \in V_4(S)} w_n(v)$$
$$= \sum_{v \in S} w_n(v) = 0.$$

This concludes the proof. □

We can extend the ($\mathbb{Z}$–valued) basepoint filtration from Section 4.2.1 to make $C_{F(n)}(S)$ a filtered complex — since we still require discs to have multiplicity 0 at X_{new}, the same argument works as in the proof of Lemma 4.3. As before, let d_i denote the differentials which decrease the filtration level by i. Since d_0 must preserve multicycles, the homology $H(C_{F(n)}(S), d_0)$ splits over the multicycles

$$H(C_{F(n)}(S), d_0) = \bigoplus_Z H(C_{F(n)}(Z), d_0).$$

We want to compute the complex $C_{F(n)}(Z)$. Recall that $C_F(Z)$ was computed to be

$$\left[\bigotimes_{e_i \in Z} R \xrightarrow{U_i} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \xrightarrow{L(v)} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \xrightarrow{Q(v)} R\right] \otimes \left[\bigotimes_{v \in V_4(S)} R \xrightarrow{L(v)} R\right].$$

We can therefore compute $C_{F(n)}(Z)$ by adding the new differentials to this complex. The only new discs in $CFK_n^-(Z)$ correspond to bigons containing X or XX basepoints (we will prove this in Lemma 4.18). For example, let e_i be an edge in Z, with x and y the two intersection points corresponding to e_i. When we weren't allowing discs to

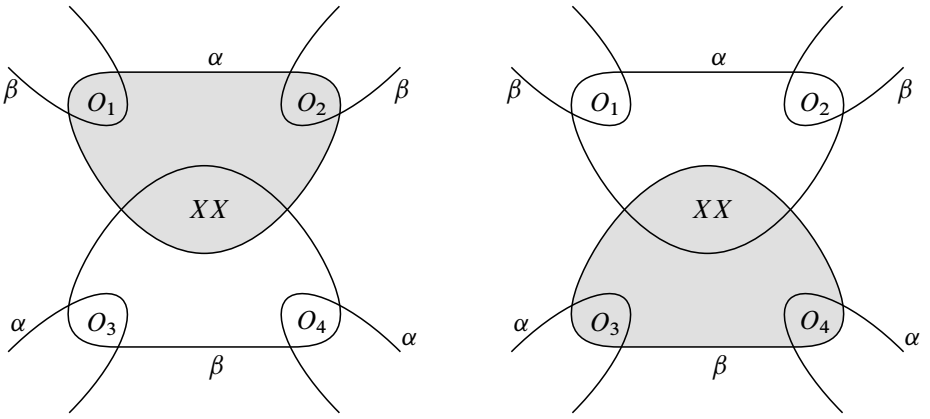


Figure 15: α–degenerations (left) and β–degenerations (right) at a four-valent vertex

Figure 16: Two new bigons from y to x

pass through X or XX, the only disc connecting x and y was the bigon containing U_i. This contributed the tensor factor of

$$R \xrightarrow{U_i} R.$$

However, when we allow discs to pass through X and XX, we get two new bigons which map from y to x, shown in Figure 16.

The type of contribution from these bigons depends on whether the endpoints of e_i are 2–valent or 4–valent. These two cases are shown in Figure 17. In either case, the contribution has a coefficient of degree n. We will denote the contribution from these new bigons at an edge e_i in Z by $p(e_i)$ (the precise polynomial will not be relevant

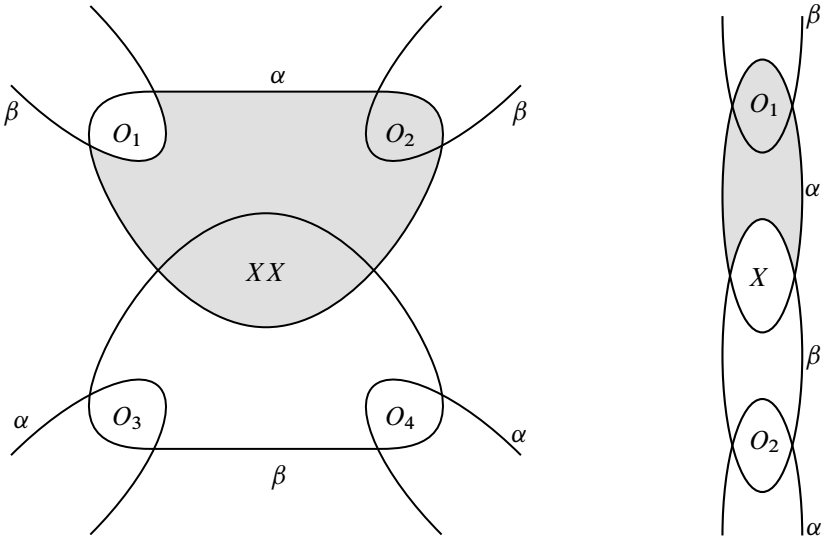


Figure 17: The two types of new bigons

for our computations). The tensor factor then becomes

$$R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R.$$

For a vertex v in $W_2(Z)$, there are two intersection points x and y corresponding to v. In $C_F(Z)$, they contributed a tensor factor of

$$R \xrightarrow{L(v)} R.$$

In $C_{F(n)}(Z)$, there is an extra differential corresponding to the bigon from y to x through X (See Figure 18). Since X carries a coefficient of $u_1(v)$, the factor becomes

$$R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R.$$

Similarly, for 4–valent vertices v in $W_4(Z)$, $C_F(Z)$ contains a tensor factor

$$R \xrightarrow{Q(v)} R.$$

In $C_{F(n)}(Z)$, there is an extra differential corresponding to the bigon through XX shown in Figure 18. The XX contributes a coefficient of $u_2(v)$, so the factor becomes

$$R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R.$$

Thus, only counting bigons, we get the following complex for $CFK_n^-(Z)$:

$$\left[\bigotimes_{e_i \in Z} R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R\right]. \tag{5}$$

We will now show that these are in fact the only differentials in the complex $CFK_n^-(Z)$.

Lemma 4.18 *All of the differentials in $CFK_n^-(Z)$ come from bigons in $\mathcal{H}(S)$.*

Proof Suppose x and y are two distinct generators of $CFK_n^-(Z)$ such that there is a Maslov index 1 homotopy class $\phi_{x,y} \in \pi_2(x, y)$ which does not pass through the basepoints p_i, and let $D(\phi_{x,y})$ denote the corresponding 2–chain in $\mathcal{H}(S)$. (See [19] for a discussion of the Maslov index.)

From (4), we see that when counting only bigons in the differential, $CFK^-(Z)$ can be written as a tensor product of Koszul complexes. Therefore, it is natural to view it as a hypercube in which the differentials from bigons correspond to oriented edges. The generators x and y are two of the vertices of this hypercube. The new bigons in $CFK_n^-(Z)$ also correspond to edges in the cube, but they have the opposite orientation

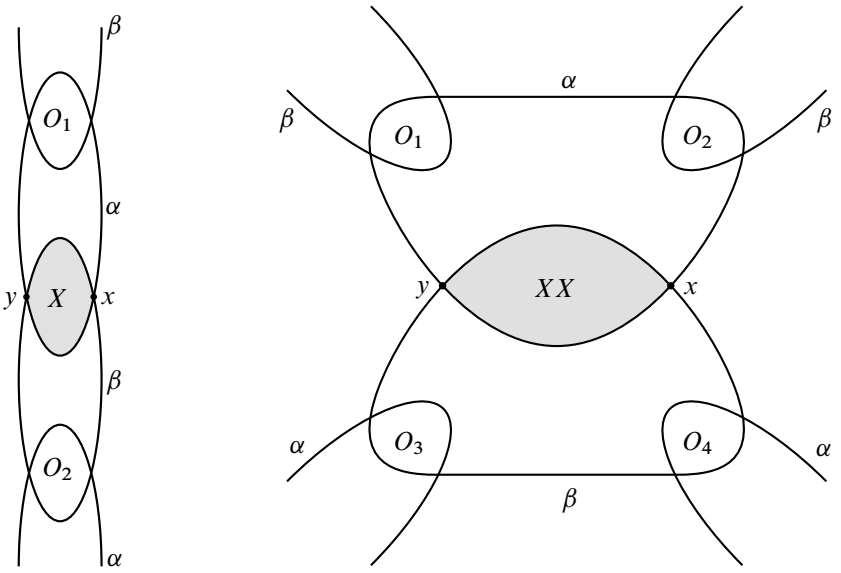


Figure 18: New differentials: the bigons passing through X (left) and XX (right)

compared with the edges coming from bigons in $CFK^-(Z)$. It is clear that this cube is fully connected in the sense that for any two vertices a and b, there is an oriented path from a to b.

Let γ be a path of minimal length from y to x — call this length l. Each edge in γ corresponds to a set of bigons (this set has either one element or two). Pick one bigon from each edge in γ, so that we have a sequence $B_1, \dots, B_l$. Since $D(\phi_{x,y})$ and $B_1, \dots, B_l$ all have Maslov index 1, the 2–chain $D(\phi_{x,y}) + B_1 + \cdots + B_l$ gives a Maslov index $l+1$ homotopy class in $\pi_2(x,x)$.

The only 2–chains which do not change the underlying generator are sums of periodic domains. The periodic domains with multiplicity 0 at the basepoints are the interiors of the α and β circles — let P_i denote these periodic domains. We can write

$$D(\phi_{x,y}) + B_1 + \cdots + B_l = \sum_{i=1}^{k} P_i.$$

We know that the sum on the right will have only positive coefficients because the total sum must have nonnegative multiplicity at every region in $\mathcal{H}(S)$, and each α or β circle contains a region which is not contained in any other α or β circles.

We see by looking at the Heegaard diagram that if two distinct B_i are contained in a single P_j, then the two B_i correspond to the same edge in the cube but with opposite orientations. Note that this is not true for arbitrary bigons in the $\mathcal{H}(S)$, but the B_i which can appear in the cube are restricted in that the two endpoints must correspond to intersection points of generators with underlying cycle Z.

Since γ is a minimal-length path in the cube, γ does not traverse any edge in both directions. Thus, no two of the B_i are contained in a single P_j, so $k \geq l$. The Maslov index of the 2–chain on the left is $l+1$, while the Maslov index of the 2–chain on the right is $2k$, because each $\alpha-$ or β–degeneration has Maslov index 2. But the equality $l+1=2k$ with $k \geq l \geq 0$ forces $l=k=1$. So we have

$$D(\phi_{x,y}) + B_1 = P_1,$$

where B_1 is a bigon and P_1 is the interior of an α or β circle. Then $D(\phi_{x,y}) = P_1 - B_1$. Since $D(\phi_{x,y})$ can have only nonnegative coefficients, $B_1 \subset P_1$. But removing a bigon from the interior of any α or β circle in $\mathcal{H}(S)$ results in another bigon, proving that $D(\phi_{x,y})$ is a bigon, as desired. □

It follows that $CFK^-_n(Z)$ is given by (5). Finally, the complex $K(S)$ gets changed to $K_n(S)$, so the whole complex for $C_{F(n)}(Z)$ can be written as

$$\left[\bigotimes_{e_i \in Z} R \underset{p(e_i)}{\overset{U_i}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_2(Z)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in W_4(Z)} R \underset{u_2(v)}{\overset{Q(v)}{\rightleftarrows}} R\right] \otimes \left[\bigotimes_{v \in V_4(S)} R \underset{u_1(v)}{\overset{L(v)}{\rightleftarrows}} R\right].$$

Now that we have our complex computed, we want to compare its homology with $H_n(S-Z)$. We will denote the differentials which do not pass through any X or XX basepoints by d_{0+}, and the new differentials by d_{0-}. Observe that with respect to the bigrading $(\mathrm{gr}_q, \mathrm{gr}_h)$ introduced in Section 4.2.3, d_{0+} has bigrading $\{2,2\}$ and d_{0-} has bigrading $\{2n,-2\}$.

Lemma 4.19 *Up to an overall grading shift, $H_*\big(H_*(C_{F(n)}(Z), d_{0+}), d^*_{0-}\big)$ is isomorphic to $H_*\big(H_*(C_n(S-Z), d_+), d^*_-\big)$.*

Proof It follows from Lemma 4.10 that $H(C_{F(n)}(Z), d_{0+}) \cong H_H(S-Z)$. To complete the proof, we need to show that d^*_{0-} corresponds to the d_- differential under this isomorphism. For vertices which are 2–valent in both S and $S-Z$ (ie

$v \in W_2(Z)$), this is obvious. The same is true for vertices which are 4–valent in both S and $S-Z$ (ie $v \in W_4(Z)$).

The only identification which is nontrivial is that the d_{0-} differential corresponding to a vertex which is 4–valent in S but 2–valent in $S-Z$ is the same as the d_- differential on the 2–valent vertex in $S-Z$. Let e_i and e_j be the outgoing edges at v and e_k and e_l the incoming edges. The multicycle Z must contain one incoming and one outgoing edge — without loss of generality, assume Z contains e_i and e_k. The coefficient of the d_- differential is given by

$$\frac{U_j^{n+1} - U_l^{n+1}}{U_j - U_l},$$

while the coefficient of the d_{0-} differential is given by

$$\frac{U_i^{n+1} + U_j^{n+1} - U_k^{n+1} - U_l^{n+1} - Q(v)u_2(v)}{U_i + U_j - U_k - U_l}.$$

Recall that to achieve the isomorphism in Lemma 4.10, we first canceled the Koszul complex on the U_p for e_p in Z, as these elements formed a regular sequence. We therefore want to show that these two coefficients are equal in $R_Z = R/\{U_p = 0 \text{ for } e_p \in Z\}$. Substituting $U_i = U_k = 0$ into the above equation and noting that this causes $Q(v)$ to be zero, we get the desired equality. □

Define $H^{\pm}(C_{F(n)}(S)) = H_*\big(H_*(C_{F(n)}(S), d_{0+}), d_{0-}^*\big)$. Since both d_{0+} and d_{0-} are homogeneous with respect to the bigrading, this homology is bigraded as well. Applying the lemma and adding in the gradings from Section 4.2.3, we see that

$$(6)\quad H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_Z H^{\pm}(C_n(S-Z))\{-2r(S-Z)+T_2(Z)-T_1(Z), 2r(Z)\}.$$

Recall from Corollary 2.7 that $H^{\pm}(C_n(S-Z))$ lies in a single horizontal grading, namely $2r(S-Z)$. Adding in the shift, the homology corresponding to a multicycle Z must lie in horizontal grading $2r(S-Z)+2r(Z) = 2r(S)$. But this does not depend on Z, so we have shown the following:

Lemma 4.20 *The homology $H^{\pm}(C_{F(n)}(S))$ lies in a single horizontal grading.*

The original differentials on $C_F(S)$ all have bigrading $\{2,2\}$. The new differentials on $CFK_n^-(S)$ have bigrading $\{2+2k(n-1), 2-4k\}$, where k is the sum of the multiplicities of the holomorphic discs at all X and XX markings. The new differentials

on $K_n(S)$ all have bigrading $\{2n, -2\}$. Thus, all differentials on $C_n(S)$ change the horizontal grading by 2 (mod 4). This implies that no induced differentials can have horizontal grading 0, which tells us that the remaining differentials on our complex are all trivial, giving us the following:

Lemma 4.21 *The total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to $H^{\pm}(C_{F(n)}(S))$.*

This isomorphism is singly graded with grading $\mathrm{gr}_n = \mathrm{gr}_q + \frac{1}{2}(n-1)\,\mathrm{gr}_h$, as the total differential on $C_{F(n)}(S)$ is homogeneous of degree $n+1$ with respect to this grading.

Going back to (6), we know that as bigraded vector spaces, we have the isomorphism

$$H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_Z H^{\pm}(C_n(S-Z))\{-2r(S-Z) + T_2(Z) - T_1(Z), 2r(Z)\}.$$

We can use Lemma 4.14 in the same way as in the previous section to convert this equation into a direct sum over labelings,

$$H^{\pm}(C_{F(n)}(S)) \cong \bigoplus_f H_1(S_{f,1}) \otimes H_n(S_{f,2})\{-2r(S_{f,2}) + T_2(S_{f,1}) - T_1(S_{f,1}), 0\}.$$

Applying the bigraded composition product formula (3), this gives an isomorphism of bigraded vector spaces

$$H^{\pm}(C_{F(n)}(S)) \cong H_{n+1}(S). \tag{7}$$

Since $H_*(C_{F(n)}(S), d) \cong H^{\pm}(C_{F(n)}(S))$ as graded vector spaces with grading gr_n, this gives an isomorphism

$$H_*(C_{F(n)}(S), d) \cong H_{n+1}(S),$$

where we are viewing $H_{n+1}(S)$ as singly graded, with grading gr_n. Since singly graded $\mathfrak{sl}_{n+1}$ homology is typically viewed with respect to the grading gr_{n+1}, this isn't quite what we want. Fortunately, since the homology is concentrated in horizontal grading $\mathrm{gr}_h = 2r(S)$, we see that $\mathrm{gr}_{n+1} = \mathrm{gr}_n + 2r(S)$.

Theorem 4.22 *The total homology $H_*(C_{F(n)}(S), d)$ is isomorphic to the homology $H_{n+1}(S)\{2r(S)\}$, where $H_*(C_{F(n)}(S), d)$ has grading gr_n and $H_{n+1}(S)$ has grading gr_{n+1}.*

Remark 4.23 The grading shift by $2r(S)$ is only an artifact of passing from the grading gr_n to gr_{n+1}, and no grading shift is needed when comparing the bigraded complexes (see (7)).

This shift appears on the HOMFLY-PT side as well in the singly graded case. When looking at the spectral sequence from $H_H(S)\langle 1\rangle$ to $H_{n+1}(S)$ induced by d_n, the grading on $H_H(S)\langle 1\rangle$ is given by gr_n, while the grading on $H_{n+1}(S)$ is given by gr_{n+1}. Thus, with respect to these gradings, there is a spectral sequence from $H_H(S)\langle 1\rangle$ to $H_{n+1}(S)\{2r(S)\}$.

Corollary 4.24 *For all $n \geq 1$, there is a spectral sequence whose E_1 page is $H_F(S)$ which converges to $H_{n+1}(S)$.*

Proof All of the original differentials on $C_F(S)$ have Alexander grading 0. The new differentials on CFK_n^- have Alexander grading $k(-n-1)$, where k is the sum of the multiplicities of the disc at the X and XX basepoints, and the new differentials on the Koszul complex have Alexander grading $-n-1$. In particular, all of the new differentials strictly decrease the Alexander grading, so it induces a filtration with respect to which the filtered homology is $H_*(C_F(S))$. Thus, the corresponding spectral sequence has E_1 page $H_F(S)$, and converges to the total homology $H(C_{F(n)}(S)) \cong H_{n+1}(S)$. □

Remark 4.25 Using the tools from this section, we can give another proof of Theorem 4.15, which does not rely on the composition product formula for HOMFLY-PT homology. From (7), we have that

$$H_*(H_*(C_{F(n)}(S), d_{0+}), d_{0-}^*) \cong H_{n+1}(S).$$

Since $H_*(C_{F(n)}(S), d_{0+})$ is precisely $H_*(C_F(S), d_0)$, this shows that there a differential d_{n+1} on $H_*(C_F(S), d_0)$ satisfying the conditions of Corollary 3.4 for $k = 1$, where d_{n+1} is given by d_{0-}^*.

4.4 Proof of Manolescu's conjecture

At this point, we have three spectral sequences: one from HOMFLY-PT homology to $\mathfrak{sl}_n$ homology, one from HOMFLY-PT homology to knot Floer homology, and one from knot Floer homology to $\mathfrak{sl}_n$ homology. Diagrammatically, this looks like

$$\begin{array}{ccc} H_H(S)\langle 1\rangle & \longrightarrow & H_F(S) \\ & \searrow \quad \swarrow & \\ & H_n(S)\{2r(S)\} & \end{array}$$

where the arrows correspond to spectral sequences. Since both HOMFLY-PT homology and knot Floer homology have spectral sequences going to $\mathfrak{sl}_n$ homology for all $n \geq 2$,

it is clear that they have a great deal in common. The conjecture of Manolescu is that they are in fact isomorphic. Since we have a spectral sequence from HOMFLY-PT homology to knot Floer homology, this is equivalent to the spectral sequence collapsing at the E_1 page. In this section, we will prove Manolescu's conjecture.

Theorem 4.26 *The* HOMFLY-PT *homology $H_H(S)\langle 1\rangle$ and knot Floer homology $H_F(S)$ are isomorphic as bigraded vector spaces.*

Proof We will start with the complex $C_{F(n)}$ from the previous section. There are two filtrations defined on this complex so far — the one induced by the Alexander grading, and the one induced by the basepoints. Consider the associated bigraded object from these two filtrations.

The differentials always change the Alexander grading by a multiple of $n+1$, so let d_{ij} denote those differentials which change the Alexander grading by $i(n+1)$ and change the basepoint grading by j.

Theorem 4.15 states that

$$H_*(C_{F(n)}(S), d_{00}) \cong H_H(S)\langle 1\rangle.$$

Since $H_F(S) \cong H_*(C_{F(n)}, d_{0*})$, our theorem is equivalent to d_{0k}^* being zero on $H_*(C_{F(n)}, d_{00})$. We will prove this by contradiction. In particular, suppose that some d_{0k}^* is nonzero, and let a denote the smallest such k.

Because a is minimal, it is clear that d_{0a}^* and d_{10}^* anticommute, as $d_{0a}^* \circ d_{10}^*$ and $d_{10}^* \circ d_{0a}^*$ are the only components of d^2 which change the basepoint filtration by a and the Alexander filtration by $n+1$. These are the differentials that we will be interested in, so we will rename them. We will write d_F instead of d_{0a}^* and d_n instead of d_{10}^*, since d_{10}^* depends on n. The differentials d_F and d_n both act on $H_*(C_{F(n)}, d_{00})$, so using the above isomorphism we will view them as acting on $H_H(S)\langle 1\rangle$.

We know from (7) that there is an isomorphism of bigraded vector spaces

$$H_*(H_H(S)\langle 1\rangle, d_n) \cong H_{n+1}(S).$$

To summarize our setup, we have a family of differentials d_n on $H_H(S)\langle 1\rangle$, each having bigrading $(2n, -2)$, such that the homology with respect to each is H_{n+1}, and there is a differential d_F on $H_H(S)\langle 1\rangle$ with bigrading $(2, 2)$ which is nontrivial and anticommutes with each d_n.

We know that the smallest horizontal grading in which $H_H(S)\langle 1\rangle$ is nontrivial is $2r(S)$, and that the homology $H_*(H_H(S)\langle 1\rangle, d_n)$ lies only in this horizontal grading. Let $h_{\min}$ be the minimal horizontal grading on which d_F is nonzero, and let x be an element of $H_H(S)\langle 1\rangle$ in bigrading $(q, h_{\min})$ for some q with $d_F(x) \neq 0$. Define $d_F(x) = y$.

We know also that $H_H(S)\langle 1\rangle$ is bounded below in quantum grading and d_n changes the quantum grading by $2n$, so choose N sufficiently large that y cannot be in the image of d_N. Since y lies in horizontal grading $h_{\min}+2$ (in particular, not $h_{\min}$), it follows that $y \in \mathrm{Ker}(d_N)$ if and only if $y \in \mathrm{Im}(d_N)$. Thus, y is not in the kernel of d_N.

But then $d_N \circ d_F(x)$ is nonzero, while $d_F \circ d_N(x)$ must be zero because $d_N(x)$ lies in horizontal grading $h_{\min}-2$, and $d_F = 0$ for all horizontal gradings less than $h_{\min}$, which contradicts the fact that d_N and d_F anticommute. □

This theorem is proved with $\mathbb{Q}$ coefficients. However, HOMFLY-PT homology over $\mathbb{Z}$ of braid graphs is known to be torsion-free. This can be seen from the MOY relations (see [11; 18]), which we won't describe explicitly here, but the argument can be summarized as follows. There is a map ι: {singular braids} $\to \mathbb{N}$, called the *complexity*, such that the HOMFLY-PT homology of any singular braid can be written in terms of the HOMFLY-PT homologies of singular braids with lower complexity. The fact that $H_H(S, \mathbb{Z})$ is $\mathbb{Z}$–torsion-free then follows from an induction argument.

Thus, HOMFLY-PT homology still satisfies the composition product formula over $\mathbb{Z}$,

$$H_H(S, \mathbb{Z})\langle 1\rangle \cong \bigoplus_Z H_H(S - Z, \mathbb{Z}).$$

In our computation of $H_*(C_F(S), d_0)$, we only canceled those differentials with coefficient ± 1, so

$$H_*(C_F(S), d_0, \mathbb{Z}) \cong \bigoplus_Z H_H(S - Z, \mathbb{Z}) \cong H_H(S, \mathbb{Z})\langle 1\rangle.$$

So, for the theorem to hold with $\mathbb{Z}$ coefficients, we need the filtered homology $H_*(C_F(S), d_0, \mathbb{Z})$ to be isomorphic to the unfiltered homology $H_*(C_F(S), d, \mathbb{Z})$, ie we need the induced higher differentials on $H_*(C_F(S), d_0, \mathbb{Z})$ to be zero. But any nonzero differential over $\mathbb{Z}$ would also be nonzero over $\mathbb{Q}$, contradicting Theorem 4.26. Thus, Theorem 4.26 is true with $\mathbb{Z}$ coefficients as well.

We relate this theorem to Manolescu's conjecture with the following corollary. Note that the original conjecture was over $\mathbb{Z}$ rather than $\mathbb{Q}$. By the previous argument, there is no $\mathbb{Z}$–torsion, so the two contexts are equivalent.

Corollary 4.27 *If S is a connected braid graph, then there is an isomorphism of* Tor *groups*

$$\mathrm{Tor}_R(R/L, R/N) \cong \mathrm{Tor}_R(R/L, R/Q)$$

as bigraded vector spaces. The bigrading on $\mathrm{Tor}_R(R/L, R/N)$ is given by (q,h), where q is the quantum grading coming from the polynomial ring and h is the homological grading, and the bigrading on $\mathrm{Tor}_R(R/L, R/Q)$ is $(q+h,h)$.

Proof The difference between our complex $C_F(S)$ and Manolescu's knot Floer complex for S is that we have added an additional unknotted component at infinity, placed the marked edge on that component and reduced that component (ie set that U_i equal to zero).

Manolescu instead placed the marked edge on the leftmost strand of the braid. Let $C_F^M(S)$ denote Manolescu's complex. By the argument at the beginning of Section 4, we know that this has the effect of doubling the homology. In particular, with respect to the (q,h) bigrading there is an isomorphism

$$H_F(S) \cong H_F^M(S) \otimes V,$$

where $V = \mathbb{Q}\{-1,-1\} \oplus \mathbb{Q}\{1,1\}$.

Similarly, the middle HOMFLY-PT homology can be viewed as the unreduced HOMFLY-PT homology of the $1-1$ tangle obtained by breaking an edge in the diagram. Let $C_H^M(S)$ denote the middle HOMFLY-PT homology of S. Then, with respect to the (q,h) grading, we have the isomorphism

$$H_H(S) \cong H_H^M(S)\{0,-1\} \oplus H_H^M(S)\{0,1\}.$$

But when we switch to the $(q+h,h)$ grading, this becomes

$$H_H(S)\langle 1\rangle \cong H_H^M(S)\langle 1\rangle \otimes V.$$

The previous theorem states that $H_F(S) \cong H_H(S)\langle 1\rangle$. Using the above arguments, this becomes

$$H_F^M(S) \otimes V \cong H_H^M(S)\langle 1\rangle \otimes V$$

as bigraded groups.

Since all of our theories are bounded in h–grading and bounded below in q–grading, the above isomorphism implies an isomorphism without tensoring with V,

$$H_F^M(S) \cong H_H^M(S)\langle 1\rangle.$$

But Manolescu showed that

$$H_F^M(S) \cong \mathrm{Tor}_R(R/L, R/N) \quad \text{and} \quad H_H^M(S) \cong \mathrm{Tor}_R(R/L, R/Q),$$

which proves the corollary. □

Another significant corollary of this result relates to the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. This is the page which was conjectured by Manolescu to give HOMFLY-PT homology. In [2], we showed that the graded Euler characteristic of the homology

$$E_2^f(D) = H_*\big(H_*(C_F(D), d_0^f), (d_1^f)^*\big)$$

is the HOMFLY-PT polynomial, where d_i^f denotes the component of the differential on $C_F(D)$ which increases the cube grading by i and preserves the basepoint filtration. In particular, we define the triple grading on this complex by the i, j and k gradings, where k denotes twice the cube grading, and

$$i = 2A - 2M - k, \quad j = 4A - 2M - k,$$

where M and A are the Maslov and Alexander gradings, respectively. With respect to this triple grading, we showed that

$$\sum_{i,j,k} (-1)^{(k-j)/2} \dim(E_2^f(D)^{i,j,k}) = P_H(aq, q, D).$$

But we have just seen that $H_*(C_F(D), d_0^f) \cong H_*(C_F(D), d_0)$, so there is a spectral sequence from $H_*\big(H_*(C_F(D), d_0^f), (d_1^f)^*\big)$ to the E_2 page $H_*\big(H_*(C_F(D), d_0), d_1^*\big)$. But all of these differentials have triple grading $\{0, 0, 2\}$, so they do not change the Euler characteristic. Thus we have shown the following:

Corollary 4.28 *Let $C_F(D)$ denote the oriented cube of resolutions complex for a braid diagram D, and let $E_2(D)$ denote the E_2 page of the spectral sequence on $C_F(D)$ induced by the cube filtration. Then the graded Euler characteristic of $E_2(D)$ with the triple grading given above is the* HOMFLY-PT *polynomial $P_H(aq, q, D)$.*

References

[1] **A S Alishahi**, **E Eftekhary**, *A refinement of sutured Floer homology*, J. Symplectic Geom. 13 (2015) 609–743 MR

[2] **N Dowlin**, *The knot Floer cube of resolutions and the composition product*, preprint (2015) arXiv

[3] **N M Dunfield**, **S Gukov**, **J Rasmussen**, *The superpolynomial for knot homologies*, Experiment. Math. 15 (2006) 129–159 MR

[4] **P Freyd**, **D Yetter**, **J Hoste**, **W B R Lickorish**, **K Millett**, **A Ocneanu**, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. 12 (1985) 239–246 MR

[5] **A Gilmore**, *Framed graphs and the non-local ideal in the knot Floer cube of resolutions*, Algebr. Geom. Topol. 15 (2015) 1239–1302 MR

[6] **M C Hughes**, *A note on Khovanov–Rozansky sl_2–homology and ordinary Khovanov homology*, J. Knot Theory Ramifications 23 (2014) art. id. 1450057 MR

[7] **F Jaeger**, *Composition products and models for the homfly polynomial*, Enseign. Math. 35 (1989) 323–361 MR

[8] **M Khovanov**, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359–426 MR

[9] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology*, Fund. Math. 199 (2008) 1–91 MR

[10] **M Khovanov**, **L Rozansky**, *Matrix factorizations and link homology, II*, Geom. Topol. 12 (2008) 1387–1425 MR

[11] **D Krasner**, *Computations and structures in* sl(n)*–link homology*, PhD thesis, Columbia University (2009) MR Available at `https://search.proquest.com/docview/205417422`

[12] **C Manolescu**, *An untwisted cube of resolutions for knot Floer homology*, Quantum Topol. 5 (2014) 185–223 MR

[13] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[14] **P Ozsváth**, **Z Szabó**, *Holomorphic disks, link invariants and the multi-variable Alexander polynomial*, Algebr. Geom. Topol. 8 (2008) 615–692 MR

[15] **P Ozsváth**, **Z Szabó**, *A cube of resolutions for knot Floer homology*, J. Topol. 2 (2009) 865–910 MR

[16] **J H Przytycki**, **P Traczyk**, *Invariants of links of Conway type*, Kobe J. Math. 4 (1988) 115–139 MR

[17] **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR Available at `https://search.proquest.com/docview/305332635`

[18] **J Rasmussen**, *Some differentials on Khovanov–Rozansky homology*, Geom. Topol. 19 (2015) 3031–3104 MR

[19] **S Sarkar**, *Maslov index formulas for Whitney n–gons*, J. Symplectic Geom. 9 (2011) 251–270 MR

[20] **S Sarkar**, *A note on sign conventions in link Floer homology*, Quantum Topol. 2 (2011) 217–239 MR

[21] **E Wagner**, *Khovanov–Rozansky graph homology and composition product*, J. Knot Theory Ramifications 17 (2008) 1549–1559 MR

Department of Mathematics, Columbia University
New York, NY, United States

ndowlin@math.columbia.edu

Received: 29 June 2017 Revised: 14 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 3887–3906

Some extensions in the Adams spectral sequence and the 51–stem

GUOZHEN WANG
ZHOULI XU

We show a few nontrivial extensions in the classical Adams spectral sequence. In particular, we compute that the 2–primary part of π_{51} is $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$. This was the last unsolved 2–extension problem left by the recent work of Isaksen and the authors through the 61–stem.

The proof of this result uses the RP^∞ technique, which was introduced by the authors to prove $\pi_{61} = 0$. This paper advertises this technique through examples that have simpler proofs than in our previous work.

55Q40

1 Introduction

The computation of the stable homotopy groups of spheres is both a fundamental and a difficult problem in homotopy theory. Recently, using Massey products and Toda brackets, Isaksen [5] extended the 2–primary Adams spectral sequence computations to the 59–stem, with a few $2, \eta, \nu$–extensions unsettled.

Based on the algebraic Kahn–Priddy theorem [7] (see also Lin [8]), the authors [12] compute a few differentials in the Adams spectral sequence, and proved that $\pi_{61} = 0$. The 61–stem result has the geometric consequence that the 61–sphere has a unique smooth structure, and it is the last odd-dimensional case. In the article [12], it took the authors more than 40 pages to introduce the method and prove one Adams differential, $d_3(D_3) = B_3$. Here B_3 and D_3 are certain elements in the 60– and 61–stem. Our notation will be consistent with [5; 12].

In this paper, we show that our technique can also be used to solve extension problems in the Adams spectral sequence. We establish a nontrivial 2–extension in the 51–stem, together with a few other extensions left unsolved by Isaksen [5]. As a result, we have the following proposition:

Proposition 1.1 *There is a nontrivial 2–extension from $h_0h_3g_2$ to gn in the 51–stem.*

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3887

We'd like to point out that this is also a nontrivial 2–extension in the Adams–Novikov spectral sequence.

Combining with Theorem 1.1 of Isaksen and Xu [6], which describes the group structure of π_{51} up to this 2–extension, we have the following corollary:

Corollary 1.2 *The 2–primary π_{51} is $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$, generated by elements that are detected by $h_3 g_2$, $P^6 h_2$ and $h_2 B_2$.*

Using a Toda bracket argument, Proposition 1.1 is deduced from the following σ–extension in the 46–stem:

Proposition 1.3 (1) *There is a nontrivial σ–extension from $h_3 d_1$ to N in the 46–stem.*

(2) *There is a nontrivial η–extension from $h_1 g_2$ to N in the 46–stem.*

As a corollary, we prove a few more extensions.

Corollary 1.4 (1) *There is a nontrivial η–extension from C to gn in the 51–stem.*

(2) *There is a nontrivial ν–extension from $h_2 h_5 d_0$ to gn in the 51–stem.*

(3) *There is a nontrivial σ–extension from $h_0^2 g_2$ to gn in the 51–stem. In particular, the element gn detects $\sigma^3 \theta_4$.*

Remark 1.5 In [5], Isaksen had an argument that implies the nonexistence of the two η–extensions on $h_1 g_2$ and C, which is contrary to our results in Proposition 1.3 and Corollary 1.4. Isaksen's argument fails because of neglect of the indeterminacy of a certain Massey product in a subtle way. For more details, see Remark 2.3.

The proof of the σ–extension in Proposition 1.3 is the major part of this article: we prove it by the RP^∞ technique as a demonstration of the effectiveness of our method.

The rest of this paper is organized as follows:

In Section 2, we deduce Proposition 1.1 and Corollary 1.4 from Proposition 1.3. We also show the two statements in Proposition 1.3 are equivalent. In Section 3, we recall some notation from [12]. We also give a brief review of how to use the RP^∞ technique to prove differentials and to solve extension problems. In Section 4, we present the proof of Proposition 1.3. In Section 5, we prove a lemma which is used in Section 4. The lemma gives a general connection that relates Toda brackets and extension problems in 2 cell complexes. In the appendix, we use cell diagrams as intuition for statements of the lemmas in Section 5.

Acknowledgement

We thank the anonymous referee for various helpful comments. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1810638.

2 The 51–stem and some extensions

We first establish the following lemma:

Lemma 2.1 *In the Adams E_2–page, we have the following Massey products in the* 46*–stem:*

$$gn = \langle N, h_1, h_2\rangle = \langle N, h_2, h_1\rangle.$$

Proof By Bruner's computation [4], there is a relation in bidegree $(t-s, s) = (81, 15)$,

$$gnr = mN.$$

We have $\mathrm{Ext}^{15,81+15} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by gnr and $h_1 x_{14,42}$. Moreover, the element gnr is not divisible by h_1, and neither of the generators is divisible by h_2.

By Tangora's computation [10], we have a Massey product in the Adams E_2–page,

$$m = \langle r, h_1, h_2\rangle.$$

Therefore,

$$gn \cdot r = m \cdot N = N \cdot \langle r, h_1, h_2\rangle = \langle N \cdot r, h_1, h_2\rangle = r \cdot \langle N, h_1, h_2\rangle$$

with zero indeterminacy. This implies

$$gn = \langle N, h_1, h_2\rangle.$$

Because of the relation $h_2 \cdot N = 0$ in $\mathrm{Ext}^{9,49+9} = 0$, we also have

$$gn \cdot r = m \cdot N = \langle r, h_1, h_2\rangle \cdot N = r \cdot \langle h_1, h_2, N\rangle.$$

This implies

$$gn = \langle N, h_2, h_1\rangle. \qquad \square$$

Based on Proposition 1.3, we prove Corollary 1.4(1):

Proof By Proposition 1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_2, h_1\rangle$$

and Moss's theorem [9] imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle \sigma^2\{d_1\}, \nu, \eta\rangle.$$

The indeterminacy of this Toda bracket is

$$\eta\cdot\pi_{50} + \sigma^2\{d_1\}\cdot\pi_5 = \eta\cdot\pi_{50},$$

since $\pi_5 = 0$. Shuffling this bracket, we have

$$\langle \sigma^2\{d_1\}, \nu, \eta\rangle \supseteq \sigma\{d_1\}\cdot\langle\sigma, \nu, \eta\rangle = 0,$$

since $\langle\sigma, \nu, \eta\rangle \subseteq \pi_{12} = 0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by η.

For filtration reasons, the only possibilities are C and h_5c_1. However, Lemma 4.2.51 of [5] states that there is no η–extension from h_5c_1 to gn. Therefore, we must have an η–extension from C to gn. □

Based on Proposition 1.3, we prove Corollary 1.4(2):

Proof By Proposition 1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_1, h_2\rangle$$

and Moss's theorem [9] imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle \sigma^2\{d_1\}, \eta, \nu\rangle.$$

The indeterminacy of this Toda bracket is

$$\nu\cdot\pi_{48} + \sigma^2\{d_1\}\cdot\pi_5 = \nu\cdot\pi_{48},$$

since $\pi_5 = 0$. Shuffling this bracket, we have

$$\langle \sigma^2\{d_1\}, \eta, \nu\rangle \supseteq \sigma\cdot\langle\sigma\{d_1\}, \eta, \nu\rangle = \sigma\{d_1\}\cdot\langle\eta, \nu, \sigma\rangle = 0,$$

since $\langle\eta, \nu, \sigma\rangle \subseteq \pi_{12} = 0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by ν.

For filtration reasons, the only possibility is $h_2h_5d_0$, which completes the proof. □

Now we prove Corollary 1.4(3), and Proposition 1.1:

Proof Lemma 4.2.31 from Isaksen's computation [5] states that the 2–extension from $h_0h_3g_2$ to gn is equivalent to the ν–extension from $h_2h_5d_0$ to gn. This proves Proposition 1.1.

It is clear that Proposition 1.1 is equivalent to Corollary 1.4(3), since σ is detected by h_3 and $\sigma^2\theta_4$ is detected by $h_0^2g_2$. (See [2; 5] for the second fact.) □

In the following lemma, we show that the two statements in Proposition 1.3 are equivalent:

Lemma 2.2 *There is a σ–extension from h_3d_1 to N if and only if there is an η–extension from h_1g_2 to N.*

Proof First note that there are relations in Ext,

$$h_3d_1 = h_1e_1, h_3e_1 = h_1g_2.$$

By Bruner's differential [3, Theorem 4.1]

$$d_3(e_1) = h_1t = h_2^2n,$$

we have Massey products in the Adams E_4–page

$$h_3d_1 = h_1e_1 = \langle h_2n, h_2, h_1\rangle, \quad h_1g_2 = h_3e_1 = \langle h_3, h_2n, h_2\rangle.$$

Then Moss's theorem implies that they converge to Toda brackets

$$\langle \nu\{n\}, \nu, \eta\rangle, \quad \langle \sigma, \nu\{n\}, \nu\rangle.$$

Therefore, the lemma follows from the shuffling

$$\sigma \cdot \langle \nu\{n\}, \nu, \eta\rangle = \langle \sigma, \nu\{n\}, \nu\rangle \cdot \eta.$$ □

We give a remark on the two η–extensions we proved:

Remark 2.3 In Lemmas 4.2.47 and 4.2.52 of [5], Isaksen showed that there are no η–extensions from h_1g_2 to N or from C to gn. Both arguments are based the statement of Lemma 3.3.45 of [5], whose proof implicitly studied the motivic Massey product

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle \ni Ph_1^3h_5e_0$$

in the 59–stem of the motivic Adams E_3–page, which therefore converges to a motivic Toda bracket. However, in the motivic Adams E_3–page, the element $Ph_1^3h_5e_0$ is in the indeterminacy of this motivic Massey product, since $Ph_1h_5e_0$ is present in the motivic E_3–page (it supports a d_3 differential). Therefore, we have

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle = \{Ph_1^3h_5e_0, 0\}$$

instead. The statement of Moss's theorem gives us the convergence of *only one* permanent cycle in the motivic Massey product; therefore, in this case, it is inconclusive.

3 The method and notation

In this section, we recall some notation from [12] and set up terminology that will be used in Section 4.

Notation 3.1 All spectra are localized at the prime 2. Suppose Z is a spectrum. Let $\mathrm{Ext}(Z)$ denote the Adams E_2–page $\mathrm{Ext}_A(\mathbb{F}_2, H_*(Z;\mathbb{F}_2))$ that converges to the 2–primary homotopy groups of Z. Here A is the mod 2 dual Steenrod algebra.

We now introduce some spectral sequence terminology. A permanent cycle is a class that does not support any nontrivial differential. A surviving cycle is a permanent cycle that is also not the target of any differential.

For spectra, let S^0 be the sphere spectrum and P_1^∞ be the suspension spectrum of RP^∞. In general, write P_n^{n+k} for the suspension spectrum of RP^{n+k}/RP^{n-1}.

Let α be a class in the stable homotopy groups of spheres. We use $C\alpha$ to denote the cofiber of α.

Definition 3.2 Let A, B, C and D be CW spectra and i and q be maps

$$A \xhookrightarrow{i} B, \quad B \xrightarrow{q} \!\!\!\!\rightarrow C.$$

We say that (A, i) is an $H\mathbb{F}_2$–subcomplex of B if the map i induces an injection on mod 2 homology. We denote an $H\mathbb{F}_2$–subcomplex by a hooked arrow as above.

We say that (C, q) is an $H\mathbb{F}_2$–quotient complex of B if the map q induces a surjection on mod 2 homology. We denote an $H\mathbb{F}_2$–quotient complex by a double-headed arrow as above.

When the maps involved are clear in the context, we also say A is an $H\mathbb{F}_2$–subcomplex of B, and C is an $H\mathbb{F}_2$–quotient complex of B.

Furthermore, we say D is an $H\mathbb{F}_2$–subquotient of B if D is an $H\mathbb{F}_2$–subcomplex of an $H\mathbb{F}_2$–quotient complex of B, or an $H\mathbb{F}_2$–quotient complex of an $H\mathbb{F}_2$–subcomplex of B.

One example is that S^1, S^3 and S^7 are $H\mathbb{F}_2$–subcomplexes of P_1^∞, due to the solution of the Hopf invariant one problem. Another example is that $\Sigma^7 C\eta$ is an $H\mathbb{F}_2$–subcomplex of P_7^9.

We use the following way to denote the elements in the Adams E_2–page of P_1^∞ and its $H\mathbb{F}_2$–subquotients. One way to compute $\mathrm{Ext}(P_1^\infty)$ is to use the algebraic Atiyah–Hirzebruch spectral sequence

$$E_1 = \bigoplus_{n=1}^{\infty} \mathrm{Ext}(S^n) \Rightarrow \mathrm{Ext}(P_1^\infty).$$

Notation 3.3 We denote any element in $\mathrm{Ext}(S^n)$ by $a[n]$, where $a \in \mathrm{Ext}(S^0)$ and n indicates that it comes from $\mathrm{Ext}(S^n)$. We will abuse notation and write the same symbol $a[n]$ for an element of $\mathrm{Ext}(P_1^\infty)$ detected by the element $a[n]$ of the Atiyah–Hirzebruch E_∞–page. Thus, there is indeterminacy in the notation $a[n]$ that is detected by Atiyah–Hirzebruch E_∞ elements in lower filtration. If $a[n]$ is the element of lowest Atiyah–Hirzebruch filtration in the Atiyah–Hirzebruch E_∞–page in a given bidegree (s,t), then $a[n]$ also is a well-defined element of $\mathrm{Ext}(P_1^\infty)$.

We use similar notation for homotopy classes.

Remark 3.4 In [11], we computed differentials in the algebraic Atiyah–Hirzebruch spectral sequence that converges to the Adams E_2–page of P_1^∞ in the range of $t < 72$.

By truncating the algebraic Atiyah–Hirzebruch spectral sequence for P_1^∞, one can also read off information about $\mathrm{Ext}(P_n^{n+k})$. For details, see [11].

Remark 3.5 Despite the indeterminacy in Notation 3.3, there is a huge advantage of it. Suppose $f\colon Q \to Q'$ is a map between two $H\mathbb{F}_2$–subquotients of P_1^∞ and there exists an element $a[n]$ which is a generator of both $\mathrm{Ext}^{s,t}(Q)$ and $\mathrm{Ext}^{s,t}(Q')$ for some

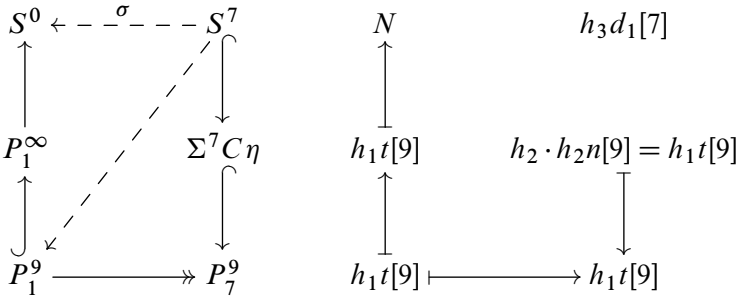
bidegree (s,t) (this implies both Q and Q' have a cell in dimension n). We must have that, with the right choices, $a[n]$ in $\mathrm{Ext}^{s,t}(Q)$ maps to $a[n]$ in $\mathrm{Ext}^{s,t}(Q')$:

$$\begin{array}{ccc} \bigoplus_{i\in I}\mathrm{Ext}(S^i) & \longrightarrow & \bigoplus_{i\in I'}\mathrm{Ext}(S^i) \\ \Vert & & \Vert \\ \mathrm{Ext}(Q) & \longrightarrow & \mathrm{Ext}(Q') \\ a[n] & \longmapsto & a[n] \end{array}$$

This property follows from the naturality of the algebraic Atiyah–Hirzebruch spectral sequence.

4 The σ–extension on h_3d_1

In this section, we prove Proposition 1.3(1). The proof can be summarized in the following "road map" with four main steps:



Here the elements in the right side of the "road map" are elements in the 46–stem of the E_∞–page of the Adams spectral sequences of the spectra in the corresponding positions.

Step 1 We show that the element $h_1t[9]$ is a permanent cycle in the Adams spectral sequence of $\Sigma^7C\eta$, and hence a permanent cycle in the Adams spectral sequence of P_7^9. This is stated as Proposition 4.3.

Step 2 Under the inclusion map $S^7 \hookrightarrow \Sigma^7C\eta$, we show that the element $h_1t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7C\eta)$. By naturality, the same statement is true after we further map it to $\pi_{46}(P_7^9)$. This is stated as Proposition 4.6.

Step 3 Under any map $S^7 \hookrightarrow P_1^9$ lifting the inclusion $S^7 \hookrightarrow P_7^9$, we show that the element $h_1 t[9]$ in $\mathrm{Ext}(P_1^9)$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(P_1^9)$. This is stated as Proposition 4.7.

Step 4 Using the inclusion map $P_1^9 \to P_1^\infty$ and the transfer map $P_1^\infty \to S^0$, we push forward the element $h_1 t[9]$ in the E_∞–page of P_1^9 to the element N in the E_∞–page of S^0. Since the composition

$$S^7 \hookrightarrow P_1^9 \to P_1^\infty \to S^0$$

is just σ, we have the desired σ–extension from $h_3 d_1$ to N in the Adams spectral sequence for S^0.

Remark 4.1 Step 2 is the essential step. Intuitively, it comes from the zigzag of the following two differentials:

$$d_3(e_1) = h_1 t$$

in the Adams spectral sequence of S^0, and

$$d_2(e_1[9]) = h_1 e_1[7] = h_3 d_1[7]$$

in the algebraic Atiyah–Hirzebruch spectral sequence of $\Sigma^7 C\eta$, which comes from the η–attaching map. Here

$$h_1 e_1 = h_3 d_1$$

is a relation in Ext. This zigzag suggested that we consider the possibility that $h_1 t[9]$ detects $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$.

We start with Step 1. Proposition 4.3 is a consequence of the following lemma:

Lemma 4.2 *The element $h_2 n[9]$ is a permanent cycle in $\Sigma^7 C\eta$, which detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map*

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

Proof The cofiber sequence

$$S^7 \xhookrightarrow{i} \Sigma^7 C\eta \xrightarrow{p} S^9 \xrightarrow{\eta} S^8$$

gives us a long exact sequence of homotopy groups

$$\pi_{43}(S^7) \xrightarrow{i_*} \pi_{43}(\Sigma^7 C\eta) \xrightarrow{p_*} \pi_{43}(S^9) \xrightarrow{\eta} \pi_{43}(S^8).$$

Since $h_2 n$ detects $\nu\{n\}$ and

$$\eta \cdot \nu\{n\} = 0,$$

there is an element α in $\pi_{43}(\Sigma^7 C\eta)$ such that $p_*\alpha = \nu\{n\}[9]$.

The element $h_2 n[9]$ in $\mathrm{Ext}(S^9)$ has Adams filtration 6, therefore by naturality, if it were not detected by $h_2 n[9]$ in $\mathrm{Ext}(\Sigma^7 C\eta)$, it would be detected by an element with Adams filtration at most 5.

From the same cofiber sequence, we have a short exact sequence on cohomology

$$0 \to H^*(S^9) \xrightarrow{p^*} H^*(\Sigma^7 C\eta) \xrightarrow{i^*} H^*(S^7) \to 0$$

and therefore a long exact sequence of Ext groups

$$\mathrm{Ext}^{s-1,t-1}(S^8) \xrightarrow{h_1} \mathrm{Ext}^{s,t}(S^7) \xrightarrow{i_\sharp} \mathrm{Ext}^{s,t}(\Sigma^7 C\eta) \xrightarrow{p_\sharp} \mathrm{Ext}^{s,t}(S^9).$$

This gives us the Adams E_2–page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \leq 6$ in Table 1.

$s \backslash t-s$	42	43
6		$h_2 n[9]$, $t[7]$
5	$h_0 p[9]$, $h_2 d_1[7]$	
4	$p[9]$	$h_0^2 h_2 h_5[9]$
3		$h_0 h_2 h_5[9]$
2		$h_2 h_5[9]$

Table 1: The Adams E_2–page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \leq 6$

The element $h_2 h_5[9]$ must support a nontrivial differential, since its image $p_\sharp(h_2 h_5[9])$ supports a d_3 differential that kills $h_0 p[9]$ in the Adams spectral sequence of S^9.

The elements $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ survive and detect homotopy classes that map to $\{h_0 h_2 h_5\}[9]$ and $\{h_0^2 h_2 h_5\}[9]$ in $\pi_{43}(S^9)$. In fact, since there is no η–extension on $h_0 h_2 h_5$ and $h_0^2 h_2 h_5$, we can choose homotopy classes in $\pi_{43}(S^9)$, which are detected by $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ and are zero after multiplying by η. Therefore, they have nontrivial preimages under the map p_* in the long exact sequence of homotopy groups. For filtration reasons, their preimages must be detected by $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ in the Adams spectral sequence of $\Sigma^7 C\eta$.

Therefore, the only possibility left is $h_2 n[9]$, which completes the proof. □

We prove Proposition 4.3 in Step 1.

Proposition 4.3 *The elements $h_2n[9]$ and $h_1t[9]$ are permanent cycles in the Adams spectral sequence of $\Sigma^7 C\eta$, and hence also in that of P^9_7.*

Proof We have a relation in Ext,

$$h_2 \cdot h_2 n = h_1 t.$$

Therefore, $h_1t[9]$ is product of permanent cycles. The second claim follows from the naturality of the Adams spectral sequences. □

For Step 2, we first show the following lemma:

Lemma 4.4 *The element $h_1t[9]$ is not a boundary in the Adams spectral sequences of $\Sigma^7 C\eta$ and P^9_7.*

Proof The element $h_1t[9]$ is hit by a d_3 differential on $e_1[9]$.

In the Adams spectral sequence of S^9, we have the Bruner differential

$$d_3(e_1[9]) = h_1 t[9].$$

However, the element $e_1[9]$ is not present in either $\mathrm{Ext}(\Sigma^7 C\eta)$ or $\mathrm{Ext}(P^9_7)$.

Therefore, by naturality, the element $h_1t[9]$ cannot be hit by any d_r differential for $r \leq 3$ in the Adams spectral sequence of $\Sigma^7 C\eta$ and P^9_7.

We have the Adams E_2–page of $\Sigma^7 C\eta$ and P^9_7 in the 46 and 47 stems for $s \leq 7$ in Table 2.

We need to rule out two candidates: $h_0h_3h_5[9]$ and $h_1h_3h_5[8]$.

In the Adams spectral sequence of S^9, we have a d_4 differential,

$$d_4(h_0 h_3 h_5[9]) = h_0^2 x[9].$$

By naturality of the quotient map to S^9, the element $h_0h_3h_5[9]$ cannot support a d_4 differential that kills $h_1t[9]$.

For the element $h_1h_3h_5[8]$, it is straightforward to check it is a permanent cycle in the Adams spectral sequence of P^8_7, and hence a permanent cycle in that of P^9_7. This rules out the candidate $h_1h_3h_5[8]$ and completes the proof. □

	$\mathrm{Ext}(\Sigma^7 C\eta)$		$\mathrm{Ext}(P_7^9)$	
$s\backslash t-s$	46	47	46	47
7	$h_1t[9]$ $h_0^2x[9]$	•	$h_1t[9]$ $h_0^2x[9]$	• •
6	$h_0x[9]$	• •	$h_0x[9]$ $h_1x[8]$	• •
5		• •	•	• •
4	•	• •	• •	• • •
3	•	$h_0h_3h_5[9]$	•	$h_0h_3h_5[9]$ $h_1h_3h_5[8]$

Table 2: The Adams E_2–page of $\Sigma^7 C\eta$ and P_7^9 in the 46 and 47 stems for $s \leq 7$

Remark 4.5 In $\mathrm{Ext}^{6,6+46}(P_7^9)$, the element $h_1x[8]$ is clearly a surviving cycle. There are two possibilities for the other element $h_0x[9]$: it is either killed by a d_3 differential from $h_0h_3h_5[9]$, or it survives and detects $\{h_1h_3h_5\}[7]$. We will leave it to the reader to figure out which way it goes.

We prove Proposition 4.6 in Step 2.

Proposition 4.6 *Under the inclusion map $S^7 \hookrightarrow \Sigma^7 C\eta$, the element $h_1t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$. By naturality, the same statement is true after we further map it to $\pi_{46}(P_7^9)$.*

Proof By Lemma 4.2 and Proposition 4.3, the element $h_2n[9]$ survives in the Adams spectral sequence of $\Sigma^7 C\eta$, and detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

By Lemma 4.4, the element $h_1t[9] = h_2 \cdot h_2n[9]$ survives and detects the homotopy class $\nu\{n\}[9] \cdot \nu$. As shown in the proof of Lemma 2.2, the element $h_3d_1 = h_1e_1$

detects an element in the Toda bracket

$$\langle \eta, \nu\{n\}, \nu \rangle.$$

Therefore, by Lemma 5.3, we have

$$\nu\{n\}[9] \cdot \nu = \langle \eta, \nu\{n\}, \nu \rangle [7] = \sigma\{d_1\}[7]$$

in $\pi_{46}(\Sigma^7 C\eta)$. □

Now we prove Step 3:

Proposition 4.7 *Under the inclusion map $S^7 \hookrightarrow P_1^9$, the element $h_1 t[9]$ in $\mathrm{Ext}(P_1^9)$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(P_1^9)$.*

The idea of the proof of Proposition 4.7 is to make use of naturality of the Adams filtrations:

$$\begin{array}{ccccc} S^7 & \hookrightarrow & P_1^9 & \twoheadrightarrow & P_7^9 \\ h_3 d_1[7] & & & & h_1 t[9] \end{array}$$

The homotopy class $\sigma\{d_1\}[7]$ is detected by $h_3 d_1[7]$ in S^7, which has Adams filtration 5, while its image in $\pi_{46}(P_7^9)$ is detected by $h_1 t[9]$ by Proposition 4.6, which has Adams filtration 7. Therefore, to prove Proposition 4.7, we only need to rule out surviving cycles in the Adams filtration 6, which also lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞–page. Note that the element $h_3 d_1[7]$ is not present in $\mathrm{Ext}(P_1^9)$.

Proof We have the Adams E_2–page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \leq 8$ in Table 3.

There are four elements in $\mathrm{Ext}^{6,6+46}(P_1^9)$:

$$P h_1 h_5[6], \quad h_0^2 g_2[2], \quad h_1 x[8], \quad h_0 x[9].$$

Remark 4.5 rules out the last two candidates, since they do not lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞–page.

	$\mathrm{Ext}(P_1^9)$		$\mathrm{Ext}(P_1^\infty)$	
$s\backslash t-s$	46	47	46	47
8	$Ph_1^3h_5[4]$	•	$Ph_1^3h_5[4]$	•
	•	•		
	•	•		
7	$Ph_1^2h_5[5]$	•	$Ph_1^2h_5[5]$	•
	$h_1t[9]$	•	$h_1t[9]$	•
	$h_0^2x[9]$	•		•
		•		
6	$Ph_1h_5[6]$	•	$Ph_1h_5[6]$	$h_1h_5d_0[1]$
	$h_0^2g_2[2]$	•	$h_0^2g_2[2]$	$h_1x[9]$
	$h_1x[8]$	•	$h_1x[8]$	
	$h_0x[9]$			
5	$h_0^3h_3h_5[8]$	•	$h_0^3h_3h_5[8]$	$h_1g_2[2]$
	•	•	•	$h_1f_1[6]$
	•	•	•	
			•	
4	•	•	$h_1^3h_5[12]$	$h_0h_4^3[2]$
		•		$g_2[3]$
		•		$f_1[7]$
		•		
3	•	•	$h_1^2h_5[13]$	
	•		•	
			•	
2			$h_1h_5[14]$	$h_2h_5[13]$
1			$h_5[15]$	

Table 3: The Adams E_2–page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \leq 8$

In the table for the transfer map in [11], we have that the element $h_0^2g_2[2]$ maps to B_1. If the image of the homotopy class $\sigma\{d_1\}[7]$ were detected by $h_0^2g_2[2]$, then we would have a σ–extension from h_3d_1 to B_1 in $\pi_{46}S^0$, which by Lemma 2.2 is equivalent to an η–extension from h_1g_2 to B_1 in $\pi_{46}S^0$. However, the proof of Lemma 4.2.47 of [5] shows the latter is not true.

The only candidate left is $Ph_1h_5[6]$. To rule it out, we notice there is a long h_0 tower in the 46 stem of P_1^∞: from $h_5[15]$ to $Ph_1^3h_5[4]$. In particular, we have

$$h_0 \cdot Ph_1h_5[6] = Ph_1^2h_5[5], \quad h_0 \cdot Ph_1^2h_5[5] = Ph_1^3h_5[4].$$

Since

$$2 \cdot \sigma\{d_1\} = 0,$$

the image of the homotopy class $\sigma\{d_1\}[7]$ must have order 2. Therefore, we only need to show the element $Ph_1^2h_5[5]$ is not a boundary. In the following Lemma 4.8, we show that the elements in Adams filtration 4 to 6 of $\mathrm{Ext}(P_1^\infty)$ are all permanent cycles. This only leaves the possibility that $h_2h_5[13]$ kills $Ph_1^3h_5[4]$, but not $Ph_1^2h_5[5]$, and hence completes the proof. □

Lemma 4.8 *The elements in Adams filtration* 4 *to* 6 *of the* 47*–stem of* $\mathrm{Ext}(P_1^\infty)$ *are all permanent cycles.*

Proof There are seven elements:

$$h_1h_5d_0[1], \quad h_1x[9], \quad h_1g_2[2], \quad h_1f_1[6], \quad h_0h_4^3[2], \quad g_2[3], \quad f_1[7].$$

The spheres S^1, S^3 and S^7 are $H\mathbb{F}_2$–subcomplexes of P_1^∞ by the solution of the Hopf invariant one problem. Since the elements $h_1h_5d_0$, g_2 and f_1 are permanent cycles in the Adams spectral sequence for S^0, The elements $h_1h_5d_0[1]$, $h_1g_2[2]$ and $f_1[7]$ are permanent cycles.

The element $h_1f_1[6] = h_0 \cdot f_1[7]$ is therefore also a permanent cycle.

It is straightforward to show that the elements $h_1g_2[2]$ and $h_0h_4^3[2]$ are permanent cycles in the Adams spectral sequence of P_1^2. By naturality, they are permanent cycles in that of P_1^∞.

For the element $h_1x[9]$, one uses the $H\mathbb{F}_2$–subcomplex of P_1^∞ which contains cells in dimensions 3, 5, 7 and 9 to show that it is a permanent cycle. In fact, by comparing the Atiyah–Hirzebruch spectral sequence with the Adams spectral sequence of this 4 cell complex, it follows from the following relations in the stable homotopy groups of spheres:

$$0 \in \eta \cdot \{h_1x\}, \quad 0 \in \langle \nu, \eta, \{h_1x\}\rangle.$$

The homotopy class $\{h_1x\}[9]$ survives in the Atiyah–Hirzebruch spectral sequence, and is detected by $h_1x[9]$ in its Adams E_2–page. In particular, $h_1x[9]$ is a permanent cycle

in the Adams spectral sequence of this 4 cell complex, and therefore also a permanent cycle in the Adams spectral sequence of P_1^∞. □

Now we prove Step 4.

Lemma 4.9 *The element $h_1 t[9]$ maps to N under the transfer map.*

Proof We check the two tables in the appendix of [11]. See [11] for more details of the Lambda algebra notation we use here. The element N is in $\mathrm{Ext}^{8,8+46}(S^0) = \mathbb{Z}/2$. Checking the table for P_1^∞, we have that

$$\mathrm{Ext}^{7,7+46}(P_1^\infty) = (\mathbb{Z}/2)^2, \text{ generated by } (5)\ 11\ 12\ 4\ 5\ 3\ 3\ 3 \text{ and } (9)\ 3\ 5\ 7\ 3\ 5\ 7\ 7,$$

which means $\mathrm{Ext}^{7,7+46}(P_1^\infty)$ is generated by $Ph_1^2h_5[5]$ and $h_1 t[9]$. Since $Ph_1^2h_5[5]$ is divisible by h_0 in $\mathrm{Ext}(P_1^\infty)$, while N is not divisible by h_0 in $\mathrm{Ext}(S^0)$, $Ph_1^2h_5[5]$ cannot map to N under the transfer map. By the algebraic Kahn–Priddy theorem, the other generator $h_1 t[9]$ has to map to N. □

5 A lemma for extensions in the Atiyah–Hirzebruch spectral sequence

Let $\alpha\colon Y \to X$ and $\beta\colon Z \to Y$ be homotopy classes of maps between spectra. Suppose that the composite $\alpha\beta = 0$. Let $C\alpha$ and $C\beta$ be the cofiber of α and β, respectively.

We have cofiber sequences

$$Y \xrightarrow{\alpha} X \xrightarrow{i_\alpha} C\alpha \xrightarrow{\partial_\alpha} \Sigma Y,$$
$$Z \xrightarrow{\beta} Y \xrightarrow{i_\beta} C\beta \xrightarrow{\partial_\beta} \Sigma Z.$$

Denote by $L^\alpha\beta$ be the set of maps in $[\Sigma Z, C\alpha]$ such that the composite

$$\Sigma Z \to C\alpha \xrightarrow{\partial_\alpha} \Sigma Y$$

is $-\Sigma\beta$. The indeterminacy of the set $L^\alpha\beta$ is

$$i_\alpha \cdot [\Sigma Z, X].$$

Similarly, denote by $L_\beta\alpha$ be the set of maps in $[C\beta, X]$ such that the composite

$$Y \xrightarrow{i_\beta} C\beta \to X$$

is α. The indeterminacy of the set $L_\beta\alpha$ is

$$[\Sigma Z, X]\cdot\partial_\beta.$$

Lemma 5.1 *The two sets of maps $L^\alpha\beta\cdot\partial_\beta$ and $i_\alpha\cdot L_\beta\alpha$ in $[C\beta, C\alpha]$ are equal.*

Proof It is clear that the indeterminacy of the two sets are given by the composition

$$i_\alpha\cdot[\Sigma Z, X]\cdot\partial_\beta.$$

We need to show that they contain one common element. We have

$$\begin{array}{ccccccccccc}
 & & Y & \xrightarrow{\alpha} & X & \xrightarrow{i_\alpha} & C\alpha & \xrightarrow{\partial_\alpha} & \Sigma Y & \xrightarrow{-\Sigma\alpha} & \Sigma X \\
 & & \| & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \| & & \\
Z & \xrightarrow{\beta} & Y & \xrightarrow{i_\beta} & C\beta & \xrightarrow{\partial_\beta} & \Sigma Z & \xrightarrow{-\Sigma\beta} & \Sigma Y & &
\end{array}$$

Take $f\in L_\beta\alpha$. Since both lines are cofiber sequences, there exists a coextension $g\in L^\alpha\beta$ such that the diagram commutes. The commutativity of the middle square gives the claim. □

Lemma 5.2 *Let*

$$W\xrightarrow{\gamma}Z\xrightarrow{\beta}Y\xrightarrow{\alpha}X$$

be a sequence of homotopy classes of maps. Suppose that $\alpha\beta=0$ and $\beta\gamma=0$. Then the two sets of maps $-L^\alpha\beta\cdot\Sigma\gamma$ and $i_\alpha\cdot L_\beta\alpha\cdot L^\beta\gamma$ in $[\Sigma W, C\alpha]$ are equal.

Proof First, the indeterminacy of the former is $i_\alpha\cdot[\Sigma Z, X]\cdot\Sigma\gamma$. The indeterminacy of the latter is

$$i_\alpha\cdot[\Sigma Z, X]\cdot\partial_\beta\cdot L^\beta\gamma + i_\alpha\cdot L_\beta\alpha\cdot i_\beta\cdot[\Sigma W, Y].$$

Note that $\partial_\beta\cdot L^\beta\gamma=-\Sigma\gamma$ and $i_\alpha\cdot L_\beta\alpha\cdot i_\beta=i_\alpha\cdot\alpha=0$. So the two sets have the same indeterminacy.

We have the following diagram:

$$\begin{array}{ccccccccccc}
W & \xrightarrow{\gamma} & Z & \xrightarrow{i_\gamma} & C\gamma & \xrightarrow{\partial_\gamma} & \Sigma W & \xrightarrow{-\Sigma\gamma} & \Sigma Z & & \\
\downarrow{\scriptstyle \Sigma^{-1}L^\beta\gamma} & & \| & & \downarrow{\scriptstyle L_\gamma\beta} & & \downarrow{\scriptstyle L^\beta\gamma} & & \| & & \\
\Sigma^{-1}C\beta & \xrightarrow{-\Sigma^{-1}\partial_\beta} & Z & \xrightarrow{\beta} & Y & \xrightarrow{i_\beta} & C\beta & \xrightarrow{\partial_\beta} & \Sigma Z & \xrightarrow{-\Sigma\beta} & \Sigma Y \\
\downarrow{\scriptstyle \Sigma^{-1}L_\beta\alpha} & & \downarrow{\scriptstyle \Sigma^{-1}L^\alpha\beta} & & \| & & \downarrow{\scriptstyle L_\beta\alpha} & & \downarrow{\scriptstyle L^\alpha\beta} & & \| \\
\Sigma^{-1}X & \xrightarrow{-\Sigma^{-1}i_\alpha} & \Sigma^{-1}C\alpha & \xrightarrow{-\Sigma^{-1}\partial_\alpha} & Y & \xrightarrow{\alpha} & X & \xrightarrow{i_\alpha} & C\alpha & \xrightarrow{\partial_\alpha} & \Sigma Y
\end{array}$$

By Lemma 5.1, with suitable choices, all the squares commute, so the claim follows. In fact, taking any choices of $L^\beta\gamma$ and $L_\beta\alpha$, Lemma 5.1 says there exist choices for $L_\gamma\beta$ and $L^\alpha\beta$ making the diagrams commute. □

Now we have the following lemma as a corollary of Lemma 5.2 when the spectra X, Y, Z and W are all spheres.

Lemma 5.3 *Let α, β and γ be maps between spheres,*

$$\alpha\colon S^{|\alpha|}\to S^0,\quad \beta\colon S^{|\alpha|+|\beta|}\to S^{|\alpha|},\quad \gamma\colon S^{|\alpha|+|\beta|+|\gamma|}\to S^{|\alpha|+|\beta|}.$$

Then, in the Atiyah–Hirzebruch spectral sequence of $C\alpha$, we have a γ–extension

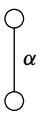
$$\beta[|\alpha|+1]\cdot\Sigma\gamma=\langle\alpha,\beta,\gamma\rangle[0].$$

Proof By definition, the set of classes represented by $\beta[|\alpha|+1]$ in the Atiyah–Hirzebruch spectral sequence is $-L^\alpha\beta$. On the other hand, by definition, $L_\beta\alpha\cdot L^\beta\gamma$ is $\langle\alpha,\beta,\gamma\rangle$, and $i_\alpha\cdot L_\beta\alpha\cdot L^\beta\gamma$ is $\langle\alpha,\beta,\gamma\rangle[0]$. So the claim follows from Lemma 5.2. □

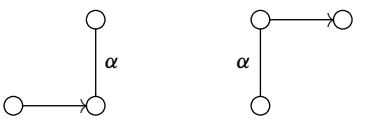
Appendix

In this appendix, we use cell diagrams as intuition for the statements of the lemmas in Section 5. It is very helpful when thinking of CW spectra. See [1; 12; 13] for example. For simplicity, we restrict to the cases when the spectra X, Y, Z and W are all spheres. For the definition of cell diagrams, see [1].

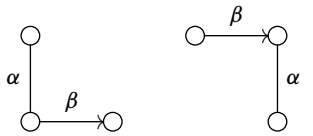
Let α and β be classes in the stable homotopy groups of spheres such that $\beta\cdot\alpha=0$. We denote the cofiber of α by



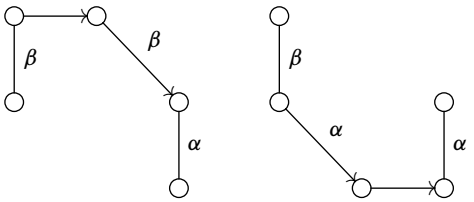
We denote the maps i_α and ∂_α by



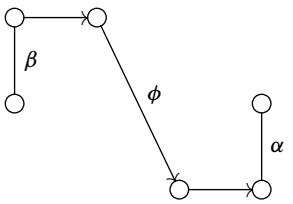
and the extension and coextension maps $L_\alpha\beta$ and $L^\alpha\beta$ by



Then Lemma 5.1 says the following two sets of maps are equal:

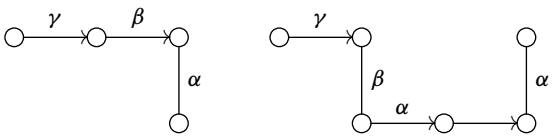


with the same indeterminacy



where $\phi \in \pi_{|\alpha|+|\beta|+1}S^0$ could be any class.

Suppose further that $\beta\cdot\gamma = 0$. Precomposing with $L^\beta\gamma$, Lemma 5.3 says that the following two sets of maps are equal:



References

[1] **M G Barratt**, **J D S Jones**, **M E Mahowald**, *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62, J. London Math. Soc. 30 (1984) 533–550 MR

[2] **M G Barratt**, **M E Mahowald**, **M C Tangora**, *Some differentials in the Adams spectral sequence, II*, Topology 9 (1970) 309–316 MR

[3] **R Bruner**, *A new differential in the Adams spectral sequence*, Topology 23 (1984) 271–276 MR

[4] **R Bruner**, *The cohomology of the mod* 2 *Steenrod algebra: a computer calculation*, preprint (2006) Available at `http://www.math.wayne.edu/~rrb/papers/cohom.pdf`

[5] **D C Isaksen**, *Stable stems*, preprint (2014) arXiv

[6] **D C Isaksen**, **Z Xu**, *Motivic stable homotopy and the stable* 51 *and* 52 *stems*, Topology Appl. 190 (2015) 31–34 MR

[7] **D S Kahn**, **S B Priddy**, *The transfer and stable homotopy theory*, Math. Proc. Cambridge Philos. Soc. 83 (1978) 103–111 MR

[8] **W H Lin**, *Algebraic Kahn–Priddy theorem*, Pacific J. Math. 96 (1981) 435–455 MR

[9] **R M F Moss**, *Secondary compositions and the Adams spectral sequence*, Math. Z. 115 (1970) 283–310 MR

[10] **M C Tangora**, *Some Massey products in* Ext, from "Topology and representation theory" (E M Friedlander, M E Mahowald, editors), Contemp. Math. 158, Amer. Math. Soc., Providence, RI (1994) 269–280 MR

[11] **G Wang**, **Z Xu**, *The algebraic Atiyah–Hirzebruch spectral sequence of real projective spectra*, preprint (2016) arXiv

[12] **G Wang**, **Z Xu**, *The triviality of the* 61*–stem in the stable homotopy groups of spheres*, Ann. of Math. 186 (2017) 501–580 MR

[13] **Z Xu**, *The strong Kervaire invariant problem in dimension* 62, Geom. Topol. 20 (2016) 1611–1624 MR

Shanghai Center for Mathematical Sciences, Fudan University
Shanghai, China

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, United States

`wangguozhen@fudan.edu.cn, xuzhouli@mit.edu`

Received: 26 July 2017 Revised: 12 July 2018

msp

Algebraic & Geometric Topology 18 (2018) 3907–3941

Dimension functions for spherical fibrations

CIHAN OKAY
ERGÜN YALÇIN

Given a spherical fibration ξ over the classifying space BG of a finite group G we define a dimension function for the m–fold fiber join of ξ, where m is some large positive integer. We show that the dimension functions satisfy the Borel–Smith conditions when m is large enough. As an application we prove that there exists no spherical fibration over the classifying space of $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ with p–effective Euler class, generalizing a result of Ünlü (2004) about group actions on finite complexes homotopy equivalent to a sphere. We have been informed that this result will also appear in upcoming work of Alejandro Adem and Jesper Grodal as a corollary of a previously announced program on homotopy group actions due to Grodal.

55M35; 55S10, 55S37

1 Introduction

This paper is motivated by a conjecture about group actions on products of spheres due to Benson and Carlson [5]. The conjecture states that the maximal rank of an elementary abelian p–group contained in a finite group is at most k if and only if there exists a finite free G–CW–complex X homotopy equivalent to a product of spheres $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$. For $k = 1$ this conjecture is proved by Swan [22]. The next case $k = 2$ is proved by Adem and Smith [2] for finite groups that do not involve $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ for any odd prime p.

An important technique developed in [2] for constructing free actions starts with a spherical fibration over BG whose Euler class is p–effective and uses fiber joins to construct a free action on a finite complex homotopy equivalent to a product of two spheres. One source of such a spherical fibration is a finite G–CW–complex $X \simeq \mathbb{S}^n$ with rank-one isotropy. Ünlü [23] proved that for $G = \mathrm{Qd}(p)$ there exists no such finite G–CW–complex. The main goal of this paper is to extend this result by showing that there exists no spherical fibration over BG with p–effective Euler class when G is $\mathrm{Qd}(p)$. We also show that $\mathrm{Qd}(p)$ cannot act freely on a finite complex homotopy

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3907

equivalent to $\mathbb{S}^n \times \mathbb{S}^n$. However, the case of the Benson–Carlson conjecture where the dimensions of the spheres are different remains open.

Given a spherical fibration $\xi\colon E \to BG$ over BG with fibers $\mathbb{S}^n$, there is an infinite-dimensional free G–space X_ξ, defined as the pullback of ξ along the universal fibration $EG \to BG$, such that the Borel construction $EG \times_G X_\xi \to BG$ is fiber homotopy equivalent to ξ. Two G–spaces X and Y are said to be hG–equivalent if there is a zig-zag sequence of G–maps between X and Y that are weak equivalences (nonequivariantly). The fiber homotopy classes of n–dimensional spherical fibrations over BG are in one-to-one correspondence with hG–equivalence classes of G–spaces that are homotopy equivalent to $\mathbb{S}^n$ (see Section A.3 for details). We will use this correspondence throughout the paper without further explanations.

Let G be a p–group and X be a finite-dimensional G–CW–complex. We write $H(-)$ for mod-p cohomology. Classical Smith theory says that if $H(X) \cong H(\mathbb{S}^n)$ for some n, then the fixed point space X^G also has the mod-p cohomology of a sphere. A systematic way of studying fixed point subspaces is to define dimension functions n_X by setting

$$H(X^K) \cong H(\mathbb{S}^{n_X(K)-1})$$

for a subgroup $K \leq G$. It is a fundamental fact that n_X satisfies certain properties called the Borel–Smith conditions. Smith theory fails for infinite-dimensional complexes. The problem is that, up to homotopy, every action can be made free by taking a product with the universal contractible free G–space EG. One way around this problem is to consider homotopy fixed points $X^{hK} = \mathrm{Map}(EK, X)^K$ instead of ordinary fixed points. An important algebraic tool for studying cohomology of homotopy fixed points is Lannes' T–functor and its variant the Fix functor. Here a technical point is that X needs to be replaced by its Bousfield–Kan p–completion $X^\wedge_p$, and the theory only works for elementary abelian p–groups. Then a theorem of Lannes relates the mod-p cohomology of homotopy fixed points $(X^\wedge_p)^{hV}$ to the algebraically defined object $\mathrm{Fix}_V(H(X_{hV}))$, where V is an elementary abelian p–subgroup of G, and $X_{hV} = EV \times_V X$.

Lannes' theory can be applied under certain conditions. We show that these conditions can be satisfied by replacing a given G–space $X \simeq \mathbb{S}^n$ with the p–completion of its m–fold join

$$X[m] = (\underbrace{X * \cdots * X}_{m})^\wedge_p.$$

For large m we prove that classical Smith theory holds for infinite-dimensional complexes, where the role of fixed points is played by homotopy fixed points.

Theorem 1.1 [1] *Let P be a finite p–group and $X \simeq (\mathbb{S}^n)^\wedge_p$ be a P–space. Then there exists a positive integer m such that $(X[m])^{hP} \simeq (\mathbb{S}^r)^\wedge_p$ for some r.*

We have been informed that this result is going to appear in Adem and Grodal [1] and is part of a program on homotopy group actions due to Grodal which was announced previously. Using this result we can define dimension functions for mod-p spherical fibrations. A mod-p spherical fibration is a fibration whose fiber has the homotopy type of a p–completed sphere. Given a mod-p spherical fibration $\xi\colon E \to BG$ and a p–subgroup $Q \leq G$, we can restrict the fibration ξ to a fibration $\xi|_{BQ}\colon E_Q \to BQ$ by taking the pullback along the inclusion map $BQ \to BG$. This corresponds to restricting the G–action on X_ξ to a Q–action via the inclusion map. We define the integer $n_{\xi[m]}(Q)$ via the weak equivalence

$$(X_\xi[m])^{hQ} \simeq (\mathbb{S}^{n_{\xi[m]}(Q)-1})^\wedge_p$$

which is a consequence of Theorem 1.1. It turns out that for m large enough, $n_{\xi[m]}$ satisfies the Borel–Smith conditions when regarded as an integer-valued function on the set of p–subgroups of G (see Theorem 4.6). The dimension function can be made independent of m by considering a rational-valued dimension function defined by

$$\mathrm{Dim}_\xi(Q) = \frac{1}{m} n_{\xi[m]}(Q) \qquad (m \gg 0)$$

for every p–subgroup $Q \leq G$.

The Euler class of a fibration is said to be p–effective if its restriction to elementary abelian p–subgroups of maximal rank is not nilpotent. This is a condition on the Euler class of a spherical fibration that is required to obtain a free action of a rank-two group on a product of two spheres using the Adem–Smith method. As an application of the dimension function that we defined, we obtain the following.

Theorem 1.2 [1] *Assume that $p > 2$. There exists no mod-p spherical fibration $\xi\colon E \to B\mathrm{Qd}(p)$ with a p–effective Euler class.*

We have been informed that this result is also going to appear in [1] and was previously announced as a theorem by Grodal. As a consequence of Theorem 1.2, we obtain that the Adem–Smith method of constructing free actions on finite complexes homotopy equivalent to a product of spheres does not work for $\mathrm{Qd}(p)$.

Another method for constructing free actions on a product of two spheres $\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ is given by Hambleton and Ünlü [14]. This method applies only to the equidimensional case ($n_1 = n_2$). The following theorem shows that this method cannot be used for $\mathrm{Qd}(p)$ either.

Theorem 1.3 *Let $G = \mathrm{Qd}(p)$. Then for any $n \geq 0$, there is no finite free G–CW–complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$.*

Therefore, if the Benson–Carlson conjecture holds, then in the construction of a complex $X \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ with free $\mathrm{Qd}(p)$–action the possibilities are narrowed down to distinct-dimensional spheres with a more exotic action.

The general theory of homotopy group actions has been considered by Adem and Grodal. They have informed us that Theorems 1.1 and 1.2 will also appear in their paper [1] under preparation. The idea of using dimension functions for studying mod-p spherical fibrations goes back to Grodal and Smith's unpublished earlier work, although an outline of their ideas can be found in the extended abstract [13]. Theorems 1.1 and 1.2 can also be thought of as corollaries of a program on homotopy group actions due to Grodal. We are grateful to Adem and Grodal for sharing their ideas with us on the subject, and we are looking forward to reading their complete account on the subject. Here we offer our proofs for Theorems 1.1 and 1.2 for completeness and to cover a gap in the existing literature. We should also mention that a result stated by Assadi [4, Corollary 4] also implies Theorems 1.1 and 1.2. Unfortunately, no proofs were provided for this result either.

The organization of the paper is as follows. In Section 2 we compute $\mathrm{Fix}(HE)$ for a fibration $\xi\colon E \to B\mathbb{Z}/p$ whose fiber has the cohomology of a sphere. Our main result Theorem 1.1 (Theorem 3.11) is proved in Section 3, where we study the space of sections of a mod-p spherical fibration over the classifying space of a p–group. The dimension function for an m–fold join of a mod-p spherical fibration is defined in Section 4. We prove the nonexistence result Theorem 1.2 (Corollary 4.9) in that section. In Section 5 we prove Theorem 1.3 (Theorem 5.1). We collect some results about mapping spaces, homotopy fixed points, and fiber joins in the appendix.

Acknowledgements We thank Alejandro Adem, Matthew Gelvin, and Jesper Grodal for their comments on the first version of this paper. Yalçin is supported by a Tübitak 1001 project (grant no. 116F194).

2 Spherical fibrations and Lannes' T–functor

In this section we compute $\mathrm{Fix}(HE)$ for a mod-p spherical fibration $E \to B\mathbb{Z}/p$. More generally, we work with fibrations where the mod-p cohomology of the fiber is isomorphic to the cohomology of a sphere. We modify the argument of [9, Chapter 3, Section 4] for the classical case, which works for group actions on finite-dimensional complexes, and use the connection between Lannes' T–functor and localization, established in [10].

2.1 Lannes' T–functor

Let $\boldsymbol{U}$ (resp. $\boldsymbol{K}$) denote the category of unstable modules (resp. unstable algebras) over the mod-p Steenrod algebra $\mathcal{A}_p$. Let V denote an elementary abelian p–group and HV the mod-p cohomology ring of V. The tensor product functor $HV \otimes -\colon \boldsymbol{U} \to \boldsymbol{U}$ has a left adjoint $T^V\colon \boldsymbol{U} \to \boldsymbol{U}$, which is called the Lannes T–functor. Let $\boldsymbol{U}(HV)$ denote the category of unstable modules M with an HV–module structure such that the multiplication map $HV \otimes M \to M$ satisfies the Cartan formula. Let $f\colon HW \to HV$ denote the map induced by a subgroup inclusion $V \subset W$. Its adjoint $\hat{f}\colon T^V HW \to \mathbb{F}_p$ is determined by a ring homomorphism $\hat{f}_0\colon (T^V HW)^0 \to \mathbb{F}_p$ in degree zero. We define

$$T^V_f(M) = \mathbb{F}_p \otimes_{(T^V HW)^0} T^V M,$$

where the $(T^V HW)^0$–module structure on $\mathbb{F}_p$ is the one determined by $\hat{f}_0$. Let S_f denote the multiplicatively closed subset of HV generated by the images of the Bocksteins of one-dimensional classes in HW that map nontrivially under f. The following is the main theorem of [10]. $\mathrm{Un}(-)$ denotes the largest unstable $\mathcal{A}_p$–submodule.

Theorem 2.1 (Dwyer–Wilkerson [10]) *Let W be an elementary abelian p–group, V a subgroup of W, and $f\colon HW \to HV$ the map induced by subgroup inclusion. Suppose that M is an object of $\boldsymbol{U}(HV)$ that is finitely generated as a module over HV. Then there is a natural isomorphism*

$$T^V_f(M) \cong \mathrm{Un}\, S_f^{-1}(M).$$

For an object M in $\boldsymbol{U}(HV)$ the Fix functor is defined by

$$\mathrm{Fix}_V(M) = \mathbb{F}_p \otimes_{T^V HV} T^V M,$$

where $\mathbb{F}_p$ is regarded as a $T^V HV$–module via the adjoint $\widehat{\varphi}\colon T^V HV \to \mathbb{F}_p$ of the identity map $\varphi\colon HV \to HV$; see [16, Section 4.4.3] for details. We record the following properties.

Proposition 2.2 *Let M be an object in $\boldsymbol{U}(HV)$. Then:*

(1) *The natural map $T^V_\varphi M \to HV \otimes \mathrm{Fix}_V M$ is an isomorphism in $\boldsymbol{U}(HV)$.*

(2) *If M is a finitely generated HV–module, then the localization of the natural map $M \to T^V_\varphi M$ with respect to S_φ is an isomorphism.*

Proof The first result is proved in [16, Proposition 4.5]. For the second result, the natural map is obtained as follows: Let $M \to HV \otimes T^V M$ denote the adjoint of the identity map $T^V M \to T^V M$. Composing this map with the unique algebra map $HV \to \mathbb{F}_p$ gives a map $M \to T^V M$. The desired map is obtained by applying the natural projection $T^V M \to T^V_\varphi M$ to the second factor. The fact that the resulting map is an isomorphism can be found in [10, Lemma 4.3, Section 5]. □

2.2 Spherical fibrations over $B\mathbb{Z}/p$

We will study fibrations $\xi\colon E \to B\mathbb{Z}/p$ where the cohomology HF of the fiber F is isomorphic to $H(\mathbb{S}^n)$ for some $n \geq 0$, and show that $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. Note that mod-p spherical fibrations satisfy this condition.

We start with recalling the mod-p cohomology ring of $\mathbb{Z}/p$. If $p = 2$, the cohomology ring $H(\mathbb{Z}/2)$ is a polynomial algebra $\mathbb{F}_2[t]$, where t is of degree one. When $p > 2$ we have $H(\mathbb{Z}/p) = \mathbb{F}_p[t] \otimes \Lambda[s]$, where s is of degree one and $t = \beta s$. Here β is the Bockstein map. The set S_φ corresponding to the identity map $\varphi\colon H\mathbb{Z}/p \to H\mathbb{Z}/p$ is generated by the Bockstein of the one-dimensional class in each case. If $S = \{1, t, t^2, \dots\}$, then localization with respect to S is the same as localization with respect to S_φ. For simplicity of notation, when $V = \mathbb{Z}/p$ we will write $T = T^V$, $T_\varphi = T^V_\varphi$, and $\mathrm{Fix} = \mathrm{Fix}_V$.

Lemma 2.3 *For an arbitrary fibration $\xi\colon E \to BV$ we have $(\mathrm{Fix}_V\, HE)^0 = 0$ if and only if $\xi^*\colon HV \to HE$ does not split in $\boldsymbol{K}$.*

Proof $(\mathrm{Fix}_V\, HE)^0$ is isomorphic to $(T^V_\varphi HE)^0$, which has an $\mathbb{F}_p$–basis Z_φ given by the set of $\boldsymbol{K}$–maps $\alpha\colon HE \to HV$ such that $\alpha\xi^*$ is the identity map on HV; see [20, Theorem 3.8.6]. □

Now we are ready to prove our main result in this section.

Theorem 2.4 *Let $\xi\colon E \to B\mathbb{Z}/p$ be a fibration such that $HF \cong H(\mathbb{S}^n)$. Then $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$.*

Proof The Serre spectral sequence of the fibration ξ has E_2–page given by

$$H(\mathbb{Z}/p) \otimes HF \Rightarrow HE,$$

which is nonzero only in two rows since $HF \cong H(\mathbb{S}^n)$. The spectral sequence is determined by the differential $d_{n+1}\colon E_2^{0,n} \to E_2^{n+1,0}$, whose image lies in the polynomial part of $H\mathbb{Z}/p$ [3, page 137]. First we assume that d_{n+1} is nonzero, ie ξ^* does not split. In this case t is nilpotent in HE. Hence the localization vanishes: $S^{-1}HE = 0$. By Theorem 2.1 we have $T_\varphi HE = 0$ and the first part of Proposition 2.2 implies that $\mathrm{Fix}\, HE = 0$. Next assume that $d_{n+1} = 0$, so that ξ^* splits. Localizing the natural map $HE \to T_\varphi HE$ with respect to S gives a diagram

$$\begin{array}{ccc} HE & \longrightarrow & T_\varphi HE \\ \downarrow & & \downarrow \\ S^{-1}HE & \xrightarrow{\cong} & S^{-1}(T_\varphi HE) \end{array} \tag{2.2.1}$$

Here the fact that the bottom map is an isomorphism is a consequence of the second part of Proposition 2.2. The right vertical monomorphism maps onto the unstable part of $S^{-1}(T_\varphi HE)$ as a consequence of Theorem 2.1 and the commutativity of the diagram. Since $HE \cong H(\mathbb{Z}/p) \otimes HF$ is a free $H(\mathbb{Z}/p)$–module generated by an element of degree n, the localization map $HE \to S^{-1}HE$ is a monomorphism. Note that in the spectral sequence multiplication by t is an isomorphism. After localizing the spectral sequence the two rows extend to negative degrees. Therefore comparing the spectral sequences we see that the localization map is an isomorphism in degrees $i \geq n$. Hence from the diagram (2.2.1) it follows that the natural map $H^i E \to (T_\varphi HE)^i \cong (H(\mathbb{Z}/p) \otimes \mathrm{Fix}(HE))^i$ is an isomorphism for $i \geq n$. Therefore

$$\bigoplus_{k=0}^{i} \mathrm{Fix}(HE)^k \cong \bigoplus_{k=0}^{i} H^k(F) = \mathbb{F}_p \oplus \mathbb{F}_p \quad \text{for } i \geq n. \tag{2.2.2}$$

One of the factors corresponds to a generator of $\mathrm{Fix}(HE)^0$, which is nonzero by Lemma 2.3. The other one corresponds to a generator of $\mathrm{Fix}(HE)$ in degree $r \leq n$. Therefore $\mathrm{Fix}(HE)$ is isomorphic to $H(\mathbb{S}^r)$, where $0 \leq r \leq n$. □

3 Cohomology of homotopy fixed points

In this section we study the homotopy fixed point space or equivalently the space of sections of a fibration by applying Lannes' results. For the relationship between the

T–functor and mapping spaces, our main references are [16] and [20]. See also the appendix for preliminaries on homotopy fixed points and space of sections.

3.1 Lannes' theorems

Lannes' T–functor gives an approximation to the cohomology of the mapping space $\mathrm{Map}(BV, Y)$, where Y is an arbitrary space. The evaluation map

$$\mathrm{ev}\colon BV \times \mathrm{Map}(BV, Y) \to Y$$

induces a map in cohomology $HY \to HV \otimes H(\mathrm{Map}(BV, Y))$ whose adjoint is $T^V HY \to H(\mathrm{Map}(BV, Y))$. The adjoint map factors through

$$\widehat{\mathrm{ev}}\colon T^V HY \to H(\mathrm{Map}(BV, Y_p^\wedge)). \tag{3.1.1}$$

In degree zero it is induced by the isomorphism $[BV, Y_p^\wedge] \to \boldsymbol{K}(HY, HV)$ defined by applying the cohomology functor [20, page 187]. It is convenient to work on a connected component associated to the homotopy class of a map $\alpha\colon BV \to Y_p^\wedge$. Let $\alpha^*\colon HY \to HV$ denote the homomorphism induced in cohomology. The component of $T^V HY$ at α^* is defined by

$$T^V(HY, \alpha^*) = \mathbb{F}_p \otimes_{(T^V HY)^0} T^V HY,$$

where the module structure on $\mathbb{F}_p$ is given by the adjoint $T^V HY \to \mathbb{F}_p$ of α^*. Then $\widehat{\mathrm{ev}}$ in (3.1.1) is the product of the maps

$$\widehat{\mathrm{ev}}_\alpha\colon T^V(HY, \alpha^*) \to H(\mathrm{Map}(BV, Y_p^\wedge)_\alpha),$$

where α runs over the homotopy classes of maps $BV \to Y_p^\wedge$.

We need the notion of freeness for the next theorem due to Lannes. Let G denote the left adjoint of the forgetful functor $\boldsymbol{K} \to \boldsymbol{E}$, where $\boldsymbol{E}$ denotes the category of graded vector spaces over $\mathbb{F}_p$. For an object $K \in \boldsymbol{K}$ let ΣK^1 denote the graded vector space isomorphic to K^1 in degree one and zero in other degrees. There is an inclusion of graded vector spaces $\Sigma K^1 \to K$. Applying G to this map and composing with the counit $GK \to K$ of the adjunction gives a canonical map

$$\chi\colon G(\Sigma K^1) \to K. \tag{3.1.2}$$

An unstable algebra K is said to be *free in degrees* ≤ 2 if χ is an isomorphism in degrees < 2 and a monomorphism in degree 2. For a more explicit definition, see [16, page 25].

Theorem 3.1 (Lannes [16, Theorem 3.2.4]) *Assume that HY and $T^V HY$ are of finite type. If $T^V HY$ is free in degrees ≤ 2, then*

$$\widehat{\mathrm{ev}}_\alpha \colon T^V(HY, \alpha^*) \to H(\mathrm{Map}(BV, Y_p^\wedge)_\alpha)$$

is an isomorphism of unstable algebras.

In the next section we will apply this theorem to mod-p spherical fibrations.

3.2 Mod-p spherical fibrations

A fibration whose fiber is homotopy equivalent to a p–completed sphere is called a mod-p spherical fibration. A source for such fibrations is the fiberwise completion of spherical fibrations. Let $\xi \colon E \to BV$ be a mod-p spherical fibration with connected fiber. There is a map of fibrations

$$\begin{array}{ccc} E & \longrightarrow & E_p^\wedge \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_p^\wedge} \\ BV & \longrightarrow & BV_p^\wedge \end{array}$$

where the horizontal maps are weak equivalences. In particular E is p–complete. Moreover, the diagram is a homotopy pullback diagram. This implies that there is a weak equivalence $\mathrm{Sec}(\xi) \to \mathrm{Sec}(\xi_p^\wedge)$ between the space of sections of these fibrations induced by the p–completion map. Therefore in applying Lannes' theory we can ignore the p–completions up to weak equivalence. We are interested in the cohomology of the space of sections $\mathrm{Sec}(\xi)$. As explained in Appendices A.1 and A.2, the space of sections $\mathrm{Sec}(\xi)$ is weakly equivalent to the homotopy fixed point space X_ξ^{hV}, where X_ξ is the V–space defined as the pullback of ξ along the universal bundle $EV \to BV$. The space of homotopy sections $\mathrm{hSec}(\xi)$ is isomorphic to $BV \times \mathrm{Sec}(\xi)$ as simplicial sets (see Section A.1). We will use Lannes' theory to study the cohomology of space of homotopy sections. Consider the diagram

$$\begin{array}{ccc} [BV, E] & \xrightarrow{\cong} & \boldsymbol{K}(HE, HV) \\ \downarrow & & \downarrow \\ [BV, BV] & \xrightarrow{\cong} & \boldsymbol{K}(HV, HV) \end{array}$$

induced by ξ, where the horizontal maps are bijections. Let Z_φ denote the subset of maps in $\boldsymbol{K}(HE, HV)$ which split ξ^* induced in cohomology. The subset of maps

in $[BV, E]$ that split ξ up to homotopy is in one-to-one correspondence with Z_φ. Then

$$T^V_\varphi HE = \prod_{\alpha^* \in Z_\varphi} T^V(HE, \alpha^*),$$

and the product of the evaluation maps $\widehat{\mathrm{ev}}_\alpha$ gives a map

$$T^V_\varphi HE \to H(\mathrm{hSec}(\xi)). \tag{3.2.1}$$

By Theorem 3.1 this map is an isomorphism of unstable algebras if $T^V_\varphi HE$ is free in degrees ≤ 2. Note that the conditions that HE and $T^V HE$ are of finite type are satisfied in this case because of the spectral sequence calculation and by Theorem 2.4.

Theorem 3.2 *Let $\xi\colon E \to BV$ be a mod-p spherical fibration and X_ξ denote the pullback of ξ along the universal fibration $EV \to BV$. Assume $\mathrm{Fix}_V(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. If $r \neq 1$, then*

$$H(X^{hV}_\xi) \cong H(\mathbb{S}^r).$$

Proof By the results in Section A.2, $X^{hV}_\xi \simeq \mathrm{Sec}(\xi)$. Since $\mathrm{hSec}(\xi) \simeq BV \times \mathrm{Sec}(\xi)$, it is enough to show that the map in (3.2.1) is an isomorphism. When $r = -1$, the result follows from Lemma 2.3. For the cases $r = 0$ and $r > 1$ we check the freeness condition. In (3.1.2) it turns out that the object $G(\Sigma K^1)$ is isomorphic to HW, where W is the $\mathbb{F}_p$–dual of K^1. Therefore χ is an isomorphism in degrees ≤ 2 if and only if $H^2 W \to K^2$ is a monomorphism. We claim that $T^V_\varphi HE$ is free in degrees ≤ 2 when $r = 0$ and $r > 1$. If $r = 0$, then the set Z_φ contains two maps α_0 and α_1, and $T^V_\varphi HE = T^V(HE, \alpha_0) \oplus T^V(HE, \alpha_1)$, where each component is isomorphic to HV. Note that for $K = HV$ the map χ is an isomorphism. Hence the freeness condition is satisfied. For $r > 1$ we have $T^V_\varphi HE = T^V(HE, \alpha)$ for a unique homotopy class of a map α. The freeness property holds since $(T^V_\varphi HE)^1 = H^1 V$ and $H^2 V \subset (T^V_\varphi HE)^2$. □

Remark 3.3 Note that in general $T^V_\varphi HE$ is not free in degrees ≤ 2 when $r = 1$. Hence in this case we cannot apply the theorem of Lannes (Theorem 3.1) to calculate the cohomology of the homotopy fixed point space. See also [16, Theorem 4.9.3].

We turn to another theorem of Lannes to study the homotopy type of the homotopy fixed point space.

Theorem 3.4 [16, Corollary 3.4.3] *Assume HY, $T^V HY$, and $H(\mathrm{Map}(BV, Y)_\alpha)$ are of finite type. Then $T^V(HY, \alpha^*) \to H(\mathrm{Map}(BV, Y)_\alpha)$ is an isomorphism of unstable algebras if and only if $(\mathrm{Map}(BV, Y)_\alpha)^\wedge_p \to \mathrm{Map}(BV, Y^\wedge_p)_\alpha$ is a homotopy equivalence.*

Using this theorem, the Fix calculation of Theorem 2.4, and Theorem 3.2 we can determine the homotopy type of the space of sections of a mod-p spherical fibration over $B\mathbb{Z}/p$.

Theorem 3.5 *Let $\xi\colon E \to B\mathbb{Z}/p$ be a mod-p spherical fibration such that $HF \cong H(\mathbb{S}^n)$ and $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$, where $r \neq 1$. Let X_ξ denote the pullback of ξ along $E\mathbb{Z}/p \to B\mathbb{Z}/p$. Then*

$$X_\xi^{h\mathbb{Z}/p} \simeq (\mathbb{S}^r)^\wedge_p,$$

where $-1 \leq r \leq n$.

Proof In Theorem 2.4 we showed that $\mathrm{Fix}(HE) \cong H(\mathbb{S}^r)$ for some $-1 \leq r \leq n$. If $r \neq 1$, then Theorem 3.2 implies that $H(\mathrm{Sec}(\xi)) \cong H(\mathbb{S}^r)$. The section space is the product of mapping spaces $\mathrm{Map}(BV, E)_\alpha$, where α is a representative of a homotopy class such that α^* lies in Z_φ. Applying Theorem 3.4 to each component we obtain a homotopy equivalence

$$\mathrm{Sec}(\xi)^\wedge_p \to \mathrm{Sec}(\xi)$$

after identifying $\mathrm{Sec}(\xi) \simeq \mathrm{Sec}(\xi^\wedge_p)$ up to homotopy. Therefore $\mathrm{Sec}(\xi) \simeq X_\xi^{h\mathbb{Z}/p}$ is a p–complete space that has the cohomology of a sphere. □

3.3 Fiber joins and the Fix functor

Next we look at the relationship between the Fix functor and fiber joins to be able to go around the problem in Theorem 3.5 when $r = 1$. Let $\xi_1\colon E_1 \to B\mathbb{Z}/p$ and $\xi_2\colon E_2 \to B\mathbb{Z}/p$ be two fibrations with fibers F_1 and F_2, respectively. We assume that $HF_i \cong H(\mathbb{S}^{n_i})$ and $\xi_1^*\colon H\mathbb{Z}/p \to HE_1$ splits.

Lemma 3.6 *The natural map*

$$\mathrm{Fix}(HE_1) \otimes \mathrm{Fix}(HE_2) \to \mathrm{Fix}\, H(E_1 \times_{B\mathbb{Z}/p} E_2)$$

is an isomorphism.

Proof Consider the pullback diagram of fibrations

$$\begin{array}{ccccc}
 & & F_1 & = & F_1 \\
 & & \downarrow & & \downarrow \\
F_2 & \longrightarrow & E_1 \times_{B\mathbb{Z}/p} E_2 & \xrightarrow{p_1} & E_1 \\
\| & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \xi_1} \\
F_2 & \longrightarrow & E_2 & \xrightarrow{\xi_2} & B\mathbb{Z}/p
\end{array}$$

We have $HE_1 = H\mathbb{Z}/p \otimes HF_1$. The differential d_{n+1} in the spectral sequence of ξ_2 is either zero or t^α. By comparing the spectral sequences we see that $H(E_1 \times_{B\mathbb{Z}/p} E_2)$ is either isomorphic to $HE_1 \otimes HF_2$ or $HE_1/(t^\alpha)$. Consider the natural map

$$\theta\colon HE_1 \otimes_{H\mathbb{Z}/p} HE_2 \to H(E_1 \times_{B\mathbb{Z}/p} E_2)$$

of unstable modules induced by p_1 and p_2. If $d_{n+1} = 0$, then the tensor product is isomorphic to $HE_1 \otimes HF_2$. If the differential is given by t^α, then it becomes $HE_1 \otimes_{H\mathbb{Z}/p} (H\mathbb{Z}/p)/(t^\alpha) \cong HE_1/(t^\alpha)$. Therefore in both cases θ is an isomorphism of unstable modules. In fact it is a morphism in $\boldsymbol{U}(H\mathbb{Z}/p)$. Then the result follows from the isomorphism $\mathrm{Fix}(M_1 \otimes_{H\mathbb{Z}/p} M_2) \cong \mathrm{Fix}(M_1) \otimes \mathrm{Fix}(M_2)$, which is valid for $\boldsymbol{U}(H\mathbb{Z}/p)$–modules [16, Theorem 4.6.2.1]. □

Proposition 3.7 *Assume that* $\mathrm{Fix}\, H(E_i) \cong H(\mathbb{S}^{r_i})$ *for some* r_i. *Then there is an isomorphism*

$$\mathrm{Fix}(H(E_1 *_{B\mathbb{Z}/p} E_2)) \cong H(\mathbb{S}^{r_1+r_2+1}).$$

Proof Consider the homotopy pushout square

$$\begin{array}{ccc} E_1 \times_{B\mathbb{Z}/p} E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_1 *_{B\mathbb{Z}/p} E_2 \end{array} \tag{3.3.1}$$

We can assume that $E_i = (X_i)_{h\mathbb{Z}/p}$ for some $\mathbb{Z}/p$–space X_i. The assignment $X \mapsto \mathrm{Fix}\, H(X_{h\mathbb{Z}/p})$ defines an equivariant cohomology theory on the category of $\mathbb{Z}/p$–spaces [16, Section 4.7]. Then associated to the pushout diagram there is a Mayer–Vietoris sequence, which breaks into short exact sequences

$$0 \to \mathrm{Fix}\, H(E_1)^q \oplus \mathrm{Fix}\, H(E_2)^q \to (\mathrm{Fix}\, H(E_1) \otimes \mathrm{Fix}\, H(E_2))^q \to \mathrm{Fix}\, H(E_1 *_{B\mathbb{Z}/p} E_2)^{q+1} \to 0,$$

where we used Lemma 3.6 for the middle term. In degree zero we need to consider the reduced cohomology groups. Compare this sequence to the Mayer–Vietoris sequence of the homotopy pushout

$$\begin{array}{ccc} \mathbb{S}^{r_1} \times \mathbb{S}^{r_2} & \longrightarrow & \mathbb{S}^{r_1} \\ \downarrow & & \downarrow \\ \mathbb{S}^{r_2} & \longrightarrow & \mathbb{S}^{r_1} * \mathbb{S}^{r_2} \end{array}$$

The result follows from $\mathbb{S}^{r_1} * \mathbb{S}^{r_2} \cong \mathbb{S}^{r_1+r_2+1}$. □

Let $\xi\colon E \to B$ be a fibration with fiber F. The fiberwise p–completion (Section A.4) of ξ is a fibration $\xi^\wedge_{p/B}\colon E^\wedge_{p/B} \to B$ whose fiber is $F^\wedge_p$. We use the following notation (Section A.5)

$$X[m] = (\underbrace{X * \cdots * X}_{m})^\wedge_p$$

and for a fibration $\xi\colon E \to B$ we define

$$E_{/B}[m] = (\underbrace{E *_B \cdots *_B E}_{m})^\wedge_{p/B}$$

and denote the associated fibration by $\xi[m]\colon E_{/B}[m] \to B$.

Corollary 3.8 *Let $X = X_\xi$ and r be defined as in Theorem 3.5. Then for all $m > 2$,*

$$(X[m])^{h\mathbb{Z}/p} \simeq \mathbb{S}^r[m].$$

Proof By Corollary A.3 we have a fiber homotopy equivalence $(X[m])_{h\mathbb{Z}/p} \simeq E_{/B\mathbb{Z}/p}[m]$. Since $X[m]$ is homotopy equivalent to a p–completed sphere, using Proposition 3.7 we obtain

$$\operatorname{Fix} H(E_{/B\mathbb{Z}/p}[m]) \cong \operatorname{Fix} H(E *_{B\mathbb{Z}/p} \cdots *_{B\mathbb{Z}/p} E) \cong H(\mathbb{S}^r[m]).$$

Therefore we can apply Theorem 3.5 to $\xi[m]$. □

Remark 3.9 According to Theorem 3.5, as long as $r \neq 1$ the statement of Corollary 3.8 holds with $m = 1$. The problem we faced for $r = 1$ can be handled by taking joins. When $r = 1$ it suffices to take $m = 2$ to obtain

$$(X[2])^{h\mathbb{Z}/p} \simeq (\mathbb{S}^3)^\wedge_p.$$

3.4 Finite p–groups

Next we extend our results to p–groups. Let P be a finite p–group and $Z \cong \mathbb{Z}/p$ be a subgroup of P contained in the center. Consider a mod-p spherical fibration $\xi\colon E \to BP$, and let $X = X_\xi$. We are interested in computing the homotopy type of the homotopy fixed point space X^{hP}. By transitivity of homotopy fixed points (Appendix A.2) we have

$$X^{hP} \simeq Y^{hP/Z},$$

where $Y = \operatorname{Map}(EP, X)^Z \simeq X^{hZ}$. By replacing X with $X[k]$ for some k and using Corollary 3.8, we can ensure that X^{hZ} is homotopy equivalent to a p–completed

sphere. Now we can consider the P/Z–space Y. But to be able to determine the homotopy type of $Y^{hP/Z}$ we may need to replace Y with $Y[l]$. At this step we need the following lemma.

Lemma 3.10 *Let X_1 and X_2 be P–spaces such that for $i = 1, 2$, we have that $X_i \simeq (\mathbb{S}^{n_i})^\wedge_p$ and $X_i^{hZ} \simeq (\mathbb{S}^{r_i})^\wedge_p$ for some $r_i > 0$. There is a weak equivalence*

$$\alpha\colon (X_1^{hZ} * X_2^{hZ})^\wedge_p \to ((X_1 * X_2)^\wedge_p)^{hZ},$$

which is induced by a map of P/Z–spaces when the homotopy fixed point spaces are interpreted as mapping spaces.

Proof We describe the map α. There is a natural map of P/Z–spaces

$$\alpha'\colon \mathrm{Map}(EP, X_1)^Z * \mathrm{Map}(EP, X_2)^Z \to \mathrm{Map}(EP, X_1 * X_2)^Z$$

defined by $\alpha'[f, g, t](z) = [f(z), g(z), t]$. Note that $\mathrm{Map}(EP, X)^Z$ is weakly equivalent to X^{hZ} via the natural map $EZ \to EP$. Hence we obtain

$$X_1^{hZ} * X_2^{hZ} \to (X_1 * X_2)^{hZ}.$$

Composing this with the natural map $(X_1 * X_2)^{hZ} \to ((X_1 * X_2)^\wedge_p)^{hZ}$, we obtain a map

$$X_1^{hZ} * X_2^{hZ} \to ((X_1 * X_2)^\wedge_p)^{hZ}.$$

Completion of this map at p gives the map α. Note that since $r_1 + r_2 + 1 > 1$ we can apply Theorem 3.5 to conclude that $((X_1 * X_2)^\wedge_p)^{hZ}$ is p–complete. To see that α is a weak equivalence it suffices to show that the map induced in mod-p cohomology is an isomorphism. A Mayer–Vietoris type of argument shows that

$$H((X_1^{hZ} * X_2^{hZ})^\wedge_p) \cong H(\mathbb{S}^{r_1+r_2+1}).$$

On the other hand, Proposition 3.7 implies that $\mathrm{Fix}\, H(X_1 * X_2)_{hZ} \cong H(\mathbb{S}^{r_1+r_2+1})$ and by Theorem 3.5 $((X_1 * X_2)^\wedge_p)^{hZ}$ is weakly equivalent to the p–completion of $\mathbb{S}^{r_1+r_2+1}$. □

Now we are ready to prove the main theorem of this section.

Theorem 3.11 *Let P be a finite p–group and $\xi\colon E \to BP$ be a mod-p spherical fibration. Then there exists a positive integer m such that $X[m]^{hP} \simeq (\mathbb{S}^r)^\wedge_p$, where $X = X_\xi$.*

Proof We will use the transitivity property (Appendix A.2) of homotopy fixed points,

$$(\mathrm{Map}(EP, X)^Z)^{hP/Z} \simeq X^{hP},$$

where $\mathrm{Map}(EP, X)^Z \simeq X^{hZ}$, and we will do induction on the order of P. We can assume X^{hZ} is nonempty, otherwise the result holds trivially. Using Corollary 3.8 we can replace X by $X[k]$ to ensure that $Y = \mathrm{Map}(EP, X[k])^Z \simeq X[k]^{hZ}$ has the homotopy type of a p–completed simply connected sphere. We regard Y as a P/Z–space. Since the order of P/Z is less than the order of P there exists, by the induction hypothesis, some l such that the homotopy fixed points $(Y[l])^{hP/Z}$ is weakly equivalent to a p–completed sphere. We claim that the homotopy fixed points of $X[kl]$ under the action of P is a p–completed sphere. To see this let $A = X[k]$. There is a weak equivalence

$$(A^{hZ})[l] \to (A[l])^{hZ}$$

that is induced by a map of P/Z–spaces when regarded as a map between the associated mapping spaces. This can be shown by using Lemma 3.10 and doing induction on l. Now consider the natural P–map

$$X[kl] \to (X[k])[l],$$

which is also a weak equivalence. Using these two maps we obtain a zig-zag of weak equivalences

$$(A^{hZ})[l] \to (A[l])^{hZ} \leftarrow X[kl]^{hZ}$$

through P/Z–maps. Thus we have

$$X[kl]^{hP} \simeq (Y[l])^{hP/Z}$$

and the result follows by induction. □

4 Dimension functions

In this section we will define dimension functions for spherical fibrations and show that they satisfy the Borel–Smith conditions after taking fiber joins.

4.1 Dimension functions

Let $C(G)$ denote the ring of integer-valued functions defined on the set of all subgroups of G that are constant on G–conjugacy classes. Let $H \leq G$ be a subgroup. Given a

finite G–CW–complex Y with $H(Y) \cong H(\mathbb{S}^n)$, the dimension function n_Y of Y is defined by $H(Y^H) \cong H(\mathbb{S}^{n_Y(H)-1})$. Thus Y gives rise to an element n_Y of $C(G)$.

We extend this definition to our situation. Let $\xi\colon E \to BG$ be a mod-p spherical fibration. For a p–subgroup $Q \leq G$ let X_Q denote the pullback $\xi|_{BQ}$ along the universal fibration $EQ \to BQ$. By Theorem 3.11 there exists an m such that $(X_Q[m])^{hQ}$ has the homotopy type of $(\mathbb{S}^{r_Q})^\wedge_p$ for some r_Q for all p–subgroups $Q \leq G$. Note that by standard properties of homotopy fixed points and Theorem 3.5, we have $r_{Q'} \leq r_Q$ if $Q \leq Q'$ and $r_Q = r_{Q'}$ if Q is conjugate to Q' in G. Let $\mathcal{S}_p(G)$ denote the set of all p–subgroups of G. We define a function

$$n_{\xi[m]}\colon \mathcal{S}_p(G) \to \mathbb{Z} \quad \text{by } n_{\xi[m]}(Q) = r_Q + 1,$$

which is constant on G–conjugacy classes and call it the dimension function associated to the fibration $\xi[m]$. Given a spherical fibration we can consider the dimension function associated to its fiberwise p–completion.

Remark 4.1 To associate a dimension function to a mod-p spherical fibration independent of m we can define a rational-valued dimension function

$$\mathrm{Dim}_\xi(Q) = \frac{1}{m} n_{\xi[m]}(Q)$$

for all p–subgroups $Q \leq G$, where m is a positive integer large enough that Theorem 3.11 holds.

4.2 Dimensions and subgroups

We will prove an important relation satisfied by the dimension functions. Let V be an elementary abelian p–group of rank two. Let $\xi\colon E \to BV$ be a mod-p spherical fibration and $X = X_\xi$. Assume that n_ξ is defined. (This can be achieved by replacing ξ with $\xi[m]$.) This means that the homotopy fixed points of X under the action of a subgroup of V is a p–completed sphere.

By the Thom isomorphism theorem for $W \leq V$ the reduced cohomology ring of the Thom space $\mathrm{Th}(\xi_W)$ of the fibration $\xi_W\colon (X^{hW})_{hV} \to BV$ is a free HV–module on a single generator $t(\xi_W)$. There is a map $X^{hV} \to X^{hW}$ defined as the composition

$$\mathrm{Map}(EV, X)^V \to \mathrm{Map}(EV, X)^W \to \mathrm{Map}(EW, X)^W$$

of the natural inclusion of the fixed points, and the map induced by $EW \to EV$. This map induces a diagram

$$\begin{array}{ccccc} (X^{hV})_{hV} & \longrightarrow & (X^{hW})_{hV} & & \\ \downarrow{\scriptstyle \xi_V} & & \downarrow{\scriptstyle \xi_W} & & \\ BV & = & BV & & \\ \downarrow & & \downarrow & & \\ \mathrm{Th}(\xi_V) & \xrightarrow{\alpha} & \mathrm{Th}(\xi_W) & \xrightarrow{\beta} & \Sigma(X_{W,V})_{hV} \end{array} \tag{4.2.1}$$

where $X_{W,V}$ is the cofiber of $X^{hV} \to X^{hW}$. The bottom row is a cofibration sequence. In the long exact sequence of cohomology groups

$$\cdots \to \widetilde{H}^i(\mathrm{Th}(\xi_W)) \xrightarrow{\alpha^*} \widetilde{H}^i(\mathrm{Th}(\xi_V)) \xrightarrow{\beta^*} H^i((X_{W,V})_{hV}) \to \cdots \tag{4.2.2}$$

we have $\alpha^*(t(\xi_W)) = e_{W,V}\, t(\xi_V)$ for some element $e_{W,V}$ in HV. Let us set $S_W = S_f$, where $f\colon HV \to HW$ is the map induced by a subgroup inclusion $W \leq V$.

Lemma 4.2 *Let $W \subset V$ be a subspace of codimension one. Then there is an isomorphism $H((X_{W,V})_{hV}) \cong HV/(e_{W,V})$.*

Proof Let Y denote the space of homotopy fixed points $\mathrm{Map}(EV, X)^W \simeq X^{hW}$. Let $V = W \times L$ be a splitting. Then $Y^{hL} \simeq X^{hV}$, and $(X^{hV})_{hV} \to (X^{hW})_{hV}$ induces a map in cohomology $H((X^{hW})_{hV}) \to H((X^{hV})_{hV})$. Note that $(X^{hW})_{hV} \simeq BW \times Y_{hL}$ and $(X^{hV})_{hV} \simeq BW \times (Y^{hL})_{hL}$. Therefore we obtain

$$HW \otimes H(Y_{hL}) \to HW \otimes H((Y^{hL})_{hL}),$$

which becomes an isomorphism after localizing with respect to S_L. This is a consequence of the isomorphism $T^L_\varphi H(Y_{hL}) \cong H((Y^{hL})_{hL})$ implied by Theorem 3.2 and the second part of Proposition 2.2 applied to $M = H(Y_{hL})$. Therefore the localization of $H((X^{hW})_{hV}) \to H((X^{hV})_{hV})$ with respect to S_V is an isomorphism. From the map between the cofiber sequences in (4.2.1) we see that the map between the cohomology rings of Thom spaces becomes an isomorphism after localizing with respect to S_V. Thus there is a diagram

$$\begin{array}{ccc} \widetilde{H}(\mathrm{Th}(\xi_W)) & \xrightarrow{\alpha^*} & \widetilde{H}(\mathrm{Th}(\xi_V)) \\ \downarrow & & \downarrow \\ S_V^{-1}\widetilde{H}(\mathrm{Th}(\xi_W)) & \xrightarrow{\cong} & S_V^{-1}\widetilde{H}(\mathrm{Th}(\xi_V)) \end{array}$$

where the vertical arrows are injective since $\widetilde{H}(\mathrm{Th}(\xi_W))$ and $\widetilde{H}(\mathrm{Th}(\xi_V))$ are HV–free. Therefore in (4.2.2) we have that α^* is injective and β^* is surjective. Then $H((X_{W,V})_{hV})$ is the quotient of the map

$$\widetilde{H}(\mathrm{Th}(\xi_W)) \to \widetilde{H}(\mathrm{Th}(\xi_V))$$

induced by $t(\xi_W) \mapsto e_{W,V}t(\xi_V)$. □

Let us simply denote $e_{W,V}$ by e_V when W is the trivial group. Let t_L denote the generator of the polynomial part of HL. We regard t_L as an element of HV via the isomorphism $HV \cong HL \otimes HW$.

Lemma 4.3 *Assume that e_V belongs to the polynomial part of HV. We have $n_X(W) > n_X(V)$ if and only if t_L divides e_V. Moreover,*

$$e_V = u \prod_W e_{W,V},$$

where $u \in \mathbb{F}_p$ is a unit and W runs over subspaces of codimension one in V such that $n_X(W) > n_X(V)$.

Proof Note that $n_X(W) > n_X(V)$ if and only if $e_{W,V} = at_L^\alpha$ for some $\alpha > 0$ and nonzero $a \in \mathbb{F}_p$. Let β be the maximal natural number such that t_L^β divides e_V. Consider the last cofiber sequences in the diagram (4.2.1) for the pair of subgroup inclusions given by $W \subset V$ and $1 \subset V$. There is a map between the cofiber sequences

$$\begin{array}{ccccc} \mathrm{Th}(\xi_V) & \longrightarrow & \mathrm{Th}(\xi_W) & \longrightarrow & \Sigma(X_{W,V})_{hV} \\ \| & & \downarrow & & \downarrow \\ \mathrm{Th}(\xi_V) & \longrightarrow & \mathrm{Th}(\xi_1) & \longrightarrow & \Sigma(X_{1,V})_{hV} \end{array}$$

We claim that the map $S_W^{-1}H(X_{hV}) \to S_W^{-1}H((X^{hW})_{hV})$ is an isomorphism. This follows from the transitivity of the Borel construction. The map $((X^{hW})_{hW})_{hL} \to (X_{hW})_{hL}$ between Borel constructions with respect to the action of L induces a map between the E_2–pages

$$H(L, S_W^{-1}H(X_{hW})) \to H\big(L, S_W^{-1}H((X^{hW})_{hW})\big)$$

of the associated spectral sequences. Then the claim follows from the isomorphism

$$S_W^{-1}H(X_{hW}) \cong S_W^{-1}H((X^{hW})_{hW}).$$

Now comparing the diagrams for $W \subset V$ and $1 \subset V$ we see that localization of $H((X_{1,V})_{hV}) \to H((X_{W,V})_{hV})$ with respect to S_W is an isomorphism. Thus by Lemma 4.2 there is an isomorphism $S_W^{-1}HV/(e_V) \cong S_W^{-1}HV/(e_{W,V})$. This forces $\alpha = \beta$. □

Proposition 4.4 *Assume e_V belongs to the polynomial part of HV. If $W_1, \dots, W_s$ denote the subspaces of codimension one in V, then*

$$n_X(1) - n_X(V) = \sum_{i=1}^{s} (n_X(W_i) - n_X(V)).$$

Proof By Lemma 4.3 a codimension one subspace W contributes to the sum on the right-hand side if and only if t_L divides e_V. Thus the result follows from comparing the degrees. Note that $|e_V| = n_X(1) - n_X(V)$ and $|e_{W,V}| = n_X(W) - n_X(V)$. □

Remark 4.5 In view of Corollary A.6, by choosing m large enough in $\xi[m]$ we can achieve the property that e_V belongs to the polynomial part of HV.

4.3 Borel–Smith functions

An element τ of $C(G)$ is called a Borel–Smith function [9, Definition 5.1] if it satisfies the following conditions:

(i) If $H \lhd K \leq G$ such that $K/H \cong \mathbb{Z}/p \times \mathbb{Z}/p$, and H_i/H denotes the cyclic subgroups, then

$$\tau(H) - \tau(K) = \sum_{i=0}^{p} (\tau(H_i) - \tau(K)).$$

(ii) If $H \lhd K \leq G$ such that $K/H \cong \mathbb{Z}/p$, where $p > 2$, then $\tau(H) - \tau(K)$ is even.

(iii) If $H \lhd L \lhd K \leq G$ such that $L/H \cong \mathbb{Z}/2$, then $\tau(H) - \tau(L)$ is even if $K/H \cong \mathbb{Z}/4$, and $\tau(H) - \tau(L)$ is divisible by 4 if K/H is a generalized quaternion group of order $\geq 2^3$.

Let $C_b(G)$ denote the additive subgroup of Borel–Smith functions in $C(G)$. We also say a function $\mathcal{S}_p(G) \to \mathbb{Z}$ constant on the G–conjugacy classes satisfies Borel–Smith conditions if it satisfies (i), (ii), and (iii) on p–subgroups.

Theorem 4.6 *There exists a positive integer m such that $n_{\xi[m]}\colon \mathcal{S}_p(G) \to \mathbb{Z}$ satisfies the Borel–Smith conditions.*

Proof Monotonicity is a consequence of Theorem 3.5. Condition (i) is proved in Proposition 4.4, where the hypothesis that e_V belongs to the polynomial part of HV holds by choosing m large enough. This is a consequence of Corollary A.6. Conditions (ii) and (iii) can be achieved by taking m large. □

When G is a finite nilpotent group, Borel–Smith functions can always be realized as dimension functions of virtual representations. Let $\mathrm{RO}(G)$ denote the Grothendieck group of real representations. There is an additive morphism $\dim\colon \mathrm{RO}(G) \to C(G)$ which sends a real representation ρ to the function which sends a subgroup H to the dimension of the fixed subspace ρ^H. Let $C_{\mathrm{rep}(G)}$ denote the image of this homomorphism. A key fact is that if G is a finite nilpotent group, then $C_b(G) = C_{\mathrm{rep}(G)}$. In the case of p–groups we can use honest representations when the Borel–Smith function is also monotone.

Theorem 4.7 [9, Theorem 5.13] *If P is a p–group and τ is a monotone Borel–Smith function, then there is a real representation ρ such that $\tau = \dim \rho$.*

Up to fiber joins the dimension function of a mod-p spherical fibration can be realized by the dimension function of a real representation.

Corollary 4.8 *Given a finite p–group P and a mod-p spherical fibration $\xi\colon E \to BP$, there is a positive integer m and a real representation ρ such that $n_{\xi[m]} = \dim \rho$.*

4.4 Proof of Theorem 1.2

A cohomology class in $H(G)$ is called *p–effective* if its restriction to maximal elementary abelian p–subgroups is not nilpotent. Let $\mathrm{Qd}(p)$ denote the semidirect product $(\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$, where the special linear group acts in the obvious way.

Corollary 4.9 *Let $p > 2$. There exists no mod-p spherical fibration $\xi\colon E \to B\mathrm{Qd}(p)$ with a p–effective Euler class.*

Proof The idea of the proof follows [23, Theorem 3.3] but here we use dimension functions for mod-p spherical fibrations. Assume that there is a fibration ξ with an effective Euler class. Consider the dimension function $n_{\xi[m]}$ for some large m. Its Euler class is still effective by Corollary A.6. Let P be a Sylow p–subgroup of $G = \mathrm{Qd}(p)$. The center $Z(P)$ is a cyclic group of order p. By Theorem 4.6 we can

choose m large enough that the dimension function of the restricted bundle $\xi[m]|_{BP}$ belongs to $C_b(P)$. Then Theorem 4.7 implies that there is a real representation ρ which realizes this dimension function. Since the Euler class is p–effective, the dimensions of subspaces of ρ fixed under a cyclic subgroup $C \leq P_i$ have the property that $\dim \rho^C = 0$ if and only if $C = Z(P)$ (see [23, Lemma 3.4]). Therefore on cyclic p–subgroups of P the dimension function $n_{\xi[m]}$ is zero only at $Z(P)$ but in G the center $Z(P)$ is conjugate to a noncentral cyclic p–subgroup C of P. Then $n_{\xi[m]}(Z(P)) = n_{\xi[m]}(C)$, but this gives a contradiction. □

5 Qd(p)–action on $\mathbb{S}^n \times \mathbb{S}^n$

In this section we prove the following theorem.

Theorem 5.1 *Let $G = \mathrm{Qd}(p)$. Then for any $n \geq 0$, there is no finite free G–CW–complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$.*

If $p = 2$, then $G = \mathrm{Qd}(2)$ is isomorphic to the symmetric group S_4, which includes A_4 as a subgroup. In this case the theorem follows from a result of Oliver [19] which says that the group A_4 does not act freely on a finite complex X homotopy equivalent to a product of two equal-dimensional spheres. Also note that for $n = 0$, the statement holds for obvious reasons. Hence it is enough to prove the theorem when p is an odd prime and $n \geq 1$.

Lemma 5.2 *Let G be a finite group generated by elements of odd order. Let X be a finite free G–CW–complex homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some $n \geq 1$. Then n is odd, and the induced G–action on $H^*(X;\mathbb{Z})$ is trivial.*

Proof It is enough to prove this for the case $G = \mathbb{Z}/p^k$, where p is an odd prime. By induction we can assume that the action of the maximal subgroup $H \leq G$ on cohomology is trivial. Consider the $G/H \cong \mathbb{Z}/p$ action on $H^*(X;\mathbb{Z})$. The only indecomposable $\mathbb{Z}$–free $\mathbb{Z}[\mathbb{Z}/p]$–modules are either 1–dimensional, $(p-1)$–dimensional, or p–dimensional [15, Theorem 2.6]. This gives that for $p > 3$, the G–action on $H^*(X;\mathbb{Z})$ is trivial. For $p = 3$, the only nontrivial module can occur in dimension n, and in this case G/H acts on $H^n(X;\mathbb{Z})$ with the action $x \to -y$ and $y \to x - y$, where x and y are generators of $H^n(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that the trace of this action is -1, so by the Lefschetz trace formula, $L(f) = 2 - (-1) = 3$ when n is odd, and $L(f) = 0 + (-1) = -1$ when n is even. In either case $L(f) \neq 0$, hence G cannot

admit a free action on X if the G/H action on homology is nontrivial. If the action is trivial, then again by the Lefschetz trace formula, n must be odd. □

The group $\mathrm{SL}_2(p)$ is generated by elements of order p. For example, we can take

$$A = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

as a set of generators. Since $\mathrm{Qd}(p) = (\mathbb{Z}/p)^2 \rtimes \mathrm{SL}_2(\mathbb{Z}/p)$ is a semidirect product of $(\mathbb{Z}/p)^2$ with $\mathrm{SL}_2(p)$, it is also generated by elements of order p. Hence we conclude the following.

Proposition 5.3 *Let $G = \mathrm{Qd}(p)$, where p is an odd prime. Suppose that there exists a finite free G–CW–complex X homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some $n \geq 1$. Then n is odd and G acts trivially on $H^*(X;\mathbb{Z})$.*

To complete the proof of Theorem 5.1, we use the Borel construction. Let $G = \mathrm{Qd}(p)$ with p odd, and let X be a finite free G–CW–complex homotopy equivalent to $\mathbb{S}^n \times \mathbb{S}^n$ for some integer $n \geq 1$. By Proposition 5.3, the induced action of G on X is trivial and $n = 2k-1$ for some $k \geq 1$. Consider the Borel fibration $X_{hG} \to BG$, where $X_{hG} = EG \times_G X$. There is an associated spectral sequence with E_2–term

$$E_2^{i,j} = H^i(G; H^j(X))$$

that converges to $H^{i+j}(X_{hG})$.

Note that since G acts freely on X we have $X_{hG} \simeq X/G$. From this one obtains that the cohomology ring $H^*(X_{hG})$ is finite-dimensional in each degree and vanishes above some degree. The first nonzero differential in the above spectral sequence takes the generators of $H^{2k-1}(X) = \mathbb{F}_p \oplus \mathbb{F}_p$ to the cohomology classes μ_1 and μ_2 in $H^{2k}(G)$. These classes are called the k–invariants of the G–space X.

For any subgroup $H \leq G$, we can restrict the above spectral sequence to the one for the action of H on X. This follows from the fact that the Borel construction is functorial. The k–invariants of this restricted action will be $\mathrm{res}^G_H \mu_1$ and $\mathrm{res}^G_H \mu_2$, where

$$\mathrm{res}^G_H \colon H^*(G) \to H^*(H)$$

denotes the homomorphism induced by inclusion of H into G.

Let V denote the (unique) normal elementary abelian subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$ in G. Let θ_1 and θ_2 denote the k–invariants of the restricted V–action on X. Note that the classes θ_i are restrictions of cohomology classes μ_1 and μ_2 in $H^{2k}(G)$. By the

Cartan–Eilenberg stable element theorem, the classes θ_i lie in the invariant subring $H^*(V)^{\mathrm{SL}_2(p)}$. This invariant ring is described in detail in [17, Proposition 1.4.1 and Claim 1.4.2]. If we write

$$H^*(V) = \mathbb{F}_p[x, y] \otimes \wedge(u, v),$$

then $H^{\mathrm{ev}}(V)^{\mathrm{SL}_2(p)} = \mathbb{F}_p[x, y]^{\mathrm{SL}_2(p)} \otimes \wedge(vu)$, where $\mathbb{F}_p[x, y]^{\mathrm{SL}_2(p)} = \mathbb{F}_p[\xi, \zeta]$ is a polynomial subalgebra generated by

$$\xi = \sum_{i=0}^{p} (x^{p-i} y^i)^{p-1} \quad \text{and} \quad \zeta = xy^p - yx^p.$$

For $i = 1, 2$, let $\theta_i = f_i(\xi, \zeta) + uvg_i(\xi, \zeta)$ for some polynomials f_i and g_i. Since θ_1 and θ_2 are integral classes, ie they are in the image of the map $H^*(V, \mathbb{Z}) \to H^*(V, \mathbb{F}_p)$ induced by mod-p reduction, we have $g_i = 0$ for $i = 1, 2$. This can be seen easily by applying the Bockstein operator $\beta\colon H^*(V) \to H^{*+1}(V)$ to the classes θ_i. Since $\beta(u) = x$ and $\beta(v) = y$, we obtain

$$0 = \beta(\theta_i) = \beta(uv)g_i = (xv - uy)g_i.$$

This gives $g_i = 0$. Hence the k–invariants θ_1 and θ_2 lie in the polynomial subalgebra $\mathbb{F}_p[\xi, \zeta]$.

Let $I = (\theta_1, \theta_2)$ be the ideal in $H^*(V)$ generated by θ_1 and θ_2. By a theorem of Carlsson [8, Corollary 7], the cohomology ring $H^*(X_{hV}) \cong H^*(X/V)$ is isomorphic to $H^*(V)/I$. In fact it is proved that $\{\theta_1, \theta_2\}$ is a homogenous system of parameters, hence it gives a regular sequence (in any order it is taken). This makes the spectral sequence collapse at the E_{n+2}–page, and gives that $H^*(X_{hV}) \cong H^*(V)/I$, which is finite-dimensional as a vector space.

For every $j \geq 0$, let $P^j\colon H^r(V) \to H^{r+2(p-1)j}(V)$ denote the Steenrod operation. From the isomorphism above, we obtain that the ideal I is closed under Steenrod operations, meaning that for every $j \geq 0$, we have $P^j(u) \in I$ for every $u \in I$. A slightly stronger Steenrod closeness condition also holds:

Lemma 5.4 *Let M denote the $\mathbb{F}_p$–vector space generated by θ_1 and θ_2. For every $m \in M$, and for $j \geq 0$, there exist α_1 and α_2 in the invariant subring $\mathbb{F}_p[\xi, \zeta]$ such that $P^j(m) = \sum_i \alpha_i \theta_i$.*

Proof Note that it is enough to show that the elements α_i can be chosen from $H^*(V)^{\mathrm{SL}_2(p)}$. Since both $P^j(m)$ and θ_i belongs to the polynomial part of the invariant

subring, this will imply α_i also belongs to $\mathbb{F}_p[\xi,\zeta]$. For this we will show that α_i can be taken as $\alpha_i = \operatorname{res}^G_V \lambda_i$ for some $\lambda_i \in H^*(G)$. Hence it is enough to show that the ideal $J \subset H^*(G)$ generated by μ_1 and μ_2 is closed under Steenrod operations, where μ_1 and μ_2 are the k–invariants of the G–action on $\mathbb{S}^n \times \mathbb{S}^n$. For this, first note that $\{\mu_1, \mu_2\}$ is a homogeneous system of parameters because its restriction to elementary abelian subgroups is a homogeneous system of parameters. If Z is the center of a Sylow p–subgroup of G, then $\operatorname{res}^G_Z \mu_i \neq 0$ for some i. Assume that $\operatorname{res}^G_Z \mu_1 \neq 0$, then by [7, Theorem 12.3.3], μ_1 is a nonzero divisor. This gives that the spectral sequence $E_2^{i,j} = H^i(G; H^j(X))$ described above collapses at E_{n+2}–page, in particular, $d_{2n+1} = 0$. Hence $H^*(G)/J$ is a module over Steenrod algebra, giving that J is closed under Steenrod operations. □

If $M \subset H^{2k}(V)$ is a subspace which lies inside the subring $\mathbb{F}_p[\xi,\zeta]$ and has the Steenrod closeness property given in the above lemma, then we say the subspace M is closed under Steenrod operations in the invariant subring.

Proposition 5.5 *Let $M \subset H^{2k}(V)$ be a nonzero subspace that lies inside the invariant subring $\mathbb{F}_p[\xi,\zeta]$. If M is closed under the Steenrod operations in the invariant subring, then M lies in the principal ideal generated by ζ.*

Proof We will use an argument similar to the argument given by Oliver [19] for A_4–actions on $\mathbb{S}^n \times \mathbb{S}^n$. A nonzero homogeneous polynomial $f \in \mathbb{F}_p[\xi,\zeta]$ can be written as a sum

$$f = \big(f_0(\xi) + f_1(\xi)\zeta + \cdots + f_t(\xi)\zeta^t\big)\zeta^s$$

for some homogeneous polynomials $f_i(\xi) = a_i \xi^{d_i}$, where f_0 and f_t are nonzero and $s \geq 0$. Note that the degree of ξ is $2p(p-1)$, the degree of ζ is $2(p+1)$, hence for dimension reasons $f_i = 0$ when p does not divide i. So we can conclude that every homogenous element f in $\mathbb{F}_p[\xi,\zeta]$ is of the form

$$f = \left(\sum_{i=0}^{t'} f_i(\xi)\zeta^{ip}\right)\zeta^s$$

for some homogenous polynomials $f_i(\xi) = a_i \xi^{d_i}$ with $a_i \in \mathbb{F}_p$ and $f_0(\xi) \neq 0$. By direct calculation, it is easy to see $P^1(\zeta) = 0$ and $P^1(\xi) = \zeta^{p-1}$. So if $P^1(f) = 0$, then

$$\left(\sum_{i=0}^{t'} P^1(f_i)\zeta^{ip}\right)\zeta^s = 0,$$

which gives $0 = P^1(f_i) = a_i d_i \xi^{d_i-1}\zeta^{p-1}$ for all i. Hence for every i, we have $d_i = pd_i'$ for some d_i'. From this we conclude that f is of the form $(f')^p\zeta^s$ for some f' when $P^1(f) = 0$. Furthermore, if the degree of f is divisible by p, then $s = ps'$ for some s', and hence in this case we have $f = (f'')^p$ for some homogeneous polynomial f'' in $\mathbb{F}_p[\xi, \zeta]$. We will use these observations in the calculations below.

Now let $M \subset H^{2k}(V)$ be a subspace as in the proposition. If p does not divide k, then each $m \in M$ can be written as $m = \zeta m'$ for some $m' \in \mathbb{F}_p[\xi, \zeta]$. In this case we have $M \subset (\zeta)$ as desired. So let us assume that p divides k. Then every $m \in M$ is of the form

$$m = \left(\sum_{i=0}^{t'} f_i(\xi)\zeta^{ip}\right)\zeta^{s'p}$$

for some homogeneous polynomials $f_i(\xi) = a_i\xi^{d_i}$ with $a_i \in \mathbb{F}_p$ and $f_0(\xi) \neq 0$. Since M is closed under Steenrod operations in the invariant subring, $P^1(m) = \sum_i \alpha_i m_i$ for some $m_i \in M$ and $\alpha_i \in \mathbb{F}_p[\xi, \zeta]$. However, there are no $2(p-1)$–dimensional classes in the invariant subring, hence we must have $P^1(m) = 0$ for every $m \in M$. By the above observation this implies that every $m \in M$ is equal to $(m')^p$ for some (unique) $m' \in H^{2k/p}(V)$. Let $M' \subset H^{2k/p}(V)$ be the subspace formed by elements m' such that $(m')^p \in M$. Note that M' also lies in the invariant subring $\mathbb{F}_p[\xi, \zeta]$.

Let $r \geq 1$ be the largest integer such that there exists a subspace $M_r \subset H^{2k/p^r}(V)$ such that M_r lies in the invariant subring $\mathbb{F}_p[\xi, \zeta]$ and every $m \in M$ is of the form $m = u^{p^r}$ for some $u \in M_r$. For every $j \geq 0$, we have

$$P^j(f^{p^b}) = \begin{cases} (P^a(f))^{p^b} & \text{if } j = ap^b, \\ 0 & \text{otherwise.} \end{cases}$$

So we have $P^{p^r}(u^{p^r}) = (P^1(u))^{p^r}$. Since M is closed under Steenrod operations in the invariant subring, we also have $P^{p^r}(m) = \sum_i \alpha_i m_i$ for some α_i in $\mathbb{F}_p[\xi, \zeta]$. Hence we have

$$(P^1(u))^{p^r} = \sum_i \alpha_i u_i^{p^r}$$

for some α_i in the invariant subring. We claim that the coefficients α_i are of the form $\alpha_i = (\alpha_i')^{p^r}$ for some α_i'.

To see this first note that P^{p^r} increases the dimension by $2p^r(p-1)$, so the dimension of α_i is $2p^r(p-1)$. If we apply P^1 to the above equation we get $0 = \sum_i P^1(\alpha_i)u_i^{p^r}$. Each element $P^1(\alpha_i)$ has degree $2(p^r+1)(p-1)$ and each element $u_i^{p^r}$ has degree at least $2(p+1)p^r$, so by dimension reasons we have $P^1(\alpha_i) = 0$ for all i. Hence

we have $\alpha_i = (\alpha_i')^p$ for some α_i'. Plugging this into the above equation and taking the p^{th} root we get

$$(P^1(u))^{p^{r-1}} = \sum_i \alpha_i' u_i^{p^{r-1}}.$$

Repeating the same argument r times, we get $\alpha_i = (\alpha_i')^{p^r}$ for some α_i' that lies in the invariant subring. This gives $P^1(u) = \sum_i \alpha_i' u_i$ for some α_i'. Since the degree of α_i' is $2(p-1)$ and there is no invariant element in that dimension, we get $\alpha_i' = 0$. This gives $P^1(u) = 0$. From this we conclude that for all u we have $u = (u')^p$ for some u'. This contradicts the assumption that r is the largest integer such that every $m \in M$ is of the form $m = u^{p^r}$ for some $u \in \mathbb{F}_p[\xi, \zeta]$. □

We are now ready to complete the proof of Theorem 5.1.

Proof of Theorem 5.1 Let $I = (\theta_1, \theta_2)$ denote the ideal generated by the k–invariants of the V–action on $X \simeq \mathbb{S}^n \times \mathbb{S}^n$. By [8, Corollary 7], there is an isomorphism

$$H^*(X/V) \cong H^*(V)/I,$$

hence $H^*(V)/I$ is a finite-dimensional vector space. Moreover the ideal I is closed under the Steenrod operations and it is of the form $M \cdot H^*(V)$, where $M \subset \mathbb{F}_p[x, y]^{\mathrm{SL}_2(p)}$ is the subspace generated by θ_1 and θ_2. By Lemma 5.4, M is closed under Steenrod operations in the invariant subring. Hence by Proposition 5.5 the ideal I is included in the principal ideal (ζ). This gives a contradiction because the fact that $H^*(V)/I$ is finite-dimensional implies that I cannot be included in a principal ideal generated by $\zeta = xy(x+y)\cdots(x+(p-1)y)$ by standard results in commutative algebra (see [8, Proposition 3]). □

Appendix

By a space we mean either a topological space or a simplicial set. The relation between the two is given by the singular simplicial set functor and the geometric realization functor. The category of simplicial sets is a model category with Quillen model structure with the usual weak equivalences and Kan fibrations. The geometric realization functor carries a Kan fibration to a Serre fibration.

A.1 Mapping spaces

Let X and Y be simplicial sets. The mapping space $\mathrm{Map}(X, Y)$ is the simplicial set whose set of n–simplices is given by $\boldsymbol{S}(\Delta[n] \times X, Y)$, and the simplicial structure

is induced by the ordinal maps $\Delta[n] \to \Delta[m]$. We will denote by $\mathrm{Map}_B(X, Y)_f$ the connected component of a map $f\colon X \to Y$, in other words the space of maps which are homotopic to f. Let $\boldsymbol{S}_{/B}$ denote the overcategory whose objects are maps $X \to B$ and whose morphisms are commutative triangles over B. Let $\mathrm{Map}_B(X, Y)$ denote the mapping space for the overcategory, which is defined to be the simplicial set with n–simplices given by the set $\boldsymbol{S}_{/B}(\Delta[n]\times X, Y)$ and with the simplicial structure defined similarly.

Let $\xi\colon E \to B$ be a Kan fibration of simplicial sets. Then ξ induces a fibration of mapping spaces

$$\xi_*\colon \mathrm{Map}(B, E) \to \mathrm{Map}(B, B)$$

whose fiber over the identity map id: $B \to B$ is the mapping space $\mathrm{Map}_B(B, E)$. We will also denote the fiber by $\mathrm{Sec}(\xi)$ and call it the space of sections of ξ. There is a pullback diagram

$$\begin{array}{ccc} \mathrm{hSec}(\xi) & \longrightarrow & \mathrm{Map}(B, E) \\ \downarrow & & \downarrow{\scriptstyle \xi_*} \\ \mathrm{Map}(B, B)_{\mathrm{id}} & \longrightarrow & \mathrm{Map}(B, B) \end{array}$$

where $\mathrm{hSec}(\xi)$ is the space of maps $B \to E$ such that the triangle

$$\begin{array}{ccc} B & \longrightarrow & E \\ \| & \swarrow_{\xi} & \\ B & & \end{array}$$

commutes up to homotopy.

In Lannes' theory we will consider fibrations over the classifying space of an elementary abelian p–group V. The classifying space BV is a simplicial group with product $BV \times BV \to BV$ induced by the product on V. The mapping space $\mathrm{Map}(BV, BV)$ is isomorphic to $\mathrm{Hom}(V, V)\times BV$ as a simplicial set [18, Proposition 25.2], and the adjoint $BV \to \mathrm{Map}(BV, BV)$ of the product map identifies BV with the identity component of the mapping space. For any X the mapping space $\mathrm{Map}(BV, X)$ has an induced action of BV. Also the simplicial monoid $\mathrm{Map}(BV, BV)$ acts by precomposition on the mapping space. The actions are equivariant with respect to the isomorphism $BV \to \mathrm{Map}(BV, BV)_{\mathrm{id}}$ of simplicial abelian groups [18, Proposition 25.3]. Given a

fibration $\xi\colon E \to BV$ in the diagram of fibrations

$$\begin{array}{ccc} BV \times \mathrm{Sec}(\xi) & \longrightarrow & \mathrm{hSec}(\xi) \\ \downarrow & & \downarrow \\ BV & \longrightarrow & \mathrm{Map}(BV, BV)_{\mathrm{id}} \end{array}$$

the horizontal arrows are isomorphisms.

A.2 Homotopy fixed points

Let G be a discrete group and X be a simplicial set with G–action, also called a G–space. The homotopy orbit space X_{hG} is the quotient $(EG \times X)/G$ under the diagonal action. The homotopy fixed point space X^{hG} is the simplicial subset $\mathrm{Map}(EG, X)^G$ of G–equivariant simplicial set maps in $\mathrm{Map}(EG, X)$. Let $f\colon X \to Y$ be a map of G–spaces which is also a weak equivalence. Then the induced maps $X_{hG} \to Y_{hG}$ and $X^{hG} \to Y^{hG}$ are weak equivalences.

Next we describe a transitivity property of homotopy fixed points [11, Lemma 10.5]. Let H be a normal subgroup of G. There is a natural action of G/H on the mapping space $\mathrm{Map}(EG, X)^H$, and there is a weak equivalence

$$X^{hG} \simeq (\mathrm{Map}(EG, X)^H)^{hG/H},$$

where $\mathrm{Map}(EG, X)^H \simeq X^{hH}$.

Let $\xi\colon E \to BG$ be a fibration with fiber F. Consider the pullback diagram

$$\begin{array}{ccc} X_\xi & \longrightarrow & E \\ \downarrow & & \downarrow \xi \\ EG & \longrightarrow & BG \end{array}$$

along the universal principal G–fibration. Since X_ξ is a free G–space, the natural map $h\colon (X_\xi)_{hG} \to (X_\xi)/G = E$ is a weak equivalence, in fact a trivial fibration. There is a map of fibrations

$$\begin{array}{ccc} X_\xi & \longrightarrow & F \\ \downarrow & & \downarrow \\ (X_\xi)_{hG} & \xrightarrow{h} & E \\ \downarrow \tilde{\xi} & & \downarrow \xi \\ BG & = & BG \end{array}$$

where $\tilde{\xi}$ is the composition $\xi \circ h$. We will usually switch from an arbitrary fibration to the natural projection $(X_\xi)_{hG} \to BG$. As a consequence we have the identifications

$$(X_\xi)^{hG} = \mathrm{Map}(EG, X_\xi)^G = \mathrm{Sec}(\tilde{\xi}) \simeq \mathrm{Sec}(\xi).$$

The equivalence $\mathrm{Sec}(\tilde{\xi}) \simeq \mathrm{Sec}(\xi)$ is a consequence of the fact that the map

$$\mathrm{Map}(BG, Y) \to \mathrm{Map}(BG, Y')$$

induced by a trivial fibration $Y \to Y'$ is also a trivial fibration [12]. Note that both spaces are fibrations over $\mathrm{Map}(BG, BG)$. Pulling back along the subspace $\mathrm{Map}(BG, BG)_{\mathrm{id}} \to \mathrm{Map}(BG, BG)$ induces the required weak equivalence between the spaces of sections.

A.3 hG–equivalence

Let $\xi_i\colon E_i \to BG$ for $i = 1, 2$ be two fibrations. A map $E_1 \to E_2$ of fibrations over BG is called a fiber homotopy equivalence if there is a homotopy inverse over BG. Let X_{ξ_i} denote the associated G–spaces. Then f induces a map $X_{\xi_1} \to X_{\xi_2}$ which is a G–homotopy equivalence. Conversely one can start with two G–spaces X_1 and X_2 and compare the fibrations associated to their Borel constructions. In this case a weaker notion of equivalence is enough. We say X_1 and X_2 are hG–equivalent if there is a zig-zag of G–maps that are also weak equivalences. Given hG–equivalent G–spaces X_1 and X_2, the products $EG \times X_1$ and $EG \times X_2$ are G–equivalent. Therefore $(X_1)_{hG}$ is fiber homotopy equivalent to $(X_2)_{hG}$. This implies that there is a one-to-one correspondence between fiber homotopy classes of fibrations over BG and hG–equivalence classes of G–spaces.

A.4 Completion at a prime

Let $X_p^\wedge$ denote the Bousfield–Kan completion of X at a prime p as defined in [6]. It comes with a natural map $X \to X_p^\wedge$. A space is called p–complete if this map is a weak equivalence. For example, the classifying space BP of a p–group is p–complete. A map $f\colon X \to Y$ induces an isomorphism $\widetilde{H}_*(f, \mathbb{F}_p)$ if and only if its p–completion $f_p^\wedge\colon X_p^\wedge \to Y_p^\wedge$ is a weak equivalence. Note that any weak equivalence between p–completed spaces is a homotopy equivalence since the p–completion of a space is a fibrant simplicial set, ie a Kan complex. Let $\xi\colon E \to B$ be a fibration with fiber F. The p–completion $\xi_p^\wedge\colon E_p^\wedge \to B_p^\wedge$ is still a fibration. The fiber lemma [6, Chapter II, 5.1] implies that if $\pi_1 B$ is a p–group and F is connected, then the fiber of $\xi_p^\wedge$ is the p–completion of F. There is also a relative version of the completion construction

which applies to a fibration $\xi\colon E \to B$, called the fiberwise completion at a prime p. We will denote the fiberwise p–completion by $\xi^\wedge_{p/B}\colon E^\wedge_{p/B} \to B$. This is a fibration whose fiber is given by $F^\wedge_p$. If ξ is a fibration over BP, then up to homotopy $\xi^\wedge_p$ can be identified with $\xi^\wedge_{p/B}$. More explicitly, there is a map of fibrations

$$\begin{array}{ccc} E^\wedge_{p/BP} & \longrightarrow & E^\wedge_p \\ \downarrow & & \downarrow \\ BP & \longrightarrow & BP^\wedge_p \end{array}$$

where the horizontal maps are weak equivalences.

Proposition A.1 *Let $\xi\colon E \to BP$ be a fibration with connected fiber and let $X = (X_\xi)^\wedge_p$ denote the p–completion of the P–space associated to ξ. Then there is a fiber homotopy equivalence*

$$\begin{array}{ccc} X_{hP} & \xrightarrow{\sim} & E^\wedge_p \\ \downarrow & & \downarrow \\ BP & = & BP \end{array}$$

Proof This result is proved in [16, Proposition 4.3.1] for an elementary abelian p–group. For a general p–group we proceed as follows. Let X_ξ denote the pullback of ξ along $EP \to BP$. It fits into a fibration sequence $F \to X_\xi \to EP$. After completion the sequence $F^\wedge_p \to (X_\xi)^\wedge_p \to EP^\wedge_p$ is a fiber sequence since $EP^\wedge_p$ is still contractible. Then the diagram

$$\begin{array}{ccc} (X_\xi)^\wedge_p & \longrightarrow & EP^\wedge_p \\ \downarrow & & \downarrow \\ E^\wedge_p & \longrightarrow & BP^\wedge_p \end{array}$$

is a homotopy pullback diagram since $E^\wedge_p \to BP^\wedge_p$ is a fibration with fiber $F^\wedge_p$ by the fiber lemma. This implies that $X = (X_\xi)^\wedge_p$ is a P–covering of $E^\wedge_p$. Thus X is a free P–space and the desired map is given by the projection $X_{hP} \to X/P$. □

A.5 Fiber joins

Next we will discuss the fiber join construction. First we look at the behavior with respect to the Borel construction and p–completion, and then we study the Euler class

of fiber joins. The fiber join of two fibrations $\xi_1\colon E_1 \to B$ and $\xi_2\colon E_2 \to B$ is defined to be the homotopy pushout of

$$\begin{array}{ccc} E_1 \times_B E_2 & \longrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & E_1 *_B E_2 \end{array} \tag{A.5.1}$$

where $E_1 \times_B E_2$ is the pullback of the maps ξ_1 and ξ_2 over B. When B is a point this definition specializes to the join construction, and we simply write $E_1 * E_2$.

Proposition A.2 *Let X and Y be G–spaces. There is a fiber homotopy equivalence*

$$\begin{array}{ccc} X_{hG} *_{BG} Y_{hG} & \xrightarrow{\sim} & (X * Y)_{hG} \\ \downarrow & & \downarrow \\ BG & = & BG \end{array}$$

Proof The map is induced by the Borel construction of the natural maps

$$X \to X * Y \leftarrow Y$$

and restricts to the identity map between the fibers. □

Given two spaces X and Y the join $X_p^\wedge * Y_p^\wedge$ is in general not p–complete [21, page 107]. For example, for spheres the p–completion of $(\mathbb{S}^n)_p^\wedge * (\mathbb{S}^m)_p^\wedge$ is given by $(\mathbb{S}^n * \mathbb{S}^m)_p^\wedge$. We use the notation

$$X[m] = (\underbrace{X * \cdots * X}_{m})_p^\wedge.$$

A fiberwise version of this definition is as follows: given a fibration $\xi\colon E \to B$ we define

$$E_{/B}[m] = (\underbrace{E *_B \cdots *_B E}_{m})_{p/B}^\wedge$$

and denote the associated fibration by $\xi[m]\colon E_{/B}[m] \to B$.

Corollary A.3 *Let P be a p–group. If ξ is the fibration $E \to BP$ associated to the Borel construction of a P–space X, then $\xi[m]$ is fiber homotopy equivalent to $(X[m])_{hP}$.*

Proof Using Proposition A.2 and induction we see that there is a fiber homotopy equivalence

$$*^n_{BP} E \to (*^n X)_{hP}$$

over BP. Let Y denote the pullback of the fiber join $*^n_{BP} E$ along $EP \to BP$. Then Y is G–equivalent to $*^n X$. Applying Proposition A.1 to $*^n X$ we obtain a fiber homotopy equivalence

$$((*^n X)^\wedge_p)_{hP} \to (*^n_{BP} E)^\wedge_p. \qquad \square$$

Remark A.4 The natural map $X[mn] \to (X[m])[n]$ is a homotopy equivalence. Similarly, $\xi[mn]$ is fiber homotopy equivalent to $(\xi[m])[n]$.

A.6 Euler class

We will study the Euler class of fiber joins of mod-p spherical fibrations. Note that the fiber join construction does not result in a mod-p spherical fibration until we fiberwise complete it at p. But the resulting fibration has a fiber whose mod-p cohomology is the mod-p cohomology of a sphere. Therefore for our purposes we consider a larger class of fibrations. Let $\xi\colon E \to B$ be a fibration whose fiber F satisfies $HF \cong H(\mathbb{S}^d)$ for some d. The Thom space $\mathrm{Th}(\xi)$ of ξ is defined to be the cofiber of the map ξ. Consider the diagram of cofibrations

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle \xi} \\ * & \longrightarrow & B \\ \downarrow & & \downarrow \\ \Sigma F & \xrightarrow{\mu} & \mathrm{Th}(\xi) \end{array}$$

The reduced cohomology ring $\widetilde{H}(\mathrm{Th}(\xi))$ is a free HB–module generated on a generator $t(\xi)$ of degree $d+1$ called the Thom class of ξ. We can take $t(\xi)$ to be the dual of the image of a chosen generator under the map $H_{d+1}(\Sigma F) \to H_{d+1}(\mathrm{Th}(\xi))$ induced by μ. The image of the Thom class under the natural map $H^{d+1}(\mathrm{Th}(\xi)) \to H^{d+1}(B)$ is called the Euler class $e(\xi)$. Alternatively, $e(\xi)$ is the image of the transgression of the generator of $H^d(F)$ in the Serre spectral sequence of ξ.

We will need an alternative description of fiber join construction. Given fibrations $\xi_i\colon E_i \to B$ with fiber F_i, where $i = 1, 2$, let us define a quotient space

$$E_1 \hat{*} E_2 = E_1 \times E_2 \times [0,1] / \sim$$

by the relations $(e_1, e_2, 0) \sim (e_1, e_2', 0)$ if $\xi_2(e_2) = \xi_2(e_2')$ and $(e_1, e_2, 1) \sim (e_1', e_2, 1)$ if $\xi_1(e_1) = \xi_1(e_1')$. Projecting onto each factor induces a fibration $\xi_1 \hat{*} \xi_2 \colon E_1 \hat{*} E_2 \to B \times B$ with fiber $F_1 * F_2$. The pullback along the diagonal inclusion $\Delta \colon B \to B \times B$ is exactly the fiber join construction

$$\begin{array}{ccc} F_1 * F_2 & = & F_1 * F_2 \\ \downarrow & & \downarrow \\ E_1 *_B E_2 & \longrightarrow & E_1 \hat{*} E_2 \\ \downarrow & & \downarrow{\scriptstyle \xi_1 \hat{*} \xi_2} \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

In more detail, the diagram (A.5.1) maps to the pullback and induces a weak equivalence between the spaces given in the two definitions.

Proposition A.5 *Let ξ_1 and ξ_2 be mod-p spherical fibrations over B. In the commutative diagram*

$$\begin{array}{ccc} \Sigma(F_1 * F_2) & \xrightarrow{\sim} & \Sigma F_1 \wedge \Sigma F_2 \\ \downarrow & & \downarrow{\scriptstyle \Sigma\mu_1 \wedge \Sigma\mu_2} \\ \mathrm{Th}(\xi_1 \hat{*} \xi_2) & \xrightarrow{\sim} & \mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2) \end{array}$$

the horizontal maps are weak equivalences.

Proof Let $D(\xi)$ denote the fiber join of the identity map $B \to B$ (regarded as a fibration) with a mod-p spherical fibration $\xi \colon E \to B$. The Thom space $\mathrm{Th}(\xi)$ can be described as the quotient $D(\xi)/B$. We have the identifications

$$\begin{aligned} \mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2) &= \frac{\mathrm{Th}(\xi_1) \times \mathrm{Th}(\xi_2)}{\mathrm{Th}(\xi_1) \vee \mathrm{Th}(\xi_2)} \\ &= \frac{D(\xi_1) \times D(\xi_2)}{(E \times D(\xi_2)) \cap (D(\xi_1) \times E)} \\ &\simeq \frac{D(\xi_1 \hat{*} \xi_2)}{\xi_1 \hat{*} \xi_2} \\ &= T(\xi \hat{*} \xi), \end{aligned}$$

which are compatible with the equivalence $\Sigma F_1 \wedge \Sigma F_2 \simeq \Sigma(F_1 * F_2)$. □

An almost immediate consequence of Proposition A.5 is that the Euler class of the fiber join of two mod-p spherical fibrations is the cup product of Euler classes of the individual fibrations.

Corollary A.6 *If ξ_1 and ξ_2 are mod-p spherical fibrations, then*

$$e(\xi_1 *_B \xi_2) = e(\xi_1)e(\xi_2).$$

Proof Proposition A.5 implies that the Thom class of $\xi_1 \hat{*} \xi_2$ is the cross product of the Thom classes of ξ_1 and ξ_2. Looking at the corresponding diagram of cohomology groups associated to the diagram

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \times B \\ \downarrow & & \downarrow \\ \mathrm{Th}(\xi_1 *_B \xi_2) & \longrightarrow & \mathrm{Th}(\xi_1) \wedge \mathrm{Th}(\xi_2) \end{array}$$

we see that the Euler class of $\xi_1 *_B \xi_2$ is the cup product of the Euler classes of ξ_1 and ξ_2. □

References

[1] **A Adem**, **J Grodal**, *Homotopical group actions and periodicity*, in preparation

[2] **A Adem**, **J H Smith**, *Periodic complexes and group actions*, Ann. of Math. 154 (2001) 407–435 MR

[3] **C Allday**, **V Puppe**, *Cohomological methods in transformation groups*, Cambridge Studies in Advanced Mathematics 32, Cambridge Univ. Press (1993) MR

[4] **A H Assadi**, *Some local-global results in finite transformation groups*, Bull. Amer. Math. Soc. 19 (1988) 455–458 MR

[5] **D J Benson**, **J F Carlson**, *Complexity and multiple complexes*, Math. Z. 195 (1987) 221–238 MR

[6] **A K Bousfield**, **D M Kan**, *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304, Springer (1972) MR

[7] **J F Carlson**, **L Townsley**, **L Valeri-Elizondo**, **M Zhang**, *Cohomology rings of finite groups*, Algebra and Applications 3, Kluwer Academic, Dordrecht (2003) MR

[8] **G Carlsson**, *On the rank of abelian groups acting freely on $(S^n)^k$*, Invent. Math. 69 (1982) 393–400 MR

[9] **T tom Dieck**, *Transformation groups*, De Gruyter Studies in Mathematics 8, de Gruyter, Berlin (1987) MR

[10] **W G Dwyer**, **C W Wilkerson**, *Smith theory and the functor T*, Comment. Math. Helv. 66 (1991) 1–17 MR

[11] **W G Dwyer**, **C W Wilkerson**, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. 139 (1994) 395–442 MR

[12] **P G Goerss**, **J F Jardine**, *Simplicial homotopy theory*, Progress in Mathematics 174, Birkhäuser, Basel (1999) MR

[13] **J Grodal**, **J H Smith**, *Algebraic models for finite G–spaces*, Oberwolfach Rep. 4 (2007) 2690–2692 MR

[14] **I Hambleton**, **O Ünlü**, *Examples of free actions on products of spheres*, Q. J. Math. 60 (2009) 461–474 MR

[15] **A Heller**, **I Reiner**, *Representations of cyclic groups in rings of integers, I*, Ann. of Math. 76 (1962) 73–92 MR

[16] **J Lannes**, *Sur les espaces fonctionnels dont la source est le classifiant d'un p–groupe abélien élémentaire*, Inst. Hautes Études Sci. Publ. Math. 75 (1992) 135–244 MR

[17] **J H Long**, *The cohomology rings of the special affine group of $\mathbb{F}_p^2$ and of* PSL$(3, p)$, PhD thesis, University of Maryland (2008) MR `https://search.proquest.com/docview/304560975`

[18] **J P May**, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies 11, Van Nostrand, Princeton, NJ (1967) MR

[19] **R Oliver**, *Free compact group actions on products of spheres*, from "Algebraic topology" (J L Dupont, I H Madsen, editors), Lecture Notes in Math. 763, Springer (1979) 539–548 MR

[20] **L Schwartz**, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, Univ. Chicago Press (1994) MR

[21] **D P Sullivan**, *Geometric topology: localization, periodicity and Galois symmetry*, *K*–Monographs in Mathematics 8, Springer (2005) MR

[22] **R G Swan**, *Periodic resolutions for finite groups*, Ann. of Math. 72 (1960) 267–291 MR

[23] **O Ünlü**, *Constructions of free group actions on products of spheres*, PhD thesis, University of Wisconsin (2004) MR `https://search.proquest.com/docview/305110769`

Department of Mathematics, Middlesex College, Western University
London, ON, Canada

Department of Mathematics, Bilkent University
Ankara, Turkey

`okay@math.ubc.ca, yalcine@fen.bilkent.edu.tr`

Received: 30 October 2017 Revised: 1 June 2018

msp

Algebraic & Geometric Topology 18 (2018) 3943–4000

Weighted sheaves and homology of Artin groups

GIOVANNI PAOLINI
MARIO SALVETTI

We expand the theory of weighted sheaves over posets, and use it to study the local homology of Artin groups. First, we use such theory to relate the homology of classical braid groups with the homology of certain independence complexes of graphs. Then, in the context of discrete Morse theory on weighted sheaves, we introduce a particular class of acyclic matchings. Explicit formulas for the homology of the corresponding Morse complexes are given, in terms of the ranks of the associated incidence matrices. We use such method to perform explicit computations for the new affine case $\widetilde{C}_n$, as well as for the cases A_n, B_n and $\widetilde{A}_n$ (which were already done before by different methods).

05E45, 20F36, 52C35

1. Introduction 3943
2. Preliminaries 3946
3. Homology of braid groups and independence complexes 3954
4. Precise matchings 3960
5. Precise matchings for Artin groups 3966
Concluding remarks 3996
References 3997

1 Introduction

The topological theory of Artin groups started in the 1960s with classical braid groups (see Fox and Neuwirth [24] and Arnold [2; 3]), and was broadened to general Artin groups in the 1970s; see Brieskorn [7], Brieskorn and Saito [8], Cohen [16], Deligne [21], Fuks [26], Gorjunov [27] and Vaĭnšteĭn [44]. It has received much attention in connection with problems in singularity theory, homotopy theory, hyperplane arrangements and combinatorics (see Paris [37] for some references), up to recent applications of some special cases to conjectures about hyperbolic groups; see Agol [1].

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.3943

Thanks to a good topological model for the $K(\pi, 1)$ space of Artin groups (see Salvetti [38; 39]), a combinatorial free resolution for these groups was found by De Concini and Salvetti [20], and several (co)homology computations were carried out by Callegaro, De Concini, Moroni, Procesi, Salvetti and Stumbo [19; 18; 14; 9; 10; 11; 12; 13] (equivalent resolutions also appeared in Squier [42], Ozornova [34] and Paolini [35]). The computational methods used in these papers are essentially based on filtering the algebraic complex and using the associated spectral sequence.

In Salvetti and Villa [41] a more combinatorial method of calculation was introduced, based on the application of discrete Morse theory to a particular class of sheaves over posets (called *weighted sheaves*). It is interesting to notice that similar ideas were considered later in Curry, Ghrist and Nanda [17], even if there the authors were mainly interested in computational aspects.

In this paper we expand the theory of weighted sheaves. First, we observe a surprising relationship between some twisted (co)homology of the classical braid groups and the (co)homology of certain *independence complexes*, which in their simplest form have already been studied in a combinatorial context (see for example Babson and Kozlov [4], Engström [22] and Kozlov [30]). This relationship is obtained in an indirect way, after localization. The exact formula we are going to prove is the following (it was announced in Salvetti [40] without proof).

Theorem 3.4 $H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \dfrac{R}{(\varphi_d)}\right)$.

Here on the left we have the homology with local coefficients of the classical braid group and on the right we have the homology of some independence complex with trivial coefficients (we indicate by φ_d the d^{th} cyclotomic polynomial).

As said before, our method is based on the notions of weighted sheaves over posets and of *weighted matchings*, which are interesting objects by themselves. In our situation, the d–weight $v_d(\sigma)$ of a simplex σ is the maximal power of the d^{th} cyclotomic polynomial which divides the Poincaré polynomial of the (finite) parabolic subgroup generated by the vertices of the simplex. We allow matchings between simplices of the same weight. Applying discrete Morse theory, one obtains a Morse complex which still computes the local homology.

Then, by applying these methods to other Coxeter graphs (in particular to finite and affine type cases), we come up with the notion of *precise matching*. In all the cases we consider, it is possible to produce acyclic weighted matchings such that the Morse complex turns

out to have a very nice property: the boundary of a critical p–simplex σ has a nonzero coefficient along a critical $(p-1)$–simplex τ if and only if $v_d(\sigma) = v_d(\tau) + 1$. Then the homology of the Morse complex reduces to the computation of the ranks of the incidence matrices among critical simplices (Theorems 4.2 and 4.4). We show that the existence of precise matchings can also be interpreted in terms of second-page collapsing of a spectral sequence which is naturally associated to the weighted sheaf (Proposition 4.3).

As we discuss at the beginning of Section 5, the existence of precise matchings for a certain Artin group has strong implications for its homology. In particular, we want to highlight the following nontrivial result:

Theorem 5.2 *Let $\boldsymbol{G_W}$ be an Artin group (corresponding to some finitely generated Coxeter group $\boldsymbol{W}$) that admits a φ–precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\boldsymbol{X_W}; R)$ does not have φ^k–torsion for $k \geq 2$.*

Here $\boldsymbol{X_W}$ (introduced in Section 2.4) is a finite CW–complex which is a deformation retract of the *orbit space* associated to $\boldsymbol{W}$, and $\pi_1(\boldsymbol{X_W}) = \boldsymbol{G_W}$. When $\boldsymbol{W}$ is finite and in a few other cases, $\boldsymbol{X_W}$ is known to be a $K(\boldsymbol{G_W}, 1)$, whereas this property is only conjectured in general; see Brieskorn [7], Callegaro, Moroni and Salvetti [13], Charney and Davis [15], Deligne [21], Hendriks [28], Lek [31], Okonek [33], Paris [37] and Salvetti [38; 39].

Theorem 5.2 can be seen as a result about the homology of the infinite cyclic covering of $\boldsymbol{X_W}$ and its monodromy. It is known that the conclusion of this theorem holds for all Artin groups of finite type, due to geometric reasons (see Remark 5.3). Our construction provides a new combinatorial condition that applies to a wider class of Artin groups (including, among others, the affine groups of type $\widetilde{A}_n$ and $\widetilde{C}_n$).

The methods we develop allow explicit and complete computations, in general in a more direct way compared to other known methods. We use them to compute the twisted homology in the case of affine Artin groups of type $\widetilde{C}_n$, obtaining the following result:

Theorem 5.20 *Let $\boldsymbol{G_W}$ be an Artin group of type $\widetilde{C}_n$. Then the φ_d–primary component of $H_*(\boldsymbol{G_W}; R)$ is trivial for d odd, and for d even is as follows:*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \leq m \leq n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k\frac{d}{2} + r$.

We also carry out complete computations in the cases A_n, B_n and $\widetilde{A}_n$; these cases were already done before by different methods [25; 19; 18; 11], but we need them to perform computations of more complicated cases.

In a recent preprint [36], the first author succeeded in constructing precise matchings for all remaining Artin groups of finite and affine type. Therefore, precise matchings seem to be a suitable and effective tool for studying the homology of Artin groups.

The paper is structured as follows. In Section 2 we recall the most important definitions and results of [41]. In particular, we introduce the general framework of weighted sheaves over posets, and then explain how it can be used to compute the homology of Artin groups. In Section 3 we study the case of braid groups (ie Artin groups of type A_n), and relate their homology with that of certain independence complexes. In Section 4 we introduce precise matchings and derive a general formula for the homology of the Morse complex. In Section 5 we apply the theory developed in Section 4 to the case of Artin groups. We construct precise matchings for Artin groups of type A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$, and use such matchings to explicitly compute the homology.

2 Preliminaries

In this section we are going to recall some definitions and constructions from [41] (see also [32]).

2.1 Weighted sheaves over posets

Let $(P, \preceq)$ be a finite poset.

Definition 2.1 Define a *sheaf of rings* over P (or a *diagram of rings* over P) as a collection

$$\{A_x \mid x \in P\}$$

of commutative rings together with a collection of ring homomorphisms

$$\{\rho_{x,y} \colon A_y \to A_x \mid x \preceq y\}$$

satisfying

$$\rho_{x,x} = \mathrm{id}_{A_x},$$
$$x \preceq y \preceq z \implies \rho_{x,z} = \rho_{x,y}\rho_{y,z}.$$

In other words, a sheaf of rings over P is a functor from P^{op} to the category of commutative rings.

Fix a PID R; usually we take $R = \mathbb{Q}[q^{\pm 1}]$. The divisibility relation

$$p_1 \,|\, p_2 \iff (p_1) \supseteq (p_2)$$

gives R the structure of a small category and R/R_* the structure of a poset (where R_* is the group of units in R) with minimum element the class of the units and maximum element 0. Any functor $w\colon (P, \preceq) \to (R, |)$, which maps every $x \in P$ to some $w(x) \in R \setminus \{0\}$, defines a sheaf over P by the collection

$$\{R/(w(x)) \mid x \in P\}$$

and

$$\{i_{x,y}\colon R/(w(y)) \to R/(w(x)) \mid x \preceq y\},$$

where $i_{x,y}$ is induced by the identity of R.

Definition 2.2 Given a poset P, a PID R and a morphism $w\colon (P, \preceq) \to (R, |)$ as above, we call the triple (P, R, w) a *weighted sheaf* over P and the coefficients $w(x)$ the *weights* of the sheaf.

Remark 2.3 Consider the *poset topology* over P, where a basis for the open sets is given by the upper intervals $\mathcal{B} = \{P_{>p},\ p \in P\}$. Let (P, R, w) be a weighted sheaf. Then one can see w as a functor from $(\mathcal{B}, \supseteq)$ to $(R, |)$ and thus a weighted sheaf defines a sheaf in the usual sense.

From now on our poset will be a simplicial complex K defined over a finite set S, with the partial ordering

$$\sigma \preceq \tau \iff \sigma \subseteq \tau.$$

We adopt the convention that K contains the empty simplex $\varnothing$. A weighted sheaf over K is given by assigning to each simplex $\sigma \in K$ a weight $w(\sigma) \in R$, with

$$\sigma \preceq \tau \implies w(\sigma) \,|\, w(\tau).$$

Let $C^0_*(K; R)$ be the 1–shifted standard algebraic complex computing the simplicial homology of K, ie

$$C^0_k(K; R) = \bigoplus_{\substack{\sigma \in K \\ |\sigma| = k}} R e^0_\sigma,$$

where e^0_σ is a generator associated to a given orientation of σ ($C^0_0(K;R) = Re^0_\varnothing$). The boundary is given by

$$\partial^0(e^0_\sigma) = \sum_{|\tau|=k-1} [\sigma:\tau]e^0_\tau,$$

where $[\sigma:\tau]$ is the incidence number (which is equal to ± 1 if $\tau \prec \sigma$, and otherwise is equal to 0).

Definition 2.4 The *weighted complex* associated to the weighted sheaf (K, R, w) is the algebraic complex $L_* = L_*(K)$ defined by

$$L_k = \bigoplus_{|\sigma|=k} \frac{R}{(w(\sigma))}\overline{e}_\sigma,$$

with boundary $\partial\colon L_k \to L_{k-1}$ induced by ∂^0:

$$\partial(a_\sigma \overline{e}_\sigma) = \sum_{\tau\prec\sigma} [\sigma:\tau] i_{\tau,\sigma}(a_\sigma)\overline{e}_\tau.$$

There is a natural projection $\pi\colon C^0_*(K;R) \to L_*$ which maps a generator e_σ in C^0_* to the generator $\overline{e}_\sigma$ in L_*.

Remark 2.5 The diagram $\{R/(w(\sigma)) \mid \sigma \in K\}$ also defines a sheaf over the poset K in the sense of Remark 2.3. The sheaf cohomology associated to the open covering given by the upper segments coincides with the homology of the weighted sheaf.

2.2 Decomposition and filtration

Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf. For any irreducible $\varphi \in R$, we define the φ–primary component $\mathcal{S}_\varphi = (K, R, w_\varphi)$ of the weighted sheaf $\mathcal{S}$ by setting

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)},$$

where $v_\varphi(\sigma)$ is the maximal r such that φ^r divides $w(\sigma)$. Since R is a PID and $w(\sigma) \neq 0$, such a maximal value exists. Notice that $\mathcal{S}_\varphi$ is a weighted sheaf. The weighted complex $(L_*)_\varphi$ associated to $\mathcal{S}_\varphi$ is called the φ–primary component of the weighted complex L_*. In degree k one has

$$(L_k)_\varphi = \bigoplus_{|\sigma|=k} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\overline{e}_\sigma.$$

The complex $(L_*)_\varphi$ has a natural increasing filtration by the subcomplexes

$$F^s(L_*)_\varphi = \bigoplus_{v_\varphi(\sigma)\le s} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\overline{e}_\sigma.$$

This filtration is associated to an increasing filtration of the simplicial complex K,

$$K_{\varphi,s} = \{\sigma \in K \mid v_\varphi(\sigma) \le s\}.$$

Then $F^s(L_*)_\varphi$ is the weighted complex associated to the weighted sheaf

$$(K_{\varphi,s}, R, w_\varphi|_{K_{\varphi,s}}).$$

Theorem 2.6 [41] *Let (K, R, w) be a weighted sheaf, with associated weighted complex L_*. For any irreducible $\varphi \in R$, there exists a spectral sequence*

$$E^r_{p,q} \Rightarrow H_*((L_*)_\varphi)$$

that abuts to the homology of the φ–primary component of the associated algebraic complex L_*. Moreover, the E^1–term

$$E^1_{p,q} = H_{p+q}(F^p/F^{p-1}) \cong H_{p+q}(K_{\varphi,p}, K_{\varphi,p-1}; R/(\varphi^p))$$

is isomorphic to the relative homology with trivial coefficients *of the simplicial pair $(K_{\varphi,p}, K_{\varphi,p-1})$.*

2.3 Discrete Morse theory for weighted complexes

Given x and y in a poset $(P, \prec)$, the notation $x \lhd y$ means that y *covers* x, ie $x \prec y$ and there is no other $z \in P$ such that $x \prec z \prec y$.

Recall the basic facts of discrete Morse theory (see for example [23; 30]). A *matching* in a poset P is a set $\mathcal{M} \subseteq P \times P$ such that $(x, y) \in M$ implies $x \lhd y$, and each $x \in P$ belongs to at most one pair of $\mathcal{M}$. An *alternating path* is a sequence

$$y_0 \rhd x_1 \lhd y_1 \rhd x_2 \lhd y_2 \rhd \cdots \rhd x_m \lhd y_m \,(\rhd x_{m+1})$$

such that each pair $x_i \lhd y_i$ belongs to $\mathcal{M}$ and no pair $x_i \lhd y_{i-1}$ belongs to $\mathcal{M}$. A *cycle* is a closed alternating path with $y_0 = y_m$. An *acyclic matching* over P is a matching with no cycles. We are going to describe a variant of discrete Morse theory which is suitable for our situation.

Definition 2.7 A *weighted acyclic matching* on a weighted sheaf (P, R, w) over P is an acyclic matching $\mathcal{M}$ on P such that

$$(x, y) \in \mathcal{M} \implies (w(x)) = (w(y))$$

(in [41] we required $w(x) = w(y)$, but the above generalization does not change the results).

The standard theory generalizes as follows. Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a finite simplicial complex K and let $\mathcal{M}$ an acyclic weighted matching. A *critical simplex* of K is a simplex σ which does not belong to any pair of $\mathcal{M}$. The following definition is equivalent to [41, Definition 3.3]:

Definition 2.8 The *Morse complex* of $\mathcal{S}$ with respect to $\mathcal{M}$ is defined as the torsion complex

$$L_*^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} \frac{R}{(w(\sigma))} \overline{e}_\sigma$$

with boundary

$$\partial^{\mathcal{M}}(\overline{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau|=|\sigma|-1}} [\sigma : \tau]^{\mathcal{M}} \overline{e}_\tau,$$

where $[\sigma : \tau]^{\mathcal{M}} \in \mathbb{Z}$ is given by the sum over all alternating paths

$$\sigma \rhd \tau_1 \lhd \sigma_1 \rhd \tau_2 \lhd \sigma_2 \rhd \cdots \rhd \tau_m \lhd \sigma_m \rhd \tau$$

from σ to τ of the quantity

$$(-1)^m [\sigma : \tau_1][\sigma_1 : \tau_1][\sigma_1 : \tau_2][\sigma_2 : \tau_2] \cdots [\sigma_m : \tau_m][\sigma_m : \tau].$$

The boundary map $\partial^{\mathcal{M}}$ is extended by R–linearity, where some care should be taken since each $\overline{e}_\tau$ lives in a component with a possibly different torsion ($\partial^{\mathcal{M}}$ is the same as the boundary map of [41]).

Theorem 2.9 [41] *Let $\mathcal{S} = (K, R, w)$ be a weighted sheaf over a simplicial complex K. Let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}$. Then there is an isomorphism*

$$H_*(L_*, \partial) \cong H_*(L_*^{\mathcal{M}}, \partial^{\mathcal{M}})$$

between the homology of the weighted complex L_* associated to $\mathcal{S}$ and the homology of the Morse complex $L_*^{\mathcal{M}}$ of $\mathcal{S}$.

Remark 2.10 If we forget about the weights, the matching $\mathcal{M}$ is in particular an acyclic matching for the simplicial complex K. Therefore, the algebraic complex $C_*^0(K;A)$, which computes the (shifted and reduced) simplicial homology of K with coefficients in some ring A, admits a "classical" algebraic Morse complex

$$C_*^0(K;A)^{\mathcal{M}} = \bigoplus_{\sigma \text{ critical}} A\overline{e}_\sigma$$

with boundary map

$$\delta^{\mathcal{M}}(\overline{e}_\sigma) = \sum_{\substack{\tau \text{ critical} \\ |\tau|=|\sigma|-1}} [\sigma:\tau]^{\mathcal{M}}\overline{e}_\tau.$$

Remark 2.11 Set $C_*^0 = C_*^0(K;R)$ and let $\pi\colon C_*^0 \to L_*$ be the natural projection. The composition with the projection $L_* \to (L_*)_\varphi$, for some fixed irreducible element $\varphi \in R$, yields a natural projection $\pi_\varphi\colon C_*^0 \to (L_*)_\varphi$. Similarly at the level of Morse complexes we have the projection $\overline{\pi}_\varphi\colon (C_*^0)^{\mathcal{M}} \to (L_*)_\varphi^{\mathcal{M}}$ which sends a generator $\overline{e}_\sigma$ in $(C_*^0)^{\mathcal{M}}$ to the generator $\overline{e}_\sigma$ in $(L_*)_\varphi^{\mathcal{M}}$. Then, by construction, the maps induced in homology by the projections π_φ and $\overline{\pi}_\varphi$ commute with the Morse isomorphisms of [30; 41]:

$$\begin{array}{ccc} H_*(C_*^0) & \xrightarrow{(\pi_\varphi)_*} & H_*((L_*)_\varphi) \\ \cong\downarrow & & \cong\downarrow \\ H_*((C_*^0)^{\mathcal{M}}) & \xrightarrow{(\overline{\pi}_\varphi)_*} & H_*((L_*)_\varphi^{\mathcal{M}}) \end{array}$$

Let $\varphi \in R$ be an irreducible element and let $\mathcal{M}$ be a weighted acyclic matching on $\mathcal{S}_\varphi$. Then the filtration on the weighted complex induces a filtration on the Morse complex,

$$F^p(L_*)_\varphi^{\mathcal{M}} = \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma)\le p}} \frac{R}{(\varphi^{v_\varphi(\sigma)})}\overline{e}_\sigma.$$

Consider also the quotient complex

$$\mathcal{F}^p(L_*)_\varphi^{\mathcal{M}} = F^p(L_*)_\varphi^{\mathcal{M}}/F^{p-1}(L_*)_\varphi^{\mathcal{M}} \cong \bigoplus_{\substack{\sigma \text{ critical} \\ v_\varphi(\sigma)=p}} \frac{R}{(\varphi^p)}\overline{e}_\sigma.$$

Theorem 2.12 [41] *Let $\mathcal{S}$ and φ be as above and let $\mathcal{M}$ be an acyclic matching for $\mathcal{S}_\varphi$. Then the E^1–page of the spectral sequence of Theorem 2.6 is identified with*

$$E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p(L_*)_\varphi^{\mathcal{M}}),$$

where $(L_*)^{\mathcal{M}}_{\varphi}$ is the Morse complex of $\mathcal{S}_\varphi$. The differential

$$d^1_{p,q}\colon E^1_{p,q} \to E^1_{p-1,q}$$

is induced by the boundary of the Morse complex, and thus it is also computed by using alternating paths.

2.4 Weighted sheaves for Artin groups

We use the constructions given above in the context of Artin groups. Let $(\boldsymbol{W}, S)$ be a Coxeter system, with $|S| = n$ finite, and let Γ be the corresponding Coxeter graph (with S as vertex set). Recall that $\boldsymbol{W}$ can always be realized as a group of reflections in some $\mathbb{R}^n$ (for example through the *Tits representation* — see [6; 43] and also [37]), so that it has the following naturally associated objects:

(i) a hyperplane arrangement in $\mathbb{R}^n$

$$\mathcal{A} = \{H \mid H \text{ is the fixed-point set of some reflection in } \boldsymbol{W}\};$$

(ii) configuration spaces

$$\boldsymbol{Y} = (\mathrm{int}(U) + i\mathbb{R}^n) \setminus \bigcup_{H\in\mathcal{A}} H_{\mathbb{C}},$$
$$\boldsymbol{Y}_{\boldsymbol{W}} = \boldsymbol{Y}/\boldsymbol{W},$$

where $U = \boldsymbol{W}\cdot\overline{C}_0$ is the *Tits cone* (here C_0 is a fixed chamber of the arrangement, and $\overline{C}_0$ is its topological closure);

(iii) a simplicial complex (defined over the finite set S)

$$K = K_{\boldsymbol{W}} = \{\sigma \subseteq S \mid \text{the parabolic subgroup } \boldsymbol{W}_\sigma \text{ generated by } \sigma \text{ is finite}\}.$$

One can define the *Artin group* $\boldsymbol{G}_{\boldsymbol{W}}$ of type $\boldsymbol{W}$ as the fundamental group $\pi_1(\boldsymbol{Y}_{\boldsymbol{W}})$. It has a presentation

$$\boldsymbol{G}_{\boldsymbol{W}} = \langle g_s,\ s\in S \mid g_s g_{s'} g_s g_{s'} \cdots = g_{s'} g_s g_{s'} g_s \cdots\rangle$$

(both the products have $m(s,s')$ factors, where $m(s,s')$ is the order of ss' in $\boldsymbol{W}$). Recall also the following results [39, Theorems 1.4 and 1.8]:

(i) The orbit space $\boldsymbol{Y}_{\boldsymbol{W}}$ deformation retracts onto a *finite* CW–complex $\boldsymbol{X}_{\boldsymbol{W}}$ given by a union

$$Q = \bigcup_{\sigma\in K_{\boldsymbol{W}}} Q_\sigma$$

of convex polyhedra with explicit identifications of their faces.

(ii) Consider the action of the Artin group $\boldsymbol{G_W}$ on the ring $R = \mathbb{Q}[q^{\pm 1}]$ given by

$$g_s \mapsto [\text{multiplication by } -q] \quad \text{for all } s \in S.$$

Then the homology $H_*(\boldsymbol{X_W}; R)$ is computed by the algebraic complex

$$C_k = \bigoplus_{\substack{\sigma \in K_{\boldsymbol{W}} \\ |\sigma| = k}} R e_\sigma$$

with boundary

$$\partial(e_\sigma) = \sum_{\tau \lhd \sigma} [\sigma : \tau] \frac{\boldsymbol{W}_\sigma(q)}{\boldsymbol{W}_\tau(q)} e_\tau,$$

where $\boldsymbol{W}_\sigma(q) = \sum_{w \in \boldsymbol{W}_\sigma} q^{l(w)}$ is the Poincaré polynomial of $\boldsymbol{W}_\sigma$.

Remark 2.13 The R–module structure on $H_*(\boldsymbol{X_W}; R)$ is given by the transformation μ_q induced by q–multiplication. If the order of μ_q is n, then the homology groups decompose into cyclic factors which are either free, of the form R^k, or torsion, of the form $R/(\varphi_d)$, where φ_d is the d^{th} cyclotomic polynomial with $d \,|\, n$. Therefore, we are interested in localizing to cyclotomic polynomials.

Theorem 2.14 [41] *To a Coxeter system $(\boldsymbol{W}, S)$ we can associate a weighted sheaf (K, R, w) over the simplicial complex $K = K_{\boldsymbol{W}}$, by setting $w(\sigma) = \boldsymbol{W}_\sigma(q)$. Also, for any cyclotomic polynomial $\varphi = \varphi_d$, the map $w_\varphi(\sigma)$ — which gives the maximal power of φ which divides $\boldsymbol{W}_\sigma(q)$ — defines a weighted sheaf (K, R, w_φ) over $K = K_{\boldsymbol{W}}$.*

The homology of the associated weighted complex is strictly related to the homology of $\boldsymbol{X_W}$. Specifically, set $C^0_* = C^0_*(K; R)$ and consider the diagonal map

$$\Delta\colon C_* \to C^0_*, \quad e_\sigma \mapsto \boldsymbol{W}_\sigma(q) e^0_\sigma.$$

By the formula for the boundary map it follows that Δ is an injective chain complex homomorphism, so there is an exact sequence of complexes

$$0 \to C_* \xrightarrow{\Delta} C^0_* \xrightarrow{\pi} L_* \to 0,$$

where

$$L_k = \bigoplus_{\substack{\sigma \in K_{\boldsymbol{W}} \\ |\sigma| = k}} \frac{R}{(\boldsymbol{W}_\sigma(q))} \overline{e}_\sigma$$

is the quotient complex. Passing to the associated long exact sequence we get

$$(1)\quad \cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \to H_k(C_*) \xrightarrow{\Delta_*} H_k(C^0_*) \xrightarrow{\pi_*} H_k(L_*) \to H_{k-1}(C_*) \xrightarrow{\Delta_*} \cdots.$$

Then the homology of L_* can be used to compute the homology of C_*.

The orbit space $\boldsymbol{Y_W}$ (and thus the CW–complex $\boldsymbol{X_W} \simeq \boldsymbol{Y_W}$) is conjectured to be a classifying space for the Artin group $\boldsymbol{G_W}$ [21; 7; 31; 37]. This conjecture was proved for Artin groups of finite type [21], for affine Artin groups of type $\widetilde{A}_n$, $\widetilde{B}_n$ and $\widetilde{C}_n$ [33; 13] and for some other families of Artin groups [28; 15]. Whenever the conjecture holds (in particular, this is true for all the cases we consider in this paper), the homology $H_*(\boldsymbol{X_W}; R)$ coincides with the twisted homology $H_*(\boldsymbol{G_W}; R)$ of the Artin group $\boldsymbol{G_W}$.

3 Homology of braid groups and independence complexes

In this section we are going to show how the twisted homology of braid groups (ie Artin groups of type A_n) is related to the homology of suitable independence complexes. Recall that a Coxeter graph of type A_n is a linear graph with n vertices, usually labeled $1, 2, \dots, n$ (see Figure 1), and that the corresponding Artin group is the braid group on $n+1$ strands (which we denote by Br_{n+1}). With a slight abuse of notation, by A_n we will sometimes indicate the graph itself.

Figure 1: A Coxeter graph of type A_n

The homology of Br_{n+1}, with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ described in the previous section, has been computed in [25; 18]. See also [10] for coefficients in $\mathbb{Z}[q^{\pm 1}]$ (but we will not address this case).

Theorem 3.1 [25; 18] *The φ_d–primary component of the twisted homology of a braid group is given by*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} = \begin{cases} R/(\varphi_d) & \text{if } n \equiv 0 \text{ or } -1 \ (\mathrm{mod}\, d), \\ 0 & \text{otherwise,} \end{cases}$$

where the nonvanishing term is in degree $(d-2)k$ if $n = dk$ or $n = dk-1$.

Recall that, if G is a graph with vertex set VG, an *independent set* of G is a subset of VG consisting of pairwise nonadjacent vertices. Also, the *independence complex* $\mathrm{Ind}(G)$ of G is the abstract simplicial complex with VG as set of vertices and whose simplices are all the nonempty independent sets of G. Thus, $\mathrm{Ind}(G)$ is the clique

complex of the complement graph of G. In contrast with the simplicial complex K introduced in Section 2, the simplicial complex $\mathrm{Ind}(G)$ does not contain the empty simplex, and the dimension of a simplex $\sigma \in \mathrm{Ind}(G)$ is given by $|\sigma|-1$. The homotopy type of $\mathrm{Ind}(A_n)$ has been computed in [30] by means of discrete Morse theory, and the result is the following:

Proposition 3.2 [30, Proposition 11.16] *We have*

$$\mathrm{Ind}(A_n) \simeq \begin{cases} S^{k-1} & \text{if } n = 3k \text{ or } n = 3k-1, \\ \{\mathrm{pt}\} & \text{if } n = 3k+1. \end{cases}$$

By comparing this with Theorem 3.1 we obtain the following relation between the homology of the independence complex of A_n and the φ_3–primary component of the twisted homology of Br_{n+1}:

Corollary 3.3 *We have*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_3} \cong \widetilde{H}_{*-2}\left(\mathrm{Ind}(A_{n-3}); \frac{R}{(\varphi_3)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients. □

In general, following [40], define the *r–independence complex* of a graph G as

$$\mathrm{Ind}_r(G) = \{\text{full subgraphs } G' \subseteq G \text{ such that each connected component of } G' \text{ has at most } r \text{ vertices}\}.$$

So $\mathrm{Ind}_r(G)$ is an abstract simplicial complex on the set of vertices VG of the graph G, which coincides with $\mathrm{Ind}(G)$ for $r=1$. The case $r=0$ also makes sense: $\mathrm{Ind}_0(G) = \varnothing$ for any graph G. We are going to prove the following generalization of Corollary 3.3:

Theorem 3.4 *We have*

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right),$$

where on the left we have local coefficients and on the right we have trivial coefficients.

From the expression of the φ_d–primary component of the local homology of the braid group (Theorem 3.1) we obtain the following consequence:

Corollary 3.5 *We have*

$$\widetilde{H}_*(\mathrm{Ind}_{d-2}(A_n)) \cong \begin{cases} \widetilde{H}_*(S^{dk-2k-1}) & \text{for } n = dk \text{ or } n = dk-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the knowledge of the twisted homology of the braid group gives the homology (with trivial coefficients) of $\mathrm{Ind}_d(A_n)$. Conversely, the knowledge of the homology of $\mathrm{Ind}_d(A_n)$ gives the twisted homology of the braid group.

The Poincaré polynomial of a Coxeter group of type A_k is given by

$$W_{A_k}(q) = [k+1]_q!, \quad \text{where } [k]_q = \frac{q^k-1}{q-1} = \prod_{\substack{d\mid k \\ d\geq 2}} \varphi_d$$

(see for example [5]). Consider now a simplex $\sigma \subseteq S \cong \{1,\dots,n\}$. Denote by $\Gamma(\sigma)$ the subgraph of A_n induced by σ. Denote by $\Gamma_1(\sigma)$, $\Gamma_2(\sigma)$, …, $\Gamma_m(\sigma)$ the connected components of $\Gamma(\sigma)$ and by n_1, n_2, … n_m their cardinalities (see Figure 2). The i^{th} connected component is a Coxeter graph of type A_{n_i}, so the entire Coxeter graph induced by σ has Poincaré polynomial

$$\begin{aligned} W_\sigma(q) &= [n_1+1]_q! \cdot [n_2+1]_q! \cdots [n_m+1]_q! \\ &= \prod_{i=1}^{m} \prod_{d\geq 2} \varphi_d^{\lfloor (n_i+1)/d \rfloor} \\ &= \prod_{d\geq 2} \varphi_d^{\sum_{i=1}^m \lfloor (n_i+1)/d \rfloor}. \end{aligned}$$

Then we are interested in the homology of the weighted complex $(L_*)_\varphi$ associated to the weighted sheaf (K, R, w_φ), where $\varphi = \varphi_d$ is a cyclotomic polynomial (with $d \geq 2$) and

$$w_\varphi(\sigma) = \varphi^{v_\varphi(\sigma)}, \quad v_\varphi(\sigma) = \sum_{i=1}^{m} \left\lfloor \frac{n_i+1}{d} \right\rfloor.$$

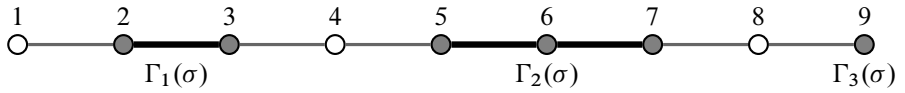


Figure 2: An example with $n = 9$ and $\sigma = \{2,3,5,6,7,9\} \in \mathrm{Ind}_3(A_9)$. In this case, $|\Gamma_1(\sigma)| = 2$, $|\Gamma_2(\sigma)| = 3$ and $|\Gamma_3(\sigma)| = 1$.

Notice that only the connected components with at least $d-1$ vertices contribute to the φ–weight. Therefore, the weighted complex $(L_*)_\varphi$ is generated by the subgraphs having at least one component with $\geq d-1$ vertices.

Theorem 3.6 *There is a weighted acyclic matching $\mathcal{M}$ on K such that the set of critical simplices is given by*

$$\begin{aligned} Cr(\mathcal{M}) &= \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1},\ |\Gamma_i| \leq d-2\} \cup \mathrm{Ind}_{d-2}(A_n) \\ &= \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\} \cup \mathrm{Ind}_{d-2}(A_n), \end{aligned}$$

where A_{d-1} is the linear graph on the vertices $n-d+2, \ldots, n$.

Proof First notice that removing the d^{th} vertex from an A_k component leaves the φ–weight unchanged:

$$v_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor = 1 + \left\lfloor \frac{k+1-d}{d} \right\rfloor = v_\varphi(A_{d-1} \sqcup A_{k-d})$$

(see Figure 3).

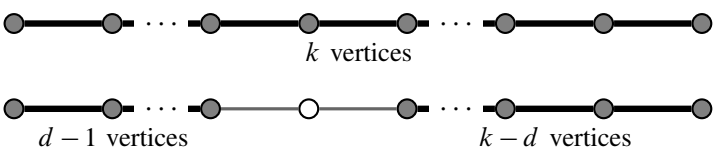


Figure 3: The φ–weight of an A_k component remains the same if we remove the d^{th} vertex, splitting A_k into $A_{d-1} \sqcup A_{k-d}$.

Let $K_0 = \mathrm{Ind}_{d-2}(A_n) \subseteq K$ (this is the set of the simplices $\sigma \in K$ such that all the connected components of the induced subgraph $\Gamma(\sigma)$ have cardinality $< d-1$). Let us define on $K' = K \setminus K_0$ the following matching $\mathcal{M}$. For $\sigma \in K'$, with $\Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_m$, set

$$i(\sigma) = \min\{i \mid |\Gamma_i| \geq d-1\}.$$

Then match σ with the simplex τ obtained by adding or removing from σ the d^{th} vertex of the component $\Gamma_{i(\sigma)}$. By the remark at the beginning of the proof, $\mathcal{M}$ is a weighted matching. We prove that it is acyclic. In fact, suppose that an alternating path contains some subpath

$$\tau \lhd \sigma \rhd \tau' \lhd \sigma'$$

with $\tau \neq \tau'$. Then either $i(\tau) < i(\tau')$ or (set $j = i(\tau) = i(\tau')$)

$$\Gamma_j(\tau) = \{a, a+1, \ldots, a+d-2\}, \quad \Gamma_j(\tau') = \{a+1, \ldots, a+d-1\}$$

for some $a \leq n-d+1$. In this case the first vertex of $\Gamma_j(\tau')$ is greater than the first vertex of $\Gamma_j(\tau)$. Therefore, an alternating path in K' cannot be closed.

The set of critical elements of $\mathcal{M}$ in K' is given by

$$\mathrm{Cr}(\mathcal{M}) = \{\sigma \mid \Gamma(\sigma) = \Gamma_1 \sqcup \cdots \sqcup \Gamma_{m-1} \sqcup A_{d-1},\ |\Gamma_i| \leq d-2\},$$

where A_{d-1} is as in the statement of the theorem. □

Proof of Theorem 3.4 Consider the matching $\mathcal{M}$ of Theorem 3.6. Since $K_0 = \mathrm{Ind}_{d-2}(A_n)$ does not contribute to the weighted complex $(L_*)_\varphi$, we concentrate on the poset $K' = K \setminus K_0$. Notice that there are no nontrivial alternating paths between critical elements of K'. Therefore, the boundaries between simplices in

$$\mathrm{Cr}'(\mathcal{M}) = \{\tau \sqcup A_{d-1} \mid \tau \in \mathrm{Ind}_{d-2}(A_{n-d})\}$$

are the same as the boundaries of $\mathrm{Ind}_{d-2}(A_{n-d})$. In the long exact sequence (1) the algebraic complex C^0_* has vanishing homology in all degrees (because K is the full simplicial complex on n vertices), so we have an isomorphism $H_{*+1}(L_*) \cong H_*(C_*)$. Therefore,

$$H_*(\mathrm{Br}_{n+1}; R)_{\varphi_d} \cong H_*(C_*)_{\varphi_d} \cong H_{*+1}(L_*)_{\varphi_d} \cong \widetilde{H}_{*-d+1}\left(\mathrm{Ind}_{d-2}(A_{n-d}); \frac{R}{(\varphi_d)}\right).$$

In the last isomorphism there is a $(d-1)$–shift in degree due to the loss of $d-1$ vertices when passing from $\mathrm{Cr}'(\mathcal{M})$ to $\mathrm{Ind}_{d-2}(A_{n-d})$; there is then a further 1–shift in degree due to the fact that in L_* a simplex σ has dimension $|\sigma|$, and in $\mathrm{Ind}_{d-2}(A_{n-d})$ it has dimension $|\sigma|-1$; finally, it is necessary to pass to reduced homology because the empty simplex is missing in $\mathrm{Ind}_{d-2}(A_{n-d})$. □

For the sake of completeness we also determine the homotopy type of the r–independence complex $\mathrm{Ind}_{d-2}(A_n)$ (obtaining again Corollary 3.5 as a consequence). This is a straightforward generalization of the case $d = 3$, proved in [30, Proposition 11.16].

Proposition 3.7 *We have*

$$\mathrm{Ind}_{d-2}(A_n) \simeq \begin{cases} S^{dk-2k-1} & \text{if } n = dk \text{ or } n = dk-1, \\ \{\mathrm{pt}\} & \text{otherwise.} \end{cases}$$

Proof We denote here for brevity $K_0 = \mathrm{Ind}_{d-2}(A_n)$. Let $n = qd + r$ be the euclidean division of n by d.

Let $P = \{c_d > c_{2d} > c_{3d} > \cdots > c_{qd} > c_*\}$ be a linearly ordered set, where c_* is the minimum element. Define a map $f\colon K_0 \to P$ as follows. If $\sigma \in K_0$ does not contain any multiples of d, then set $f(\sigma) = c_*$. Otherwise, set $f(\sigma) = c_{jd}$ if $jd \in \sigma$ but $j'd \notin \sigma$ for $j' < j$. Clearly f is a poset map (if we remove a vertex from σ, j increases or remains the same).

In $f^{-1}(c_d)$ consider the matching $\mathcal{M}_1 = \{(\sigma \setminus \{1\}, \sigma \cup \{1\}) \mid \sigma \in f^{-1}(c_d)\}$, which is justified by

$$\sigma \in f^{-1}(c_d) \implies \{2, \dots, d-1\} \not\subseteq \sigma.$$

Similarly, in $f^{-1}(c_{jd})$ for $j \leq q$, consider the matching

$$\mathcal{M}_j = \big\{(\sigma \setminus \{(j-1)d+1\},\, \sigma \cup \{(j-1)d+1\}) \mid \sigma \in f^{-1}(c_{jd})\big\},$$

justified by

$$\sigma \in f^{-1}(c_{jd}) \implies \{(j-1)d+2, \dots, jd-1\} \not\subseteq \sigma.$$

Each $\mathcal{M}_j$ is acyclic in $f^{-1}(c_{jd})$, thus

$$\mathcal{M} = \bigcup_{j=1}^{q} \mathcal{M}_j$$

is an acyclic matching on K_0 by the patchwork theorem (see [29, Lemma 4.2; 30, Theorem 11.10] for example). Since each $\mathcal{M}_j$ is a perfect matching on $f^{-1}(c_{jd})$ for $j = 1, \dots, q$ and $K_1 = f^{-1}(c_*)$ is a subcomplex of K_0, it follows that K_0 deformation retracts onto K_1.

Notice now that K_1 is the join of q copies of $\mathrm{Ind}_{d-2}(A_{d-1}) \cong S^{d-3}$ and one copy of $\mathrm{Ind}_{d-2}(A_r)$. For $r = 0$ we get

$$K_0 \cong (S^{d-3})^{*q} \cong S^{q(d-3)+q-1} = S^{qd-2q-1}$$

and for $r = d-1$ we get

$$K_0 \cong (S^{d-3})^{*(q+1)} \cong S^{(q+1)(d-3)+q} = S^{(q+1)d-2(q+1)-1},$$

which give the first case in the corollary. If r is not 0 or $d-1$ then $\mathrm{Ind}_{d-2}(A_r)$ is a full simplex Δ^{r-1} on r vertices, and therefore K_0 is contractible. □

4 Precise matchings

In this section we are going to introduce the notion of *precise matching* on a weighted sheaf. The motivation comes from the study of the twisted homology of some families of Artin groups, as we will show in the subsequent sections. For now we go back to the general framework of weighted sheaves over simplicial complexes (later we will specialize in the case of Artin groups).

Assume from now on that the PID R contains some field $\mathbb{K}$. Our main case of interest is $\mathbb{K}=\mathbb{Q}$ and $R=\mathbb{Q}[q^{\pm 1}]$. Let $\mathcal{S}=(K,R,w)$ be a weighted sheaf over the finite simplicial complex K, with associated weighted complex L_*. Given a fixed irreducible element φ of R, let $\mathcal{S}_\varphi$ be the φ–primary component of $\mathcal{S}$ and let $(L_*)_\varphi$ be its associated weighted complex. Let $\mathcal{M}$ be a weighted acyclic matching for $\mathcal{S}_\varphi$.

Let $G^{\mathcal{M}}$ be the incidence graph of the corresponding Morse complex: the vertices of $G^{\mathcal{M}}$ are the critical simplices of K, and there is an (oriented) edge $\sigma\to\tau$ whenever $[\sigma:\tau]^{\mathcal{M}}$ is not 0 in R (or equivalently in $\mathbb{K}$), where $[\sigma:\tau]^{\mathcal{M}}\in\mathbb{Z}$ is the incidence number between σ and τ in the Morse complex of K. In other words, there is an edge $\sigma\to\tau$ if $[\sigma:\tau]^{\mathcal{M}}$ is not a multiple of $\operatorname{char}\mathbb{K}=\operatorname{char}R$. When $\mathbb{K}=\mathbb{Q}$, this simply means that $[\sigma:\tau]^{\mathcal{M}}\neq 0$.

Let $\mathcal{I}$ be the set of connected components of $G^{\mathcal{M}}$ (computed ignoring the orientation of the edges). Recall that $v_\varphi(\sigma)$ is the maximal $k\in\mathbb{N}$ such that φ^k divides $w(\sigma)$.

Definition 4.1 The matching $\mathcal{M}$ is *φ–precise* (or simply *precise*) if, for any edge $\sigma\to\tau$ of $G^{\mathcal{M}}$, we have that $v_\varphi(\sigma)=v_\varphi(\tau)+1$.

In other words, $\mathcal{M}$ is precise if, for any two simplices σ and τ lying in the same connected component $i\in\mathcal{I}$, the following relation holds:

$$v_\varphi(\sigma)-v_\varphi(\tau)=|\sigma|-|\tau|.$$

Equivalently the quantity $|\sigma|-v_\varphi(\sigma)$, as a function of σ, is constant within a fixed connected component of $G^{\mathcal{M}}$. This definition is motivated by the fact that precise matchings exist in many cases of interest (we will see this in Section 5), and that the homology of the Morse complex is much simpler to compute (and takes a particularly nice form) when the matching is precise. The name "precise" has been chosen because for a generic matching one only has $v_\varphi(\sigma)\geq v_\varphi(\tau)$ (when $\sigma\to\tau$ is an edge of $G^{\mathcal{M}}$), and we require $v_\varphi(\sigma)$ to be precisely $v_\varphi(\tau)+1$.

Assume from now on in this section that $\mathcal{M}$ is a φ–precise matching. To simplify the notation, set $(A_*, \partial) = ((L_*)^{\mathcal{M}}_{\varphi}, \partial^{\mathcal{M}})$ and $(V_*, \delta) = (C^0_*(K, \mathbb{K})^{\mathcal{M}}, \delta^{\mathcal{M}})$. Our aim is to derive a formula for the homology of the Morse complex A_*. Since the differential δ vanishes between simplices in different connected components of $G^{\mathcal{M}}$, the complex (V_*, δ) splits as

$$(V_*, \delta) = \bigoplus_{i \in \mathcal{I}} (V^i_*, \delta^i),$$

where

$$V^i_* = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \mathbb{K}\overline{e}_\sigma$$

and the boundary map $\delta^i\colon V^i_* \to V^i_*$ is the restriction of δ to V^i_*. The differential ∂ of A_* is induced by δ, and thus it also vanishes between simplices in different connected components. Therefore, we have an analogous splitting for (A_*, ∂),

$$(A_*, \partial) = \bigoplus_{i \in \mathcal{I}} (A^i_*, \partial^i),$$

where

$$A^i_* = \bigoplus_{\substack{\sigma \text{ critical} \\ \sigma \in i}} \frac{R}{(w_\varphi(\sigma))}\overline{e}_\sigma$$

and the boundary map $\partial^i\colon A^i_* \to A^i_*$ is simply the restriction of ∂ to A^i_*.

Fix now a connected component $i \in \mathcal{I}$. Since $\mathcal{M}$ is precise, there exists some $k \in \mathbb{Z}$ (which depends on i) such that $v_\varphi(\sigma) = |\sigma| + k$ for all $\sigma \in i$. Therefore, in degree m we have

$$A^i_m = \bigoplus_{\sigma \in \mathrm{Cr}^i_m} \frac{R}{(\varphi^{m+k})}\overline{e}_\sigma = \Bigg(\bigoplus_{\sigma \in \mathrm{Cr}^i_m} \mathbb{K}\overline{e}_\sigma \Bigg) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})} = V^i_m \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})},$$

where Cr^i_m is the set of critical simplices σ such that $\sigma \in i$ and $|\sigma| = m$. By construction the boundary $\partial^i_m\colon A^i_m \to A^i_{m-1}$ factors accordingly:

$$\partial^i_m = \delta^i_m \otimes_{\mathbb{K}} \pi_m,$$

where

$$\pi_m\colon \frac{R}{(\varphi^{m+k})} \to \frac{R}{(\varphi^{m+k-1})}$$

is the projection induced by the identity $R \to R$. Since $\operatorname{im}\delta^i_{m+1} \subseteq \ker\delta^i_m$, each V^i_m splits (as a vector space over $\mathbb{K}$) as a direct sum of linear subspaces

$$V^i_m = W^i_{m,1} \oplus W^i_{m,2} \oplus W^i_{m,3},$$

where $W^i_{m,1} = \operatorname{im}\delta^i_{m+1}$ and $W^i_{m,1} \oplus W^i_{m,2} = \ker\delta^i_m$. Then

$$\begin{aligned}\ker(\partial^i_m) &= \ker(\delta^i_m \otimes_{\mathbb{K}} \pi_m)\\ &= \left(\left(W^i_{m,1} \oplus W^i_{m,2}\right) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}\right) \oplus \left(W^i_{m,3} \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})}\right);\\ \operatorname{im}(\partial^i_{m+1}) &= \operatorname{im}(\delta^i_{m+1} \otimes_{\mathbb{K}} \pi_{m+1}) = W^i_{m,1} \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}.\end{aligned}$$

Therefore, the homology of (A^i_*, δ^i) is given, as an R–module, by

$$\begin{aligned}H_m(A^i_*) &= \frac{\ker(\partial^i_m)}{\operatorname{im}(\partial^i_{m+1})}\\ &= \left(W^i_{m,2} \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}\right) \oplus \left(W^i_{m,3} \otimes_{\mathbb{K}} \frac{(\varphi^{m+k-1})}{(\varphi^{m+k})}\right)\\ &\cong \left(H_m(V^i_*, \delta^i) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k})}\right) \oplus \left(\mathbb{K}^{\operatorname{rk}\delta^i_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)}\right).\end{aligned}$$

In the last isomorphism we used the fact that $\dim W^i_{m,3} = \dim V^i_m - \dim\ker\delta^i_m = \operatorname{rk}\delta^i_m$.

Recall that the previous formula holds for a fixed connected component $i \in \mathcal{I}$, and k depends on i. Since we now need to take the direct sum over the connected components, let k_i be the value of k for the component i.

Theorem 4.2 *The homology of $(L_*)_\varphi$ is given, as an R–module, by*

$$H_m((L_*)_\varphi, \partial) \cong \left(\bigoplus_{i\in\mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})}\right) \oplus \left(\mathbb{K}^{\operatorname{rk}\delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)}\right).$$

Proof By Theorem 2.9, $H_m((L_*)_\varphi, \partial) \cong H_m((L_*)^{\mathcal{M}}_\varphi, \partial^{\mathcal{M}})$. Using what we have done in this section, we have that

$$\begin{aligned}H_m((L_*)^{\mathcal{M}}_\varphi, \partial^{\mathcal{M}}) &= H_m(A_*, \partial)\\ &= \bigoplus_{i\in\mathcal{I}} H_m(A^i_*, \partial)\\ &\cong \left(\bigoplus_{i\in\mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})}\right) \oplus \left(\left(\bigoplus_{i\in\mathcal{I}} \mathbb{K}^{\operatorname{rk}\delta^i_m}\right) \otimes_{\mathbb{K}} \frac{R}{(\varphi)}\right)\\ &\cong \left(\bigoplus_{i\in\mathcal{I}} H_m(V^i_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_i})}\right) \oplus \left(\mathbb{K}^{\operatorname{rk}\delta_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)}\right).\end{aligned}$$

□

Let us see how the existence of a precise matching can be interpreted in terms of the spectral sequence associated to the weighted sheaf (see Theorem 2.6).

Proposition 4.3 *If a φ–precise matching $\mathcal{M}$ exists, then the spectral sequence $E^r_{p,q}$ associated to the weighted sheaf $\mathcal{S}_\varphi$ collapses at the E^2–page.*

Proof By Theorem 2.12, the E^1–page can be computed through the Morse complex of our matching $\mathcal{M}$:

$$E^1_{p,q} \cong H_{p+q}(\mathcal{F}^p(L_*)^{\mathcal{M}}_\varphi),$$

and the differential $d^1_{p,q}$ is induced by the boundary of the Morse complex. The spectral sequence then splits as a direct sum over the connected components of $G^{\mathcal{M}}$,

$$E^r_{p,q} = \bigoplus_{i\in\mathcal{I}} E^{r,i}_{p,q},$$

where $E^{1,i}_{p,q} \cong H_{p+q}(\mathcal{F}^p A^i_*)$. Since the matching is precise, for $m \neq p-k_i$ we have

$$\mathcal{F}^p A^i_m = F^p A^i_m / F^{p-1} A^i_m = 0.$$

This means that the page $E^{0,i}_{p,q} \cong E^{1,i}_{p,q}$ is nontrivial only in the row $q=-k_i$, and the entire spectral sequence $E^r_{p,q}$ collapses at the E^2–page. □

What we have done so far in this section assumed φ to be some fixed irreducible element of the PID R. In order to recover the full homology of L_* we need to let φ vary among all equivalence classes of irreducible elements of R modulo the units. Suppose from now on we have a φ–precise matching $\mathcal{M}_\varphi$ on $\mathcal{S}_\varphi$ for each φ. The following result follows immediately from Theorem 4.2, provided that we add a "φ" subscript (or superscript) to all the quantities that depend on the matching $\mathcal{M}_\varphi$.

Theorem 4.4 *The homology of L_* is given, as an R–module, by*

$$H_m(L_*,\partial) \cong \bigoplus_\varphi \left(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V^{\varphi,i}_*) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \right) \oplus \left(\mathbb{K}^{\operatorname{rk}\delta^\varphi_m} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \qquad \square$$

For later applications, we finally need to study how the isomorphism of Theorem 4.4 behaves with respect to the projection $\pi\colon C^0_* \to L_*$ of Remark 2.11. This projection is the direct sum over φ of the projections

$$\pi_\varphi\colon C^0_* \to (L_*)_\varphi.$$

Instead of studying the induced map $(\pi_\varphi)_*\colon H_m(C_*^0) \to H_m((L_*)_\varphi)$, we study the map

$$(\overline{\pi}_\varphi)_*\colon H_m((C_*^0)^{\mathcal{M}}) \to H_m((L_*)_\varphi^{\mathcal{M}})$$

between the Morse complexes (here $\mathcal{M} = \mathcal{M}_\varphi$ is a precise matching which depends on φ). For $i \in \mathcal{I}_\varphi$, let $\pi_i\colon (C_*^0)^{\mathcal{M}} \to V_*^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq (C_*^0)^{\mathcal{M}}$ be the projection on the subcomplex corresponding to the connected component i, and let $(\pi_i)_*$ be the map induced in homology. Let $[c] \in H_m((C_*^0)^{\mathcal{M}})$, for some cycle $c \in \ker \delta^{\mathcal{M}} \subseteq (C_m^0)^{\mathcal{M}}$. Applying the map $(\overline{\pi}_\varphi)_*\colon H_m((C_*^0)^{\mathcal{M}}) \to H_m((L_*)_\varphi^{\mathcal{M}})$ we obtain

$$(\overline{\pi}_\varphi)_*([c]) = (\overline{\pi}_\varphi)_*\Bigg(\sum_{i\in\mathcal{I}_\varphi} (\pi_i)_*([c])\Bigg) = \sum_{i\in\mathcal{I}_\varphi} (\overline{\pi}_\varphi)_*((\pi_i)_*([c])).$$

Applying the isomorphism of Theorem 4.2, this element is sent to

$$\sum_{i\in\mathcal{I}_\varphi} ((\pi_i)_*([c])) \otimes_{\mathbb{K}} [1] \in \bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

We are going to use these computations to prove the following result, which describes the kernel and the cokernel of π_*:

Proposition 4.5 *The cokernel of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$ is given by*

$$\operatorname{coker} \pi_* \cong \bigoplus_\varphi \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta_m^\varphi}.$$

In addition, the kernel of π_* is a free R–module isomorphic to $H_m(C_*^0)$.

Proof Throughout the proof, consider the following R–modules identified one with each other, without explicitly mentioning the isomorphisms between them:

$$\begin{aligned} H_m(L_*) &\cong \bigoplus_\varphi H_m((L_*)_\varphi) \\ &\cong \bigoplus_\varphi H_m((L_*)_\varphi^{\mathcal{M}}) \\ &\cong \bigoplus_\varphi \Bigg(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \Bigg) \oplus \left(\mathbb{K}^{\operatorname{rk}\delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right). \end{aligned}$$

Recall that the matching $\mathcal{M}$ depends on φ, although we write $\mathcal{M}$ instead of $\mathcal{M}_\varphi$ in order to make the notation more readable. Also recall that the isomorphisms $H_m((L_*)_\varphi) \cong H_m((L_*)_\varphi^{\mathcal{M}})$ occur in the commutative diagram of Remark 2.11.

We want to show that the image of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$ is given by

$$\operatorname{im}\pi_* = \bigoplus_{\varphi}\Bigg(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}\Bigg) \subseteq H_m(L_*).$$

Let $\psi_\varphi\colon H_*(C_*^0) \to H_*((L_*)_\varphi^{\mathcal{M}})$ be the map defined as the composition

$$H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}}) \xrightarrow{(\overline{\pi}_\varphi)_*} H_*((L_*)_\varphi^{\mathcal{M}}).$$

By commutativity of the diagram of Remark 2.11, the image of $\pi_* = \bigoplus_\varphi (\pi_\varphi)_*$ is the same as the image of

$$\bigoplus_{\varphi} \psi_\varphi\colon H_*(C_*^0) \to \bigoplus_{\varphi} H_*((L_*)_\varphi^{\mathcal{M}}).$$

We have already proved that, for any $[c] \in H_m((C_*^0)^{\mathcal{M}})$,

$$(\overline{\pi}_\varphi)_*([c]) = \sum_{i\in\mathcal{I}_\varphi} ((\pi_i)_*([c])) \otimes_R [1] \in \bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})},$$

which means in particular that

$$\operatorname{im}(\overline{\pi}_\varphi)_* \subseteq \bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}.$$

Therefore, we immediately have the inclusion

$$\operatorname{im}(\pi_*) \subseteq \sum_{\varphi} \operatorname{im}\psi_\varphi = \sum_{\varphi} \operatorname{im}(\overline{\pi}_\varphi)_* \subseteq \bigoplus_{\varphi}\Bigg(\bigoplus_{i\in\mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}\Bigg).$$

To prove the opposite inclusion, we show that any element of the form

$$[c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})}$$

is in the image of π_* (for any fixed φ and i). To do so, choose $\alpha \in R$ such that $\alpha \equiv 1 \pmod{\varphi^{m+k_{\varphi,i}}}$ and $\alpha \equiv 0 \pmod{\eta^{m+k_{\eta,j}}}$ for any irreducible element $\eta \neq \varphi$ which divides some weight $w(\sigma)$ and for any connected component $j \in \mathcal{I}_\eta$ (there is only a finite number of such η up to multiplication by units, because K is finite). The element $c \otimes_{\mathbb{K}} \alpha$ is a cycle in $V_m^{\varphi,i} \otimes_{\mathbb{K}} R \subseteq C_m^0(K,\mathbb{K})^{\mathcal{M}} \otimes_{\mathbb{K}} R \cong (C_m^0)^{\mathcal{M}}$. Then, if $[\widetilde{c}]$ is the preimage of $[c \otimes_{\mathbb{K}} \alpha]$ under the isomorphism $H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}})$, we

have that

$$\psi_\varphi([\widetilde{c}]) = (\overline{\pi}_\varphi)_*([c \otimes_{\mathbb{K}} \alpha]) = [c] \otimes_{\mathbb{K}} [1] \in H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})},$$
$$\psi_\eta([\widetilde{c}]) = (\overline{\pi}_\eta)_*([c' \otimes_{\mathbb{K}} \alpha]) = 0 \quad \text{for any } \eta \neq \varphi,$$

where $[c']$ is the image of $[c]$ under the isomorphism

$$H_*((C_*^0)^{\mathcal{M}_\varphi}) \xrightarrow{\cong} H_*(C_*^0) \xrightarrow{\cong} H_*((C_*^0)^{\mathcal{M}_\eta}).$$

Therefore, $[c] \otimes_{\mathbb{K}} [1]$ is in the image of π_*. We have thus proved that

$$\operatorname{im} \pi_* = \bigoplus_\varphi \Bigg(\bigoplus_{i \in \mathcal{I}_\varphi} H_m(V_*^{\varphi,i}) \otimes_{\mathbb{K}} \frac{R}{(\varphi^{m+k_{\varphi,i}})} \Bigg).$$

Then the cokernel of π_* can be easily computed:

$$\operatorname{coker} \pi_* = \frac{H_m(L_*)}{\operatorname{im} \pi_*} = \bigoplus_\varphi \left(\mathbb{K}^{\operatorname{rk} \delta_m^\varphi} \otimes_{\mathbb{K}} \frac{R}{(\varphi)} \right) = \bigoplus_\varphi \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_m^\varphi}.$$

The R–module $H_m(C_*^0) \cong H_m(C_*^0(K;\mathbb{K})) \otimes_{\mathbb{K}} R$ is free and finitely generated, because $H_m(C_*^0(K;\mathbb{K}))$ is a finite-dimensional vector space over $\mathbb{K}$ (recall that K is a finite simplicial complex). The kernel of π_* is a submodule of $H_m(C_*^0)$, so it is itself a free R–module with lower or equal rank. Let $[c_1], \dots, [c_k]$ be an R–base of $H_m(C_*^0)$. Consider the nonzero ideal

$$I = \bigcap_\varphi \bigcap_{i \in \mathcal{I}_\varphi} (\varphi^{m+k_i}) \subseteq R,$$

where φ varies among the (finitely many) irreducible elements which divide some weight $w(\sigma)$ (for $\sigma \in K$). Fix any nonzero element $\alpha \in I$. Then the elements $\alpha[c_1], \dots, \alpha[c_k]$ generate a free submodule of $\ker \pi_*$ of rank $k = \operatorname{rk} H_m(C_*^0)$. Therefore, $\ker \pi_*$ and $H_m(C_*^0)$ have the same rank, so they are isomorphic as R–modules. □

5 Precise matchings for Artin groups

Consider now the case of Artin groups, as in Section 2.4. For a Coxeter system (W, S) we have constructed a weighted sheaf $\mathcal{S} = (K, R, w)$ with

$$K = \{\sigma \subset S \mid \text{the parabolic subgroup } W_\sigma \text{ is finite}\}$$

and $R = \mathbb{Q}[q^{\pm 1}]$. The associated weighted complex L_* fits into the short exact sequence

$$0 \to C_* \xrightarrow{\Delta} C_*^0 \xrightarrow{\pi} L_* \to 0,$$

which gives rise to the long exact sequence (1):

$$\cdots \xrightarrow{\pi_*} H_{k+1}(L_*) \to H_k(C_*) \xrightarrow{\Delta_*} H_k(C_*^0) \xrightarrow{\pi_*} H_k(L_*) \to H_{k-1}(C_*) \xrightarrow{\Delta_*} \cdots.$$

In order to compute $H_*(C_*) = H_*(\boldsymbol{X_W}; R)$, we split this long exact sequence into the short exact sequences

$$0 \to \operatorname{coker} \pi_* \to H_m(C_*) \xrightarrow{\Delta_*} \ker \pi_* \to 0,$$

where on the left we have the cokernel of $\pi_*\colon H_{m+1}(C_*^0) \to H_{m+1}(L_*)$ and on the right we have the kernel of $\pi_*\colon H_m(C_*^0) \to H_m(L_*)$. Since $\ker \pi_*$ is a free R–module, these short exact sequences split:

$$H_m(C_*) \cong \operatorname{coker} \pi_* \oplus \ker \pi_*.$$

Recall that the only irreducible elements of R that occur in the factorization of the weights are the cyclotomic polynomials φ_d for $d \geq 2$. As in Section 4, suppose from now on that we have constructed a φ–precise matching $\mathcal{M}_\varphi$ for each cyclotomic polynomial $\varphi = \varphi_d$ (with $d \geq 2$). Then we have an explicit description of $\operatorname{coker} \pi_*$ and $\ker \pi_*$ thanks to Proposition 4.5, and we obtain the following result:

Theorem 5.1 *Under the above hypothesis, the homology of $\boldsymbol{X_W}$ with coefficients in the representation $R = \mathbb{Q}[q^{\pm 1}]$ is given by*

$$H_m(\boldsymbol{X_W}; R) \cong \left(\bigoplus_{\varphi} \left(\frac{R}{(\varphi)} \right)^{\oplus \operatorname{rk} \delta_{m+1}^{\varphi}} \right) \oplus H_m(C_*^0). \qquad \square$$

In particular, the term $H_m(C_*^0)$ gives the free part of the homology, and the other direct summands give the torsion part. The torsion part actually takes a very particular form, and we are going to highlight this in the following result:

Theorem 5.2 *Let $\boldsymbol{G_W}$ be an Artin group that admits a φ–precise matching for all cyclotomic polynomials $\varphi = \varphi_d$ (with $d \geq 2$). Then the homology $H_*(\boldsymbol{X_W}; R)$ does not have φ^k–torsion for $k \geq 2$.* $\square$

We are particularly interested in Artin groups of finite and affine type. When $\boldsymbol{G_W}$ is an Artin group of finite type with n generators, K is the full simplicial complex on $S \cong \{1,\dots,n\}$ and therefore C^0_* has trivial homology in every dimension. Thus, the formula of Theorem 5.1 reduces to

$$H_m(\boldsymbol{X_W}; R) \cong \bigoplus_{\varphi} \left(\frac{R}{(\varphi)}\right)^{\oplus \operatorname{rk} \delta^{\varphi}_{m+1}}.$$

When $\boldsymbol{G_W}$ is an Artin group of affine type with $n+1$ generators, K is obtained from the full simplicial complex on $S \cong \{0,1,\dots,n\}$ by removing the single top-dimensional simplex. Then we have

$$H_m(\boldsymbol{X_W}; R) \cong \begin{cases} R & \text{for } m = n, \\ \bigoplus_{\varphi} (R/(\varphi))^{\oplus \operatorname{rk} \delta^{\varphi}_{m+1}} & \text{for } m < n. \end{cases}$$

Remark 5.3 When $\boldsymbol{G_W}$ is an Artin group of finite type, the corresponding reflection arrangement of hyperplanes $\mathcal{A}$ is finite. In this case it is well known that there is an R–module isomorphism between the twisted homology $H_*(\boldsymbol{X_W}; R)$ and the homology with constant coefficients $H_*(F; \mathbb{Q})$ of the Milnor fiber F of $\mathcal{A}$ [9]. The q–multiplication on the homology of $\boldsymbol{X_W}$ corresponds to the action of the monodromy operator on the homology of F. If $N = |\mathcal{A}|$, the square of the defining polynomial of the arrangement is $\boldsymbol{W}$–invariant, thus the order of the monodromy of the Milnor fibration divides $2N$. It follows that the polynomial $q^{2N} - 1$ must annihilate the homology. Since $q^{2N} - 1$ is square-free in characteristic 0, the homology cannot have φ^k–torsion for $k \geq 2$. So the conclusion of Theorem 5.2 is not surprising in the case of Artin groups of finite type.

In the rest of this paper we are going to construct precise matchings for Artin groups of type A_n, B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$. For each of these cases we are then going to: describe the critical simplices with respect to the constructed matching; find all alternating paths and incidence numbers between critical simplices; determine the ranks $\operatorname{rk} \delta^{\varphi}_*$ and use Theorem 5.1 to compute the homology $H_*(\boldsymbol{X_W}; R)$. The final results are stated in Theorem 3.1 (for A_n), Theorem 5.13 (for B_n), Theorem 5.16 (for $\widetilde{A}_n$) and Theorem 5.20 (for $\widetilde{C}_n$).

Let us first introduce some notation. We will have $S = \{1,2,\dots,n\}$ for the finite cases (A_n and B_n), and $S = \{0,1,2,\dots,n\}$ for the affine cases ($\widetilde{A}_n$ and $\widetilde{C}_n$). A simplex $\sigma \subseteq S$ will be also represented as a string of bits $\epsilon_i \in \{0,1\}$ (for $i \in S$), where $\epsilon_i = 1$

if $i \in S$ and $\epsilon_i = 0$ if $i \notin S$. For example, if $S = \{1, 2, 3, 4\}$, the string representation of $\sigma = \{1, 2, 4\}$ is 1101. Also, for $\sigma \subseteq S$ and $v \in S$, let

$$\sigma \veebar v = \begin{cases} \sigma \cup \{v\} & \text{if } v \notin \sigma, \\ \sigma \setminus \{v\} & \text{if } v \in \sigma; \end{cases}$$

it can be regarded as the bitwise xor between the string representation of σ and the string with $\epsilon_v = 1$ and $\epsilon_i = 0$ for $i \neq v$.

5.1 Case A_n

Many properties of the homology $H_*(X_W; R)$ in the case A_n have been thoroughly discussed in Section 3. Using precise matchings we are going to obtain a new proof of the formula for the homology of braid groups (Theorem 3.1).

Let $S = \{1, 2, \ldots, n\}$ be the set of vertices of the Coxeter graph of type A_n, as in Figure 1 on page 3954. In this case K is the full simplicial complex on S. Fix now an integer $d \geq 2$ and set $\varphi = \varphi_d$. The φ–weight of a simplex $\sigma \in K$ has been computed in Section 3 and is as follows:

$$v_\varphi(\sigma) = \sum_{i=1}^{m} \omega_\varphi(A_{n_i}),$$

where n_i is the size of the i^{th} (linear) connected component of the subgraph induced by σ and $\omega_\varphi(A_k)$ stands for the φ–weight of a connected component of type A_k, given by

$$\omega_\varphi(A_k) = \left\lfloor \frac{k+1}{d} \right\rfloor.$$

Fix also an integer f with $0 \leq f \leq d-1$. Let $K^A_{n,f} \subseteq K$ be

$$K^A_{n,f} = \{\sigma \in K \mid 1, 2, \ldots, f \in \sigma\}.$$

Notice that $K^A_{n,0} = K$, and $K^A_{n,f}$ is not a subcomplex of K for $f \geq 1$ (but it is still a subposet of K, so it makes sense to define a matching on it). We are going to construct a φ–precise matching on $K^A_{n,f}$. In particular, for $f = 0$, we will get a φ–precise matching for K. The precise matchings on $K^A_{n,f}$ for $f \geq 1$ will become useful when treating the cases B_n, $\widetilde{A}_n$ and $\widetilde{C}_n$.

For a fixed d, the matching will be constructed recursively in n and f for $n \geq 0$ and $0 \leq f \leq d-1$. We will write $K_{n,f}$ for $K^A_{n,f}$ throughout Section 5.1. The matching is as follows:

(a) If $\{1,\dots,d-1\}\subseteq\sigma$ then match σ with $\sigma\veebar d$ (unless $n=d-1$, in which case σ is critical). Here σ is matched with a simplex, which also occurs in case (a). Notice that for $f=d-1$ case (a) always applies, thus in the subsequent cases we can assume $f\leq d-2$.

(b) Otherwise, if $n=f$ then σ is critical.

(c) Otherwise, if $f+1\in\sigma$ then match σ with $\sigma\setminus\{f+1\}$. Notice that $\sigma\setminus\{f+1\}$ occurs in case (d).

(d) Otherwise, if $\{f+2,\dots,d-1\}\not\subseteq\sigma$ then match σ with $\sigma\cup\{f+1\}$. Notice that $\sigma\cup\{f+1\}$ occurs in case (c).

(e) We are left with the simplices σ such that $\{1,\dots,f,f+2,\dots,d-1\}\subseteq\sigma$ and $f+1\notin\sigma$. If we ignore the vertices $1,\dots,f+1$ we are left with the simplices on the vertex set $\{f+2,\dots,n\}$ which contain $f+2,\dots,d-1$; relabeling the vertices, these are the same as the simplices on the vertex set $\{1,\dots,n-f-1\}$ which contain $1,\dots,d-2-f$. Then construct the matching recursively as in $K_{n-f-1,d-2-f}$.

Example 5.4 For $n=5$, $d=3$ and $f=1$, $K_{n,f}$ contains $2^4=16$ simplices of which 14 are matched and 2 are critical. For instance, consider the simplex $\sigma=\{1,4,5\}$. Case (e) applies because $1\in\sigma$ and $2\notin\sigma$; the recursion requires us to consider the new simplex $\sigma'=\{2,3\}\in K_{3,0}$. Again case (e) applies because $1\notin\sigma'$ and $2\in\sigma'$; the recursion requires us to consider the new simplex $\sigma''=\{1,2\}\in K_{2,1}$. Finally, case (a) applies because $\{1,2\}\subseteq\sigma''$, and σ'' is critical because $\sigma''=\{1,2\}$. Therefore, $\sigma=\{1,4,5\}\in K_{5,1}$ is also critical. See Table 1 for an explicit description of the matching for $K_{5,1}$, $d=3$.

Remark 5.5 A peculiarity of this matching is that, if $\sigma'\to\tau'$ is in the matching, then $\sigma'=\tau'\cup\{v\}$ with $v\equiv f+1$ or $v\equiv 0\pmod d$. This can be easily checked by induction.

Lemma 5.6 *The matching described above is an acyclic weighted matching on $K_{n,f}$.*

Proof We proceed in two parts:

Part 1 (the matching is acyclic) The proof is by induction on n, the case $n=0$ being trivial. For $n\leq f$, in $K_{n,f}$ there are either 1 or 0 simplices. Assume from now on $n>f$. Let $P=\{p_{\mathrm{a}},p_{\mathrm{e}},p_{\mathrm{c,d}}\}$ be a three-element totally ordered poset, with the order

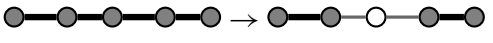
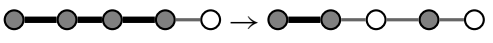
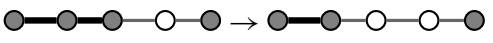
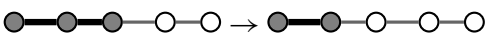
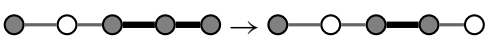
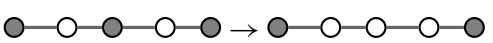
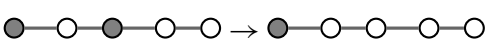
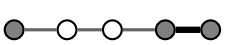
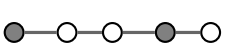
Simplices	$v_\varphi(\sigma)$	Step
	2	(a)
	1	(a)
	1	(a)
	1	(a)
	1	(e) $\rightsquigarrow$ (a)
	0	(e) $\rightsquigarrow$ (c)/(d)
	0	(e) $\rightsquigarrow$ (c)/(d)
 (critical)	1	(e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (a)
 (critical)	0	(e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (e) $\rightsquigarrow$ (b)

Table 1: Matching in the case A_n with $n = 5$, $d = 3$, $f = 1$. The last columns indicates the case where simplices occur. When case (e) is reached, the arrow "$\rightsquigarrow$" indicates how the recursion continues (after $K_{5,1}$, the recursion involves $K_{3,0}$, $K_{2,1}$ and $K_{0,0}$).

given by $p_{\mathrm{a}} > p_{\mathrm{e}} > p_{\mathrm{c,d}}$. Consider the map $\eta\colon K_{n,f} \to P$ which sends $\sigma \in K_{n,f}$ to the p_{x} such that σ occurs in case (x) (here cases (c) and (d) are united). For instance, $\eta(\sigma) = p_{\mathrm{a}}$ if and only if $\{1, \dots, d-1\} \subseteq \sigma$. Notice that the map η is compatible with the matching, ie two matched simplices lie in the same fiber of η.

Let us prove that η is a poset map. Given two simplices $\sigma \geq \tau$, we want to prove that $\eta(\sigma) \geq \eta(\tau)$. If $\eta(\tau) = p_{\mathrm{a}}$ then $\{1, \dots, d-1\} \subseteq \tau \subseteq \sigma$, so $\eta(\sigma) = p_{\mathrm{a}}$ also. If $\eta(\tau) = p_{\mathrm{e}}$ then $\{1, \dots, f, f+2, \dots, d-1\} \subseteq \tau \subseteq \sigma$, thus $\eta(\sigma) \in \{p_{\mathrm{a}}, p_{\mathrm{e}}\}$. Finally, if $\eta(\tau) = p_{\mathrm{c,d}}$, there is nothing to prove.

Since η is a poset map, our matching is acyclic if and only if it is acyclic on each fiber of η. The restriction of the matching to $\eta^{-1}(p_{\mathrm{a}})$ consists of edges $\sigma \to \tau$ with $\sigma = \tau \cup \{d\}$, so there is no alternating path of length ≥ 4 and the matching is acyclic. The restriction to $\eta^{-1}(p_{\mathrm{c,d}})$ is acyclic for the same reason (it is always the vertex $f+1$ which is added or removed). The restriction to $\eta^{-1}(p_{\mathrm{e}})$ is acyclic by induction.

Part 2 (the matching is weighted) The proof is by induction on n, the case $n = 0$ being trivial. In case (a), $v_\varphi(\sigma) = v_\varphi(\sigma \veebar d)$ by what we have already said in Section 3 (see the proof of Theorem 3.6). In cases (c) and (d), both the simplices σ and $\sigma \veebar (f+1)$ do not contain $\{1, \dots, d-1\}$. Thus, the (possibly empty) connected component of 1 has size $\leq d-2$ and does not contribute to the weight. Therefore, $v_\varphi(\sigma) = v_\varphi(\sigma \veebar (f+1))$. Finally, suppose that σ and τ are simplices that occur in case (e). Let $\hat{\sigma}, \hat{\tau} \in K_{n-f-1,\, d-2-f}$ be the simplices obtained ignoring the vertices $1, \dots, f+1$, as described above. Since $f \leq d-2$ we have that

$$v_\varphi(\sigma) = v_\varphi(\tau) \quad \text{if and only if} \quad v_\varphi(\hat{\sigma}) = v_\varphi(\hat{\tau}),$$

so we are done by induction on n. □

The critical simplices of the matching on $K_{n,f}$ are quite simple to describe. If $f \leq d-2$, $n > f$ and $n \equiv f$ or $-1 \pmod d$, there are two critical simplices, one of weight 1 and one of weight 0. If $n = f$ there is one critical simplex (in fact there is only one simplex in $K_{n,f}$). In all the other cases the matching has no critical simplices. We are going to prove this in the following theorem. See Table 2, and Figures 4 and 5, for an explicit description of the critical simplices. Notice in particular that, when there are two critical simplices, one is a face of the other in $K_{n,f}$.

Case		Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$
$n > f$, $f \leq d-2$	$n = kd+f$ (Figure 4)	$(1^f 01^{d-2-f}0)^{k-1} 1^f 01^{d-1}$	$n-2k+1$	1
		$(1^f 01^{d-2-f}0)^{k} 1^f$	$n-2k$	0
	$n = kd-1$ (Figure 5)	$(1^f 01^{d-2-f}0)^{k-1} 1^{d-1}$	$n-2k+2$	1
		$(1^f 01^{d-2-f}0)^{k-1} 1^f 01^{d-2-f}$	$n-2k+1$	0
$n = f$		1^f	n	1 or 0

Table 2: Description of the critical simplices for A_n

Theorem 5.7 (critical simplices in case A_n) *The critical simplices for the matching on $K_{n,f}$ are those listed in Table 2. In particular, the matching is always precise, and has no critical simplices if $n \not\equiv f, -1 \pmod d$ or $n > f = d-1$. In addition, when there are two critical simplices there is only one alternating path between them (the trivial one).*

Proof We proceed in two parts:

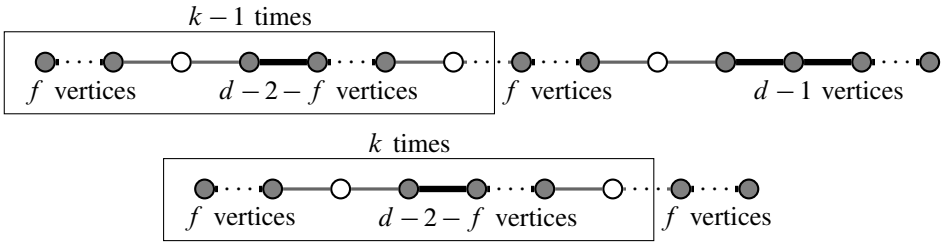


Figure 4: Critical simplices for $f \leq d-2$ and $n = kd+f$ $(k \geq 1)$

Part 1 (critical simplices) The proof is by induction on n, the case $n = 0$ being trivial (there is one critical simplex for $f = 0$, namely the empty simplex $\varnothing$, and no critical simplices for $f > 0$). Let $\sigma \in K_{n,f}$ be a simplex. Let us consider each of the five cases that can occur in the construction of the matching:

(a) We have $\{1, \dots, d-1\} \subseteq \sigma$. Then σ is critical if and only if $n = d-1$. If $n = d-1$ then σ is indeed listed in Table 2, as the first of the two critical simplices in the case $n = kd-1$ (here $k = 1$). Conversely, the only simplex of Table 2 which contains $\{1, \dots, d-1\}$ is the first one of the case $n = kd-1$ when $k = 1$.

(b) We have $n = f$ and $\sigma = \{1, \dots, f\}$. Then σ is critical, and it is indeed listed as the second of the two critical simplices in the case $n = kd+f$ (here $k = 0$).

(c) We have $\{1, \dots, f+1\} \subseteq \sigma$. Then σ is not critical. The only listed simplex which contains $\{1, \dots, f+1\}$ is the first one of the case $n = kd-1$ for $k = 1$, but it is equal to $\{1, \dots, d-1\}$ and so it must be different from σ.

(d) We have $f+1 \notin \sigma$ and $\{f+2, \dots, d-1\} \not\subseteq \sigma$. Then σ is not critical. It is easy to check that all the listed simplices τ with $n > f$ and $f+1 \notin \tau$ satisfy $\{f+2, \dots, d-1\} \subseteq \tau$, so none of them can be equal to σ.

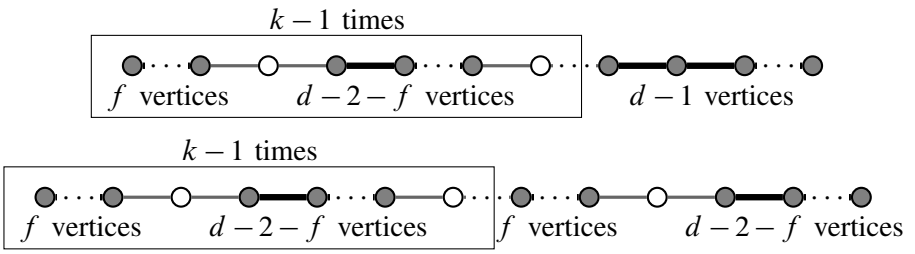


Figure 5: Critical simplices for $f \leq d-2$ and $n = kd-1$ $(k \geq 1)$

(e) We have $\{1,\dots,f,f+2,\dots,d-1\}\subseteq\sigma$ and $f+1\notin\sigma$. Then σ is critical if and only if the simplex $\widehat{\sigma}\in K_{n-f-1,d-2-f}$ is critical, where $\widehat{\sigma}$ is constructed from σ by ignoring the first $f+1$ vertices. By induction, $\widehat{\sigma}$ is critical if and only if it is listed in Table 2. By taking the simplices of Table 2 for $K_{n-f-1,d-2-f}$ and adding 1^f0 at the beginning, one obtains exactly the simplices of Table 2 for $K_{n,f}$ (but the two cases are exchanged). This concludes the induction argument.

For fixed n and f, the quantity $|\sigma|-v_\varphi(\sigma)$ is constant among the critical simplices σ. More precisely, it is equal to $n-2k$ for $n=kd+f$ (except for $f=d-1$) and to $n-2k+1$ for $n=kd-1$. Therefore, the matching is precise.

Part 2 (alternating paths) We want to prove that the only alternating path between critical simplices is the trivial one. Consider the case $n=kd+f$ for $k\geq 1$ (the case $n=kd-1$ is analogous). Let σ and τ be the two critical simplices (the ones of Figure 4). Notice that $\sigma=\tau\cup\{n-f\}$. To establish a contradiction suppose we have a nontrivial alternating path

$$\sigma\rhd\tau_1\lhd\sigma_1\rhd\tau_2\lhd\sigma_2\rhd\cdots\rhd\tau_m\lhd\sigma_m\rhd\tau,\quad m\geq 1.$$

Since $\tau_1\neq\tau$, we must have $\sigma=\tau_1\cup\{v_1\}$ with $v_1\neq n-f$. Notice that $n-f$ is the only vertex $v\in\sigma$ which satisfies $v\equiv f+1$ or $v\equiv 0 \pmod d$. Therefore, by Remark 5.5, the vertex v_1 will never be recovered throughout the alternating path. This is a contradiction since $v_1\in\tau$. □

As a consequence we can compute the homology $H_*(\boldsymbol{G_W};R)$ for $\boldsymbol{G_W}$ of type A_n. This gives a proof of Theorem 3.1.

Proof of Theorem 3.1 (homology in case A_n) We simply need to apply the formula given by Theorem 5.1 using our precise matching on $K=K_{n,0}$. Since $f=0$, there are critical simplices for $n=kd$ or $n=kd-1$. The boundary δ^φ_{m+1} is nontrivial only for

$$\begin{cases} m=n-2k=k(d-2) & \text{if } n=kd,\\ m=n-2k+1=k(d-2) & \text{if } n=kd-1.\end{cases}$$

In both cases for $m=k(d-2)$ we have $\operatorname{rk}\delta^\varphi_{m+1}=1$, and all the other boundaries are trivial. Theorem 3.1 follows. □

Remark 5.8 A relation with independence complexes can be found for $K_{n,f}$ also when $1\leq f\leq d-2$. Indeed, choosing a suitable weighted matching (similar to the

one of Theorem 3.6), the set of critical simplices of positive weight in $K_{n,f}$ is

$$\{\sigma \sqcup A_{d-1} \mid \sigma \in \mathrm{Ind}_{d-2}(A_{n-d}) \cap K_{n-d,f}\}.$$

5.2 Case B_n

Consider now a Coxeter graph of type B_n, as in Figure 6. In this case K is again the full simplex on vertices $\{1, 2, \dots, n\}$. The Poincaré polynomial of a Coxeter group of type B_k is given by

$$W_{B_k}(q) = [2k]_q!! = \prod_{i=1}^{n} [2i]_q = \prod_{\varphi} \varphi^{\omega_\varphi(B_k)},$$

where, for a given cyclotomic polynomial $\varphi = \varphi_d$, we have

$$\omega_\varphi(B_k) = \begin{cases} \lfloor k/d \rfloor & \text{if } d \text{ is odd,} \\ \lfloor (k/\frac{d}{2}) \rfloor & \text{if } d \text{ is even;} \end{cases}$$

see for example [5]. Then we can compute the φ–weight of any simplex $\sigma \in K$ by looking at the subgraph $\Gamma(\sigma)$ induced by σ. Let $\Gamma_1(\sigma), \dots, \Gamma_m(\sigma)$ be the connected components of $\Gamma(\sigma)$, with cardinality $n_1, \dots, n_m$, respectively, where $\Gamma_1(\sigma)$ is the (possibly empty) component that contains the vertex $1 \in S$. Then

$$v_\varphi(\sigma) = \omega_\varphi(B_{n_1}) + \sum_{i=2}^{m} \omega_\varphi(A_{n_i}).$$

The situation is quite different depending on the parity of d. If d is odd, the φ–weight of a B_{k+1} component is equal to the φ–weight of an A_k component:

$$\omega_\varphi(B_{k+1}) = \left\lfloor \frac{k+1}{d} \right\rfloor = \omega_\varphi(A_k) \quad (d \text{ odd}).$$

For this reason it is possible to construct a very simple matching on K: match any simplex $\sigma \in K$ with $\sigma \trianglerighteq 1$.

Figure 6: A Coxeter graph of type B_n

Lemma 5.9 *The matching constructed above for d odd is an acyclic weighted matching on K, with no critical simplices.*

Proof Clearly the matching is acyclic and there are no critical simplices. Let us prove that the matching is weighted. Let $\sigma \in K$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\sigma \veebar 1)$ is the leftmost connected component, which in one case is of type B_{k+1} and in the other case is of type A_k. Therefore, $v_\varphi(\sigma) = v_\varphi(\sigma \veebar 1)$. □

Suppose from now on that d is even. The simplicial complex K is partitioned as

$$K = \bigsqcup_{q \geq 0} K_q, \quad \text{where } K_q = \left\{ \sigma \in K \;\middle|\; \left\lfloor \frac{|\Gamma_1(\sigma)|}{d/2} \right\rfloor = q \right\}.$$

Here $\Gamma_1(\sigma)$ is the (possibly empty) connected component of $\Gamma(\sigma)$ which contains the vertex 1. Notice that each K_q is a subposet of K, but not a subcomplex in general. For a given simplex $\sigma \in K_q$, let $|\Gamma_1(\sigma)| = q\frac{d}{2} + r$ with $0 \leq r < \frac{d}{2}$. The matching on K_q is as follows:

(a) If $r \geq 1$ $\left(\text{ie } q\frac{d}{2}+1 \in \sigma\right)$ then match σ with $\sigma \setminus \left\{q\frac{d}{2}+1\right\}$.

(b) If $r = 0$ $\left(\text{ie } q\frac{d}{2}+1 \notin \sigma\right)$ and $\left\{q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\right\} \not\subseteq \sigma$, then match σ with $\sigma \cup \left\{q\frac{d}{2}+1\right\}$ (unless $n = q\frac{d}{2}$, in which case σ is critical).

(c) We are left with the simplices σ for which neither (a) nor (b) apply, ie with $r = 0$ and $\left\{q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\right\} \subseteq \sigma$. Ignore the first $q\frac{d}{2}+1$ vertices and relabel the remaining ones from 1 to $n - q\frac{d}{2} - 1$, so that we are left exactly with the simplices of $K^A_{n-qd/2-1,\, d/2-1}$. Then construct the matching on $K^A_{n-qd/2-1,\, d/2-1}$ as in Section 5.1.

Putting together the matchings on each K_q, we obtain a matching on the full simplicial complex K.

Example 5.10 For $n = 4$ and $d = 4$, the simplicial complex K contains $2^4 = 16$ simplices of which 12 are matched and 4 are critical. For instance, consider $\sigma = \{1, 2\} \in K$. Then $q = 1$ and $r = 0$. Since $4 \notin \sigma$, case (b) occurs. Therefore, σ is matched with $\sigma \cup \{3\} = \{1, 2, 3\}$. See Table 3 for an explicit description of the matching in this case.

Lemma 5.11 *The matching constructed above for d even is an acyclic weighted matching on K.*

Proof We proceed in two parts:

Part 1 (the matching is acyclic) The map $K \to (\mathbb{N}, \leq)$ which sends σ to $q = \left\lfloor |\Gamma_1(\sigma)| / \frac{d}{2} \right\rfloor$ is a poset map compatible with the matching, and its fibers are exactly

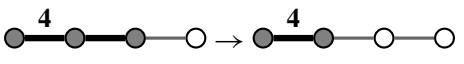
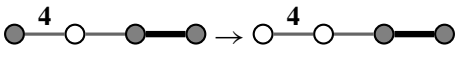
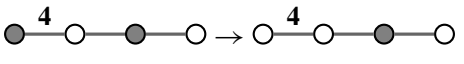
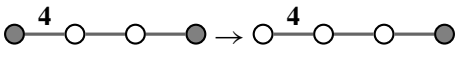
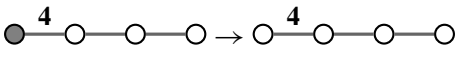
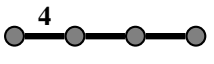
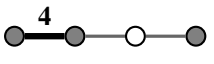
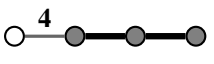
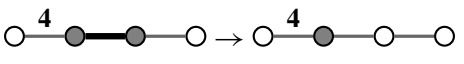
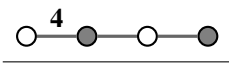
Simplices	$v_\varphi(\sigma)$	q	Step
	1	1	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
	0	0	(a)/(b)
 (critical)	2	2	(b)
 (critical)	1	1	(c)
 (critical)	1	0	(c)
	0	0	(c)
 (critical)	0	0	(c)

Table 3: Matching in the case B_n with $n = 4$ and $d = 4$

the subsets K_q for $q \in \mathbb{N}$. Therefore, we only need to prove that the matching on each fiber K_q is acyclic.

Let $P = \{p_{\mathrm{c}}, p_{\mathrm{a,b}}\}$ be a two-element totally ordered poset with $p_{\mathrm{c}} > p_{\mathrm{a,b}}$. For a fixed $q \in \mathbb{N}$, consider the map $\eta\colon K_q \to P$ which sends σ to the p_{x} such that σ occurs in case (x) (here cases (a) and (b) are united). Clearly η is compatible with the matching. We want to prove that it is a poset map, and for this we only need to show that if $\eta(\tau) = p_{\mathrm{c}}$ and $\tau \le \sigma$ then $\eta(\sigma) = p_{\mathrm{c}}$ also. We have that $\{1, \dots, q\frac{d}{2}, q\frac{d}{2}+2, \dots, (q+1)\frac{d}{2}\} \subseteq \tau \subseteq \sigma$. The simplex σ cannot contain the vertex $q\frac{d}{2}+1$, because otherwise we would have $\sigma \in K_{q+1}$. Therefore, $\eta(\sigma) = p_{\mathrm{c}}$.

On the fiber $\eta^{-1}(p_{\mathrm{a,b}})$ the matching is acyclic because the same vertex $q\frac{d}{2}+1$ is always added or removed. On the fiber $\eta^{-1}(p_{\mathrm{c}})$ the matching is acyclic by Lemma 5.6. Therefore, the entire matching on K_q is acyclic.

Part 2 (the matching is weighted) Let $\sigma \in K_q$ be a simplex which occurs either in case (a) or case (b). We want to show that $v_\varphi(\sigma) = v_\varphi\big(\sigma \veebar \big(q\frac{d}{2}+1\big)\big)$. Suppose

Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
$\sigma_q = 1^{qd/2}(01^{d/2-1})^{k-q-2}01^{d-1}$	$n-k+q+1$	$q+1$	$0 \le q \le k-2$
$\sigma'_q = 1^{qd/2}(01^{d/2-1})^{k-q}$	$n-k+q$	q	$0 \le q \le k$

Table 4: Description of the critical simplices for B_n, where d is even and $n = k\frac{d}{2}$

without loss of generality that σ occurs in case (a), ie $r \ge 1$ and $q\frac{d}{2}+1 \in \sigma$. Let $\tau = \sigma \setminus \{q\frac{d}{2}+1\}$. The only difference between $\Gamma(\sigma)$ and $\Gamma(\tau)$ is that $\Gamma(\sigma)$ has a $B_{qd/2+r}$ component, whereas $\Gamma(\tau)$ has a $B_{qd/2}$ component and an A_{r-1} component instead. Since $1 \le r < \frac{d}{2}$, we have that

$$\omega_\varphi(B_{q\frac{d}{2}+r}) = \omega_\varphi(B_{q\frac{d}{2}}) = q \quad \text{and} \quad \omega_\varphi(A_{r-1}) = 0$$

(the second equality holds because $r-1 \le \frac{d}{2}-1 \le d-2$). Therefore, $v_\varphi(\sigma) = v_\varphi(\tau)$.

For simplices occurring in case (c), the matching only involves changes in the connected components not containing the first vertex (ie connected components of type A_k). Therefore, Lemma 5.6 applies. □

Now we are going to describe the critical simplices. The matching has no critical simplices when n is not a multiple of $\frac{d}{2}$. On the other hand, if $n = k\frac{d}{2}$, we have two families of critical simplices: σ_q (for $0 \le q \le k-2$) and σ'_q (for $0 \le q \le k$). See Table 4 and Figure 7 for the definition of these simplices. For instance, in Example 5.10, the critical simplices are $\sigma_0 = \{2,3,4\}$, $\sigma'_2 = \{1,2,3,4\}$, $\sigma'_1 = \{1,2,4\}$ and $\sigma'_0 = \{1,3\}$. See also Table 5, where the critical simplices are listed by dimension.

Theorem 5.12 (critical simplices in case B_n) *The critical simplices for the matching on K are those defined in Table 4. In particular, the matching is always precise, and*

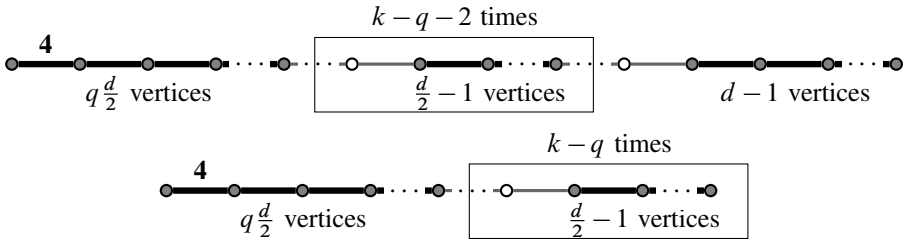


Figure 7: Critical simplices for B_n, where d is even and $n = k\frac{d}{2}$. Above is the simplex σ_q $(0 \le q \le k-2)$ and below is the simplex σ'_q $(0 \le q \le k)$.

$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	Simplices
n	k	σ'_k
$n-1$	$k-1$	σ_{k-2}, σ'_{k-1}
$n-2$	$k-2$	σ_{k-3}, σ'_{k-2}
$\vdots$	$\vdots$	$\vdots$
$n-k+1$	1	σ_0, σ'_1
$n-k$	0	σ'_0

Table 5: Critical simplices for B_n by dimension, where d is even and $n = k\frac{d}{2}$

has no critical simplices if d is odd or if d is even and $n \not\equiv 0 \left(\mathrm{mod}\, \frac{d}{2}\right)$. In addition, if d is even and $n \equiv 0 \left(\mathrm{mod}\, \frac{d}{2}\right)$, the incidence numbers between the critical simplices in the Morse complex are as follows:

$$\begin{aligned}
[\sigma_q : \sigma_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)d/2}, \\
[\sigma_q : \sigma'_q]^{\mathcal{M}} &= (-1)^{(k-1)(d/2-1)+q}, \\
[\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k(d/2-1)+q}, \\
[\sigma'_q : \sigma'_{q-1}]^{\mathcal{M}} &= (-1)^{(q-1)d/2}.
\end{aligned}$$

Proof We proceed in two parts:

Part 1 (critical simplices) For d odd there is nothing to prove. Suppose from now on that d is even. For a given simplex $\sigma \in K$, let us consider each of the three cases that can occur in the construction of the matching.

(a) We have $q\frac{d}{2} + 1 \in \sigma$, and σ is not critical. Indeed, none of the simplices of Table 4 contains the vertex $q\frac{d}{2} + 1$.

(b) We have $q\frac{d}{2} + 1 \notin \sigma$ and $\{q\frac{d}{2} + 2, \dots, (q+1)\frac{d}{2}\} \not\subseteq \sigma$. In this case σ is critical if and only if $n = q\frac{d}{2}$. Indeed, the only simplex of this type in Table 4 is σ'_k $\left(\text{which occurs for } q = k, \text{ ie } n = q\frac{d}{2}\right)$.

(c) In the remaining case, we end up with the matching on $K^A_{n-qd/2-1,\, d/2-1}$. By Theorem 5.7, this matching admits critical simplices if and only if

$$n - q\frac{d}{2} - 1 \equiv \frac{d}{2} - 1 \ (\mathrm{mod}\, d) \quad \text{or} \quad n - q\frac{d}{2} - 1 \equiv -1 \ (\mathrm{mod}\, d),$$

ie for $n \equiv 0 \pmod{\frac{d}{2}}$. Notice that if $f = \frac{d}{2} - 1$ then $d - 2 - f = \frac{d}{2} - 1$ also. Therefore, again by Theorem 5.7, for $n \equiv 0 \pmod{\frac{d}{2}}$ the critical simplices are exactly the ones listed in Table 4.

For a fixed $n = k\frac{d}{2}$, the quantity $|\sigma| - v_\varphi(\sigma)$ is constant among the critical simplices and is equal to $n - k$. Thus, the matching is precise.

Part 2 (incidence numbers) We are going to show that there is exactly one alternating path for each of the four pairs. Notice that if $\sigma \to \tau$ is in the matching of K_q, then $\sigma = \tau \cup \{v\}$ with $v \equiv 1 \pmod{\frac{d}{2}}$ and $v \geq q\frac{d}{2} + 1$. In particular, if at a certain point of an alternating path one removes a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, then that vertex will never be added again. But all the critical simplices contain every vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$, so any alternating path between critical simplices cannot ever drop a vertex v with $v \not\equiv 1 \pmod{\frac{d}{2}}$.

Let us now consider each of the four pairs.

(σ_q, σ_{q-1}) We have that $\sigma_q = \sigma_{q-1} \cup \{v\}$ where $v = (q-1)\frac{d}{2} + 1$, so there is the trivial alternating path $\sigma_q \to \sigma_{q-1}$ which contributes to the incidence number $[\sigma_q : \sigma_{q-1}]^{\mathcal{M}}$ by

$$[\sigma_q : \sigma_{q-1}] = (-1)^{|\{w \in \sigma_q | w < v\}|} = (-1)^{(q-1)d/2}.$$

Suppose by contradiction that there exists some other (nontrivial) alternating path

$$\sigma_q \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-1}, \quad m \geq 1.$$

Let $\sigma_q = \tau_1 \cup \{u\}$. By the previous considerations, we must have $u \equiv 1 \pmod{\frac{d}{2}}$. If $u = (k-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma'_q$, but this is not possible since alternating paths stop at critical simplices. Similarly, if $u = (q-1)\frac{d}{2} + 1$ then we would stop at $\tau_1 = \sigma_{q-1}$. Therefore, we must have $u \leq (q-2)\frac{d}{2} + 1$. But then $\tau_1 \in K_{q'}$ with $q' \leq q - 2$, and by induction all the subsequent simplices in the alternating path must lie in

$$\bigcup_{q'' \leq q-2} K_{q''}.$$

In particular, $\sigma_{q-1} \in K_{q''}$ for some $q'' \leq q - 2$, but this is a contradiction since $\sigma_{q-1} \in K_{q-1}$.

(σ_q, σ'_q) This case is similar to the previous one, except for the fact that $\sigma_q = \sigma'_q \cup \{v\}$ for $v = (k-1)\frac{d}{2} + 1$. So the only alternating path is the trivial one, which contributes

to the incidence number by

$$[\sigma_q : \sigma'_q] = (-1)^{|\{w \in \sigma_q | w < v\}|} = (-1)^{qd/2+(k-q-1)(d/2-1)} = (-1)^{(k-1)(d/2-1)+q}.$$

$(\sigma'_q, \sigma'_{q-1})$ This case is also similar to the previous ones. Here we have $\sigma'_q = \sigma'_{q-1} \cup \{v\}$ with $v = (q-1)\frac{d}{2} + 1$, so the contribution to the incidence number due to the trivial alternating path is

$$[\sigma'_q : \sigma'_{q-1}] = (-1)^{|\{w \in \sigma'_q | w < v\}|} = (-1)^{(q-1)d/2}.$$

$(\sigma'_q, \sigma_{q-2})$ This case is more complicated because the only alternating path is non-trivial. Suppose we have an alternating path

$$\sigma'_q \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q-2}, \quad m \geq 1$$

(m must be at least 1 because σ_{q-2} is not a face of σ'_q in K). Let $\sigma'_q = \tau_1 \cup \{v\}$, with $v \equiv 1 \pmod{\frac{d}{2}}$. If $v = (q-1)\frac{d}{2} + 1$ then $\tau_1 = \sigma'_{q-1}$ and we must stop because σ'_{q-1} is already critical. If $v \leq (q-3)\frac{d}{2} + 1$ we fall into some $K_{q'}$ with $q' \leq q-3$ and it is not possible to reach $\sigma_{q-2} \in K_{q-2}$. Therefore, necessarily $v = (q-2)\frac{d}{2} + 1$. Then τ_1 is matched with

$$\rho_1 = \tau_1 \cup \left\{q\frac{d}{2} + 1\right\}.$$

Then again, $\tau_2 = \rho_1 \setminus \{v_2\}$ and the only possibility for v_2 (in order to have $\tau_2 \neq \tau_1$ and not to fall into some $K_{q'}$ with $q' \leq q-3$) is $v_2 = (q-1)\frac{d}{2} + 1$. Proceeding by induction we obtain that

$$\tau_i = \rho_{i-1} \setminus \left\{(q-3+i)\frac{d}{2} + 1\right\},$$
$$\rho_i = \tau_i \cup \left\{(q-1+i)\frac{d}{2} + 1\right\}.$$

The path stops at $\tau_{k-q+1} = \sigma_{q-2}$, and its length is $m = k-q$. See Table 6 for an example. The contribution of this path to the incidence number is

$$(-1)^{k-q}[\sigma'_q : \tau_1][\rho_1 : \tau_1] \cdots [\rho_{k-q} : \tau_{k-q}][\rho_{k-q} : \sigma_{q-2}],$$

where

$$[\rho_{i-1} : \tau_i] = (-1)^{(q-2)d/2+(i-1)(d/2-1)},$$
$$[\rho_i : \tau_i] = (-1)^{(q-2)d/2+(i-1)(d/2-1)+d-1}$$
$$= (-1)^{(q-2)d/2+(i-1)(d/2-1)+1}$$

(these formulas also hold for $\rho_0 = \sigma'_q$ and $\tau_{k-q+1} = \sigma_{q-2}$). Then

$$\begin{aligned}[\sigma'_q : \sigma_{q-2}]^{\mathcal{M}} &= (-1)^{k-q}\Bigg(\prod_{i=1}^{k-q}[\rho_{i-1} : \tau_i][\rho_i : \tau_i]\Bigg)[\rho_{k-q} : \sigma_{q-2}] \\ &= (-1)^{k-q}(-1)^{k-q}(-1)^{(q-2)d/2+(k-q)(d/2-1)} \\ &= (-1)^{k(d/2-1)+q}.\end{aligned}$$

□

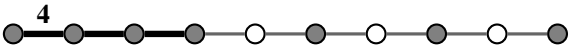
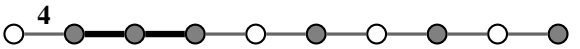
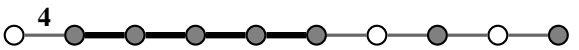
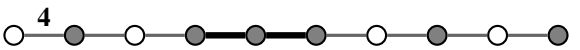
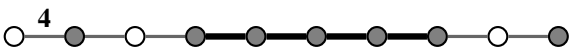
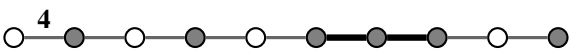
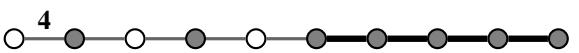
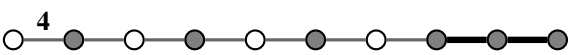
Simplices	$v_\varphi(\sigma)$
$\sigma'_2 =$ 	2
$\tau_1 =$ 	1
$\rho_1 =$ 	1
$\tau_2 =$ 	1
$\rho_2 =$ 	1
$\tau_3 =$ 	1
$\rho_3 =$ 	1
$\sigma_0 =$ 	1

Table 6: The only alternating path from σ'_q to σ_{q-2} for $n = 10$, $d = 4$, $k = 5$ and $q = 2$

Having a complete description of a precise matching on K, we can now compute the homology $H_*(\boldsymbol{G_W}; R)$ for $\boldsymbol{G_W}$ of type B_n. We recover the result of [19].

Theorem 5.13 (homology in case B_n [19]) *For an Artin group $\boldsymbol{G_W}$ of type B_n, we have*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} R/(\varphi_d) & \text{if } d \text{ is even, } n = k\frac{d}{2} \text{ and } n-k \le m \le n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof For a fixed $\varphi = \varphi_d$, we need to compute the boundary δ^{φ}_{m+1}. Assume that $n = k\frac{d}{2}$, otherwise there are no critical simplices. In top dimension ($m+1 = n$) we

have $\operatorname{rk}\delta_n^\varphi = 1$, because $[\sigma'_k : \sigma'_{k-1}] \neq 0$. For $m \leq n-k-1$ the boundary δ_{m+1}^φ vanishes, because there are no critical simplices in dimension $\leq n-k-1$. For $m = n-k$ we have $\operatorname{rk}\delta_{m+1}^\varphi = 1$, because $[\sigma'_1 : \sigma'_0] \neq 0$. Finally, for $n-k+1 \leq m \leq n-2$, we have (set $q = m-n+k$)

$$\operatorname{rk}\delta_{m+1}^\varphi = \operatorname{rk}\begin{pmatrix} [\sigma_q : \sigma_{q-1}][\sigma'_{q+1} : \sigma_{q-1}] \\ [\sigma_q : \sigma'_q][\sigma'_{q+1} : \sigma'_q] \end{pmatrix} = \operatorname{rk}\begin{pmatrix} (-1)^{(q-1)d/2}(-1)^{k(d/2-1)+q+1} \\ (-1)^{(k-1)(d/2-1)+q}(-1)^{qd/2} \end{pmatrix} = 1,$$

because

$$\det\begin{pmatrix} (-1)^{(q-1)d/2} & (-1)^{k(d/2-1)+q+1} \\ (-1)^{(k-1)(d/2-1)+q} & (-1)^{qd/2} \end{pmatrix} = (-1)^{d/2} - (-1)^{d/2} = 0.$$

To summarize, we have $\operatorname{rk}\delta_{m+1}^\varphi = 1$ for $n-k \leq m \leq n-1$, and $\operatorname{rk}\delta_{m+1}^\varphi = 0$ otherwise. We conclude applying Theorem 5.1. □

5.3 Case $\widetilde{A}_n$

Consider now the case of affine Artin groups of type $\widetilde{A}_n$ (see Figure 8 for a picture of the corresponding Coxeter graph). Recall that, for affine Artin groups, the simplicial complex K consists of all the simplices $\sigma \subseteq S = \{0, \dots, n\}$ except for the full simplex $\sigma = \{0, \dots, n\}$. For any $\sigma \in K$, the induced subgraph $\Gamma(\sigma)$ consists only of connected components of type A_k. The weight $v_\varphi(\sigma)$ is then computed as in Section 5.1.

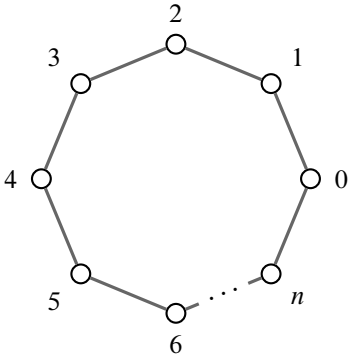


Figure 8: A Coxeter graph of type $\widetilde{A}_n$

If σ is a simplex of K then σ misses at least one vertex of $\{0, 1, \dots, n\}$. Define h to be the first vertex missing from σ reading counterclockwise from 0. Then σ belongs to a unique

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \dots, h-1\} \subseteq \sigma\}.$$

In other words, the subsets K_h form a partition of K,

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Let $h = qd + r$ with $0 \le r \le d-1$, and let $m = n - qd$. Consider the following clockwise relabeling of the vertices: $r-1$ becomes 1, $r-2$ becomes 2, …, 0 becomes r, n becomes $r+1$, …, $h+2$ becomes $m-1$, $h+1$ becomes m (the vertices r, $r+1$, …, h are forgotten). This relabeling induces a poset isomorphism

$$K_h \xrightarrow{\cong} K^A_{m,r}.$$

Then equip K_h with the pullback of the matching on $K^A_{m,r}$ defined in Section 5.1 (for $h = n$ the only simplex in K_h is critical).

Lemma 5.14 *The matching defined above is an acyclic weighted matching on K.*

Proof Consider the map $\eta\colon K \to (\mathbb{N}, \le)$ which sends $\sigma \in K$ to $\min\{h \in \mathbb{N} \mid h \notin \sigma\}$. Then η is a poset map with fibers $\eta^{-1}(h) = K_h$. The matching on each $K_h \cong K^A_{m,r}$ is acyclic by Lemma 5.6, therefore, the whole matching on K is acyclic.

Fix $h \in \{0, \dots, n\}$. The bijection $K_h \xrightarrow{\cong} K^A_{m,r}$ is such that, if $\sigma \mapsto \widehat{\sigma}$, then $v_\varphi(\sigma) = v_\varphi(\widehat{\sigma}) + q$. Indeed, $\Gamma(\sigma)$ is obtained from $\Gamma(\widehat{\sigma})$ by adding qd vertices to one (possibly empty) connected component, and this increases the weight by q. Then, since the matching on $K^A_{m,r}$ is weighted, its pullback also is. □

Theorem 5.15 (critical simplices in case $\widetilde{A}_n$) *The critical simplices for the matching on K are those listed in Table 7. The only nontrivial incidence numbers between critical cells are*

$$\begin{aligned} [\tau_q : \tau'_q]^{\mathcal{M}} &= \pm 1 \quad (\text{for } n = kd + r \text{ with } 0 \le r \le d-2), \\ [\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} &= \pm 1, \\ [\overline{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} &= \pm 1 \quad (\text{for } n = kd - 1). \end{aligned}$$

In particular, the matching is precise.

Proof We proceed in three parts:

Part 1 (critical simplices) For $n \not\equiv -1 \pmod d$, the matching on K_h has critical simplices only for $h \equiv n \pmod d$. If we write $n = kd + r$ with $0 \le r \le d-2$, then

Case	Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
$n = kd + r$	τ_q	$n-2(k-q)+1$	$q+1$	$0 \le q \le k-1$
$\not\equiv -1 \pmod d$	τ'_q	$n-2(k-q)$	q	$0 \le q \le k$
	$\sigma_{q,r}$	$n-2(k-q)+2$	$q+1$	$0 \le q \le k-2, 0 \le r \le d-2$
$n = kd-1$	$\sigma'_{q,r}$	$n-2(k-q)+1$	q	$0 \le q \le k-1, 0 \le r \le d-2$
	$\overline{\sigma} = 1^n 0$	n	k	

Simplices
$\tau_q = 1^{qd+r}01^{d-1}0(1^r01^{d-2-r}0)^{k-q-1}$
$\tau'_q = 1^{qd+r}0(1^r01^{d-2-r}0)^{k-q}$
$\sigma_{q,r} = 1^{qd+r}01^{d-1}0(1^{d-2-r}01^r0)^{k-q-2}1^{d-2-r}0$
$\sigma'_{q,r} = 1^{qd+r}0(1^{d-2-r}01^r0)^{k-q-1}1^{d-2-r}0$

Table 7: Description of the critical simplices for $\widetilde{A}_n$. Below, the binary string notation is used.

$K_{qd+r} \cong K^A_{n-qd,r}$ has two critical simplices for $0 \le q \le k-1$ and one critical simplex for $q = k$. These simplices are the ones listed in Table 7.

Suppose now that $n = kd-1$. For any $h = qd+r$, with $0 \le r \le d-2$, the complex $K_{qd+r} \cong K^A_{n-qd,r}$ has two critical simplices because $n - qd \equiv -1 \pmod d$. Moreover, for $h = n$ the complex $K_n \cong K^A_{d-1,d-1}$ has one critical simplex. In the remaining cases the matching on K_h has no critical simplices. Again, the critical simplices are those listed in Table 7.

Part 2 (incidence numbers for $n = kd+r$) We want to find the incidence numbers between critical simplices of consecutive dimensions. We start with the case $n = kd+r$, with $0 \le r \le d-2$. First, let us look for alternating paths from τ_q to τ'_q (for $0 \le q \le k-1$). Set $h = qd+r$. Suppose we have one such path,

$$\tau_q \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau'_q.$$

If at some point a vertex $v \in \{0, \dots, h-1\}$ is removed, then the path falls into some $K_{h'}$ with $h' < h$ and can never return in K_h. Therefore, the path must be entirely contained in $K_h \cong K^A_{n-qd,r}$, and by Theorem 5.7 it must be the trivial path $\tau_q \rhd \tau'_q$. Thus, $[\tau_q : \tau'_q] = \pm 1$.

The other pairs of critical simplices in consecutive dimensions are (τ'_{q+1}, τ_q) for $0 \le q \le k-1$. There is a trivial path $\tau'_{q+1} \rhd \tau_q$ which consists in removing the vertex $h = qd + r$, and contributes to the incidence number by

$$[\tau'_{q+1} : \tau_q] = (-1)^h.$$

Suppose we have some other (nontrivial) alternating path

$$\tau'_{q+1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \tau_q, \quad m \ge 1. \tag{2}$$

Let $\rho_m = \tau_q \cup \{v\}$. If $v = h$ then $\rho_m = \tau'_{q+1}$, which is excluded. Then v must be one of the other vertices that do not belong to τ_q. They are of the form $v = n - sd$ (for $0 \le s \le k-q-1$) or $v = n - (d-1-r) - sd$ (for $0 \le s \le k-q-2$). If $v = n - (d-1-r) - sd$, then ρ_m is matched with $\rho_m \cup \{n - (s+1)d\}$. This is impossible because ρ_m must be matched with some $\zeta_m \lhd \rho_m$. Similarly, if $v = n - sd$ with $0 \le s \le k-q-2$, then ρ_m is matched with $\rho_m \cup \{n - (d-1-r) - sd\}$ and not with some $\zeta_m \lhd \rho_m$. The only remaining possibility is $v = n - sd$ with $s = k-q-1$, ie $v = n - (k-q-1)d = (q+1)d + r$. In this case ρ_m is matched with $\zeta_m = \rho_m \setminus \{qd + 2r + 1\}$. Going on with the same argument, we have exactly one way to continue the alternating path (from right to left in (2)), and we eventually end up with τ'_{q+1}. From left to right, the obtained alternating path is the following:

$$\begin{aligned}
\zeta_1 &= \tau'_{q+1} \setminus \{qd + 2r + 1\},\\
\rho_1 &= \zeta_1 \cup \{n\},\\
\zeta_2 &= \rho_1 \setminus \{qd + r\},\\
\rho_2 &= \zeta_2 \cup \{n - (d-1-r)\},\\
\zeta_3 &= \rho_2 \setminus \{n\},\\
\rho_3 &= \zeta_3 \cup \{n - d\},\\
\zeta_4 &= \rho_3 \setminus \{n - (d-1-r)\},\\
&\vdots\\
\zeta_m &= \rho_{m-1} \setminus \{(q+1)d + 2r + 1\},\\
\rho_m &= \zeta_m \cup \{qd + 2r + 1\},\\
\tau_q &= \rho_m \setminus \{(q+1)d + r\}.
\end{aligned}$$

The length of the path is $m = 2(k-q)$. Apart from $qd + r$, which is the vertex in $\tau'_{q+1} \setminus \tau_q$, the other vertices are added and removed exactly once during the path. If a certain v is added in some ρ_i and removed in some ζ_j (the removal might possibly

come before the addition), then

$$[\rho_i : \zeta_i][\rho_{j-1} : \zeta_j] = \begin{cases} 1 & \text{if } v = n, \\ -1 & \text{otherwise.} \end{cases}$$

This is true because, except for $v = n$, between the addition and the removal of v exactly one vertex u with $u < v$ has been added/removed. Namely, between the addition and the removal of a vertex $v = jd + 2r + 1$ (for $q \le j \le k-1$) the vertex $u = jd + r$ is added/removed, and between the addition of the removal of a vertex $v = jd + r$ (for $q + 1 \le j \le k - 1$) the vertex $u = (j-1)d + 2r + 1$ is added. Therefore, the alternating path (2) contributes to the incidence number by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1} = (-1)^{h+1}.$$

Finally, the incidence number is given by

$$[\tau'_{q+1} : \tau_q]^{\mathcal{M}} = (-1)^h + (-1)^{h+1} = 0.$$

Part 3 (incidence numbers for $n = kd - 1$) Consider a generic alternating path starting from $\overline{\sigma} = \{0, \dots, n-1\}$,

$$\overline{\sigma} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \zeta_{m+1}.$$

Let $\overline{\sigma} = \zeta_1 \cup \{v\}$. If $n - d + 1 \le v \le n - 1$ then $\zeta_1 = \sigma'_{k-1,r}$ for $r = n - 1 - v$. Therefore, we have trivial alternating paths from $\overline{\sigma}$ to any of the $\sigma'_{k-1,r}$. If $v \le n - d$ then $\zeta_1 \in K_h$ with $h \le n - d = (k-1)d - 1$. None of these K_h's contains critical simplices ζ_{m+1} with $|\zeta_{m+1}| = n-1$, and the alternating path cannot return in any $K_{h'}$ with $h' \ge (k-1)d$. Thus, there are no other alternating paths from $\overline{\sigma}$ to critical simplices of K. Then the nontrivial incidence numbers involving $\overline{\sigma}$ are

$$[\overline{\sigma} : \sigma'_{k-1,r}]^{\mathcal{M}} = \pm 1.$$

Consider now a generic alternating path from σ_{q,r_1} to σ'_{q,r_2},

$$\sigma_{q,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma'_{q,r_2}.$$

Let $\rho_m = \sigma'_{q,r_2} \cup \{v\}$. Adding v to σ'_{q,r_2} causes the creation of a connected component Γ_i with $|\Gamma_i| \equiv -1 \pmod{d}$. This means that ρ_m is matched with a simplex of higher dimension, or is not matched at all (this happens for $v = (q+1)d - 1$). Therefore, the alternating path must be trivial, and it occurs only for $r_1 = r_2$. Then the nontrivial incidence numbers of the form $[\sigma_{q,r_1} : \sigma'_{q,r_2}]^{\mathcal{M}}$ are

$$[\sigma_{q,r} : \sigma'_{q,r}]^{\mathcal{M}} = \pm 1.$$

Finally, consider a generic alternating path from σ'_{q+1,r_1} to σ_{q,r_2},

$$\sigma'_{q+1,r_1} \rhd \zeta_1 \lhd \rho_1 \rhd \zeta_2 \lhd \rho_2 \rhd \cdots \rhd \zeta_m \lhd \rho_m \rhd \sigma_{q,r_2}. \tag{3}$$

As before, we work backwards. Let $\rho_m = \sigma_{q,r_2} \cup \{v\}$. Apart from the choices $v = qd + r$ and $v = (q+1)d + r$, in all other cases ρ_m has a connected component of size $\equiv -1 \pmod d$ and this prevents the continuation of the alternating path. For $v = qd + r$ we obtain $\rho_m = \sigma'_{q+1,r_2}$, so we have a trivial alternating path from σ'_{q+1,r_2} to σ_{q,r_2}. For $v = (q+1)d + r$, iterating the same argument, we have exactly one way to continue the alternating path (from right to left in (3)) and we end up with σ'_{q+1,r_2}. From left to right, the path is as follows (set $r = r_2$):

$$\begin{aligned}
\zeta_1 &= \sigma'_{q+1,r} \setminus \{(q+1)d - 1\},\\
\rho_1 &= \zeta_1 \cup \{n\},\\
\zeta_2 &= \rho_1 \setminus \{qd + r\},\\
\rho_2 &= \zeta_2 \cup \{n - (d - 1 - r)\},\\
\zeta_3 &= \rho_2 \setminus \{n\},\\
\rho_3 &= \zeta_3 \cup \{n - d\},\\
\zeta_4 &= \rho_3 \setminus \{n - (d - 1 - r)\},\\
&\vdots\\
\zeta_m &= \rho_{m-1} \setminus \{(q+2)d - 1\},\\
\rho_m &= \zeta_m \cup \{(q+1)d - 1\},\\
\sigma_{q,r} &= \rho_m \setminus \{(q+1)d + r\}.
\end{aligned}$$

The length of the path is $m = 2(k - q) - 1$. As happened in the case $n = kd + r$, the contribution of this alternating path to the incidence number is given by

$$(-1)^m \cdot (-1)^{m-1} \cdot [\zeta_2 : \rho_1] = (-1)^{qd+r+1}.$$

The contribution of the trivial path $\sigma'_{q+1,r} \rhd \sigma_{q,r}$ is given by $(-1)^{qd+r}$. Therefore,

$$[\sigma'_{q+1,r} : \sigma_{q,r}]^{\mathcal{M}} = (-1)^{qd+r} + (-1)^{qd+r+1} = 0. \qquad \square$$

We are now able to recover the result of [12] about the homology $H_*(\boldsymbol{G_W}; R)$ when $\boldsymbol{G_W}$ is an Artin group of type $\widetilde{A}_n$.

Theorem 5.16 (homology in case $\widetilde{A}_n$ [12]) *For an Artin group $\boldsymbol{G_W}$ of type $\widetilde{A}_n$, we have*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus d-1} & \text{if } n = kd-1 \text{ and } m = n-2i+1 \ (1 \le i \le k), \\ R/(\varphi_d) & \text{if } n = kd+r \text{ and } m = n-2i \ (1 \le i \le k), \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \le r \le d-2$ in the second case.

Proof We apply Theorems 5.1 and 5.15. For $n = kd + r$ $(0 \le r \le d-2)$, the boundary map δ^{φ}_{m+1} has rank 1 when $m+1 = n-2(k-q)+1$ (for $0 \le q \le k-1$, due to τ_q); it has rank 0 otherwise. For $n = kd-1$, the boundary map δ^{φ}_{m+1} has rank $d-1$ if $m+1 = n-2(k-q)+2$ (for $0 \le q \le k-2$, due to the simplices $\sigma_{q,r}$) or if $m+1 = n$ (due to $\overline{\sigma}$); it has rank 0 otherwise. □

5.4 Case $\widetilde{C}_n$

The last case we consider is that of Artin groups of type $\widetilde{C}_n$. The corresponding Coxeter graph Γ is shown in Figure 9. As in the $\widetilde{A}_n$ case, the simplicial complex K consists of all the simplices $\sigma \subseteq S = \{0, \dots, n\}$ except for the full simplex $\sigma = \{0, \dots, n\}$. For any $\sigma \in K$, the subgraph $\Gamma(\sigma)$ of Γ splits as a union of connected components of type B_k (those containing the first or the last vertex) and of type A_k (the remaining ones).

Figure 9: A Coxeter graph of type $\widetilde{C}_n$

For each m with $1 \le m \le n$, let K^B_m be the full simplicial complex on $\{1, \dots, m\}$, endowed with the weight function of B_m (see Section 5.2). We are going to construct a precise matching for the case $\widetilde{C}_n$ using the precise matching on K^B_m.

For $h \in \{0, \dots, n\}$ set

$$K_h = \{\sigma \in K \mid h \notin \sigma \text{ and } \{0, \dots, h-1\} \subseteq \sigma\}.$$

As in the $\widetilde{A}_n$ case, since every simplex $\sigma \in K$ misses at least one vertex, the subsets K_h form a partition of K,

$$K = \bigsqcup_{h=0}^{n} K_h.$$

Construct a matching on a fixed K_h as follows. Ignore the vertices $0, \dots, h$ and relabel the remaining ones from right to left: n becomes 1, $n-1$ becomes 2, $\dots$, $h+1$ becomes $n-h$. This induces a poset isomorphism

$$K_h \xrightarrow{\cong} K^B_{n-h}.$$

Then equip K_h with the pullback of the matching on K^B_{n-h} constructed in Section 5.2 (for $h = n$ the single simplex in K_h is critical).

Remark 5.17 For d odd we are simply matching σ with $\sigma \trianglerighteq n$ for all σ except $\sigma = \{0, \dots, n-1\}$.

Lemma 5.18 *The matching defined above is an acyclic weighted matching on K.*

Proof As in the proof of Lemma 5.14, the subsets K_h are the fibers of a poset map and the matching is acyclic on each $K_h \cong K^B_{n-h}$.

If a simplex $\sigma \in K_h$ is sent to $\widehat{\sigma}$ by the isomorphism $K_h \xrightarrow{\cong} K^B_{n-h}$, then $v_\varphi(\sigma) = v_\varphi(\widehat{\sigma}) + \omega_\varphi(B_h)$. Since the matching on K^B_{n-h} is weighted, the matching on K_h also is. □

Theorem 5.19 (critical simplices in case $\widetilde{C}_n$) *The critical simplices for the matching on K are those listed in Table 8. In particular, the matching is precise. In addition the only nontrivial incidence numbers between critical simplices in the Morse complex are as follows $\big($for d even and $n = k\frac{d}{2} + r\big)$:*

$$\begin{aligned}
[\sigma_{q_1,q_2} : \sigma_{q_1,q_2-1}]^{\mathcal{M}} &= (-1)^{\alpha},\\
[\sigma_{q_1,q_2} : \sigma_{q_1-1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+1},\\
[\sigma_{q_1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+d/2+1},\\
[\sigma'_{q_1,q_2} : \sigma_{q_1,q_2-2}]^{\mathcal{M}} &= (-1)^{\beta+1},\\
[\sigma'_{q_1,q_2} : \sigma_{q_1-2,q_2}]^{\mathcal{M}} &= (-1)^{\beta},\\
[\sigma'_{q_1,q_2} : \sigma'_{q_1,q_2-1}]^{\mathcal{M}} &= (-1)^{\alpha+1},\\
[\sigma'_{q_1,q_2} : \sigma'_{q_1-1,q_2}]^{\mathcal{M}} &= (-1)^{\beta+d/2},
\end{aligned}$$

where $\alpha = (k-q_2)\big(\frac{d}{2}-1\big) + q_1 + r + \frac{d}{2}$ and $\beta = q_1\frac{d}{2} + r$.

Proof As we have already said, for d odd there is exactly one critical simplex. Suppose from now on that d is even. Let $n = k\frac{d}{2} + r$, with $0 \le r \le \frac{d}{2} - 1$.

Case	Simplices	$\lvert\sigma\rvert$	$v_\varphi(\sigma)$	
d odd	$\overline{\sigma} = 1^n 0$	n	1 or 0	
d even	σ_{q_1,q_2}	$n-k+q_1+q_2+1$	q_1+q_2+1	$0 \le q_1+q_2 \le k-2$
	σ'_{q_1,q_2}	$n-k+q_1+q_2$	q_1+q_2	$0 \le q_1+q_2 \le k$

Simplices
$\sigma_{q_1,q_2} = 1^{q_1 d/2+r} 0 1^{d-1} 0 (1^{d/2-1}0)^{k-q_1-q_2-2} 1^{q_2 d/2}$
$\sigma'_{q_1,q_2} = 1^{q_1 d/2+r} 0 (1^{d/2-1}0)^{k-q_1-q_2} 1^{q_2 d/2}$

Table 8: Description of the critical simplices for $\widetilde{C}_n$. When d is even, set $n = k\frac{d}{2} + r$.

Part 1 (the critical simplices) In K_h there are critical simplices if and only if $n - h \equiv 0 \pmod{\frac{d}{2}}$, ie when $h = q_1\frac{d}{2} + r$ for some q_1 (with $0 \le q_1 \le k$). By Theorem 5.12 there are two families of critical simplices: σ_{q_1,q_2} (for $0 \le q_1+q_2 \le k-2$) and σ'_{q_1,q_2} (for $0 \le q_1+q_2 \le k$), as shown in Table 8 and Figure 10. In a fixed dimension $m = n-k+l$ $(0 \le l \le k)$ there are $2l+1$ critical simplices if $l \le k-1$ and l critical simplices if $l = k$. See Table 9.

Part 2 (paths ending in σ_{q_1,q_2}) Consider a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ_{q_1,q_2},

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma_{q_1,q_2}.$$

Let $\rho_m = \sigma_{q_1,q_2} \cup \{v\}$. If $v = q_1\frac{d}{2} + r$ then $\rho_m = \sigma'_{q_1+2,\,q_2}$ and the alternating path stops. If $v = n - q_2\frac{d}{2}$ then the alternating path stops at $\rho_m = \sigma_{q_1,\,q_2+1}$. Suppose

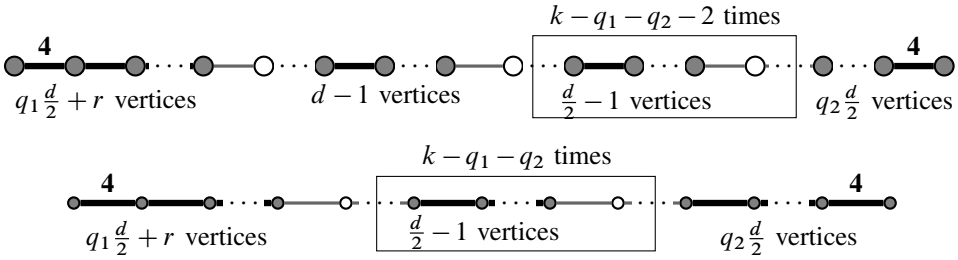


Figure 10: Critical simplices for $\widetilde{C}_n$, with d even and $n = k\frac{d}{2} + r$. The diagram for the simplex σ_{q_1,q_2} is at the top and the diagram for the simplex σ'_{q_1,q_2} is below it.

$q_1+q_2\le k-3$, otherwise there are no more cases. If $q_1\frac{d}{2}+r+d<v<n-q_2\frac{d}{2}$, then $\Gamma(\rho_m)$ has at least one connected component of size $d-1$ and therefore ρ_m is matched with a simplex of higher dimension; thus the path stops without having reached a critical simplex. If $v=q_1\frac{d}{2}+r+d$ then ρ_m is matched with

$$\tau_m=\rho_m\setminus\Big\{q_1\frac{d}{2}+r+\frac{d}{2}\Big\}=1^{q_1d/2+r}01^{d/2-1}01^{d-1}0(1^{d/2-1}0)^{k-q_1-q_2-3}01^{q_2d/2}$$

in the binary string notation. From here the path can continue in many ways. Let $\rho_{m-1}=\tau_m\cup\{w\}$.

- If $w=q_1\frac{d}{2}+r$, we end up with $\rho_{m-1}=\sigma_{q_1+1,\,q_2}$.
- If $w=q_1\frac{d}{2}+r+\frac{d}{2}$, we would be going back to ρ_m.
- If $w>q_1\frac{d}{2}+r+\frac{3}{2}d$, then $\Gamma(\rho_{m-1})$ has at least one connected component of size $d-1$ and therefore ρ_{m-1} is matched with a simplex of higher dimension.
- If $w=q_1\frac{d}{2}+r+\frac{3}{2}d$, then ρ_{m-1} is matched with

$$\begin{aligned}\tau_{m-1}&=\rho_{m-1}\setminus\Big\{q_1\frac{d}{2}+r+d\Big\}\\&=1^{q_1d/2+r}01^{d/2-1}01^{d/2-1}01^{d-1}0(1^{d/2-1}0)^{k-q_1-q_2-4}01^{q_2d/2}.\end{aligned}$$

By induction, repeating the same argument as above, this path can be continued in exactly one way and it eventually arrives at the critical simplex $\sigma'_{q_1,\,q_2+2}$. The path has length $m=k-q_1-q_2-2$ and is as follows:

$$\begin{aligned}\tau_1&=\sigma'_{q_1,\,q_2+2}\setminus\Big\{n-q_2\frac{d}{2}\Big\},\\\rho_1&=\tau_1\cup\Big\{n-q_2\frac{d}{2}-d\Big\},\\\tau_2&=\rho_1\setminus\Big\{n-q_2\frac{d}{2}-\frac{d}{2}\Big\},\\\rho_2&=\tau_2\cup\Big\{n-q_2\frac{d}{2}-\frac{3}{2}d\Big\},\\\tau_3&=\rho_2\setminus\Big\{n-q_2\frac{d}{2}-d\Big\},\\&\vdots\\\rho_{m-1}&=\tau_{m-1}\cup\Big\{q_1\frac{d}{2}+r+d\Big\},\\\tau_m&=\rho_{m-1}\setminus\Big\{q_1\frac{d}{2}+r+\frac{3}{2}d\Big\},\\\rho_m&=\tau_m\cup\Big\{q_1\frac{d}{2}+r+\frac{d}{2}\Big\},\\\sigma_{q_1,q_2}&=\rho_m\setminus\Big\{q_1\frac{d}{2}+r+d\Big\}.\end{aligned}$$

Simplices	
$\sigma_{l-1,0}$	$\sigma'_{l,0}$
$\sigma_{l-2,1}$	$\sigma'_{l-1,1}$
$\vdots$	$\vdots$
$\sigma_{1,l-2}$	$\sigma'_{2,l-2}$
$\sigma_{0,l-1}$	$\sigma'_{1,l-1}$
	$\sigma'_{0,l}$

Table 9: Critical simplices for $\widetilde{C}_n$ in dimension $m = n-k+l$ $(0 \le l \le k)$, where d is even and $n = k\frac{d}{2} + r$. For $l = k$ only the second column occurs.

Part 3 (paths ending in σ'_{q_1,q_2}) Consider now a generic alternating path starting from any critical cell ρ_0 and ending in a critical cell of the form σ'_{q_1,q_2},

$$\rho_0 \rhd \tau_1 \lhd \rho_1 \rhd \tau_2 \lhd \rho_2 \rhd \cdots \rhd \tau_m \lhd \rho_m \rhd \sigma'_{q_1,q_2}. \tag{4}$$

As usual, let $\rho_m = \sigma'_{q_1,q_2} \cup \{v\}$. For the same reasons as above, there are only three possibilities: $v = q_1\frac{d}{2} + r$, $v = q_1\frac{d}{2} + r + \frac{d}{2}$ and $v = n - q_2\frac{d}{2}$; in all the other cases, ρ_m is matched with a simplex of higher dimension. If $v = q_1\frac{d}{2} + r$ then the path ends (to the left in (4)) at $\rho_m = \sigma'_{q_1+1,\,q_2}$. If $v = q_1\frac{d}{2} + r + \frac{d}{2}$ then the path ends at σ_{q_1,q_2}. Finally, if $v = n - q_2\frac{d}{2}$ then the path ends at $\sigma'_{q_1,\,q_2+1}$.

Part 4 (incidence numbers) We have seven families of incidence numbers to compute, each coming from one of the alternating paths we have found:

From σ'_{q_1+2,q_2} to σ_{q_1,q_2} The alternating path is trivial and consists in removing the vertex $v = q_1\frac{d}{2} + r$, so

$$[\sigma'_{q_1+2,\,q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} = (-1)^{|\{w \in \sigma_{q_1,q_2} | w < v\}|} = (-1)^{q_1 d/2 + r},$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma_{q_1-2,\,q_2}]^{\mathcal{M}} = (-1)^{q_1 d/2 + r} = (-1)^{\beta}.$$

From σ_{q_1,q_2+1} to σ_{q_1,q_2} Again the path is trivial, and it consists in removing $v = n - q_2\frac{d}{2}$. Therefore,

$$\begin{aligned}[\sigma_{q_1,\,q_2+1} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{|\{w \in \sigma_{q_1,q_2} | w < v\}|} \\ &= (-1)^{n - q_2 d/2 - (k - q_1 - q_2 - 1)}\end{aligned}$$

$$\begin{aligned}&= (-1)^{kd/2+r-q_2d/2-k+q_1+q_2+1}\\&= (-1)^{(k-q_2)(d/2-1)+r+q_1+1},\end{aligned}$$

which implies

$$[\sigma_{q_1,q_2} : \sigma_{q_1,q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)(d/2-1)+r+q_1+1} = (-1)^{\alpha}.$$

From σ_{q_1+1,q_2} to σ_{q_1,q_2} The path consists in removing $q_1\frac{d}{2}+r$, adding $q_1\frac{d}{2}+r+\frac{d}{2}$ and removing $q_1\frac{d}{2}+r+d$. Therefore,

$$\begin{aligned}[\sigma_{q_1+1,q_2} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)(-1)^{q_1d/2+r}(-1)^{q_1d/2+r+d/2-1}(-1)^{q_1d/2+r+d-1}\\&= (-1)^{q_1d/2+r+d/2+1},\end{aligned}$$

which implies

$$[\sigma_{q_1,q_2} : \sigma_{q_1-1,q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)d/2+r+d/2+1} = (-1)^{\beta+1}.$$

From σ'_{q_1,q_2+2} to σ_{q_1,q_2} The path is the one we explicitly wrote at the end of Part 2. Notice that from ρ_{i-1} to τ_i one removes the vertex $v_i = n-(q_2-i+1)\frac{d}{2}$, and from τ_i to ρ_i one adds the vertex $v'_i = n-(q_2-i-1)\frac{d}{2}$. Therefore,

$$[\rho_{i-1} : \tau_i][\rho_i : \tau_i] = (-1)^{|\{w\in\tau_i \,|\, v'_i<w<v_i\}|} = (-1)^{d-1} = -1.$$

Then we can compute the incidence number in the Morse complex:

$$\begin{aligned}[\sigma'_{q_1,q_2+2} : \sigma_{q_1,q_2}]^{\mathcal{M}} &= (-1)^m\left(\prod_{i=1}^{m}[\rho_{i-1} : \tau_i][\rho_i : \tau_i]\right)[\sigma_{q_1,q_2} : \rho_m]\\&= (-1)^m(-1)^m(-1)^{q_1d/2+r+(d-1)}\\&= (-1)^{q_1d/2+r+1},\end{aligned}$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma_{q_1,q_2-2}]^{\mathcal{M}} = (-1)^{q_1d/2+r+1} = (-1)^{\beta+1}.$$

From σ'_{q_1+1,q_2} to σ'_{q_1,q_2} The path is trivial and consists of removing $v = q_1\frac{d}{2}+r$. Then

$$[\sigma'_{q_1+1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1d/2+r},$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma'_{q_1-1,q_2}]^{\mathcal{M}} = (-1)^{(q_1-1)d/2+r} = (-1)^{\beta+d/2}.$$

From σ_{q_1,q_2} to σ'_{q_1,q_2} The path is trivial and consists in removing $v = q_1\frac{d}{2} + r + \frac{d}{2}$. Then

$$[\sigma_{q_1,q_2} : \sigma'_{q_1,q_2}]^{\mathcal{M}} = (-1)^{q_1 d/2 + r + d/2 - 1} = (-1)^{\beta + d/2 + 1}.$$

From σ'_{q_1,q_2+1} to σ'_{q_1,q_2} The path is trivial and consists in removing $v = n - q_2\frac{d}{2}$. Then

$$\begin{aligned}[\sigma'_{q_1,q_2+1} : \sigma'_{q_1,q_2}]^{\mathcal{M}} &= (-1)^{n - q_2 d/2 - (k - q_1 - q_2)} \\ &= (-1)^{kd/2 + r - q_2 d/2 - k + q_1 + q_2} \\ &= (-1)^{(k-q_2)(d/2-1) + r + q_1},\end{aligned}$$

which implies

$$[\sigma'_{q_1,q_2} : \sigma'_{q_1,q_2-1}]^{\mathcal{M}} = (-1)^{(k-q_2+1)(d/2-1)+r+q_1} = (-1)^{\alpha+1}. \qquad \square$$

We can finally compute the homology $H_*(\boldsymbol{G_W}; R)$ for Artin groups of type $\widetilde{C}_n$.

Theorem 5.20 (homology in case $\widetilde{C}_n$) *Let $\boldsymbol{G_W}$ be an Artin group of type $\widetilde{C}_n$. Then the φ_d–primary component of $H_*(\boldsymbol{G_W}; R)$ is trivial for d odd, and for d even is as follows:*

$$H_m(\boldsymbol{G_W}; R)_{\varphi_d} \cong \begin{cases} (R/(\varphi_d))^{\oplus m+k-n+1} & \text{if } n-k \le m \le n-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = k\frac{d}{2} + r$.

Proof In order to apply Theorem 5.1 we need to find the rank of the boundary maps δ^{φ}_{m+1} of the Morse complex. For d odd there is only one critical cell, thus all the boundaries vanish. Suppose from now on that d is even, and let $n = k\frac{d}{2} + r$. In order to have a nontrivial boundary δ^{φ}_{m+1} we must have at least one critical simplex both in dimension m and in dimension $m+1$, thus $m = n - k + l$ with $0 \le l \le k-1$ (see Table 9).

Case 1 ($l \le k-2$) We are going to prove that, for $l \le k-2$, a basis for the image of $\delta = \delta^{\varphi}_{m+1}$ is given by

$$\mathcal{B} = \{\delta\sigma_{q_1,q_2} \mid q_1 + q_2 = l\}.$$

By Theorem 5.19 we obtain the following formula for $\delta\sigma_{q_1,q_2}$:

$$\delta\sigma_{q_1,q_2} = (-1)^{\alpha}\sigma_{q_1,q_2-1} + (-1)^{\beta+1}\sigma_{q_1-1,q_2} + (-1)^{\beta+d/2+1}\sigma'_{q_1,q_2},$$

where $\alpha = (k-q_2)\left(\frac{d}{2}-1\right)+q_1+r+\frac{d}{2}$ and $\beta = q_1\frac{d}{2}+r$. In this formula the σ_{q_1,q_2-1} (resp. σ_{q_1-1,q_2}) term vanishes if $q_2=0$ (resp. $q_1=0$). The term σ'_{q_1,q_2} appears in $\delta\sigma_{q_1,q_2}$ but not in any other element of $\mathcal{B}$, thus $\mathcal{B}$ is a linearly independent set. In addition, if $q_1+q_2=l+1$, we have that

$$\begin{aligned}\delta\sigma'_{q_1,q_2} &= (-1)^{\beta+1}\sigma_{q_1,q_2-2}+(-1)^{\beta}\sigma_{q_1-2,q_2}+(-1)^{\alpha+1}\sigma'_{q_1,q_2-1}\\ &\qquad +(-1)^{\beta+d/2}\sigma'_{q_1-1,q_2}\\ &= (-1)^{\alpha+\beta+d/2}\big((-1)^{\alpha+d/2+1}\sigma_{q_1,q_2-2}+(-1)^{\beta+1}\sigma_{q_1-1,q_2-1}\\ &\qquad +(-1)^{\beta+d/2+1}\sigma'_{q_1,q_2-1}\big)\\ &\quad +(-1)^{d/2+1}\big((-1)^{\alpha+1}\sigma_{q_1-1,q_2-1}+(-1)^{\beta+d/2+1}\sigma_{q_1-2,q_2}\\ &\qquad +(-1)^{\beta+1}\sigma'_{q_1-1,q_2}\big)\\ &= (-1)^{\alpha+\beta+d/2}\delta\sigma_{q_1,q_2-1}+(-1)^{d/2+1}\delta\sigma_{q_1-1,q_2}.\end{aligned}$$

Therefore, $\mathcal{B}$ generates the image of δ^{φ}_{m+1}. Thus, $\operatorname{rk}\delta^{\varphi}_{m+1}=|\mathcal{B}|=l+1$ for $l\le k-2$.

Case 2 $(l=k-1)$ For $l=k-1$, ie $m=n-1$, the situation is a bit different because there are no critical simplices of the form σ_{q_1,q_2} with $q_1+q_2=l$. However, we can still define

$$\epsilon_{q_1,q_2}=(-1)^{\alpha}\sigma_{q_1,q_2-1}+(-1)^{\beta+1}\sigma_{q_1-1,q_2}+(-1)^{\beta+d/2+1}\sigma'_{q_1,q_2}$$

for $q_1+q_2=l$, and

$$\mathcal{B}'=\{\epsilon_{q_1,q_2}\mid q_1+q_2=l\}.$$

The term σ'_{q_1,q_2} appears in ϵ_{q_1,q_2} but not in any other element of $\mathcal{B}'$, thus $\mathcal{B}'$ is a linearly independent set. As above we have that, for $q_1+q_2=l+1$,

$$\delta\sigma'_{q_1,q_2}=(-1)^{\alpha+\beta+d/2}\epsilon_{q_1,q_2-1}+(-1)^{d/2+1}\epsilon_{q_1-1,q_2}.$$

Then $\mathcal{B}'$ generates (and so it is a basis of) the image of δ^{φ}_{n}.

We have proved that, for $0\le l\le k-1$, the rank of δ^{φ}_{m+1} is equal to $l+1=m+k-n+1$. Then we conclude by applying Theorem 5.1. □

Concluding remarks

Future work will focus on other families of Artin groups. In particular, it seems that precise matchings can be constructed in all finite and affine cases (see [36]), possibly allowing explicit homology computations.

The methods developed in this paper are particularly powerful when the coefficients are over a PID, but Artin groups with roots of different lengths (eg B_n, $\widetilde{B}_n$ and $\widetilde{C}_n$) also admit natural representations over polynomial rings with more than one variable. We believe that some of our theory can be extended to such cases, and this can also be the aim of future work.

Acknowledgements

The authors are grateful to the anonymous referee for his/her useful comments and suggestions.

This work was partially supported by Ministero dell'Istruzione, dell'Università e della Ricerca and by University of Pisa, Project no. PRA_67.

References

[1] **I Agol**, *The virtual Haken conjecture*, Doc. Math. 18 (2013) 1045–1087 MR

[2] **V I Arnold**, *The cohomology ring of the colored braid group*, Mat. Zametki 5 (1969) 227–231 MR In Russian; translated in Math. Notes 5 (1969) 138–140 and reprinted in "Collected works", vol. II, Springer (2014) 183–186

[3] **V I Arnold**, *On some topological invariants of algebrac functions*, Trudy Moskov. Mat. Obšč. 21 (1970) 27–46 MR In Russian; translated in Trans. Moscow Math. Soc 21 (1970) 30–52 and reprinted in "Collected works", vol. II, Springer (2014) 199–221

[4] **E Babson**, **D N Kozlov**, *Proof of the Lovász conjecture*, Ann. of Math. 165 (2007) 965–1007 MR

[5] **A Björner**, **F Brenti**, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics 231, Springer (2005) MR

[6] **N Bourbaki**, *Éléments de mathématique, XXXIV: Groupes et algèbres de Lie*, Actualités Scientifiques et Industrielles 1337, Hermann, Paris (1968) MR

[7] **E Brieskorn**, *Sur les groupes de tresses (d'après V I Arnold)*, from "Séminaire Bourbaki, 1972/1973", Lecture Notes in Math. 317, Springer (1973) exposé 401, 21–44 MR

[8] **E Brieskorn**, **K Saito**, *Artin-Gruppen und Coxeter-Gruppen*, Invent. Math. 17 (1972) 245–271 MR

[9] **F Callegaro**, *On the cohomology of Artin groups in local systems and the associated Milnor fiber*, J. Pure Appl. Algebra 197 (2005) 323–332 MR

[10] **F Callegaro**, *The homology of the Milnor fiber for classical braid groups*, Algebr. Geom. Topol. 6 (2006) 1903–1923 MR

[11] **F Callegaro**, **D Moroni**, **M Salvetti**, *Cohomology of affine Artin groups and applications*, Trans. Amer. Math. Soc. 360 (2008) 4169–4188 MR

[12] **F Callegaro**, **D Moroni**, **M Salvetti**, *Cohomology of Artin groups of type $\widetilde{A}_n$, B_n and applications*, from "Groups, homotopy and configuration spaces" (N Iwase, T Kohno, R Levi, D Tamaki, J Wu, editors), Geom. Topol. Monogr. 13, Geom. Topol. Publ., Coventry (2008) 85–104 MR

[13] **F Callegaro**, **D Moroni**, **M Salvetti**, *The $K(\pi,1)$ problem for the affine Artin group of type $\widetilde{B}_n$ and its cohomology*, J. Eur. Math. Soc. 12 (2010) 1–22 MR

[14] **F Callegaro**, **M Salvetti**, *Integral cohomology of the Milnor fibre of the discriminant bundle associated with a finite Coxeter group*, C. R. Math. Acad. Sci. Paris 339 (2004) 573–578 MR

[15] **R Charney**, **M W Davis**, *The $K(\pi,1)$–problem for hyperplane complements associated to infinite reflection groups*, J. Amer. Math. Soc. 8 (1995) 597–627 MR

[16] **F R Cohen**, *The homology of C_{n+1}–spaces, $n \geq 0$*, from "The homology of iterated loop spaces", Lecture Notes in Mathematics 533, Springer (1976) 207–351 MR

[17] **J Curry**, **R Ghrist**, **V Nanda**, *Discrete Morse theory for computing cellular sheaf cohomology*, Found. Comput. Math. 16 (2016) 875–897 MR

[18] **C De Concini**, **C Procesi**, **M Salvetti**, *Arithmetic properties of the cohomology of braid groups*, Topology 40 (2001) 739–751 MR

[19] **C De Concini**, **C Procesi**, **M Salvetti**, **F Stumbo**, *Arithmetic properties of the cohomology of Artin groups*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28 (1999) 695–717 MR

[20] **C De Concini**, **M Salvetti**, *Cohomology of Artin groups*, Math. Res. Lett. 3 (1996) 293–297 MR

[21] **P Deligne**, *Les immeubles des groupes de tresses généralisés*, Invent. Math. 17 (1972) 273–302 MR

[22] **A Engström**, *Complexes of directed trees and independence complexes*, Discrete Math. 309 (2009) 3299–3309 MR

[23] **R Forman**, *Morse theory for cell complexes*, Adv. Math. 134 (1998) 90–145 MR

[24] **R Fox**, **L Neuwirth**, *The braid groups*, Math. Scand. 10 (1962) 119–126 MR

[25] **E V Frenkel**, *Cohomology of the commutator subgroup of the braid group*, Funktsional. Anal. i Prilozhen. 22 (1988) 91–92 MR In Russian; translated in Funct. Anal. Appl. 22 (1988) 248–250

[26] **D B Fuks**, *Cohomology of the braid group* mod 2, Funkcional. Anal. i Priložen. 4 (1970) 62–73 MR In Russian; translated in Funct. Anal. Appl. 4 (1970) 143–151

[27] **V V Gorjunov**, *The cohomology of braid groups of series C and D and certain stratifications*, Funktsional. Anal. i Prilozhen. 12 (1978) 76–77 MR In Russian; translated in Funct. Anal. Appl. 12 (1978) 139–140

[28] **H Hendriks**, *Hyperplane complements of large type*, Invent. Math. 79 (1985) 375–381 MR

[29] **J Jonsson**, *Simplicial complexes of graphs*, Lecture Notes in Mathematics 1928, Springer (2008) MR

[30] **D Kozlov**, *Combinatorial algebraic topology*, Algorithms and Computation in Mathematics 21, Springer (2008) MR

[31] **H van der Lek**, *The homotopy type of complex hyperplane complements*, PhD thesis, Katholieke Universiteit te Nijmegen (1983) `https://repository.ubn.ru.nl/handle/2066/148301`

[32] **D Moroni**, **M Salvetti**, **A Villa**, *Some topological problems on the configuration spaces of Artin and Coxeter groups*, from "Configuration spaces" (A Bjorner, F Cohen, C De Concini, C Procesi, M Salvetti, editors), CRM Series 14, Ed. Norm., Pisa (2012) 403–431 MR

[33] **C Okonek**, *Das $K(\pi, 1)$–Problem für die affinen Wurzelsysteme vom Typ A_n, C_n*, Math. Z. 168 (1979) 143–148 MR

[34] **V Ozornova**, *Discrete Morse theory and a reformulation of the $K(\pi, 1)$–conjecture*, Comm. Algebra 45 (2017) 1760–1784 MR

[35] **G Paolini**, *On the classifying space of Artin monoids*, Comm. Algebra 45 (2017) 4740–4757 MR

[36] **G Paolini**, *On the local homology of Artin groups of finite and affine type*, preprint (2017) arXiv

[37] **L Paris**, *$K(\pi, 1)$ conjecture for Artin groups*, Ann. Fac. Sci. Toulouse Math. 23 (2014) 361–415 MR

[38] **M Salvetti**, *Topology of the complement of real hyperplanes in $\mathbb{C}^N$*, Invent. Math. 88 (1987) 603–618 MR

[39] **M Salvetti**, *The homotopy type of Artin groups*, Math. Res. Lett. 1 (1994) 565–577 MR

[40] **M Salvetti**, *Some combinatorial constructions and relations with Artin groups*, from "Combinatorial methods in topology and algebra" (B Benedetti, E Delucchi, L Moci, editors), Springer INdAM 12, Springer (2015) 121–126 MR

[41] **M Salvetti**, **A Villa**, *Combinatorial methods for the twisted cohomology of Artin groups*, Math. Res. Lett. 20 (2013) 1157–1175 MR

[42] **C C Squier**, *The homological algebra of Artin groups*, Math. Scand. 75 (1994) 5–43 MR

[43] **E B Vinberg**, *Discrete linear groups that are generated by reflections*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971) 1072–1112 MR In Russian; translated in Math. USSR-Izv. 5 (1971) 1083–1119

[44] **F V Weinstein**, *Cohomologies of braid groups*, Funktsional. Anal. i Prilozhen. 12 (1978) 72–73 MR In Russian; translated in Funct. Anal. Appl. 12 (1978) 135–137

Scuola Normale Superiore
Pisa, Italy

Department of Mathematics, University of Pisa
Pisa, Italy

giovanni.paolini@sns.it, salvetti@dm.unipi.it

Received: 4 November 2017 Revised: 19 March 2018

msp

Algebraic & Geometric Topology 18 (2018) 4001–4035

Equivariant complex bundles, fixed points and equivariant unitary bordism

ANDRÉS ÁNGEL
JOSÉ MANUEL GÓMEZ
BERNARDO URIBE

We study the fixed points of the universal G–equivariant complex vector bundle of rank n and obtain a decomposition formula in terms of twisted equivariant universal complex vector bundles of smaller rank. We use this decomposition to describe the fixed points of the complex equivariant K–theory spectrum and the equivariant unitary bordism groups for adjacent families of subgroups.

19L47, 19L50, 55N22, 57R77, 57R85

1 Introduction

In this article decomposition formulas for equivariant K–theory and geometric equivariant bordism of stably almost complex manifolds are obtained under suitable hypotheses. The underlying main technical idea behind such decompositions is a splitting formula for equivariant complex vector bundles first obtained by Gómez and Uribe [10] for the particular case of finite groups. We generalize this splitting formula for the general case of compact Lie groups and apply it to obtain the decompositions of equivariant K–theory and equivariant unitary bordism mentioned above.

More precisely, suppose that G is a compact Lie group that fits in a short exact sequence of compact Lie groups $1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$. Let X be a compact G–space such that A acts trivially on X. In the first part of this article we study G–equivariant complex vector bundles $p\colon E \to X$. Since A acts trivially on X, the fibers of E can be seen as A–representations. By decomposing E into A–isotypical pieces we obtain a splitting of E as an A–equivariant vector bundle in the form $\bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E) \cong E$. Here $\mathbb{V}_\tau$ denotes the trivial A–vector bundle $\pi_1\colon X \times V_\tau \to X$ associated to an irreducible representation $\tau\colon A \to U(V_\tau)$ and $\mathrm{Irr}(A)$ denotes the set of isomorphism classes of complex irreducible A–representations. This splitting is one of A–vector bundles and not one of G–vector bundles since in general the bundles $\mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ may not possess the structure of a G–vector bundle

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4001

(see Example 3.5). A key technical observation of this work is that, up to isomorphism, the direct sum $\bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ can be rearranged using the different orbits of the action of Q on $\mathrm{Irr}(A)$ to obtain a decomposition of E in terms of G–vector bundles. This way a splitting of E as a G–equivariant vector bundle is obtained in Theorem 3.6. This result plays a key role in this paper.

Given an irreducible representation $\rho\colon A \to U(V_\rho)$ we can obtain in a natural way a central extension of the form $1 \to \mathbb{S}^1 \to \widetilde{Q}_\rho \to Q_\rho \to 1$, where $Q_\rho = G_\rho/A$ and $G_\rho = \{g \in G \mid g\cdot\rho \cong \rho\}$ (see Sections 2 and 3 for definitions). It turns out that each of the pieces in the splitting formula given in Theorem 3.6 can be used to define a twisted form of an equivariant K–theory, and as a consequence the following result is obtained:

Corollary 3.7 *Let G be a compact Lie group and X a G–space on which the normal subgroup A acts trivially. Then there is a natural isomorphism*

$$K^*_G(X) \cong \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} {}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X),$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations and $Q_\rho = G_\rho/A$.

In the above formula, ${}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X)$ denotes a twisted form of Q_ρ–equivariant K–theory. This result generalizes a similar decomposition obtained in [10] for the particular case of finite groups.

On the other hand, the decomposition obtained in Theorem 3.6 can be carried out at the level of the universal G–equivariant complex bundle of rank n, denoted by $\gamma_G U(n) \to B_G U(n)$. Here $B_G U(n)$ is the classifying space of G–equivariant rank n complex vector bundles. Applying this decomposition to the restriction of $\gamma_G U(n)$ to $B_G U(n)^A$, we obtain an N_A/A–equivariant homotopy equivalence with a product of classifying spaces parametrized by the orbits of the action of the normalizer N_A on the set of nontrivial irreducible representations of A. This result is also one of the main results of this article and is summarized in Theorem 4.1.

The second part of this article adds to the understanding of the geometric equivariant bordism groups of stably almost complex manifolds with boundary, whenever the isotropy groups of the interior of the manifold differ by one conjugacy class of subgroups from the isotropies of the boundary. The equivariant version of the bordism theories was developed by Conner and Floyd in their monumental work [5; 6] and the unitary equivariant bordism theory was developed by Stong [21] among others. A compact G–equivariant

manifold is unitary if the tangent bundle may be stabilized with trivial real bundles, thus becoming isomorphic to a G–equivariant complex vector bundle. The bordism group of unitary G–equivariant manifolds is denoted by Ω_*^G and the product of manifolds makes Ω_*^G into a ring and moreover an Ω_*–module. The calculation of the Ω_*–module structure of Ω_*^G has been elusive and very little is known whenever G is not abelian. Whenever G is abelian it is known that Ω_*^G is zero in odd degrees and a free Ω_*–module in even degrees, (see Comezaña [4, Theorem 5.3] and Ossa [18, Theorem 1]), and the question remains open whether this is also the case whenever G is not abelian.

The main calculational tool to understand Ω_*^G is to restrict attention to unitary manifolds on which the isotropy groups at each point lie on a prescribed family of subgroups of G. For a pair of families $(\mathcal{F}, \mathcal{F}')$ of subgroups of G with $\mathcal{F}' \subset \mathcal{F}$, denote by $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ the bordism classes of unitary G–manifolds with boundary $(M, \partial M)$ such that the isotropy groups of the points in M lie on $\mathcal{F}$ and the isotropy groups of the points of the boundary ∂M lie on $\mathcal{F}'$. Whenever the families differ by the set of groups conjugate to a fixed group A they are called adjacent. Whenever A is normal in G and the pair of families $(\mathcal{F}, \mathcal{F}')$ is adjacent differing by A, the bordism class of a manifold $(M, \partial M)$ in $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ is equivalent to the bordism class of the disk bundle of the tubular neighborhood of the fixed-point set M^A in M. Therefore, we may keep the information of the normal bundle by a map from M^A to the classifying space of G–equivariant complex vector bundles over trivial A–spaces. Hence, the unitary G–equivariant bordism groups for adjacent families can be written in terms of nonequivariant unitary bordism groups of a product of certain classifying spaces. As a consequence of Theorem 3.6 the following decomposition of G–equivariant bordisms is obtained. This theorem is the last main result in this article and is a new result for compact Lie groups that are not abelian.

Theorem 5.6 *Suppose that G is a compact Lie group and let A be a closed normal subgroup of G. If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of G differing by A, then*

$$\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0 \le 2k \le n - \dim(G/A)} \Omega_{n-2k}^{G/A}\{\{1\}\}\left(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k,A)} B_{G/A}U(\overline{P})\right),$$

where $\{1\}$ is the family of subgroups of G/A which only contains the trivial group.

In the above theorem $B_{G/A}U(\overline{P})$ denotes a product of classifying spaces for different twisted equivariant vector bundles (see Section 5 for the precise definition).

In the last section we use the previous theorem to determine the Ω_*–module structure of $\Omega_*^{D_{2p}}$, where D_{2p} is the dihedral group of order $2p$ with p an odd prime. We show that $\Omega_*^{D_{2p}}$ is a free Ω_*–module in even degrees and zero in odd degrees.

This paper is organized as follows: In Section 2 we review some preliminaries related to central extensions and twisted equivariant K–theory. In Section 3 we prove Theorem 3.6 and obtain Corollary 3.7 as a consequence. In Section 4 we calculate the homotopy type of the fixed-points space $B_G U(n)^A$ for a closed subgroup A of G. Section 5 is dedicated to studying geometric G–equivariant bordism and Theorem 5.6 is proved there. Finally, in Section 6 some applications are considered.

Acknowledgements

The authors acknowledge and thank the financial support provided by the Max Planck Institute for Mathematics and the Office of External Activities of the ICTP through Network NT8. Ángel acknowledges and thanks the financial support of the grant P12.160422.004/01–FAPA *Andrés Ángel* from Vicedecanatura de Investigaciones de la Facultad de Ciencias de la Universidad de los Andes, Colombia. Gómez and Uribe acknowledge and thank the support of COLCIENCIAS through grant numbers FP44842-617-2014 and FP44842-013-2018 of the Fondo Nacional de Financiamiento para la Ciencia, la Tecnología y la Innovación. Uribe acknowledges and thanks the financial support provided by the Alexander von Humboldt Foundation.

2 Preliminaries

2.1 Central extensions and representations

Suppose that we have an exact sequence of compact Lie groups

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

and let $\rho\colon A \to U(V_\rho)$ be a complex, finite-dimensional, irreducible representation of A. Since A is normal in G, the group G acts on the left on the set $\mathrm{Hom}(A, U(V_\rho))$ of homomorphisms from A to $U(V_\rho)$ by the equation

$$(g \cdot \chi)(a) := \chi(g^{-1} a g)$$

for $\chi \in \mathrm{Hom}(A, U(V_\rho))$ and $a \in A$. Also, the unitary group $U(V_\rho)$ acts on the right on $\mathrm{Hom}(A, U(V_\rho))$ by conjugation by the equation

$$(\chi \cdot M)(a) := M^{-1} \chi(a) M$$

for $\chi \in \mathrm{Hom}(A, U(V_\rho))$ and $M \in U(V_\rho)$. Note further that this left G–action on $\mathrm{Hom}(A, U(V_\rho))$ commutes with the right $U(V_\rho)$–action.

In this section we are going to show that if the representation ρ is such that $g \cdot \rho \cong \rho$ for every $g \in G$, then we can associate to ρ a central extension of G by $\mathbb{S}^1$ and that this central extension can be thought as an obstruction for the existence of an extension $\widetilde{\rho}\colon G \to U(V_\rho)$ of the representation ρ. For this notice that the projective unitary group $\mathrm{PU}(V_\rho) := U(V_\rho)/Z(U(V_\rho)) = U(V_\rho)/\mathbb{S}^1$ can be identified with the inner automorphisms of $U(V_\rho)$ via the map $p(M) = \mathrm{Ad}_M$, where $\mathrm{Ad}_M(N) = MNM^{-1}$ for $M \in U(V_\rho)$.

Lemma 2.1 *Suppose that for all $g \in G$ the irreducible representation $g \cdot \rho$ is isomorphic to ρ. Then there is a unique homomorphism $f\colon G \to \mathrm{PU}(V_\rho)$ making the following diagram commutative:*

$$\begin{array}{ccc} A & \xrightarrow{\iota} & G \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle f} \\ U(V_\rho) & \xrightarrow{p} & \mathrm{PU}(V_\rho) \end{array}$$

Proof Suppose that $g \in G$. Note that the representation $g \cdot \rho\colon A \to U(V_\rho)$, defined by $(g \cdot \rho)(a) = \rho(g^{-1}ag)$ for $a \in A$ and $g \in G$, also has V_ρ for its underlying vector space. By Schur's lemma we know that $\mathrm{Hom}_{U(V_\rho)}(g \cdot \rho, \rho) \cong \mathbb{C}$, thus there is only one inner automorphism $f(g^{-1}) \in \mathrm{Inn}(U(V_\rho))$ of $U(V_\rho)$ such that $g \cdot \rho = f(g^{-1}) \circ \rho$. Whenever $g \in A$, we have that $g \cdot \rho = \mathrm{Ad}_{\rho(g)^{-1}} \circ \rho$ and therefore we set $f(g) = \mathrm{Ad}_{\rho(g)}$ whenever $g \in A$.

For $h, g \in G$ we know that $(hg \cdot \rho) = h \cdot (g \cdot \rho)$, thus implying that $f((hg)^{-1}) \circ \rho = h \cdot (f(g^{-1}) \circ \rho) = f(g^{-1}) \circ f(h^{-1}) \circ \rho$ and therefore $f((hg)^{-1}) = f(g^{-1}) \circ f(h^{-1})$. Hence, f is a homomorphism and by definition it is unique. □

Suppose now that we have an irreducible representation $\rho\colon A \to U(V_\rho)$ such that $g \cdot \rho \cong \rho$ for every $g \in G$. Let $f\colon G \to \mathrm{PU}(V_\rho)$ be the homomorphism constructed in the previous lemma, so that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & G \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle f} \\ U(V_\rho) & \xrightarrow{p} & \mathrm{PU}(V_\rho) \end{array}$$

Recall that the natural projection map

$$1 \to \mathbb{S}^1 \to U(V_\rho) \xrightarrow{p} \mathrm{PU}(V_\rho) \to 1$$

defines a central extension of $\mathrm{PU}(V_\rho)$ by $\mathbb{S}^1$. Define the Lie group $\widetilde{G}_\rho := f^*U(V_\rho)$ as the pullback of $U(V_\rho)$ under the homomorphism f, so that we obtain a central extension of Lie groups

$$1 \to \mathbb{S}^1 \to \widetilde{G}_\rho \xrightarrow{\tau_\rho} G \to 1.$$

If we denote by $\widetilde{f}\colon \widetilde{G}_\rho \to U(V_\rho)$ the induced homomorphism, we obtain the following commutative diagram in the category of Lie groups:

(1)
$$\begin{array}{ccccc}
 & & \mathbb{S}^1 & & \mathbb{S}^1 \\
 & & \downarrow & & \downarrow \\
A & \xrightarrow{\tilde{\iota}} & \widetilde{G}_\rho & \xrightarrow{\widetilde{f}} & U(V_\rho) \\
\downarrow & = & \downarrow & & \downarrow \\
A & \xrightarrow{\iota} & G & \xrightarrow{f} & \mathrm{PU}(V_\rho)
\end{array}$$

In the above diagram the vertical sequences are $\mathbb{S}^1$–central extensions and the homomorphism $\tilde{\iota}\colon A \to \widetilde{G}_\rho$ is the unique homomorphism such that $\rho = \widetilde{f} \circ \tilde{\iota}$. Since A is normal in G and $\widetilde{G}_\rho$ is a central extension of G, we have that $\tilde{\iota}(A)$ is also normal in $\widetilde{G}_\rho$. Therefore, the quotient $\widetilde{G}_\rho/\tilde{\iota}(A)$ is a Lie group and we denote it by

$$\widetilde{Q}_\rho := \widetilde{G}_\rho/\tilde{\iota}(A)$$

since it depends only on ρ, and it fits into the diagram

(2)
$$\begin{array}{ccccc}
 & & \mathbb{S}^1 & & \mathbb{S}^1 \\
 & & \downarrow & & \downarrow \\
A & \xrightarrow{\tilde{\iota}} & \widetilde{G}_\rho & \xrightarrow{\tilde{\pi}} & \widetilde{Q}_\rho \\
\| & & \downarrow & & \downarrow \\
A & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q
\end{array}$$

where the horizontal sequences are exact, the vertical are $\mathbb{S}^1$–central extensions and the square on the right-hand side is a pullback square.

Proposition 2.2 *Consider the short exact sequence $1 \to A \to G \to Q \to 1$ of compact Lie groups and $\rho\colon A \to U(V_\rho)$ an irreducible representation of A such that its isomorphism class is invariant under the G–action, namely that $(g \cdot \rho) \cong \rho$ for all $g \in G$. Then the representation ρ may be extended to an irreducible representation $\widetilde{\rho}\colon G \to U(V_\rho)$ if and only if the $\mathbb{S}^1$–central extension $\widetilde{Q}_\rho$ is trivial, that is, $\widetilde{Q}_\rho$ is isomorphic to $Q \times \mathbb{S}^1$ as Lie groups.*

Proof If $\widetilde{Q}_\rho$ is trivial as an $\mathbb{S}^1$–central extension, then $\widetilde{G}_\rho$ must also be trivial as an $\mathbb{S}^1$–central extension, that is, $\widetilde{G}_\rho \cong G \times \mathbb{S}^1$. Therefore, there is a homomorphism $\sigma\colon G \to \widetilde{G}_\rho$ compatible with the quotient homomorphism $\widetilde{G}_\rho \to G$ whose composition $\widetilde{\rho} := \widetilde{f} \circ \sigma\colon G \to U(V_\rho)$ is the desired extension of ρ.

Conversely, if $\widetilde{\rho}\colon G \to U(V_\rho)$ extends the homomorphism ρ then $\widetilde{\rho}$ defines a homomorphism $\sigma\colon G \to \widetilde{G}_\rho$ compatible with the quotient homomorphism $\widetilde{G}_\rho \to G$, thus making $\widetilde{G}$ a trivial $\mathbb{S}^1$–central extension. It follows that $\widetilde{Q}_\rho$ is also trivial as an $\mathbb{S}^1$–central extension. □

Remark 2.3 Recall that isomorphism classes of $\mathbb{S}^1$–central extensions of Q are in one-to-one correspondence with elements in $H^3(BQ, \mathbb{Z})$ (see [2, Proposition 6.3]). By the previous proposition we may say that the obstruction for the existence of the extension $\widetilde{\rho}\colon G \to U(V_\rho)$ of the irreducible representation $\rho\colon A \to U(V_\rho)$ is the cohomology class $[\widetilde{Q}_\rho] \in H^3(BQ, \mathbb{Z})$ which encodes the information of the $\mathbb{S}^1$–central extension $\mathbb{S}^1 \to \widetilde{Q}_\rho \to Q$. In [10, Section 1] the obstruction of the existence of extensions of representations $\rho\colon A \to U(V_\rho)$ was studied for the case of finite groups and the obstruction was explicitly described in terms of cocycles.

2.2 Twisted equivariant K–theory

Next we recall the definition of twisted equivariant K–theory that we will use throughout this article. For this suppose that Q is a compact Lie group and let

$$1 \to \mathbb{S}^1 \to \widetilde{Q} \xrightarrow{\tau} Q \to 1$$

be an $\mathbb{S}^1$–central extension of Q. Let X be a Q–space and endow it with the action of $\widetilde{Q}$ induced by the Q–action. Consider the set of isomorphism classes of $\widetilde{Q}$–vector bundles $p\colon E \to X$ on which the elements $z \in \mathbb{S}^1$ act by the scalar multiplication of z^{-1}; denote this set by ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. The set ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$ is a semigroup under direct sum of vector bundles and we define the twisted equivariant K–group ${}^{\widetilde{Q}}K^0_Q(X)$ as the Grothendieck construction applied to ${}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. For $n > 0$ the twisted groups

${}^{\widetilde{Q}}K^n_Q(X)$ are defined as ${}^{\widetilde{Q}}\widetilde{K}^0_Q(\Sigma^n X_+)$. We call the groups ${}^{\widetilde{Q}}K^*_Q(X)$ the $\widetilde{Q}$–twisted Q–equivariant K–theory groups of X. Notice that ${}^{\widetilde{Q}}K^*_Q(X)$ is naturally a module over $R(Q)$. The twisted groups ${}^{\widetilde{Q}}K^0_Q(X)$ can alternatively be defined as follows. The action of $\mathbb{S}^1$ on X obtained by restricting the $\widetilde{Q}$–action is trivial. Therefore, we obtain a natural map $K^0_{\widetilde{Q}}(X) \to K^0_{\mathbb{S}^1}(X) \cong K^0(X) \otimes R(\mathbb{S}^1)$. Composing this with the restriction map $K^0(X) \otimes R(\mathbb{S}^1) \to R(\mathbb{S}^1)$ we obtain a natural map $K^0_{\widetilde{Q}}(X) \to R(\mathbb{S}^1)$ and ${}^{\widetilde{Q}}K^0_Q(X)$ can also be defined as the inverse image of the subgroup generated by the $\mathbb{S}^1$–representations on which a scalar z acts by multiplication of z^{-1}. This description can also be used to define ${}^{\widetilde{Q}}K^n_Q(X)$. The cohomology class that classifies the twist is the image of the class $[\widetilde{Q}] \in H^3(BQ, \mathbb{Z})$ that corresponds to the central extension $\widetilde{Q}$ under the canonical map $H^3_Q(*, \mathbb{Z}) = H^3(BQ, \mathbb{Z}) \to H^3_Q(X, \mathbb{Z})$.

Remark 2.4 In the literature it is more common to encounter a different (but equivalent) definition of this twisted form of equivariant K–theory (see [1, Definition 7.1] for example). Suppose that we are given a central extension $1 \to \mathbb{S}^1 \to \widetilde{Q} \xrightarrow{\tau} Q \to 1$. We can also consider the set ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$ of isomorphism classes of $\widetilde{Q}$–vector bundles over $p\colon E \to X$ on which an element $z \in \mathbb{S}^1$ acts by the scalar multiplication of z. The set ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$ is also a semigroup under direct sum of vector bundles and we can also define a twisted form of K–theory, which we will denote by ${}^{\widetilde{Q}^+}K^0_Q(X)$, as the Grothendieck construction applied to ${}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$. For $n > 0$ the twisted groups ${}^{\widetilde{Q}^+}K^n_Q(X)$ can be defined in a similar way as above. These two twisted forms of equivariant K–theory are naturally isomorphic, as we show next. By definition it suffices to prove the case $n = 0$. Let $p\colon E \to X$ be a $\widetilde{Q}$–vector bundle such that a central element $z \in \mathbb{S}^1$ acts by the scalar multiplication of z^{-1}, so that $[E] \in {}^{\widetilde{Q}}\mathrm{Vec}_Q(X)$. Let $\mathrm{Hom}(E, \underline{\mathbb{C}})$ be the vector bundle dual to E, where $\underline{\mathbb{C}}$ denotes the trivial $\widetilde{Q}$–vector bundle $\pi_1\colon X \times \mathbb{C} \to X$. If $\phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_x$ and $q \in \widetilde{Q}$, then the action of q on ϕ is the element $q \cdot \phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_{qx}$ defined by $(q \cdot \phi)(v) = \phi(q^{-1} \cdot v)$ for every $v \in E_{qx}$. With this action $\mathrm{Hom}(E, \underline{\mathbb{C}})$ is a $\widetilde{Q}$–vector bundle. If $z \in \mathbb{S}^1$ is a central element and $\phi \in \mathrm{Hom}(E, \underline{\mathbb{C}})_x$, then as the action of z in E is given by scalar multiplication of z^{-1}, we have

$$(z \cdot \phi)(v) = \phi(z^{-1} \cdot v) = \phi(zv) = z\phi(v)$$

for every $v \in E_x$. This shows that $\mathrm{Hom}(E, \underline{\mathbb{C}})$ is a $\widetilde{Q}$–equivariant vector bundle on which the central factor $\mathbb{S}^1$ acts by multiplication of scalars on the fibers, so that $[\mathrm{Hom}(E, \underline{\mathbb{C}})] \in {}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X)$. The assignment

$$ {}^{\widetilde{Q}}\mathrm{Vec}_Q(X) \xrightarrow{\cong} {}^{\widetilde{Q}^+}\mathrm{Vec}_Q(X), \quad [E] \mapsto [\mathrm{Hom}(E, \underline{\mathbb{C}})], $$

is an isomorphism of semigroups. After applying the Grothendieck construction we obtain an isomorphism ${}^{\widetilde{Q}}K^0_Q(X) \xrightarrow{\cong} {}^{\widetilde{Q}^+}K^0_Q(X)$. Throughout this article we will work with the twisted form of equivariant K–theory constructed using vector bundles on which the elements of the central factor $\mathbb{S}^1$ act by multiplication of their inverse; these are the bundles that appear naturally in our work.

3 Equivariant K–theory with prescribed fibers

The goal of this section is to generalize the decomposition of G–equivariant K–theory obtained in [10, Theorem 3.2] to the case of compact Lie groups.

To start, assume that G is a compact Lie group and let A be a normal subgroup of G, so that we have an extension of compact Lie groups

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1,$$

where $Q = G/A$. Let G act on a compact space X in such a way that A acts trivially on X. Assume that $p\colon E \to X$ is a G–equivariant complex vector bundle. We can give E a Hermitian metric that is invariant under the action of G; in particular, this metric is A–invariant. If we see $p\colon E \to X$ as an A–vector bundle then, as the action of A on X is trivial, by [20, Proposition 2.2] we have a natural isomorphism of A–vector bundles

$$\beta\colon \bigoplus_{[\tau]\in \mathrm{Irr}(A)} \mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E) \xrightarrow{\cong} E, \quad v \otimes f \mapsto f(v).$$

In the above equation $\mathrm{Irr}(A)$ denotes the set of isomorphism classes of complex irreducible A–representations and, if $\tau\colon A \to U(V_\tau)$ is an irreducible A–representation, then $\mathbb{V}_\tau$ denotes the trivial A–vector bundle $\pi_1\colon X \times V_\tau \to X$. The decomposition of the vector bundle E provided above is a decomposition as an A–equivariant bundle and not as a G–equivariant bundle. Furthermore, the summands $\mathbb{V}_\tau \otimes \mathrm{Hom}_A(\mathbb{V}_\tau, E)$ that appear in this decomposition do not in general have the structure of a G–vector bundle in such a way that the map β is G–equivariant. Following [10], we have the next definition.

Definition 3.1 Suppose that $\rho\colon A \to U(V_\rho)$ is a complex irreducible representation and that $1 \to A \to G \to Q \to 1$ is a short exact sequence of compact Lie groups. A (G, ρ)–equivariant vector bundle over X is a G–vector bundle $p\colon E \to X$ such that the map

$$\beta\colon \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E, \quad v \otimes f \mapsto f(v),$$

is an isomorphism of A–vector bundles.

A (G,ρ)–equivariant vector bundle is a G–equivariant vector bundle $p\colon E\to X$ such that for every $x\in X$ the A–representation E_x is isomorphic to a direct sum of copies of the representation ρ. Notice that if $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle then for every $g\in G$ we have $g\cdot\rho\cong\rho$, so that (G,ρ)–equivariant vector bundles can only exist when this happens. We can define a direct summand of the equivariant K–theory using (G,ρ)–equivariant vector bundles. For this let $\mathrm{Vec}_{G,\rho}(X)$ denote the set of isomorphism classes of (G,ρ)–equivariant vector bundles, where two (G,ρ)–equivariant vector bundles are isomorphic if they are isomorphic as G–vector bundles. Notice that if E_1 and E_2 are two (G,ρ)–equivariant vector bundles then so is the direct sum $E_1\oplus E_2$. Therefore, $\mathrm{Vec}_{G,\rho}(X)$ is a semigroup.

Definition 3.2 Assume that G acts on a compact space X in such a way that A acts trivially on X. We define $K^0_{G,\rho}(X)$, the (G,ρ)–equivariant K–theory of X, as the Grothendieck construction applied to $\mathrm{Vec}_{G,\rho}(X)$. For $n>0$ the group $K^n_{G,\rho}(X)$ is defined as $\widetilde{K}^0_{G,\rho}(\Sigma^n X_+)$, where as usual X_+ denotes the space X with an added basepoint.

Following [10] we relate the (G,ρ)–equivariant K–theory of X with a suitable twisted form of equivariant K–theory as defined in Section 2.2. For this suppose that $\rho\colon A\to U(V_\rho)$ is an irreducible representation and let $f\colon G\to \mathrm{PU}(V_\rho)$ be the homomorphism associated to ρ as constructed in Lemma 2.1. Consider $\widetilde{G}_\rho = f^*U(V_\rho)$ and $\widetilde{Q}_\rho := \widetilde{G}_\rho/\tilde{\imath}(A)$, so that we have a commutative diagram of central extensions as in diagram (2).

Theorem 3.3 *Let X be a G–space such that A acts trivially on X. Assume that $g\cdot\rho\cong\rho$ for every $g\in G$. If $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle, then $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ has the structure of a $\widetilde{Q}_\rho$–vector bundle on which the elements of the central factor $\mathbb{S}^1$ act by multiplication by their inverse. Moreover, the assignment*

$$[E]\mapsto[\mathrm{Hom}_A(\mathbb{V}_\rho,E)]$$

defines a natural one-to-one correspondence between isomorphism classes of (G,ρ)–equivariant vector bundles over X and isomorphism classes of $\widetilde{Q}_\rho$–equivariant vector bundles over X for which the elements of the central $\mathbb{S}^1$ act by multiplication of their inverse.

Proof Suppose $p\colon E\to X$ is a (G,ρ)–equivariant vector bundle. Then $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ is a complex vector bundle over X. Next we give $\mathrm{Hom}_A(\mathbb{V}_\rho,E)$ an action of $\widetilde{G}_\rho$ on which $\tilde{\imath}(A)$ acts trivially.

Take $\phi \in \mathrm{Hom}_A(\mathbb{V}_\rho, E)_x$ and $\widetilde{g} \in \widetilde{G}_\rho$, and define $\widetilde{g} \bullet \phi \in \mathrm{Hom}_A(\mathbb{V}_\rho, E)_{g\cdot x}$ by

$$(\widetilde{g} \bullet \phi)(v) = g\phi(\widetilde{f}(\widetilde{g})^{-1}v),$$

where $\widetilde{g}$ projects to g in G and $\widetilde{f}\colon \widetilde{G}_\rho \to U(V_\rho)$ is the homomorphism defined in diagram (1). The action is a composition of continuous maps and therefore it is continuous. It is straightforward to check that it is a homomorphism.

Now let us take $a \in A$ and consider the action of $\tilde{\iota}(a)$ on ϕ. In this case we have

$$(\tilde{\iota}(a) \bullet \phi)(v) = a\phi\big(\widetilde{f}(\tilde{\iota}(a))^{-1}v\big) = a\phi(\rho(a)^{-1}v) = \phi(v),$$

which implies that $\tilde{\iota}(a) \bullet \phi = \phi$. Hence, the action of $\tilde{\iota}(A)$ on $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is trivial and therefore there is an induced action of $\widetilde{Q}_\rho = \widetilde{G}_\rho/\tilde{\iota}(A)$ on $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ compatible with the action of Q on X.

Now, if $\lambda \in \ker(\widetilde{G}_\rho \to G)$, the action becomes

$$(\lambda \bullet \phi)(v) = \phi(\widetilde{f}(\lambda)^{-1}v) = \lambda^{-1}\phi(v),$$

which implies that $\mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a $\widetilde{Q}_\rho$–equivariant bundle where the elements of $\mathbb{S}^1$ act by multiplication by their inverse.

Now let us take a $\widetilde{Q}_\rho$–equivariant bundle $F \to X$ where the elements of $\mathbb{S}^1$ act by multiplication by their inverse. Consider the vector bundle $\mathbb{V}_\rho \otimes F$ and define a $\widetilde{G}_\rho$–action in the following way: for $\widetilde{g} \in \widetilde{G}_\rho$ and $v \otimes e \in \mathbb{V}_\rho \otimes F$, let the action be

$$\widetilde{g} \cdot (v \otimes e) := (\widetilde{f}(\widetilde{g})v) \otimes (\widetilde{\pi}(\widetilde{g}) \cdot e),$$

where $\widetilde{\pi}\colon \widetilde{G}_\rho \to \widetilde{Q}_\rho$ is the homomorphism induced by $\pi\colon G \to Q$. This is clearly a continuous $\widetilde{G}_\rho$–action and for $\lambda \in \ker(\widetilde{G}_\rho \to G)$ we have that

$$\lambda \cdot (v \otimes e) := (\widetilde{f}(\lambda)v) \otimes (\widetilde{\pi}(\lambda) \cdot e) = \lambda v \otimes \lambda \cdot e = \lambda v \otimes \lambda^{-1}e = v \otimes e,$$

which implies that the action factors through $G = \widetilde{G}_\rho/\mathbb{S}^1$.

Let us see now what the action looks like once restricted to A. Take $a \in A$ and consider the element $\tilde{\iota}(a) \in \widetilde{G}_\rho$. The action of a on $v \otimes e$ becomes

$$\tilde{\iota}(a) \cdot (v \otimes e) = \big(\widetilde{f}(\tilde{\iota}(a))v\big) \otimes \big(\widetilde{\pi}(\tilde{\iota}(a)) \cdot e\big) = \rho(a)v \otimes e,$$

which implies that the action of A on the fibers of $\mathbb{V}_\rho \otimes F$ is determined by the representation ρ. Hence, $\mathbb{V}_\rho \otimes F$ is a (G, ρ)–equivariant vector bundle.

If E is a (G,ρ)–equivariant vector bundle then $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is also a (G,ρ)–equivariant vector bundle and the canonical map

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E, \quad v \otimes \phi \mapsto \phi(v),$$

is by definition an isomorphism of vector bundles. Note that the map is moreover G–equivariant: for $\widetilde{g} \in \widetilde{G}_\rho$ which projects to $g \in G$ we have that

$$g \cdot (v \otimes \phi) = \widetilde{g} \cdot (v \otimes \phi) = (\widetilde{f}(\widetilde{g})v) \otimes (\widetilde{g} \bullet \phi),$$

whose evaluation becomes

$$(\widetilde{g} \bullet \phi)(\widetilde{f}(\widetilde{g})v) = g\phi(\widetilde{f}(\widetilde{g})^{-1}\widetilde{f}(\widetilde{g})v) = g\phi(v),$$

thus implying that the canonical evaluation map is a G–equivariant isomorphism.

Finally, if F is a $\widetilde{Q}_\rho$–equivariant bundle where the elements of $\mathbb{S}^1$ act by multiplication by their inverse, we may consider the canonical isomorphism of vector bundles

$$F \to \mathrm{Hom}_A(\mathbb{V}_\rho, \mathbb{V}_\rho \otimes F), \quad e \mapsto \phi_e\colon v \mapsto v \otimes e.$$

It is straightforward to check that it is moreover an isomorphism of $\widetilde{Q}_\rho$–equivariant vector bundles.

We conclude that the inverse map of the assignment $[E] \mapsto [\mathrm{Hom}_A(\mathbb{V}_\rho, E)]$ is precisely the map defined by the assignment $[F] \mapsto [\mathbb{V}_\rho \otimes F]$. The theorem follows. □

Theorem 3.3 provides the following identification of the (G,ρ)–equivariant K–groups of Definition 3.1 with the twisted groups ${}^{\widetilde{Q}_\rho}K^*_Q(X)$ defined in Section 2.2.

Corollary 3.4 *Let G be a compact Lie group and X be a compact G–space such that the normal subgroup A of G acts trivially on X. Assume furthermore that $\rho\colon A \to U(V_\rho)$ is an irreducible representation whose isomorphism class is fixed by G, that is, $g \cdot \rho \cong \rho$ for every $g \in G$. Then the homomorphism*

$$K^*_{G,\rho}(X) \xrightarrow{\cong} {}^{\widetilde{Q}_\rho}K^*_Q(X), \quad [E] \mapsto [\mathrm{Hom}_A(\mathbb{V}_\rho, E)],$$

is a natural isomorphism of $R(Q)$–modules. The inverse map is $F \mapsto \mathbb{V}_\rho \otimes F$.

Whenever the isomorphism class of the A–representation ρ is not fixed by the whole group G we need to be more careful. Define

$$G_\rho := \{g \in G \mid g \cdot \rho \cong \rho\} \quad \text{and} \quad Q_\rho := G_\rho/A$$

and call $\widetilde{G}_\rho$ and $\widetilde{Q}_\rho$ the $\mathbb{S}^1$–central extensions which measure the obstruction for the extension of ρ to G_ρ constructed in Proposition 2.2.

Now consider a G–vector bundle E over the compact G–space X on which A acts trivially. We know that as a A–vector bundle we have the isomorphism

$$\bigoplus_{\rho\in \mathrm{Irr}(A)} \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \xrightarrow{\cong} E \tag{3}$$

given by the evaluation, where ρ runs over the set of isomorphism classes of irreducible A–representations. Each of the vector bundles $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ is a G_ρ–vector bundle. However, it is not possible in general to provide each factor $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ with the structure of a G–vector bundle in such a way that the isomorphism (3) is an isomorphism of G–equivariant vector bundles as the G–action intertwines the vector bundles associated to irreducible A–representations which are related by the action of G. To illustrate this issue we explore the following example:

Example 3.5 Suppose that $G = D_8$ is the dihedral group generated by the elements a and b with relations $a^4 = b^2 = 1$ and $bab = a^{-1}$, and let $A = \langle a\rangle = \mathbb{Z}/4$, so that we have a short exact sequence

$$1 \to A \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$$

with $Q = G/A = \{1, \tau\} = \mathbb{Z}/2$, where $\tau = [b] \in Q$. Let $\rho\colon A \to \mathbb{S}^1$ be the representation given by $\rho(a) = e^{2\pi i/4} = i$. In this example we have $\mathrm{Irr}(\mathbb{Z}/4) = \{\mathbf{1}, \rho, \rho^2, \rho^3\}$ and the action of Q on $\mathrm{Irr}(\mathbb{Z}/4)$ is such that τ permutes the isomorphism classes of the representations ρ and ρ^3. Let $V_\rho := \mathbb{C}$ equipped with the A–representation ρ and $V_{\rho^3} := \mathbb{C}$ equipped with the A–representation ρ^3. Consider the balanced product $E := D_8 \times_A V_\rho$ seen as a D_8–equivariant vector bundle over $D_8/A = \{*, *'\}$. Note that, as a D_8–equivariant vector bundle, E is isomorphic to the bundle $V_\rho \sqcup V_{\rho^3}$, where the action of b maps V_ρ to V_{ρ^3} and vice versa using the explicit isomorphisms with $\mathbb{C}$. Here we see V_ρ as a bundle over $\{*\}$ and V_{ρ^3} as a bundle over $\{*'\}$. In this case we have

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \cong V_\rho \sqcup \{*'\},$$
$$\mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E) \cong \{*\} \sqcup V_{\rho^3}.$$

Hence,

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \oplus \mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E) \cong V_\rho \sqcup \{*'\} \oplus \{*\} \sqcup V_{\rho^3} \cong E$$

as D_8–equivariant vector bundles. However, the factors

$$\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \quad \text{and} \quad \mathbb{V}_{\rho^3} \otimes \mathrm{Hom}_A(\mathbb{V}_{\rho^3}, E)$$

do not possess a structure of a D_8–vector bundle that is compatible with the above isomorphism.

Suppose now that $p\colon E \to X$ is a G–vector bundle over the compact G–space X on which A acts trivially. As our next step we show that the factors in the decomposition described in formula (3) can be arranged in a suitable way to obtain a decomposition of E as a G–vector bundle. Choosing representatives $\{g_i\}_i$ for each class in G/G_ρ, we know that the image of the evaluation map

$$\bigoplus_i \mathbb{V}_{g_i\cdot\rho} \otimes \mathrm{Hom}_A(\mathbb{V}_{g_i\cdot\rho}, E) \to E$$

becomes a G–equivariant vector bundle. Notice that G/G_ρ is finite since G_ρ contains the connected component of the identity of G and G is compact; this follows from the fact that G is acting on the discrete set $\mathrm{Irr}(A)$. Now, in order to define the bundle above in a coordinate-free fashion, we need to promote the G_ρ–equivariant bundle $\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)$ over X to a G–equivariant bundle over the same space X. This construction was called *multiplicative induction* in [3, Section 4] and here we will recall its properties.

Let G be a compact Lie group and H a closed subgroup. The restriction functor r_H^G from G–spaces to H–spaces which restricts the action to H has a left adjoint which maps a H–space Y to the G–space $G\times_H Y$, thus giving a homeomorphism

$$\mathrm{map}(G\times_H Y, X)^G \cong \mathrm{map}(Y, r_H^G X)^H$$

for any G–space X. This left adjoint is additive, but in general it is not multiplicative. A right adjoint for the restriction functor r_H^G can be defined on an H–space Y as the G–space of H–equivariant maps from G to Y,

$$m_H^G(Y) := \mathrm{map}(G, Y)^H,$$

where G is considered as an H–space via left multiplication. The G–action on $m_H^G(Y)$ is given by $(g\cdot f)(k) := f(kg)$. A map of H–spaces $\phi\colon Y_1 \to Y_2$ induces a map of G–spaces $m_H^G(\phi)\colon m_H^G(Y_1) \to m_H^G(Y_2)$ by composition. In this case there is a homeomorphism

$$\mathrm{map}(r_H^G(X), Y)^H \cong \mathrm{map}(X, m_H^G(Y))^G$$

for any G–space X and any H–space Y. The maps are defined by

$$\begin{aligned}
\mathrm{map}(X, m_H^G(Y))^G &\to \mathrm{map}(r_H^G(X), Y)^H, \quad F \mapsto (x \mapsto F(x)(1_G)),\\
\mathrm{map}(r_H^G(X), Y)^H &\to \mathrm{map}(X, m_H^G(Y))^G, \quad f \mapsto m_H^G(f)\circ p_H^G,
\end{aligned}$$

where $p_H^G\colon X \to m_H^G(r_H^G(X))$ is defined by the equation

$$(p_H^G(x))(g) = gx$$

and is the unit of the adjunction.

The functor m_H^G is called multiplicative because

$$m_H^G(Y_1 \times Y_2) \cong m_H^G(Y_1) \times m_H^G(Y_2)$$

for any H–spaces Y_1 and Y_2.

Note also that the space $m_H^G(Y)$ is homeomorphic to the space $\Gamma(G \times_H Y, G/H)$ of sections of the projection map $G \times_H Y \to G/H$, endowed with the G–action given by $(g \cdot \sigma)(kH) = g\sigma(g^{-1}kH)$, where σ is any section. In the case that G/H is finite the space $m_H^G(Y)$ is homeomorphic to the product of $|G : H|$ copies of Y.

Let us now consider the G–equivariant bundle

$$m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E)) \to m_{G_\rho}^G(X)$$

and construct the pullback bundle

$$(p_{G_\rho}^G)^*\big(m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big) \to X.$$

The G–equivariant bundle $(p_{G_\rho}^G)^*\big(m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big)$ is endowed with a natural G–equivariant map to E, defined by the restriction of the natural map

$$\mathrm{map}(G, \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))^{G_\rho} \to E, \quad \phi \mapsto \mathrm{ev}(\phi(1_G)),$$

induced by the evaluation map $\mathrm{ev}\colon \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \to E$, $\mathrm{ev}(v \otimes f) = f(v)$. Therefore, we have constructed the desired G–vector bundle over X.

Theorem 3.6 *Let G be a compact Lie group and E a G–equivariant complex vector bundle over the compact G–space X. If the action on X by the normal subgroup A of G is trivial, then the following decomposition formula is an isomorphism of G–equivariant bundles:*

$$\bigoplus_{\rho \in G \setminus \mathrm{Irr}(A)} (p_{G_\rho}^G)^*\big(m_{G_\rho}^G(\mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E))\big) \xrightarrow{\cong} E,$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations.

Note that when the Lie group G is connected, then $G_\rho = G$ for all ρ and the decomposition simplifies to the isomorphism

$$\bigoplus_{\rho\in \mathrm{Irr}(A)} \mathbb{V}_\rho \otimes \mathrm{Hom}_A(\mathbb{V}_\rho, E) \xrightarrow{\cong} E$$

of G–equivariant bundles. This is also the case, for example, when G is abelian.

The decomposition formula of the equivariant vector bundle E induces an isomorphism in K–theory as follows:

Corollary 3.7 *Let G be a compact Lie group and X a G–space on which the normal subgroup A acts trivially. Then there is a natural isomorphism*

$$K^*_G(X) \cong \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} {}^{\widetilde{Q}_\rho}K^*_{Q_\rho}(X),$$

where ρ runs over representatives of the orbits of the G–action on the set of isomorphism classes of irreducible A–representations and $Q_\rho = G_\rho/A$.

Proof The isomorphism follows from the isomorphism

$$\bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} K^*_{G_\rho,\rho}(X) \xrightarrow{\cong} K^*_G(X), \qquad \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} E_\rho \mapsto \bigoplus_{\rho\in G\setminus \mathrm{Irr}(A)} (p^G_{G_\rho})^*(m^G_{G_\rho} E_\rho),$$

and Corollary 3.4. □

4 The decomposition at the level of classifying spaces

In this section we will write the results of the previous section at the level of the classifying space of G–equivariant complex vector bundles. This will show us how the spectrum of G–equivariant K–theory decomposes at the fixed-point set of each subgroup.

The universal bundles for twisted equivariant K–theory associated to central extensions of the group G are constructed as follows. Consider a central extension

$$1 \to \mathbb{S}^1 \to \widetilde{G} \to G \to 1$$

of the compact Lie group G by $\mathbb{S}^1$. Let $\widetilde{\boldsymbol{C}}^\infty$ denote the direct sum of countably many copies of all irreducible $\widetilde{G}$–representations on which elements in $\mathbb{S}^1 = \ker(\widetilde{G} \to G)$ act by scalar multiplication of their inverse. Let ${}^{\widetilde{G}}B_G U(n)$ denote the Grassmannian

of n–dimensional complex subspaces of $\widetilde{\boldsymbol{C}}^{\infty}$ and denote by ${}^{\widetilde{G}}\gamma_G U(n)$ the universal n–plane bundle over ${}^{\widetilde{G}}B_G U(n)$. The complex vector bundle

$$\mathbb{C}^n \to {}^{\widetilde{G}}\gamma_G U(n) \to {}^{\widetilde{G}}B_G U(n) \tag{4}$$

is the universal $\widetilde{G}$–twisted G–equivariant complex vector bundle of rank n, and therefore for a finite G–CW complex X we have

$$ {}^{\widetilde{G}}\mathrm{Vec}^n_G(X) \cong [X, {}^{\widetilde{G}}B_G U(n)]_G.$$

Note that since $\mathbb{S}^1$ acts by multiplication of the inverse of scalars, its action on the Grassmannian of complex n–planes is trivial, and therefore the $\widetilde{G}$–action on ${}^{\widetilde{G}}B_G U(n)$ reduces to a G–action. If $V \subset \widetilde{\boldsymbol{C}}^{\infty}$ is a finite-dimensional complex $\widetilde{G}$–subrepresentation, then there is a map

$$ {}^{\widetilde{G}}\gamma_G U(n) \oplus \mathbb{V} \to {}^{\widetilde{G}}\gamma_G U(n+|V|)$$

which induces a map $\iota_V\colon {}^{\widetilde{G}}B_G U(n) \to {}^{\widetilde{G}}B_G U(n+|V|)$ at the level of the classifying spaces. The colimit

$$ {}^{\widetilde{G}}B_G U := \operatorname*{colim}_{V \subset \widetilde{\boldsymbol{C}}^{\infty}} \bigsqcup_{n \geq 0} {}^{\widetilde{G}}B_G U(n) \tag{5}$$

is the classifying space for reduced $\widetilde{G}$–twisted G–equivariant complex K–theory,

$$ {}^{\widetilde{G}}\widetilde{K}^0_G(X) \cong [X, {}^{\widetilde{G}}B_G U]_G$$

for X a finite G–CW complex.

Whenever the extension is trivial $\widetilde{G} \cong \mathbb{S}^1 \times G$, the spaces ${}^{\mathbb{S}^1\times G}B_G U(n)$ classify G–equivariant $U(n)$–principal bundles, and therefore we may write $B_G U(n) := {}^{\mathbb{S}^1\times G}B_G U(n)$ and $B_G U := {}^{\mathbb{S}^1\times G}B_G U$, thus having that, for X a compact G–space,

$$ \widetilde{K}^0_G(X) \cong [X, B_G U]_G.$$

Furthermore, the vector bundles $\gamma_G U(n) := {}^{\mathbb{S}^1\times G}\gamma_G U(n)$ are the universal G–equivariant complex vector bundles of rank n.

Suppose now that A is a closed subgroup of the compact Lie group G. Consider the fixed-point set $B_G U(n)^A$ and the restriction $\gamma_G U(n)|_{B_G U(n)^A}$ of the universal vector bundle. Denote by N_A the normalizer of A in G and by $W_A = N_A/A$ the quotient. Therefore, $\gamma_G U(n)|_{B_G U(n)^A} \to B_G U(n)^A$ is an N_A–equivariant vector bundle such that A acts trivially on the base space. In this way we are in the situation

of the previous section for the short exact sequence $1 \to A \to N_A \to W_A \to 1$. Take $\rho \in \mathrm{Irr}(A)$. By Theorem 3.3 the bundle $\mathrm{Hom}_A(\mathbb{V}_\rho, \gamma_G U(n)|_{B_G U(n)^A})$ is a $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundle, but since the space $B_G U(n)^A$ is not necessarily connected, it may not have constant rank. Therefore, in order to construct a universal N_A–equivariant complex bundle over spaces with trivial A–actions using universal $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundles, we need to work with bundles of all ranks. We claim the following result:

Theorem 4.1 *There is a W_A–equivariant homotopy equivalence*

$$\coprod_{n=0}^{\infty} B_G U(n)^A \simeq \prod_{\rho \in W_A \backslash \mathrm{Irr}(A)} m^{W_A}_{(W_A)_\rho}\Bigg(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\Bigg).$$

The stable version is

$$B_G U^A \simeq \prod_{\rho \in W_A \backslash \mathrm{Irr}(A)} m^{W_A}_{(W_A)_\rho}({}^{(\widetilde{W}_A)_\rho} B U_{(W_A)_\rho} U)$$

as W_A–spaces.

Proof To start notice that by [17, Chapter V, Lemma 4.7 and Chapter VII, Theorem 2.4] it follows that $B_G U(n)^A$ classifies N_A–equivariant complex vector bundles of rank n over A–trivial N_A–spaces. Therefore, $\bigsqcup_{n=0}^{\infty} B_G U(n)^A$ classifies N_A–equivariant complex bundles (of any rank) over spaces with trivial A–actions. The theorem will follow by Theorem 3.6 since both sides classify N_A–equivariant complex bundles over spaces with trivial A–actions. Let us define the maps.

Since $\mathrm{Hom}_A(\mathbb{V}_\rho, \gamma_G U(n)|_{B_G U(n)^A})$ is a $(\widetilde{W}_A)_\rho$–twisted $(W_A)_\rho$–equivariant complex bundle, there is a $(W_A)_\rho$–equivariant classifying map

$$f_\rho \colon B_G U(n)^A \to \coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)$$

which induces a W_A–equivariant map

$$m^{W_A}_{(W_A)_\rho}(f_\rho) \circ p^{W_A}_{(W_A)_\rho} \colon B_G U(n)^A \to m^{W_A}_{(W_A)_\rho}\Bigg(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho} B_{(W_A)_\rho} U(n_\rho)\Bigg).$$

This constructs the map from left to right.

For the map from the right to the left, we know from Theorem 3.3 that

$$\coprod_{n_\rho=0}^{\infty} \mathbb{V}_\rho \otimes {}^{(\widetilde{W}_A)_\rho}\gamma_{W_A} U(n_\rho)$$

is an $(N_A)_\rho$–equivariant complex bundle and

$$m_{(N_A)_\rho}^{N_A}\Big(\coprod_{n_\rho=0}^{\infty} \mathbb{V}_\rho \otimes {}^{(\widetilde{W}_A)_\rho}\gamma_{W_A} U(n_\rho)\Big)$$

is an N_A–equivariant complex bundle. The product over $\rho \in W_A \backslash \operatorname{Irr}(A)$ is also an N_A–equivariant complex bundle and therefore there is a classifying map

$$\prod_{\rho\in W_A\backslash \operatorname{Irr}(A)} m_{(W_A)_\rho}^{W_A}\Big(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho} U(n_\rho)\Big) \to \coprod_{n=0}^{\infty} B_G U(n)^A.$$

The homotopy equivalence follows from Theorem 3.6. The homotopy equivalence of the stable version follows from Corollary 3.7. □

Remark 4.2 Distributing the product over union, we obtain a homeomorphism

$$\prod_{\rho\in W_A\backslash \operatorname{Irr}(A)} m_{(W_A)_\rho}^{W_A}\Big(\coprod_{n_\rho=0}^{\infty} {}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho} U(n_\rho)\Big) \cong \coprod_{n=0}^{\infty} \coprod_{\sum n_\rho|\rho|=n} \prod_{\rho\in\operatorname{Irr}(A)} {}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho} U(n_\rho),$$

where $|\rho|$ denotes the complex dimension of the representation ρ. We note that the expression on the right-hand side is not canonically a W_A–space, therefore we induce the W_A–action on the right-hand side from the one of the expression on the left.

Let $\mathcal{P}(n, A)$ be the set of arrays $P = (n_\rho)_{\rho\in\operatorname{Irr}(A)}$ such that

$$\sum_{\rho\in\operatorname{Irr}(A)} n_\rho|\rho| = n.$$

Restricting to G–equivariant complex bundles of rank n over G–spaces, one gets the W_A–homotopy equivalence

$$B_G U(n)^A \simeq \coprod_{P\in\mathcal{P}(n,A)} \prod_{\rho\in\operatorname{Irr}(A)} ({}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho} U(n_\rho)),$$

where W_A acts on $\mathcal{P}(n, A)$ by permuting the arrays of numbers according to its action on $\operatorname{Irr}(A)$, and the isotropy subgroups $(W_A)_\rho$ act on the appropriate coordinate space ${}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho} U(n_\rho)$.

Whenever G is abelian G acts trivially on $\mathrm{Irr}(A)$ and therefore we recover the homotopy equivalence of G/A–spaces

$$B_G U(n)^A \simeq \coprod_{P\in\mathcal{P}(n,A)} \prod_{\rho\in\mathrm{Irr}(A)} B_{G/A} U(n_\rho)$$

that appeared in [11, Proposition 4.3]. Note that in this case $\sum_\rho n_\rho = n$ since all irreducible representations of A are 1–dimensional.

Whenever the normalizer of A is connected, the right-hand side simplifies as

$$B_G U(n)^A \simeq \coprod_{P\in\mathcal{P}(n,A)} \prod_{\rho\in\mathrm{Irr}(A)} \widetilde{W}_{A\rho} B_{W_A} U(n_\rho),$$

where $\widetilde{W}_{A\rho}$ is the $\mathbb{S}^1$–central extension of W_A that depends on ρ.

5 Equivariant unitary bordism

The decomposition of equivariant complex vector bundles on fixed-point sets carried out in the previous sections is a key ingredient in the calculation of equivariant unitary bordisms for families. Conner and Floyd in their monumental work on the study of the bordism groups [5; 6] introduced the use of families of subgroups in order to restrict the bordisms to manifolds whose isotropy groups lie in a prescribed family.

We will concentrate on the tangentially stably equivariant unitary bordism groups Ω^G_*, which will be called the geometric G–equivariant unitary bordisms. The explicit definition of these homology groups and their stable versions can be found in [12, Section 2; 11, Definition 3.1]. Let us recall the main ingredients. In all of what follows, G will be a compact Lie group.

Definition 5.1 Let M be a smooth G–manifold. A tangentially stably almost complex G–structure on M is a complex G–structure on $TM \oplus \underline{\mathbb{R}}^k$ for some $k \geq 0$, where $\underline{\mathbb{R}}^k$ denotes the trivial bundle $M \times \mathbb{R}^k$ over M with trivial G–action; that is, there exists a G–equivariant complex bundle ξ over M such that $TM \oplus \underline{\mathbb{R}}^k \cong \xi$ as G–equivariant real vector bundles. Two tangentially stably almost complex G–structures are identified if after stabilization with further G–trivial $\underline{\mathbb{C}}$ summands the structures become G–homotopic through complex G–structures.

With this definition, if H is a closed subgroup of G then the fixed points M^H also have a tangentially stably almost complex N_H–structure. Moreover, an N_H–tubular neighborhood around M^H in M possesses an N_H–complex structure by

[11, Proposition 3.2]; let us see why. The bundle $TM|_{M^H}$ contains $T(M^H)$ as a subbundle and

$$TM|_{M^H} = T(M^H) \oplus \nu(M^H, M),$$

where $\nu(M^H, M)$ is the normal bundle of M^H in M. Also $T(M^H) = (TM|_{M^H})^H$ is an N_H–equivariant real vector bundle.

Given the tangentially stably almost complex G–structure ξ, we have that $\xi|_{M^H}$ is a complex vector bundle over M^H with a complex vector subbundle ξ^H, and also

$$\xi^H \cong (TM \oplus \underline{\mathbb{R}}^k)^H = (TM|_{M^H})^H \oplus \underline{\mathbb{R}}^k = T(M^H) \oplus \underline{\mathbb{R}}^k.$$

Therefore, M^H has a tangentially stably almost complex N_H–structure.

Now, since

$$\xi|_{M^H} \cong TM|_{M^H} \oplus \underline{\mathbb{R}}^k = T(M^H) \oplus \nu(M^H, M) \oplus \underline{\mathbb{R}}^k,$$

the normal bundle $\nu(M^H, M)$ is isomorphic to the quotient bundle $\xi|_{M^H}/\xi^H$, which is an N_H–complex bundle on M^H, thus showing that the normal bundle of M^H on M possesses an N_H–complex structure.

Definition 5.2 For a cofibration of G–spaces $Y \to X$ the geometric G–equivariant unitary bordism groups $\Omega_n^G(X, Y)$ are defined as G–bordism classes of singular tangentially stably almost complex n–dimensional G–manifolds $(M^n, \partial M^n) \to (X, Y)$.

When G is trivial, a tangentially stably almost complex structure is the same as a normally stably almost complex structure and $\Omega_n^{\{1\}}(X, Y)$ is the usual unitary bordism of the pair (X, Y).

One way to study the equivariant bordism groups is through the study of the equivariant bordism groups of manifolds M whose isotropies lie in a fixed family of subgroups of G. This way of studying equivariant bordism groups was developed by Conner and Floyd [6, Section 5] and it is currently one of the most useful techniques for calculating the equivariant bordism groups.

A family of subgroups $\mathcal{F}$ of G is a set (possibly empty) consisting of subgroups of G which is closed under taking subgroups and under conjugation. Denote by $E\mathcal{F}$ the classifying space for the family $E\mathcal{F}$, a G–space which is terminal in the category of $\mathcal{F}$–numerable G–spaces [9, Section 1, Theorem 6.6], and which is characterized by the following properties on fixed-point sets: $E\mathcal{F}^H \simeq *$ if $H \in \mathcal{F}$ and $E\mathcal{F}^H = \varnothing$ if $H \notin \mathcal{F}$.

Given families of subgroups $\mathcal{F}' \subset \mathcal{F}$ of G the induced map $E\mathcal{F}' \to E\mathcal{F}$ can be constructed so that it is a G–cofibration.

Following tom Dieck [8, page 310], we can define equivariant unitary bordism groups for families $\Omega_*^G[\mathcal{F}, \mathcal{F}']$ as follows. Given a G–space X, $\Omega_*^G[\mathcal{F}, \mathcal{F}'](X)$ is defined as

$$\Omega_*^G[\mathcal{F}, \mathcal{F}'](X) := \Omega_*^G(X \times E\mathcal{F}, X \times E\mathcal{F}').$$

Alternatively, we may define the geometric G–equivariant unitary bordism groups $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X, A)$ in a geometric way, as was done in [21, Section 2]. We recall the definition of the absolute unitary bordism groups $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$ for completeness.

An $(\mathcal{F}, \mathcal{F}')$–*free geometric unitary bordism element* of X is an equivalence class of a triple $(M, \partial M, f)$, where M is an n–dimensional G–manifold endowed with tangentially stably almost complex G–structure which is moreover $\mathcal{F}$–free; this means that all isotropy groups $G_m = \{g \in G \mid gm = m\}$ for $m \in M$ belong to $\mathcal{F}$, ∂M is $\mathcal{F}'$–free and $f\colon M \to X$ is a G–equivariant map. Two triples $(M, \partial M, f)$ and $(M', \partial M', f')$ are equivalent if there exists a G–manifold V that is $\mathcal{F}$–free such that $\partial V = M \cup M' \cup V^+$, and $M \cap V^+ = \partial M$, $M' \cap V^+ = \partial M'$, $M \cap M' = \varnothing$, $V^+ \cap (M \cup M') = \partial V^+$ and V^+ is $\mathcal{F}'$–free, together with a G–equivariant map $F\colon V \to X$ that restricts to f on M and to f' on M'.

Definition 5.3 The set of equivalence classes of $(\mathcal{F}, \mathcal{F}')$–free geometric unitary bordism elements of X, consisting of classes $(M, \partial M, f)$ where the dimension of M is n, and under the operation of disjoint union, forms an abelian group, denoted by $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$. We refer to these groups as the geometric unitary bordisms of X restricted to the pair of families $\mathcal{F}' \subset \mathcal{F}$. The equivalence class corresponding to the triple $(M, \partial M, f)$ will be denoted by $[M, \partial M, f]$.

Notice that if N is a stably almost complex closed manifold, we can define

$$[N] \cdot [M, \partial M, f] := [N \times M, N \times \partial M, f \circ \pi_M],$$

thus making $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$ a module over the unitary bordism ring Ω_*.

The covariant functor $\Omega_*^G\{\mathcal{F}, \mathcal{F}'\}$ defines a G–equivariant homology theory; see [21, Proposition 2.1]. A natural transformation $\mu\colon \Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \to \Omega_n^G[\mathcal{F}, \mathcal{F}'](X)$ can be defined as in [8, Satz 3] in the following way. Suppose that $(M, \partial M, f)$ is a representative of an element in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$. Since the inclusion $E\mathcal{F}' \subset E\mathcal{F}$ is

a G–cofibration, ∂M is $\mathcal{F}'$–free and M is $\mathcal{F}$–free, there is a G–equivariant map $k\colon M \to E\mathcal{F}$ such that $k(\partial M) \subset E\mathcal{F}'$. The G–equivariant map

$$(f,k)\colon M \to X \times E\mathcal{F}, \quad m \mapsto (f(m), k(m)),$$

maps ∂M into $X \times E\mathcal{F}'$ and therefore (f,k) becomes an element in $\Omega_n^G[\mathcal{F},\mathcal{F}'](X)$.

Proposition 5.4 *The natural transformation*

$$\mu\colon \Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X) \to \Omega_n^G[\mathcal{F},\mathcal{F}'](X), \quad [M,\partial M, f] \mapsto [(M,\partial M,(f,k))],$$

is an isomorphism.

The proof of this proposition follows the same lines as the one done by tom Dieck [8, Satz 3] in the case of an equivariant unoriented bordism. We will not reproduce the proof here.

The long exact sequence of the pair $(X \times E\mathcal{F}', X \times E\mathcal{F})$ becomes

$$\cdots \to \Omega_n^G\{\mathcal{F}'\}(X) \to \Omega_n^G\{\mathcal{F}\}(X) \to \Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X) \to \Omega_{n-1}^G\{\mathcal{F}'\}(X) \to \cdots,$$

where $\Omega_n^G\{\mathcal{F}\}(X) = \Omega_n^G\{\mathcal{F},\varnothing\}(X)$ is the bordism group of $\mathcal{F}$–free tangentially stably almost complex closed manifolds with an equivariant map to X. Note that for finite G, if $\mathcal{F} = \{e\}$ then $\Omega_n^G\{\{1\}\}(X)$ is the bordism group of tangentially stably complex closed manifolds with a free G–action and an equivariant map to X, which can be identified with the usual unitary bordism group $\Omega_n(EG \times_G X)$.

Similarly, for three families of representations $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3$, we have the corresponding long exact sequence of a triple

$$\cdots \to \Omega_n^G\{\mathcal{F}_2,\mathcal{F}_1\}(X) \to \Omega_n^G\{\mathcal{F}_3,\mathcal{F}_1\}(X) \to \Omega_n^G\{\mathcal{F}_3,\mathcal{F}_2\}(X) \to \Omega_{n-1}^G\{\mathcal{F}_2,\mathcal{F}_1\}(X) \to \cdots.$$

Following the same argument as in [6, Lemma 5.2] we can obtain the next lemma.

Lemma 5.5 *Let $(M^n,\partial M^n, f)$ be a $(\mathcal{F},\mathcal{F}')$–free geometric unitary bordism element of X and W^n a compact manifold with boundary regularly embedded in the interior of M^n and invariant under the G–action. If $G_m \in \mathcal{F}'$ for all $m \in M^n \backslash W^n$, then $[M^n,\partial M^n, f] = [W^n,\partial W_n, f|_{W^n}]$ in $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}(X)$.*

A pair of families $\mathcal{F}' \subset \mathcal{F}$ of subgroups of G is said to be an *adjacent pair of families of groups* if $\mathcal{F}\backslash\mathcal{F}' = (A)$, where A is a subgroup of G and (A) is the set of subgroups conjugate to A in G. We then say that $\mathcal{F}$ and $\mathcal{F}'$ differ by A. Notice that if A is a normal subgroup of G, then a pair of families $\mathcal{F}$ and $\mathcal{F}$ differ by A precisely if $\mathcal{F} = \mathcal{F}' \sqcup \{A\}$. Moreover, in this case, if M is a G–manifold such that $G_m \in \mathcal{F}$ for every $m \in M$, then the fixed-point set M^A has a free action of G/A.

Building on the notation of Theorem 4.1 and Remark 4.2, we denote by $\overline{\mathcal{P}}(n, A)$ the set of arrays $\overline{P} = (n_\rho)_{\rho\in \mathrm{Irr}(A),\, \rho\neq 1}$ of nonnegative integers, where the number n_1 associated to the trivial representation is not considered, such that

$$\sum_{\rho\in \mathrm{Irr}(A),\, \rho\neq 1} n_\rho |\rho| = n.$$

In the above equation n_ρ is a nonnegative integer and $|\rho|$ denotes the complex dimension of the representation ρ. Suppose now that A is a closed and normal subgroup of G. For any such partition $\overline{P}$ we define the space

$$B_{G/A}U(\overline{P}) := \prod_{\rho\in \mathrm{Irr}(A),\rho\neq 1} {}^{(\widetilde{G/A})_\rho} B_{(G/A)_\rho} U(n_\rho)$$

with the G/A–action induced by the homeomorphism shown in Remark 4.2.

As an application of Theorem 4.1 we obtain the following decomposition formula for the geometric G–equivariant unitary equivariant bordism groups. This decomposition is well known for the case of a compact abelian Lie group but is new for the case of nonabelian groups. Since we follow the same line of argument as in the abelian case we only sketch part of its proof.

Theorem 5.6 *Suppose that G is a compact Lie group and let A be a closed normal subgroup of G. If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of G differing by A, then*

$$\Omega^G_n\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0\leq 2k\leq n-\dim(G/A)} \Omega^{G/A}_{n-2k}\{\{1\}\}\Big(X^A \times \bigsqcup_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{G/A}U(\overline{P})\Big),$$

where $\{1\}$ is the family of subgroups of G/A which only contains the trivial group.

Proof We are going to define an isomorphism Φ between these groups. Suppose that $f\colon M \to X$ is an $(\mathcal{F}, \mathcal{F}')$–free geometric unitary bordism of X such that

$$[M, \partial M, f] \in \Omega^G_n\{\mathcal{F}, \mathcal{F}'\}(X).$$

Let $M^A = M_1^A \cup \cdots \cup M_l^A$ be a decomposition of disjoint manifolds, where each M_j^A is an $n_1(j)$–dimensional manifold which is moreover connected. Using the G–equivariant tubular neighborhood theorem, we may find pairwise disjoint tubular neighborhoods U_j of M_j^A in M which are diffeomorphic to the G–manifolds $D(\nu_j)$ through the diffeomorphisms $\phi_j\colon U_j \xrightarrow{\cong} D(\nu_j)$, where $D(\nu_j)$ denotes the unit disk bundle of the G–equivariant normal bundle $\nu_j \to M_j^A$ of the inclusion $M_j^A \subset M$ for $j = 1, \ldots, l$. By Lemma 5.5 we know that

$$[M, \partial M, f] = \sum_{j=1}^{l} [U_j, \partial U_j, f|_{U_j}] = \sum_{j=1}^{l} [D(\nu_j), S(\nu_j), f|_{U_j} \circ \phi_j^{-1}]$$

in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$. The bundle $\nu_j \to M_j^A$ is a complex G–equivariant bundle with the property that the trivial A–representation does not appear on the fibers. Let $r_j = \operatorname{rank}_{\mathbb{C}}(\nu_j)$, so that $n = n_1^j + 2r_j$ for each $j = 1, \ldots, l$ with n_1^j the dimension of M_j^A. By Theorem 4.1 we know that the bundle ν_j is classified by a G/A–equivariant map

$$\kappa_j\colon M_j^A \to \coprod_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}).$$

Now set $f_j\colon M_j^A \to X^A$ to be the restriction of f to M_j^A, and define the product map

$$(f_j, \kappa_j)\colon M_j^A \to X^A \times \coprod_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}).$$

Notice that $0 \le \dim(G/A) \le n_1^j$ and thus the class $[M_j^A, \varnothing, (f_j, \kappa_j)]$ defines an element in $\Omega_{n-2r_j}^{G/A}\{\{1\}\}(X^A \times \coprod_{\overline{P} \in \overline{\mathcal{P}}(r_j, A)} B_{G/A} U(\overline{P}))$ for $j = 1, \ldots, l$. We define

$$\Phi([M, \partial M, f]) := \sum_{j=1}^{l} [M_j^A, \varnothing, (f_j, \kappa_j)].$$

We claim that map Φ is an isomorphism. To see that Φ is surjective, suppose that

$$\left[Y, \varnothing, \varphi\colon Y \to X^A \times \coprod_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})\right] \in \Omega_{n-2k}^{G/A}\{\{1\}\}\left(X^A \times \coprod_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})\right).$$

Let $p\colon E \to Y$ be the G–equivariant complex vector bundle defined by the map $\pi_2 \circ \varphi\colon Y \to \coprod_{\overline{P} \in \overline{\mathcal{P}}(k, A)} B_{G/A} U(\overline{P})$ as is given in Theorem 4.1. Since Y is a closed manifold with a free G/A–action and with a tangentially stably almost complex G/A–structure, the closed unit disk of the bundle $D(E)$ is an n–dimensional manifold endowed with a tangentially stably complex G–structure. Moreover, the boundary

$S(E)$ of $D(E)$ is $\mathcal{F}'$–free since the trivial A–representation does not appear on the fibers of E. Denoting by $\psi\colon D(E) \to X$ the composition of the maps

$$D(E) \to Y \to X^A \hookrightarrow X,$$

where the first is the projection on the base, the second is $\pi_1 \circ \varphi$ and the third is the inclusion, we see that the bordism class $[D(E),\ S(E),\ \psi\colon D(E) \to X]$ lives in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X)$ and, by construction,

$$\Phi([D(E),\ S(E),\ \psi\colon D(E) \to X]) = [Y, \varnothing, \varphi].$$

This proves the surjectivity of Φ. The injectivity of Φ can be proved in a similar way as in the case of a compact abelian Lie group using Theorem 4.1. □

To be able to extend the previous theorem to general subgroups that are not necessarily normal, we consider $G = N_A$ and extend N_A–bordisms to G–bordisms with the change of groups formula [11, Lemma 3.4]: For any subgroup H of a finite group G, we have an isomorphism

$$\begin{aligned} \Omega_*^H(X) &\xrightarrow{\cong} \Omega_*^G(G \times_H X), \\ [M,\ \partial M,\ f\colon M \to X] &\longmapsto [G \times_H M,\ \partial(G \times_H M),\ G \times_H f\colon G \times_H M \to G \times_H X]. \end{aligned}$$

Corollary 5.7 *If $(\mathcal{F}, \mathcal{F}')$ is an adjacent pair of families of subgroups of the finite group G differing by the subgroup A, then*

$$\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}(X) \cong \bigoplus_{0 \le 2k \le n} \Omega_{n-2k}^{W_A}\{\{1\}\}(X^A \times \bigsqcup_{\overline{P} \in \overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})),$$

where $\{1\}$ is the family of subgroups of W_A which only contains the trivial group.

Proof We just need to note that $M^A \cap M^{gAg^{-1}} = \varnothing$ whenever g does not belong to N_A. Therefore, we can choose an N_A–equivariant tubular neighborhood U of M^A in M such that its G–orbit $G \cdot U$ is a G–equivariant tubular neighborhood of $G \cdot M^A$ and such that

$$G \times_{N_A} U \to G \cdot U, \quad [(g, u)] \mapsto gu,$$

is a G–equivariant diffeomorphism. Hence, we have an isomorphism

$$\begin{aligned} \Omega_*^G\{\mathcal{F}, \mathcal{F}'\}(X) &\xrightarrow{\cong} \Omega_*^{N_A}\{\mathcal{F}|_{N_A}, \mathcal{F}'|_{N_A}\}(X), \\ [M,\ \partial M,\ f\colon M \to X] &\longmapsto [U,\ \partial U,\ f|_U\colon U \to X], \end{aligned}$$

which composed with the isomorphism of Theorem 5.6 for the group N_A and its normal subgroup A provides the desired result. □

Let us use the decomposition formula given in Corollary 5.7 of the equivariant bordism groups for adjacent families in the case of finite groups to give an alternative proof of Theorem 1.1 in [19]. Let $(\mathcal{F}, \mathcal{F}')$ be an adjacent pair of families of subgroups of the finite group G differing by the subgroup A, and consider the restriction map from the G–bordisms to A–bordisms. Let $\mathcal{A}$ denote the family of all subgroups of A and let $\mathcal{P}$ denote the family of all subgroups of A besides A itself. The restriction of G–manifolds to A–manifolds gives a homomorphism

$$r_A^G\colon \Omega_n^G\{\mathcal{F}, \mathcal{F}'\} \to \Omega_n^A\{\mathcal{A}, \mathcal{P}\}^{W_A},$$

which lies in the W_A–invariants since the action of the inner automorphisms of G in $\Omega_n^G\{\mathcal{F}, \mathcal{F}'\}$ is trivial (see [5, Section 20]) and the restriction map is N_A–equivariant.

Applying the fixed-point construction done in Theorem 5.6 to both sides of the homomorphism above we obtain the following diagram with horizontal isomorphisms:

$$\begin{array}{ccc} \Omega_n^G\{\mathcal{F}, \mathcal{F}'\} & \xrightarrow{\cong} & \displaystyle\bigoplus_{0\le 2k\le n} \Omega_{n-2k}^{W_A}\{\{1\}\}\Big(\coprod_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big) \\ \Big\downarrow{\scriptstyle r_A^G} & & \Big\downarrow{\scriptstyle r_{\{1\}}^{W_A}} \\ \Omega_n^A\{\mathcal{A}, \mathcal{P}\}^{W_A} & \xrightarrow{\cong} & \displaystyle\bigoplus_{0\le 2k\le n} \Omega_{n-2k}\Big(\coprod_{\overline{P}'\in\overline{\mathcal{P}}(k,A)} BU(\overline{P}')\Big)^{W_A}. \end{array}$$

Note that $B_{W_A}U(\overline{P})$ is a model for $BU(\overline{P})$ and therefore we may take $BU(\overline{P}) := B_{W_A}U(\overline{P})$.

If we tensor with the ring Z_P of P–local integers, where P is the collection of primes which do not divide the order of the group, the right vertical map

$$\Omega_{n-2k}^{W_A}\{\{1\}\}\Big(\coprod_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big)\otimes Z_P \cong \Omega_{n-2k}\Big(\coprod_{\overline{P}\in\overline{\mathcal{P}}(k,A)} B_{W_A}U(\overline{P})\Big)^{W_A}\otimes Z_P$$

induces an isomorphism. Therefore, the restriction map

$$r_A^G\colon \Omega_n^G\{\mathcal{F}, \mathcal{F}'\}\otimes Z_P \to \Omega_n^A\{\mathcal{A}, \mathcal{P}\}^{W_A}\otimes Z_P$$

becomes an isomorphism (see [19, Proposition 3.1]). The spaces $B_{W_A}U(\overline{P})$ are products of $BU(j)$'s and therefore the bordism groups $\Omega_*(B_{W_A}U(\overline{P}))$ are zero in odd degrees and Ω_*–free in even degrees. Since

$$\Omega_*({}^{(\widetilde{W}_A)_\rho}B_{(W_A)_\rho}U(n_\rho))$$

is $(W_A)_\rho$–invariant, the action of W_A on

$$\bigoplus_{\overline{P}\in\overline{\mathcal{P}}(k,A)} \Omega_{n-2k}(B_{W_A}U(\overline{P}))$$

permutes the generators and therefore the W_A invariants are also Ω_*–free. Hence, we conclude that the bordism groups $\Omega_n^G\{\mathcal{F},\mathcal{F}'\}\otimes Z_P$ for adjacent families are $\Omega_*\otimes Z_P$–free in even degrees and zero in odd degrees. Therefore, the short exact sequences

$$0\to\Omega_*^G\{\mathcal{F}'\}\otimes Z_P\to\Omega_*^G\{\mathcal{F}\}\otimes Z_P\to\Omega_*^G\{\mathcal{F},\mathcal{F}'\}\otimes Z_P\to 0$$

are all split for all pairs of families of subgroups of G, $\Omega_*^G\otimes Z_P$ is an $\Omega_*\otimes Z_P$–free module and there is a canonical isomorphism

$$\Omega_*^G\otimes Z_P\cong\bigoplus_{(A)}\Omega_*^A\{\mathcal{A},\mathcal{P}\}^{W_A}\otimes Z_P,$$

where (A) runs over the set of conjugacy classes of subgroups of G (see Theorem 1.1 of [19]).

6 Applications

In this section we use Corollary 5.7 to calculate the Ω_*–module structure of the equivariant unitary bordism groups of the dihedral groups of order $2p$, where p is an odd prime number.

Let $D_{2p}=\langle a,b\mid a^p=b^2=1,\ bab=a^{-1}\rangle$ denote the dihedral group of order $2p$. Notice that $\langle a\rangle\cong\mathbb{Z}/p$ is a normal subgroup of D_{2p} and we have an extension of groups

$$1\to\mathbb{Z}/p\xrightarrow{i}D_{2p}\xrightarrow{\pi}\mathbb{Z}/2\to 1.$$

If we write $\mathbb{Z}/2=\{1,\tau\}$, where $\tau^2=1$, then the map $j(\tau)=b$ defines a splitting of the previous short exact sequence and thus $D_{2p}\cong\mathbb{Z}/p\rtimes\mathbb{Z}/2$.

Let us recall what we know about the unitary bordism groups of free D_{2p}–actions. Denote by S_-^{2i-1} the sphere with the antipodal $\mathbb{Z}/2$–action and by $S_{\lambda^l}^{2k-1}$ the spheres

with the action of $\mathbb{Z}/p$ given by multiplying by $\lambda = e^{2\pi l/p}$. By [13, Corollary 2.5] the bordism classes of the D_{2p}–free unitary manifolds defined by the balanced products $D_{2p} \times_{\mathbb{Z}/2} S^{2i-1}_{-}$ and $D_{2p} \times_{\mathbb{Z}/p} S^{4k-1}_{\lambda^l}$ with $\mathbb{Z}/2 \cong \langle b\rangle$ and $\mathbb{Z}/p \cong \langle a\rangle$ form a generating set of $\widetilde{\Omega}^{D_{2p}}_*\{\{1\}\}$ as an Ω_*–module. In [13, Theorem. 2.6] it is shown that the map

$$i_* \oplus j_*\colon (\widetilde{\Omega}^{\mathbb{Z}/p}_*\{\{1\}\})^{\mathbb{Z}/2} \oplus \widetilde{\Omega}^{\mathbb{Z}/2}_*\{\{1\}\} \xrightarrow{\cong} \widetilde{\Omega}^{D_{2p}}_*\{\{1\}\} \tag{6}$$

induced by the balanced products is an isomorphism of Ω_*–modules, where

$$i_*\big(\tfrac{1}{2}([S^{4k-1}_{\lambda^l}] + [S^{4k-1}_{\lambda^{p-l}}])\big) = [D_{2p} \times_{\mathbb{Z}/p} S^{4k-1}_{\lambda^l}],\ j_*[S^{2i-1}_{-}] = [D_{2p} \times_{\mathbb{Z}/2} S^{2i-1}_{-}].$$

Here $\widetilde{\Omega}^G_*\{\{1\}\} := \widetilde{\Omega}_*(BG)$ denotes the reduced bordism groups of BG; that is, $\widetilde{\Omega}_*(BG)$ is the kernel of the augmentation map $\Omega_*(BG) \to \Omega_*$.

Using [15, Theorem 3] we see that since $H^n(BD_{2p};\mathbb{Z}) = 0$ for all $n \geq 1$ odd, the projective dimension of $\Omega_*(BD_{2p})$ over Ω_* is at most 1. Since $\Omega_*(BD_{2p})$ contains torsion elements we conclude that it is not an Ω_*–projective module and thus it has projective dimension 1 over Ω_*.

For what follows we will use the notation $\Omega^G_+\{\mathcal{F}\}(X) := \bigoplus_{n \text{ even}} \Omega^G_n\{\mathcal{F}\}(X)$ and $\Omega^G_-\{\mathcal{F}\}(X) := \bigoplus_{n \text{ odd}} \Omega^G_n\{\mathcal{F}\}(X)$ for any family $\mathcal{F}$ of subgroups of a finite group G. Therefore, $\Omega^G_*\{\mathcal{F}\}(X) = \Omega^G_+\{\mathcal{F}\}(X) \oplus \Omega^G_-\{\mathcal{F}\}(X)$, and similarly for pairs of families. With this notation we have that $\Omega^{D_{2p}}_+\{\{1\}\} \cong \Omega_+$ and $\Omega^{D_{2p}}_-\{\{1\}\}$ is all torsion. Moreover, we can identify Ω_* with Ω_+ as $\Omega_- = 0$.

The following theorem was originally proved in [16] and we offer here a simpler proof which makes use of the results of the previous sections.

Theorem 6.1 *The unitary bordism group $\Omega^{D_{2p}}_*$ is a free Ω_*–module on even-dimensional generators.*

Proof Consider the families $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ of subgroups of D_{2p} defined by

$$\begin{aligned}
\mathcal{F}_0 &:= \{\{1\}\},\\
\mathcal{F}_1 &:= \{\{1\}, \langle a\rangle\},\\
\mathcal{F}_2 &:= \{\{1\}, \langle a\rangle, \langle b\rangle, \langle aba^{-1}\rangle, \dots, \langle a^{p-1}ba^{1-p}\rangle\} = \text{all} \setminus \{D_{2p}\},\\
\mathcal{F}_3 &:= \text{all}.
\end{aligned}$$

The proof of the theorem will be based on the following facts that will be proved later:

- The unitary bordism group $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module on even-dimensional generators.
- The unitary bordism group $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is a free Ω_*–module.
- The boundary map $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective.
- Both $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ and $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ have projective dimension 1 as modules over Ω_*.
- The boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective.

Using these facts we can prove the theorem as follows. Since $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is trivial, the long exact sequence associated to the families $\mathcal{F}_0\subset\mathcal{F}_1\subset\mathcal{F}_3$ becomes

$$0\to\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\beta}\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\xrightarrow{\delta}\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to 0.$$

Since $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective, the previous exact sequence yields the short exact sequence

$$0\to\operatorname{Im}(\beta)\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\xrightarrow{\delta}\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to 0.$$

We know that $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ has projective dimension 1 as a module over Ω_*. Since $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module, we conclude, by Schanuel's lemma, that $\operatorname{Im}(\beta)$ must be a projective Ω_*–module and hence free by [7, Proposition 3.2]. On the other hand, using the long exact sequence given above we obtain the short exact sequence

$$0\to\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\beta}\operatorname{Im}(\beta)\to 0.$$

As both $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ and $\operatorname{Im}(\beta)$ are free modules over Ω_* we conclude that $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module as well. Moreover, since the boundary map $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective, then $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is trivial. Hence, $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module on even-dimensional generators.

Now, the long exact sequence associated to the families $\mathcal{F}_0\subset\mathcal{F}_3$ becomes

$$0\to\Omega_+\to\Omega_+^{D_{2p}}\xrightarrow{\gamma}\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\partial}\Omega_-^{D_{2p}}\{\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\to 0$$

since $\Omega_+^{D_{2p}}\{\mathcal{F}_0\}\cong\Omega_+$ and $\Omega_-\{\mathcal{F}_3,\mathcal{F}_0\}=0$. We know that the boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective and thus we conclude that $\Omega_-^{D_{2p}}$ is zero. On the other hand, using the previous long exact sequence we obtain the short exact sequence

$$0\to\operatorname{Im}(\gamma)\to\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}\xrightarrow{\partial}\Omega_-^{D_{2p}}\{\mathcal{F}_0\}\to 0.$$

In this short exact sequence we know that $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ has projective dimension 1 as an Ω_*–module and that $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_0\}$ is a free Ω_*–module. By Schanuel's lemma we conclude that $\mathrm{Im}(\gamma)$ is a projective Ω_*–module and hence free by [7, Proposition 3.2]. Finally, using the short exact sequence

$$0 \to \Omega_+ \to \Omega_+^{D_{2p}} \xrightarrow{\gamma} \mathrm{Im}(\gamma) \to 0$$

we conclude that $\Omega_+^{D_{2p}}$ is a free Ω_*–module because Ω_+ and $\mathrm{Im}(\gamma)$ are free as well. The theorem follows. □

Let us now check each one of the facts listed above.

Lemma 6.2 *The unitary bordism group $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is a free Ω_*–module on even-dimensional generators.*

Proof Consider the adjacent pairs of families $(\mathcal{F}_3,\mathcal{F}_2)$ and $(\mathcal{F}_2,\mathcal{F}_1)$ with $A = D_{2p}$ in the first case and $A = \langle b\rangle$ in the second. Since both D_{2p} and $\langle b\rangle$ are their own normalizers in D_{2p}, in both cases the group W_A is trivial. By Corollary 5.7 we know that both $\Omega_*^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_2\}$ and $\Omega_*^{D_{2p}}\{\mathcal{F}_2,\mathcal{F}_1\}$ are isomorphic to unitary bordism groups of copies of $BU(k)$'s and therefore free Ω_*–modules on even-dimensional generators. The long exact sequence associated to the families $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3$ implies that $\Omega_-^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ is trivial and the short exact sequence

$$0 \to \Omega_+^{D_{2p}}\{\mathcal{F}_2,\mathcal{F}_1\} \to \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\} \to \Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_2\} \to 0$$

implies that the middle term is also a free Ω_*–module. □

Lemma 6.3 *The unitary bordism group $\Omega_+^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is a free Ω_*–module.*

Proof By Corollary 5.7 we know that

$$\Omega_*^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\} \cong \Omega_*^{\mathbb{Z}/2}\{\{1\}\}\Bigg(\coprod_{n_1,n_2,\dots,n_{p-1}\in\mathbb{N}} BU(n_1)\times\cdots\times BU(n_{p-1})\Bigg),$$

where the number n_l parametrizes the rank of the irreducible representation of $\mathbb{Z}/p$ given by multiplication of $e^{2\pi l/p}$. The action of $\mathbb{Z}/2$ interchanges the coordinates

$$BU(n_1)\times\cdots\times BU(n_{p-1}) \to BU(n_{p-1})\times\cdots\times BU(n_1),$$
$$(x_1,\dots,x_{p-1}) \mapsto (x_{p-1},\dots,x_1),$$

and it only has fixed points whenever $n_l = n_{p-l}$ for all $1 \leq l \leq \frac{1}{2}(p-1)$. Therefore,

$$\Omega_*^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\} \cong M_* \oplus N_*,$$

where M_* is isomorphic to a direct sum of unitary bordism groups of copies of $BU(k)$'s (thus a free Ω_*–module) and

$$N_* := \bigoplus_{n_1, n_2, \ldots, n_{(p-1)/2} \in \mathbb{N}} \Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2),$$

where $X := \prod_{l_1}^{l=(p-1)/2} BU(n_l)$ and $\mathbb{Z}/2$ acts on X^2 by permutation of the coordinates. Next we study $\Omega_*(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$. The second page of the Atiyah–Hirzebruch spectral sequence becomes

$$E^2_{s,t} \cong H_s(\mathbb{Z}/2, \Omega_t(X^2))$$

and therefore $E^2_{s,\text{odd}} = 0$ and $E^2_{2k,t} = 0$ for $k > 0$. Whenever $s = 0$ we have that the groups $E^2_{0,*}$ are the $\mathbb{Z}/2$–coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$. By [14, Propositions 4.3.2 and 4.3.3] we know that $\Omega_*(BU(k))$ is a free Ω_*–module with basis

$$\{\alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_s} \mid 1 \leq \alpha_{j_1} \leq \cdots \leq \alpha_{j_s},\ s \leq k\},$$

where the degree of $\alpha_{j_1}\cdots\alpha_{j_s}$ is $2(j_1 + \cdots + j_s)$, therefore it follows that the $\mathbb{Z}/2$–coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$ form a free Ω_*–module. Since the odd columns and the even rows are trivial, the vertical axis is a free Ω_*–module and the other components of the first quadrant are $\mathbb{Z}/2$–torsion, the spectral sequence collapses on the second page. This implies that $\Omega_+(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is isomorphic to the coinvariants $\Omega_*(X^2)_{\mathbb{Z}/2}$, and therefore $\Omega_+(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ is a free Ω_*–module. Hence, N_+ is a free Ω_*–module, and therefore $\Omega_+^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is a free Ω_*–module. □

Lemma 6.4 *The boundary map $\delta\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is surjective.*

Proof Following the argument of the proof of the previous lemma it is enough to show that there are elements in $\Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_1\}$ whose boundary correspond in $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ to the generators of $\Omega_-(E\mathbb{Z}/2 \times_{\mathbb{Z}/2} X^2)$ as Ω_*–module for $X = \prod_{l_1}^{l=(p-1)/2} BU(n_l)$.

The bordism group $\Omega_*(X)$ is generated as an Ω_*–module by unitary manifolds $M \to X$ (see [14, Propositions 4.3.2 and 4.3.3]) and therefore the trivial $\mathbb{Z}[\mathbb{Z}/2]$–submodule of $\Omega_*(X^2)$ is generated as an Ω_*–module by the manifolds $M^2 \to X^2$.

We claim that $\Omega_-(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$ is generated as an Ω_*–module by the unitary manifolds

$$S_-^{2i-1}\times_{\mathbb{Z}/2}M^2\to E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2.$$

This follows from the following argument. Consider the maps

$$S_-^{2i-1}\times_{\mathbb{Z}/2}M^2\to E\mathbb{Z}/2\times_{\mathbb{Z}/2}M^2\to E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2,$$

where the first one is induced by the inclusion $S_-^{2i-1}\to S_-^\infty=E\mathbb{Z}/2$ and the second is induced by the map $M^2\to X^2$. If the dimension of M is n, the composition of the maps in homology

$$\begin{aligned}H_{2i+2n-1}(S_-^{2i-1}\times_{\mathbb{Z}/2}M^2)&\to H_{2i+2n-1}(E\mathbb{Z}/2\times_{\mathbb{Z}/2}M^2)\\&\to H_{2i+2n-1}(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)\end{aligned}$$

sends the volume form $[S_-^{2i-1}\times_{\mathbb{Z}/2}M^2]$ to the $\mathbb{Z}/2$–torsion class in the group $H_{2i+2n-1}(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$ which corresponds in $E^2_{2i-1,2n}\cong H_{2i-1}(\mathbb{Z}/2,H_{2n}(X^2))$ of the Serre spectral sequence to the class in $H_{2i-1}(\mathbb{Z}/2,\mathbb{Z}[M^2])\cong\mathbb{Z}/2$. Therefore, the homology classes in $H_-(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$ defined by the volume forms of the unitary manifolds $S_-^{2i-1}\times_{\mathbb{Z}/2}M^2$ generate the homology in odd degrees. This implies that the Thom homomorphism $\mu\colon\Omega_*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)\to H_*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$ is surjective and that the bordism spectral sequence collapses. We conclude that the unitary manifolds $S_-^{2i-1}\times_{\mathbb{Z}/2}M^2\to E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2$ generate $\Omega_-(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$ as an Ω_*–module.

Now consider the complex vector bundle $E\to M$ of rank $n_1+\cdots+n_{(p-1)/2}$ that the map $M\to X=\prod_{l_1}^{l=(p-1)/2}BU(n_l)$ defines, with the appropriate induced action of $\mathbb{Z}/p=\langle a\rangle$ on the fibers. Take the manifold $S_-^{2i-1}\times D(E\times E)$, where $D(E\times E)$ is the disk bundle of $E^2\to M^2$, and define the D_{2p}–action on it as follows: for $(x,y,z)\in S_-^{2i-1}\times D(E\times E)$, let $a\cdot(x,y,z):=(x,ay,a^{-1}z)$ and $b\cdot(x,y,z):=(-x,z,y)$. The class of the D_{2p}–manifold $S_-^{2i-1}\times D(E\times E)$ lies in $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ and corresponds to the class of $S_-^{2i-1}\times_{\mathbb{Z}/2}M^2\to E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2$ in $\Omega_*(E\mathbb{Z}/2\times_{\mathbb{Z}/2}X^2)$. The class of the D_{2p}–manifold $D(\mathbb{C}^i_-)\times D(E\times E)$ lies in $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}$ and its boundary its the class of $S_-^{2i-1}\times D(E\times E)$ in $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$. Hence, the boundary map $\Omega_+^{D_{2p}}\{\mathcal{F}_3,\mathcal{F}_1\}\to\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ is surjective and the lemma follows. □

Lemma 6.5 *As modules over Ω_*, both $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ and $\Omega_-^{D_{2p}}\{\mathcal{F}_1,\mathcal{F}_0\}$ have projective dimension 1.*

Proof Notice that $\Omega_*^{D_{2p}}\{\mathcal{F}_0\} = \Omega_*(BD_{2p})$ and thus it has projective dimension 1 over Ω_*. On the other hand, using the previous lemma we conclude that the Thom map corresponding to $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ is surjective and thus Proposition 4 of [15] implies that $\Omega_-^{D_{2p}}\{\mathcal{F}_1, \mathcal{F}_0\}$ has projective dimension at most 1 as a module over Ω_*. The projective dimension of this module is 1 because it also contains torsion elements. □

Lemma 6.6 *The boundary map* $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_0\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ *is surjective.*

Proof By the isomorphism described in formula (6) we know that the bordism classes $[D_{2p} \times_{\mathbb{Z}/2} S_-^{2i-1}]$ and $[D_{2p} \times_{\mathbb{Z}/p} S_{\lambda^l}^{4k-1}]$ generate $\Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ as an Ω_*-module. Let $D(\mathbb{C}_-^i)$ and $D(\mathbb{C}_{\lambda^l}^{2k})$ denote the disks of the representations of $\mathbb{Z}/2$ and $\mathbb{Z}/p$, respectively, whose boundaries are S_-^{2i-1} and $S_{\lambda^l}^{4k-1}$. The manifolds $D_{2p}\times_{\mathbb{Z}/2} D(\mathbb{C}_-^i)$ and $D_{2p}\times_{\mathbb{Z}/p} D(\mathbb{C}_{\lambda^l}^{2k})$ are both $(\mathcal{F}_3, \mathcal{F}_0)$–free, and their boundaries are $D_{2p} \times_{\mathbb{Z}/2} S_-^{2i-1}$ and $D_{2p} \times_{\mathbb{Z}/p} S_{\lambda^l}^{4k-1}$, respectively. Therefore, the boundary map $\partial\colon \Omega_+^{D_{2p}}\{\mathcal{F}_3, \mathcal{F}_0\} \to \Omega_-^{D_{2p}}\{\mathcal{F}_0\}$ is surjective and the lemma follows. □

References

[1] **A Adem**, **Y Ruan**, *Twisted orbifold K–theory*, Comm. Math. Phys. 237 (2003) 533–556 MR

[2] **M Atiyah**, **G Segal**, *Twisted K–theory and cohomology*, from "Inspired by S S Chern" (P A Griffiths, editor), Nankai Tracts Math. 11, World Sci., Hackensack, NJ (2006) 5–43 MR

[3] **M Bix**, **T tom Dieck**, *Characteristic numbers of G–manifolds and multiplicative induction*, Trans. Amer. Math. Soc. 235 (1978) 331–343 MR

[4] **G Comezaña**, *Calculations in complex equivariant bordism* MR Chapter XXVIII in [17]

[5] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergeb. Math. Grenzgeb. 33, Academic, New York (1964) MR

[6] **P E Conner**, **E E Floyd**, *Maps of odd period*, Ann. of Math. 84 (1966) 132–156 MR

[7] **P E Conner**, **L Smith**, *On the complex bordism of finite complexes*, Inst. Hautes Études Sci. Publ. Math. 37 (1969) 117–221 MR

[8] **T tom Dieck**, *Orbittypen und äquivariante Homologie, I*, Arch. Math. (Basel) 23 (1972) 307–317 MR

[9] **T tom Dieck**, *Transformation groups*, De Gruyter Studies in Mathematics 8, de Gruyter, Berlin (1987) MR

[10] **J M Gómez**, **B Uribe**, *A decomposition of equivariant K–theory in twisted equivariant K–theories*, Internat. J. Math. 28 (2017) art. id. 1750016 MR

[11] **J P C Greenlees**, **J P May**, *Localization and completion in complex bordism* MR Chapter XXVI in [17]

[12] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR

[13] **M Kamata**, **H Minami**, *Bordism groups of dihedral groups*, J. Math. Soc. Japan 25 (1973) 334–341 MR

[14] **S O Kochman**, *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs 7, Amer. Math. Soc., Providence, RI (1996) MR

[15] **P S Landweber**, *Complex bordism of classifying spaces*, Proc. Amer. Math. Soc. 27 (1971) 175–179 MR

[16] **C Lazarov**, *Actions of groups of order pq*, Trans. Amer. Math. Soc. 173 (1972) 215–230 MR

[17] **J P May**, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics 91, Amer. Math. Soc., Providence, RI (1996) MR

[18] **E Ossa**, *Unitary bordism of abelian groups*, Proc. Amer. Math. Soc. 33 (1972) 568–571 MR

[19] **R J Rowlett**, *The fixed-point construction in equivariant bordism*, Trans. Amer. Math. Soc. 246 (1978) 473–481 MR

[20] **G Segal**, *Equivariant K–theory*, Inst. Hautes Études Sci. Publ. Math. 34 (1968) 129–151 MR

[21] **R E Stong**, *Complex and oriented equivariant bordism*, from "Topology of manifolds" (J C Cantrell, J C H Edwards, editors), Markham, Chicago (1970) 291–316 MR

AÁ: *Departamento de Matemáticas, Universidad de los Andes*
Bogotá, Colombia

AÁ, BU: *Departamento de Matemáticas y Estadística, Universidad del Norte*
Barranquilla, Colombia

JMG: *Escuela de Matemáticas, Universidad Nacional de Colombia sede Medellín*
Medellin, Colombia

ajangel@uninorte.edu.co, jmgomez0@unal.edu.co,
bjongbloed@uninorte.edu.co

https://sites.google.com/a/unal.edu.co/jmgomez0/,
https://sites.google.com/site/bernardouribejongbloed/

Received: 15 November 2017 Revised: 8 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 4037–4068

Detecting a subclass of torsion-generated groups

EMILY STARK

We classify the groups quasi-isometric to a group generated by finite-order elements within the class of one-ended hyperbolic groups which are not Fuchsian and whose JSJ decomposition over two-ended subgroups does not contain rigid vertex groups. To do this, we characterize which JSJ trees of a group in this class admit a cocompact group action with quotient a tree. The conditions are stated in terms of two graphs we associate to the degree refinement of a group in this class. We prove there is a group in this class which is quasi-isometric to a Coxeter group but is not abstractly commensurable to a group generated by finite-order elements. Consequently, the subclass of groups in this class generated by finite-order elements is not quasi-isometrically rigid. We provide necessary conditions for two groups in this class to be abstractly commensurable. We use these conditions to prove there are infinitely many abstract commensurability classes within each quasi-isometry class of this class that contains a group generated by finite-order elements.

20F65; 20E08, 20F55, 57M07, 57M20

1 Introduction

The large-scale geometry type of a finitely generated group does not depend on whether the group contains elements of finite order; every finitely generated group is quasi-isometric to a group that contains torsion. However, quasi-isometry classes that contain groups generated by finite-order elements are distinguishable. Torsion-generated groups, such as Coxeter groups, play an important role in geometric group theory. Background is given by Davis [18]. An interesting problem is to determine which finitely generated groups are quasi-isometric to a group generated by finite-order elements. In this paper, we solve this problem within a certain class of hyperbolic groups.

A natural approach to this problem begins by decomposing the group using a graph of groups decomposition. Dunwoody [20] proved every finitely presented group admits a maximal splitting as a graph of groups with finite edge groups, and Papasoglu and Whyte [30] proved for an infinite-ended finitely presented group, the set of quasi-isometry classes of the one-ended vertex groups in this graph of groups decomposition is a complete quasi-isometry invariant. Therefore, an infinite-ended finitely presented

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4037

group is quasi-isometric to a group generated by finite-order elements if and only if each of the one-ended vertex groups in this graph of groups decomposition is quasi-isometric to a group generated by finite-order elements. Thus, for finitely presented groups, the problem reduces to the case the group is one-ended.

Rips and Sela [32] proved if G is a one-ended finitely presented group that is not Fuchsian, then there is a canonical graph of groups decomposition of G, called the *JSJ decomposition of G*, with edge groups that are 2–ended and vertex groups of three types: 2–ended, *maximally hanging Fuchsian* (MHF), and quasiconvex *rigid* vertex groups not of the first two types. In this paper, we follow the language and structure of the JSJ decomposition due to Bowditch [5] for one-ended hyperbolic groups that are not Fuchsian. We characterize the groups quasi-isometric to a group generated by finite-order elements within the class $\mathcal{C}$ of 1–ended hyperbolic groups that are not Fuchsian and whose JSJ decomposition does not contain rigid vertex groups.

The isomorphism type of the Bass–Serre tree of the JSJ decomposition of a group in $\mathcal{C}$ is a complete quasi-isometry invariant, as shown by Malone [29] for a subclass of groups in $\mathcal{C}$ called geometric amalgams of free groups and by Cashen and Martin [9] in the general setting; see also related work of Dani and Thomas [17]. Furthermore, there is a one-to-one correspondence between isomorphism types of JSJ trees of groups in $\mathcal{C}$ and (equivalence classes of) certain finite matrices, called *degree refinements*, which are algorithmically computed from the JSJ decomposition.

A group generated by finite-order elements does not surject onto $\mathbb{Z}$. Consequently, if a group $G \in \mathcal{C}$ is generated by finite-order elements, then the underlying graph of the JSJ decomposition of G is a tree; we call this graph the *JSJ graph of G*. Conversely, Dani, Stark and Thomas [15, Theorem 1.16] proved if a quasi-isometry class in $\mathcal{C}$ contains a group whose JSJ graph is a tree, then the quasi-isometry class contains a right-angled Coxeter group. Therefore, classifying the quasi-isometry classes within $\mathcal{C}$ which contain a group generated by finite-order elements is equivalent to classifying the JSJ trees of a group in $\mathcal{C}$ which admit a cocompact group action with quotient a tree. To accomplish this, we introduce two graphs associated to the degree refinement of a group $G \in \mathcal{C}$: the *graph of blocks* of G and the *augmented graph of blocks* of G. The graph of blocks of G has vertex set in one-to-one correspondence with orbits of vertices in the JSJ tree of G under the action of the full isometry group of the tree. The augmented graph of blocks of G is the graph with fewest vertices and degree refinement equivalent to the degree refinement of G. See Section 3 for more details. The first main result of the paper is the following:

Theorem 1.1 *Let $G \in \mathcal{C}$. The following are equivalent:*

(1) *The group G is quasi-isometric to a right-angled Coxeter group.*

(2) *The group G is quasi-isometric to a group generated by finite-order elements.*

(3) *The group G is quasi-isometric to a group with JSJ graph a tree.*

(4) *The degree refinement of G satisfies the two conditions:*

  (M1) *The graph of blocks of G is a tree.*

  (M2) *The augmented graph of blocks of G has no 2–cycles at even distance bounded by Type I vertices.*

The equivalence of conditions (1)–(3) was established in [15]; see Section 4.3. We prove here that (1)–(3) are equivalent to (4), a pair of conditions defined in Section 3 that can be easily verified. Thus, from the underlying graph of the JSJ decomposition of a group in $\mathcal{C}$, one may determine whether conditions (1)–(3) hold. The graphs in condition (4) are quasi-isometry invariants (Corollary 3.8), and we prove conditions (M1) and (M2) hold for a group in $\mathcal{C}$ with JSJ graph a tree. Conversely, if conditions (M1) and (M2) hold, we perform a finite series of moves on the augmented graph of blocks of the group to obtain a finite tree with an equivalent degree refinement.

Two groups are *abstractly commensurable* if the groups contain finite-index subgroups that are isomorphic; two finitely generated groups that are abstractly commensurable are quasi-isometric. Hence, the result in Theorem 1.1 gives a partial answer to [15, Question 1.3], which asks which geometric amalgams of free groups are abstractly commensurable to a right-angled Coxeter group. However, the next result proves the set of groups quasi-isometric to right-angled Coxeter groups in $\mathcal{C}$ is strictly larger than the set of groups abstractly commensurable to a right-angled Coxeter group in $\mathcal{C}$.

Theorem 1.2 *There exists $G \in \mathcal{C}$ for which the following holds:*

(1) *The group G is quasi-isometric to a right-angled Coxeter group.*

(2) *The group G is not abstractly commensurable to any group with JSJ graph a tree. In particular, G is not abstractly commensurable to any group generated by finite-order elements.*

To prove Theorem 1.2, we construct G in Construction 5.7 as the fundamental group of a union of surfaces with boundary glued together along their boundary components. While the JSJ tree of $G \cong \pi_1(X)$ has a finite quotient which is a tree, there is an asymmetry in the Euler characteristics of certain subsurfaces in the space X that yields the commensurability result.

A class of groups $\mathcal{G}$ is *quasi-isometrically rigid* if every group quasi-isometric to a group in $\mathcal{G}$ is abstractly commensurable to a group in $\mathcal{G}$. (A slightly different notion of quasi-isometric rigidity requires every group quasi-isometric to a group in $\mathcal{G}$ to be *virtually isomorphic* to a group in $\mathcal{G}$. These notions are equivalent within $\mathcal{C}$ as groups in $\mathcal{C}$ are virtually torsion-free; see Haïssinsky, Paoluzzi and Walsh [22, Observation 3.1] and the book by Druțu and Kapovich [19] for background.) The class of Coxeter groups is not quasi-isometrically rigid. For example, Burger and Mozes [7] provided examples of (nonhyperbolic) infinite simple groups which act geometrically on the product of two finite-valence trees. Such groups have no finite-index subgroups, yet these groups are quasi-isometric to the direct product of two free groups of rank greater than one, and, hence, are quasi-isometric to a right-angled Coxeter group with defining graph a complete bipartite graph with vertex sets of size greater than two. The class of groups $\mathcal{C}$ is quasi-isometrically rigid by the construction of the JSJ decomposition given by Bowditch [5]; see [22, Observation 3.1] for a related result. Theorem 1.2 proves the set of right-angled Coxeter groups within the class $\mathcal{C}$ is not quasi-isometrically rigid, and we also have the following corollary:

Corollary 1.3 *The subclass of groups in $\mathcal{C}$ which have JSJ graph a tree is not quasi-isometrically rigid. The subclass of groups in $\mathcal{C}$ which are generated by finite-order elements is not quasi-isometrically rigid.*

There are classes of right-angled Coxeter groups which are quasi-isometrically rigid; simple examples include virtually free right-angled Coxeter groups and right-angled Coxeter groups which act properly and cocompactly by isometries on the hyperbolic plane; see Casson and Jungreis [10], Gabai [21] and Tukia [35]. Determining the classes of right-angled Coxeter groups which are quasi-isometrically rigid is an interesting problem. A natural focus is classes for which quasi-isometry invariants or classification is known; see Behrstock, Hagen and Sisto [3; 2], Caprace [8], Charney and Sultan [12], Dani and Thomas [16], Haulmark, Nguyen and Tran [23] and Levcovitz [28; 27].

The abstract commensurability classification within $\mathcal{C}$ remains open. This classification may be an important step in resolving [15, Question 1.3]. Partial results are given by Crisp and Paoluzzi [13], Malone [29], the author [34] and Dani, Stark and Thomas [15]. These known results impose strong conditions on the subclass of groups considered: for example, they require the diameter of the JSJ graph to be at most 4. In Propositions 6.3 and 6.8 we prove two necessary conditions for commensurability for any geometric amalgam of free groups in $\mathcal{C}$ with JSJ graph a tree. The commensurability invariants are the

commensurability classes of two vectors whose entries record the sum of the Euler characteristics of certain vertex groups. As a consequence, we prove the following theorem:

Theorem 1.4 *There are infinitely many abstract commensurability classes within every quasi-isometry class in $\mathcal{C}$ that contains a group generated by finite-order elements.*

The conclusion of Theorem 1.4 is not surprising based on the previous results on commensurability for groups in $\mathcal{C}$. The novelty of the result comes from removing the hypothesis that the JSJ graphs have small diameter. Moreover, previous results strongly use a topological rigidity theorem of Lafont [25], which applies only to geometric amalgams of free groups in $\mathcal{C}$. The result in Proposition 6.3 does not require this hypothesis.

The main questions addressed in this paper, whether a given group is quasi-isometric or abstractly commensurable to a right-angled Coxeter group (or, more generally, a group generated by finite-order elements), may be viewed as coarse versions of the problem of determining whether a given group is a right-angled Coxeter group (or a group generated by finite-order elements); see, for example, Charney, Ruane, Stambaugh and Vijayan [11] and Cunningham, Eisenberg, Piggott and Ruane [14].

Outline

Preliminaries are given in Section 2. The degree refinement of a group in $\mathcal{C}$ and related graphs are defined in Section 3. Theorem 1.1 is proven in Section 4. Section 5 contains the proof of Theorem 1.2. Section 6 contains the proof of necessary conditions for commensurability. Theorem 1.4 is proven in Section 7.

Acknowledgements

The author is thankful for helpful discussions with Pallavi Dani, Misha Kapovich, Michah Sageev, Anne Thomas and Genevieve Walsh. The author is grateful to the anonymous referee for thoughtful comments and corrections.

The author is thankful for support from the Azrieli Foundation. The author was partially supported at the Technion by a Zuckermann STEM Leadership Postdoctoral Fellowship.

2 Preliminaries

2.1 Graph theory

In this section, we record relevant graph-theoretic terminology and establish notation. Most graphs we consider are unoriented, and we view these graphs as CW–complexes.

Let $\Lambda = (V(\Lambda), E(\Lambda))$ be a graph, where $V(\Lambda)$ is the vertex set of Λ and $E(\Lambda)$ is the edge set of Λ. If $e = (u, v) \in E(\Lambda)$, we say e is *incident* to the vertices u and v, and we say u and v are *adjacent* vertices. The *valence* of a vertex u is the number of edges incident to u. A graph is *bipartite* if $V(\Lambda)$ is the disjoint union of two nonempty subsets $V(\Lambda) = V_1 \sqcup V_2$ such that every edge of Λ is incident to exactly one element of V_1 and exactly one element of V_2. A *tree* is a connected graph that does not contain an embedded cycle. An *oriented graph* Λ consists of a vertex set $V(\Lambda)$, an edge set $E(\Lambda)$ and maps $i\colon E(\Lambda) \to V(\Lambda)$ and $t\colon E(\Lambda) \to V(\Lambda)$ such that $e = (i(e), t(e))$ for each edge $e \in E(\Lambda)$; we refer to $i(e)$ as the *initial vertex* of e and $t(e)$ as the *terminal vertex* of e. If $S \subset V(\Lambda)$, the subgraph of Λ *induced by* the vertices in S is the subgraph whose vertex set is S and whose edge set consists of all edges in $E(\Lambda)$ that have both endpoints in S.

2.2 JSJ decomposition and the class of groups considered

Definition 2.1 A *graph of groups* $\mathcal{G}$ is a graph $\Lambda = (V(\Lambda), E(\Lambda))$ with a *vertex group* G_v for each $v \in V(\Lambda)$, an *edge group* G_e for each $e \in E(\Lambda)$, and *edge maps*, which are injective homomorphisms $\Theta_e^{\pm}\colon G_e \to G_{\pm e}$ for each $e = (-e, +e) \in E(\Lambda)$. The graph Λ is called the *underlying graph* of $\mathcal{G}$.

A *graph of spaces* associated to a graph of groups $\mathcal{G}$ is a space X with a graph Λ constructed from a pointed *vertex space* (X_v, x_v) for each $v \in V(\Lambda)$ with $\pi_1(X_v, x_v) = G_v$, a pointed *edge space* (X_e, x_e) for each $e \in E(\Lambda)$ such that $\pi_1(X_e, x_e) = G_e$, and maps $\theta_e^{\pm}\colon (X_e, x_e) \to (X_{\pm e}, x_{\pm e})$ such that $(\theta_e^{\pm})_* = \Theta_e^{\pm}$. The space X is the union

$$\Bigg(\bigsqcup_{v \in V(\Lambda)} X_v \bigsqcup_{e \in E(\Lambda)} (X_e \times [-1, 1]) \Bigg) \Big/ \{(x, \pm 1) \sim \theta_e^{\pm}(x) \mid (x, \pm 1) \in X_e \times [-1, 1]\}.$$

The *fundamental group* of the graph of groups $\mathcal{G}$ is $\pi_1(X)$. A group G *splits as graph of groups* if G is the fundamental group of a nontrivial graph of groups.

Definition 2.2 A *Fuchsian group* is a nonelementary finitely generated group which acts properly discontinuously on the hyperbolic plane $\mathbb{H}^2$.

Remark 2.3 The action of a Fuchsian group G on the hyperbolic plane need not be faithful, but the kernel of the action is finite. This kernel is the unique maximal finite normal subgroup of G. Thus, the quotient $X = \mathbb{H}^2/G$ is a canonically defined (*ineffective*) orbifold. In particular, X admits a decomposition as a finite cell complex

such that each cell σ of X is equipped with a finite isotropy group K_σ which is isomorphic to the stabilizer of each lift of σ in $\mathbb{H}^2$.

Definition 2.4 Let G be a Fuchsian group so that $G \cong \pi_1^{\mathrm{orb}}(X)$, where $X = \mathbb{H}^2/G$ is an (ineffective) orbifold. Realize X with a cell decomposition such that each cell σ of X has a well-defined isotropy subgroup K_σ. The *Euler characteristic of* G, denoted by $\chi(G)$, is

$$\chi(G) = \chi(X) = \sum_{\sigma \text{ cell in } X} (-1)^{\dim(\sigma)} \frac{1}{|K_\sigma|}.$$

Lemma 2.5 *Let G and H be Fuchsian groups. If H is an index-d subgroup of G, then $d \cdot \chi(G) = \chi(H)$.*

For background on orbifolds, see Kapovich [24] and Ratcliffe [31]. The proof of Lemma 2.5 is also given by the more general theory presented by Brown [6, Chapter IX-7].

Definition 2.6 A *bounded Fuchsian group* is a Fuchsian group that is convex cocompact but not cocompact. The convex core of the quotient is a compact orbifold with nonempty boundary consisting of a disjoint union of compact 1–orbifolds. The *peripheral subgroups* are the maximal two-ended subgroups which project to the fundamental groups of the boundary 1–orbifolds. A *hanging Fuchsian* subgroup H is a virtually free quasiconvex subgroup together with a collection of *peripheral* two-ended subgroups, which arise from an isomorphism of H with a bounded Fuchsian group. A *full quasiconvex subgroup* of a group G is a subgroup that is not a finite-index subgroup of any strictly larger subgroup of G.

Theorem 2.7 [5, Theorem 0.1] *Let G be a one-ended hyperbolic group that is not Fuchsian. There is a canonical* ***JSJ decomposition*** *of G as the fundamental group of a graph of groups such that each edge group is 2–ended and each vertex group is either* (1) *2–ended,* (2) *maximal hanging Fuchsian, or* (3) *a maximal quasiconvex subgroup not of type* (2)*. These types are mutually exclusive, and no two vertices of the same type are adjacent. Every vertex group is a full quasiconvex subgroup. Moreover, the edge groups that connect to any given vertex group of type* (2) *are precisely the peripheral subgroups of that group.*

Definition 2.8 Let G be a one-ended hyperbolic group that is not Fuchsian. The *JSJ tree* of G is the Bass–Serre tree of the JSJ decomposition of G. The *JSJ graph* of G is the underlying graph of the JSJ decomposition of G.

Definition 2.9 (class of groups considered) Let $\mathcal{C}$ denote the class of one-ended hyperbolic groups which are not Fuchsian and for which the JSJ decomposition has no vertex groups of type (3).

Remark 2.10 If $G \in \mathcal{C}$, then the JSJ graph of G is bipartite.

2.3 Hyperbolic P–manifolds

We make use of the following subclass of groups in $\mathcal{C}$. An example of a space defined below is given in Figure 3.

Definition 2.11 A *2–dimensional hyperbolic P–manifold X* is a space with a graph of spaces decomposition over a finite oriented graph Λ with the following properties:

(1) The underlying graph Λ is bipartite with vertex set $V(\Lambda) = V_1 \sqcup V_2$ and edge set $E(\Lambda)$ such that each edge $e \in E(\Lambda)$ has $i(e) \in V_1$ and $t(e) \in V_2$.

(2) For each $u \in V_1$, the vertex space X_u is a copy of the circle S^1, and x_u is a point on X_u. For each $v \in V_2$, the vertex space X_v is a connected surface with negative Euler characteristic and nonempty boundary, and x_v is a point on X_v.

(3) For each edge $e \in E(\Lambda)$, the edge space X_e is a copy of the circle S^1, and x_e is a point on X_e. If $e = (-e, +e)$ with $-e = i(e)$ and $+e = t(e)$, the map $\theta_e^-\colon (X_e, x_e) \to (X_{i(e)}, x_{i(e)})$ is a homeomorphism, and the map $\theta_e^+\colon (X_e, x_e) \to (X_{t(e)}, x_{t(e)})$ is a homeomorphism onto a boundary component of $X_{t(e)}$.

(4) Each vertex $u \in V_1$ has valence at least three. Given any vertex $v \in V_2$, for each boundary component B of X_v, there exists an edge e with $t(e) = v$ such that the associated edge map identifies X_e with B. The valence of v is the number of boundary components of X_v.

The fundamental group of a 2–dimensional hyperbolic P–manifold is a *geometric amalgam of free groups* and is a group in the class $\mathcal{C}$. If X is a 2–dimensional hyperbolic P–manifold, a *connected subsurface in X* is the union of a surface X_v with $v \in V_2$ together with $\{X_e \times [-1, 1] \mid e \text{ is incident to } v\}$, a set of annuli. (This subsurface is homeomorphic to X_v and its boundary components are *branching curves* on X.) A *subsurface in X* is a nonempty finite union of connected subsurfaces in X.

Remark 2.12 If $X = X(\Lambda)$ is a 2–dimensional hyperbolic P–manifold, then the JSJ graph of the geometric amalgam of free groups $\pi_1(X)$ is the (unoriented) bipartite graph Λ. For details, see [29, Section 4.1].

Remark 2.13 "P–manifold" is short for "piecewise-manifold". Lafont [25] refers to 2–dimensional hyperbolic P–manifolds as *simple, thick, 2–dimensional hyperbolic P–manifolds*; we omit the extra adjectives for ease of exposition. In [15], these spaces are referred to as *surface amalgams*.

3 Degree refinement and related graphs

Angluin [1, Section 6] proved two finite graphs have isomorphic universal covers if and only if the graphs have equivalent degree refinements, which is a matrix defined below; see also Leighton [26]. Malone [29] extended this work by defining the degree refinement for a group $G \in \mathcal{C}$ and proving this matrix encodes the isomorphism type of the JSJ tree of G. Examples of the definitions given in this section appear in Figure 1.

Definition 3.1 The *degree partition* of a graph Λ is a partition of the vertices of Λ into the minimum number of blocks $M_1, \dots, M_n$ such that there exist constants m_{ij} such that for each i and j with $1 \le i \le n$ and $1 \le j \le n$, each vertex in M_i is connected via m_{ij} edges to M_j. The *degree refinement* of Λ is the $n \times n$ matrix $M = (m_{ij})$.

Definition 3.2 Two degree refinements M and M' are *equivalent* if they have the same size and there exists a permutation matrix P such that $M' = PMP^T$.

The permutation matrix in the above definition accounts for possibly relabeling the blocks of the degree partition. There is a correspondence between isomorphism types of trees and equivalence classes of matrices given as follows.

Theorem 3.3 [26, Section 2; 29, Theorem 2.32] *To each matrix M with entries in $\mathbb{N} \cup \{\infty\}$ there is a unique tree T up to graph isomorphism such that M is the degree refinement of T. Conversely, to each tree T there is a unique matrix M with entries in $\mathbb{N} \cup \{\infty\}$ up to the equivalence defined in Definition 3.2 such that M is the degree refinement of T.*

Definition 3.4 If $G \in \mathcal{C}$, then the *degree refinement of G* is the degree refinement of the JSJ tree of G. Alternatively, the degree refinement of G can be constructed from the JSJ graph for G as follows, which was shown by Malone [29, Section 2.5], extending [26, Section 2].

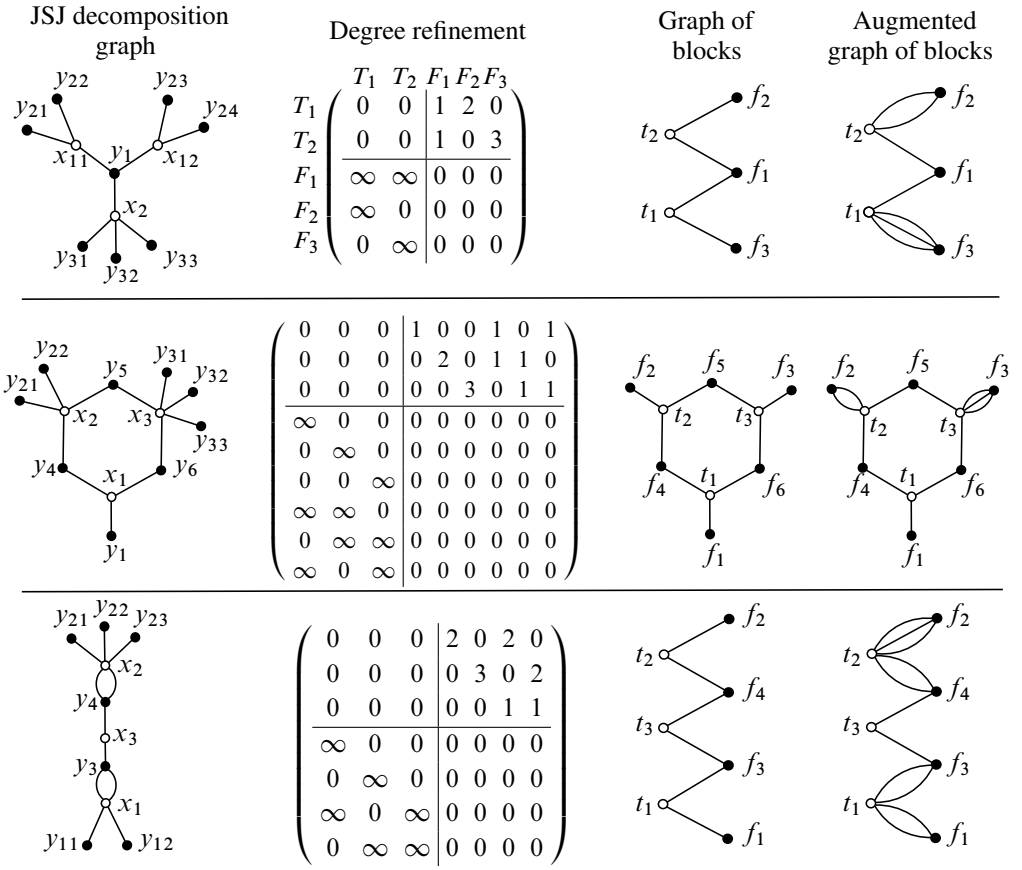


Figure 1: Three examples. White vertices correspond to 2–ended vertex groups, and black vertices correspond to maximal hanging Fuchsian vertex groups. Vertices x_i and x_{ij} are in degree partition block T_i, and vertices y_i and y_{ij} are in degree partition block F_i. In each matrix the rows and columns corresponding to the T_i come first, then those for the F_i. Any group in $\mathcal{C}$ with a JSJ decomposition graph of the lower two types is not quasi-isometric to any group generated by finite-order elements.

Suppose G has JSJ graph Λ with vertex set $V_1 \sqcup V_2$ where each vertex group G_u for $u \in V_1$ is two-ended, and each vertex group G_v for $v \in V_2$ is maximal hanging Fuchsian.

Let $r \in V(\Lambda)$. The *augmented valence of* r is the valence of any lift of r in the JSJ tree. More specifically, suppose $s \in V(\Lambda)$. Let $\iota(r,s) = \infty$ if $r \in V_2$ and r is adjacent to s. Let $\iota(r,s) = k$ if $r \in V_1$ and r is adjacent to s via k edges. Let $\iota(r,s) = 0$ otherwise. The augmented valence of the vertex r is equal to $\sum_{s \in V(\Lambda)} \iota(r,s)$.

Perform the following steps to compute the degree refinement:

Step 1 Partition the vertices of $V(\Lambda)$ into blocks according to their augmented valence.

Step 2 Refine the partition so that two vertices r and r' remain in the same block M_i if and only if $\sum_{s\in M_j} \iota(r,s) = \sum_{s\in M_j} \iota(r',s)$ for all $j \neq i$.

Step 3 Repeat Step 2 recursively until no further partitioning is possible to obtain the *degree partition of* Λ.

The degree refinement is the matrix $M = (m_{ij})$ where $m_{ij} = \iota(r_i, r_j)$ for $r_i \in M_i$ and $r_j \in M_j$. The process is finite since $V(\Lambda)$ is finite.

Malone [29] proved the following theorem for geometric amalgams of free groups using techniques of Behrstock and Neumann [4]. Cashen and Martin [9] proved the remaining cases.

Theorem 3.5 [29, Theorem 4.14; 9, Theorem 4.9] *Let $G, G' \in \mathcal{C}$. The groups G and G' are quasi-isometric if and only if the degree refinement of G is equivalent to the degree refinement of G'.*

Bipartite graphs considered in this paper arise as a JSJ graph of a group in $\mathcal{C}$, which leads to the following definition of the degree refinement for a bipartite graph. We caution the reader that this is not the same as the usual notion of degree refinement for the graph (without a specified bipartite structure).

Definition 3.6 If Λ is a bipartite graph with $V(\Lambda) = V_1 \sqcup V_2$, define the *degree refinement of* Λ to be the degree refinement of a group with JSJ graph Λ as defined in Definition 3.4. Similarly, define the *degree partition of* Λ to be the degree partition of a group with JSJ graph Λ as defined in Definition 3.4.

Suppose in the degree partition of Λ, the vertices in V_1 are contained in blocks $T_1, \dots, T_n$ and the vertices in V_2 are in blocks $F_1, \dots F_m$. (We use "T" for two-ended and "F" for hanging Fuchsian.) Let M be the degree refinement defined above. Let n_{ij} be the entry in the degree refinement corresponding to the blocks T_i and F_j. The *graph of blocks of* M, denoted by $\Gamma_{\mathcal{B}}$, has vertex set $\{t_1, \dots, t_n, f_1, \dots, f_m\}$ and an edge $\{t_i, f_j\}$ if and only if $n_{ij} > 0$. The *augmented graph of blocks of* M, denoted by Γ_0, has vertex set $\{t_1, \dots, t_n, f_1, \dots, f_m\}$ and n_{ij} edges from t_i to n_j for $1 \leq i \leq n$ and $1 \leq j \leq m$.

If $G \in \mathcal{C}$ with JSJ graph Λ as above, then the *graph of blocks of* G is the graph of blocks of the degree refinement of G; the *augmented graph of blocks of* G is the augmented graph of blocks of the degree refinement of G. Similarly, if Λ is a bipartite graph, then the *graph of blocks of* Λ is the graph of blocks of any group $G \in \mathcal{C}$ with JSJ graph Λ; the *augmented graph of blocks of* Λ is the augmented graph of blocks of G.

Remark 3.7 The bipartite graph Γ_0 defined in Definition 3.6 has degree refinement M.

The next corollary follows from Theorem 3.5.

Corollary 3.8 *Let $G, G' \in \mathcal{C}$, let $\Gamma_{\mathcal{B}}$ and $\Gamma'_{\mathcal{B}}$ denote the graphs of blocks for G and G', respectively, and let Γ_0 and Γ'_0 denote the augmented graphs of blocks of G and G', respectively. If G and G' are quasi-isometric, then $\Gamma_{\mathcal{B}} \cong \Gamma'_{\mathcal{B}}$ and $\Gamma_0 \cong \Gamma'_0$.*

4 Characterization up to quasi-isometry

4.1 Obstructions

In this section, we prove that conditions (1)–(3) of Theorem 1.1 imply condition (4). The first lemma generalizes [15, Example 8.1]. An example of a group that does not satisfy the hypothesis of Lemma 4.1 is given in the middle example in Figure 1; an example of a group which does not satisfy the hypothesis of Lemma 4.3 is given in the bottom example in Figure 1.

Lemma 4.1 *Suppose $G \in \mathcal{C}$ has JSJ graph a tree T. Then the graph of blocks of G is a tree.*

Proof Suppose $G \in \mathcal{C}$ has JSJ graph a tree T, and let $\Gamma_{\mathcal{B}}$ be the graph of blocks of G. Suppose towards a contradiction that $\Gamma_{\mathcal{B}}$ is not a tree. Then $\Gamma_{\mathcal{B}}$ contains an embedded cycle γ. Without loss of generality, suppose the vertices in the cycle γ are labeled $\{t_1, f_1, t_2, \dots, t_k, f_k\}$ with t_i adjacent to f_i and f_{i-1}, with indices taken mod k, and likewise for f_i. Let $T_1, \dots T_k, F_1, \dots F_k$ be the corresponding blocks in the degree partition of T. Consider the subgraph of T induced by the vertices in $\bigcup_{i=1}^{k} T_i \cup F_i$. Choose a connected component C of the subgraph; then C must be a finite tree. However, each vertex in C has valence at least two. That is, if $v_i \in F_i$, then v_i is adjacent to some vertices $u_i \in T_i$ and $u_{i+1} \in T_{i+1}$, with indices taken mod k. Since the cycle γ is embedded, $T_i \cap T_{i+1} = \varnothing$; so, $u_i \neq u_{i+1}$. Similarly, if $v_i \in T_i$, v_i has valence at least two, a contradiction. □

Definition 4.2 Let $G \in \mathcal{C}$ with degree refinement and related graphs as defined in Definition 3.4. We say *the augmented graph of blocks of G has no 2–cycles at even distance bounded by Type I vertices* if for any embedded path $t_1, f_1, t_2, \dots, f_{k-1}, t_k$ in Γ_0 with $t_i \in V_1$ and $f_j \in V_2$, if $n_{11} > 1$, then $n_{ij} > 1$ only if $i = j$.

Lemma 4.3 *Suppose $G \in \mathcal{C}$ has JSJ graph a tree. Then the augmented graph of blocks of G has no 2–cycles at even distance bounded by Type I vertices.*

Proof Suppose $G \in \mathcal{C}$ has JSJ graph a tree. By Lemma 4.1, the graph of blocks of G is a tree. Thus, in the notation of Definition 4.2, $n_{ij} \geq 1$ only if $i = j$ or $i = j+1$. Suppose towards a contradiction that $n_{\ell,\ell-1} > 1$ for some ℓ with $1 \leq \ell \leq k$. Let $T_1, \dots, T_\ell \subset V_1$ and $F_1, \dots, F_{\ell-1} \subset V_2$ be the corresponding blocks in the degree partition of T. Consider the subgraph of T induced by the vertices in $\left(\bigcup_{i=1}^{\ell} T_i\right) \cup \left(\bigcup_{i=1}^{\ell-1} F_i\right)$. Choose a connected component C of this subgraph; then C must be a finite tree. However, each vertex in C has valence at least two. That is, for $v_i \in F_i$, $1 \leq i \leq \ell - 1$, v_i is adjacent to some vertices $u_i \in T_i$ and $u_{i+1} \in T_{i+1}$ with $u_i \neq u_{i+1}$ since $T_i \cap T_{i+1} = \varnothing$. Likewise, $v_i \in T_i$ has valence two for $2 \leq i \leq \ell - 1$. Since $n_{11}, n_{\ell,\ell-1} > 1$, if $v_i \in T_i$ for $i = 1, \ell$, then v_i has valence at least two. Thus, each vertex of C has valence at least two, a contradiction since C is a finite tree. □

Remark 4.4 In the notation of Definition 4.2, a group in $\mathcal{C}$ may have 2–cycles at even distance bounded by Type II vertices. An example is given in the top of Figure 1.

4.2 Construction

Outline 4.5 Let M be a degree refinement of a group in $\mathcal{C}$ that satisfies conditions (M1) and (M2) of Theorem 1.1. Let $\Gamma_{\mathcal{B}}$ be the graph of blocks of M, and let Γ_0 be the augmented graph of blocks of M. We will describe a finite process to construct a finite bipartite tree with degree refinement M (as in Definition 3.6). The bipartite graph Γ_0 has degree refinement M, but, in general, Γ_0 is not a tree. We will perform a finite series of moves on Γ_0 to produce a finite tree. The moves on the graph Γ_0 recursively unwrap the cycles of length two in Γ_0 so that each move preserves the degree refinement.

An image of the following definition appears in Figure 2.

Definition 4.6 (split an MHF vertex) Let Λ be a bipartite graph with $V(\Lambda) = V_1 \sqcup V_2$ as defined in Definition 3.6. Let $t \in V_1$ and $f \in V_2$, and suppose Λ has $r > 0$ edges

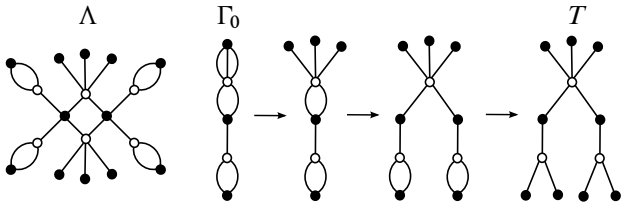


Figure 2: Example: Λ is the JSJ graph of a group in $\mathcal{C}$ that satisfies conditions (M1) and (M2) of Theorem 1.1, and Γ_0 is the augmented graph of blocks of Λ. Between the graphs Γ_0 and T, the *split a vertex* move was performed four times to produce a finite tree with the same degree refinement as Λ and Γ_0.

$e_1, \dots, e_r$ with endpoints $\{t, f\}$. Let $m_i \in \Lambda$ be the midpoint of the edge e_i, and suppose that $\bigcup_{i=1}^r m_i$ separates Λ into two components. Then $\Lambda - \bigcup_{i=1}^r \mathrm{Int}(e_i)$ has two components, where $\mathrm{Int}(e_i)$ denotes the interior of the edge e_i. Let $C \subset \Lambda$ be the component containing f, and let $C' \subset \Lambda$ be the component containing t. $\big($So, $\Lambda = C \cup C' \cup \big(\bigcup_{i=1}^r e_i\big).\big)$ Define Λ' to be the following finite graph, which is obtained by *splitting* f into r vertices. For $1 \le i \le r$, let C_i be a graph isomorphic to C, and let $\phi_i \colon C \to C_i$ be a graph isomorphism. Let $f_i = \phi_i(f)$. Let Λ' be the graph formed by the union of C', $\bigcup_{i=1}^r C_i$, and r edges $e'_1, \dots, e'_r$, where e'_i has one endpoint $t \in C'$ and the other endpoint $f_i \in C_i$. Let $p \colon \Lambda' \to \Lambda$ be the projection that is the identity on C', maps e'_i to e_i and C_i to C by an isomorphism.

Lemma 4.7 *The degree refinement of Λ is equivalent to the degree refinement of Λ', where Λ and Λ' are the graphs defined in Definition 4.6.*

Proof Suppose $T_1, \dots, T_n, F_1, \dots, F_m$ are the blocks in the degree partition of Λ. As in Definition 3.6, there exist constants n_{ij} with $1 \le i \le n$ and $1 \le j \le m$ such that each vertex in T_i is adjacent via n_{ij} edges to F_j and if $n_{ij} > 0$, each vertex in F_j is adjacent via at least one edge to T_i. There exists a partition of the vertices of Λ' into blocks $T'_1, \dots, T'_n, F'_1, \dots, F'_m$, where $T'_i = p^{-1}(T_i)$ and $F'_i = p^{-1}(F_i)$, where p is the projection map in Definition 4.6. By construction, since adjacencies are affected only between f and t, each vertex in T'_i is adjacent via n_{ij} edges to F'_j and if $n_{ij} > 0$, each vertex in F'_j is adjacent via at least one edge to T'_i. □

Construction 4.8 Let Γ_0 be the augmented graph of blocks of M as in Outline 4.5. Suppose $V(\Gamma_0) = V_1 \sqcup V_2$ as in Definition 3.4. The following construction produces

a finite tree T with degree refinement M. Since Γ_0 has degree refinement M, if Γ_0 is a tree, no additional moves are necessary. Otherwise, suppose Γ_0 is not a tree. Let $t \in V_1$ be such that there exists $f \in V_2$ and $r > 1$ edges $e_1, \dots, e_r$ connecting t and f. By condition (M1), removing the interiors of the edges in this collection disconnects the graph Γ_0. Let $C \subset \Lambda - \bigcup_{i=1}^{r} \mathrm{Int}(e_i)$ be the component containing f. Let $D = C \cup \bigcup_{i=1}^{r} e_i$.

Define a height function $h\colon V(D) \to \mathbb{N}$ by $h(v) = d(v,t)$. Then $h(v) \in 2\mathbb{Z}$ if and only if $v \in V_1$. Furthermore, if there exists $t' \in D \cap V_1$ and $f' \in D \cap V_2$ and $r' > 1$ edges connecting t' and f', then $h(f') = h(t') + 1$; otherwise, condition (M2) would be violated.

Perform a series of moves recursively to split vertices (as in Definition 4.6) of D at height $1, 3, 5, \dots$, and so on. Let D^k denote the graph obtained after vertices at height k have been split. As in Definition 4.6, there are projections $D^k \to D^{k-2}$ for $k \geq 3$ and $D_1 \to D$. Denote the composition of these maps as the projection $p_k\colon D^k \to D$ for $k \geq 1$. In an abuse of notation, we use t to denote the unique vertex in $p_k^{-1}(t) \in D^k$. Let $h_k\colon D^k \to \mathbb{N}$ be given by $h_k(v) = d(v,t)$. Then $h_k(v) = h(p_k(v))$ for all $v \in D^k$. Let $D^k_{\leq k}$ be the subgraph of D^k induced by vertices with height $\leq k$. Then $D^k_{\leq k}$ is a finite tree. Since D is a finite graph, there exists $N \in \mathbb{N}$ such that $h(v) < N$ for all $v \in V(D)$. Thus, after finitely many moves, the resulting graph D^N is a tree.

The moves on Γ_0 are the identity on $\Gamma_0 \setminus D$, a graph which may still contain (finitely many) embedded cycles of length two. For each set of additional cycles, the above procedure can be performed. By condition (M1) $\Gamma_{\mathcal{B}}$ is a tree, so these cycles of length 2 are the only embedded cycles in Γ_0. Hence, after finitely many steps, the resulting graph is a tree. In addition, by Lemma 4.7, the resulting graph has the same degree refinement as Γ_0, as desired.

Proposition 4.9 *If $G \in \mathcal{C}$ satisfies conditions* (M1) *and* (M2) *of Theorem 1.1, then G is quasi-isometric to a group with JSJ graph a tree.*

Proof Let M be the degree refinement of G, and let Γ_0 be the augmented graph of blocks of M. Then Γ_0 has degree refinement M, and by Construction 4.8, there exists a finite tree T with degree refinement M. Let $G' \in \mathcal{C}$ be any group with JSJ graph T. By Theorem 3.5, the groups G and G' are quasi-isometric. □

4.3 Characterization

We collect the above conditions and constructions to prove one of the main theorems of the paper.

Theorem 4.10 *Let $G \in \mathcal{C}$. The following are equivalent:*

(1) *The group G is quasi-isometric to a right-angled Coxeter group.*

(2) *The group G is quasi-isometric to a group generated by finite-order elements.*

(3) *The group G is quasi-isometric to a group with JSJ graph a tree.*

(4) *The degree refinement of G satisfies the two conditions:*

(M1) *The graph of blocks of G is a tree.*

(M2) *The augmented graph of blocks of G has no 2–cycles at even distance bounded by Type I vertices.*

Proof Let $G \in \mathcal{C}$. We first show (3) and (4) are equivalent. To prove (3) implies (4), let $G' \in \mathcal{C}$ be a group with JSJ graph a tree which is quasi-isometric to G. Let $\Gamma_{\mathcal{B}}$ and $\Gamma'_{\mathcal{B}}$ be the graphs of blocks of G and G', respectively, and let Γ_0 and Γ'_0 be the augmented graphs of blocks of G and G', respectively. By Corollary 3.8, $\Gamma_{\mathcal{B}} \cong \Gamma'_{\mathcal{B}}$ and $\Gamma_0 \cong \Gamma'_0$. Therefore, by Lemmas 4.1 and 4.3, conditions (M1) and (M2) hold. By Proposition 4.9, (4) implies (3).

Clearly, (1) implies (2) and (2) implies (3). Suppose $G \in \mathcal{C}$ is quasi-isometric to a group with JSJ graph a tree. Then G is quasi-isometric to a geometric amalgam of free groups with JSJ graph a tree. Thus, by [15, Theorem 1.16], G is quasi-isometric to a right-angled Coxeter group, so (3) implies (1), concluding the proof. □

5 Commensurability classes

5.1 The structure of finite-index subgroups

The structure of subgroups of a graph of groups is described by Scott and Wall in [33, Section 3]. In this subsection, we record the facts and constructions relevant to this paper.

As described in Section 2.2, the JSJ decomposition $\mathcal{G}$ of a group $G \in \mathcal{C}$ is a splitting of G as the fundamental group of a graph of groups. There is a finite CW–complex X

which is a graph of spaces associated to $\mathcal{G}$ and such that $G \cong \pi_1(X)$. Any finite-index subgroup $H \leq G$ is the fundamental group of a graph of spaces Y which finitely covers X. Thus, H splits as a graph of groups. Moreover, the graph of groups splitting of H associated to this graph of spaces Y is the JSJ decomposition of H. The details are as follows.

Proposition 5.1 *Let $G \in \mathcal{C}$ and $H \leq G$ be a finite-index subgroup. Let $\mathcal{G}$ be the JSJ decomposition of G with underlying graph Λ. The subgroup H is the fundamental group of a graph of groups $\mathcal{H}$ associated to the graph of spaces Y described above. If Γ is the underlying graph of $\mathcal{H}$, then for each $w \in V(\Gamma)$, there exists $v \in V(\Lambda)$ and $g_w \in G$ such that $H_w = H \cap g_w G_v g_w^{-1}$ and H_w is a finite-index subgroup of $g_w G_v g_w^{-1}$. The graph of groups $\mathcal{H}$ is the JSJ decomposition of H.*

Proof All statements except the last sentence of the proposition are given in Section 3 of [33]; it remains to show that $\mathcal{H}$ is the JSJ decomposition of H. Indeed, suppose $w \in V(\Gamma)$ is such that $H_w = H \cap g_w G_v g_w^{-1}$. By the construction of the JSJ decomposition of G due to Bowditch [5], the subgroup $g_w G_v g_w^{-1}$ is the stabilizer in G of a distinguished subset $A \subset \partial_\infty G$ in the visual boundary of G, called either a *necklace* or a *jump*, depending on whether the group G_v is maximally hanging Fuchsian or 2–ended, respectively. (See [5, Section 5].) Since H is a finite-index subgroup of G, the inclusion of H in G induces a homeomorphism from the visual boundary of H to the visual boundary of G. Thus, H_w is the stabilizer in H of the subset $A \subset \partial_\infty H \cong \partial_\infty G$. Therefore, H_w is a vertex group in the JSJ decomposition of H. By the same reasoning, the adjacencies between vertex groups in $\mathcal{G}$ yield the appropriate adjacencies between vertex groups in $\mathcal{H}$. Therefore, $\mathcal{H}$ is the JSJ decomposition of H. □

Notation 5.2 Suppose $G, G' \in \mathcal{C}$ are abstractly commensurable. Let $\mathcal{G}$ and $\mathcal{G}'$ be the JSJ decompositions of G and G', respectively. Suppose $\mathcal{G}$ and $\mathcal{G}'$ have underlying graphs Λ and Λ', respectively, with $V(\Lambda) = V_1 \sqcup V_2$ and $V(\Lambda') = V_1' \sqcup V_2'$. Suppose each vertex group G_v and $G'_{v'}$ is 2–ended for $v \in V_1$ and $v' \in V_1'$, respectively; suppose each vertex group G_v and $G'_{v'}$ is maximally hanging Fuchsian for each $v \in V_2$ and $v' \in V_2'$, respectively. Let $H \leq G$ and $H' \leq G'$ be subgroups of finite-index with $H \cong H'$. Let $\mathcal{H}$ and $\mathcal{H}'$ be the JSJ decompositions of H and H', respectively. Suppose $\mathcal{H}$ and $\mathcal{H}'$ have underlying graphs Γ and Γ', respectively, with $V(\Gamma) = W_1 \sqcup W_2$ and $V(\Gamma') = W_1' \sqcup W_2'$. Suppose each vertex group H_w and $H'_{w'}$ is 2–ended for $w \in W_1$ and $w' \in W_1'$, respectively; suppose each vertex group H_w and $H'_{w'}$ is maximally hanging Fuchsian for each $w \in W_2$ and $w' \in W_2'$, respectively.

Suppose in the degree partition of Λ the vertices in V_1 are contained in blocks $T_1(G), \dots, T_n(G)$, and the vertices in V_2 are contained in blocks $F_1(G), \dots, F_m(G)$. All groups in $\{G, G', H, H'\}$ are quasi-isometric. Hence, by Theorem 3.5, in the degree partition of Λ' the vertices in V_1' may be partitioned into blocks $T_1(G'), \dots, T_n(G')$ and the vertices in V_2' may be partitioned into blocks $F_1(G'), \dots, F_m(G')$ such that the resulting degree refinement matrix for Λ' is equal to the degree refinement matrix for Λ (without permuting the indices of the $T_i(G')$ and $F_i(G')$). Similarly, assume the degree partition of Γ into blocks $T_1(H), \dots, T_n(H), F_1(H), \dots, F_m(H)$ and the degree partition of Γ' into blocks $T_1(H'), \dots, T_n(H'), F_1(H'), \dots, F_m(H')$ also yield degree refinement matrices equal to the degree refinement matrix of Λ and Λ' (without permuting the indices of the blocks).

The degree partitions of the vertices of the JSJ graphs yield natural partitions of the vertex groups in the JSJ decompositions. We will use the notation

$$\begin{aligned} T_i^G &= \{G_v \mid v \in T_i(G)\}, & F_i^G &= \{G_v \mid v \in F_i(G)\}, \\ T_i^{G'} &= \{G'_{v'} \mid v' \in T_i(G')\}, & F_i^{G'} &= \{G'_{v'} \mid v' \in F_i(G')\}, \\ T_i^H &= \{H_w \mid w \in T_i(H)\}, & F_i^H &= \{H_w \mid w \in F_i(H)\}, \\ T_i^{H'} &= \{H'_{w'} \mid w' \in T_i(H')\}, & F_i^{H'} &= \{H'_{w'} \mid w' \in F_i(H')\}. \end{aligned}$$

Lemma 5.3 *Let $H \leq G$ be a finite-index subgroup with the notation defined above. Let $w \in V(\Gamma)$ be such that there exists $v \in V(\Lambda)$ and $g_w \in G$* (by Proposition 5.1) *be such that $H_w = H \cap g_w G_v g_w^{-1}$. Then $G_v \in F_i^G$ if and only if $H_w \in F_i^H$. Similarly, $G_v \in T_i^G$ if and only if $H_w \in T_i^H$.*

Proof Since H is a finite-index subgroup of G, the inclusion of H in G induces a homeomorphism from the visual boundary of H to the visual boundary of G. Hence, H and G have isomorphic JSJ trees by the construction of the JSJ tree due to Bowditch. Denote this tree by T. The subgroup H_w stabilizes the same vertex of T as $g_w G_v g_w^{-1}$, and the conclusion of the lemma follows. □

To study finite covers of a 2–dimensional hyperbolic P–manifold X, one often considers the full preimage of a singular curve on X or a subsurface of X. Algebraically, this corresponds to considering the following subset of vertex groups:

Definition 5.4 For $v \in V(\Lambda)$, let

$$\mathcal{H}_v = \{H_w \leq H \mid w \in V(\Gamma) \text{ and there exists } g_w \in G \text{ such that } H_w = H \cap g_w G_v g_w^{-1}\}.$$

Define $\mathcal{H}'_v$ similarly.

We will make use of the following two elementary observations, which follow from the discussion in [33, Section 3].

Lemma 5.5 *Suppose $H \leq G$ is a subgroup of index d. For each $v \in V(\Lambda)$,*

$$\sum_{H_w \in \mathcal{H}_v} [G_v : H_w] = d.$$

Lemma 5.6 *If $w, w' \in V(\Gamma)$ are adjacent, $H_w = H \cap g_w G_v g_w^{-1}$, and $H_{w'} = H \cap g_{w'} G_{v'} g_{w'}^{-1}$ for some $g_w, g_{w'} \in G$ and $v, v' \in V(\Lambda)$, then v and v' are adjacent.*

5.2 Quasi-isometric rigidity does not hold for the subclass of torsion-generated groups

Construction 5.7 Let $G \cong \pi_1(X) \in \mathcal{C}$ be the fundamental group of the 2–dimensional hyperbolic P–manifold X shown in Figure 3. The group G has JSJ decomposition with JSJ graph Λ. The vertices in Figure 3 are labeled so that $G_{v_K} \cong \pi_1(K)$, where K is a connected subsurface or branching curve in the space X. The JSJ graph for G has the degree partition

$$\begin{aligned}
&T_1(G) = \{v_\alpha, v_\beta\}, \quad F_1(G) = \{v_A\}, && F_4(G) = \{v_{D_1}, \dots, v_{D_4}\},\\
&T_2(G) = \{v_\gamma\}, \quad F_2(G) = \{v_B\}, && F_5(G) = \{v_{E_1}, \dots, v_{E_5}\},\\
&T_3(G) = \{v_\delta\}, \quad F_3(G) = \{v_{C_1}, v_{C_2}, v_{C_1'}, v_{C_2'}\},
\end{aligned}$$

The degree refinement for G is the matrix in Figure 3.

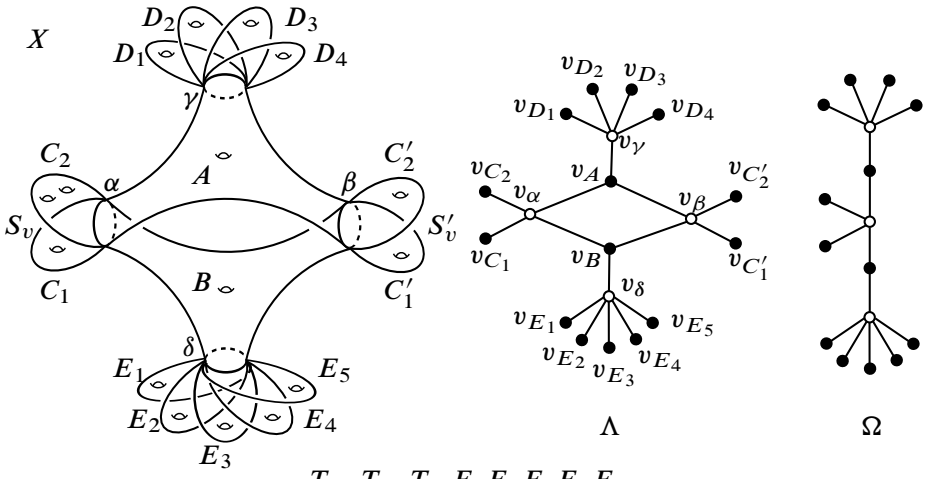
Theorem 5.8 *Let $G \cong \pi_1(X) \in \mathcal{C}$ be as given in Construction 5.7. The following hold:*

(1) *The group G is quasi-isometric to a right-angled Coxeter group.*

(2) *The group G is not abstractly commensurable to any group with JSJ graph a tree. In particular, G is not abstractly commensurable to any group generated by finite-order elements.*

We first prove (1) in the next lemma.

Lemma 5.9 *The group G is quasi-isometric to a right-angled Coxeter group.*



	T_1	T_2	T_3	F_1	F_2	F_3	F_4	F_5
T_1	0	0	0	1	1	2	0	0
T_2	0	0	0	1	0	0	4	0
T_3	0	0	0	0	1	0	0	5
F_1	∞	∞	0	0	0	0	0	0
F_2	∞	0	∞	0	0	0	0	0
F_3	∞	0	0	0	0	0	0	0
F_4	0	∞	0	0	0	0	0	0
F_5	0	0	∞	0	0	0	0	0

Figure 3: The group $\pi_1(X)$ has JSJ decomposition graph Λ and degree refinement the matrix shown. The group $\pi_1(X)$ is quasi-isometric to a right-angled Coxeter group with JSJ graph Ω, but $\pi_1(X)$ is not abstractly commensurable to any group generated by finite-order elements, as shown in Theorem 5.8.

Proof Let G' be a geometric amalgam of free groups with JSJ graph Ω shown in Figure 3, where the vertex groups associated to the white vertices are infinite cyclic and the vertex groups associated to the black vertices are maximally hanging Fuchsian. The graph Ω is a tree; so, by [15, Theorem 1.16], G' is quasi-isometric to a right-angled Coxeter group. The group G' also has degree refinement the matrix shown. Therefore, by Theorem 3.5, the groups G and G' are quasi-isometric. □

Outline 5.10 We outline the proof of Theorem 5.8(2) in the case of a simplifying assumption. Suppose towards a contradiction that G is abstractly commensurable to a group G' with JSJ graph a tree and such that $G' \cong \pi_1(X')$ where X' is a 2–dimensional hyperbolic P–manifold. In this setting, by [25, Theorem 1.2], there

exist finite covers $p\colon Y \to X$ and $p'\colon Y' \to X'$, where Y and Y' are 2–dimensional hyperbolic P–manifolds, and there exists a homeomorphism $f\colon Y \to Y'$ inducing an isomorphism between finite-index subgroups of G and G'.

The vertices $\{v_\alpha, v_A, v_\beta, v_B\}$ form a cycle in the JSJ graph Λ for G. The full preimage of $A \cup B \cup \alpha \cup \beta$ in the cover $Y \to X$ yields (not necessarily disjoint) cycles in the JSJ graph for $\pi_1(Y)$ and hence in the JSJ graph for $\pi_1(Y') \cong \pi_1(Y)$. We show these cycles cannot project to a tree in the JSJ graph for $\pi_1(X')$. The first step is to show that $p'(f(p^{-1}(S_v))) \cap p'(f(p^{-1}(S'_v))) = \varnothing$, where S_v and S'_v are the closed surfaces in X labeled in Figure 3. This claim holds since the vertex groups in F_3^G are exactly $\{\pi_1(C_1), \pi_1(C_2), \pi_1(C'_1), \pi_1(C'_2)\}$ and the ratio of the Euler characteristic of the subsurfaces in S_v is different from the ratio of the Euler characteristic of the subsurfaces of S'_v. Consequently, $p'(f(p^{-1}(\alpha))) \cap p'(f(p^{-1}(\beta))) = \varnothing$. Since $\pi_1(A) \in F_1^G$ and $\pi_1(B) \in F_2^G$, Lemma 5.3 implies $p'(f(p^{-1}(A))) \cap p'(f(p^{-1}(B))) = \varnothing$. Each subsurface or curve in X' in

$$\{p'(f(p^{-1}(A))), p'(f(p^{-1}(B))), p'(f(p^{-1}(\alpha))), p'(f(p^{-1}(\beta)))\}$$

corresponds to a vertex in a set V' contained in the JSJ graph for $\pi_1(X')$. The vertices in V' are adjacent to at least two other vertices in V' by the above arguments, and this yields a cycle in the JSJ graph, a contradiction. The proof in full generality given below translates these topological ideas to the algebraic setting.

Proof of Theorem 5.8 The proof of (1) is given in Lemma 5.9. To prove (2), suppose towards a contradiction that G is abstractly commensurable to a group G' with JSJ graph a tree Λ'. Suppose $H \leq G$ and $H' \leq G'$ are finite-index subgroups which are isomorphic. Since G is a geometric amalgam of free groups, the groups H and H' are geometric amalgams of free groups. Hence, $H \cong \pi_1(Y)$ and $H' \cong \pi_1(Y')$, where Y and Y' are 2–dimensional hyperbolic P–manifolds. By [25, Theorem 1.2] there exists a homeomorphism $f\colon Y \to Y'$ inducing an isomorphism $\Phi\colon H \to H'$. Suppose $p\colon Y \to X$ is a finite covering map.

Let $\mathcal{S}$ be the full preimage of S_v in Y, and let $\mathcal{S}'$ be the full preimage of S'_v in Y, where $S_v = C_1 \cup C_2$ and $S'_v = C'_1 \cup C'_2$ are the closed surfaces in X labeled in Figure 3. The spaces $\mathcal{S}$ and $\mathcal{S}'$ are each a disjoint collection of connected closed surfaces and $\mathcal{S} \cap \mathcal{S}' = \varnothing$ since $S_v \cap S'_v = \varnothing$. The lifts of the branching curve α on X partition each surface Σ in $\mathcal{S}$ as $\Sigma = Z_1 \cup Z_2$ so that $p(Z_i) = C_i$. Similarly, the lifts of the branching curve β on X partition each surface Σ' in $\mathcal{S}'$ as $\Sigma' = Z'_1 \cup Z'_2$, where

$p(Z_i') = C_i'$. Each branching curve in Y that intersects Σ is incident to exactly one subsurface in each of Z_1 and Z_2. Hence, Z_1 covers C_1 by the same degree that Z_2 covers C_2. A similar argument holds for Σ'. Moreover, since $f\colon Y \to Y'$ is a homeomorphism,

$$\text{(5-1)} \qquad \frac{\chi(f(Z_1))}{\chi(f(Z_2))} = \frac{\chi(Z_1)}{\chi(Z_2)} = \frac{\chi(C_1)}{\chi(C_2)} \quad \text{and} \quad \frac{\chi(f(Z_1'))}{\chi(f(Z_2'))} = \frac{\chi(Z_1')}{\chi(Z_2')} = \frac{\chi(C_1')}{\chi(C_2')}.$$

Let $H'_\Sigma \cong \pi_1(f(\Sigma)) \leq H'$ and $H'_{\Sigma'} \cong \pi_1(f(\Sigma')) \leq H'$. We now set notation. The subgroup H'_Σ is generated by a union of vertex groups in $T_1^{H'}$ and $F_3^{H'}$ in the JSJ decomposition of H' by Lemma 5.3. Suppose $H'_\Sigma = \langle H'_{w_1}, \dots, H'_{w_r}, H'_{x_1}, \dots, H'_{x_s}\rangle$, where $w_i \in F_3(H')$ for $i \in \{1,\dots,r\}$ and $x_j \in T_1(H')$ for $j \in \{1,\dots,s\}$. Similarly, $H'_{\Sigma'} = \langle H'_{w_1'}, \dots, H'_{w_{r'}'}, H'_{x_1'}, \dots, H'_{x_{s'}'}\rangle$, where $w_i' \in F_3(H')$ for $i \in \{1,\dots,r'\}$ and $x_j' \in T_1(H')$ for $j \in \{1,\dots,s'\}$. Assume that the fundamental group of every branching curve in Y' that intersects $f(\Sigma)$ is included in the set $\{H'_{x_j}\}_{j=1}^{s}$, and assume that the fundamental group of every branching curve in Y' that intersects $f(\Sigma')$ is included in the set $\{H'_{x_j'}\}_{j=1}^{s'}$. That is, the generating sets above are not minimal since these branching curves are boundary curves of surfaces whose fundamental groups are contained in the sets $\{H'_{w_i}\}_{i=1}^{r}$ and $\{H'_{w_i'}\}_{i=1}^{r'}$. By Proposition 5.1 and Lemma 5.3, there are vertices $v_i, v_i' \in F_3(G')$ and $y_j, y_j^j \in T_1(G')$ and elements $g_{w_i}, g_{w_i'}, g_{x_j}, g_{x_j'} \in G'$ such that

$$H'_{w_i} = H' \cap g_{w_i} G'_{v_i} g_{w_i}^{-1}, \quad H'_{x_j} = H' \cap g_{x_j} G'_{y_j} g_{x_j}^{-1},$$
$$H'_{w_i'} = H' \cap g_{w_i'} G'_{v_i'} g_{w_i'}^{-1}, \quad H'_{x_j'} = H' \cap g_{x_j'} G'_{y_j'} g_{x_j'}^{-1}.$$

The surfaces $f(\Sigma)$ and $f(\Sigma')$ are disjoint in the space Y', so

$$\{w_1,\dots,w_r,x_1,\dots,x_s\} \cap \{w_1',\dots,w_{r'}',x_1',\dots,x_{s'}'\} = \varnothing.$$

Claim $\{v_1,\dots,v_r,y_1,\dots,y_s\} \cap \{v_1',\dots,v_{r'}',y_1',\dots,y_{s'}'\} = \varnothing.$

Proof of Claim Since the fundamental group of every branching curve in Y' that intersects $f(\Sigma)$ or $f(\Sigma')$ is included in the set $\{H_{x_j}\}_{j=1}^{s} \cup \{H_{x_{j'}}\}_{j=1}^{s'}$, if $v_i = v_\ell'$ for some i,ℓ, then $y_j = y_k'$ for some j,k. Suppose towards a contradiction $y_j = y_k'$ for some $j \in \{1,\dots,s\}$ and $k \in \{1,\dots,s'\}$.

We first show recursively this assumption implies

$$\{v_1,\dots,v_r,y_1,\dots,y_s\} = \{v_1',\dots,v_{r'}',y_1',\dots,y_{s'}'\}.$$

Each vertex $x_j \in \{x_i\}_{i=1}^{s}$ is adjacent to exactly two vertices $w, w' \in \{w_i\}_{i=1}^{r}$. By the structure of the degree refinement for H', each vertex $x_j \in \{x_i\}_{i=1}^{s}$ is not adjacent to any other vertices in $F_3(H')$. An analogous statement holds for the vertices $\{x'_j\}_{j=1}^{s'}$ and $\{w'_i\}_{i=1}^{r'}$. Since the degree refinement for G' is the same as the degree refinement for H', an analogous statement also holds for the pairs $(\{v_i\}_{i=1}^{r}, \{y_j\}_{j=1}^{s})$ and $(\{v'_i\}_{i=1}^{r'}, \{y'_j\}_{j=1}^{s'})$. Therefore, if $y_j = y'_k$, then the two vertices $v, v' \in F_3(G')$ incident to $y_j = y'_k$ are in the sets $\{v_i\}_{i=1}^{r}$ and $\{v'_i\}_{i=1}^{r'}$. Since $f(\Sigma)$ and $f(\Sigma')$ are closed surfaces, for each vertex w_i and w'_i with $i \in \{1, \dots, r\}$ and $i' \in \{1, \dots, r'\}$, each vertex in $T_1(H')$ incident to w_i is in the set $\{x_j\}_{j=1}^{s}$ and each vertex in $T_1(H')$ incident to w'_i is in the set $\{x'_j\}_{j=1}^{s'}$. Thus, an analogous statement holds for the pairs $(\{v_i\}, \{y_i\})$ and $(\{v'_i\}, \{y'_i\})$. So, each vertex in $T_1(G')$ incident to either v or v' is in the sets $\{y_i\}_{i=1}^{r}$ and $\{y'_i\}_{i=1}^{r'}$. The above argument can then be applied to these vertices incident to v and v'. Each pair of vertex sets $(\{w_i\}, \{x_j\})$, $(\{w'_i\}, \{x'_j\})$, $(\{v_i\}, \{y_j\})$ and $(\{v'_i\}, \{y'_j\})$ spans a connected subgraph in either Γ' or Λ'. Therefore, these arguments can be repeated to conclude $\{v_1, \dots, v_r, y_1, \dots, y_s\} = \{v'_1, \dots, v'_{r'}, y'_1, \dots, y'_{s'}\}$.

The partitions of the surfaces $f(\Sigma) = f(Z_1) \cup f(Z_2)$ and $f(\Sigma') = f(Z'_1) \cup f(Z'_2)$ yield partitions of the vertices $\{w_i\}_{i=1}^{r} = \mathcal{Z}_1 \sqcup \mathcal{Z}_2$ and $\{w'_i\}_{i=1}^{r'} = \mathcal{Z}'_1 \sqcup \mathcal{Z}'_2$. Hence, there are partitions $\{v_i\}_{i=1}^{r} = \mathcal{V}_1 \sqcup \mathcal{V}_2$ and $\{v'_i\}_{i=1}^{r'} = \mathcal{V}'_1 \sqcup \mathcal{V}'_2$, where $v_i \in \mathcal{V}_j$ if H_{w_i} is the fundamental group of a subsurface in $f(Z_j)$, and likewise for v'_i. Each vertex $x_j \in \{x_k\}_{k=1}^{s}$ is adjacent to exactly one vertex in $\mathcal{Z}_1$ and exactly one vertex in $\mathcal{Z}_2$, and an analogous statement holds for $x'_j \in \{x'_k\}_{k=1}^{s'}$. Hence, each vertex y_j for $j \in \{1, \dots, s\}$ is adjacent to exactly one vertex in each of $\mathcal{V}_1$ and $\mathcal{V}_2$, and the same holds for y'_j for $j \in \{1, \dots, s'\}$. Therefore, by the conclusion of the previous paragraph, either $\mathcal{V}_1 = \mathcal{V}'_1$ and $\mathcal{V}_2 = \mathcal{V}'_2$ or $\mathcal{V}_1 = \mathcal{V}'_2$ and $\mathcal{V}_2 = \mathcal{V}'_1$. Assume $\mathcal{V}_1 = \mathcal{V}'_1$ and $\mathcal{V}_2 = \mathcal{V}'_2$; the other case is similar.

Since each y_j with $j \in \{1, \dots, s\}$ is incident to exactly one vertex in each of $\mathcal{V}_1$ and $\mathcal{V}_2$, by the description of finite-index subgroups in [33, Section 3] and Lemma 2.5, there exists $d \in \mathbb{N}$ such that

$$d \cdot \sum_{v \in \mathcal{V}_1} \chi(G'_v) = \chi(f(Z_1)) \quad \text{and} \quad d \cdot \sum_{v \in \mathcal{V}_2} \chi(G'_v) = \chi(f(Z_2)).$$

Similarly, there exists $d' \in \mathbb{N}$ such that

$$d' \cdot \sum_{v \in \mathcal{V}'_1} \chi(G'_v) = \chi(f(Z'_1)) \quad \text{and} \quad d' \cdot \sum_{v \in \mathcal{V}'_2} \chi(G'_v) = \chi(f(Z'_2)).$$

Therefore, since $\mathcal{V}_1 = \mathcal{V}_1'$ and $\mathcal{V}_2 = \mathcal{V}_2'$,

$$\frac{\chi(f(Z_1))}{\chi(f(Z_2))} = \frac{\sum_{v\in\mathcal{V}_1}\chi(G_v')}{\sum_{v\in\mathcal{V}_2}\chi(G_v')} = \frac{\sum_{v\in\mathcal{V}_1'}\chi(G_v')}{\sum_{v\in\mathcal{V}_2'}\chi(G_v')} = \frac{\chi(f(Z_1'))}{\chi(f(Z_2'))}.$$

So, by (5-1), $\chi(C_1)/\chi(C_2) = \chi(C_1')/\chi(C_2')$, a contradiction. Therefore,

$$\{v_1,\dots,v_r,y_1,\dots,y_s\}\cap\{v_1',\dots,v_{r'}',y_1',\dots,y_{s'}'\} = \varnothing. \qquad \square$$

To conclude the proof of the theorem, let $\mathcal{H}_\alpha' \subset V(\Gamma')$ and $\mathcal{H}_\beta' \subset V(\Gamma')$ be the sets of vertices in Γ' whose vertex groups are the fundamental group of a component of $f(p^{-1}(\alpha))$ and $f(p^{-1}(\beta))$, respectively, where α and β are the singular curves in X labeled in Figure 3. Let $\mathcal{G}_\alpha' \subset \Lambda'$ and $\mathcal{G}_\beta' \subset \Lambda'$ be the set of vertices of Λ' whose vertex groups contain, as a finite-index subgroup, H_w' for $w \in \mathcal{H}_\alpha'$ and $w \in \mathcal{H}_\beta'$, respectively. The above arguments imply $\mathcal{G}_\alpha' \cap \mathcal{G}_\beta' = \varnothing$. Indeed, every lift of α in Y is contained in some surface Σ in $\mathcal{S}$, and every lift of β in Y is contained in some surface Σ' in $\mathcal{S}'$. Let $\mathcal{H}_A' \subset V(\Gamma')$ and $\mathcal{H}_B' \subset V(\Gamma')$ be the set of vertices in Γ' whose vertex groups are the fundamental group of a component of $f(p^{-1}(A))$ and $f(p^{-1}(B))$, respectively, where A and B are the subsurfaces in X shown in Figure 3. Let $\mathcal{G}_A' \subset \Lambda'$ and $\mathcal{G}_B' \subset \Lambda'$ be the set of vertices in Λ' whose vertex groups contain, as a finite-index subgroup, H_w' for $w \in \mathcal{H}_A'$ and $w \in \mathcal{H}_B'$, respectively. Since $\pi_1(A) \in F_1^G$ and $\pi_1(B) \in F_2^G$, by Lemma 5.3, $\mathcal{G}_A' \cap \mathcal{G}_B' = \varnothing$. Every vertex in $\mathcal{H}_A'$ is adjacent to a vertex in $\mathcal{H}_\alpha'$ and a vertex in $\mathcal{H}_\beta'$ and vice versa. Similarly, every vertex in $\mathcal{H}_B'$ is adjacent to a vertex in $\mathcal{H}_\alpha'$ and a vertex in $\mathcal{H}_\beta'$ and vice versa. Therefore, by Lemma 5.6, every vertex in $\mathcal{G}_A'$ and $\mathcal{G}_B'$ is adjacent to a vertex in $\mathcal{G}_\alpha'$ and a vertex in $\mathcal{G}_\beta'$ and vice versa. So, there is a cycle in Λ', a contradiction. Therefore, G is not abstractly commensurable to any group with JSJ graph a tree. $\square$

6 Necessary conditions for commensurability

6.1 Block Euler characteristic vector

Definition 6.1 Let $G \in \mathcal{C}$, and let Λ be the JSJ graph of G so that $V(\Lambda) = V_1 \sqcup V_2$ as in Definition 3.4. Suppose in the degree partition of Λ, the vertices in V_1 are contained in blocks $T_1,\dots,T_n$ and the vertices in V_2 are contained in blocks $F_1,\dots,F_m$. Let $\chi_i = \sum_{G_v\in F_i^G}\chi(G_v)$. Suppose the blocks $\{F_i\}_{i=1}^m$ are indexed so that $\chi_i \geq \chi_j$ for $i \geq j$. The *block Euler characteristic vector of* G is $(\chi_1,\dots,\chi_m)$.

Definition 6.2 Vectors $v, v' \in \mathbb{R}^n$ are *commensurable* if there exist nonzero integers $K, K' \in \mathbb{Z}$ such that $Kv = K'v'$.

Proposition 6.3 *If $G, G' \in \mathcal{C}$ are abstractly commensurable, then the block Euler characteristic vector of G is commensurable to the block Euler characteristic vector of G'.*

Proof Let $(\chi_1, \dots, \chi_m)$ and $(\chi'_1, \dots, \chi'_m)$ be the block Euler characteristic vectors of G and G', respectively, where $\chi_i = \sum_{G_v \in F_i^G} \chi(G_v)$ and $\chi'_i = \sum_{G'_{v'} \in F_i^{G'}} \chi(G'_{v'})$. Suppose $H \leq G$ and $H' \leq G'$ are finite-index subgroups with $H \cong H'$. Suppose $[G : H] = d$ and $[G' : H'] = d'$. By Lemma 5.5, for every $v \in V(\Lambda)$ and $v' \in V(\Lambda')$,

$$d \cdot \chi(G_v) = \sum_{H_w \in \mathcal{H}_v} \chi(H_w) \quad \text{and} \quad d' \cdot \chi(G'_{v'}) = \sum_{H'_{w'} \in \mathcal{H}'_{v'}} \chi(H'_{w'}).$$

Furthermore, by Lemma 5.3,

$$\bigcup_{v \in F_i^G} \mathcal{H}_v = F_i^H \quad \text{and} \quad \bigcup_{v' \in F_i^{G'}} \mathcal{H}'_{v'} = F_i^{H'}.$$

Therefore,

$$d \cdot \sum_{G_v \in F_i^G} \chi(G_v) = \sum_{H_w \in F_i^H} \chi(H_w) = \sum_{H'_{w'} \in F_i^{H'}} \chi(H'_{w'}) = d' \cdot \sum_{G'_v \in F_i^{G'}} \chi(G'_v).$$

So, $d(\chi_1, \dots, \chi_m) = d'(\chi'_1, \dots, \chi'_m)$. □

6.2 Matching Euler characteristic vector

Suppose $G \in \mathcal{C}$ has JSJ decomposition with underlying graph Λ with $V(\Lambda) = V_1 \sqcup V_2$ as in Definition 3.4. If all vertices in V_1 have the same valence, then the block Euler characteristic vector of G has one entry. Hence, the block Euler characteristic vector does not distinguish commensurability classes in this setting. So, we define a finer invariant in this subsection to deal with this case. Results in this section are proved only within the subclass of geometric amalgams of free groups.

If Γ is a graph, a *matching* in Γ is a collection of disjoint edges whose vertex set is exactly the vertex set of Γ. This notion extends to subsurfaces in a 2–dimensional hyperbolic P–manifold, and a necessary criterion for commensurability can be stated in these terms.

Definition 6.4 Let X be a 2–dimensional hyperbolic P–manifold. A *matching* in X is a (not necessarily connected) subsurface in X (see Definition 2.11) whose boundary is exactly the set of branching curves in X. In particular, a matching does not contain any branching curves in its interior, and each branching curve is incident to exactly one connected subsurface with boundary in the matching. A *maximal matching* in X is a matching which has the greatest Euler characteristic of any matching in X.

Lemma 6.5 (existence of a matching) *Let X be a possibly disconnected hyperbolic P–manifold over a finite forest, and suppose each branching curve in X is incident to exactly n subsurfaces. Then X admits a matching.*

Proof Let X be a possibly disconnected hyperbolic P–manifold over a finite forest T, and suppose each branching curve in X is incident to exactly n subsurfaces. Suppose X has branching curves $c_1, \dots, c_m$. To build a matching $\mathcal{M}$ of X, choose $\Sigma_1 \subset X$, a subsurface of X, and let $\Sigma_1 \in \mathcal{M}$. Without loss of generality, Σ_1 has boundary $c_1, \dots, c_k$ for some $k \leq m$. Let $X_1 \subset X$ be the set of subsurfaces in X which have at least one boundary component in $\{c_1, \dots, c_k\}$. Without loss of generality, the set of boundary curves of subsurfaces in X_1 is $\{c_1, \dots, c_{k+\ell}\}$. Let $X_1' = \overline{X \setminus X_1}$, a hyperbolic P–manifold over a finite forest. Since T is a finite forest, each curve in $\{c_{k+1}, \dots, c_{k+\ell}\}$ has degree $n-1 > 1$ in X_1'. So, it is possible to add to $\mathcal{M}$ one subsurface in X_1' incident to each curve in $\{c_{k+1}, \dots, c_{k+\ell}\}$. Repeat this procedure with each new surface chosen for $\mathcal{M}$ in the place of Σ_1 to produce a matching $\mathcal{M}$ in finitely many steps. □

Example 6.6 There are 2–dimensional hyperbolic P–manifolds for which each branching curve has the same degree and for which there does not exist a matching. Indeed, let X be a 2–dimensional hyperbolic P–manifold whose fundamental group has JSJ graph Ω shown in Figure 4. The white vertices in Ω correspond to 2–ended vertex groups; the black vertices correspond to maximally hanging Fuchsian vertex groups.

Definition 6.7 Let X be a 2–dimensional hyperbolic P–manifold with underlying graph a tree, and suppose each branching curve in X is incident to exactly n subsurfaces. Let $\mathcal{M}_1$ be a maximal matching in X, and let $\mathcal{M}_i$ be a maximal matching in $X \setminus \operatorname{Int}\big(\bigcup_{k=1}^{i-1} \mathcal{M}_i\big)$ for $i = 2, \dots, n$. (Note that in a slight abuse of notation, we still refer to the image of the branching curves of X in $X \setminus \operatorname{Int}\big(\bigcup_{k=1}^{n-2} \mathcal{M}_i\big)$ as "branching" even though at this step the space is a manifold; similarly for $X \setminus \operatorname{Int}\big(\bigcup_{k=1}^{n-1} \mathcal{M}_i\big)$.)

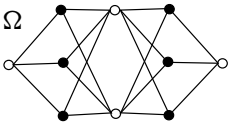


Figure 4: The JSJ graph Ω of a 2–dimensional hyperbolic P–manifold whose fundamental group does not admit a matching

The *matching Euler characteristic vector* of X is

$$(\chi(\mathcal{M}_1), \chi(\mathcal{M}_2), \dots, \chi(\mathcal{M}_n)).$$

The proof of the following proposition generalizes [34, Proposition 3.3.2] and [15, Proposition 6.4].

Proposition 6.8 *Suppose X and X' are 2–dimensional hyperbolic P–manifolds, the underlying graphs of X and X' are trees, and each branching curve in X and X' is incident to exactly n subsurfaces with boundary. Let v and v' be the matching Euler characteristic vectors of X and X', respectively. If $\pi_1(X)$ and $\pi_1(X')$ are abstractly commensurable, then v and v' are commensurable vectors.*

Proof Suppose that X and X' are 2–dimensional hyperbolic P–manifolds, the underlying graphs of X and X' are trees, and each branching curve in X and X' is incident to n branching curves. Let $v = (\chi(\mathcal{M}_1), \dots, \chi(\mathcal{M}_n))$ and $v' = (\chi(\mathcal{M}'_1), \dots, \chi(\mathcal{M}'_n))$ be the matching Euler characteristic vectors of X and X', respectively. Suppose that $\pi_1(X)$ and $\pi_1(X')$ are abstractly commensurable. We seek to prove that v and v' are commensurable.

Since $\pi_1(X)$ and $\pi_1(X')$ are abstractly commensurable, there are finite covering spaces $p\colon Y \to X$ and $p'\colon Y' \to X'$ and a homeomorphism $f\colon Y \to Y'$ by [25, Theorem 1.2]. Suppose that p is a degree D cover and p' is a degree D' cover. We will show $Dv = D'v'$.

Suppose that

$$\chi(\mathcal{M}_1) = \cdots = \chi(\mathcal{M}_s) > \chi(\mathcal{M}_{s+1}) \geq \cdots \geq \chi(\mathcal{M}_n),$$
$$\chi(\mathcal{M}'_1) = \cdots = \chi(\mathcal{M}'_t) > \chi(\mathcal{M}'_{t+1}) \geq \cdots \geq \chi(\mathcal{M}'_n).$$

Without loss of generality, $D\chi(\mathcal{M}_1) \geq D'\chi(\mathcal{M}'_1)$ and if $D\chi(\mathcal{M}_1) = D'\chi(\mathcal{M}'_1)$ then $s \geq t$.

Consider the matching $f(p^{-1}(\mathcal{M}_1)) = \{S'_1, \dots, S'_r\} \subset Y'$, a disjoint collection of connected subsurfaces of Y' whose boundary is exactly the set of branching curves of Y'. Let $\{c_1, \dots, c_m\}$ be the set of branching curves of X'. Each surface $S'_i \in f(p^{-1}(\mathcal{M}_1))$ covers a subsurface S_{I_i} of X', where S_{I_i} has boundary $\{c_j \mid j \in I_i\}$ for some $I_i \subset \{1, \dots, m\}$. Suppose S'_i covers S_{I_i} by degree d_{I_i} with $1 \leq d_{I_i} \leq D'$.

Claim
$$\sum_{i=1}^{r} d_{I_i} \cdot \chi(S_{I_i}) \leq D' \chi(\mathcal{M}'_1).$$

Proof of Claim The inequality holds by the definition of maximal matching if $D' = 1$. In general, the surfaces in $\{S_{I_i}\}_{i=1}^{r}$ need not form a matching of X'. However, we show that if these surfaces are counted with the right multiplicity, then they can be partitioned into D' matchings of X'.

If $D' > 1$, we will partition the surfaces in the set

$$\mathcal{S} = \{\underbrace{S_{I_1}, \dots, S_{I_1}}_{d_{I_1}}, \underbrace{S_{I_2}, \dots, S_{I_2}}_{d_{I_2}}, \dots, \underbrace{S_{I_r}, \dots, S_{I_r}}_{d_{I_r}}\}$$

into D' matchings of X', called $\mathcal{N}_1, \dots, \mathcal{N}_{D'}$. The sum of the Euler characteristics of the surfaces in the D' matchings constructed is equal to the left-hand side of the inequality in the claim. The conclusion of the claim will follow by the definition of maximal matching. For $i \in \{1, \dots, r\}$, we call the value d_{I_i} the *weight* of the surface S_{I_i}. The weight of S_{I_i} records the contribution of $\chi(S_{I_i})$ to the Euler characteristic of $f(p^{-1}(\mathcal{M}_1))$. If c_j is a boundary curve of the surface S_{I_i}, we say d_{I_i} is the weight at c_j coming from S_{I_i}. The total weight at the curve c_j is $\sum_{j \in I_i} d_{I_i} = D'$ since $p'\colon Y' \to X'$ is a degree D' cover.

Construct the set of D' matchings recursively. To construct the first matching $\mathcal{N}_1$ built out of surfaces in $\mathcal{S}$, choose a surface $S_{I_i} \in \mathcal{S}$. The surface S_{I_i} is incident to curves $\{c_j \mid j \in I_i\}$. View the underlying graph of X as a bipartite graph with white vertices corresponding to branching curves and black vertices corresponding to surfaces with boundary. Let c be a curve corresponding to a white vertex at distance two (in the graph) from c_j for some $j \in I_i$ and at distance three (in the graph) from the black vertex corresponding to S_{I_i}, if such a curve exists. There is a surface in $\mathcal{S}$ incident to c and not to c_j. That is, both curves c and c_j have total weight D' coming from a collection of surfaces incident to these curves. Some of the weight at c_j comes from S_{I_i}, so the (unique) surface S incident to both c_j and c has weight less than D'. Thus, some of

the weight at c comes from a surface S_{I_k} different from S. Let $S_{I_i} \cup S_{I_k} \in \mathcal{N}_1$. Since the underlying graph of X is a tree, this selection may be continued (finitely many times) to build the matching $\mathcal{N}_1$ of surfaces contained in $\mathcal{S}$.

Build the remaining $D'-1$ matchings $\mathcal{N}_2, \dots, \mathcal{N}_{D'}$ similarly. First form the set $\mathcal{S}_1$ by removing from $\mathcal{S}$ one copy of each surface contained in $\mathcal{N}_1$. Then, the matching $\mathcal{N}_2$ may be chosen analogously to $\mathcal{N}_1$, where each branching curve in X' now has weight $D'-1$ associated to it. This process may be continued to build the D' matchings $\mathcal{N}_1, \dots, \mathcal{N}_{D'}$. Furthermore, by construction and since $\mathcal{M}'_1$ is a maximal matching of X',

$$\sum_{i=1}^{r} d_{I_i} \cdot \chi(S_{I_i}) = \chi(\mathcal{N}_1) + \cdots + \chi(\mathcal{N}_{D'}) \leq D'\chi(\mathcal{M}'_1),$$

concluding the proof of the claim. □

Thus,

$$D\chi(\mathcal{M}_1) = \chi(f(p^{-1}(\mathcal{M}_1))) = \sum_{i=1}^{r} d_{I_i} \cdot \chi(S_{I_i}) \leq D'\chi(\mathcal{M}'_1),$$

where the last inequality is given by the claim above. Since $D\chi(\mathcal{M}_1) \geq D'\chi(\mathcal{M}'_1)$ by assumption, $D\chi(\mathcal{M}_1) = D'\chi(\mathcal{M}'_1)$. Each branching curve in Y' is incident to exactly s connected surfaces in $f(p^{-1}(\mathcal{M}_1)) \cup \cdots \cup f(p^{-1}(\mathcal{M}_s))$. Thus,

$$p'\big(f(p^{-1}(\mathcal{M}_1)) \cup \cdots \cup f(p^{-1}(\mathcal{M}_s))\big)$$

must have in its image at least s surfaces in the matchings $\mathcal{M}'_1, \dots, \mathcal{M}'_n$; so, $t \geq s$. Therefore, $D\chi(\mathcal{M}_i) = D'\chi(\mathcal{M}'_i)$ for $1 \leq i \leq s = t$ and $\bigcup_{i=1}^{s} f(p^{-1}(\mathcal{M}_i)) = \bigcup_{i=1}^{s} p^{-1}(\mathcal{M}_i)$. So, the above argument can be repeated (at most finitely many times) with the remaining matchings in X and X' of strictly smaller Euler characteristic, proving the proposition. □

7 Quasi-isometry versus abstract commensurability

Theorem 7.1 *There are infinitely many abstract commensurability classes within every quasi-isometry class in $\mathcal{C}$ that contains a group with JSJ graph a tree.*

Proof Let $\mathcal{Q}$ be a quasi-isometry class in $\mathcal{C}$ that contains a group with JSJ graph a tree T. Let $G \in \mathcal{Q}$ be a geometric amalgam of free groups with JSJ graph T such that $G \cong \pi_1(X)$ for some 2–dimensional hyperbolic P–manifold X. Choose a subsurface

$\Sigma \subset X$. Exchange Σ with a surface Σ_g with the same number of boundary components as Σ and with genus $g \geq 1$. Let X_g denote the resulting 2–dimensional hyperbolic P–manifold, and let $G_g \cong \pi_1(X_g)$. The groups G_g and G are quasi-isometric for every $g \geq 1$, but G_g and G_h are abstractly commensurable if and only if $g = h$. Indeed, if not all branching curves of X_g have the same degree, let v_g be the block Euler characteristic vector of G_g, and if all branching curves of X_g have the same degree, let v_g be the matching Euler characteristic vector of G_g. Changing the Euler characteristic of one subsurface of X_g changes the commensurability type of the vector v_g, hence the claim follows from Propositions 6.3 and 6.8. □

Remark 7.2 The proof above applies to any quasi-isometry class in $\mathcal{C}$ for which the degree refinement contains more than one row coming from maximal hanging Fuchsian vertex groups, since in this case the block Euler characteristic vector has more than one entry.

References

[1] **D Angluin**, *Local and global properties in networks of processors*, from "Proceedings of the twelfth annual ACM Symposium on Theory of Computing" (R E Miller, S Ginsburg, W A Burkhard, R J Lipton, editors), ACM, New York (1980) 82–93

[2] **J Behrstock**, **M F Hagen**, **A Sisto**, *Quasiflats in hierarchically hyperbolic spaces*, preprint (2017) arXiv

[3] **J Behrstock**, **M F Hagen**, **A Sisto**, *Thickness, relative hyperbolicity, and randomness in Coxeter groups*, Algebr. Geom. Topol. 17 (2017) 705–740 MR

[4] **J A Behrstock**, **W D Neumann**, *Quasi-isometric classification of graph manifold groups*, Duke Math. J. 141 (2008) 217–240 MR

[5] **B H Bowditch**, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998) 145–186 MR

[6] **K S Brown**, *Cohomology of groups*, Graduate Texts in Mathematics 87, Springer (1982) MR

[7] **M Burger**, **S Mozes**, *Lattices in product of trees*, Inst. Hautes Études Sci. Publ. Math. 92 (2000) 151–194 MR

[8] **P-E Caprace**, *Buildings with isolated subspaces and relatively hyperbolic Coxeter groups*, Innov. Incidence Geom. 10 (2009) 15–31 MR Correction in 14 (2015) 77–79

[9] **C H Cashen**, **A Martin**, *Quasi-isometries between groups with two-ended splittings*, Math. Proc. Cambridge Philos. Soc. 162 (2017) 249–291 MR

[10] **A Casson**, **D Jungreis**, *Convergence groups and Seifert fibered* 3*–manifolds*, Invent. Math. 118 (1994) 441–456 MR

[11] **R Charney**, **K Ruane**, **N Stambaugh**, **A Vijayan**, *The automorphism group of a graph product with no SIL*, Illinois J. Math. 54 (2010) 249–262 MR

[12] **R Charney**, **H Sultan**, *Contracting boundaries of* CAT(0) *spaces*, J. Topol. 8 (2015) 93–117 MR

[13] **J Crisp**, **L Paoluzzi**, *Commensurability classification of a family of right-angled Coxeter groups*, Proc. Amer. Math. Soc. 136 (2008) 2343–2349 MR

[14] **C Cunningham**, **A Eisenberg**, **A Piggott**, **K Ruane**, *Recognizing right-angled Coxeter groups using involutions*, Pacific J. Math. 284 (2016) 41–77 MR

[15] **P Dani**, **E Stark**, **A Thomas**, *Commensurability for certain right-angled Coxeter groups and geometric amalgams of free groups*, Groups Geom. Dyn. 12 (2018) 1273–1341 MR

[16] **P Dani**, **A Thomas**, *Divergence in right-angled Coxeter groups*, Trans. Amer. Math. Soc. 367 (2015) 3549–3577 MR

[17] **P Dani**, **A Thomas**, *Bowditch's JSJ tree and the quasi-isometry classification of certain Coxeter groups*, J. Topol. 10 (2017) 1066–1106 MR

[18] **M W Davis**, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series 32, Princeton Univ. Press (2008) MR

[19] **C Druţu**, **M Kapovich**, *Geometric group theory*, American Mathematical Society Colloquium Publications 63, Amer. Math. Soc., Providence, RI (2018) MR

[20] **M J Dunwoody**, *The accessibility of finitely presented groups*, Invent. Math. 81 (1985) 449–457 MR

[21] **D Gabai**, *Convergence groups are Fuchsian groups*, Ann. of Math. 136 (1992) 447–510 MR

[22] **P Haïssinsky**, **L Paoluzzi**, **G Walsh**, *Boundaries of Kleinian groups*, Illinois J. Math. 60 (2016) 353–364 MR

[23] **M Haulmark**, **H Nguyen**, **H C Tran**, *On boundaries of relatively hyperbolic right-angled Coxeter groups*, preprint (2017) arXiv

[24] **M Kapovich**, *Hyperbolic manifolds and discrete groups*, Progress in Mathematics 183, Birkhäuser, Boston, MA (2001) MR

[25] **J-F Lafont**, *Diagram rigidity for geometric amalgamations of free groups*, J. Pure Appl. Algebra 209 (2007) 771–780 MR

[26] **F T Leighton**, *Finite common coverings of graphs*, J. Combin. Theory Ser. B 33 (1982) 231–238 MR

[27] **I Levcovitz**, *A quasi-isometry invariant and thickness bounds for right-angled Coxeter groups*, preprint (2017) arXiv

[28] **I Levcovitz**, *Divergence of* CAT(0) *cube complexes and Coxeter groups*, Algebr. Geom. Topol. 18 (2018) 1633–1673 MR

[29] **W Malone**, *Topics in geometric group theory*, PhD thesis, The University of Utah (2010) MR Available at `https://search.proquest.com/docview/305212500`

[30] **P Papasoglu**, **K Whyte**, *Quasi-isometries between groups with infinitely many ends*, Comment. Math. Helv. 77 (2002) 133–144 MR

[31] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics 149, Springer (1994) MR

[32] **E Rips**, **Z Sela**, *Cyclic splittings of finitely presented groups and the canonical JSJ decomposition*, Ann. of Math. 146 (1997) 53–109 MR

[33] **P Scott**, **T Wall**, *Topological methods in group theory*, from "Homological group theory" (C T C Wall, editor), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979) 137–203 MR

[34] **E Stark**, *Abstract commensurability and quasi-isometry classification of hyperbolic surface group amalgams*, Geom. Dedicata 186 (2017) 39–74 MR

[35] **P Tukia**, *Homeomorphic conjugates of Fuchsian groups*, J. Reine Angew. Math. 391 (1988) 1–54 MR

Department of Mathematics, Technion - Israel Institute of Technology
Haifa, Israel

`emily.stark@campus.technion.ac.il`

Received: 19 November 2017 Revised: 31 July 2018

msp

Algebraic & Geometric Topology 18 (2018) 4069–4091

Cohomology of symplectic groups and Meyer's signature theorem

DAVE BENSON
CATERINA CAMPAGNOLO
ANDREW RANICKI
CARMEN ROVI

Meyer showed that the signature of a closed oriented surface bundle over a surface is a multiple of 4, and can be computed using an element of $H^2(\mathsf{Sp}(2g,\mathbb{Z}),\mathbb{Z})$. If we denote by $1\to\mathbb{Z}\to\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}\to\mathsf{Sp}(2g,\mathbb{Z})\to 1$ the pullback of the universal cover of $\mathsf{Sp}(2g,\mathbb{R})$, then by a theorem of Deligne, every finite index subgroup of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ contains $2\mathbb{Z}$. As a consequence, a class in the second cohomology of any finite quotient of $\mathsf{Sp}(2g,\mathbb{Z})$ can at most enable us to compute the signature of a surface bundle modulo 8. We show that this is in fact possible and investigate the smallest quotient of $\mathsf{Sp}(2g,\mathbb{Z})$ that contains this information. This quotient $\mathfrak{H}$ is a nonsplit extension of $\mathsf{Sp}(2g,2)$ by an elementary abelian group of order 2^{2g+1}. There is a central extension $1\to\mathbb{Z}/2\to\widetilde{\mathfrak{H}}\to\mathfrak{H}\to 1$, and $\widetilde{\mathfrak{H}}$ appears as a quotient of the metaplectic double cover $\mathsf{Mp}(2g,\mathbb{Z})=\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}/2\mathbb{Z}$. It is an extension of $\mathsf{Sp}(2g,2)$ by an almost extraspecial group of order 2^{2g+2}, and has a faithful irreducible complex representation of dimension 2^g. Provided $g\geqslant 4$, the extension $\widetilde{\mathfrak{H}}$ is the universal central extension of $\mathfrak{H}$. Putting all this together, in Section 4 we provide a recipe for computing the signature modulo 8, and indicate some consequences.

20J06; 20C33, 55R10

1 Introduction

Let $\Sigma_g\to M\to\Sigma_h$ be an oriented surface bundle over a surface. This is determined by a homotopy class of maps $\Sigma_h\to B\mathsf{Aut}^+(\Sigma_g)$. If $g\geqslant 2$ then the connected components of $\mathsf{Aut}^+(\Sigma_g)$ are contractible (Corollary 19 in Luke and Mason [25]; see also Earle and Eells [9] and Hamstrom [19]), and $\pi_0\,\mathsf{Aut}^+(\Sigma_g)=\Gamma_g$ is the (orientation-preserving) mapping class group of Σ_g. So $B\mathsf{Aut}^+(\Sigma_g)\simeq B\Gamma_g$, and the bundle is classified by a homotopy class of maps $\Sigma_h\to B\Gamma_g$, or equivalently by the monodromy homomorphism

$$\pi_1(\Sigma_h)=\langle a_1,b_1,\dots,a_h,b_h\mid[a_1,b_1]\cdots[a_h,b_h]=1\rangle\to\Gamma_g.$$

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4069

Now Γ_g acts on $H^1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, preserving the symplectic form given by cup product into $H^2(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}$. So we have a map $\Gamma_g \to \mathsf{Sp}(2g, \mathbb{Z})$, which is surjective. Composing, we obtain a map

$$\chi\colon \pi_1(\Sigma_h) \to \Gamma_g \to \mathsf{Sp}(2g, \mathbb{Z}),$$

and an induced map in cohomology

$$\chi^*\colon H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}) \to H^2(\pi_1(\Sigma_h), \mathbb{Z}).$$

Meyer [26] constructed a 2–cocycle τ on $\mathsf{Sp}(2g, \mathbb{Z})$ such that

$$\text{signature}(M) = \langle \chi^*[\tau], [\Sigma_h] \rangle \in 4\mathbb{Z} \subseteq \mathbb{Z},$$

with $[\tau] = 4$ in $H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$ for $g \geqslant 3$, and $[\tau]/4 = 1$ in $H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$, classifying the universal central extension of $\mathsf{Sp}(2g, \mathbb{Z})$. Let

$$([\tau]/4)_2 \in H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}/2) \cong \mathbb{Z}/2$$

be the mod 2 reduction. The mod 2 residue

$$\text{signature}(M)/4 = \langle \chi^*[\tau]/4, [\Sigma_h] \rangle = \langle \chi^*([\tau]/4)_2, [\Sigma_h] \rangle \in \mathbb{Z}/2$$

was identified by Rovi [31] with the Arf–Kervaire invariant of a Pontryagin squaring operation.

Our main purpose in this paper is to construct a normal subgroup $\mathfrak{K}$ of $\mathsf{Sp}(2g, \mathbb{Z})$ for $g \geq 1$ with finite quotient $\mathfrak{H} = \mathsf{Sp}(2g, \mathbb{Z})/\mathfrak{K}$ of shape $2^{2g+1} \cdot \mathsf{Sp}(2g, 2)$ (for notation describing group extensions, see Section 5.2 of the introduction to the Atlas [5]). Let $p\colon \mathsf{Sp}(2g, \mathbb{Z}) \to \mathfrak{H}$ be the projection. There is a nonzero element $c \in H^2(\mathfrak{H}, \mathbb{Z}/2)$ (for $g \geqslant 4$ we have $H^2(\mathfrak{H}, \mathbb{Z}/2) \cong \mathbb{Z}/2$ but there are extraneous summands inflated from $H^2(\mathsf{Sp}(2g, 2), \mathbb{Z}/2)$ for small g) which classifies a nonsplit double cover $\widetilde{\mathfrak{H}}$ of $\mathfrak{H}$. The inflation $p^*(c) = [\tau/4]_2$ in $H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z}/2) \cong \mathbb{Z}/2$ classifies the metaplectic double cover $\mathsf{Mp}(2g, \mathbb{Z})$ of $\mathsf{Sp}(2g, \mathbb{Z})$. Now p factors through $\mathsf{Sp}(2g, \mathbb{Z}/4)$, so that we obtain as a consequence that $\text{signature}(M)/4 \in \mathbb{Z}/2$ only depends on the $\mathbb{Z}/4$–coefficient monodromy $\chi_4\colon \pi_1(\Sigma_h) \to \mathsf{Sp}(2g, \mathbb{Z}/4)$ (this was already proved by a different method by Korzeniewski [23]). The sequence of group homomorphisms

$$\mathsf{Sp}(2g, \mathbb{Z}) \to \mathsf{Sp}(2g, \mathbb{Z}/4) \to \mathfrak{H} \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])/\{\pm 1\}$$

lifts to a sequence of double covers

$$\mathsf{Mp}(2g, \mathbb{Z}) \to \widetilde{\mathsf{Sp}(2g, \mathbb{Z}/4)} \to \widetilde{\mathfrak{H}} \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}]),$$

which is used in the recipe of Section 4 for the signature modulo 8. The faithful representation $\widetilde{\mathfrak{H}} \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])$ is investigated in Benson [2].

Denote by $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ the central extension obtained by pulling back the universal cover of $\mathsf{Sp}(2g,\mathbb{R})$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{R})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{R}) & \longrightarrow & 1
\end{array}$$

Then for $g \geqslant 4$ the group $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is the universal central extension of $\mathsf{Sp}(2g,\mathbb{Z})$, while for $g = 3$ there is an extra copy of $\mathbb{Z}/2$ coming from the fact that $\mathsf{Sp}(6,2)$ has an exceptional double cover (see Lemma 6.11). Note also that the centre of $\mathsf{Sp}(2g,\mathbb{Z})$ has order two. The centre of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is twice as big as the subgroup $\mathbb{Z}$ displayed above; it is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2$ if g is even, and $\mathbb{Z}$ if g is odd.

A theorem of Deligne [6] implies that the group $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is not residually finite. Every subgroup of finite index contains the subgroup $2\mathbb{Z}$. To rephrase, every finite quotient of $\widetilde{\mathsf{Sp}(2g,\mathbb{Z})}$ is in fact a finite quotient of the metaplectic double cover $\mathsf{Mp}(2g,\mathbb{Z})$ of $\mathsf{Sp}(2g,\mathbb{Z})$ defined by

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 2\mathbb{Z} & = & 2\mathbb{Z} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathsf{Sp}(2g,\mathbb{Z})} & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathsf{Mp}(2g,\mathbb{Z}) & \longrightarrow & \mathsf{Sp}(2g,\mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

As a consequence, if we compose χ with the map to a finite quotient of $\mathsf{Sp}(2g,\mathbb{Z})$, we lose information about the signature; the best we can hope to do is compute the signature modulo 8. We shall discuss this in greater detail elsewhere.

An outline of the paper is as follows. We describe the subgroup $\mathfrak{K} \leqslant \mathsf{Sp}(2g, \mathbb{Z})$ and quotient $\mathfrak{H} = \mathsf{Sp}(2g, \mathbb{Z})/\mathfrak{K}$ in Section 2. Their properties are described in Theorem 2.2, and the proof occupies much of the rest of the paper. Section 3 contains background and references on extraspecial and almost extraspecial groups, and there we explain what this has to do with the structure of $\mathfrak{H}$ and its double cover $\widetilde{\mathfrak{H}}$. We describe a faithful unitary representation $\widetilde{\mathfrak{H}} \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])$, which inflates to a representation $\mathsf{Mp}(2g, \mathbb{Z}) \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])$, and which is investigated in greater detail in [2]. Section 4 uses this representation to give a recipe for computing the signature modulo 8 of a surface bundle over a surface. The rest of the paper consists of cohomology computations. In preparation for this, in Section 5 we discuss the Lie algebra of the symplectic group. We show that as a module, it is isomorphic to the divided square of the natural module, and we discuss the submodule structure. This enables us in Section 6 to exploit the five-term sequence to compute $H_2(\mathfrak{H})$ and $H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n))$ for $n \geqslant 2$. Provided that $g \geqslant 4$, these are isomorphic to $\mathbb{Z}/2$.

Acknowledgements Campagnolo acknowledges support by the Swiss National Science Foundation, grant number PP00P2-128309/1, and by the German Science Foundation via the Research Training Group 2229, under which this research was started and then completed.

2 The subgroup $\mathfrak{K} \leqslant \mathsf{Sp}(2g, \mathbb{Z})$ and the main theorem

Denote by J the $2g \times 2g$ matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Regarding J as a symplectic form, we have

$$\mathsf{Sp}(2g, \mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} = X \;\middle|\; X^t J X = J \right\}.$$

Since $J^{-1} = -J$, a matrix is symplectic if and only if its transpose is symplectic. Writing out the above condition explicitly, a matrix is symplectic if and only if

(i) AB^t and CD^t are symmetric and $AD^t - BC^t = I$, or equivalently

(ii) A^tC and B^tD are symmetric and $A^tD - C^tB = I$.

We write $\mathsf{Sp}(2g, 2)$ for the matrices satisfying the same conditions over $\mathbb{F}_2$, and note that reduction modulo two $\mathsf{Sp}(2g, \mathbb{Z}) \to \mathsf{Sp}(2g, 2)$ is surjective (Newman and Smart [28]).

We write $\Gamma(2g,N) \leqslant \mathsf{Sp}(2g,\mathbb{Z})$ for the *principal congruence subgroup* consisting of symplectic matrices which are congruent to the identity modulo N. We write $\Gamma(2g,N,2N)$ for the *Igusa subgroup* [22] of $\Gamma(2g,N)$ consisting of the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where the entries of $\mathsf{Diag}(AB^t)$ and $\mathsf{Diag}(CD^t)$ are divisible by $2N$, or equivalently where the entries of $\mathsf{Diag}(A^tC)$ and $\mathsf{Diag}(B^tD)$ are divisible by $2N$. If $N=1$, this is the *theta subgroup*, also known as the *symplectic quadratic group*, and denoted $\mathsf{Sp}^{\mathfrak{q}}(2g,\mathbb{Z})$. It is the inverse image in $\mathsf{Sp}(2g,\mathbb{Z})$ of the orthogonal subgroup $\mathsf{O}^+(2g,2) \leqslant \mathsf{Sp}(2g,2)$.

Definition 2.1 We write $\mathfrak{K}$ for the subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$ consisting of matrices

$$\begin{pmatrix} I+2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I+2\mathrm{d} \end{pmatrix} \in \mathsf{Sp}(2g,\mathbb{Z})$$

satisfying the following:

(i) The vectors of diagonal entries $\mathsf{Diag}(\mathrm{b})$ and $\mathsf{Diag}(\mathrm{c})$ are even.

(ii) The trace $\mathsf{Tr}(a)$ is even.

Thus we have $\Gamma(2g,4) \leqslant \mathfrak{K} \leqslant \Gamma(2g,2)$ and $|\Gamma(2g,2):\mathfrak{K}| = 2^{2g+1}$. The interpretation of the subgroup $\mathfrak{K}$ is that it is the inverse image in $\mathsf{Sp}(2g,\mathbb{Z})$ of the largest subspace of $\Gamma(2g,2)/\Gamma(2g,4)$ on which the quadratic form in Theorem 2.2(iv) is identically zero.

Our main theorem is as follows. We assume that $g \geqslant 4$ for the purpose of simplifying the statements. In the appendix we include statements for all values of g. The main difference for low values of g is that the cohomology of $\mathsf{Sp}(2g,2)$ in degrees one and two contributes some further annoying complications.

Theorem 2.2 *Let $g \geqslant 4$.*

(i) *$\mathfrak{K}$ is a normal subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$. We write $\mathfrak{H}$ for the quotient $\mathsf{Sp}(2g,\mathbb{Z})/\mathfrak{K}$.*

(ii) *The quotient $\Gamma(2g,2)/\mathfrak{K} \leqslant \mathfrak{H}$ is an elementary abelian 2–group $(\mathbb{Z}/2)^{2g+1}$.*

(iii) *The extension*

$$1 \to (\mathbb{Z}/2)^{2g+1} \to \mathfrak{H} \to \mathsf{Sp}(2g,2) \to 1$$

does not split.

(iv) *The group $(\mathbb{Z}/2)^{2g+1}$ supports an invariant quadratic form $\mathfrak{q}$ given by*

$$\mathfrak{q}\begin{pmatrix} I+2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I+2\mathrm{d} \end{pmatrix} = \mathsf{Tr}(\mathrm{a}) + \langle \mathsf{Diag}(\mathrm{b}), \mathsf{Diag}(\mathrm{c}) \rangle$$

(see Remark 5.3 for the definition of the pointy brackets here).

(v) *The action of* $\mathsf{Sp}(2g,2)$ *on* $(\mathbb{Z}/2)^{2g+1}$ *described by the extension in* (iii) *gives the exceptional isomorphism* $\mathsf{Sp}(2g,2) \cong \mathsf{O}(2g+1,2)$*, the orthogonal group of the quadratic form* $\mathfrak{q}$*.*

(vi) *We have* $H^2(\mathfrak{H}, \mathbb{Z}/2) \cong \mathbb{Z}/2$*, and an associated central extension*

$$1 \to \mathbb{Z}/2 \to \widetilde{\mathfrak{H}} \to \mathfrak{H} \to 1.$$

(vii) *For* $n \geqslant 2$*, the inflation map* $H^2(\mathfrak{H}, \mathbb{Z}/2) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}/2^n), \mathbb{Z}/2)$ *is an isomorphism.*

(viii) *The nonzero element of* $H^2(\mathsf{Sp}(2g,\mathbb{Z})/\mathfrak{K}, \mathbb{Z}/2)$ *inflates to the reduction modulo two of* $\frac{1}{4}[\tau]$ *as an element of* $H^2(\mathsf{Sp}(2g,\mathbb{Z}), \mathbb{Z}/2)$*.*

(ix) *Restricting the central extension of* $\mathfrak{H}$ *to the subgroup* $\Gamma(2g,2)/\mathfrak{K}$ *gives an almost extraspecial group* $2^{1+(2g+1)} \leqslant \widetilde{\mathfrak{H}}$*.*

The proof of this theorem occupies the rest of the paper.

3 Extraspecial and almost extraspecial groups

For background on extraspecial and almost extraspecial groups, we refer the reader to Section I.5.5 of Gorenstein [15] and Section III.13 of Huppert [21], as well as the papers of Bouc and Mazza [3], Carlson and Thévenaz [4], Glasby [12], Griess [16], Hall and Higman [17], Lam and Smith [24], Quillen [30], Schmid [36], Stancu [37], and the letter from Isaacs to Diaconis reproduced in the appendix of Diaconis [8].

The cohomology ring $H^*((\mathbb{Z}/2)^n, \mathbb{Z}/2)$ is a polynomial ring in generators $z_1, \dots, z_n$ of degree one. Thus

$$H^1((\mathbb{Z}/2)^n, \mathbb{Z}/2) = \mathsf{Hom}((\mathbb{Z}/2)^n, \mathbb{Z}/2)$$

is an n–dimensional vector space spanned by the linear forms $z_1, \dots, z_n$. An element of degree two is therefore a quadratic form $\mathfrak{q}$ on $(\mathbb{Z}/2)^n$. Letting $\mathfrak{b}$ be the associated bilinear form $(\mathbb{Z}/2)^n \times (\mathbb{Z}/2)^n \to \mathbb{Z}/2$, we have

$$\mathfrak{q}(x+y) = \mathfrak{q}(x) + \mathfrak{q}(y) + \mathfrak{b}(x,y).$$

In the corresponding central extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^n \to 1$$

the role played by $\mathfrak{q}$ and $\mathfrak{b}$ is as follows. If x and y are elements of $(\mathbb{Z}/2)^n$, choose preimages $\widehat{x}$ and $\widehat{y}$ in E. Then as elements of the central $\mathbb{Z}/2$, we have $\widehat{x}^2 = \mathfrak{q}(x)$ and $[\widehat{x}, \widehat{y}] = \mathfrak{b}(x, y)$.

Definition 3.1 We say that a quadratic form $\mathfrak{q}$ is *nonsingular* if the radical $\mathfrak{b}^\perp$ of the associated bilinear form $\mathfrak{b}$ is $\{0\}$, and *nondegenerate* if $\mathfrak{b}^\perp \cap \mathfrak{q}^{-1}(0) = \{0\}$.

If $\mathfrak{q}$ is nonsingular then $n = 2g$ is even; in this case there are two isomorphism classes of quadratic forms, distinguished by the Arf invariant. The corresponding groups E defined by the central extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^{2g} \to 1$$

are called *extraspecial 2–groups*, and are characterised by the properties

$$\Phi(E) = [E, E] = Z(E) \cong \mathbb{Z}/2.$$

The two isomorphism classes of extraspecial groups are denoted 2_+^{1+2g} (Arf invariant zero) and 2_-^{1+2g} (Arf invariant one).

If $\mathfrak{q}$ is singular but nondegenerate then $n = 2g + 1$ is odd; in this case there is one isomorphism class of quadratic forms. The corresponding groups E defined by the central extension

$$1 \to \mathbb{Z}/2 \to E \to (\mathbb{Z}/2)^{2g+1} \to 1$$

are called *almost extraspecial groups*. The central product of $\mathbb{Z}/4$ with an extraspecial group of either isomorphism type of order 2^{1+2g} gives the almost extraspecial group of order $2^{1+(2g+1)}$.

If G is a group, we write $\mathsf{Aut}(G)$ for the group of automorphisms of G, we write $\mathsf{Out}(G)$ for the group of outer automorphisms, and we write $\mathsf{Inn}(G)$ for the group of inner automorphisms. These fit into short exact sequences

$$1 \to Z(G) \to G \to \mathsf{Inn}(G) \to 1 \quad \text{and} \quad 1 \to \mathsf{Inn}(G) \to \mathsf{Aut}(G) \to \mathsf{Out}(G) \to 1.$$

Writing the automorphism groups of the extraspecial and almost extraspecial groups as extensions of the outer by the inner automorphisms in this way, we have sequences

$$\begin{aligned} 1 &\to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2_+^{1+2g}) \to \mathsf{O}^+(2g, 2) \to 1, \\ 1 &\to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2_-^{1+2g}) \to \mathsf{O}^-(2g, 2) \to 1, \\ 1 &\to (\mathbb{Z}/2)^{2g} \to \mathsf{Aut}(2^{1+(2g+1)}) \to \mathsf{Sp}(2g, 2) \times \mathbb{Z}/2 \to 1, \end{aligned} \tag{3.2}$$

which do not split provided $g \geqslant 4$. It is the last case that is of interest to us: in this case the extra factor of $\mathbb{Z}/2$ in the outer automorphism group $\mathsf{Out}(E)$ acts by inverting the central element of order four, and for $g \geqslant 3$ the derived subgroup $\mathsf{Out}(E)'$ is $\mathsf{Sp}(2g, 2)$.

It was proved by Griess [16] using representation theory that in each case, there is an extension of the extraspecial group by its outer automorphism group, and of the almost extraspecial group by the subgroup of index two in its outer automorphism group.

We are interested in the almost extraspecial case. In this case, what Griess proved (part (b) of Theorem 5 of [16]) is that there is a group, which he denotes H_0, of shape $2^{1+(2g+1)}\mathsf{Sp}(2g, 2)$, with the following properties. The normal 2–subgroup $O_2(H_0)$ is the almost extraspecial group $2^{1+(2g+1)}$, and the quotient $H_0/Z(H_0)$ is isomorphic to the subgroup of index two in $\mathsf{Aut}(2^{1+(2g+1)})$.

Dempwolff [7] proved that for $g \geqslant 2$ there is a unique isomorphism class of nonsplit extensions of $\mathsf{Sp}(2g, 2)$ by an elementary abelian group $(\mathbb{Z}/2)^{2g}$ with nontrivial action. We shall combine the results of Griess and Dempwolff to show that the group $\mathfrak{H}$ of Theorem 2.2 is isomorphic to the quotient of Griess' group H_0 by the central subgroup of order two. This in turn allows us to compute $H^2(\mathfrak{H}, \mathbb{Z}/2)$ and relate it to $H^2(\mathsf{Sp}(2g, \mathbb{Z}), \mathbb{Z})$.

There is another approach to this, which we describe in a separate paper [2]. This avoids the use of the theorems of Griess and Dempwolff, replacing them with a computation showing that the group $\widetilde{\mathfrak{H}}$ has a Curtis–Tits–Steinberg type presentation. This approach is closely related to the action of $\widetilde{\mathfrak{H}}$ on a certain 2^g–dimensional space of theta functions, and shows that the following defines a projective representation $\sigma\colon \mathsf{Sp}(2g, \mathbb{Z}) \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])/\{\pm I\}$ with kernel $\mathfrak{K}$, and then induces a 2^g–dimensional representation $\widetilde{\mathfrak{H}} \to \mathsf{U}(2^g, \mathbb{Q}[\mathrm{i}])$.

The underlying vector space for the representation has as a basis the vectors e_w for $w \in \{0, 1\}^g$. In the following matrices, we regard $\det A$, which is really an element of $(\mathbb{Z}/4)^\times = \{1, -1\}$, as being either $+1$ or -1 in $\mathbb{C}$, and $\sqrt{\det A}$ is either 1 or i:

$$\sigma\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\colon e_w \mapsto \mathrm{i}^{w^t B w} e_w,$$

$$\sigma\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}\colon e_w \mapsto \sqrt{\det A}\, e_{(A^t)^{-1}w},$$

$$\sigma\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\colon e_w \mapsto \frac{1}{(1-\mathrm{i})^g} \sum_{w'} (-1)^{w^t w'} e_{w'}.$$

Note that $\mathsf{Sp}(2g,\mathbb{Z})$ is generated by these elements, but it is not at all obvious that the relations in $\mathsf{Sp}(2g,\mathbb{Z})$ hold up to sign for the linear transformations listed here; this is proved in [2]. Note also that in the first formula above, the matrix B may be interpreted as having diagonal entries in $\mathbb{Z}/4$ and off-diagonal entries in $\mathbb{Z}/2$, so that it represents a quadratic form on $(\mathbb{Z}/2)^g$, taking values in $\mathbb{Z}/4$.

Further references for the representation described here include Funar and Pitsch [11], Glasby [12], Gocho [13; 14], Nebe, Rains and Sloane [27], Runge [32; 33; 34] and Tsushima [41].

4 Signature modulo eight

Given an oriented surface bundle over a surface $\Sigma_g \to M \to \Sigma_h$, recall that there is an associated map $\chi\colon \pi_1(\Sigma_h) \to \mathsf{Sp}(2g,\mathbb{Z})$. Composing with $\sigma\colon \mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])/\{\pm I\}$, we obtain a map

$$\phi\colon \pi_1(\Sigma_h) = \langle a_1, b_1, \dots, a_h, b_h \mid [a_1,b_1]\cdots[a_h,b_h] = 1\rangle \to \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])/\{\pm I\}.$$

Now the commutators $[\phi(a_i),\phi(b_i)]$ are well defined in $\mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}])$, since changing the sign on $\phi(a_i)$ or $\phi(b_i)$ changes the sign twice in the commutator. Since the product of the commutators is in the kernel of ϕ, we have

$$[\phi(a_1),\phi(b_1)]\cdots[\phi(a_h),\phi(b_h)] = \pm I \in \mathsf{U}(2^g,\mathbb{Q}[\mathrm{i}]).$$

Theorem 4.1 *We have*

$$[\phi(a_1),\phi(b_1)]\cdots[\phi(a_h),\phi(b_h)] = \begin{cases} I & \text{if and only if signature}(M) \equiv 0 \pmod 8,\\ -I & \text{if and only if signature}(M) \equiv 4 \pmod 8.\end{cases}$$

Remarks 4.2 (1) As a method of computation, this theorem is not very useful, because of the large size of the matrices involved. Endo [10] provided a much more efficient and purely algebraic method for computing the signature, and not just modulo 8. On the other hand, there are consequences of the theorem that are not very apparent from the point of view of Endo's method.

(2) The following is a consequence of the theta function point of view, and will be discussed in a separate paper [2]. Let $\mathsf{Sp}^{\mathsf{q}}(2g,\mathbb{Z})$ be the theta subgroup of $\mathsf{Sp}(2g,\mathbb{Z})$. If the image of χ lies in $\mathsf{Sp}^{\mathsf{q}}(2g,\mathbb{Z})$ then we have signature$(M) \equiv 0 \pmod 8$. In particular, this holds if the action of $\pi_1(\Sigma_h)$ on $H^1(\Sigma_g,\mathbb{Z}/2)$ is trivial. This proves a special case of the Klaus–Teichner conjecture; see the introduction to [18] for details.

(3) Consider next the subgroup consisting of the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathsf{Sp}(2g, \mathbb{Z})$ such that the entries of C are even, and those of $\mathsf{Diag}(C)$ are divisible by four. If the image of χ lies in this subgroup then again we have $\text{signature}(M) \equiv 0 \pmod 8$. This will be proved in [2].

5 Symplectic groups and their Lie algebras

Let R be a commutative ring, and $\mathsf{Sp}(2g, R)$ be the symplectic group of dimension $2g$ over R. Explicitly, this consists of matrices X with entries in R, and satisfying $X^t J X = J$, where J is the symplectic form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I is a $g \times g$ identity matrix. Denoting by V_R a free R–module of rank g, and setting

$$W_R = V_R^* = \mathsf{Hom}_R(V_R, R),$$

the matrices X act on $U_R = V_R \oplus W_R$, preserving the skew-symmetric bilinear form

$$\langle\ ,\ \rangle \colon U_R \times U_R \to R$$

given by

$$\langle (v, w), (v', w') \rangle = w'(v) - w(v').$$

For the action of matrices in $\mathsf{Sp}(2g, R)$, we regard (v, w) as a column vector of length $2g$ with entries in R. The skew-symmetric bilinear form induces an isomorphism from U_R to U_R^* sending u to $\langle u,\ \rangle$. If $R = \mathbb{F}_2$, we shall write U, V and W instead of $U_{\mathbb{F}_2}$, $V_{\mathbb{F}_2}$ and $W_{\mathbb{F}_2}$.

The Lie algebra $\mathfrak{sp}(2g, R)$ consists of matrices Y with entries in R, and satisfying

$$JY + Y^t J = 0.$$

Thus

$$Y = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & -\mathrm{a}^t \end{pmatrix},$$

where b and c are symmetric. To say that b is symmetric is to say that as an element of

$$\mathsf{Hom}_R(W_R, V_R) \cong V_R \otimes_R V_R$$

it is invariant under the transposition swapping the two tensor factors. Thus b is an element of the divided square $D^2(V_R)$ (which may not be identified with the symmetric

square $S^2(U_R)$ unless 2 happens to be invertible in R, which will not be the case for us). Similarly, we have $\mathrm{c} \in D^2(W_R)$ and

$$\mathrm{a} \in \mathsf{Hom}_R(V_R, V_R) \cong V_R \otimes_R W_R.$$

Putting this together, we see that

$$Y \in D^2(V_R) \oplus D^2(W_R) \oplus (V_R \otimes_R W_R) \cong D^2(U_R).$$

Thus, as a module for $\mathsf{Sp}(2g, R)$, we have identified the Lie algebra $\mathfrak{sp}(2g, R)$ with the divided square of the natural module. More abstractly, if $u \in U_R$ then the symmetric tensor $u \otimes u$ is identified with the endomorphism sending x to $\langle u, x\rangle u$. Polarising, this identifies $u\otimes u' + u'\otimes u$ with the endomorphism of U_R sending x to $\langle u, x\rangle u' + \langle u', x\rangle u$. We have therefore proved the following.

Theorem 5.1 *For any commutative ring R, we have isomorphisms*

$$\mathfrak{sp}(2g, R) \cong D^2(U_R) \cong R^{g(2g+1)}.$$

The first isomorphism is an isomorphism of $\mathsf{Sp}(2g, R)$–modules, while the second is an isomorphism of R–modules.

We are interested in the group $\mathsf{Sp}(2g, \mathbb{Z}/4)$. This sits in a short exact sequence

$$1 \to \mathfrak{sp}(2g, 2) \to \mathsf{Sp}(2g, \mathbb{Z}/4) \to \mathsf{Sp}(2g, 2) \to 1.$$

The elementary abelian 2–subgroup is identified with $\mathfrak{sp}(2g, 2)$, the symplectic Lie algebra over $\mathbb{F}_2$, and consists of the matrices $I + 2Y$ with $Y \in \mathfrak{sp}(2g, 2)$. These have the form

$$\begin{pmatrix} I + 2\mathrm{a} & 2\mathrm{b} \\ 2\mathrm{c} & I - 2\mathrm{a}^t \end{pmatrix},$$

with b and c symmetric. We have a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^2(U) & \longrightarrow & D^2(U) & \longrightarrow & U & \longrightarrow & 0 \\ & & & & \big\downarrow\cong & & & & \\ & & & & \mathfrak{sp}(2g, 2) & & & & \end{array}$$

where $\Lambda^2(U)$ is spanned by elements of the form $u \otimes u' + u' \otimes u$. As a submodule of $\mathfrak{sp}(2g, 2)$, this consists of the matrices where $\mathsf{Diag}(\mathrm{b}) = \mathsf{Diag}(\mathrm{c}) = 0$. The quotient U

corresponds to the diagonal entries in b and c. Thus the above short exact sequence can be thought of as a short exact sequence of groups

$$1 \to \Gamma(2g,2,4)/\Gamma(2g,4) \to \Gamma(2g,2)/\Gamma(2g,4) \to \Gamma(2g,2)/\Gamma(2g,2,4) \to 1.$$

More generally, we have short exact sequences

$$1 \to \mathfrak{sp}(2g,2) \to \mathsf{Sp}(2g,\mathbb{Z}/2^{n+1}) \to \mathsf{Sp}(2g,\mathbb{Z}/2^n) \to 1,$$

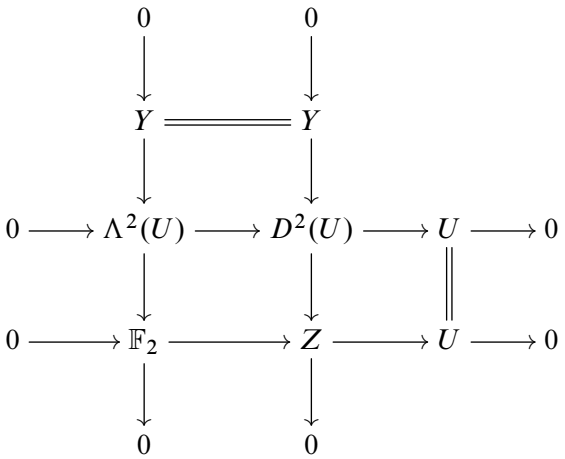
and

$$1 \to \Gamma(2g,2^n,2^{n+1})/\Gamma(2g,2^{n+1}) \to \Gamma(2g,2^n)/\Gamma(2g,2^{n+1}) \to \Gamma(2g,2^n)/\Gamma(2g,2^n,2^{n+1}) \to 1.$$

Proposition 5.2 *As modules over* $\mathsf{Sp}(2g,2)$*, for* $g \geqslant 1$ *and* $n \geqslant 1$*, we have*

$$\Gamma(2g,2^n)/\Gamma(2g,2^n,2^{n+1}) \cong U \quad \text{and} \quad \Gamma(2g,2^n,2^{n+1})/\Gamma(2g,2^{n+1}) \cong \Lambda^2(U).$$

Now the symplectic form on U gives us a map $\Lambda^2(U) \to \mathbb{F}_2$, sending $u \otimes u' + u' \otimes u$ to $\langle u, u' \rangle$. We write Y for the kernel of this map, and we write Z for $D^2(U)/Y$, an $\mathbb{F}_2$–vector space of dimension $2g+1$. Putting these together, we have the following diagram of modules:



We claim that the symplectic form on U lifts to a nondegenerate orthogonal form on Z, invariant under $\mathsf{Sp}(2g,2)$. The quadratic form $Z \to \mathbb{F}_2$ is given by

$$\mathfrak{q}(u \otimes u) = 0 \quad \text{and} \quad \mathfrak{q}(u \otimes u' + u' \otimes u) = \langle u, u' \rangle,$$

and the associated symmetric bilinear form is

$$\begin{aligned}\mathfrak{b}(u\otimes u, u'\otimes u') &= \langle u, u'\rangle,\\ \mathfrak{b}(u\otimes u' + u'\otimes u, u''\otimes u'') &= 0,\\ \mathfrak{b}(u\otimes u' + u'\otimes u, u''\otimes u''' + u'''\otimes u'') &= 0.\end{aligned}$$

A priori, these are a quadratic form and associated bilinear form on $D^2(U)$. But they clearly vanish identically on Y, and define a nondegenerate but singular quadratic form and associated bilinear form on Z. These are invariant under $\mathsf{Sp}(2g,2)$, which is therefore the orthogonal group on $Z \cong {\mathbb{F}_2}^{2g+1}$, displaying the isomorphism

$$\mathsf{Sp}(2g,2) \cong \mathsf{O}(2g+1,2).$$

Remark 5.3 Translating back from $D^2(U)$ to $\mathfrak{sp}(2g,2)$, the quadratic and bilinear form are given as follows:

$$\begin{aligned}\mathfrak{q}\begin{pmatrix}\mathsf{a} & \mathsf{b}\\ \mathsf{c} & -\mathsf{a}^t\end{pmatrix} &= \mathsf{Tr}(\mathsf{a}) + \langle \mathsf{Diag}(\mathsf{b}), \mathsf{Diag}(\mathsf{c})\rangle,\\ \mathfrak{b}\left(\begin{pmatrix}\mathsf{a} & \mathsf{b}\\ \mathsf{c} & -\mathsf{a}^t\end{pmatrix}, \begin{pmatrix}\mathsf{a}' & \mathsf{b}'\\ \mathsf{c}' & -\mathsf{a}'^t\end{pmatrix}\right) &= \langle \mathsf{Diag}(\mathsf{b}), \mathsf{Diag}(\mathsf{c}')\rangle + \langle \mathsf{Diag}(\mathsf{b}'), \mathsf{Diag}(\mathsf{c})\rangle.\end{aligned}$$

Here, the pointy brackets denote the standard inner product on ${\mathbb{F}_2}^g$ given by multiplying corresponding coordinates and summing.

The normal subgroup $\mathfrak{K}$ described in Section 2 is the inverse image of

$$Y \leqslant \mathfrak{sp}(2g,2) \leqslant \mathsf{Sp}(2g, \mathbb{Z}/4)$$

under the quotient map $\mathsf{Sp}(2g,\mathbb{Z}) \to \mathsf{Sp}(2g,\mathbb{Z}/4)$. Thus there is a short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g,2) \to 1$$

and the subgroup $Z \cong (\mathbb{Z}/2)^{2g+1}$ may be viewed as the orthogonal module ${\mathbb{F}_2}^{2g+1}$ for $\mathsf{Sp}(2g,2)$ via conjugation.

Remark 5.4 The submodule structure of the $\mathbb{F}_2\mathsf{Sp}(2g,2)$–modules $\Lambda^2(U)$ of dimension $g(2g-1)$ and $D^2(U) \cong \mathfrak{sp}(2g,2)$ of dimension $g(2g+1)$ can be described explicitly as follows (see also Hiss [20]). There is a map $\Lambda^2(U) \to \mathbb{F}_2$ corresponding to the symplectic form, given by

$$u\otimes u' + u'\otimes u \mapsto \langle u, u'\rangle.$$

There is a dual map $\mathbb{F}_2 \to \Lambda^2(U)$ coming from the fact that the representation $\Lambda^2(U)$ is self-dual. In terms of the natural bases $v_1, \dots, v_g$ of V and $w_1, \dots, w_g$ of W, this is given by

$$1 \mapsto \sum_i (v_i \otimes w_i + w_i \otimes v_i).$$

If $g = 1$ then $\Lambda^2(U) \cong \mathbb{F}_2$ is one-dimensional, $Y = 0$, and $Z = D^2(U)$ decomposes as a direct sum $\mathbb{F}_2 \oplus U$.

If $g \geqslant 2$ is even then the composite $\mathbb{F}_2 \to \Lambda^2(U) \to \mathbb{F}_2$ is zero, and the quotient of the kernel by the image is a simple module S of dimension $g(2g-1)-2$. Thus $\Lambda^2(U)$ is uniserial (ie it has a unique composition series) with composition factors $\mathbb{F}_2$, S, $\mathbb{F}_2$.

If $g \geqslant 3$ is odd, then the composite is nonzero, and $\Lambda^2(U)$ decomposes as a direct sum of a trivial module $\mathbb{F}_2$ and a simple module S of dimension $g(2g-1)-1$.

In both cases with $g \geqslant 2$, the unique maximal submodule of $D^2(U)$ is $\Lambda^2(U)$. We can therefore draw diagrams for the structure of $D^2(U) \cong \mathfrak{sp}(2g, 2)$ as follows:

$g = 1$: $\mathbb{F}_2 \oplus U$ $\qquad$ $g \geqslant 2$ even: $\begin{array}{c} U \\ | \\ \mathbb{F}_2 \\ | \\ S \\ | \\ \mathbb{F}_2 \end{array}$ $\qquad$ $g \geqslant 3$ odd: $\begin{array}{ccc} & U & \\ \mathbb{F}_2 & & S \end{array}$

For $g \geqslant 2$, the quotient Z of $D^2(U)$ has structure

$$\begin{array}{c} U \\ | \\ \mathbb{F}_2 \end{array}$$

and this is the orthogonal module for $\mathsf{Sp}(2g, 2) \cong \mathsf{O}(2g+1, 2)$. The submodule Y is S for $g \geqslant 3$ odd, it is a nonsplit extension

$$0 \to \mathbb{F}_2 \to Y \to S \to 0$$

for $g \geqslant 2$ even, and $Y = 0$ for $g = 1$.

Lemma 5.5 (i) *$H_0(\mathsf{Sp}(2g, 2), Y) = 0$ and $H_0(\mathsf{Sp}(2g, 2), U) = 0$ for $g \geqslant 1$.*

(ii) *$H_0(\mathsf{Sp}(2g, 2), Z) = 0$ and $H_0(\mathsf{Sp}(2g, 2), \mathfrak{sp}(2g, 2)) = 0$ for $g \geqslant 2$.*

Proof This follows immediately from the structure of Y, U, Z, and $\mathfrak{sp}(2g,2)$ as $\mathsf{Sp}(2g,2)$–modules given in the above remark, since these modules admit no nontrivial homomorphisms to $\mathbb{F}_2$ with trivial action. □

6 Computations in degree two homology and cohomology

Lemma 6.1 (i) *$H_2(\mathsf{Sp}(2g,2)) = 0$ for $g \geqslant 4$ and $H_2(\mathsf{Sp}(6,2)) \cong \mathbb{Z}/2$.*

(ii) *$H^2(\mathsf{Sp}(2g,2), \mathbb{Z}/2) = 0$ for $g \geqslant 4$ and $H^2(\mathsf{Sp}(6,2), \mathbb{Z}/2) \cong \mathbb{Z}/2$.*

Proof (i) This is computed in the paper of Steinberg [40].

(ii) This follows from the universal coefficient theorem, because $\mathsf{Sp}(2g,2)$ is perfect for $g \geqslant 3$. □

Lemma 6.2 *$H_0\big(\mathsf{Sp}(2g,\mathbb{Z}/2^n), H_1(\Gamma(2g,2^n))\big) = 0$ for $n \geqslant 1$ and $g \geqslant 2$.*

Proof Proposition 10.1 of Sato [35] computes $H_1(\Gamma(2g,N))$, the abelianisation of $\Gamma(2g,N)$, finding that the derived subgroup is $\Gamma(2g,N^2)$ if N is odd, and $\Gamma(2g,N^2,2N^2)$ if N is even.

Taking $N = 2^n$, this gives

$$H_1(\Gamma(2g,2^n)) \cong \Gamma(2g,2^n)\,/\,\Gamma(2g,2^{2n},2^{2n+1}).$$

As modules over $\mathsf{Sp}(2g,\mathbb{Z}/2^n)$ we have

$$\Gamma(2g,2^n)\,/\,\Gamma(2g,2^{2n}) \cong \mathfrak{sp}(2g,\mathbb{Z}/2^n) \quad \text{and} \quad \Gamma(2g,2^{2n})\,/\,\Gamma(2g,2^{2n},2^{2n+1}) \cong U$$

(see Section 5). This gives us a short exact sequence

$$0 \to U \to H_1(\Gamma(2g,2^n)) \to \mathfrak{sp}(2g,\mathbb{Z}/2^n) \to 0. \tag{6.3}$$

We also have short exact sequences

$$0 \to \mathfrak{sp}(2g,2) \to \mathfrak{sp}(2g,\mathbb{Z}/2^n) \to \mathfrak{sp}(2g,\mathbb{Z}/2^{n-1}) \to 0.$$

By Lemma 5.5(ii), for $g \geqslant 2$, we have

$$H_0(\mathsf{Sp}(2g,\mathbb{Z}/2^n), \mathfrak{sp}(2g,2)) \cong H_0(\mathsf{Sp}(2g,2), \mathfrak{sp}(2g,2)) = 0,$$

and so by induction on n and right exactness of H_0, we have

$$H_0(\mathsf{Sp}(2g,\mathbb{Z}/2^n), \mathfrak{sp}(2g,\mathbb{Z}/2^n)) = 0.$$

Finally, by Lemma 5.5(i) we have

$$H_0(\mathsf{Sp}(2g, \mathbb{Z}/2^n), U) \cong H_0(\mathsf{Sp}(2g, 2), U) = 0.$$

Therefore, using right exactness of H_0 on the sequence (6.3), the lemma is proved. □

Proposition 6.4 *For $n \geqslant 1$ and $g \geqslant 2$,*

(i) *the map $H_2(\mathsf{Sp}(2g, \mathbb{Z})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n))$ is surjective, and*

(ii) *the map $H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^{n+1})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n))$ is surjective.*

Proof (i) The short exact sequence

$$1 \to \Gamma(2g, 2^n) \to \mathsf{Sp}(2g, \mathbb{Z}) \to \mathsf{Sp}(2g, \mathbb{Z}/2^n) \to 1$$

gives rise to a five-term sequence in homology

$$H_2(\mathsf{Sp}(2g, \mathbb{Z})) \to H_2(\mathsf{Sp}(2g, \mathbb{Z}/2^n)) \to H_0\big(\mathsf{Sp}(2g, \mathbb{Z}/2^n), H_1(\Gamma(2g, 2^n))\big) \to H_1(\mathsf{Sp}(2g, \mathbb{Z})) \to H_1(\mathsf{Sp}(2g, \mathbb{Z}/2^n)) \to 0.$$

The proposition therefore follows immediately from Lemma 6.2.

(ii) This is similar, observing that $H_1(\Gamma(2g, 2^n) / \Gamma(2g, 2^{n+1})) \cong \mathfrak{sp}(2g, 2)$, so that by Lemma 5.5(ii) we have

$$H_0\big(\mathsf{Sp}(2g, \mathbb{Z}/2^n), H_1(\Gamma(2g, 2^n) / \Gamma(2g, 2^{n+1}))\big) = 0. \qquad \square$$

Corollary 6.5 *For $n \geqslant 1$ and $g \geqslant 3$, the map*

$$H^2(\mathsf{Sp}(2g, \mathbb{Z}/2^n), A) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}), A)$$

is injective for any abelian group of coefficients A with trivial action.

Proof This follows directly from Proposition 6.4 together with the universal coefficient theorem for cohomology, as the groups $\mathsf{Sp}(2g, \mathbb{Z})$ and $\mathsf{Sp}(2g, \mathbb{Z}/2^n)$ are perfect for $g \geqslant 3$. □

Proposition 6.6 *For $g \geqslant 2$, the maps $H_2(\mathsf{Sp}(2g, \mathbb{Z}/4)) \to H_2(\mathfrak{H}) \to H_2(\mathsf{Sp}(2g, 2))$ are surjective.*

Proof For the first map, we use the five-term sequence for the short exact sequence

$$1 \to Y \to \mathsf{Sp}(2g, \mathbb{Z}/4) \to \mathfrak{H} \to 1,$$

and the computation

$$H_0(\mathfrak{H}, H_1(Y)) = H_0(\mathsf{Sp}(2g, 2), Y) = 0$$

given in Lemma 5.5. Note that Y is an elementary abelian 2–group, so $H_1(Y) \cong Y$.

The computation for the second map is similar, using the short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g, 2) \to 1$$

and the computation $H_0(\mathsf{Sp}(2g, 2), Z) = 0$ given in Lemma 5.5. □

Corollary 6.7 *For $g \geqslant 3$, the inflation map $H^2(\mathfrak{H}, A) \to H^2(\mathsf{Sp}(2g, \mathbb{Z}/4), A)$ is injective for any abelian group of coefficients A with trivial action.*

Proof This follows directly from Proposition 6.6 and the universal coefficient theorem for cohomology, as the groups $\mathsf{Sp}(2g, \mathbb{Z}/4)$ are perfect for $g \geqslant 3$, hence all their quotients are perfect as well. □

Proposition 6.8 *For $g \geqslant 4$, the group $\mathfrak{H} = \mathsf{Sp}(2g, \mathbb{Z})/\mathfrak{K} \cong \mathsf{Sp}(2g, \mathbb{Z}/4)/Y$ is isomorphic to the quotient $\overline{H}_0$ of the group H_0 of Griess* (described in Section 3) *by its central subgroup of order two.*

Proof Examine the extension

$$\textbf{(6.9)} \quad 1 \to \Gamma(2g, 2)/\Gamma(2g, 2, 4) \to \mathsf{Sp}(2g, \mathbb{Z}/4)/\Gamma(2g, 2, 4) \to \mathsf{Sp}(2g, 2) \to 1.$$

This is nonsplit, since the element of order two in $\mathsf{Sp}(2g, 2)$ which swaps the first basis vectors of L and L^* and fixes the remaining basis vectors does not lift to an element of order two in $\mathsf{Sp}(2g, \mathbb{Z}/4)/\Gamma(2g, 2, 4)$.

Let E be the almost extraspecial group $O_2(H_0)$ of shape $2^{1+(2g+1)}$. The action of $\mathsf{Sp}(2g, 2)$ on $\Gamma(2g, 2)/\Gamma(2g, 2, 4) \cong U$ is the same as the action of $\mathsf{Out}(E)'$ on $\mathsf{Inn}(E)$ (see (3.2)), namely the natural symplectic module. It follows from the main theorem of Dempwolff [7] that

$$H^2\big(\mathsf{Sp}(2g, 2), \Gamma(2g, 2)/\Gamma(2g, 2, 4)\big)$$

is one-dimensional. Thus $\mathsf{Sp}(2g, \mathbb{Z}/4)/\Gamma(2g, 2, 4)$ is isomorphic to the group $\mathsf{Aut}(E)'$.

Since $\Gamma(2g,4) \subseteq \mathfrak{K} \subseteq \Gamma(2g,2,4)$ it follows that the short exact sequence

$$1 \to Z \to \mathfrak{H} \to \mathsf{Sp}(2g,2) \to 1$$

also does not split. We have $\Gamma(2g,2,4)/\mathfrak{K} \cong \mathbb{Z}/2$, and since $g \geqslant 4$, by Lemma 6.1 we have $H^2(\mathsf{Sp}(2g,2),\mathbb{Z}/2) = 0$. So $H^2\big(\mathsf{Sp}(2g,2),\Gamma(2g,2,4)/Y\big) = 0$, and hence $H^2(\mathsf{Sp}(2g,2),Z)$ is at most one-dimensional. Since we have a nonsplit extension (6.9), it is exactly one-dimensional. The modules $E/[E,E]$ and Z for $\mathsf{Sp}(2g,2)$ are both isomorphic to the natural orthogonal module of dimension $2g+1$, so it follows that $\mathfrak{H}$ is isomorphic to $\overline{H}_0$. □

Remark 6.10 In the case $g = 3$, Proposition 6.8 is still true, but needs a bit more work. The group $H^2(\mathsf{Sp}(6,2),\mathbb{Z}/2)$ is one-dimensional by Lemma 6.1, and we are left with the nasty possibility that $\mathfrak{H} = \mathsf{Sp}(6,\mathbb{Z}/4)/Y$ is isomorphic to a quotient of the pullback of $\overline{H}_0 \to \mathsf{Sp}(6,2)$ and $\widetilde{\mathsf{Sp}(6,2)} \to \mathsf{Sp}(6,2)$ by the diagonal central element of order two. In order to prove that $\mathfrak{H}$ is really isomorphic to $\overline{H}_0$ and not this other group, it suffices to construct a matrix representation of a double cover of $\mathfrak{H}$ of dimension 8. Explicit matrices for this representation were given in Section 3. On the other hand, the smallest faithful irreducible complex representation in the case of the other possibility has dimension 64. It is worth noticing, though, that it does not matter which possibility is true, if we just wish to prove the next theorem.

Lemma 6.11 *$H_2(\mathsf{Sp}(2g,\mathbb{Z})) \cong \mathbb{Z}$ for $g \geqslant 4$ and $H_2(\mathsf{Sp}(2g,\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z}/2$ for $g = 3$.*

Proof See for example Stein [39, Theorem 2.2] for $g = 3$, and Behr [1, Korollar 3.2], together with Stein [38, Theorem 5.3 and Remark 5 following Corollary 5.5], for $g \geqslant 4$. See also Putman [29, Theorem 5.1] for a different proof in the case $g \geqslant 4$. □

Theorem 6.12 *For $g \geqslant 3$, we have $H_1(\mathfrak{H}) = 0$. For $g \geqslant 4$, we have $H_2(\mathfrak{H}) \cong \mathbb{Z}/2$, and for $g = 3$, we have $H_2(\mathfrak{H}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. The map*

$$H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n)) \to H_2(\mathfrak{H})$$

is an isomorphism for $n \geqslant 2$.

Proof The computation of the abelianisation $H_1(\mathfrak{H})$ is straightforward. It follows from Propositions 6.4 and 6.6 that for $n \geqslant 2$ the maps

$$\begin{aligned}\textbf{(6.13)}\quad H_2(\mathsf{Sp}(2g,\mathbb{Z})) &\to H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n))\\ &\to H_2(\mathsf{Sp}(2g,\mathbb{Z}/4)) \to H_2(\mathfrak{H}) \to H_2(\mathsf{Sp}(2g,2))\end{aligned}$$

are surjective, and from Deligne's theorem (see the introduction) that the kernel of the first map contains every element divisible by two. Consulting Lemma 6.11, we see that $H_2(\mathsf{Sp}(2g,\mathbb{Z}/2^n))$ and $H_2(\mathfrak{H})$ are quotients of the groups given.

By Proposition 6.8, there is a nontrivial element of $H_2(\mathfrak{H},\mathbb{Z}/2)$ which is killed by the map to $H_2(\mathsf{Sp}(2g,2))$. Namely, the central extension $\widetilde{\mathfrak{H}}\to\mathfrak{H}$ is not inflated from $\mathsf{Sp}(2g,2)$ because the kernel of $\widetilde{\mathfrak{H}}\to\mathsf{Sp}(2g,2)$ is the nonabelian group E.

Comparing the value of $H_2(\mathsf{Sp}(2g,\mathbb{Z}))$ given in Lemma 6.11 with the value of $H_2(\mathsf{Sp}(2g,2))$ given in Lemma 6.1, the theorem follows. □

Corollary 6.14 *We have $H^2(\mathfrak{H},\mathbb{Z}/2)\cong\mathbb{Z}/2$. For $g\geqslant 3$ and $n\geqslant 2$, the map*

$$H^2(\mathfrak{H},A)\to H^2(\mathsf{Sp}(2g,\mathbb{Z}/2^n),A)$$

is an isomorphism for any abelian group of coefficients A with trivial action.

Proof This follows from Theorem 6.12 and the universal coefficient theorem. □

Appendix Summary of homology and cohomology groups

Values of the homology and cohomology groups are summarized in Tables 1 and 2.

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_g	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8$	$\mathbb{Z}/2$
$\mathsf{Sp}(2g,\mathbb{Z})$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/8$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2g,\mathbb{Z})$	0	$\begin{cases}\mathbb{Z}\oplus\mathbb{Z}/2\\ \mathbb{Z}\end{cases}$	$\mathbb{Z}$	$\begin{cases}\mathbb{Z}/8\oplus\mathbb{Z}/2\\ \mathbb{Z}/8\end{cases}$	$\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2\\ \mathbb{Z}/2\end{cases}$
$\mathsf{Sp}(2g,\mathbb{Z}/4)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2g,\mathbb{Z}/4)$	0	$\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2\\ \mathbb{Z}/4\end{cases}$	0	$\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2\\ \mathbb{Z}/4\end{cases}$	$\begin{cases}\mathbb{Z}/2\oplus\mathbb{Z}/2\\ \mathbb{Z}/2\end{cases}$
$\mathfrak{H}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(2g,2)$	0	0	0	0	0

Table 1: Values for $g\geq 4$. The group $\mathfrak{H}$ is $\mathsf{Sp}(2g,\mathbb{Z}/4)/Y$. When the value is expressed in cases, the first case is for g even and the second for g odd.

Group	$H_1(-)$	$H_2(-)$	$H^2(-,\mathbb{Z})$	$H^2(-,\mathbb{Z}/8)$	$H^2(-,\mathbb{Z}/2)$
Γ_3	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathsf{Sp}(6,\mathbb{Z})$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$P\mathsf{Sp}(6,\mathbb{Z})$	0	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/8\oplus\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(6,\mathbb{Z}/4)$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$P\mathsf{Sp}(6,\mathbb{Z}/4)$	0	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathfrak{H}=\mathsf{Sp}(6,\mathbb{Z}/4)/Y$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	0	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$
$\mathsf{Sp}(6,2)$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
Γ_2	$\mathbb{Z}/10$	$\mathbb{Z}/2$	$\mathbb{Z}/10$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
$\mathsf{Sp}(4,\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/8\oplus(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$
$P\mathsf{Sp}(4,\mathbb{Z})$	$\mathbb{Z}/2$	$\mathbb{Z}\oplus(\mathbb{Z}/2)^2$	$\mathbb{Z}\oplus\mathbb{Z}/2$	$\mathbb{Z}/8\oplus(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^4$
$\mathsf{Sp}(4,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
$P\mathsf{Sp}(4,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^4$
$\mathfrak{H}=\mathsf{Sp}(4,\mathbb{Z}/4)/Y$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
$\mathsf{Sp}(4,2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
Γ_1	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$	$\mathbb{Z}/4$	$\mathbb{Z}/2$
$\mathsf{Sp}(2,\mathbb{Z})$	$\mathbb{Z}/12$	0	$\mathbb{Z}/12$	$\mathbb{Z}/4$	$\mathbb{Z}/2$
$P\mathsf{Sp}(2,\mathbb{Z})$	$\mathbb{Z}/6$	0	$\mathbb{Z}/6$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$\mathsf{Sp}(2,\mathbb{Z}/4)$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$P\mathsf{Sp}(2,\mathbb{Z}/4)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$
$\mathfrak{H}=\mathsf{Sp}(2,\mathbb{Z}/4)/Y$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/4\oplus\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$
$\mathsf{Sp}(2,2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

Table 2: Values for $1\leqslant g\leqslant 3$

References

[1] **H Behr**, *Explizite Präsentation von Chevalley-gruppen über Z*, Math. Z. 141 (1975) 235–241 MR

[2] **D J Benson**, *Theta functions and a presentation of* $2^{1+(2g+1)}\mathsf{Sp}(2g,2)$, preprint (2017) `https://tinyurl.com/DJBenson-papers`

[3] **S Bouc**, **N Mazza**, *The Dade group of (almost) extraspecial p–groups*, J. Pure Appl. Algebra 192 (2004) 21–51 MR

[4] **J F Carlson**, **J Thévenaz**, *Torsion endo-trivial modules*, Algebr. Represent. Theory 3 (2000) 303–335 MR

[5] **J H Conway**, **R T Curtis**, **S P Norton**, **R A Parker**, **R A Wilson**, *Atlas of finite groups: maximal subgroups and ordinary characters for simple groups*, Oxford Univ. Press, Eynsham (1985) MR

[6] **P Deligne**, *Extensions centrales non résiduellement finies de groupes arithmétiques*, C. R. Acad. Sci. Paris Sér. A-B 287 (1978) A203–A208 MR

[7] **U Dempwolff**, *Extensions of elementary abelian groups of order 2^{2n} by $S_{2n}(2)$ and the degree 2–cohomology of $S_{2n}(2)$*, Illinois J. Math. 18 (1974) 451–468 MR

[8] **P Diaconis**, *Threads through group theory*, from "Character theory of finite groups" (M L Lewis, G Navarro, D S Passman, T R Wolf, editors), Contemp. Math. 524, Amer. Math. Soc., Providence, RI (2010) 33–47 MR

[9] **C J Earle**, **J Eells**, *A fibre bundle description of Teichmüller theory*, J. Differential Geometry 3 (1969) 19–43 MR

[10] **H Endo**, *A construction of surface bundles over surfaces with non-zero signature*, Osaka J. Math. 35 (1998) 915–930 MR

[11] **L Funar**, **W Pitsch**, *Finite quotients of symplectic groups vs mapping class groups*, preprint (2016) `http://www.maths.ed.ac.uk/~v1ranick/papers/funarpitsch1.pdf`

[12] **S P Glasby**, *On the faithful representations, of degree 2^n, of certain extensions of 2–groups by orthogonal and symplectic groups*, J. Austral. Math. Soc. Ser. A 58 (1995) 232–247 MR

[13] **T Gocho**, *The topological invariant of three-manifolds based on the* U(1) *gauge theory*, Proc. Japan Acad. Ser. A Math. Sci. 66 (1990) 237–239 MR

[14] **T Gocho**, *The topological invariant of three-manifolds based on the* U(1) *gauge theory*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 39 (1992) 169–184 MR

[15] **D Gorenstein**, *Finite groups*, Harper & Row, New York (1968) MR

[16] **R L Griess, Jr**, *Automorphisms of extra special groups and nonvanishing degree* 2 *cohomology*, Pacific J. Math. 48 (1973) 403–422 MR

[17] **P Hall**, **G Higman**, *On the p–length of p–soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. 6 (1956) 1–42 MR

[18] **I Hambleton**, **A Korzeniewski**, **A Ranicki**, *The signature of a fibre bundle is multiplicative mod* 4, Geom. Topol. 11 (2007) 251–314 MR

[19] **M-E Hamstrom**, *Homotopy groups of the space of homeomorphisms on a* 2*–manifold*, Illinois J. Math. 10 (1966) 563–573 MR

[20] **G Hiss**, *Die adjungierten Darstellungen der Chevalley–Gruppen*, Arch. Math. (Basel) 42 (1984) 408–416 MR

[21] **B Huppert**, *Endliche Gruppen, I*, Grundl. Math. Wissen. 134, Springer (1967) MR

[22] **J-i Igusa**, *On the graded ring of theta-constants*, Amer. J. Math. 86 (1964) 219–246 MR

[23] **A Korzeniewski**, *On the signature of fibre bundles and absolute Whitehead torsion*, PhD thesis, University of Edinburgh (2005) `http://hdl.handle.net/1842/12106`

[24] **T Y Lam**, **T Smith**, *On the Clifford–Littlewood–Eckmann groups: a new look at periodicity mod* 8, Rocky Mountain J. Math. 19 (1989) 749–786 MR

[25] **R Luke**, **W K Mason**, *The space of homeomorphisms on a compact two-manifold is an absolute neighborhood retract*, Trans. Amer. Math. Soc. 164 (1972) 275–285 MR

[26] **W Meyer**, *Die Signatur von Flächenbündeln*, Math. Ann. 201 (1973) 239–264 MR

[27] **G Nebe**, **E M Rains**, **N J A Sloane**, *The invariants of the Clifford groups*, Des. Codes Cryptogr. 24 (2001) 99–121 MR

[28] **M Newman**, **J R Smart**, *Symplectic modulary groups*, Acta Arith. 9 (1964) 83–89 MR

[29] **A Putman**, *The Picard group of the moduli space of curves with level structures*, Duke Math. J. 161 (2012) 623–674 MR

[30] **D Quillen**, *The mod* 2 *cohomology rings of extra-special* 2*–groups and the spinor groups*, Math. Ann. 194 (1971) 197–212 MR

[31] **C Rovi**, *The nonmultiplicativity of the signature modulo* 8 *of a fibre bundle is an Arf–Kervaire invariant*, Algebr. Geom. Topol. 18 (2018) 1281–1322 MR

[32] **B Runge**, *On Siegel modular forms, I*, J. Reine Angew. Math. 436 (1993) 57–85 MR

[33] **B Runge**, *On Siegel modular forms, II*, Nagoya Math. J. 138 (1995) 179–197 MR

[34] **B Runge**, *Codes and Siegel modular forms*, Discrete Math. 148 (1996) 175–204 MR

[35] **M Sato**, *The abelianization of the level d mapping class group*, J. Topol. 3 (2010) 847–882 MR

[36] **P Schmid**, *On the automorphism group of extraspecial* 2*–groups*, J. Algebra 234 (2000) 492–506 MR

[37] **R Stancu**, *Almost all generalized extraspecial p–groups are resistant*, J. Algebra 249 (2002) 120–126 MR

[38] **M R Stein**, *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math. 93 (1971) 965–1004 MR

[39] **M R Stein**, *The Schur multipliers of* $\mathrm{Sp}_6(\mathbb{Z})$, $\mathrm{Spin}_8(\mathbb{Z})$, $\mathrm{Spin}_7(\mathbb{Z})$, *and* $F_4(\mathbb{Z})$, Math. Ann. 215 (1975) 165–172 MR

[40] **R Steinberg**, *Generators, relations and coverings of algebraic groups, II*, J. Algebra 71 (1981) 527–543 MR

[41] **K Tsushima**, *On a decomposition of Bruhat type for a certain finite group*, Tsukuba J. Math. 27 (2003) 307–317 MR

Institute of Mathematics, University of Aberdeen
Aberdeen, United Kingdom

Department of Mathematics, Karlsruhe Institute of Technology
Karlsruhe, Germany

School of Mathematics, University of Edinburgh
Edinburgh, United Kingdom

Department of Mathematics, Indiana University Bloomington
Bloomington, IN, United States

d.j.benson@abdn.ac.uk, caterina.campagnolo@kit.edu, crovi@indiana.edu

Received: 26 November 2017 Revised: 30 May 2018

msp

Algebraic & Geometric Topology 18 (2018) 4093–4107

On periodic groups of homeomorphisms of the 2–dimensional sphere

JONATHAN CONEJEROS

We prove that every finitely generated group of homeomorphisms of the 2–dimensional sphere all of whose elements have a finite order which is a power of 2 and is such that there exists a uniform bound for the orders of the group elements is finite. We prove a similar result for groups of area-preserving homeomorphisms without the hypothesis that the orders of group elements are powers of 2 provided there is an element of even order.

20F50, 37B05, 37E30, 37E45, 57S25

1 Introduction

Despite some remaining open questions, there is a very complete understanding of group actions on 1–manifolds (see Ghys [3] and Navas [13]). However, when passing to the 2–dimensional setting, many natural and fundamental questions remain unsolved. One of the most striking ones is related to the Burnside problem.

Recall that Burnside (see [1]) proved that every finitely generated linear group all of whose elements have finite order and such that there exists a uniform bound for the orders of the group elements is actually finite. This result has been extended to some other contexts, but fails in general, as is shown by classical examples due to Golod (see [4]). Later, Ol'shanskii (see [14]), Ivanov (see [8]), and Lysenok (see [12]) exhibited many other examples of infinite, finitely generated groups all of whose elements have a finite order which is bounded by a uniform constant. The case of groups of homeomorphisms is particularly interesting. The following question seems to be folklore: Does there exist an infinite, finitely generated group of homeomorphisms of the 2–dimensional sphere all of whose elements have finite order? Some progress on this question has been made by Guelman and Liousse [5; 6] (provided there is a finite orbit for the action and — in some cases — that all maps involved are of class C^1), and Hurtado [7] (provided the action is by C^∞ volume-preserving diffeomorphisms and there is a uniform bound for the orders of the group elements). The main result

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4093

of this paper yields a new positive result for actions by homeomorphisms under some hypothesis on the orders of group elements. For short, in what follows, we will call *periodic* a group in which all elements have finite order, we will say that such a group is a 2–*group* if the orders of group elements are powers of 2. Also, we will say that a periodic group has *uniformly bounded order* if there exists a uniform bound for the orders of its group elements. Our main result is the following:

Theorem A *Let G be a finitely generated 2–group of homeomorphisms of the 2–dimensional sphere. Suppose that G has uniformly bounded order. Then G is finite.*

We note that the composition of two orientation-reversing homeomorphisms preserves orientation. We deduce that the subgroup of orientation-preserving homeomorphisms has at most index 2 in the group G above. Moreover, Schreier's lemma states that any finite-index subgroup in a finitely generated group is finitely generated. Hence, in order to prove Theorem A, it is enough to show that a finitely generated 2–group of *orientation-preserving* homeomorphisms of the 2–dimensional sphere is finite provided there is a uniform bound for the orders of the group elements. As a first step to proving this, we will show the next result, which is interesting by itself.

Theorem B *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G acts with a global fixed point. Then G is finite and cyclic.*

The second step in the proof is the following:

Theorem C *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has a finite orbit. Then G is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.*

As a by product of our methods, we obtain the following result for groups of area-preserving homeomorphisms:

Theorem D *Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has uniformly bounded order and contains an element of even order. Then G is finite.*

As above, in order to prove Theorem D, it is enough to show an analog of Theorem C in this setting.

Theorem E *Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Suppose that G has a finite orbit. Then G is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.*

Acknowledgements The preparation of this article was funded by the FONDECYT postdoctoral grant No. 3170455 entitled "Algunos problemas para grupos de homeomorfismos de superficies". I thank A Navas for several useful discussions, suggestions and corrections.

2 Preliminary results

2.1 Local rotation set

In this section, we introduce the notion of local rotation introduced by F Le Roux (see [11]). Since local dynamics (more precisely, the dynamics around a fixed point) does not fit into a compact framework, we consider only rotation numbers of "good orbits". This means that, in order to get a definition of a rotation set which is invariant under conjugacy, we consider only recurrent points close to the fixed point.

Let h be a homeomorphism of the plane $\mathbb{R}^2$ that preserves the orientation and fixes the vector $\mathbf{0} := (0,0) \in \mathbb{R}^2$. We will denote by $\widetilde{\mathbb{A}} = \mathbb{R} \times (0, +\infty)$ the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$. Let $\widetilde{\pi}\colon \widetilde{\mathbb{A}} \to \mathbb{R}^2 \setminus \{\mathbf{0}\}$ be the corresponding universal covering map and $p_1\colon \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ the projection on the first coordinate. Let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$. We say that the *rotation number (around $\mathbf{0}$)* of a h–recurrent point $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ under $\widetilde{h}$ is well defined and equal to $\rho_{\mathbf{0}}(\widetilde{h}, x) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ if for every sequence of integers $(n_k)_{k\in\mathbb{N}}$ which goes to $+\infty$ such that $(h^{n_k}(x))_{k\in\mathbb{N}}$ converges to x, the sequence $(\rho_{n_k}(\widetilde{h}, x))_{k\in\mathbb{N}}$, defined as

$$\rho_{n_k}(\widetilde{h}, x) := \frac{1}{n_k}(p_1(\widetilde{h}^{n_k}(\widetilde{x})) - p_1(\widetilde{x})),$$

where $\widetilde{x}$ is a point in $\widetilde{\pi}^{-1}(x)$, converges to $\rho_{\mathbf{0}}(\widetilde{h}, x)$. Notice that this definition does not depend on the choice of $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

The *local rotation set (around the fixed point $\mathbf{0}$) of $\widetilde{h}$*, which we denote by $\rho_{\mathbf{0}}(\widetilde{h})$, is the set of all rotation numbers of recurrent points of h.

We have the following properties (see [11] for more details):

(1) The rotation numbers of a recurrent point and, consequently, the local rotation set, are invariant under (local) oriented topological conjugacy. More precisely, if φ is a homeomorphism of $\mathbb{R}^2$ that preserves the orientation and fixes $\mathbf{0} \in \mathbb{R}^2$ and $\widetilde{\varphi}$ is a lift of φ to $\widetilde{\mathbb{A}}$, then

$$\rho_{\mathbf{0}}(\widetilde{\varphi}^{-1}\widetilde{h}\widetilde{\varphi}) = \rho_{\mathbf{0}}(\widetilde{h}).$$

(2) For every $p, q \in \mathbb{Z}$, we have $\rho_{\mathbf{0}}(\widetilde{h}^q + (p, 0)) = q\rho_{\mathbf{0}}(\widetilde{h}) + p$. A similar formula holds for the rotation number of a recurrent point.

2.2 Periodic, orientation-preserving homeomorphisms of the 2–dimensional sphere

We say that an orientation-preserving homeomorphism g of the 2–dimensional sphere is *periodic* if its order is finite, that is, if there exists an integer q such that $g^q = \mathrm{Id}$. We recall that Keréjártó proved that every periodic, orientation-preserving homeomorphism of the 2–dimensional sphere is conjugate to a rotation (see [9; 2]). Formally, we have the following proposition:

Proposition 2.1 *Let g be a periodic, orientation-preserving homeomorphism of the 2–dimensional sphere. Then there exist an orientation-preserving homeomorphism h of the 2–dimensional sphere which has the same fixed points as g and a rotation R such that $hgh^{-1} = R$. In particular, if g is nontrivial, then it has exactly two fixed points, and every point that is not fixed is periodic. Additionally, the periods of all nonfixed, periodic points are the same.*

2.3 Local rotation set for periodic homeomorphisms

In our setting, let g be a periodic homeomorphism that preserves the orientation of $\mathbb{S}^2$ and fixes a point $z \in \mathbb{S}^2$. Considering a chart ϕ centered at z, we have that $h = \phi g \phi^{-1}$ is a periodic homeomorphism of the plane $\mathbb{R}^2$ that preserves the orientation and fixes the vector $\phi(z) = \mathbf{0} \in \mathbb{R}^2$. Let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$ the universal covering of $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, which, as before, we identify with $\mathbb{R} \times (0, +\infty)$. Suppose that h has order q, that is, q is the smallest positive integer such that $h^q = \mathrm{Id}$. We write $\mathrm{ord}(h) = q$.

Then there exists an integer p such that, for every $\widetilde{x} \in \widetilde{\mathbb{A}}$,

$$\widetilde{h}^q(\widetilde{x}) = \widetilde{x} + (p, 0).$$

It is not hard to prove that p and q are coprime. Also, every point is recurrent for h and has a rotation number around $\mathbf{0}$ equal to p/q. Because of the invariance under conjugacy by $\rho_{\mathbf{0}}(\widetilde{h})$ (property (1) above), this number does not depend on the choice of the chart ϕ. Hence, by property (2) above, we can associate to our periodic homeomorphism g a unique "local rotation number" around z, defined as

$$\rho_{\mathrm{loc},z}(g) = \frac{p}{q} \pmod 1 \in \mathbb{T}^1.$$

Remark 1 Clearly, if $\rho_{\mathrm{loc},z}(g) = 0$, then g is the identity.

Given $z \in \mathbb{S}^2$, we will denote by $\mathrm{Homeo}^+(\mathbb{S}^2; z)$ the group of all homeomorphisms of $\mathbb{S}^2$ that preserve the orientation and fix z. By the discussion above, the "local rotation number map" is well defined for a periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. We have the following properties:

Lemma 2.2 *Let g be an element of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$.*

(1) *The local rotation set around z is invariant under (local) oriented topological conjugacy. More precisely, if φ belongs to $\mathrm{Homeo}^+(\mathbb{S}^2; z)$, then*

$$\rho_{\mathrm{loc},z}(\varphi^{-1} g \varphi) = \rho_{\mathrm{loc},z}(g).$$

(2) *For every $q \in \mathbb{Z}$, we have $\rho_{\mathrm{loc},z}(g^q) = q\rho_{\mathrm{loc},z}(g)$.*

The first nontrivial observation concerning the local rotation set is the following:

Proposition 2.3 *Let G_0 be a periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. The local rotation number map defined on G_0 is a group homomorphism into $\mathbb{T}^1$ if and only if G_0 is abelian.*

Proof Let f and g be two elements in G_0. We recall that $[f, g] := fgf^{-1}g^{-1}$ denotes the commutator of f and g. If $\rho_{\mathrm{loc},z}$ is a group homomorphism, then $\rho_{\mathrm{loc},z}([f, g])$ is null, which implies that $[f, g] = \mathrm{Id}$ (see Remark 1). As f and g are arbitrary, G_0 is abelian.

Conversely, assume that G_0 is abelian, and let f and g be two elements in G_0. Consider a chart ϕ centered at z, and let $h_1 := \phi f \phi^{-1}$ and $h_2 := \phi g \phi^{-1}$ be the conjugate homeomorphisms. Both h_1 and h_2 are periodic homeomorphisms of the plane $\mathbb{R}^2$ that preserve the orientation and fix the vector $\phi(z) = \mathbf{0} \in \mathbb{R}^2$. Since

h_1 and h_2 commute, we can consider commuting lifts $\widetilde{h}_1$ and $\widetilde{h}_2$ of h_1 and h_2, respectively. Suppose that $\rho_{\mathbf{0}}(\widetilde{h}_1) = p'/q'$ and $\rho_{\mathbf{0}}(\widetilde{h}_2) = p/q$. Then, for every $\widetilde{x}$ and $\widetilde{x}'$ in $\widetilde{\mathbb{A}}$, we have

$$\widetilde{h}_2^q(\widetilde{x}) = \widetilde{x} + (p, 0) \quad \text{and} \quad \widetilde{h}_1^{q'}(\widetilde{x}') = \widetilde{x}' + (p', 0).$$

Thus,

$$(\widetilde{h}_1\widetilde{h}_2)^{q'q}(\widetilde{x}) = \widetilde{h}_1^{q'q}(\widetilde{h}_2^{q'q}(\widetilde{x})) = \widetilde{h}_2^{q'q}(\widetilde{x}) + (qp', 0) = \widetilde{x} + (q'p, 0) + (qp', 0).$$

Therefore,

$$\rho_{\mathrm{loc},z}(fg) = \frac{q'p + qp'}{q'q} = \frac{p}{q} + \frac{p'}{q'} = \rho_{\mathrm{loc},z}(f) + \rho_{\mathrm{loc},z}(g).$$

This shows that $\rho_{\mathrm{loc},z}$ is a group homomorphism. □

2.4 Consequences for abelian, periodic subgroups of orientation-preserving homeomorphisms of $\mathbb{S}^2$ that fix a point

From Proposition 2.3, we know that the "local rotation number map" is an injective group homomorphism for abelian, periodic subgroups of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$ (see Remark 1). So $\rho_{\mathrm{loc},z}$ gives an isomorphism with its image, a periodic subgroup of $\mathbb{R}/\mathbb{Z}$.

We deduce the following results:

Lemma 2.4 *Let A be an abelian, periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. Let a and b be two elements of A with $\mathrm{ord}(a) = \mathrm{ord}(b)$. Then there exists an integer $i \in \{1, \dots, \mathrm{ord}(a) - 1\}$ such that $b = a^i$. In particular, there exists at most one element of order 2 in A.*

Lemma 2.5 *If A is a finite, abelian subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$, then A is cyclic.*

3 Burnside problem for 2–groups of homeomorphisms of $\mathbb{S}^2$: particular cases

In order to prove Theorem A, it is enough to prove that every finitely generated 2–group G of *orientation-preserving* homeomorphisms of the 2–dimensional sphere is finite. This is the purpose of the following two subsections. Our proof consists in first considering the case where G has a global fixed point and later the case where G has a finite orbit. Finally, we settle the general case, and for this we prove that the group G contains only a finite number of involutions.

3.1 The case where the group has a global fixed point

In this section we will prove Theorem B, that is, every finitely generated 2–group of orientation-preserving homeomorphisms of the 2–dimensional sphere that has a global fixed point is finite and cyclic. The idea of the proof is as follows: Notice that a (nontrivial) 2–group G_0 always contains involutions, that is, elements of order 2. The key step consists in proving that, in our case, there is a unique involution in G_0. This implies that such an involution must belong to the center of G_0, that is, it commutes with each element of G_0. Since we are assuming that G_0 is a 2–group, we can deduce that G_0 is abelian using the following property (see Proposition 3.6 below): if f and g^2 in G_0 commute, then f and g commute. Finally, using that G_0 is finitely generated, we can conclude that G_0 is finite, and hence cyclic by Lemma 2.5.

We start with a lemma that follows from classical properties of the local rotation set around a fixed point (Lemma 2.2).

Lemma 3.1 *Let g be a finite-order element in* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *Suppose that g is conjugate* (by an element in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$) *to its inverse. Then* $g^2 = \mathrm{Id}$.

Proof By hypothesis, there exists $\varphi \in \mathrm{Homeo}^+(\mathbb{S}^2; z)$ such that $\varphi^{-1} g \varphi = g^{-1}$. By Lemma 2.2,

$$\rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}(g^{-1}) = -\rho_{\mathrm{loc},z}(g).$$

This implies that $0 = 2\rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}(g^2)$. Since g has finite order, it must satisfy $g^2 = \mathrm{Id}$. □

Proposition 3.2 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *Let σ and σ' be two elements of order 2 in G_0. Then σ and σ' commute.*

Proof We know that, in any group, $\sigma\sigma'$ is conjugate (by σ) to $\sigma'\sigma$. Indeed,

$$\sigma'\sigma = (\sigma^{-1}\sigma)\sigma'\sigma = \sigma^{-1}(\sigma\sigma')\sigma.$$

Since σ and σ' have order 2, we have that $(\sigma\sigma')^{-1} = \sigma'\sigma$. By Lemma 3.1, it follows that $\sigma\sigma'$ has order 2. Since σ, σ', and $\sigma\sigma'$ have order 2, we deduce (using an argument due to Burnside) that

$$\mathrm{Id} = (\sigma\sigma')^2 = \sigma\sigma'\sigma\sigma' = \sigma\sigma'\sigma^{-1}\sigma'^{-1} := [\sigma, \sigma'].$$

This implies that σ and σ' commute. □

We deduce the following properties:

Proposition 3.3 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *Then G_0 has at most one element of order* 2.

Proof Suppose that σ and σ' are two elements of order 2 in G_0. By the previous proposition, the group generated by σ and σ' is an abelian, periodic subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. By Lemma 2.4, we deduce that $\sigma = \sigma'$. □

Corollary 3.4 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *If $\sigma \in G_0$ has order* 2, *then σ belongs to the center of G_0.*

Proof Let g be an element of G_0. Since $g\sigma g^{-1}$ has order 2, by Proposition 3.3 we deduce that $g\sigma g^{-1} = \sigma$. This implies that g and σ commute. □

Lemma 3.5 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *Let f and g be two elements in G_0. Suppose that f and g^2 commute. Then, for every $i \in \{0, \dots, \mathrm{ord}(f) - 1\}$, the element $[f^i, g]$, the commutator of f^i and g, satisfies $[f^i, g]^2 = \mathrm{Id}$. Moreover, $[g, f^i]^2 = \mathrm{Id}$.*

Proof Since f and g^2 commute, we have that f^i and g^2 commute, that is, $f^i g^2 = g^2 f^i$. Hence,

$$g^{-1} f^{-i} = g f^{-i} g^{-2}.$$

Consequently,

$$[g, f^i] = g f^i g^{-1} f^{-i} = g f^i (g f^{-i} g^{-2}) = g(f^i g f^{-i} g^{-1}) g^{-1} = g[f^i, g] g^{-1}.$$

Since $[f^i, g]^{-1} = [g, f^i]$, it follows from Lemma 3.1 that $[f^i, g]^2 = \mathrm{Id}$. Finally, notice that $[g, f^i] = [f^i, g]^{-1}$. Hence, we deduce that $[g, f^i]^2 = \mathrm{Id}$. □

Proposition 3.6 *Let G_0 be a periodic subgroup of* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. *Let f and g be two elements in G_0. If f and g^2 commute, then f and g commute.*

Proof By the previous lemma, we have that $[f, g]^2 = \mathrm{Id}$ and so $g[f, g]^2 = g$, that is,

$$\begin{aligned} g &= g(fgf^{-1}g^{-1})(fgf^{-1}g^{-1}) \\ &= (gf)(gf^{-1}g^{-1}f)g(f^{-1}g^{-1}) \\ &= (gf)[g, f^{-1}]g(gf)^{-1}. \end{aligned}$$

Applying the local rotation map and using its invariance under (local) topological conjugacy, we obtain

$$\rho_{\mathrm{loc},z}([g, f^{-1}]g) = \rho_{\mathrm{loc},z}(g).$$

Since $[g, f^{-1}]$ commutes with g (Corollary 3.4), the local rotation map restricted to the group generated by $[g, f^{-1}]$ and g is a group homomorphism (Proposition 2.3). Thus,

$$\rho_{\mathrm{loc},z}([g, f^{-1}]) + \rho_{\mathrm{loc},z}(g) = \rho_{\mathrm{loc},z}([g, f^{-1}]g) = \rho_{\mathrm{loc},z}(g).$$

Therefore, $\rho_{\mathrm{loc},z}([g, f^{-1}]) = 0$, and so $[g, f^{-1}] = \mathrm{Id}$. This contradiction proves that f and g commute. □

End of the proof of Theorem B Let G_0 be a finitely generated 2–group contained in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. Let f and g be two elements in G_0. Since G_0 is a 2–group, g has order 2^{p+1} for a certain integer $p \geq 0$, and then g^{2^p} has order 2. It follows from Corollary 3.4 that g^{2^p} and f commute. Applying the previous proposition, we obtain that $g^{2^{p-1}}$ and f commute. Iterating this argument, we get that f and g commute. Therefore, G_0 is an abelian, finitely generated group, and so it is finite. Finally, we deduce from Lemma 2.5 that G_0 is cyclic. □

3.2 The case where the group has a finite orbit

In this section, we prove Theorem C, that is, every finitely generated 2–group G of orientation-preserving homeomorphisms of the 2–dimensional sphere which has a finite orbit is finite. Moreover, if G has a finite orbit of cardinality 2, then it is either a cyclic or a dihedral group.

Proof of Theorem C Let z_0 be a point with finite G–orbit. We write $\mathcal{O}_G(z_0) = \{z_0, z_1, \ldots, z_n\}$, where, for every $i \in \{0, \ldots, n\}$, $z_i = g_i(z_0)$ for some $g_i \in G$. We denote by $\mathrm{Sta}_G(z_0)$, the stabilizer in G of z_0, that is, the set

$$\mathrm{Sta}_G(z_0) := \{g \in G : g(z_0) = z_0\}.$$

We first have, by Theorem B, that $\mathrm{Sta}_G(z_0)$ is a finite cyclic group. Finally, we conclude that G is finite, by proving that $G = \bigcup_{i=0}^{n} g_i(\mathrm{Sta}_G(z_0))$. Indeed, if $g \in G$, since $g(z_0) \in \mathcal{O}_G(z_0)$ there exists an integer $i \in \{0, \ldots, n\}$ such that $g(z_0) = g_i(z_0)$. Hence, $g_i^{-1} g \in \mathrm{Sta}_G(z_0)$ and then $g \in g_i(\mathrm{Sta}_G(z_0))$. This proves that G is finite. Now suppose that G has a finite orbit of cardinality 2. We will prove, in this case, G is either a cyclic or a dihedral finite group. Let z be a point with G–orbit of cardinality 2.

We write $\mathcal{O}_G(z) = \{z, z'\}$. We will consider the subgroup G_0 of homeomorphisms that fix both z and z'.

Lemma 3.7 *The group G_0 is an index-2, normal subgroup of G. In particular, G_0 is a finitely generated 2–group contained in* $\mathrm{Homeo}^+(\mathbb{S}^2; z)$.

Proof It is easy to check that G_0 is normal in G. Moreover, notice that if σ and σ' are in $G \setminus G_0$, then $\sigma\sigma'$ is in G_0. Hence, G_0 has index 2 in G. Moreover, Schreier's lemma states that any finite-index subgroup in a finitely generated group is finitely generated. Hence, as G is a finitely generated 2–group, we deduce that G_0 is a finitely generated 2–group contained in $\mathrm{Homeo}^+(\mathbb{S}^2; z)$. □

Lemma 3.8 *Every $g \in G \setminus G_0$ has order 2.*

Proof If $g \in G \setminus G_0$, then $g(z) = z'$ and $g(z') = z$. We deduce that $g^2(z) = z$. As the local rotation number of g is a singleton, we deduce that g has order 2 (see Proposition 2.1). □

End of the proof of Theorem C By Theorem B, we know that G_0 is a finite cyclic group. If $G = G_0$, then G is finite and cyclic. Otherwise, let g_0 in G be a generator of G_0, and let $g \in G \setminus G_0$. Consider Γ the subgroup of G generated by g_0 and g. We claim that $\Gamma = G$. Indeed, if g' is any element in $G \setminus G_0$, then $gg' \in G_0$, hence $g' \in \Gamma$. Moreover, as gg_0 and g do not belong to G_0, we have that gg_0 and g have order 2 (by the previous lemma). Hence, we have $gg_0gg_0 = \mathrm{Id} = g^2$, which yields $gg_0g^{-1} = g_0^{-1}$. It follows that G is a dihedral group. □

4 Burnside problem for 2–groups of homeomorphisms of the 2–dimensional sphere

In this section, we prove Theorem A, that is, every finitely generated 2–group G of homeomorphisms of the 2–dimensional sphere for which there is a uniform bound for the orders of the group elements is finite. Recall that a nontrivial 2–group always contains involutions, that is, elements of order 2. Let $\mathrm{Inv}(G) := \{g \in G \setminus \mathrm{Id} : g^2 = \mathrm{Id}\}$, and let $Z(\sigma)$ be the centralizer of σ in G, that is, $Z(\sigma) = \{g \in G : g\sigma = \sigma g\}$. In order to prove Theorem A, we start by proving, using Theorem C, that, for every $\nu \in \mathrm{Inv}(G)$, the set $Z(\nu) \cap \mathrm{Inv}(G)$ is finite (following the proof of Theorem A, this is the only part where we use the existence of a uniform bound for the orders of the group elements).

Then we will prove that the set $\mathrm{Inv}(G)$ is finite. Since each $g \in G \setminus \{\mathrm{Id}\}$ has exactly two fixed points (by Proposition 2.1), we obtain that the union of fixed points of the involutions is also finite. Moreover, this set is nonempty and G–invariant, and has finite cardinality. We deduce that G is finite by Theorem C. For $\sigma \in G$, let us denote by $\mathrm{Fix}(\sigma)$ the set of all fixed points of σ.

Proposition 4.1 *Let G be a finitely generated 2–group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. Suppose that G has uniformly bounded order. Then the following assertions hold:*

(1) *If $\nu \in \mathrm{Inv}(G)$, then the set $Z(\nu) \cap \mathrm{Inv}(G)$ is finite.*

(2) *The set $\mathrm{Inv}(G)$ is finite.*

Proof Let us prove (1). Suppose that there is an infinite sequence $\nu, \nu_1, \dots, \nu_n, \dots$ contained in $Z(\nu) \cap \mathrm{Inv}(G)$. Fix an integer $n \geq 1$. The group G_n generated by $\nu, \nu_1, \dots, \nu_n$ is finitely generated, periodic, and preserves the set of fixed points of ν (because each ν_i commutes with ν). Then, by Theorem C, the group G_n is finite and either cyclic or dihedral. Moreover,

$$\{\mathrm{Id}\} \subset G_0 \subset \cdots \subset G_n \cdots .$$

Since we are assuming that elements in G have uniformly bounded order, this sequence must stabilize at some integer n_0. That is, for every integer $n \geq n_0$, one has $G_n = G_{n_0}$. This proves that $Z(\nu) \cap \mathrm{Inv}(G)$ is finite.

Let us prove (2). Suppose that there exists an infinite sequence of involutions $\sigma_1, \sigma_2, \dots, \sigma_n, \dots$ contained in G. For every integer n, the group generated by σ_1 and σ_n is either cyclic or dihedral, because $\sigma_1\sigma_n$ has finite order. Therefore, it contains an involution ν_n that commutes with σ_1 and σ_n ($\nu_n = \sigma_1$ in the cyclic case, and $\nu_n = (\sigma_1\sigma_n)^{\mathrm{ord}(\sigma_1\sigma_n)/2}$ in the dihedral case). Since $Z(\sigma_1) \cap \mathrm{Inv}(G)$ is finite (by (1)), we can suppose (by passing to a subsequence of $(\sigma_n)_{n\in\mathbb{N}}$) that $\nu_n = \nu$ for every integer n. This implies that ν commutes with all σ_n, and hence the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ is contained in $Z(\nu) \cap \mathrm{Inv}(G)$. But this last set is finite by (1). This contradiction proves that the set $\mathrm{Inv}(G)$ is finite. □

End of the proof of Theorem A Assume G is nontrivial. Applying Proposition 4.1, we obtain that the set $\mathrm{Inv}(G)$ is finite. Since each $g \in G \setminus \{\mathrm{Id}\}$ has exactly two fixed points (by Proposition 2.1), we obtain that the set

$$F := \bigcup_{\sigma \in \mathrm{Inv}(G)} \mathrm{Fix}(\sigma)$$

is also finite. As $\mathrm{Fix}(g\sigma g^{-1}) = g(\mathrm{Fix}(\sigma))$ and $g\sigma g^{-1}$ is an involution, the set F is nonempty and G–invariant, and has finite cardinality. We deduce that G is finite by Theorem C. □

5 Burnside problem for area-preserving homeomorphisms of the 2–dimensional sphere

In this section, we prove Theorem D, that is, every finitely generated, periodic group of area-preserving homeomorphisms of the 2–dimensional sphere having uniformly bounded order and an element of even order — equivalently, of order two — is finite. As in the case of a 2–group, we start by proving Theorem E (which is the analog of Theorem C in the area-preserving setting). Then using Theorem E we deduce that Proposition 4.1 holds in the area-preserving case (in the case where the set $\mathrm{Inv}(G)$ is nonempty). We then finish the proof of Theorem D in the same way as that of Theorem A. In order to prove Theorem E, we first introduce the rotation set for a homeomorphism of the open annulus.

5.1 Rotation set for a homeomorphism of the open annulus

Let $\mathbb{A} = \mathbb{T}^1 \times \mathbb{R}$ be the open annulus and $\widetilde{\mathbb{A}} := \mathbb{R} \times \mathbb{R}$ its universal covering. We denote by $\widetilde{\pi}\colon \widetilde{\mathbb{A}} \to \mathbb{A}$ the corresponding universal covering map and $p_1\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the projection on the first coordinate. By the two-point compactification, one can identify $\mathbb{A}$ to the punctured sphere $\mathbb{S}^2 \setminus \{N, S\}$, where N and S are two distinct points of $\mathbb{S}^2$ (the north and south poles). The Lebesgue measure on $\mathbb{S}^2$ induces a probability measure on $\mathbb{A}$, which we still call the Lebesgue measure and denote by Leb.

Let h be a homeomorphism of $\mathbb{A}$ that is isotopic to the identity, and let $\widetilde{h}$ be a lift of h to $\widetilde{\mathbb{A}}$. Following [10], we say that the *rotation number* of an h–recurrent point $x \in \mathbb{A}$ under $\widetilde{h}$ is well defined and equal to $\rho(\widetilde{h}, x) \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ if, for every sequence of integers $(n_k)_{k\in\mathbb{N}}$ which goes to $+\infty$ such that $(h^{n_k}(x))_{k\in\mathbb{N}}$ converges to x, the sequence $(\rho_{n_k}(\widetilde{h}, x))_{k\in\mathbb{N}}$, defined as

$$\rho_{n_k}(\widetilde{h}, x) := \frac{1}{n_k}(p_1(\widetilde{h}^{n_k}(\widetilde{x})) - p_1(\widetilde{x})),$$

where $\widetilde{x}$ is a point in $\widetilde{\pi}^{-1}(x)$, converges to $\rho(\widetilde{h}, x)$. Again, this definition does not depend on the choice of $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

Assume that h preserves a probability measure μ on $\mathbb{A}$. We say that the *rotation number* of $\widetilde{h}$ (with respect to μ) is well defined and equal to $\rho(\widetilde{h}, \mu)$ if

(1) μ–almost every point $x \in \mathbb{A}$ has a rotation number $\rho(\widetilde{h}, x)$, and

(2) the function $x \mapsto \rho(\widetilde{h}, x)$ is μ–integrable, with

$$\rho(\widetilde{h}, \mu) := \int_{\mathbb{A}} \rho(\widetilde{h}, x)\, d\mu.$$

Notice that, by the Birkhoff ergodic theorem, we have

$$\rho(\widetilde{h}, \mu) := \int_{\mathbb{A}} \rho_1(\widetilde{h}, x)\, d\mu,$$

where $\rho_1(\widetilde{h}, x) = p_1(\widetilde{h}(\widetilde{x})) - p_1(\widetilde{x})$, with $\widetilde{x} \in \widetilde{\pi}^{-1}(x)$.

5.2 Rotation set for periodic homeomorphisms of $\mathbb{S}^2$

In our setting, let g be a periodic, orientation-preserving homeomorphism of $\mathbb{S}^2$ that preserves the Lebesgue measure. We know that if g is nontrivial, then it fixes two distinct points N and S of $\mathbb{S}^2$. As in the local case, we can associate to our periodic homeomorphism g a unique "rotation number" on the open annulus $\mathbb{A}_{N,S} := \mathbb{S}^2 \setminus \{N, S\}$, defined as

$$\rho_{\mathbb{A}_{N,S}}(g) := \int_{\mathbb{A}_{N,S}} \rho_1(g, x)\, d\mathrm{Leb} \in \mathbb{T}^1.$$

Remark 2 If $\rho_{\mathbb{A}_{N,S}}(g) = 0$, then g is the identity.

Given two distinct points N and S of $\mathbb{S}^2$, we will denote by $\mathrm{Homeo}_0(\mathbb{A}_{N,S})$ the group of all homeomorphisms of $\mathbb{S}^2$ that preserve the orientation and fix both N and S. As in the local case we have the following result:

Proposition 5.1 *Let G_0 be a periodic subgroup of $\mathrm{Homeo}_0(\mathbb{A}_{N,S})$. The rotation number map defined on G_0 is a group homomorphism into $\mathbb{T}^1$ if and only if G_0 is abelian.*

5.3 Proof of Theorems D and E

We start by proving Theorem E.

Proof of Theorem E Let G be a finitely generated periodic group of area-preserving homeomorphisms of the 2–dimensional sphere. Let z be a point with G–orbit of cardinality 2. We write $\mathcal{O}_G(z) = \{z, z'\}$. We consider the subgroup G_0 of homeomorphisms that fix both z and z'. By Lemma 3.7, the group G_0 is an index-2,

normal subgroup of G. In particular G_0 is a finitely generated periodic group contained in $\mathrm{Homeo}_0(\mathbb{A}_{z,z'})$ all of whose elements preserve the Lebesgue measure. Since the rotation number is a group homomorphism in the area-preserving case (see Lemma 5.2 below), we can invoke an analog of Proposition 5.1 to conclude that G_0 is abelian.

Lemma 5.2 *Let G_0 be a subgroup of $\mathrm{Homeo}_0(\mathbb{A}_{z,z'})$. Suppose each element of G_0 preserves the Lebesgue measure. Then the rotation map is a group homomorphism.*

Proof Let f and g be two elements of G_0. We have that

$$\begin{aligned}\rho_{\mathbb{A}_{z,z'}}(fg) &= \int_{\mathbb{A}_{z,z'}} \rho_1(fg,x)\, d\mathrm{Leb}(x)\\ &= \int_{\mathbb{A}_{z,z'}} \rho_1(f,g(x))\, d\mathrm{Leb}(x) + \int_{\mathbb{A}_{z,z'}} \rho_1(g,x)\, d\mathrm{Leb}(x)\\ &= \int_{\mathbb{A}_{z,z'}} \rho_1(f,y)\, d\mathrm{Leb}(y) + \int_{\mathbb{A}_{z,z'}} \rho_1(g,x)\, d\mathrm{Leb}(x)\\ &= \rho_{\mathbb{A}_{z,z'}}(f) + \rho_{\mathbb{A}_{z,z'}}(g).\end{aligned}$$

This shows that $g \mapsto \rho_{\mathbb{A}_{z,z'}}(g)$ is a group homomorphism. □

Since G_0 is finitely generated, periodic and abelian, we deduce that it is finite. Moreover, by an analog of Lemma 2.5 (using the rotation number instead of the local rotation set), we deduce that G_0 is cyclic. The proof finishes as the proof of Theorem C. □

Now we can prove Theorem D.

Proof of Theorem D The proof is a straightforward adaptation of the proof of Theorem A. Let G be a finitely generated periodic group of orientation-preserving homeomorphisms of $\mathbb{S}^2$. Suppose that each element of G preserves the Lebesgue measure, that G has at least one element of even order, and that G has uniformly bounded order. Let $\mathrm{Inv}(G) := \{g \in G \setminus \mathrm{Id} : g^2 = \mathrm{Id}\}$. Notice that G always contains involutions. Indeed, if $g^{2p} = \mathrm{Id}$ for some integer p, then $g^p \in \mathrm{Inv}(G)$. Applying Proposition 4.1 (using Theorem E instead of Theorem C), we obtain that the set $\mathrm{Inv}(G)$ is finite. The proof follows as the proof of Theorem A. □

References

[1] **W Burnside**, *On an unsettled question in the theory of discontinuous groups*, Quart. J. Math. 33 (1902) 230–238

[2] **A Constantin**, **B Kolev**, *The theorem of Kerékjártó on periodic homeomorphisms of the disc and the sphere*, Enseign. Math. 40 (1994) 193–204 MR

[3] **E Ghys**, *Groups acting on the circle*, Enseign. Math. 47 (2001) 329–407 MR

[4] **E S Golod**, *On nil-algebras and finitely approximable p–groups*, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964) 273–276 MR In Russian; translated in "Fourteen papers on logic, algebra, complex variables and topology", Amer. Math. Soc. Transl. Ser. (2) 48, Amer. Math. Soc., Providence RI (1965) 103–106

[5] **N Guelman**, **I Liousse**, *Burnside problem for measure preserving groups and for 2–groups of toral homeomorphisms*, Geom. Dedicata 168 (2014) 387–396 MR

[6] **N Guelman**, **I Liousse**, *Burnside problem for groups of homeomorphisms of compact surfaces*, Bull. Braz. Math. Soc. 48 (2017) 389–397 MR

[7] **S Hurtado**, *The Burnside problem for* $\mathrm{Diff}_{\mathrm{Vol}}(\mathbb{S}^2)$, preprint (2016) arXiv

[8] **S V Ivanov**, *The free Burnside groups of sufficiently large exponents*, Internat. J. Algebra Comput. 4 (1994) 1–308 MR

[9] **B von Kerékjártó**, *Über die periodischen Transformationen der Kreisscheibe und der Kugelfläche*, Math. Ann. 80 (1919) 36–38 MR

[10] **P Le Calvez**, *Rotation numbers in the infinite annulus*, Proc. Amer. Math. Soc. 129 (2001) 3221–3230 MR

[11] **F Le Roux**, *L'ensemble de rotation autour d'un point fixe*, Astérisque 350, Soc. Math. France, Paris (2013) MR

[12] **I G Lysënok**, *Infinite Burnside groups of even period*, Izv. Ross. Akad. Nauk Ser. Mat. 60 (1996) 3–224 MR In Russian; translated in Izv. Math. 60 (1996) 453–654

[13] **A Navas**, *Groups of circle diffeomorphisms*, Univ. Chicago Press (2011) MR

[14] **A Y Ol'shanskiĭ**, *The Novikov–Adyan theorem*, Mat. Sb. 118 (1982) 203–235 MR In Russian; translated in Math. USSR-Sb. 46 (1983) 203–236

Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile
Santiago, Chile

jonathan.conejeros@usach.cl

Received: 1 December 2017 Revised: 7 June 2018

msp

Algebraic & Geometric Topology 18 (2018) 4109–4142

Algebraic and topological properties of big mapping class groups

PRIYAM PATEL
NICHOLAS G VLAMIS

Let S be an orientable, connected topological surface of infinite type (that is, with infinitely generated fundamental group). The main theorem states that if the genus of S is finite and at least 4, then the isomorphism type of the pure mapping class group associated to S, denoted by $\mathrm{PMap}(S)$, detects the homeomorphism type of S. As a corollary, every automorphism of $\mathrm{PMap}(S)$ is induced by a homeomorphism, which extends a theorem of Ivanov from the finite-type setting. In the process of proving these results, we show that $\mathrm{PMap}(S)$ is residually finite if and only if S has finite genus, demonstrating that the algebraic structure of $\mathrm{PMap}(S)$ can distinguish finite- and infinite-genus surfaces. As an independent result, we also show that $\mathrm{Map}(S)$ fails to be residually finite for any infinite-type surface S. In addition, we give a topological generating set for $\mathrm{PMap}(S)$ equipped with the compact-open topology. In particular, if S has at most one end accumulated by genus, then $\mathrm{PMap}(S)$ is topologically generated by Dehn twists, otherwise it is topologically generated by Dehn twists along with handle shifts.

20E26, 37E30, 57M07, 57S05

1 Introduction

A surface is of finite type if its fundamental group is finitely generated; otherwise, it is of infinite type. Throughout, all surfaces are assumed to be connected, orientable, and to have compact (possibly empty) boundary.

The *mapping class group*, denoted by $\mathrm{Map}(S)$, of a surface S is the group of orientation-preserving homeomorphisms of S up to isotopy, where we require all homeomorphisms as well as every stage of an isotopy to fix ∂S pointwise. The algebraic and geometric structure of mapping class groups of finite-type surfaces is generally well understood. In contrast, very little is known about *big* mapping class groups, ie mapping class groups of infinite-type surfaces.

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4109

Big mapping class groups arise naturally from the study of group actions on surfaces (eg Calegari [12]), taut foliations of 3–manifolds (eg Cantwell and Conlon [14]), and the Artinization of groups (eg Funar and Kapoudjian [18; 19]). (See Calegari [13] for a detailed discussion of these connections.) There has been a recent trend aimed at understanding infinite-type surfaces and their mapping class groups. For instance, recent work has investigated big mapping class group actions on hyperbolic graphs (eg Aramayona, Fossas and Parlier [2], Bavard [6] and Durham, Fanoni and Vlamis [16]).

This article focuses on the algebraic structure of big mapping class groups. Our methods use the language of topological groups and initiates the study of big mapping class groups in this category. The motivation for our work is a type of algebraic rigidity question for mapping class groups:

Question 1.1 Does the isomorphism type of $\mathrm{Map}(S)$ determine the topology of S? In particular, if $\mathrm{Map}(S)$ is isomorphic to $\mathrm{Map}(S')$, then are S and S' homeomorphic?

Remark Since the original submission of this article, Question 1.1 has been answered in the affirmative by Bavard, Dowdall and Rafi [7]. Their techniques are different than those used in this article in the proof of Theorem 1. In particular, we use a topological-group-theoretic characterization of Dehn twists to prove Theorem 1, and Bavard, Dowdall and Rafi use an algebraic characterization of Dehn twists to obtain their result.

With the exception of finitely many sporadic cases in low complexity, we can see that Question 1.1 has a positive answer in the finite-type setting. To do so, observe that the rank of the center of $\mathrm{Map}(S)$, the virtual cohomological dimension of $\mathrm{Map}(S)$ (see Harer [21]), and the rank of a maximal abelian subgroup in $\mathrm{Map}(S)$ (see Birman, Lubotzky and McCarthy [11]) together determine the genus, number of punctures, and the number of boundary components of S and thus its topological type.

Observe that $\mathrm{Map}(S)$ is countable if and only if S is of finite type. Therefore, the isomorphism type of $\mathrm{Map}(S)$ can distinguish between finite- and infinite-type surfaces. This reduces the question to considering specifically infinite-type surfaces, where little is known. The invariants mentioned for the finite-type setting, with the exception of the rank of the center, are all infinite for big mapping class groups.

In progress towards this question, we change our focus to a natural subgroup of $\mathrm{Map}(S)$. In particular, we will work with the *pure mapping class group*, denoted by $\mathrm{PMap}(S)$,

consisting of mapping classes acting trivially on the topological ends of S (see Section 2.2 for the definition of ends). In this setting we address the same question and provide a partial answer:

Theorem 1 *Let S be a surface whose genus is finite and at least 4. If S' is a surface, then any isomorphism between $\mathrm{PMap}(S)$ and $\mathrm{PMap}(S')$ is induced by a homeomorphism.*

If S has empty boundary, then the *extended* mapping class group, denoted by $\mathrm{Map}^{\pm}(S)$, is the degree-2 extension of $\mathrm{Map}(S)$ that includes orientation-reversing mapping classes; otherwise, we set $\mathrm{Map}^{\pm}(S) = \mathrm{Map}(S)$. Setting $S' = S$ in Theorem 1 yields a generalization of a celebrated result of Ivanov [25] for finite-type mapping class groups:

Corollary 2 *If S is a borderless surface whose genus is finite and at least 4, then the natural monomorphism from $\mathrm{Map}^{\pm}(S)$ to $\mathrm{Aut}(\mathrm{PMap}(S))$ is an isomorphism.*

An essential aspect of the proof of Theorem 1, just as in Ivanov's work, is to give a characterization of Dehn twists (Proposition 3.7), which allows us to conclude that an isomorphism of pure mapping class groups must send Dehn twists to Dehn twists. To obtain this characterization, we introduce a nonstandard topology τ_w on $\mathrm{PMap}(S)$, described in detail in Section 3. In Corollary 5.4, we discuss the structure of $\mathrm{Out}(\mathrm{PMap}(S))$; in particular, it is isomorphic to the automorphism group of a countable boolean algebra.

Because of Theorem 1, it is natural for one to expect that $\mathrm{PMap}(S)$ is a characteristic subgroup of $\mathrm{Map}(S)$. If this is the case, then Corollary 2 can be extended to $\mathrm{Map}(S)$ to again match the finite-type setting. We state these as conjectures with the same assumptions as in Theorem 1; however, we expect these conjectures and Corollary 2 to hold in general.

Conjecture 1.2 *Let S be a surface whose genus is finite and at least 4.*

(1) *$\mathrm{PMap}(S)$ is a characteristic subgroup of $\mathrm{Map}(S)$ and $\mathrm{Map}^{\pm}(S)$.*

(2) *If S is borderless, then $\mathrm{Aut}(\mathrm{Map}^{\pm}(S)) = \mathrm{Aut}(\mathrm{Map}(S)) = \mathrm{Map}^{\pm}(S)$.*

(This conjecture has been subsequently proven — in greater generality — by Bavard, Dowdall and Rafi [7].)

The first step in proving Theorem 1 is to distinguish finite- and infinite-genus surfaces via the algebraic structure of their associated pure mapping class groups. In order to do this, we investigate whether big mapping class groups are residually finite. As these results are of independent interest, we include Theorem 3 as a summary of the results in Section 4. Recall that a group is *residually finite* if the intersection of its proper finite-index normal subgroups is trivial.

Theorem 3 *Let S be any surface.*

(1) $\mathrm{PMap}(S)$ *is residually finite if and only if S has finite genus.*

(2) $\mathrm{Map}(S)$ *is residually finite if and only if S is of finite type.*

Note that the finite-type cases are handled by Grossman [20]. Using the structure of τ_w, we show that $\mathrm{PMap}(S)$ is residually finite whenever S has finite genus (Proposition 4.1). We then introduce the infinite-stranded braid group B_∞, show that it fails to be residually finite (Corollary 4.3), and prove that it embeds in every finite-genus big mapping class group (Proposition 4.4). Thus, $\mathrm{Map}(S)$ is not residually finite when S is of infinite type and has finite genus. Proposition 4.6 shows that $\mathrm{PMap}_c(S)$, the subgroup of $\mathrm{Map}(S)$ consisting of mapping classes with compact support, has no finite quotients when S has infinite genus. This fact is almost immediate from a theorem of Paris [31] and implies that when S has infinite genus, $\mathrm{Map}(S)$ and $\mathrm{PMap}(S)$ are not residually finite.

The reason $\mathrm{PMap}(S)$, when S is a finite-genus surface, behaves differently than other big mapping class groups is the existence of forgetful homomorphisms to finite-type pure mapping class groups. We use these forgetful homomorphisms to build the topology τ_w on $\mathrm{PMap}(S)$ mentioned above (see Section 3). Understanding basic topological properties of τ_w is the key to characterizing Dehn twists and understanding the residual properties of $\mathrm{PMap}(S)$. The construction of τ_w naturally leads to an inverse limit of finite-type pure mapping class groups, which is the completion (as uniform spaces) of the associated big pure mapping class group. This inverse limit viewpoint is thoroughly discussed in Section 7.

As Dehn twists and the topology of τ_w play a critical role in our understanding of $\mathrm{PMap}(S)$, it is natural to understand which of these properties hold in the more standard compact-open topology. Equipping the group of homeomorphisms of a surface with the compact-open topology, we give $\mathrm{Map}(S)$ the corresponding quotient topology. We let τ_q denote the restriction of this topology to $\mathrm{PMap}(S)$. Note that if S is of finite

type, then τ_q is the discrete topology. From the definition of τ_w and the Dehn–Lickorish theorem, it follows that Dehn twists topologically generate $(\mathrm{PMap}(S), \tau_w)$.

In the following theorem, we see that Dehn twists topologically generate $(\mathrm{PMap}(S), \tau_q)$ if S has finite genus; however, this can fail in the infinite-genus setting. Intuitively, a topological end of a surface is *accumulated by genus* if there is a sequence of handles converging to the end (see Section 2.2 for the precise definition). In order to build a topological generating set in the infinite-genus setting we introduce the notion of a *handle shift* in Section 6. Roughly speaking, a handle shift is a homeomorphism which applies a $+1$ shift to a $\mathbb{Z}$–indexed collection of handles in the surface. This notion allows us to build a topological generating set for $(\mathrm{PMap}(S), \tau_q)$:

Theorem 4 *The set of Dehn twists topologically generate* $(\mathrm{PMap}(S), \tau_q)$ *if and only if* S *has at most one end accumulated by genus. If* S *has at least two ends accumulated by genus, then the set of Dehn twists together with the set of handle shifts topologically generate* $(\mathrm{PMap}(S), \tau_q)$.

1.1 Outline

In Section 2 we give the necessary background, which focuses on the structure of the space of ends of an infinite-type surface.

In Section 3, we define the topology τ_w, show it is Hausdorff (Lemma 3.4), and investigate the closure of the set of Dehn twists (Lemma 3.6), allowing us to characterize Dehn twists (Proposition 3.7). Section 4 is dedicated to investigating residual finiteness.

Given the results of Section 4, Section 5 focuses on finite-genus surfaces. In this section, we combine the characterization of Dehn twists from Section 3 and results regarding homomorphisms between finite-type mapping class groups to conclude the proof of Theorem 1 in Propositions 5.1 and 5.3.

In Section 6, we introduce the notion of a handle shift and prove Theorem 4. In Section 7, we discuss inverse limits of finite-type pure mapping class groups and finish the paper by showing the proper containment $\tau_w \subset \tau_q$ in Section 7.1.

Acknowledgements

The authors would like to thank Javier Aramayona, Ara Basmajian, Daryl Cooper, David Fernández-Bretón, Chris Leininger, Darren Long, and Scott Schneider for several helpful conversations. The authors also thank the referee for helpful comments.

Vlamis was supported in part by NSF RTG grant 1045119.

2 Background

2.1 Curves

A *simple closed curve* in a surface S is the image of a topological embedding $\mathbb{S}^1 \hookrightarrow S$. A simple closed curve is *trivial* if it is homotopic to a point; it is *peripheral* if it either is homotopic to a boundary component or bounds a once-punctured disk in S; it is *essential* if it is neither trivial nor peripheral; it is *separating* if its complement is disconnected and *nonseparating* otherwise. If c is a simple closed curve in S, then we let $[c]$ denote its isotopy class in S. Given two isotopy classes of simple closed curves $[c]$ and $[d]$, their *geometric intersection number*, denoted by $i([c],[d])$, is defined to be

$$i([c],[d]) = \min\{|c' \cap d'| : [c'] = [c] \text{ and } [d'] = [d]\}.$$

When is it clear from context, we will conflate a simple close curve with its isotopy class.

Recent work of Hernández, Morales and Valdez [22] assures us that, just as in the finite-type setting, simple closed curves play an essential role in the study of big mapping class groups. In particular, they extend the Alexander method to infinite-type surfaces [22, Theorem 1.1], yielding the following lemma:

Lemma 2.1 [22, Corollary 1.2] *Suppose S is an infinite-type surface without boundary. If an element of $\mathrm{Map}^{\pm}(S)$ fixes the isotopy class of every simple closed curve, then it is trivial.*

We will need one final notion: if $F \subset S$ is a subsurface, then we call F *essential* if it is connected and the inclusion $F \hookrightarrow S$ induces a monomorphism $\mathrm{PMap}(F) \hookrightarrow \mathrm{PMap}(S)$, where a homeomorphism of F is extended to S by the identity.

2.2 Ends

In order to work with infinite-type surfaces, it is essential to understand their space of ends, which we now define.

Definition 2.2 Let S be an infinite-type surface and fix an exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of S by compact subsurfaces, that is, each K_i is a compact surface of S, $K_i \subset K_{i+1}$, and $\bigcup_{i\in\mathbb{N}} K_i = S$.

- An *end* of S is a sequence $\{U_i\}_{i\in\mathbb{N}}$ such that each U_i is a connected component of $S \smallsetminus K_i$ and $U_{i+1} \subset U_i$ for all $i \in \mathbb{N}$.

- Define U_i^* to be the set of all ends that contain U_i, where U_i is a connected component of $S \smallsetminus K_i$. The *space of ends* of S, denoted by $\mathrm{Ends}(S)$, consists — as a set — of the ends of S. It is equipped with the topology generated by sets of the form U_i^*.

Note that the construction of $\mathrm{Ends}(S)$ is independent of the choice of exhaustion.

The space $\mathrm{Ends}(S)$ is Hausdorff, totally disconnected, second countable, and compact (see [1, Chapter 1, Sections 36–37]); hence, it can be realized as a closed subset of the Cantor set. An end of S comes in two types:

- An end $\{U_i\}$ is *accumulated by genus* if U_i has infinite genus for all i.
- An end $\{U_i\}$ is *planar* if U_i is planar for all but finitely many i.

In addition, an end is *isolated* if it is an isolated point in the topology given above. We will use the word *puncture* to denote an isolated planar end.

Kerékjártó [36] showed that the homeomorphism type of an orientable infinite-type surface is determined by the quadruple

$$(g, b, \mathrm{Ends}_\infty(S), \mathrm{Ends}(S)),$$

where $g \in \mathbb{N} \cup \{0, \infty\}$ is the genus of S, $b \in \mathbb{N}$ is the number of boundary components, and $\mathrm{Ends}_\infty(S) \subset \mathrm{Ends}(S)$ is the subset of ends accumulated by genus (see Richards [34] for a proof).

An immediate consequence of the classification of infinite-type surfaces is:

Lemma 2.3 (definition of $\overline{S}$) *When S has finite genus, there exists a unique — up to homeomorphism — compact surface, which we label $\overline{S}$, containing S such that $\overline{S} \smallsetminus S$ is homeomorphic to* $\mathrm{Ends}(S)$.

The surface $\overline{S}$ given by Lemma 2.3 is the Freudenthal compactification of S. Note that the Freudenthal compactification exists for any surface, but is not a manifold in general.

Given the definition of $\mathrm{Ends}(S)$ it is clear that a homeomorphism of S induces a homeomorphism of $\mathrm{Ends}(S)$. As any isotopy must fix the ends of S, we see that there is an induced action of $\mathrm{Map}(S)$ on $\mathrm{Ends}(S)$. The *pure mapping class group* $\mathrm{PMap}(S)$ is the kernel of this action.

Proposition 2.4 [8, Proposition I.1] *Let $\mathbb{D}$ denote the unit disk and let Π be a totally disconnected compact subset contained in the interior of $\mathbb{D}$. Given a homeomorphism $f\colon \Pi \to \Pi$ there exists a homeomorphism $\widehat{f}\colon \mathbb{D} \to \mathbb{D}$ and a regular neighborhood N of $\partial\mathbb{D}$ such that $\widehat{f}|_\Pi = f$ and $\widehat{f}|_N = \mathrm{id}$.*

Observe that since a regular neighborhood of the boundary of $\mathbb{D}$ is fixed pointwise, it follows that $\widehat{f}$ is orientation-preserving. This yields:

Corollary 2.5 *If S has finite genus, then the induced homomorphism $\mathrm{Map}(S) \to \mathrm{Homeo}(\mathrm{Ends}(S))$ is surjective.*

3 The initial topology

For the entirety of this section, we require S to have finite genus and be of infinite type. In this setting, we will introduce a topology on $\mathrm{PMap}(S)$ that is suited to our algebraic study of the group. Define the set $\Lambda(S)$ to be the set of finite subsets of $\mathrm{Ends}(S)$, that is,

$$\Lambda(S) = \{\lambda \subset \mathrm{Ends}(S) : |\lambda| < \infty\}.$$

When it is clear from context, we will simply write Λ. For each $\lambda \in \Lambda$, we define the surface $S_\lambda = \overline{S} \smallsetminus \lambda$. The inclusion $i_\lambda\colon S \hookrightarrow S_\lambda$ induces a homomorphism $\varphi_\lambda\colon \mathrm{PMap}(S) \to \mathrm{PMap}(S_\lambda)$. This homomorphism is the forgetful homomorphism, where one forgets all ends of S not in λ.

The *initial topology*, denoted by τ_w, on $\mathrm{PMap}(S)$ is the initial topology with respect to the family of maps $\{\varphi_\lambda\}_{\lambda\in\Lambda}$, that is, it is the coarsest topology such that φ_λ is continuous for each $\lambda \in \Lambda$, where we equip $\mathrm{PMap}(S_\lambda)$ with the discrete topology. If at this point the reader is inclined to think about inverse limits, we discuss this viewpoint in Section 7.

For the rest of the section our goal is to present two key lemmas (Lemmas 3.4 and 3.6) regarding the algebraic structure of $\mathrm{PMap}(S)$. Before starting, we need to understand the center of $\mathrm{PMap}(S)$, which we denote by $Z(\mathrm{PMap}(S))$. Recall that the pure mapping class group of any finite-type surface is generated by Dehn twists (see [17, Section 4.4.4]). Combining this fact with the definition of the initial topology above, we have the following lemma as an immediate corollary:

Lemma 3.1 $(\mathrm{PMap}(S), \tau_w)$ *is topologically generated by Dehn twists.*

Lemma 3.2 *An element of* $\mathrm{PMap}(S)$ *is central if and only if it fixes the isotopy class of every simple closed curve.*

Proof Let $f \in \mathrm{PMap}(S)$. Recall that if c is a simple closed curve in S, then $f \cdot T_c \cdot f^{-1} = T_{f(c)}$. Now assume that f fixes the isotopy class of every simple closed curve. It follows that $f \cdot T_c \cdot f^{-1} = T_{f(c)} = T_c$ for any simple closed curve c. Therefore, f commutes with every Dehn twist in $\mathrm{PMap}(S)$ and hence, by Lemma 3.1, f commutes with every element of a topological generating set for $\mathrm{PMap}(S)$. It is an easy exercise to show that multiplication in $(\mathrm{PMap}(S), \tau_w)$ is continuous. Therefore, we can conclude that $f \in Z(\mathrm{PMap}(S))$.

Now assume that f is central. It follows that $T_c = f \cdot T_c \cdot f^{-1} = T_{f(c)}$ and hence $f([c]) = [c]$ for every simple closed curve c in S. □

Lemma 3.3 $Z(\mathrm{PMap}(S))$ *is generated by the Dehn twists about the boundary components of* S. *In particular, if* S *is borderless then* $\mathrm{PMap}(S)$ *has trivial center.*

Proof Let $G < \mathrm{PMap}(S)$ be the subgroup generated by the Dehn twists about the boundary components of S.

Let R be the surface obtained from S by gluing a once-punctured disk to each component of ∂S. Let $\iota\colon S \hookrightarrow R$ be the associated inclusion and $\iota_*\colon \mathrm{PMap}(S) \to \mathrm{PMap}(R)$ the induced homomorphism. By construction, ι_* is surjective. To see this, note that ι_* is continuous and every Dehn twist in $\mathrm{PMap}(R)$ is an image of a Dehn twist in $\mathrm{PMap}(S)$.

As ι_* is surjective, it follows that if $f \in Z(\mathrm{PMap}(S))$, then $\iota_*(f) \in Z(\mathrm{PMap}(R))$. By Lemma 3.2, $\iota_*(f)$ fixes the isotopy class of every simple closed curve; therefore, as R is borderless, we can apply Lemma 2.1 to see that $\iota_*(f)$ is trivial.

A direct replacement of the finite-type Alexander method with the infinite-type Alexander method [22] in the proof of [17, Theorem 3.18] implies $G = \ker \iota_*$; in particular, $f \in G$. Therefore, $Z(\mathrm{PMap}(S)) \subset G$.

Now, let $f \in G$. Then f fixes the isotopy class of every simple closed curve. Lemma 3.2 allows us to conclude that f is central and $G \subset Z(\mathrm{PMap}(S))$. □

Lemma 3.4 τ_w *is Hausdorff.*

Proof To show the Hausdorff property, it is enough to show that any point can be separated from the identity with open sets. By the definition of τ_w and continuity of left multiplication, it is enough to show that

$$\bigcap_{\lambda\in\Lambda} \ker \varphi_\lambda = \{\mathrm{id}\}.$$

Let $f \in \mathrm{PMap}(S)$ be a nontrivial element. If f is central in $\mathrm{PMap}(S)$, then by Lemma 3.3 it is a product of Dehn twists about boundary components and hence the same is true for $\varphi_\lambda(f)$. In particular, f is not an element of $\ker\varphi_\lambda$ whenever $|\lambda| > 1$. Assuming f is not central, by Lemma 3.2, there exists an essential simple closed curve a in S such that $f(a)$ is not homotopic to a. Choose an essential finite-type surface $F \subset S$ so that $\mathrm{genus}(F) = \mathrm{genus}(S)$ and both a and $f(a)$ are contained in F. Let $\lambda \in \Lambda$ be such that the intersection of λ with each component of $\overline{S} \smallsetminus F$ is nonempty. It follows that $i_\lambda|_F$ is an embedding of F into S_λ as an incompressible surface; in particular, $i_\lambda(f(a))$ and $i_\lambda(a)$ are not homotopic in S_λ. It follows that $\varphi_\lambda(f)$ does not fix the isotopy class of $i_\lambda(a)$ in S_λ; hence, $f \notin \ker\varphi_\lambda$. □

The proof of Lemma 3.4 tells us that $\mathrm{PMap}(S)$ inherits the residual properties of finite-type pure mapping class groups. In particular, Lemma 3.4 combined with the work of Grossman [20] implies that $\mathrm{PMap}(S)$ is residually finite. This is discussed in more depth in Section 4. In addition, it follows from the inverse limit construction in Section 7 and Lemma 3.4 that $(\mathrm{PMap}(S), \tau_w)$ is a topological group (see Proposition 7.1).

The goal of the next key lemma (Lemma 3.6) is to detect Dehn twists in $\mathrm{PMap}(S)$. In the case of finite-type surfaces, the main idea behind Ivanov's results in [25] on the automorphisms of mapping class groups is to algebraically characterize Dehn twists. Given an automorphism preserving Dehn twists, one can then build a map on the collection of simple closed curves in order to determine the isotopy class of a homeomorphism. Building off the literature surrounding homomorphisms between mapping class groups, we take a similar approach in Section 5.

Given a simple closed curve a in S let $[a]$ denote its isotopy class in S and let $[a]_\lambda$ denote the isotopy class of $i_\lambda(a)$ in S_λ. For the sake of the argument below, we record an obvious lemma:

Lemma 3.5 *For any $\lambda \in \Lambda$,*

$$i([a],[b]) \geq i([a]_\lambda,[b]_\lambda),$$

where a and b are any two simple closed curves in S.

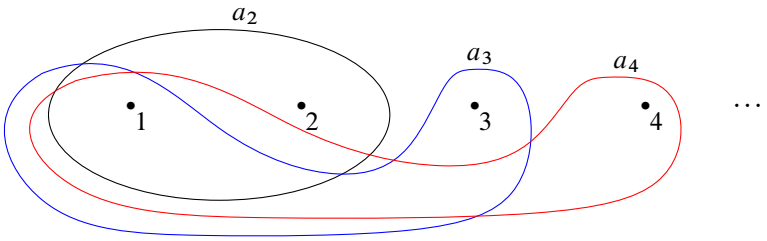


Figure 1: The sequence of Dehn twists $\{T_{a_n}\}_{n=2}^{\infty}$ in $\mathrm{PMap}(S)$, where $S = \mathbb{C} \smallsetminus \mathbb{N}$, converges to the identity in τ_w, but does not converge in τ_q to any homeomorphism.

Let $\mathcal{D}$ denote the collection of Dehn twists in $\mathrm{PMap}(S)$ and let $\mathcal{D}_\lambda$ denote the collection of Dehn twists in $\mathrm{PMap}(S_\lambda)$. The subgroup of $\mathrm{PMap}(S)$ generated by $\mathcal{D}$ is the group of compactly supported mapping classes, denoted by $\mathrm{PMap}_c(S)$. We will use this notation throughout the article.

Lemma 3.6 *The closure of $\mathcal{D}$ in τ_w is $\overline{\mathcal{D}} = \mathcal{D} \cup \{\mathrm{id}\}$.*

We note that this statement also holds in τ_q; we give a proof in Section 6 (see Proposition 6.4). Additionally, the reader should be warned that the subtlety in the proof of the lemma lies in the fact that the stabilizer of an isotopy class of a curve is not open in τ_w. This failure for a stabilizer of a curve to be open follows from the proof of Proposition 7.6 and is demonstrated by the example illustrated in Figure 1, which shows an example of a (nonintuitive) sequence of Dehn twists converging to the identity.

Proof Fix an exhaustion $K_1 \subset K_2 \subset \cdots$ of S by essential finite-type genus(S)–subsurfaces. Observe, for later in the proof, that the direct limit of the groups $\{\mathrm{PMap}(K_n)\}_{n\in\mathbb{N}}$ is $\mathrm{PMap}_c(S)$.

We first see that the identity is in the closure of $\mathcal{D}$. Recall that the sets of the form $\ker \varphi_\lambda$ for $\lambda \in \Lambda$ form a neighborhood basis for the identity in $(\mathrm{PMap}(S), \tau_w)$. Given $\lambda \in \Lambda$ we can find a clopen subset U of $\mathrm{Ends}(S)$ disjoint from λ and a disk $D \subset \overline{S}$ such that $U \subset D$. The Dehn twist about ∂D is then an element in $\mathcal{D} \cap \ker \varphi_\lambda$. It follows that $\mathcal{D}$ intersects every neighborhood of the identity and hence the identity is in $\overline{\mathcal{D}}$.

As τ_w is not first countable when $\mathrm{Ends}(S)$ is uncountable (see Lemma 7.4), it is necessary for the remainder of the proof for us to work with nets as opposed to sequences; however, the reader will lose no intuition by replacing the word net with sequence.

Let I be a directed set and let $\{T_{c_i}\}_{i\in I}$ be a net in $\mathcal{D}$, where T_{c_i} denotes the Dehn twist about the simple closed curve c_i. Assume the net converges to $f \in (\mathrm{PMap}(S), \tau_w)$. Further assuming that f is not the identity, we will then show that f is a Dehn twist.

We first claim that f is an element of $\mathrm{PMap}_c(S)$. Suppose not. Then, by Lemma 3.3, f is not in the center of $\mathrm{PMap}(S)$, so there exists an essential simple closed curve a with $[f(a)] \neq [a]$. Choose $N \in \mathbb{N}$ such that both a and $f(a)$ can be homotoped into K_N. For each $n \in \mathbb{N}$ with $n > N$ we can find b_n homotopic into the complement of K_n such that $[f(b_n)] \neq [b_n]$. Fix $n \in \mathbb{N}$ with $n > N$. Then there exists $m \in \mathbb{N}$ with $m > n$ and where b_n and $f(b_n)$ are homotopic into K_m. Choose $\lambda \in \Lambda$ such that λ intersects each component of $\overline{S} \smallsetminus K_m$.

Given the choice of λ, we have $[f(a)]_\lambda \neq [a]_\lambda$ and $[f(b_n)]_\lambda \neq [b_n]_\lambda$. It follows that $\varphi_\lambda(f)$ is nontrivial and thus a Dehn twist, call it T, as our net is in $\mathcal{D}$. By construction,

$$T([a]_\lambda) = \varphi_\lambda(f)([a]_\lambda) \neq [a]_\lambda \quad \text{and} \quad T([b_n]_\lambda) = \varphi_\lambda(f)([b_n]_\lambda) \neq [b_n]_\lambda.$$

Choose a simple closed curve c in K_m such that $T = T_{[c]_\lambda}$. Then

- $i([c]_\lambda, [a]_\lambda) > 0$,
- $i([c]_\lambda, [b_n]_\lambda) > 0$, and
- $i([a]_\lambda, [b_n]_\lambda) = 0$,

where the first two inequalities follow from the fact that T fixes neither $[a]_\lambda$ nor $[b_n]_\lambda$. The last equality comes from the fact that a and b_n are disjoint combined with Lemma 3.5. A direct application of [17, Proposition 3.4] gives the inequality

$$i(T([a]_\lambda), [b_n]_\lambda) = i([c]_\lambda, [a]_\lambda)\, i([c]_\lambda, [b_n]_\lambda) > 0.$$

Another application of Lemma 3.5 yields

$$i([f(a)], [b_n]) \geq i([f(a)]_\lambda, [b_n]_\lambda) = i(T([a]_\lambda), [b_n]_\lambda) > 0.$$

As n was arbitrary, we see that $f(a)$ must exit every compact set of S; hence, it cannot be compact and f is not a homeomorphism.

It follows there exists $n \in \mathbb{N}$ such that $f \in \mathrm{PMap}(K_n) < \mathrm{PMap}_c(S)$. Let $m \in \mathbb{N}$ such that $m > n$ and K_n is contained in the interior of K_m. Choose $\lambda \in \Lambda$ such that λ intersects each component of $\overline{S} \smallsetminus K_m$ nontrivially. We have that φ_λ restricted to $\mathrm{PMap}(K_n)$ is injective and induced by the inclusion $i_\lambda|_{K_n}\colon K_n \hookrightarrow \overline{S} \smallsetminus \lambda$; in particular, as $f \in \mathrm{PMap}(K_n)$ and $\varphi_\lambda(f)$ is a Dehn twist, we must have that f is a Dehn twist. □

As a direct consequence, we give a characterization of Dehn twists. For $\lambda \in \Lambda$, let $\overline{\mathcal{D}}_\lambda$ be the union of the Dehn twists in $\mathrm{PMap}(S_\lambda)$ with the identity, so $\overline{\mathcal{D}}_\lambda = \varphi_\lambda(\overline{\mathcal{D}})$.

Proposition 3.7 *$f \in \overline{\mathcal{D}}$ if and only if $\varphi_\lambda(f) \in \overline{\mathcal{D}}_\lambda$ for every $\lambda \in \Lambda$.*

Proof The forward direction is trivial. For the reverse, assume $f \in \mathrm{PMap}(S)$ such that $\varphi_\lambda(f) \in \overline{\mathcal{D}}_\lambda$ for every $\lambda \in \Lambda$. For each $\lambda \in \Lambda$ choose $T_\lambda \in \overline{\mathcal{D}}$ such that $\varphi_\lambda(T_\lambda) = \varphi_\lambda(f)$. This implies that the net $\{T_\lambda\}_{\lambda\in\Lambda}$ converges to f; hence, $f \in \overline{\mathcal{D}}$ by Lemma 3.6. □

4 Residual finiteness

A group is *residually finite* if the intersection of all of its finite-index proper normal subgroups is trivial. As noted earlier, the work of Grossman [20] tells us that mapping class groups of finite-type surfaces are residually finite. As this property is inherited by subgroups it follows that pure mapping class groups are also residually finite.

In this section, we will prove Theorem 3 in a series of propositions. We break the section into two subsections handling the finite- and infinite-genus cases separately.

4.1 Finite genus

As a start, we properly record the result mentioned in Section 3:

Proposition 4.1 *If S has finite genus, then $\mathrm{PMap}(S)$ is residually finite.*

Proof Let $f \in \mathrm{PMap}(S)$ be an arbitrary nontrivial element. Then by Lemma 3.4 there exists $\lambda \in \Lambda$ such that $f \notin \ker \varphi_\lambda$. Further, since $\mathrm{PMap}(S_\lambda)$ is residually finite [20] there exists a homomorphism $\psi\colon \mathrm{PMap}(S_\lambda) \to G$, where G is a finite group and $f \notin \ker(\psi \circ \varphi_\lambda)$. □

In a slight change of perspective, we will focus on the full mapping class group $\mathrm{Map}(S)$ for the remainder of this subsection.

Let B_n denote the n–stranded braid group, which is also the mapping class group of the n–punctured disk. The classical presentation for B_n is

$$B_n = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j|\geq 2 \text{ and } 1\leq i,j\leq n-1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1\leq i\leq n-2 \end{array} \right\rangle.$$

With this presentation we see that there are natural inclusions $\iota_{n,m}\colon B_n \hookrightarrow B_m$ whenever $n \leq m$. In particular $\langle B_n, \iota_{n,m}\rangle$ is a directed system of groups and we define B_∞ to be the direct limit. The presentation for B_∞ is immediate from that of B_n, namely,

$$B_\infty = \left\langle \{\sigma_i : i \in \mathbb{N}\} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \text{ and } i, j \in \mathbb{N}, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } i \in \mathbb{N} \end{array} \right\rangle.$$

We will proceed by proving that B_∞ is not residually finite and that it is a subgroup of every finite-genus big mapping class group.

Proposition 4.2 *Every finitely generated quotient of B_∞ is cyclic.*

Proof Let $f\colon B_\infty \to G$ be a surjective homomorphism to a finitely generated group. We can then find some N such that $f|_{B_N}$ is surjective. For $m > N$, σ_m commutes with every element of B_N; hence, $f(\sigma_m)$ is in the center of G for every $m > N$. In particular, $f(\sigma_N)$ commutes with $f(\sigma_{N+1})$. From the relations, it is easy to see that if $f(\sigma_i)$ commutes with $f(\sigma_{i+1})$, then $f(\sigma_i) = f(\sigma_{i+1})$. It now follows that G is generated by $f(\sigma_1)$. □

The abelianization of B_∞ is isomorphic to $\mathbb{Z}$; in particular, from Proposition 4.2, it follows that every finite quotient of B_∞ factors through its abelianization. Thus the commutator subgroup is in the intersection of every proper finite-index normal subgroup of B_∞. This shows:

Corollary 4.3 *B_∞ is not residually finite.*

Proposition 4.4 *Let S be an infinite-type surface. If S has finite genus, then B_∞ is a subgroup of* $\mathrm{Map}(S)$.

In fact, Proposition 4.4 holds whenever S has infinitely many planar ends, but we will not use the full generality of the result in what follows.

Proof The proof splits into two cases: $\mathrm{Ends}(S)$ has either infinitely or finitely many isolated points. Let us start by assuming that $\mathrm{Ends}(S)$ has infinitely many isolated points. As $\mathrm{Ends}(S)$ is separable, the set of isolated ends must be countably infinite, so we can choose a labeling $\{p_1, p_2, \dots\}$ of the isolated ends. For each $n \in \mathbb{N}$ let $D_n \subset S$ be an n–punctured disk satisfying

(1) $\mathrm{Ends}(D_n) = \{p_1, \dots, p_n\}$, and

(2) $D_n \subset D_{n+1}$.

It follows that D_n is an essential subsurface of S; in particular, the embedding $D_n \hookrightarrow S$ induces an injection $\mathrm{Map}(D_n) \hookrightarrow \mathrm{Map}(S)$. Recall that $\mathrm{Map}(D_n) \cong B_n$. By the universal property of direct limits, the inclusions $D_n \hookrightarrow D_{n+1}$ induce an injective homomorphism $B_\infty \hookrightarrow \mathrm{Map}(S)$ as desired.

Now assume $\mathrm{Ends}(S)$ has a finite number of isolated points $\{p_1, \dots, p_n\}$. It follows that $C = \mathrm{Ends}(S) \smallsetminus \{p_1, \dots, p_n\}$ is perfect. Let $D \subset \overline{S}$ be a disk such that $D \cap F = \varnothing$ and $D \cap C = C'$, where C' is a clopen subset of C. It follows from Brouwer's topological characterization of the Cantor set (see [27, Theorem 7.4]) that C' is homeomorphic to the Cantor set. The work of Funar and Kapoudjian [18, Section 7] implies $B_\infty < \mathrm{Map}(D \smallsetminus C')$; thus, $B_\infty < \mathrm{Map}(S)$ as $D \smallsetminus C'$ is an essential subsurface. □

Combining Corollary 4.3 and Proposition 4.4, we have:

Corollary 4.5 *If S is a finite-genus surface of infinite type, then $\mathrm{Map}(S)$ is not residually finite.*

4.2 Infinite genus

In this subsection we will prove the remaining pieces of Theorem 3 involving infinite-genus surfaces.

Proposition 4.6 *If S has infinite genus, then $\mathrm{PMap}_c(S)$ has no finite quotients.*

This has an immediate corollary:

Corollary 4.7 $\mathrm{PMap}_c(S)$ *is perfect and does not virtually surject onto $\mathbb{Z}$.*

Mapping class groups of finite-type surfaces are known to be perfect [32] (except for some low-complexity cases). Motivated by Kazhdan's property (T), Ivanov conjectured that mapping class groups of finite-type surfaces do not virtually surject onto $\mathbb{Z}$ (see [28, Problem 2.11.A]). (Equivalently, this says $H_1(G, \mathbb{Q}) = 0$ for any finite-index subgroup G of $\mathrm{Map}(S)$.) Putman and Wieland [33] have proven stability conditions for this conjecture; it would be fascinating if the stability could manifest into a statement about big mapping class groups.

As a second corollary, we have our desired result in relation to Theorem 3:

Corollary 4.8 *Neither* $\mathrm{Map}(S)$ *nor* $\mathrm{PMap}(S)$ *is residually finite when S has infinite genus.*

The heavy lifting in the proof of Proposition 4.6 is done by a theorem of Paris [31], which says that when S is a finite-type genus-g surface, the minimal index of a proper subgroup of $\mathrm{PMap}(S)$ is bounded below by a linear function in g. (The work of Berrick, Gebhardt and Paris [9] gives the exact value as an exponential function in g.)

Proof of Proposition 4.6 Let $\{F_n\}_{n\in\mathbb{N}}$ be an exhaustion of S by essential finite-type surfaces. Then $\mathrm{genus}(F_n)\to\infty$ as $n\to\infty$. Let m_n denote the minimal index of a subgroup in $\mathrm{PMap}(F_n)$. Then $\lim_{n\to\infty} m_n = \infty$ [31].

Suppose N is a proper normal subgroup of $\mathrm{PMap}_c(S)$ with finite index k. As $\mathrm{PMap}_c(S)$ is the direct limit of $\{\mathrm{PMap}(F_n)\}_{n\in\mathbb{N}}$, we have that there exists $M\in\mathbb{N}$ such that $\mathrm{PMap}(F_m)$ is not a subgroup of N for all $m>M$ (otherwise N would not be proper). It follows that for $m>M$ the intersection $N\cap\mathrm{PMap}(F_m)$ is a proper normal subgroup of $\mathrm{PMap}(F_m)$ with index at most k. However, this implies that the sequence $\{m_n\}_{n\in\mathbb{N}}$ is bounded, a contradiction. □

5 Morphisms

The goal of this section is to finish the proof of Theorem 1 with Propositions 5.1 and 5.3. We then deduce corollaries about the outer automorphism group. The tools in this section combine the results on the initial topology in Section 3 with the literature on homomorphisms between mapping class groups (namely, [4; 15]). By Proposition 4.1 and Corollary 4.8, the algebraic structure of $\mathrm{PMap}(S)$ detects whether S has finite or infinite genus. For the remainder of the section, we focus on finite-genus surfaces.

Proposition 5.1 *Let S and S' be finite-genus surfaces. If the genus of S is at least 3 and strictly greater than that of S', then there is no monomorphism from $\mathrm{PMap}(S)$ to $\mathrm{PMap}(S')$.*

Proof Suppose S and S' are surfaces satisfying

$$\mathrm{genus}(S) > \max\{3, \mathrm{genus}(S')\}$$

and that $\psi\colon \mathrm{PMap}(S)\to\mathrm{PMap}(S')$ is a homomorphism. Let $F\subset S$ be an essential finite-type subsurface with $\mathrm{genus}(F)=\mathrm{genus}(S)$ and $\iota\colon \mathrm{PMap}(F)\hookrightarrow\mathrm{PMap}(S)$ the inclusion homomorphism. For each $\lambda\in\Lambda(S')$, we obtain a homomorphism $\psi_\lambda\colon \mathrm{PMap}(F)\to\mathrm{PMap}(S'_\lambda)$ defined by the composition $\psi_\lambda=\varphi_\lambda\circ\psi\circ\iota$, where φ_λ is the forgetful map defined in Section 3. Because F and S'_λ are finite-type and $\mathrm{genus}(F)>\max\{3,\mathrm{genus}(S'_\lambda)\}$, it follows from [4, Proposition 7.1] that ψ_λ is trivial.

Further, as λ was arbitrary, we see that

$$\psi \circ \iota(\mathrm{PMap}(F)) \subset \bigcap_{\lambda \in \Lambda(S')} \ker \varphi_\lambda,$$

implying that $\iota(\mathrm{PMap}(F)) \subset \ker \psi$ by Lemma 3.4; hence, ψ is not injective. □

Continuing with our progress towards Theorem 1, we can now assume that if $\mathrm{PMap}(S)$ and $\mathrm{PMap}(S')$ are isomorphic, then they have the same genus. We have one additional lemma before finishing the final proposition towards Theorem 1.

Lemma 5.2 *Let S and S' be surfaces such that their genera are finite, equal, and at least 4. If $\Psi\colon \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ is an isomorphism, then Ψ preserves Dehn twists.*

Proof Let T_c denote the Dehn twist along a simple closed curve c. Choose an essential finite-type surface $F \subset S$ such that $c \subset F$ and $\mathrm{genus}(F) = \mathrm{genus}(S)$. The inclusion $F \hookrightarrow S$ induces a homomorphism $\iota\colon \mathrm{PMap}(F) \to \mathrm{PMap}(S)$. For each $\lambda \in \Lambda(S')$ we define $\iota_\lambda\colon \mathrm{PMap}(F) \to \mathrm{PMap}(S'_\lambda)$ to be the composition $\varphi_\lambda \circ \Psi \circ \iota$, where again φ_λ is the forgetful map defined in Section 3. If ι_λ is nontrivial, then [4, Theorem 1.1] implies ι_λ is induced by an inclusion $F \hookrightarrow S'_\lambda$ (note that this is where the lower bound on genus is required[1]).

In particular, ι_λ sends Dehn twists in $\mathrm{PMap}(F)$ into $\overline{\mathcal{D}}_\lambda$. As T_c is the image of a Dehn twist in $\mathrm{PMap}(F)$ under ι, we have that $\varphi_\lambda \circ \Psi(T_c) \in \overline{\mathcal{D}}_\lambda$ for each $\lambda \in \Lambda(S')$. By Proposition 3.7, $\Psi(T_c)$ is a Dehn twist. □

Proposition 5.3 *Let S and S' be surfaces. If the genera of S and S' are finite, equal, and at least 4, then any isomorphism between $\mathrm{PMap}(S)$ and $\mathrm{PMap}(S')$ is induced by a homeomorphism.*

Proof Let $\Psi\colon \mathrm{PMap}(S) \to \mathrm{PMap}(S')$ be an isomorphism. Observe that the centers $Z(\mathrm{PMap}(S))$ and $Z(\mathrm{PMap}(S'))$ of $\mathrm{PMap}(S)$ and $\mathrm{PMap}(S')$, respectively, are isomorphic, and therefore, S and S' have the same number of boundary components by Lemma 3.3. By applying Lemma 5.2 to both Ψ and Ψ^{-1}, we see that Ψ restricts to an isomorphism from $\mathrm{PMap}_c(S)$ to $\mathrm{PMap}_c(S')$.

[1] The statement of [4, Theorem 1.1] requires the genus to be at least 6; however, the remark immediately following the theorem statement in the article states that this requirement can be decreased to 4 under our assumptions.

Let $F_1 \subset F_2 \subset \cdots$ be an exhaustion of S by essential finite-type surfaces such that $\operatorname{genus}(F_n) = \operatorname{genus}(S)$ and $\partial S \subset \partial F_n$ for each $n \in \mathbb{N}$. Recall that essential surfaces are connected by definition. It follows that the direct limit of the groups $\{\mathrm{PMap}(F_n)\}_{n\in\mathbb{N}}$ with respect to the natural inclusions is $\mathrm{PMap}_c(S)$.

Fix $n \in \mathbb{N}$ and let $d_1, \ldots, d_m$ be a collection of essential simple closed curves in F_n such that the set $\{T_{d_1}, \ldots, T_{d_m}\}$ of Dehn twists generates $\mathrm{PMap}(F_n)$. For each $i \in \{1, \ldots, m\}$ let d_i' be the simple closed curve in S' such that $\Psi(T_{d_i}) = T_{d_i'}$; the existence of d_i' is guaranteed by Lemma 5.2. Let R' be a regular neighborhood of $d_1' \cup \cdots \cup d_m'$. It is possible that one or more boundary components of R' bounds either a disk or a once-punctured disk in S'. If this is the case, glue the associated disk or once-punctured disk onto these boundary components; we call the resulting surface R_n.

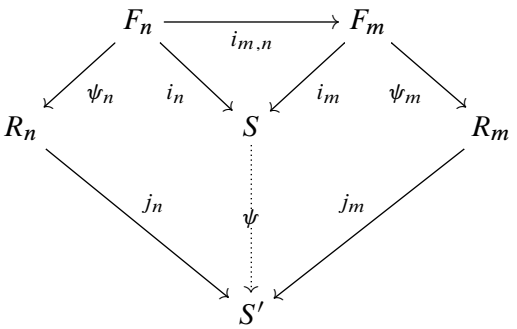
Observe that since any Dehn twist about a boundary component of S' can be written as a product of Dehn twists about curves contained in R_n, we can conclude that $\partial S' \subset \partial R_n$. We now claim that the surface R_n is connected: Indeed, observe that d_i and d_j are disjoint if and only if d_i' and d_j' are disjoint. Using that Ψ is an isomorphism, the above fact follows from the fact that T_{d_i} and T_{d_j} commute if and only if d_i and d_j are disjoint. It now follows that $d_1' \cup \cdots \cup d_m'$ is connected as $d_1 \cup \cdots \cup d_m$ is connected; hence, R_n is connected.

As $\operatorname{genus}(F_n) = \operatorname{genus}(S) = \operatorname{genus}(S')$, we have (again by [4, Proposition 7.1]) that $\operatorname{genus}(R_n) = \operatorname{genus}(F_n)$. Now if $f \in \mathrm{PMap}(S')$ is in the centralizer of $\mathrm{PMap}(R_n)$, then f commutes with $T_{d_i'}$ for each $i \in \{1, \ldots, m\}$; in particular, f is in the centralizer of $\Psi(\mathrm{PMap}(F_n))$. Conversely, if $f \in \mathrm{PMap}(S')$ is in the centralizer of $\Psi(\mathrm{PMap}(F_n))$, then again f commutes with the Dehn twist $T_{d_i'}$ for every $i \in \{1, \ldots, m\}$; therefore, as the collection of curves $\{d_1', \ldots, d_m'\}$ is filling in R_n, we can conclude that f is in the centralizer of $\mathrm{PMap}(R_n)$. The above discussion also implies that an element in the centralizer of $\mathrm{PMap}(R_n)$ is supported in the complement of the interior of R_n; therefore, by Lemma 3.3, the center of the centralizer of $\mathrm{PMap}(R_n)$ is generated by the boundary components of R_n. In particular, the rank of the center of the centralizer of $\mathrm{PMap}(R_n)$ is the number of boundary components of R_n (the same holds for F_n and $\mathrm{PMap}(F_n)$). Now the center of the centralizer of $\mathrm{PMap}(F_n)$ and that of $\Psi(\mathrm{PMap}(F_n))$ are isomorphic; it follows from the above that F_n and R_n have the same number of boundary components.

Let Z_n denote the center of $\mathrm{PMap}(F_n)$. From the preceding discussion, we have established that the center of the centralizer of $\mathrm{PMap}(R_n)$ is the center of $\mathrm{PMap}(R_n)$.

The same holds for $\mathrm{PMap}(F_n)$ in $\mathrm{PMap}(S)$. Further, we can conclude that $\Psi(Z_n)$ is the center of $\mathrm{PMap}(R_n)$. This implies that Ψ induces a monomorphism from $\mathrm{PMap}(F_n)/Z_n$ to $\mathrm{PMap}(R_n)/\Psi(Z_n)$. Observe that $\mathrm{PMap}(F_n)/Z_n$ is the pure mapping class group of the surface obtained by capping off the boundary components of F_n with once-punctured disks and the same is true for $\mathrm{PMap}(R_n)/\Psi(Z_n)$ and R_n. Combining [15, Theorem 1.2.7] with the fact that F_n and R_n have the same number of boundary components, we see they also have the same number of punctures; in particular, F_n and R_n are homeomorphic.

We can now apply [4, Theorem 1.1] to see that $\Psi_n := \Psi|_{\mathrm{PMap}(F_n)}$ is induced by an embedding $\psi_n\colon F_n \hookrightarrow R_n$; it follows from our discussion thus far that ψ_n is in fact a homeomorphism and Ψ_n an isomorphism. As $\Psi|_{\mathrm{PMap}_c(S)}$ is an isomorphism to $\mathrm{PMap}_c(S')$, it follows that $\mathrm{PMap}_c(S')$ is the direct limit of groups $\{\mathrm{PMap}(R_n)\}_{n\in\mathbb{N}}$. It also follows that $R_1 \subset R_2 \subset \cdots$ is an exhaustion of S' by essential finite-type surfaces: indeed, if not, then there exists an essential simple closed curve in the complement of $\bigcup_{n\in\mathbb{N}} R_n$, but this would imply that the Dehn twist about this curve is not an element of $\bigcup_{n\in\mathbb{N}} \mathrm{PMap}(R_n)$, a contradiction. We can now conclude that S' is the direct limit of the spaces $\{R_n\}_{n\in\mathbb{N}}$ with respect to inclusion. Note that S is the direct limit of the spaces $\{F_n\}_{n\in\mathbb{N}}$ as well. Given the construction of ψ_n, we can build the following commutative diagram where each indexed i and j is the natural inclusion and the natural numbers n and m satisfy $m > n$:



The existence of the unique continuous map $\psi\colon S \to S'$ is guaranteed by the universal property of direct limits. If we repeat the above process starting with Ψ^{-1}, we see that ψ is the desired homeomorphism between S and S'. □

Observe that after the immediate application of Lemma 5.2 in the proof Proposition 5.3, the proof reduces to proving the analogous statement for $\mathrm{PMap}_c(S)$. The inspiration for this portion comes from [3, Proposition 9.1], in which the authors show that

an automorphism of $\mathrm{PMap}_c(S)$ — in the special case where S has finite genus and $\mathrm{Ends}(S)$ is a Cantor set — preserves Dehn twists.

As pointed out in the introduction, Corollary 2 is an immediate consequence of Proposition 5.3 and Lemma 3.3, that is, for a borderless surface S we have

$$\mathrm{Aut}(\mathrm{PMap}(S)) = \mathrm{Map}^{\pm}(S)$$

when $4 \leq \mathrm{genus}(S) < \infty$. We point out an alternative proof: By Lemma 5.2, an automorphism permutes the Dehn twists and hence induces an automorphism of the curve graph (two curves are disjoint if and only if their associated Dehn twists commute). In an recent paper [23], it is shown that the automorphism group of the curve graph is the extended mapping class group. One can then follow a — by now — standard argument to obtain Corollary 2. (This is the route taken in the recent work [7] in which Question 1.1 is answered.)

As a corollary to Corollary 2, we see the topological structure of $\mathrm{Ends}(S)$ becoming apparent in the algebra of $\mathrm{PMap}(S)$.

Corollary 5.4 *If S is a borderless surface whose genus is finite and at least 4, then $\mathrm{Out}(\mathrm{PMap}(S))$ is isomorphic to $\mathrm{Homeo}(\mathrm{Ends}(S)) \times \mathbb{Z}/2\mathbb{Z}$.*

Proof Corollary 2.5 tells us the homomorphism $\mathrm{Map}(S) \to \mathrm{Homeo}(\mathrm{Ends}(S))$ is surjective and by extension $\mathrm{Map}^{\pm}(S) \to \mathrm{Homeo}(\mathrm{Ends}(S))$ is also surjective. It follows that there exists an order-two orientation-reversing homeomorphism fixing all the ends of S; let $f \in \mathrm{Map}^{\pm}(S)$ be such an element. It follows that

$$\mathrm{Map}^{\pm}(S) = \mathrm{Map}(S) \rtimes \mathbb{Z}/2\mathbb{Z},$$

where the $\mathbb{Z}/2\mathbb{Z}$ factor is identified with $\langle f \rangle$. It easily follows from Corollary 2 and Corollary 2.5 that

$$\mathrm{Out}(\mathrm{PMap}(S)) = \mathrm{Map}^{\pm}(S)/\,\mathrm{PMap}(S) = \mathrm{Homeo}(\mathrm{Ends}(S)) \times \mathbb{Z}/2\mathbb{Z}. \qquad \square$$

Given Corollary 5.4, it is natural to ask if the isomorphism type of $\mathrm{Out}(\mathrm{PMap}(S))$ determines the homeomorphism type of $\mathrm{Ends}(S)$. This is equivalent to asking: given a closed subset E of the Cantor set, does the isomorphism type of $\mathrm{Homeo}(E)$ determine the homeomorphism type of E?

Note that there is an easy counterexample: if $E = C \sqcup X$ where X is a one-point space and C is a Cantor space, then $\mathrm{Homeo}(C) \cong \mathrm{Homeo}(E)$. It was originally

conjectured by Monk [30] that this was the only counterexample; however, another counterexample was given by McKenzie [29, Theorem 6]. On the other hand, McKenzie [29, Theorem 5] showed the conjecture holds in a large subclass; namely, if the closure of the isolated points in E is clopen, then $\mathrm{Homeo}(E)$ determines E. (The work of Monk and McKenzie is in the category of boolean algebras; one can translate to the language above using Stone's representation theorem.)

In light of Theorem 1, it is natural to pose the following:

Problem 5.5 Find correspondences between algebraic invariants of $\mathrm{PMap}(S)$ and topological invariants of S.

As an example, we highlight the work of Monk:

Proposition 5.6 [30, Corollary 2.1 and Theorems 3 and 5] *Let E be a closed subset of the Cantor set with infinite cardinality.*

(1) $\mathrm{Homeo}(E)$ *is simple if and only if E is homeomorphic to either a Cantor space or the disjoint union of a Cantor space and the one-point space.*

(2) *If $m \in \mathbb{N}$ with $1 < m < \infty$, then E is homeomorphic to $C \sqcup \{p_1, \dots, p_m\}$ if and only if $\mathrm{Homeo}(E) \cong \mathrm{Sym}_m \times \mathrm{Homeo}(C)$, where C is a Cantor space.*

(3) $\mathrm{Homeo}(E)$ *has exactly two proper normal subgroups if and only if E is homeomorphic to the one-point compactification of $\mathbb{N}$.*

6 The compact-open topology

We begin by restating the definition of the mapping class group: Equip $\mathrm{Homeo}(S)$ with the compact-open topology. Then

$$\mathrm{Map}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S)),$$

where $\mathrm{Homeo}^+(S, \partial S)$ is the group of orientation-preserving homeomorphisms of S fixing ∂S pointwise. With this definition, $\mathrm{Map}(S)$ comes equipped with a natural topology, namely, the corresponding quotient topology, giving $\mathrm{Map}(S)$ the structure of a topological group. If S is a finite-type surface, then this is the discrete topology on $\mathrm{Map}(S)$ (one way to see this is as a consequence of the Alexander method [17, Proposition 2.8]). We set τ_q to be the corresponding subspace topology on $\mathrm{PMap}(S)$. It follows from Lemma 2.1 that the identity component of $\mathrm{Homeo}^+(S)$ is closed when S is borderless. This implies that the one-point sets in $(\mathrm{PMap}(S), \tau_q)$

are closed; it is a standard exercise to show that in a topological group this implies the group is Hausdorff. (A similar argument works for surfaces with boundary.)

Moreover, it is metrizable:

Proposition 6.1 $(\mathrm{PMap}(S), \tau_q)$ *is metrizable.*

Proof As $\mathrm{Homeo}(S, \partial S)$ is first countable and $\mathrm{Map}^{\pm}(S)$ is the image under an open quotient map, it follows that $(\mathrm{PMap}(S), \tau_q)$ is first countable. By the Birkhoff–Kakutani theorem [10; 26], every first-countable Hausdorff topological group is metrizable. □

Recall that for a finite-type surface, a combination of the Dehn–Lickorish theorem and the Birman exact sequence (see [17, Chapter 4]) tells us that $\mathrm{PMap}(S)$ is generated by Dehn twists. In the infinite-type setting this of course cannot be true as $\mathrm{PMap}(S)$ is uncountable; however, building off the result in the finite-type setting we are able to obtain topological generating sets for $\mathrm{PMap}(S)$. The remainder of this section is dedicated to proving Theorem 4.

Before we get to the proof of Theorem 4, we need to introduce a class of homeomorphisms contained in $\mathrm{PMap}(S)$. Consider the surface Σ defined by taking the surface $\mathbb{R} \times [0, 1]$, removing the interior of each disk of radius $\frac{1}{4}$ with center of the form $\left(n, \frac{1}{2}\right)$, for each $n \in \mathbb{Z}$, and attaching a torus with one boundary component to the boundary of each such disk. Let σ be the homeomorphism of Σ that behaves like $(x, y) \mapsto (x+1, y)$ on the interior of Σ and tapers to the identity in a neighborhood of $\partial\Sigma$ (see Figure 2). One should think of σ as obtained by sliding the n^{th}–handle $\left($in the disk with center $\left(n, \frac{1}{2}\right)\right)$ horizontally until it reaches the position of the $(n+1)^{\text{th}}$–handle.

Given a surface S with at least two ends accumulated by genus, consider an embedding of Σ into S inducing an injection $\mathrm{Ends}(\Sigma) \hookrightarrow \mathrm{Ends}(S)$. The homeomorphism σ can then be extended to all of S by the identity. The isotopy class of the extension σ defines an element of $\mathrm{PMap}(S)$; we call such an element a *handle shift*.

In the proof of Proposition 6.2, we will make use of the notion of a pants decomposition: a *pants decomposition* of a surface S is a maximal collection $\mathcal{P}$ of pairwise disjoint, pairwise nonisotopic, essential simple closed curves with the property that any simple closed curve on S intersects at most finitely curves in $\mathcal{P}$. The complement of the union of curves in a pants decomposition is a disjoint union of surfaces each of which is homeomorphic to a thrice-punctured sphere.

Towards Theorem 4 we prove the following proposition.

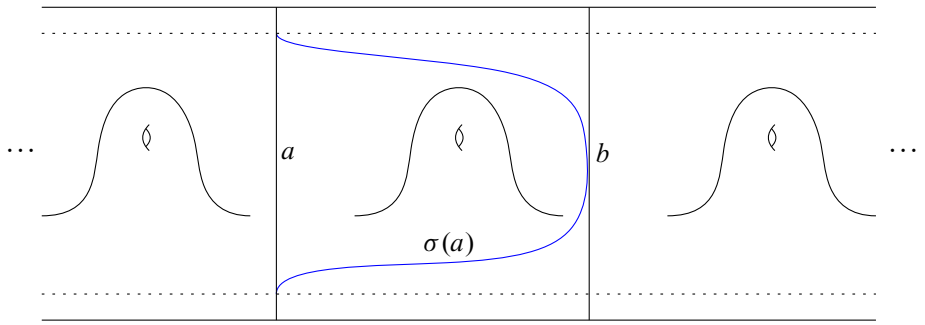


Figure 2: The surface Σ. Outside the dotted lines, the homeomorphism σ is the identity. The arcs $\sigma(a)$ and b are homotopic inside Σ.

Proposition 6.2 *The set of Dehn twists topologically generate* $(\mathrm{PMap}(S), \tau_q)$ *when* S *has at most one end accumulated by genus. If* S *has at least two ends accumulated by genus, then the set of Dehn twists together with the set of handle shifts topologically generate* $(\mathrm{PMap}(S), \tau_q)$.

Proof Let $\mathcal{D}$ denote the collection of Dehn twists in $\mathrm{PMap}(S)$. Since Dehn twists generate finite-type pure mapping class groups and $\mathrm{PMap}_c(S)$ is a direct limit of finite-type pure mapping class groups, it follows that $\langle \mathcal{D} \rangle = \mathrm{PMap}_c(S)$. In light of the finite-type results, we will assume S is of infinite type. When S has at least two ends accumulated by genus, let $\mathrm{PMap}_h(S)$ denote the subgroup of $\mathrm{PMap}(S)$ generated by Dehn twists and handle shifts.

Take $f \in \mathrm{PMap}(S)$ to be an arbitrary element. Abusing notation, we will conflate f with a representative homeomorphism. Fix a pants decomposition $\mathcal{P}$ of S. Let $\{P_1, P_2, \dots\}$ be the components of the complement of $\bigcup_{\gamma \in \mathcal{P}} \gamma$ in S, labeled so that the surface $R_n = \bigcup_{i=1}^{n} \overline{P}_i$ is connected, where $\overline{P}_i$ is the closure of P_i in S. Choose a component, call it a, of ∂P_1 and let F be a finite-type essential surface such that

- both a and $f(a)$ are contained in F,
- each component of ∂F is separating, and
- each component of $\overline{S \smallsetminus F}$ intersects $\mathrm{Ends}(S)$ nontrivially.

When S has at most one end accumulated by genus, we additionally require F to be such that

- at most one component of $S \smallsetminus F$ has positive genus.

By the classification of surfaces, the $\mathrm{PMap}(S)$–orbit of a, up to isotopy, is determined by its partition of $\mathrm{Ends}(S)$, its partition of ∂S, and the topological type of its complements. (This is true for any surface and any simple closed curve in the surface.) Observe that the construction of F guarantees that a and $f(a)$ must induce the same partition of ∂F and $\mathrm{Ends}(F)$ as well.

First, if a is nonseparating, then both a and $f(a)$ are nonseparating in F, so there exists $g_a \in \mathrm{PMap}(F) < \mathrm{PMap}(S)$ such that $g_a(a) = f(a)$. In addition, we require that g_a fixes an orientation on a.

Now suppose a is separating. In the case where S has at most one end accumulated by genus, there is a component U of $S \smallsetminus a$ that has finite genus. Let $W = U \cap F$ and $V = f(U) \cap F$. Note that since S has at most one end accumulated by genus, $S \smallsetminus F$ has at most one component with positive genus, and when S is a surface of finite genus, F can be chosen to have the same genus as S. It follows that $\mathrm{genus}(W) = \mathrm{genus}(V)$, and therefore, the components of $F \smallsetminus a$ and $F \smallsetminus f(a)$ have the same topological type. In particular, there exists $g_a \in \mathrm{PMap}(F) < \mathrm{PMap}(S)$ such that $g_a(a) = f(a)$. In both of these cases, g_a is supported on a finite-type surface and hence is contained in $\mathrm{PMap}_c(S)$.

Since $g_a^{-1} \circ f$ fixes a and both f and g_a fix $\mathrm{Ends}(S)$ pointwise, $g_a^{-1} \circ f$ restricts to a homeomorphism of each component of $S \smallsetminus a$. Let $b \neq a$ be another component of ∂P_1, and let S' be the component of $S \smallsetminus a$ containing b. We can then repeat the above process with S replaced by S' and f by $g_a^{-1} \circ f|_{S'}$. If P_1 has a third boundary component, then we repeat this process with this final component. In either case, we have built a homeomorphism $g_1 \in \mathrm{PMap}_c(S)$ such that $g_1(P_1) = f(P_1)$ and $g_1^{-1} \circ f$ fixes each boundary component of P_1. Thus, $f_1 = g_1^{-1} \circ f$ restricts to the identity on $P_1 = R_1$.

Let S_2 be the component of $S \smallsetminus P_1$ containing P_2. Following the same reasoning as above and working in $\mathrm{PMap}(S_2)$, we can find a product of Dehn twists in $\mathrm{PMap}(S_2)$, call it g_2', such that $g_2'^{-1} \circ f_1'$ is the identity when restricted to P_2, where f_1' is the restriction of f_1 to S_2. Choose an element g_2'' in the stabilizer of P_1 in $\mathrm{PMap}_c(S)$ whose restriction to S_2 agrees with g_2'. Defining $g_2 = g_1 \circ g_2'' \in \mathrm{PMap}_c(S)$ we have that $g_2^{-1} \circ f$ restricts to the identity on R_2 after composing with Dehn twists about the components of ∂P_1 if necessary.

Continuing in this fashion, we construct $g_n \in \mathrm{PMap}_c(S)$ such that $g_n^{-1} \circ f$ restricted to R_n is the identity. As R_n is an exhaustion of S, we see that $g_n^{-1} \circ f$ converges to the identity; hence, $g_n \to f$ as desired.

Now suppose that S has at least two ends accumulated by genus. Note that we may no longer be able to choose F to satisfy the fourth bullet point above. Let a be a separating curve with each component of $S \smallsetminus a$ being infinite genus and let U be a component of $F \smallsetminus a$. It is possible that $\operatorname{genus}(V) \neq \operatorname{genus}(W)$, where $W = U \cap F$ and $V = f(U) \cap F$. If this is the case, there exists a handle shift $h \in \mathrm{PMap}_h(S)$ such that $h(a)$ induces the same partition of ∂F and $\operatorname{Ends}(F)$ as a and $f(a)$, where $h(a) \subset F$, and such that the genera of $h(U) \cap F$ and V are equal. Thus, there exists $g_a \in \mathrm{PMap}(F) < \mathrm{PMap}(S)$ such that $g_a \circ h(a) = f(a)$. Of course, $g_a \circ h \in \mathrm{PMap}_h(S)$, so we can now repeat the argument above to produce $k_n \in \mathrm{PMap}_h(S)$ such that $k_n \to f$. □

The next proposition shows that introducing handle shifts is essential when S has at least two ends accumulated by genus.

Proposition 6.3 *If S has at least two ends accumulated by genus, then $\mathrm{PMap}_c(S)$ is not dense in $(\mathrm{PMap}(S), \tau_q)$.*

Proof We begin by describing a property of all elements of $\mathrm{PMap}_c(S)$, and then show that handle shifts do not have this property and cannot be approximated by those elements that do. Choose a separating simple closed curve a such that each component of $S \smallsetminus a$ has infinite genus. Let f be an arbitrary element of $\mathrm{PMap}_c(S)$. Let F be an essential finite-type subsurface of S so that $f \in \mathrm{PMap}(F) < \mathrm{PMap}(S)$ and F satisfies the first three bullet points in the proof of Proposition 6.2. Let U be a component of $F \smallsetminus a$. Then U and $f(U)$ contain the same components of ∂F. Further, $\operatorname{genus}(U) = \operatorname{genus}(f(U))$, so either $[f(a)] = [a]$ or $i([a], [f(a)]) > 0$.

Now, let b be a simple closed curve so that a and b cobound a genus-1 subsurface of S. It is not hard to see that there exists a handle shift $h \in \mathrm{PMap}(S)$ with $h(a) = b$. Thus, no element in $\mathrm{PMap}_c(S)$ can agree with h on the compact set a, since $b = h(a)$ and a are distinct and disjoint; hence, $\mathrm{PMap}_c(S)$ is not dense in $(\mathrm{PMap}(S), \tau_q)$. □

Propositions 6.2 and 6.3 together imply Theorem 4.

In Lemma 3.6, we gave the closure of the set of Dehn twists in τ_w. In the finite-genus setting, the analogous result holds for τ_q and is obtained by combining Lemma 3.6 and Proposition 7.6. Below we give a proof in general, regardless of whether the genus of S is finite or infinite:

Proposition 6.4 *The closure of $\mathcal{D}$ in τ_q is $\overline{\mathcal{D}} = \mathcal{D} \cup \{\mathrm{id}\}$.*

Proof Let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of S by essential finite-type subsurfaces. We first see that the identity is in $\overline{\mathcal{D}}$ (which is a repeated argument from the beginning of Lemma 3.6). Let b_n be a simple closed curve contained in the complement of K_n. Then the sequence $\{T_{b_n}\}_{n\in\mathbb{N}} \subset \mathcal{D}$ converges to the identity in $(\mathrm{PMap}(S), \tau_q)$.

Following the proof outline of Lemma 3.6, we want to show that if a sequence $\{T_{a_i}\}_{i\in\mathbb{N}} \subset \mathcal{D}$ converges to a nonidentity element $f \in (\mathrm{PMap}(S), \tau_q)$, then f is compactly supported.

Assume that $f \notin \mathrm{PMap}_c(S)$. It follows that there exists a natural number N such that for every integer n with $n \geq N$ there exists an integer i_n such that the curve a_{i_n}, up to isotopy, is contained in neither K_n nor its complement and intersects K_N nontrivially.

By possibly increasing i_n, we may assume that the curve a_{i_n} has nontrivial intersection with a single boundary component of ∂K_N, call it b. For each curve a_{i_n}, let b_n be a component of K_n such that $i(a_{i_n}, b_n) > 0$. As the compact-open topology agrees with the topology of compact convergence, there exists $J \in \mathbb{N}$ such that, up to isotopy, $f(b) = T_{a_j}(b)$ for all $j > J$. Apply [17, Proposition 3.4] to obtain

$$i(f(b), b_n) = i(T_{a_{i_n}}(b), b_n) = i(a_{i_n}, b)\, i(a_{i_n}, b_n) > 0,$$

whenever $i_n > \max\{J, N\}$. It follows that $f(b)$ leaves every compact set; hence, it must not be compact and f cannot be a homeomorphism. Therefore, $f \in \mathrm{PMap}_c(S)$.

We can now find $m \in \mathbb{N}$ such that all but finitely many of the T_{a_i} are elements of $\mathrm{PMap}(K_m) < \mathrm{PMap}_c(S)$. The subspace topology on $\mathrm{PMap}(K_m)$ is discrete, so the sequence $\{T_{a_i}\}_{i\in\mathbb{N}}$ is eventually constant; in particular, f is an element of $\overline{\mathcal{D}}$. □

Note that from the above proof we see that a sequence in $\mathcal{D}$ converges to a Dehn twist in τ_q if and only if it is eventually constant. This is not the case in τ_w: Consider the curves a_n in Figure 1 from Section 3 with the modification that the curve a_n partitions off $\{0, 1, n\}$ in $\mathbb{N} \cup \{0\}$. The sequence $\{T_{a_n}\} \subset \mathcal{D}$ would then converge to T_{a_1} in $\big(\mathrm{PMap}(\mathbb{C} \smallsetminus (\mathbb{N} \cup \{0\})), \tau_w\big)$, but would not converge in τ_q.

7 Inverse limits

For the entirety of this section, S will be a finite-genus surface. To simplify notation, for $\lambda \in \Lambda(S)$, we let

$$\begin{aligned} G_\lambda &= \mathrm{PHomeo}^+(\overline{S}, \partial S, \lambda) \\ &= \{f \in \mathrm{Homeo}^+(\overline{S}, \partial S) : f(p) = p \text{ for every } p \in \lambda\} \end{aligned}$$

and

$$\begin{aligned} G &= \mathrm{PHomeo}^+(\overline{S}, \partial S, \mathrm{Ends}(S)) \\ &= \{f \in \mathrm{Homeo}^+(\overline{S}, \partial S) : f(p) = p \text{ for every } p \in \mathrm{Ends}(S)\}. \end{aligned}$$

Equip $\mathrm{Homeo}(\overline{S}, \partial S)$ with the compact-open topology and equip G and each G_λ with the subspace topology.

Observe that $\mathrm{Homeo}(S, \partial S)$ and $\mathrm{Homeo}(\overline{S}, \partial S, \mathrm{Ends}(S))$ are isomorphic. Furthermore, it is clear that the compact-open topology on the latter group must be at least as fine as the former. However, the classical work of Arens [5, Theorem 3] tells us that the compact-open topology on the former group is the strongest admissible topology making it a topological group; hence, the two groups are topologically isomorphic. In other words, we may treat the ends of S as marked points.

The inclusion maps $\iota_{\lambda,\mu}\colon G_\mu \hookrightarrow G_\lambda$, whenever $\lambda \subseteq \mu$, make $(\{G_\lambda\}, \{\iota_{\lambda,\mu}\})$ an inverse system. We can then set G_L to be the inverse limit and define $\iota_\lambda\colon G_L \to G_\lambda$ to be the projections. The initial topology on G with respect to the collection $\{\iota_\lambda\}$ gives G_L the structure of a topological group; this is the standard topology given to an inverse limit of topological groups. With this topology, we see that G_L is topologically isomorphic to the intersection of the G_λ, each endowed with the compact-open topology, so

$$G_L \cong \bigcap_{\lambda \in \Lambda} G_\lambda = G.$$

For the remainder of the section we will identify G and G_L.

As $\mathrm{PMap}(S) = \pi_0(G)$, we want to investigate whether $\mathrm{PMap}(S)$ has the structure of an inverse limit. Similar to the above, the maps $\iota_{\lambda,\mu}$ induce forgetful maps $\varphi_{\lambda,\mu}\colon \mathrm{PMap}(S_\mu) \to \mathrm{PMap}(S_\lambda)$.

This defines an inverse system $(\{\mathrm{PMap}(S_\lambda)\}, \{\varphi_{\lambda,\mu}\})$ and we can take the inverse limit

$$L(S) = \varprojlim_{\lambda \in \Lambda} \mathrm{PMap}(S_\lambda).$$

Let $\pi_\lambda\colon L(S) \to \mathrm{PMap}(S_\lambda)$ be the projections defining $L(S)$. Giving $\mathrm{PMap}(S_\lambda)$ the discrete topology for each $\lambda \in \Lambda$, we equip $L(S)$ with the initial topology with respect to $\{\pi_\lambda\}_{\lambda \in \Lambda}$. The universal property of inverse limits gives a continuous homomorphism $\Phi\colon (\mathrm{PMap}(S), \tau_w) \to L(S)$.

Proposition 7.1 *Φ is a topological embedding. Further, Φ is a topological isomorphism if and only if S is of finite type.*

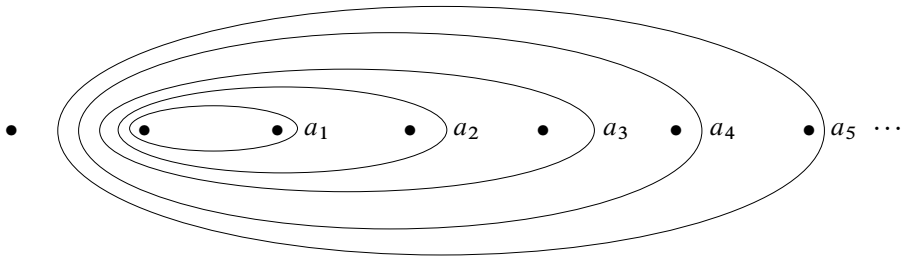


Figure 3: The surface $S = \mathbb{C} \smallsetminus \mathbb{Z}$. The sequence of Dehn twists $\{T_{a_n}\}_{n\in\mathbb{N}}$ converges in $L(S)$, but not in $(\mathrm{PMap}(S), \tau_w)$.

Proof Lemma 3.4 implies that τ_w separates points and hence Φ is injective. It is clear from the definition that τ_w agrees with the subspace topology when $\mathrm{PMap}(S)$ is viewed as a subset of $L(S)$. Note that if S is of finite type, then $L(S) = \mathrm{PMap}(S)$, so for the remainder we may assume that S is of infinite type.

We now show that Φ fails to be onto. We will show this in the case $S = \mathbb{C} \smallsetminus \mathbb{Z}$; the general case can be ascertained by replacing each integer in $\mathbb{Z}$ with a clopen set of $\mathrm{Ends}(S)$ in the general case.

For each $n \in \mathbb{N}$, let a_n be the boundary of a regular neighborhood of the line segment on the real axis connecting 1 and $n+1$ as in Figure 3. Let $T_n \in \mathrm{PMap}(S)$ denote the Dehn twist about a_n. For each $\lambda \in \Lambda$, the sequence $\{\varphi_\lambda(T_n)\}$ is eventually constant; call this limit Dehn twist $T_\lambda \in \mathrm{PMap}(S_\lambda)$. It follows that the sequence $\{T_n\}_{n\in\mathbb{N}}$ converges to $\overline{T} \in L(S)$, where $\overline{T}$ has coordinates $\varphi_\lambda(\overline{T}) = T_\lambda$.

We claim $\overline{T}$ is not in $\mathrm{PMap}(S)$; we proceed by contradiction. Suppose $\overline{T} \in \mathrm{PMap}(S)$ and conflate $\overline{T}$ with a representative homeomorphism. Let b_n be the boundary of the ball centered at the origin with radius $n + \frac{1}{2}$. Using Lemma 3.5 and a similar argument as in the proof of Lemma 3.6, we see that $i(\overline{T}(b_1), b_n) > 0$. It follows that $\overline{T}(b_1)$ cannot be compact, a contradiction. □

We point out the contrast between the pure homeomorphism group of S being an inverse limit and the pure mapping class group failing to do so. It is natural to ask if the topologies τ_w and τ_q agree as the compact-open topology and the initial topologies agree on the level of homeomorphism groups. Proposition 7.6 tells us that this also fails to be the case (when S is of infinite type).

Now recall that a topological group is *Polish* if it is separable and completely metrizable. Such groups have many nice properties and are studied extensively in descriptive set

theory. Of particular interest is the work of Rosendal [35], which provides a theory of coarse geometry for Polish groups that are not locally compact.

Proposition 7.2 *$L(S)$ is Polish if and only if the cardinality of $\mathrm{Ends}(S)$ is countable.*

Before getting to the proof of Proposition 7.2, we separate out two lemmas that are interesting to us in their own right. Observe that by the definition of the initial topology and the Dehn–Lickorish theorem, the first lemma is immediate.

Lemma 7.3 *$L(S)$ is topologically generated by Dehn twists.*

Lemma 7.4 *If the cardinality of $\mathrm{Ends}(S)$ is uncountable, then $(\mathrm{PMap}(S), \tau_w)$ is not first countable.*

The proof mimics the standard proof that an uncountable product of discrete topological spaces is not first countable.

Proof Suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a countable neighborhood basis of the identity in $\mathrm{PMap}(S)$. For each $n \in \mathbb{N}$ there exists a finite set $\Lambda_n \subset \Lambda$ such that

$$V_n := \bigcap_{\lambda\in\Lambda_n} \ker\varphi_\lambda \subset U_n.$$

Therefore, $\{V_n\}_{n\in\mathbb{N}}$ is a countable neighborhood basis for the identity.

Define $C \subset \mathrm{Ends}(S)$ such that $\lambda \subset C$ if and only if there exists $n \in \mathbb{N}$ with $\lambda \in \Lambda_n$. As C is countable, we can find $\mu \in \Lambda$ such that $\mu \cap C = \varnothing$ and $|\mu| \geq 4$. We claim that $\ker\varphi_\mu$ is a neighborhood of the identity not containing any of the V_n. We proceed by contradiction: Suppose that $V_n \subset \ker\varphi_\mu$. Pick a simple closed curve c in S bounding a disk D in $\overline{S}$ disjoint from each $\lambda \in \Lambda_n$ and satisfying $|D \cap \mu| = 2$. The Dehn twist T_c about c satisfies $T_c \in V_n$ and $T_c \notin \ker\varphi_\mu$. Thus, there is no countable neighborhood basis of the identity. □

Proof of Proposition 7.2 If S is finite-type, then $L(S)$ is equal to $\mathrm{PMap}(S)$ and has the discrete topology, so the result is immediate. Suppose now that $\mathrm{Ends}(S)$ is countably infinite. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \Lambda$ be cofinal, so that

$$L(S) = \varprojlim \mathrm{PMap}(S_{\lambda_n}).$$

We can then define the metric $d\colon L(S) \to \mathbb{R}$: for any two elements $f, g \in L(S)$, set $d(f,g) = 1$ if $\pi_{\lambda_n}(f) \neq \pi_{\lambda_n}(g)$ for every $n \in \mathbb{N}$, otherwise set

$$d(f,g) = \min\{2^{-n} : \pi_{\lambda_n}(f) = \pi_{\lambda_n}(g)\}.$$

It is a routine exercise to check that d is in fact a complete metric and induces the initial topology on $L(S)$. Lemma 7.3 tells us that $L(S)$ is separable; hence, it is Polish. To finish the proof, Lemma 7.4 tells us that $L(S)$ is not metrizable if the cardinality of $\mathrm{Ends}(S)$ is uncountable. □

Corollary 7.5 *If the cardinality of* $\mathrm{Ends}(S)$ *is countably infinite, then* $L(S)$ *is homeomorphic to the Baire space* $\mathbb{N}^{\mathbb{N}}$.

(Another realization of the Baire space is the space of irrational numbers as a subspace of the real line.)

Proof First observe that $L(S)$ is a subspace of a direct product of discrete spaces; hence, it has a basis of clopen sets, so it is zero-dimensional. Given that $L(S)$ is zero-dimensional and Polish, to show that $L(S)$ is homeomorphic to the Baire space it is enough to show that every compact set has empty interior; this utilizes the topological characterization of the Baire space by Alexandrov and Urysohn (see [27, Theorem 7.7]). As in the proof of Proposition 7.2, let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \Lambda$ be a cofinal subset. With this setup, the sets of the form $\pi_n^{-1}(g)$, where $\pi_n = \pi_{\lambda_n}$ and $g \in \mathrm{PMap}(S_{\lambda_n})$, give a basis for the topology of $L(S)$.

Let $C \subset L(S)$ be compact. Assume that C has nonempty interior. Then there exists $n \in \mathbb{N}$ and $g \in \mathrm{PMap}(S_{\lambda_n})$ such that $\pi_n^{-1}(g) \subset C$. As $\pi_n^{-1}(g)$ is closed and C is compact, we must have that $\pi_n^{-1}(g)$ is compact. But, then $\pi_{n+1}(\pi_n^{-1}(g))$ is compact as π_{n+1} is continuous. This is a contradiction as $\pi_{n+1}(\pi_n^{-1}(g))$ has infinite cardinality in a discrete space. □

Even when $\mathrm{Ends}(S)$ is uncountable, $L(S)$, as an inverse limit of discrete spaces, has the structure of a complete uniform space [24, Exercise IV.8]. Hence, in either case, $L(S)$ can be viewed as the completion of $\mathrm{PMap}(S)$.

7.1 Comparing topologies

Recall from Lemma 7.4 that if the cardinality of $\mathrm{Ends}(S)$ is uncountably infinite, then $(\mathrm{PMap}(S), \tau_w)$ is not metrizable, so, in this case, $\tau_w \neq \tau_q$; moreover, we have:

Proposition 7.6 *If S is of finite type, then $\tau_w = \tau_q$; otherwise, there is a strict containment $\tau_w \subset \tau_q$.*

Proof We have already noted that when S is of finite type both τ_q and τ_w are discrete. Let us assume from now on that S is of infinite type. In the above notation, for $\lambda \in \Lambda$, let $\psi_\lambda\colon G \to \mathrm{PMap}(S_\lambda)$ be the homomorphism factoring through G_λ. As ψ_λ is continuous for each $\lambda \in \Lambda$, we have that the projection homomorphism $q\colon G \to (\mathrm{PMap}(S), \tau_w)$ is continuous. By the definition of τ_q, this shows that $\tau_w \subseteq \tau_q$.

By the remark preceding the statement of Proposition 7.6, we may assume that the cardinality of $\mathrm{Ends}(S)$ is countable. To finish the proof, we will find a sequence that converges in τ_w but not in τ_q. Let $K_1 \subset K_2 \subset \cdots$ be an exhaustion of S by essential finite-type surfaces each of whose genus agrees with S. Choose a sequence of homeomorphisms $\widetilde{g}_n \in G$ such that $\widetilde{g}_n$ restricts to the identity on K_n, which implies that $\widetilde{g}_n \to 1$ in G. If $g_n = q(\widetilde{g}_n)$, then $g_n \to 1$ in τ_q, hence in τ_w as well.

For each $\lambda \in \Lambda$, let n_λ be the largest integer n such that λ intersects each component of the complement of K_n in $\overline{S}$. In addition, for each $\lambda \in \Lambda$, choose an isotopy class of an essential simple closed curve in S, denoted by c_λ, such that c_λ is trivial in S_λ and $c_\lambda \cap K_{n_\lambda}$ fills K_{n_λ}. Let T_λ denote the Dehn twist about c_λ and define $h_\lambda \in \mathrm{PMap}(S)$ to be $T_\lambda \circ g_{n_\lambda}$.

The map $\Lambda \to \mathrm{PMap}(S)$ defined by $\lambda \mapsto h_\lambda$ gives a net in $\mathrm{PMap}(S)$. Now, the collection of sets of the form $U_\mu = \ker \varphi_\mu$ for $\mu \in \Lambda$ form a neighborhood subbasis for the identity in τ_w. If $\lambda \supset \mu$, then we claim $h_\lambda \in U_\mu$. First, observe that $K_{n_\mu} \subset K_{n_\lambda}$, so that $\widetilde{g}_{n_\lambda}$ restricted to K_{n_μ} is the identity. It follows that $\varphi_\mu(g_{n_\lambda})$ is trivial. Also,

$$\varphi_\mu(T_\lambda) = \varphi_{\mu,\lambda} \circ \varphi_\lambda(T_\lambda)$$

is the identity as $T_\lambda \in \ker \varphi_\lambda$ by definition. This shows that $h_\lambda \in U_\mu$ for all $\lambda \supset \mu$. In particular, the net $\{h_\lambda\}_{\lambda \in \Lambda}$ converges to the identity in τ_w.

We claim that this net does not converge in τ_q. Let

$$U = \{f \in \mathrm{PMap}(S) : f = q(\widetilde{f}), \text{ where } \widetilde{f}(K_1) = K_1\}.$$

Then U is an open neighborhood of the identity in τ_q. Indeed, let V be an open regular neighborhood of K_1. Then U is the image, under q, of

$$U(K_1, V) = \{\widetilde{f} \in G : \widetilde{f}(K_1) \subset V\},$$

which is a basis element for the compact-open topology on G. Further, as q is both a homomorphism and a quotient map, it is open; hence U is open. Now observe that $h_\lambda \notin U$ for any λ; this follows from the fact that given any simple closed curve $c \subset K_1$, we have $h_\lambda(c) = T_\lambda(c)$, which intersects ∂K_1 nontrivially. □

References

[1] **L V Ahlfors**, **L Sario**, *Riemann surfaces*, Princeton Mathematical Series 26, Princeton Univ. Press (1960) MR

[2] **J Aramayona**, **A Fossas**, **H Parlier**, *Arc and curve graphs for infinite-type surfaces*, Proc. Amer. Math. Soc. 145 (2017) 4995–5006 MR

[3] **J Aramayona**, **L Funar**, *Asymptotic mapping class groups of closed surfaces punctured along Cantor sets*, preprint (2017) arXiv

[4] **J Aramayona**, **J Souto**, *Homomorphisms between mapping class groups*, Geom. Topol. 16 (2012) 2285–2341 MR

[5] **R Arens**, *Topologies for homeomorphism groups*, Amer. J. Math. 68 (1946) 593–610 MR

[6] **J Bavard**, *Hyperbolicité du graphe des rayons et quasi-morphismes sur un gros groupe modulaire*, Geom. Topol. 20 (2016) 491–535 MR

[7] **J Bavard**, **S Dowdall**, **K Rafi**, *Isomorphisms between big mapping class groups*, preprint (2017) arXiv

[8] **F Béguin**, **S Crovisier**, **F Le Roux**, *Construction of curious minimal uniquely ergodic homeomorphisms on manifolds: the Denjoy–Rees technique*, Ann. Sci. École Norm. Sup. 40 (2007) 251–308 MR

[9] **A J Berrick**, **V Gebhardt**, **L Paris**, *Finite index subgroups of mapping class groups*, Proc. Lond. Math. Soc. 108 (2014) 575–599 MR

[10] **G Birkhoff**, *A note on topological groups*, Compositio Math. 3 (1936) 427–430 MR

[11] **J S Birman**, **A Lubotzky**, **J McCarthy**, *Abelian and solvable subgroups of the mapping class groups*, Duke Math. J. 50 (1983) 1107–1120 MR

[12] **D Calegari**, *Circular groups, planar groups, and the Euler class*, from "Proceedings of the Casson Fest" (C Gordon, Y Rieck, editors), Geom. Topol. Monogr. 7, Geom. Topol. Publ., Coventry (2004) 431–491 MR

[13] **D Calegari**, *Mapping class groups: the next generation*, blog post (2014) Available at `https://tinyurl.com/geometry-and-the-imagination`

[14] **J Cantwell**, **L Conlon**, *Endperiodic automorphisms of surfaces and foliations*, preprint (2010) arXiv

[15] **F Castel**, *Geometric representations of the braid groups*, Astérisque 378, Soc. Math. France, Paris (2016) MR

[16] **M G Durham**, **F Fanoni**, **N G Vlamis**, *Graphs of curves on infinite-type surfaces with mapping class group actions*, preprint (2016) arXiv

[17] **B Farb**, **D Margalit**, *A primer on mapping class groups*, Princeton Mathematical Series 49, Princeton Univ. Press (2012) MR

[18] **L Funar**, **C Kapoudjian**, *On a universal mapping class group of genus zero*, Geom. Funct. Anal. 14 (2004) 965–1012 MR

[19] **L Funar**, **C Kapoudjian**, *The braided Ptolemy–Thompson group is finitely presented*, Geom. Topol. 12 (2008) 475–530 MR

[20] **E K Grossman**, *On the residual finiteness of certain mapping class groups*, J. London Math. Soc. 9 (1974/75) 160–164 MR

[21] **J L Harer**, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. 84 (1986) 157–176 MR

[22] **J H Hernández**, **I Morales**, **F Valdez**, *The Alexander method for infinite-type surfaces*, preprint (2017) arXiv

[23] **J H Hernández**, **I Morales**, **F Valdez**, *Isomorphisms between curve graphs of infinite-type surfaces are geometric*, preprint (2017) arXiv

[24] **J R Isbell**, *Uniform spaces*, Mathematical Surveys 12, Amer. Math. Soc., Providence, RI (1964) MR

[25] **N V Ivanov**, *Automorphisms of Teichmüller modular groups*, from "Topology and geometry: Rohlin seminar" (O Y Viro, editor), Lecture Notes in Math. 1346, Springer (1988) 199–270 MR

[26] **S Kakutani**, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. 12 (1936) 82–84 MR

[27] **A S Kechris**, *Classical descriptive set theory*, Graduate Texts in Mathematics 156, Springer (1995) MR

[28] **R Kirby**, *Problems in low dimensional manifold theory*, Proc. Sympos. Pure Math. XXXII, Amer. Math. Soc., Providence, RI (1978) 273–312 MR

[29] **R McKenzie**, *Automorphism groups of denumerable Boolean algebras*, Canad. J. Math. 29 (1977) 466–471 MR

[30] **J D Monk**, *On the automorphism groups of denumerable Boolean algebras*, Math. Ann. 216 (1975) 5–10 MR

[31] **L Paris**, *Small index subgroups of the mapping class group*, J. Group Theory 13 (2010) 613–618 MR

[32] **J Powell**, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. 68 (1978) 347–350 MR

[33] **A Putman**, **B Wieland**, *Abelian quotients of subgroups of the mappings class group and higher Prym representations*, J. Lond. Math. Soc. 88 (2013) 79–96 MR

[34] **I Richards**, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963) 259–269 MR

[35] **C Rosendal**, *Coarse geometry of topological groups*, preprint (2017) Available at `https://tinyurl.com/Rosendal-Coarse-Geometry17`

[36] **B von Kerékjártó**, *Vorlesungen über Topologie, I: Flächentopologie*, Grundl. Math. Wissen. 8, Springer (1923)

Department of Mathematics, University of California, Santa Barbara
Santa Barbara, CA, United States

Department of Mathematics, Queens College, CUNY
Flushing, NY, United States

`patel@math.ucsb.edu, nicholas.vlamis@qc.cuny.edu`

`http://www.patelp.com, http://math.nickvlamis.com`

Received: 1 December 2017 Revised: 23 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 4143–4160

Equivariant cohomology Chern numbers determine equivariant unitary bordism for torus groups

ZHI LÜ
WEI WANG

We show that the integral equivariant cohomology Chern numbers completely determine the equivariant geometric unitary bordism classes of closed unitary G–manifolds, which gives an affirmative answer to a conjecture posed by Guillemin, Ginzburg and Karshon (*Moment maps, cobordisms, and Hamiltonian group actions*, Remark H.5 in Appendix H.3), where G is a torus. As a further application, we also obtain a satisfactory solution of their Question (A) (Appendix H.1.1) on unitary Hamiltonian G–manifolds. Our key ingredients in the proof are the universal toric genus defined by Buchstaber, Panov and Ray and the Kronecker pairing of bordism and cobordism. Our approach heavily exploits Quillen's geometric interpretation of homotopic unitary cobordism theory. Moreover, this method can also be applied to the study of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism and can still derive the classical result of tom Dieck.

55N22, 57R20, 57R85, 57R91

1 Introduction and main results

1.1 Background

In his seminal work [35], R Thom introduced the unoriented bordism theory, which corresponds to the infinite orthogonal group O. Since then, various other bordism theories, which correspond to subgroups $\mathbb{G}$ of the orthogonal group O as structure groups of stable tangent bundles or stable normal bundles of compact smooth manifolds, have been studied and established (eg see Milnor [29], Novikov [30] and Wall [36] and, for more details, see Landweber [24] and Stong [33]). When $\mathbb{G}$ is chosen to be SO (resp. U, SU, etc), the corresponding bordism theory is often called the oriented (resp. unitary, special unitary, etc) bordism theory. One of the main results in these bordism theories is that a closed manifold bounds if and only if certain characteristic numbers vanish. For example, when $\mathbb{G}$ is O or U, the corresponding characteristic numbers will be Stiefel–Whitney numbers or Chern numbers. Another main result is that the bordism ring $\Omega_*^{\mathbb{G}}$ associated to the given subgroup $\mathbb{G}$ can also be described quite

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4143

completely, where $\Omega_*^{\mathbb{G}}$ consists of the bordism classes of all closed smooth manifolds with $\mathbb{G}$–structure (ie whose stable tangent bundles admit $\mathbb{G}$ as structure group). If $\mathbb{G}$ is O (resp. SO, U, etc), then a compact smooth manifold with $\mathbb{G}$–structure is often called an unoriented (resp. oriented, unitary, etc) manifold. Homotopy-theoretic bordism theories $M\mathbb{G}_*(X)$ (which is isomorphic to $\Omega_*^{\mathbb{G}}(X)$ by the Pontryagin–Thom construction) and cobordism theories $M\mathbb{G}^*(X)$ of a topological space X are due to M Atiyah [2], where $M\mathbb{G}$ is the corresponding Thom spectrum of $\mathbb{G}$ and $\Omega_*^{\mathbb{G}}(X)$ is the geometric bordism ring generated by the bordism classes of singular manifolds $f\colon M \to X$, where each M is a smooth closed manifold with $\mathbb{G}$–structure. Note that when X is a point, $\Omega_*^{\mathbb{G}}(X)$ is just $\Omega_*^{\mathbb{G}}$. These two kinds of theories are generalized homology and cohomology theories.

In the early 1960s, Conner and Floyd [12; 13; 14] began the study of geometric equivariant bordism theory by combining the ideas of bordism theory and transformation groups. They studied geometric equivariant unoriented, oriented and unitary bordism theories for smooth closed manifolds with periodic diffeomorphisms, and the subject has also continued to further develop by extending their ideas or combining with various ideas and theories from other research areas since then. For example, tom Dieck [15] introduced and studied the homotopy-theoretic equivariant bordisms $M\mathbb{G}_*^G(X)$ and cobordisms $M\mathbb{G}^*_G(X)$ for a G–space X, where G is a compact Lie group and, recently, Buchstaber, Panov and Ray [9] introduced and studied the universal toric genus in a geometric manner. The geometric equivariant bordism ring $\Omega_*^{\mathbb{G},G}(X)$ for a G–space X can also be defined in a natural way, and it is formed by singular G–manifolds $f\colon M \to X$, where each M is a smooth closed G–manifold with $\mathbb{G}$–structure such that the G–action preserves the $\mathbb{G}$–structure (in this case, we say that M admits a G–equivariant $\mathbb{G}$–structure). When X is a point, $\Omega_*^{\mathbb{G},G}(X)$ will be denoted by $\Omega_*^{\mathbb{G},G}$, and it is an equivariant analogue of $\Omega_*^{\mathbb{G}}$. It should be pointed out that in the above definition, the G–equivariant $\mathbb{G}$–structure of a smooth closed G–manifold with $\mathbb{G}$–structure is equipped on the stably G–equivariant tangent bundle of M rather than the stably G–equivariant normal bundle of the embedding of M in some G–representation because there is a substantial difference between both notions (see Hanke [19]). In this paper we will pay more attention to the case in which $\mathbb{G} = U$ (ie G–unitary bordism). A nice survey of the development of G–unitary bordism is contained in the introduction of Buchstaber, Panov and Ray [9].

Comparing with the nonequivariant case, a natural question arises and is stated as follows:

Question 1.1 *What kinds of equivariant characteristic numbers determine the equivariant geometric bordism class of a smooth closed G–manifold with $\mathbb{G}$–structure?*

As far as the authors know, the above question is far from solved. Most known works with respect to the above question have considered mainly the cases $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $(U, T^k \times \mathbb{Z}_m)$, (SO, finite group) (eg see Bix and tom Dieck [6], Guillemin, Ginzburg and Karshon [18], Hattori [20], Khare [21], Lü and Tan [27], Lee and Wasserman [25], Sinha [32], Stong [34] and tom Dieck [16; 17]). In the following, we mainly give an investigation in the cases $(\mathbb{G}, G) = (U, T^k)$ and $(O, \mathbb{Z}_2^k)$.

Tom Dieck first investigated the above question. In his series of papers [15; 16; 17], by combining the geometric approach of Conner and Floyd [13; 14] and the K–theory approach developed by Atiyah, Bott, Segal and Singer [3; 4; 5], tom Dieck introduced the bundling transformation which further develops ideas of Boardman [7] and Conner [11], and proved a series of integrality theorems in equivariant (co)bordism theory. Furthermore, he gave an answer to the above question in the cases $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $(U, T^k \times \mathbb{Z}_m)$, which is stated as follows:

Theorem 1.2 (tom Dieck) *When $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$, $\beta \in \Omega_*^{O,(\mathbb{Z}_2)^k}$ is zero if and only if all equivariant Stiefel–Whitney numbers of β vanish. When $(\mathbb{G}, G) = (U, T^k \times \mathbb{Z}_m)$, $\beta \in \Omega_*^{U,G}$ is zero if and only if all equivariant K–theoretic Chern numbers of β vanish.*

In their book [18, Appendix H], Guillemin, Ginzburg and Karshon discussed the problem of calculating the ring $\mathcal{H}_*^G$ of equivariant Hamiltonian bordism classes of all unitary Hamiltonian G–manifolds with integral equivariant cohomology classes $\frac{1}{2\pi}[\omega - \Phi]$, where G is a torus. With respect to the determination of the ring $\mathcal{H}_*^G$, they posed three series of questions, the first one of which is stated as follows:

Question 1.3 [18, Appendix H.1.1, Question (A)] *Do mixed equivariant characteristic numbers form a full system of invariants of Hamiltonian bordism?*

On this question, Guillemin, Ginzburg and Karshon constructed a monomorphism

$$\Sigma_G \colon \mathcal{H}_*^G \to \Omega_{*+2}^{U,G},$$

so that Question 1.3 is equivalent to asking if the integral equivariant cohomology Chern numbers determine the equivariant geometric unitary bordism classes for the

ring $\Omega_*^{U,G}$. They showed that a closed unitary G–manifold M with only isolated fixed points represents the zero element in $\Omega_*^{U,G}$ if and only if all integral equivariant cohomology Chern numbers of M vanish, which gives a partial solution of Question 1.3. Furthermore, they posed the following conjecture without the restriction of isolated fixed points.

Conjecture 1.4 [18, Appendix H.3, Remark H.5] *Let G be a torus. Then $\beta \in \Omega_*^{U,G}$ is zero if and only if all integral equivariant cohomology Chern numbers of β vanish.*

For this detecting problem, it is natural to use the localization theorem to analyze the topological data of fixed-point sets, which has been used successfully in the case when the fixed point sets are isolated; see Guillemin, Ginzburg and Karshon [18]. For the general case, we found that the analysis of fixed data seems to be not easy to handle and we turned to consider the geometric presentation of the image $\Phi([M]) \in MU_*(BG)$ of *the universal toric genus* instead, which admits a natural geometric representation described by Buchstaber, Panov and Ray [9]. In particular, by using the Kronecker pairing, our argument can directly be reduced to a simpler calculation.

1.2 Main results

The motivation of this paper is mainly supplied by the work of Guillemin, Ginzburg and Karshon as mentioned above. The geometric description of the universal toric genus developed by Buchstaber, Panov and Ray [9] provides us much more insight, so that we can carry out our work by utilizing the Kronecker pairing of bordism and cobordism. This leads us to give an affirmative answer to Conjecture 1.4, which is stated as follows.

Theorem 1.5 *Conjecture 1.4 holds.*

As a further consequence, we obtain a satisfactory solution of Question 1.3.

Corollary 1.6 *Mixed equivariant characteristic numbers determine equivariant Hamiltonian bordism.*

It is interesting that our approach above can also be applied to the detecting problem of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism, and, in particular, it can still derive the classical result of tom Dieck in the case $(\mathbb{G}, G) = (O, (\mathbb{Z}_2)^k)$ of Theorem 1.2 by replacing the Boardman map by Kronecker pairing in the original proof of tom Dieck [16].

This paper is organized as follows. In Section 2, we shall review Quillen's geometric interpretation of homotopic unitary cobordism theory first, which is of essential importance, so that the Kronecker pairing between bordism and cobordism can be calculated in a geometric way. We then review *the universal toric genus* defined by Buchstaber, Panov and Ray, which can be expressed in terms of Quillen's geometric interpretation. We also recollect some necessary facts about equivariant Chern classes and equivariant Chern numbers. Then we shall give the proof of Theorem 1.5 in Section 3, which is more geometric. In particular, our approach in the unitary case can also be carried out in the study of $(\mathbb{Z}_2)^k$–equivariant unoriented bordism, as we shall see in the final part of Section 3.

Acknowledgements

The authors would like to thank Peter Landweber for his comments and suggestions on an earlier version of this paper. We are also grateful to the referee for carefully reading this manuscript and providing a number of valuable thoughts and helpful suggestions, which led to this version.

Lü was supported by grants from NSFC (No. 11371093, No. 11431009 and No. 11661131004). Wang was supported by a grant from NSFC (No. 1130 1335).

2 Preliminaries

2.1 Geometric interpretation of elements in $MU^*(X)$ and Kronecker pairing

Given a topological space X, let $MU_*(X)$ and $MU^*(X)$ be the complex (homotopic) bordism and cobordism of X, which are defined as

$$MU_*(X) = \lim_{l\to\infty}[S^{2l+*}, X_+ \wedge MU(l)]$$

and

$$MU^*(X) = \lim_{l\to\infty}[S^{2l-*} \wedge X_+, MU(l)],$$

respectively, where X_+ denotes the union of X and a disjoint point, and $MU(l)$ denotes the Thom space of the universal complex l–dimensional vector bundle over $BU(l)$.

Geometrically, it is very well known that elements of $MU_n(X)$ are regarded as bordism classes of maps $M \to X$ of stably complex closed n–dimensional manifolds M to X since $MU_n(X) \cong \Omega^U_n(X)$.

On the geometric interpretation of elements in $MU^*(X)$, Quillen [31, Proposition 1.2] showed that for a manifold X, $MU^{\pm n}(X)$ is isomorphic to the group formed by cobordism classes of proper *complex-oriented maps* $f\colon M \to X$ of dimension $\mp n$, where $\dim X - \dim M = \pm n$. Recall that when n is even, a *complex-oriented map* $f\colon M \to X$ of dimension $\mp n$ is a composition of maps of manifolds

$$M \xrightarrow{i} \mathbb{E} \xrightarrow{\pi} X, \tag{1}$$

where $\pi\colon \mathbb{E} \to X$ is a complex vector bundle over X and $i\colon M \to \mathbb{E}$ is an embedding endowed with, a complex structure on its normal bundle. When n is odd, the complex orientation of f will be defined in a same way as above with $\mathbb{E}$ replaced by $\mathbb{E} \times \mathbb{R}$ in (1).

It is very well known that as a generalized cohomology and homology theory, $MU^*(X)$ and $MU_*(X)$ admit a canonical Kronecker pairing

$$\langle\,,\rangle\colon MU^n(X) \otimes MU_m(X) \to MU_{m-n},$$

where $MU_{m-n} = MU_{m-n}(\mathrm{pt}) \cong \Omega^U_{m-n}$.

This Kronecker pairing can be calculated in a geometric way as follows. For example, if $\alpha \in MU^{-n}(X)$ is represented by a smooth fiber bundle of closed smooth stably complex manifolds $E \to X$ with $\dim E - \dim X = n$ and $\beta \in MU_m(X)$ can be represented by a smooth map $f\colon M \to X$, then the Kronecker pairing $\langle \alpha, \beta\rangle \in \Omega^U_{m-n}$ is the bordism class of the pullback $\tilde{f}^*(E) = E \times_X M$, as shown in the diagram

$$\begin{array}{ccc} E \times_X M & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

(see [8, Construction D.3.4]).

2.2 The universal toric genus

Based upon the works of tom Dieck, Krichever and Löffler (see [15; 17; 23; 26]), Buchstaber, Panov and Ray [9] defined the *universal toric genus* Φ in a geometric manner, which is a ring homomorphism from the geometric unitary T^k–bordism ring to the complex cobordism ring of the classifying space BT^k

$$\Phi\colon \Omega_*^{U,T^k} \to MU^*(BT^k).$$

Note that, as an Ω_*^U–algebra, $MU^*(BT^k)$ is isomorphic to $\Omega_*^U[\![u_1,\dots,u_k]\!]$, where u_i is the cobordism Chern class $c_1^{MU}(\xi_i)$ of the conjugate Hopf bundle ξ_i over the i^{th} factor of BT^k for $i=1,\dots,k$, so $MU^*(BT^k)$ can be replaced by $\Omega_*^U[\![u_1,\dots,u_k]\!]$.

Let $[M]_{T^k}\in\Omega_n^{U,T^k}$ be an element represented by a closed unitary T^k–manifold M. Buchstaber, Panov and Ray showed that $\Phi([M]_{T^k})$ can be defined to be the cobordism class of the complex oriented map $\pi\colon ET^k\times_{T^k}M\to BT^k$. More precisely, choose a T^k–equivariant embedding $i\colon M\hookrightarrow V$ into a unitary T^k–representation space V; then the Borelification of i gives a complex-oriented map

$$\pi_l\colon ET^k(l)\times_{T^k}M\hookrightarrow ET^k(l)\times_{T^k}V\to BT^k(l),$$

which determines a cobordism class α_l in $MU^{-n}(BT^k(l))$, where $BT^k(l)=(\mathbb{C}P^l)^k$ and $ET^k(l)=(S^{2l+1})^k$. Since $BT^k=\bigcup_l BT^k(l)$ and $ET^k=\bigcup_l ET^k(l)$, these cobordism classes α_l form an inverse system. This produces a class $\alpha=\varprojlim\alpha_l$ in $MU^{-n}(BT^k)$, which is represented geometrically by the complex-oriented map

$$\pi\colon ET^k\times_{T^k}M\hookrightarrow ET^k\times_{T^k}V\to BT^k.$$

Then $\Phi([M]_{T^k})$ is defined as this limit α.

The following result is essentially due to Löffler [26] and Comezaña [10], as noted by Hanke [19].

Proposition 2.1 *The ring homomorphism $\Phi\colon\Omega_*^{U,T^k}\to MU^*(BT^k)$ is injective.*

Corollary 2.2 *If M is a closed unitary T^k–manifold, then M is null-bordant in Ω_*^{U,T^k} if and only if the complex-oriented map $\pi\colon ET^k\times_{T^k}M\to BT^k$ represents the zero element in $MU^*(BT^k)$.*

2.3 Equivariant Chern classes and equivariant Chern numbers

Let M be a closed unitary T^k–manifold. Then applying the Borel construction to the stable tangent bundle TM of M gives a vector bundle $ET^k\times_{T^k}TM$ over $ET^k\times_{T^k}M$.

Definition 2.3 The *total equivariant cohomology Chern class* of M is defined to be the total Chern class of the vector bundle $ET^k\times_{T^k}TM$ over $ET^k\times_{T^k}M$, ie

$$\begin{aligned}c^{T^k}(M)&:=c(ET^k\times_{T^k}TM)\\&=1+c_1(ET^k\times_{T^k}TM)+c_2(ET^k\times_{T^k}TM)+\cdots\\&=1+c_1^{T^k}(M)+c_2^{T^k}(M)+\cdots\in H^*(ET^k\times_{T^k}M)=H^*_{T^k}(M).\end{aligned}$$

Let $p\colon M \to \mathrm{pt}$ be the constant map, and let $p_!^{T^k}\colon H^*(ET^k \times_{T^k} M) \to H^*(BT^k)$ be *the equivariant Gysin map* induced by p. Then, the *integral equivariant cohomology Chern numbers* of M are defined to be

$$c_\omega^{T^k}[M]_{T^k} := p_!^{T^k}(c_\omega^{T^k}(M)),$$

each of which is a homogeneous polynomial of degree $2|\omega| - \dim M$ in the polynomial ring $H^*(BT^k) = \mathbb{Z}[x_1, \dots, x_k]$ with $\deg x_i = 2$, where $\omega = (i_1, i_2, \dots, i_s)$ is a partition of weight $|\omega| = i_1 + i_2 + \cdots + i_s$ and $c_\omega^{T^k}(M)$ denotes the product $c_{i_1}^{T^k}(M)c_{i_2}^{T^k}(M)\cdots c_{i_s}^{T^k}(M)$.

In particular, by definition, when $\dim M$ *is odd*, one has $c_\omega^{T^k}[M]_{T^k} = 0$ for all ω. Hence, in what follows, it is safe to assume $\dim M = 2n$ most of the time.

Note that the map $\pi\colon ET^k \times_{T^k} M \to BT^k$ is just the Borelification of $p\colon M \to \mathrm{pt}$, so $p_!^{T^k}$ is often replaced by the Gysin map $\pi_!$ in this paper.

3 Proof of the main theorem and equivariant unoriented bordism

3.1 Determining equivariant cohomology Chern numbers by ordinary Chern numbers

Let M be a $2n$–dimensional closed unitary T^k–manifold; we consider the equivariant cohomology Chern numbers $c_\omega^{T^k}[M]_{T^k} \in H^*(BT^k;\mathbb{Z})$ for the partition $\omega = (i_1, i_2, \dots, i_s)$ of weight $|\omega| = i_1 + i_2 + \cdots + i_s$. Since $H^*(BT^k;\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \dots, x_k]$ with $\deg x_i = 2$, the equivariant Chern number $c_\omega^{T^k}[M]_{T^k}$ admits the form

$$(*) \qquad c_\omega^{T^k}[M]_{T^k} = \sum_J n_J^\omega x^J,$$

where the k–tuples of natural numbers $J = (j_1, j_2, \dots, j_k)$ range over all k–partitions of weight $|\omega| - n$ and x^J means $x_1^{j_1}x_2^{j_2}\cdots x_k^{j_k}$, with the coefficients $n_J^\omega \in \mathbb{Z}$. It follows that to determine the equivariant cohomology Chern numbers $c_\omega^{T^k}[M]_{T^k}$ it is equivalent to determine the coefficient n_J^ω for each $J = (j_1, j_2, \dots, j_k)$.

The coefficient n_J^ω is closely related to the ordinary Chern numbers of certain manifolds. Indeed, for $J = (j_1, j_2, \dots, j_k)$, let $\left(\prod_{i=1}^k S^{2j_i+1}\right) \times_{T^k} M$ be the pullback of the

natural inclusion $\prod_{i=1}^{k}\mathbb{C}P^{j_i}\hookrightarrow BT^k$:

$$\begin{array}{ccc}\left(\prod_{i=1}^{k}S^{2j_i+1}\right)\times_{T^k}M & \xrightarrow{\tilde{f}_J} & ET^k\times_{T^k}M\\ \downarrow{\scriptstyle \pi_J} & & \downarrow{\scriptstyle \pi}\\ \prod_{i=1}^{k}\mathbb{C}P^{j_i} & \xrightarrow{\quad f_J\quad} & BT^k\end{array}$$

First, we find that:

Lemma 3.1 *For fixed ω and J,*

$$n_J^{\omega}x^J=(\pi_J)_!\left(c_{\omega}\left(\left(\prod_{i=1}^{k}S^{2j_i+1}\right)\times_{T^k}TM\right)\right),$$

where $(\pi_J)_!$ is the Gysin map induced by π_J.

Proof By the definition of equivariant Chern numbers,

$$c_{\omega}^{T^k}[M]_{T^k}=\pi_!(c_{\omega}(ET^k\times_{T^k}TM))=\sum_J n_J^{\omega}x^J\in H^*(BT^k;\mathbb{Z}).$$

Since $\left(\prod_{i=1}^{k}S^{2j_i+1}\right)\times_{T^k}TM=\tilde{f}_J^*(ET^k\times_{T^k}TM)$ and Gysin maps are commutative in the pullback square, ie $f_J^*\pi_!=(\pi_J)_!\tilde{f}_J^*$, we have

$$\begin{aligned}n_J^{\omega}x^J&=f_J^*\pi_!(c_{\omega}(ET^k\times_{T^k}TM))\\&=(\pi_J)_!\tilde{f}_J^*c_{\omega}(ET^k\times_{T^k}TM)\\&=(\pi_J)_!\left(c_{\omega}\left(\left(\prod_{i=1}^{k}S^{2j_i+1}\right)\times_{T^k}TM\right)\right).\end{aligned}$$

□

Let $\omega=(i_1,i_2,\dots,i_s)$ and $\omega'=(i_1',i_2',\dots,i_s')$ be two partitions; we say $\omega'\leqslant\omega$ provided $i_j'\leqslant i_j$ for each j and we say $\omega'<\omega$ if $\omega'\leqslant\omega$ and $\omega'\neq\omega$. We denote by $\omega\pm\omega'$ the partition $(i_1\pm i_1',i_2\pm i_2',\dots,i_s\pm i_s')$. Let $J=(i_1,\dots,i_k)$ and $J'=(i_1',\dots,i_k')$ be two k–partitions; we also define $J'\leqslant J$, $J'<J$ and $J\pm J'$ in the same way.

For each partition ω and each k–partition J, the corresponding coefficient n_J^{ω} is determined by the ordinary Chern number $c_{\omega}\left[\left(\prod_{i=1}^{k}S^{2j_i+1}\right)\times_{T^k}M\right]$ and the coefficients $n_{J'}^{\omega'}$ with $J'<J$ and $\omega'<\omega$. More precisely, we have:

Proposition 3.2 *When* $\dim M = 2n > 0$, *one has*

$$c_\omega\left[\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M\right] = n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\,\omega''<\omega,\,J'<J}} n_{J'}^{\omega'} m_{(J-J')}^{\omega''},$$

where $m_{J-J'}$ *is the coefficient of* $x^{J-J'}$ *in the Chern class*

$$c_{\omega''}\Big(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\Big) \in H^*\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i};\mathbb{Z}\right) = \mathbb{Z}[x_1,x_2,\dots,x_k]/(x_1^{j_1+1},\dots,x_k^{j_k+1}).$$

Proof The stable tangent bundle of $\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M$ admits a decomposition

$$T\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M\right) = \left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} TM\right) \oplus \pi_J^* T\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\right).$$

By the Cartan formula, we have

$$\begin{aligned} c_\omega&\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M\right)\\ &= c_\omega\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} TM\right) + \pi_J^* c_\omega\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\right)\\ &\quad + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega}} c_{\omega'}\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} TM\right)\pi_J^* c_{\omega''}\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\right). \end{aligned}$$

Applying $(\pi_J)_!$ to both sides, we obtain

$$\begin{aligned} &c_\omega\left[\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M\right]x^J\\ &= (\pi_J)_! c_\omega\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} M\right)\\ &= (\pi_J)_!\left(c_\omega\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} TM\right)\right) + (\pi_J)_!\left(\pi_J^* c_\omega\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\right)\right)\\ &\quad + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega}} (\pi_J)_!\left(c_{\omega'}\left(\left(\prod_{i=1}^{k} S^{2j_i+1}\right)\times_{T^k} TM\right)\pi_J^* c_{\omega''}\left(\prod_{i=1}^{k}\mathbb{C}P^{j_i}\right)\right) \end{aligned}$$

$$= n_J^\omega x^J + (\pi_J)_!(1)c_\omega\Big(\prod_{i=1}^k \mathbb{C}P^{j_i}\Big) + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega,J'<J}} (\pi_J)_!\Big(c_{\omega'}\Big(\Big(\prod_{i=1}^k S^{2j_i+1}\Big)\times_{T^k} TM\Big)\Big)c_{\omega''}\Big(\prod_{i=1}^k \mathbb{C}P^{j_i}\Big),$$

where the first equality uses that $(\pi_J)_!$ is isomorphic on the top cohomology and the last uses that $(\pi_J)_!(1) = 0$ when $\dim M > 0$.

Note that $f_J^*\colon H^*(BT^k;\mathbb{Z}) \to H^*\big(\prod_{i=1}^k \mathbb{C}P^{j_i};\mathbb{Z}\big)$ is an isomorphism up to dimension $2|J|$, so we have

$$\begin{aligned}(\pi_J)_!\Big(c_{\omega'}\Big(\Big(\prod_{i=1}^k S^{2j_i+1}\Big)\times_{T^k} TM\Big)\Big) &= f_J^*(\pi_! c_{\omega'}(ET^k\times_{T^k} TM))\\ &= f_J^*\Big(\Big(\sum n_{J'}^{\omega'}\Big)x^{J'}\Big) \qquad \text{(by } (*)\text{)}\\ &= \sum n_{J'}^{\omega'}x^{J'},\end{aligned}$$

because $f_J^*(x^{J'}) = x^{J'}$ when $J' \leqslant J$. Write $c_{\omega''}\big(\prod_{i=1}^k \mathbb{C}P^{j_i}\big) = \sum_{J''} m_{J''}^{\omega''}x^{J''}$ and we have

$$\begin{aligned}c_\omega\Big[\Big(\prod_{i=1}^k S^{2j_i+1}\Big)\times_{T^k} M\Big]x^J &= n_J^\omega x^J + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega,J'<J}} \Big(\sum n_{J'}^{\omega'}x^{J'}\Big)\Big(\sum_{J''} m_{J''}^{\omega''}x^{J''}\Big)\\ &= \Big(n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega,J'<J}} n_J^{w'}m_{J-J'}^{\omega''}\Big)x^J.\end{aligned}$$

Therefore, by comparing the coefficients of both sides, one has the equation

$$c_\omega\Big[\Big(\prod_{i=1}^k S^{2j_i+1}\Big)\times_{T^k} M\Big] = n_J^\omega + \sum_{\substack{\omega'+\omega''=\omega\\ \omega'<\omega,\omega''<\omega,J'<J}} n_{J'}^{\omega'}m_{(J-J')}^{\omega''}. \qquad \square$$

Remark When the dimension of M is odd, all Chern numbers and equivariant Chern numbers are zero for dimensional reasons.

Clearly, one has:

Corollary 3.3 *$c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]\in(n^{\omega'}_{J'})\subseteq\mathbb{Z}$ with $\omega'\leqslant\omega$ and $J'=(j'_1,\dots,j'_k)\leqslant J$, where $(n^{\omega'}_{J'})$ is the ideal in $\mathbb{Z}$ generated by the integers $n^{\omega'}_{J'}$.*

On the other hand, when $\dim M=2n>0$, the coefficient n^ω_J of the equivariant Chern number $c^{T^k}_\omega[M]_{T^k}$ is determined by the ordinary Chern numbers

$$c_{\omega'}\bigg[\bigg(\prod_{i=1}^k S^{2j'_i+1}\bigg)\times_{T^k} M\bigg]$$

of the manifolds $\big(\prod_{i=1}^k S^{2j'_i+1}\big)\times_{T^k} M$ with partition $\omega'\leqslant\omega$ and $J'\leqslant J$. Indeed, we have:

Proposition 3.4 *$n^\omega_J\in\big(c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]\big)\subset\mathbb{Z}$ with $\widetilde\omega\leqslant\omega$ and $\widetilde J\leqslant J$, where the ideal $\big(c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]\big)$ in $\mathbb{Z}$ is generated by the integers $c_{\widetilde\omega}\big[\big(\prod_{i=1}^k S^{2\tilde j_i+1}\big)\times_{T^k} M\big]$.*

Proof Note that when the weight satisfies $|\omega|<n$, we have $n^\omega_J=0$ for all J. When $|\omega|=n$, the corresponding k–partition is $J_0=(0,\dots,0)$ and we have $n^\omega_{J_0}=c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]$.

When $|\omega|>n$, by Proposition 3.2 we see

$$n^\omega_J-c_\omega\bigg[\bigg(\prod_{i=1}^k S^{2j_i+1}\bigg)\times_{T^k} M\bigg]\in(n^{\omega'}_{J'})\subseteq\mathbb{Z}$$

with $\omega'<\omega$ and $J'<J$. When $|\omega'|<|\omega|$, by induction on the weight $|\omega|$ we obtain that

$$(n^{\omega'}_{J'})\subset\bigg(c_{\omega''}\bigg[\bigg(\prod_{i=1}^k S^{2j''_i+1}\bigg)\times_{T^k} M\bigg]\bigg)\subseteq\mathbb{Z},$$

with $\omega''\leqslant\omega'$ and $J''\leqslant J'$. Hence, we have

$$n^\omega_J\in c_\omega\bigg[\bigg(\prod_{i=1}^k S^{2j_i+1}\bigg)\times_{T^k} M\bigg]+(n^{\omega'}_{J'})\subset(c_{\widetilde\omega}\bigg[\bigg(\prod_{i=1}^k S^{2\tilde j_i+1}\bigg)\times_{T^k} M\bigg])\subseteq\mathbb{Z}$$

with $\widetilde\omega\leqslant\omega$ and $\widetilde J\leqslant J$. □

Theorem 3.5 *The equivariant Chern number $c^{T^k}_\omega[M]_{T^k}$ equals 0 for all partitions ω if and only if the ordinary Chern number $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]$ equals 0 for all ω and all $J=(j_1,j_2,\dots,j_k)$.*

Proof When $\dim M = 0$, M is the disjoint union of some isolated points with trivial torus action, and the statement certainly holds. When $\dim M$ is odd, clearly all $c_\omega^{T^k}[M]_{T^k}$ are 0 and all $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big]$ are 0 for dimensional reasons.

Now we assume $\dim M = 2n > 0$, and suppose $c_\omega^{T^k}[M]_{T^k} = 0$ for all ω; then $n_{\widetilde{J}}^{\widetilde{\omega}} = 0$ for all $\widetilde{\omega}$ and $\widetilde{J}$. By Proposition 3.4, $c_\omega\big[\big(\prod_{i=1}^k S^{2j_i+1}\big)\times_{T^k} M\big] \in (n_{\widetilde{J}}^{\widetilde{\omega}}) = (0) \subseteq \mathbb{Z}$. The inverse direction is similar. □

3.2 Proof of Theorem 1.5

Let M be a closed unitary T^k–manifold. Since the universal toric genus

$$\Phi\colon \Omega_*^{U,T^k} \to MU^*(BT^k)$$

is injective, it is sufficient to show that the image $\Phi([M]_{T^k}) = 0$ if and only if all integral equivariant cohomology Chern numbers of M vanish.

Geometrically, the image $\Phi([M]_{T^k}) \in MU^*(BT^k)$ is represented by the map

$$\pi\colon ET^k \times_{T^k} M \to BT^k.$$

The Kronecker pairing

$$MU^*(BT^k) \otimes MU_*(BT^k) \to MU_*$$

defines an isomorphism

$$MU^*(BT^k) \cong \mathrm{Hom}_{MU_*}(MU(BT^k), MU_*),$$

because $MU^*(BT^k)$ is a free MU_*–module in even-dimensional generators.

Therefore, the image $\Phi([M]_{T^k})$ is zero if and only if its evaluation with all the MU_* generators of $MU_*(BT^k)$ is zero.

According to [1; 22], the natural inclusion maps $\mathbb{C}P^n \hookrightarrow BT^1$ are the MU_*–generators of $MU_*(BT^1)$, so one can take the natural inclusion maps $\prod_{i=1}^k \mathbb{C}P^{j_i} \hookrightarrow BT^k$ as MU_*–generators of $MU_*(BT^k)$.

By Quillen's construction, the Kronecker pairing of

$$\Phi([M]_{T^k}) \quad \text{and} \quad \left[\prod_{i=1}^k \mathbb{C}P^{j_i} \to BT^k\right]$$

defines the bordism class of the manifold

$$\left\langle \Phi([M]_{T^k}), \left[\prod_{i=1}^{k} \mathbb{C}P^{j_i} \to BT^k \right] \right\rangle = \left[\left(\prod_{i=1}^{k} S^{2j_i+1} \right) \times_{T^k} M \right] \in MU_*.$$

Therefore, $\Phi([M]_{T^k})$ is zero if and only if $\left[\left(\prod_{i=1}^{k} S^{2j_i-1} \right) \times_{T^k} M \right] = 0 \in MU_*$ for all the inclusions $\prod_{i=1}^{k} \mathbb{C}P^{j_i} \hookrightarrow BT^k$ for all $J = (j_1, j_2, \ldots, j_k)$.

On the other hand, by Theorem 3.5, we see the equivariant Chern numbers $c_\omega^{T^k}[M]_{T^k}$ are 0 if and only if the ordinary Chern numbers $c_\omega\left[\left(\prod_{i=1}^{k} S^{2j_i-1} \right) \times_{T^k} M \right]$ are 0 for all ω and all $J = (j_1, j_2, \ldots, j_k)$.

Thus, we conclude that the equivariant Chern number $c_\omega^{T^k}[M]$ is 0 for all ω if and only if $\Phi([M]) = 0$. Hence, Theorem 1.5 has been proved.

Together with the above arguments, we conclude:

Theorem 3.6 *Let M be a closed unitary T^k–manifold. Then the following statements are equivalent:*

(1) *M is null-bordant in Ω_*^{U,T^k}.*

(2) *All integral equivariant cohomology Chern numbers of M vanish.*

(3) *For any partition $J = (j_1, \ldots, j_k)$ with each $j_i \geq 0$, $\left(\prod_{i=1}^{k} S^{2j_i+1} \right) \times_{T^k} M$ is null-bordant in Ω_*^U.*

(4) *The fibration $\pi\colon ET^k \times_{T^k} M \to BT^k$ is null-cobordant in $MU^*(BT^k)$.*

3.3 Equivariant unoriented bordism

Our approach above can also be carried out in the case of equivariant unoriented bordism. Let $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$ (or $\Omega_*^{O,(\mathbb{Z}_2)^k}$) be the ring formed by the equivariant bordism classes of all unoriented closed smooth $(\mathbb{Z}_2)^k$–manifolds, and let $MO^*(X)$ be the unoriented (homotopic) cobordism ring of a topological space X, ie

$$MO^*(X) = \lim_{l \to \infty} [S^{l-*} \wedge X_+, MO(l)].$$

Then Quillen's geometric approach on $MU^*(X)$ can be carried out in the case of $MO^*(X)$, so that the Kronecker pairing

$$\langle\ ,\ \rangle\colon MO^*(X) \otimes MO_*(X) \to MO_* \cong \mathfrak{N}_*$$

can be calculated in a geometric way similar to the unitary case as in Section 2.1 (see [8, Constructions D.2.8 and D.3.4]), where $\mathfrak{N}_*$ is the nonequivariant Thom unoriented bordism ring. In [16], tom Dieck showed that there is also a monomorphism

$$\Phi_{\mathbb{R}}\colon \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \to MO^*(B(\mathbb{Z}_2)^k).$$

It is well known that $MO^*(B(\mathbb{Z}_2)^k) = \varprojlim MO^*(B(\mathbb{Z}_2)^k(n))$, where $B(\mathbb{Z}_2)^k(n) = (\mathbb{R}P^n)^k$ and $B(\mathbb{Z}_2)^k = \bigcup_n B(\mathbb{Z}_2)^k(n)$. Then, applying the finite approximation method yields that for a class $[M]_{(\mathbb{Z}_2)^k} \in \mathfrak{N}_*^{(\mathbb{Z}_2)^k}$, the image $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k})$ is geometrically represented by the fibration $\pi\colon E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \to B(\mathbb{Z}_2)^k$.

Theorem 3.7 *Let M be a closed smooth $(\mathbb{Z}_2)^k$–manifold. Then the following statements are equivalent:*

(1) *M is null-bordant in $\mathfrak{N}_*^{(\mathbb{Z}_2)^k}$.*

(2) *All equivariant Stiefel–Whitney numbers of M vanish.*

(3) *For any partition $I = (j_1, \dots, j_k)$ with each $j_i \geq 0$, $\left(\prod_{i=1}^k S^{j_i+1}\right) \times_{(\mathbb{Z}_2)^k} M$ is null-bordant in $\mathfrak{N}_*$.*

(4) *The fibration*

$$\pi\colon E(\mathbb{Z}_2)^k \times_{(\mathbb{Z}_2)^k} M \to B(\mathbb{Z}_2)^k$$

is null-cobordant in $MO^*(B(\mathbb{Z}_2)^k)$.

Proof The proof follows closely that of Theorem 3.6 in a very similar way. We leave it as an exercise to the reader. □

Remark Theorem 3.7 tells us that $(\mathbb{Z}_2)^k$–equivariant unoriented bordism is determined by the $(\mathbb{Z}_2)^k$–equivariant Stiefel–Whitney numbers. This result is essentially due to tom Dieck with an argument involving the Boardman map

$$B\colon MO^*(B(\mathbb{Z}_2)^k) \to H^*(B(\mathbb{Z}_2)^k; \mathbb{Z}_2) \,\widehat{\otimes}\, H_*(BO; \mathbb{Z}_2)$$

(see [16]). Here we exactly replace the use of the Boardman map by the Kronecker pairing.

It is well known that as an $\mathfrak{N}_*$–algebra, $MO^*(B(\mathbb{Z}_2)^k)$ is isomorphic to the ring $\mathfrak{N}_*[\![v_1, \dots, v_k]\!]$, where v_i denotes the first cobordism Stiefel–Whitney class of the canonical line bundle over the i^{th} factor of $B(\mathbb{Z}_2)^k$ for $i = 1, \dots, k$. Thus, in the

sense of Buchstaber, Panov and Ray [9], we can also define the $(\mathbb{Z}_2)^k$–*equivariant universal genus*, which is an $\mathfrak{N}_*$–algebra homomorphism,

$$\Phi_{\mathbb{R}}\colon \mathfrak{N}_*^{(\mathbb{Z}_2)^k} \to \mathfrak{N}_*[\![v_1,\dots,v_k]\!],$$

by $\Phi_{\mathbb{R}}([M]_{(\mathbb{Z}_2)^k}) = \sum_{\omega=(j_1,\dots,j_k)} g_\omega^{\mathbb{R}}(M)v^\omega$, where $v^\omega = v_1^{j_1}\cdots v_k^{j_k}$ for $\omega = (j_1,\dots,j_k)$ with $j_i \geq 0$, and $g_\omega^{\mathbb{R}}(M) \in \mathfrak{N}_*$ with $\dim g_\omega^{\mathbb{R}}(M) = |\omega| + \dim M$.

The determination of the coefficients $g_\omega^{\mathbb{R}}(M)$ depends upon the choice of a dual basis in $MO_*(B(\mathbb{Z}_2)^k)$ to the basis $\{v^\omega\}$ in $MO^*(B(\mathbb{Z}_2)^k)$. Buchstaber, Panov and Ray's approach in the unitary case provides us much more insight on the determination of coefficients $g_\omega^{\mathbb{R}}(M)$. In our case, since it was shown in [28, Corollary 6.4] that any real Bott manifold is null-bordant in $\mathfrak{N}_*$, we may employ the real Bott manifolds to realize a dual basis in $MO_*(B(\mathbb{Z}_2)^k)$ of the basis $\{v^\omega\}$ in $MO^*(B(\mathbb{Z}_2)^k)$, so that the coefficients $g_\omega^{\mathbb{R}}(M)$ can be represented by manifolds $G_\omega^{\mathbb{R}}(M) = (S^1)^\omega \times_{(\mathbb{Z}_2)^\omega} M$, where the action of $(\mathbb{Z}_2)^\omega = (\mathbb{Z}_2)^{j_1}\times\cdots\times(\mathbb{Z}_2)^{j_k}$ on $(S^1)^\omega = (S^1)^{j_1}\times\cdots\times(S^1)^{j_k}$ is coordinatewise, and on M via the representation $(g_{1,1},\dots,g_{1,j_1};\dots;g_{k,1},\dots,g_{k,j_k}) \mapsto (g_{1,j_1},\dots,g_{k,j_k})$. The argument is close to that of the unitary case in [8, Section 9.2].

References

[1] **J F Adams**, *Stable homotopy and generalised homology*, Univ. Chicago Press (1974) MR

[2] **M F Atiyah**, *Bordism and cobordism*, Proc. Cambridge Philos. Soc. 57 (1961) 200–208 MR

[3] **M F Atiyah**, **R Bott**, *A Lefschetz fixed point formula for elliptic complexes, II: Applications*, Ann. of Math. 88 (1968) 451–491 MR

[4] **M F Atiyah**, **G B Segal**, *The index of elliptic operators, II*, Ann. of Math. 87 (1968) 531–545 MR

[5] **M F Atiyah**, **I M Singer**, *The index of elliptic operators, I*, Ann. of Math. 87 (1968) 484–530 MR

[6] **M Bix**, **T tom Dieck**, *Characteristic numbers of G–manifolds and multiplicative induction*, Trans. Amer. Math. Soc. 235 (1978) 331–343 MR

[7] **J M Boardman**, *Stable homotopy theory, VI*, mimeographed notes, University of Warwick (1966)

[8] **V M Buchstaber**, **T E Panov**, *Toric topology*, Mathematical Surveys and Monographs 204, Amer. Math. Soc., Providence, RI (2015) MR

[9] **V Buchstaber**, **T Panov**, **N Ray**, *Toric genera*, Int. Math. Res. Not. 2010 (2010) 3207–3262 MR

[10] **G Comezaña**, *Calculations in complex equivariant bordism*, from "Equivariant homotopy and cohomology theory" (J P May, editor), CBMS Regional Conference Series in Mathematics 91, Amer. Math. Soc., Providence, RI (1996) 333–352 MR

[11] **P E Conner**, *Seminar on periodic maps*, Lecture Notes in Mathematics 46, Springer (1967) MR

[12] **P E Conner**, **E E Floyd**, *Differiable periodic maps*, Bull. Amer. Math. Soc. 68 (1962) 76–86 MR

[13] **P E Conner**, **E E Floyd**, *Differentiable periodic maps*, Ergeb. Math. Grenzgeb. 33, Springer (1964) MR

[14] **P E Conner**, **E E Floyd**, *Periodic maps which preserve a complex structure*, Bull. Amer. Math. Soc. 70 (1964) 574–579 MR

[15] **T tom Dieck**, *Bordism of G–manifolds and integrality theorems*, Topology 9 (1970) 345–358 MR

[16] **T tom Dieck**, *Characteristic numbers of G–manifolds, I*, Invent. Math. 13 (1971) 213–224 MR

[17] **T tom Dieck**, *Characteristic numbers of G–manifolds, II*, J. Pure Appl. Algebra 4 (1974) 31–39 MR

[18] **V Guillemin**, **V Ginzburg**, **Y Karshon**, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs 98, Amer. Math. Soc., Providence, RI (2002) MR

[19] **B Hanke**, *Geometric versus homotopy theoretic equivariant bordism*, Math. Ann. 332 (2005) 677–696 MR

[20] **A Hattori**, *Equivariant characteristic numbers and integrality theorem for unitary T^n–manifolds*, Tôhoku Math. J. 26 (1974) 461–482 MR

[21] **S S Khare**, *Characteristic numbers for oriented singular G–bordism*, Indian J. Pure Appl. Math. 13 (1982) 637–642 MR

[22] **S O Kochman**, *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs 7, Amer. Math. Soc., Providence, RI (1996) MR

[23] **I M Krichever**, *Obstructions to the existence of S^1–actions: bordisms of branched coverings*, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976) 828–844 MR In Russian; translated in Math. USSR-Izv. 10 (1976) 783–797

[24] **P S Landweber**, *A survey of bordism and cobordism*, Math. Proc. Cambridge Philos. Soc. 100 (1986) 207–223 MR

[25] **C N Lee**, **A G Wasserman**, *Equivariant characteristic numbers*, from "Proceedings of the second conference on compact transformation groups, I" (H T Ku, L N Mann, J L Sicks, J C Su, editors), Lecture Notes in Math. 298, Springer (1972) 191–216 MR

[26] **P Löffler**, *Bordismengruppen unitärer Torusmannigfaltigkeiten*, Manuscripta Math. 12 (1974) 307–327 MR

[27] **Z Lü**, **Q Tan**, *Equivariant Chern numbers and the number of fixed points for unitary torus manifolds*, Math. Res. Lett. 18 (2011) 1319–1325 MR

[28] **Z Lü**, **Q Tan**, *Small covers and the equivariant bordism classification of* 2*–torus manifolds*, Int. Math. Res. Not. 2014 (2014) 6756–6797 MR

[29] **J Milnor**, *On the cobordism ring* Ω^* *and a complex analogue, I*, Amer. J. Math. 82 (1960) 505–521 MR

[30] **S P Novikov**, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Dokl. Akad. Nauk SSSR 132 (1960) 1031–1034 MR In Russian; translated in Soviet Math. Dokl. 1 (1960) 717–720

[31] **D Quillen**, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. 7 (1971) 29–56 MR

[32] **D P Sinha**, *Computations of complex equivariant bordism rings*, Amer. J. Math. 123 (2001) 577–605 MR

[33] **R E Stong**, *Notes on cobordism theory*, Mathematical Notes 7, Princeton Univ. Press (1968) MR

[34] **R E Stong**, *Equivariant bordism and Smith theory, IV*, Trans. Amer. Math. Soc. 215 (1976) 313–321 MR

[35] **R Thom**, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954) 17–86 MR

[36] **C T C Wall**, *Determination of the cobordism ring*, Ann. of Math. 72 (1960) 292–311 MR

School of Mathematical Sciences, Fudan University
Shanghai, China

College of Information Technology, Shanghai Ocean University
Shanghai, China

zlu@fudan.edu.cn, weiwang@amss.ac.cn

Received: 10 December 2017 Revised: 23 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 4161–4185

Spaces of orders of some one-relator groups

JUAN ALONSO
JOAQUÍN BRUM

We show that certain left-orderable groups admit no isolated left orders. The groups we consider are cyclic amalgamations of a free group with a general left-orderable group, the HNN extensions of free groups over cyclic subgroups, and a particular class of one-relator groups. In order to prove the results about orders, we develop perturbation techniques for actions of these groups on the line.

06F15, 20F60

1 Introduction

A group G is said to be left-orderable if it admits a total order invariant by left translations. Left-orderability of groups is a wide and active topic of research (see Ghys [8], Clay and Rolfsen [5], Deroin, Navas and Rivas [7] and Kopytov and Medvedev [11]). Within this theory, an important object of study is the *space of left orders* $\mathcal{LO}(G)$ of a left-orderable group G. This is the set of left orders on G endowed with a natural topology that makes it a Hausdorff, totally disconnected and compact space; see Sikora [18]. For a countable group G, this implies that $\mathcal{LO}(G)$ is a Cantor set exactly when it contains no isolated orders. In addition to that, Linnell showed that this space is either finite or uncountable [12], and Tararin classified the groups that have finitely many left orders [11, Theorem 5.2.1]. In light of these results, one of the main interesting problems concerning the topology of the space of left orders is to determine which left-orderable groups admit isolated orders.

This problem turns out to be a complex one, as shown by the partial results that have been obtained. For virtually solvable groups, Rivas and Tessera gave a complete description of these spaces, showing that they are either finite or Cantor sets [17]. Free products of left-orderable groups admit no isolated orders — see Rivas [16] — and neither do cyclic amalgams of free groups; see Alonso, Brum and Rivas [1]. On the other hand, $\mathbb{F}_n \times \mathbb{Z}$ admits isolated orders if and only if n is even; see Malicet, Mann, Rivas and Triestino [13].

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4161

Here we provide a generalization of the results about amalgams in [1].

Theorem 1.1 *Let $G = \mathbb{F}_n *_{\mathbb{Z}} H$ be a left-orderable group with $n \geq 2$. Then G has no isolated orders.*

The orderability of G in Theorem 1.1 is equivalent to that of H since the amalgamating subgroup is cyclic, as shown by Bludov and Glass [3]. The hypothesis that $n \geq 2$ is necessary; for example, the torus knot groups $\langle a, b \mid a^m = b^k \rangle$ have isolated orders; see Navas [15] and Ito [9].

The situation for general amalgamated products is more complex, even when the amalgamating subgroups are cyclic. Ito [10] constructs a wide class of amalgamated products of the form $G *_{\mathbb{Z}} H$ that do have isolated orders. The groups G and H considered in [10] both have isolated orders. Theorem 1.1 points in the other direction, for amalgams $G *_{\mathbb{Z}} H$ where one of the factors is a nonabelian free group.

The results and techniques used in [1] suggest the question:

Question 1.2 Can a one-relator group generated by three or more elements have an isolated order?

We obtain some partial results in this direction. The next one can also be seen as a generalization of Theorem 1.1 in [1], this time dealing with the case of an HNN extension.

Theorem 1.3 *Let $G = \langle t, x_1, \dots, x_n \mid t w_1 t^{-1} = w_2 \rangle$ with $n \geq 2$ and nontrivial $w_1, w_2 \in \langle x_1, \dots, x_n \rangle$. Then G has no isolated orders.*

One-relator groups are left-orderable unless they have torsion — see Brodskiĭ [4] — and they only have torsion when the relation is a proper power. Thus the groups in Theorem 1.3 are left-orderable. Again, the result does not hold for $n = 1$. This can be seen for the Klein bottle group $\langle a, b \mid aba^{-1} = b^{-1} \rangle$, which has finitely many orders; see Deroin, Navas and Rivas [7].

Following Question 1.2, we consider one-relator groups with more complex relations. The techniques we developed seem well adjusted to the case where the relation contains only positive powers of some given generator t. Namely, $G = \langle t, x_1, \dots, x_n \mid t w_1 \cdots t w_k = 1 \rangle$ for $n \geq 2$ and $w_i \in \langle x_1, \dots, x_n \rangle$. If $k = 1$ then G is free, and for $k = 2$ we can change the presentation to $G = \langle s, x_1, \dots, x_n \mid s^2 w_1^{-1} w_2 = 1 \rangle$, which is an amalgam covered in Theorem 1.1. Our techniques allow us to obtain the case $k = 3$, which turns out to be quite nontrivial.

Theorem 1.4 *Let $G = \langle t, x_1, \dots, x_n \mid tw_1tw_2tw_3 = 1\rangle$ with $n \geq 2$ and $w_i \in \langle x_1, \dots, x_n\rangle$ for $i = 1, 2, 3$. Assume that $w_i \neq w_j$ for some i and j. Then G has no isolated orders.*

Remark 1.5 If $w_1 = w_2 = w_3$ then the group G in Theorem 1.4 has torsion and therefore is not left-orderable.

The conclusion of Theorem 1.4 does not hold when $n = 1$, as Dehornoy shows in [6, Proposition 8.1] that the groups $\langle t, x \mid tx^p tx^p tx^{-1}\rangle$ for $p \geq 1$ have isolated left orders.

We would conjecture that the groups of the form $G = \langle t, x_1, \dots, x_n \mid tw_1 \cdots tw_k = 1\rangle$ with $n \geq 2$ do not have isolated orders in general. However, we meet technical obstructions in our approach when $k \geq 4$.

The method for proving these theorems involve the close relationship between left orders and actions on the line that has been introduced in [14]. A countable group G is left-orderable if and only if it admits a faithful action by orientation-preserving homeomorphisms of the line. Indeed, a left order on G induces an action on the line via the construction called *dynamical realization* (see Section 2A). Furthermore, the topology of $\mathcal{LO}(G)$ is related to rigidity properties on the space of such actions. More precisely, an isolated order induces an action that is *structurally stable* (also called *rigid*); see [1] for the definition. In [1] it is shown that cyclic amalgams of free groups have no rigid actions on the line, implying they have no isolated orders. Here we do not deal with rigid actions in order to avoid more technicalities; instead we use a weaker version of this principle, namely Proposition 2.2.

The proofs of our three theorems follow the same rough strategy. We start from the dynamical realization of a given order and construct an arbitrarily small perturbation that has nontrivial stabilizer on the orbit of 0, then we use Proposition 2.2 to conclude. The groups in our theorems contain free subgroups, namely the factor $\mathbb{F}_n$ in Theorem 1.1, and the subgroups generated by $x_1, \dots, x_n$ in Theorems 1.3 and 1.4 (see the Freiheitsatz in Baumslag [2]), and the strategy in all cases is to perturb the action restricted to these free subgroups in a way that can be extended to an action of the whole group. Our main technical tool to achieve that is Lemma 3.3, that gives a way to perturb actions of free groups creating nontrivial stabilizers, while controlling the behaviour of a particular element and of finitely many partial orbits.

We would like to remark on another application of our techniques, which we use as a lemma but is also important on its own right. Let $\mathrm{Rep}(G, H)$ denote the set of group representations of G in H. For each $w \in \mathbb{F}_n$, and any group H, we can define a *word map* from $\mathrm{Rep}(\mathbb{F}_n, H)$ to H that associates a representation ρ to $\rho(w) \in H$. We study the case when $H = \mathrm{Homeo}_+(\mathbb{R})$, the group of orientation-preserving homeomorphisms of the real line.

Lemma 1.6 *Let $w \in \mathbb{F}_n$, $w \neq 1$ and $g \in \mathrm{Homeo}_+(\mathbb{R})$. Then*

$$V_w(g) = \{\rho \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R})) : \rho(w) = g\}$$

is nonempty.

This is to say that the word map for $\mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$ is surjective for any nontrivial $w \in \mathbb{F}_n$.

Acknowledgements

The authors want to thank Cristóbal Rivas for his invitation to Santiago de Chile and for several fruitful conversations, and the referees for many useful suggestions.

2 Preliminaries

2A Left orders and actions on the line

A left order on a group G is a total order $<$ satisfying that if $f, g, h \in G$ are such that $f < h$, then $gf < gh$. If G admits a left order we say that G is *left-orderable*. The reader unfamiliar with this notion may wish to consult [5; 7; 11]. The groups under consideration in this paper are left-orderable, as we already mentioned in Section 1.

A natural topology can be defined on the set $\mathcal{LO}(G)$ of all left orders on G, making it a compact and totally disconnected space. A local base at a left order $<$ is given by the sets

$$V_{g_1,\dots,g_n} := \{<' \in \mathcal{LO}(G) \mid 1 <' g_i\},$$

where $\{g_1, \dots, g_n\}$ runs over all finite subsets of $<$–positive elements of G. In particular, a left order $<$ is isolated in $\mathcal{LO}(G)$ if and only if there is a finite set $S \subset G$ such that $<$ is the only left order satisfying

$$\mathrm{id} < s \quad \text{for every } s \in S.$$

When the group is countable this topology is metrizable [5; 7; 18]. For instance, if G is finitely generated and B_n denotes the ball of radius n with respect to a finite generating set, then we can declare that $\operatorname{dist}(<_1, <_2) = 1/n$ if B_n is the largest ball in which $<_1$ and $<_2$ coincide.

When the group G is countable, for every left order $<$ on G one can attach a fixed-point-free action $\rho\colon G \to \operatorname{Homeo}_+(\mathbb{R})$ that models the left translation action of G on $(G, <)$, in the sense that

$$f < g \iff \rho(f)(0) < \rho(g)(0). \tag{1}$$

This is the so-called *dynamical realization* of $<$ (which is unique up to conjugation), and 0 is sometimes called the *basepoint*; see [5; 7; 8].

On the other hand, any representation $\rho\colon G \to \operatorname{Homeo}_+(\mathbb{R})$ defines a *partial* left invariant order on G through (1). This is a total order exactly when the stabilizer of 0 under ρ is trivial.

Given a group G we consider the set $\operatorname{Rep}(G, \operatorname{Homeo}_+(\mathbb{R}))$ of group representations from G to $\operatorname{Homeo}_+(\mathbb{R})$ endowed with the pointwise convergence. That is, ρ_n converges to ρ if and only if $\rho_n(g)$ converges to $\rho(g)$ for all $g \in G$, where the convergence $\rho_n(g) \to \rho(g)$ is given by the compact–open topology: for every positive ε and for every compact set $K \subset M$, there is n_0 such that $n \geq n_0$ implies

$$\sup_{x \in K} |\rho_n(g)(x) - \rho(g)(x)| \leq \varepsilon.$$

Remark 2.1 The convergence of $\rho_n \to \rho$ in $\operatorname{Rep}(G, \operatorname{Homeo}_+(\mathbb{R}))$ is equivalent to requiring that $\rho_n(g) \to \rho(g)$ for every g in a generating set of G.

The next result is a way to relate the topologies of $\mathcal{LO}(G)$ and $\operatorname{Rep}(G, \operatorname{Homeo}_+(\mathbb{R}))$, and will be the key tool for proving all our theorems.

Proposition 2.2 *Let $\rho \in \operatorname{Rep}(G, \operatorname{Homeo}_+(\mathbb{R}))$ be the dynamical realization of a total left order $<$ on G* (in the sense of (1)). *If ρ can be arbitrarily approximated by representations that have nontrivial stabilizers on the orbit of 0, then $<$ is not isolated in $\mathcal{LO}(G)$.*

Proof Let $F \subseteq G$ be a finite set with $1 < g$ for every $g \in F$. We take a neighbourhood V of ρ such that if $\rho' \in V$ then $0 < \rho'(g)(0)$ for $g \in F$. By our hypothesis, there

exists $\rho' \in V$ that has nontrivial stabilizer on the orbit of 0. This induces a partial left order $\prec$ on G. Since $H = \mathrm{Stab}_{\rho'}(0)$ is a subgroup of a left-orderable group, it is also left-orderable and has at least two different left orders. By the convex extension procedure (see [7, Section 2.1]) we can extend $\prec$ to at least two different total left orders $<_1$ and $<_2$ on G, satisfying $1 <_i g$ for $g \in F$ and $i = 1, 2$. One of them must be different from $<$. □

2B Conjugacy and roots in $\mathrm{Homeo}_+(\mathbb{R})$

Here we present some facts and constructions on line homeomorphisms that will be needed in the sequel. Given $\phi \in \mathrm{Homeo}_+(\mathbb{R})$ we define the sets

$$\mathrm{Fix}(\phi) = \{x \mid \phi(x) = x\},$$
$$\mathrm{Inc}(\phi) = \{x \mid \phi(x) > x\},$$
$$\mathrm{Decr}(\phi) = \{x \mid \phi(x) < x\}.$$

These sets help us study the conjugacy class of ϕ. If $\psi\phi_1\psi^{-1} = \phi_2$, then ψ induces bijections between the corresponding sets for ϕ_1 and ϕ_2. (Namely, $\psi(\mathrm{Fix}(\phi_1)) = \mathrm{Fix}(\phi_2)$ and so on). On the other hand, two homeomorphisms ϕ_1 and ϕ_2 are conjugated in $\mathrm{Homeo}_+(\mathbb{R})$ if there exists $\psi \in \mathrm{Homeo}_+(\mathbb{R})$ that maps $\mathrm{Fix}(\phi_1)$ to $\mathrm{Fix}(\phi_2)$ and $\mathrm{Inc}(\phi_1)$ to $\mathrm{Inc}(\phi_2)$ (and so maps $\mathrm{Decr}(\phi_1)$ to $\mathrm{Decr}(\phi_2)$). With this in mind, we define a *weak conjugation* as follows:

Definition 2.3 For ψ, ϕ_1 and ϕ_2 homeomorphisms of the real line, we will say that ψ is a *weak conjugation* from ϕ_1 to ϕ_2 if

- $\psi(\mathrm{Fix}(\phi_1)) = \mathrm{Fix}(\phi_2)$, and
- $\psi(\mathrm{Inc}(\phi_1)) = \mathrm{Inc}(\phi_2)$.

Additionally, if for an interval I we have that $\psi\phi_1(x) = \phi_2\psi(x)$ for all $x \in I$, we will say that the weak conjugation ψ is *strong* on I.

Observe that conjugacy and weak conjugacy classes are identical, but it is much easier to find/build weak conjugations rather than true conjugating elements. We will need a result that allows us to pass from a weak conjugation to a conjugation, while respecting the parts in which the weak conjugation is strong. This is Lemma 2.7 in [1], which we state below.

Lemma 2.4 *Let $\psi, \phi_1, \phi_2 \in \mathrm{Homeo}_+(\mathbb{R})$. If ψ is a weak conjugation from ϕ_1 to ϕ_2 that is strong on a interval I, then there exists a conjugation $\bar{\psi}$ from ϕ_1 to ϕ_2 such that*

- *$\bar{\psi}(x) = \psi(x)$ for every $x \in I$, and*
- *$\bar{\psi}(x) = \psi(x)$ for every $x \in \mathrm{Fix}(\phi_1)$.*

Moreover, $\bar{\psi}$ agrees with ψ over $I \cup \phi_1(I)$.

For the proof, we refer to [1].

Remark 2.5 We will use Lemma 2.4 in a slightly stronger form, where ϕ_1 and ϕ_2 are homeomorphisms of arbitrary intervals I and J, respectively. The map $\psi\colon I \to J$ is a weak conjugation, following a straightforward adaptation of Definition 2.3. This stronger version of Lemma 2.4 is obtained as a corollary through conjugation.

We will also need to take square roots of homeomorphisms under composition.

Lemma 2.6 *Every $h \in \mathrm{Homeo}_+(\mathbb{R})$ has a square root.*

Proof Translations on $\mathbb{R}$ clearly have square roots. For $h \in \mathrm{Homeo}_+(\mathbb{R})$ define ψ so that $\mathrm{Fix}(\psi) = \mathrm{Fix}(h)$, and if I is a connected component of $\mathbb{R} - \mathrm{Fix}(h)$ then $\psi|_I$ is a square root of $h|_I$, which exists because $h|_I$ is conjugated to a translation. It is easy to check that $\psi^2 = h$. □

Remark 2.7 In the proof above it is clear that if $q \in \mathrm{Fix}(h)$ and ψ_0 is a square root of $h|_{(-\infty,q)}$, we can choose ψ as an extension of ψ_0. We can also adapt Lemma 2.6 for a homeomorphism $h\colon (-\infty, q_1] \to (-\infty, q_2]$, obtaining $\psi\colon (-\infty, q_3] \to (-\infty, q_4]$ with ψ^2 a restriction of h. This can be done by extending h to a homeomorphism of $\mathbb{R}$ and then restricting ψ to a suitable interval.

3 Key technical tools

Let $w \in \mathbb{F}_n = \langle x_1, \dots, x_n \rangle$ be a reduced word, and write $w = a_m \cdots a_1$ with $a_j \in \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$. We define $w_0 = e$ and $w_j = a_j \cdots a_1$ for $0 < j \leq m$. If $\rho \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$ and $x \in \mathbb{R}$, we will be interested in the sequence

$$S(\rho, w, x) = (\rho(w_0)(x), \dots, \rho(w_m)(x)),$$

which we call the *trajectory* of x by w under the action ρ.

Trajectories will play a key role in our perturbation techniques. On one hand, we will construct new representations by "extending" prefixed trajectories. That is, we will first define some arbitrary, but suitable, sequence $S = (s_0, \dots, s_m) \in \mathbb{R}^{m+1}$ and then find a representation that realizes S as a trajectory by w.

On the other hand, we will need to perturb the representations while keeping in mind the effect on certain trajectories. To do that, for each generator x_i we will need to look at the minimum point from which we can perturb the map $\rho(x_i)$ without changing the trajectory $S(\rho, w, x)$. That point is the largest one where we apply the generator x_i in the trajectory.

Taking into account that we will work with "trajectories" *before* realizing them by representations, it will be useful to make the relevant definitions in a combinatorial context, without reference to a specific representation.

Definition 3.1 Take a word $w \in \mathbb{F}_n = \langle x_1, \dots, x_n \rangle$ with $|w| = m$, a sequence $S = (s_0, \dots, s_m) \in \mathbb{R}^{m+1}$ and $i \in \{1, \dots, n\}$. Write $w = a_m \cdots a_1$ with $a_j \in \{x_1^{\pm 1}, \dots, x_n^{\pm 1}\}$. We define

$$D_w(S, i) = \{s_j : a_{j+1} = x_i \text{ or } a_j = x_i^{-1}\}$$

and

$$d_w(S, i) = \max D_w(S, i).$$

When x_i does not appear in w, we set $D_w(S, i) = \varnothing$ and $d_w(S, i) = -\infty$.

We remark that when $S = S(\rho, w, x)$ is an actual trajectory, the set $D_w(S, i)$ is the subset of S where we apply the generator x_i in the trajectory. Look at Figure 1 for an example.

We find it enlightening to think of a trajectory as a graph, as shown in Figure 1. This can also be formulated for general sequences. With the notation of Definition 3.1, we associate a *trajectory graph* to $S = (s_0, \dots, s_m)$ as follows: The vertex set is just $\{s_0, \dots, s_m\}$, and we put in an edge for every $j = 1, \dots, m$ connecting s_{j-1} to s_j. This edge is labelled and oriented according to a_j. Namely, if $a_j = x_i^{\epsilon}$ then the edge label is x_i, and its orientation depends on ϵ: when $\epsilon = 1$ it is oriented from s_{j-1} towards s_j, and the reverse for $\epsilon = -1$.

Looking at the trajectory graph of S, we can regard $D_w(S, i)$ as the set of vertices which have an outgoing edge marked with x_i.

The following observation will be useful in the proof of most of the heavier results. It comes up naturally by looking at trajectory graphs.

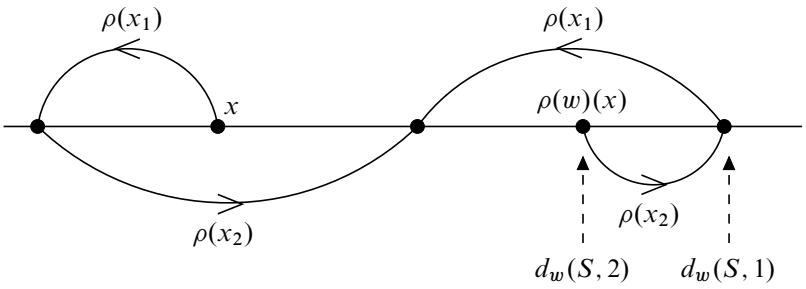


Figure 1: A possible trajectory $S = S(\rho, w, x)$ for $w = x_2^{-1}x_1^{-1}x_2x_1$ in which we can see the sets $D_w(S,i)$ as the starting points of the arrows marked with $\rho(x_i)$

Remark 3.2 For any $\rho \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$, $w \in \mathbb{F}_n$ and $p \in \mathbb{R}$, we have that $S(\rho, w^{-1}, \rho(w)(p))$ is just $S(\rho, w, p)$ traversed backwards. Thus,

$$D_w(S(\rho, w, p), i) = D_{w^{-1}}\big(S(\rho, w^{-1}, \rho(w)(p)), i\big)$$

for all $i = 1, \dots, n$.

Our first result is the surjectivity of the word map, namely Lemma 1.6. Its proof will illustrate the technique of defining a representation by prefixing some trajectories.

Proof of Lemma 1.6 We will prove it for the case $\mathrm{Fix}(g) = \varnothing$; the general case reduces to this by setting $\mathrm{Fix}(g)$ as the global fixed points of ρ. We can further assume that $g(x) > x$ for every $x \in \mathbb{R}$, the other case being analogous.

It suffices to find a representation ρ_0 such that $\rho_0(w)(x) > x$ for all $x \in \mathbb{R}$, since then we can conjugate ρ_0 to obtain a representation ρ with $\rho(w) = g$. The same reasoning allows us to exchange w by one of its conjugates in $\mathbb{F}_n$, so we may assume that w is cyclically reduced.

Let $m = |w|$ and set $S_r = (mr, mr+1, \dots, m(r+1))$ for all $r \in \mathbb{Z}$. In other words, $S_r = (s_{r,0}, \dots, s_{r,m})$ is the sequence defined by $s_{r,j} = mr + j$. Notice that S_r ends where S_{r+1} begins. We will give a representation ρ_0 that has all S_r as trajectories by w. This is enough, since then we shall have that $\rho_0(w^r)(0) = mr \to \pm\infty$ as $n \to \pm\infty$, proving that $\rho_0(w)(x) > x$ for all $x \in \mathbb{R}$.

Fix some $r \in \mathbb{Z}$. For each $i \in \{1, \dots, n\}$, we define a map g_i on $D_w(S_r, i)$ by the formula $g_i(s_{r,j-1}) = s_{r,j}$ if $a_j = x_i$ and $g_i(s_{r,j}) = s_{r,j-1}$ if $a_j = x_i^{-1}$. These are indeed well-defined maps, and are also injective, due to the fact that S_r has no

repetitions and w is reduced (admits no cancellations). This can be seen by considering the trajectory graph for S_r: the formula for g_i is just defined by the arrows (oriented edges) marked by x_i, and we notice that no vertex admits two incoming or two outgoing edges of the same label.

Assuming that w is cyclically reduced allows us to join, for fixed $i \in \{1, \dots, n\}$, all the maps g_i corresponding to every $r \in \mathbb{Z}$ (we abuse notation by omitting r from it). This gives rise to a well-defined map g_i on $\bigcup_{r\in\mathbb{Z}} D_w(S_r, i)$, as can be shown by the same argument as for the trajectory graphs, where the fact that w is cyclically reduced is used for the common vertices of different sequences, namely $s_{r,m} = s_{r+1,0}$.

We will show that $g_i\colon \bigcup_{r\in\mathbb{Z}} D_w(S_r, i) \to \mathbb{R}$ are increasing (ie $g_i(x) < g_i(y)$ for $x < y$), and therefore can be extended to $\mathbb{R}$ as homeomorphisms (eg linear interpolation). This defines the maps $\rho_0(x_i)$ that give the desired representation ρ_0. Notice that if a generator x_i is not present in w, we can choose $\rho_0(x_i)$ freely.

Now we see that g_i is increasing: Take $x < y$ in $X_i = \bigcup_{r\in\mathbb{Z}} D_w(S_r, i)$. The construction of the sequences S_r gives that $g_i(x) = x + \epsilon(x)$ for $x \in X_i$ and $\epsilon(x) = \pm 1$. This shows $g_i(x) < g_i(y)$ directly for $y - x \geq 3$, and using injectivity for $y - x = 2$. Also notice that the union of the trajectory graphs for S_r admits no closed edge-paths. So if both x and $x + 1$ belong in X_i, then we cannot have $\epsilon(x) > \epsilon(x + 1)$, which gives $g_i(x) < g_i(y)$ for $y - x = 1$. □

The following is the main technical lemma. It refines Lemma 3.1 in [1]. As we mentioned earlier, its aim is to give a perturbation of a representation of $\mathbb{F}_n$ in $\mathrm{Homeo}_+(\mathbb{R})$ that achieves many desired properties: having nontrivial stabilizer in the orbit of a given point, preserving some given trajectories of the original action, and controlling the dynamics near $+\infty$ of a specified element $w \in \mathbb{F}_n$.

Lemma 3.3 *Let $\rho \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$, $p, p_1, \dots, p_k \in \mathbb{R}$ and $w, v_1, \dots, v_k \in \mathbb{F}_n$, with w cyclically reduced. Let*

$$d_{ij} = d_{v_j}(S(\rho, v_j, p_j), i) \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, k$$

and

$$d_i = d_w(S(\rho, w, p), i) \quad \text{for } i = 1, \dots, n.$$

Assume that $\rho(w)(p) \neq p$, and that there exists i_0 with $d_{i_0} \geq \max_j d_{i_0 j}$.

Then there exists $\rho' \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$ such that:

(1) $\rho'(x_i)$ *agrees with* $\rho(x_i)$ *on* $(-\infty, m_i]$, *where* $m_i = \max\{d_i, d_{i1}, \dots, d_{ik}\}$ *for each* $i = 1, \dots, n$.

(2) $S(\rho', v_j, p_j) = S(\rho, v_j, p_j)$ *for every* $j = 1, \dots, k$.

(3) $\rho'(w)$ *agrees with* $\rho(w)$ *on* $(-\infty, p]$ *and* $\rho'(w)(x) \neq x$ *for* $x \geq p$.

(4) p *has nontrivial stabilizer under* ρ'.

Remark 3.4 In the statement of Lemma 3.3, (2) and the first part of (3) can be deduced from (1). This is straightforward from the definitions. Nevertheless, we state these consequences explicitly as they are important in the applications.

The hypothesis of Lemma 3.3 is a bit involved. The following gives an easier way to verify it that will suffice in most cases.

Lemma 3.5 *In the notation of Lemma 3.3, if* $p \geq m = \max\{S(\rho, v_j, p_j) : j = 1, \dots, k\}$ *then there exists* i_0 *with* $d_{i_0} \geq \max_j d_{i_0 j}$.

Proof Write $w = a_m \cdots a_1$ and $a_1 = x_{i_0}^{\epsilon}$. We claim this i_0 works.

According to the sign of ϵ, either p or $\rho(a_1)(p)$ belongs to $D_w(S(\rho, w, p), i_0)$. If p does (when $\epsilon = 1$), then $d_{i_0} \geq p \geq m \geq d_{i_0 j}$ for every j.

Otherwise, $\epsilon = -1$ and $\rho(a_1)(p) \leq d_{i_0}$. If the assertion were not true, then $\rho(a_1)(p) \leq d_{i_0} < d_{i_0 j} \leq m \leq p$ for some j. But then we should have that $\rho(x_{i_0})(d_{i_0 j}) > p$. This is a contradiction since $\rho(x_{i_0})(d_{i_0 j})$ belongs to $S(\rho, v_j, p_j)$, so it should be less than m. □

The proof of Lemma 3.3 gets very technical. We point out that Theorems 1.1, 1.3 and 1.4 can be derived from the statement of Lemma 3.3 without using facts that come up in its proof.

Proof of Lemma 3.3 For simplicity, we will assume that $\rho(w)(p) > p$. Otherwise we exchange w for w^{-1} and p for $\rho(w)(p)$. This can be done without altering the hypotheses by Remark 3.2.

The strategy of the proof follows the same idea we used for Lemma 1.6. We will first define some suitable sequences S_r for $r = 1, 2, \dots$ that will become the trajectories $S(\rho', w, \rho'(w^r)(p))$ for the desired representation ρ'. Those sequences will be used to define the maps $\rho'(x_i)$ in some discrete sets, and then it will be possible to extend them

by interpolation. Choosing the S_r carefully will allow us to achieve the objectives in the statement: make these extensions possible with $\rho'(x_i)$ and $\rho(x_i)$ agreeing on $(-\infty, m_i]$, make $\rho'(w^r)(p)$ tend to $+\infty$ (thus $\rho'(w)$ will have no fixed points after p), and also make $\rho'(z)(p) = p$ for some nontrivial $z \in \mathbb{F}_n$ (nontrivial stabilizer).

Write $w = a_m \cdots a_1$, with $m = |w|$. Recall that $m_i = \max\{d_i, d_{i1}, \ldots, d_{ik}\}$, and let $M = \max S(\rho, w, p) \cup \bigcup_{j=1}^{k} S(\rho, v_j, p_j)$.

Step 1 (definition of S_1) We define $S_1 = (s_{1,0}, \ldots, s_{1,m})$ as follows:

Set $s_{1,0} = \rho(w)(p)$ and, for $0 < j \leq m$, let

$$s_{1,j} = \rho(a_j)(s_{1,j-1}) \quad \text{while} \quad \begin{cases} s_{1,j-1} \leq m_i & \text{if } a_j = x_i, \\ s_{1,j-1} \leq \rho(x_i)(m_i) & \text{if } a_j = x_i^{-1}. \end{cases}$$

We get to $s_{1,l}$, the last element we can define by that process. Then we choose $s_{1,l+1} > M$, and set $s_{1,j+1} = s_{1,j} + 1$ for every $j = l+1, \ldots, m$.

The above definition of the $s_{1,j}$ amounts to saying that S_1 agrees with $S(\rho, w, \rho(w)(p))$ for as long as $s_{1,j}$ can be defined using the $\rho(x_i)$ restricted to $(-\infty, m_i]$. When that is no longer possible, we have freedom to pick the next $s_{1,j}$ without contradicting (1). We will do so to help us achieve (3), in a similar fashion as in Lemma 1.6. Figure 2 gives an example of this process. The next claim says that the last part of this definition really takes place.

Claim $l < m$.

Proof Recall the hypothesis that there is some i_0 with $d_{i_0} \geq \max_j d_{i_0 j}$. Also recall this means $d_{i_0} = m_{i_0}$. Notice that $d_w(S(\rho, w, x), i)$ is increasing in x, as it is a maximum of increasing homeomorphisms. Since we are assuming $\rho(w)(p) > p$, this gives us that

$$d_w(S(\rho, w, \rho(w)(p)), i_0) > d_w(S(\rho, w, p), i_0) = d_{i_0} = m_{i_0}.$$

On the other hand, if $l = m$, we would have that $S_1 = S(\rho, w, \rho(w)(p))$ and every point in $D_w\big(S(\rho, w, \rho(w)(p)), i\big)$ would be less than m_i for every i, for that is what is needed for the "while" condition to hold through $j = 1, \ldots, m$. This contradicts what we just obtained for i_0. □

Step 2 (definition of S_r for $r \geq 2$) The idea is to make a suitable S_2 that will help us create a nontrivial stabilizer, and define S_r for $r \geq 3$ in a similar way as in Lemma 1.6,

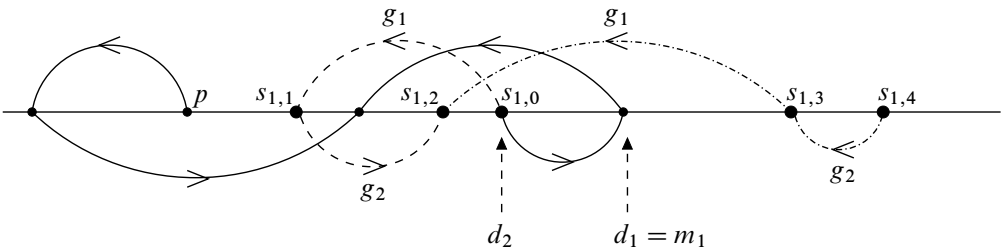


Figure 2: Step 1: we show the construction of the sequence S_1 for the example in Figure 1, assuming that $d_1 = m_1$ (the sequences $S(\rho, v_j, p_j)$ are not drawn in the picture). Here $l = 2$. Step 3: we draw the arrows defining the g_i on $D_w(S_1, i)$.

in order to obtain (3). We remark that S_r must begin where S_{r-1} ends. We need to distinguish two cases according to the form of w. They will only differ in S_2 and possibly in S_3.

Case A Suppose $n = 2$ and $w = [x_1, x_2]^u$, where $[x_1, x_2] = x_2^{-1}x_1^{-1}x_2x_1$ and $m = 4u$. This case applies for the other commutators of x_1 and x_2 as well, possibly exchanging the generators, or replacing them by their inverses.

We define $S_2 = (s_{2,0}, \dots, s_{2,4u})$ as $s_{2,0} = s_{1,m}$, $s_{2,1} = s_{1,m} + 3$, $s_{2,2} = s_{1,m} + 4$, $s_{2,3} = s_{1,m} + 1$, $s_{2,4} = s_{1,m} + 2$ and, if $u > 1$, we set $s_{2,5} = s_{1,m} + 5$ and $s_{2,j+1} = s_{2,j} + 1$ for $j \in \{5, \dots, 4u - 1\}$.

For $r \geq 3$ we define $S_r = (s_{r,0}, \dots, s_{r,4u})$ by $s_{r,0} = s_{r-1,4u}$ and $s_{r,j+1} = s_{r,j} + 1$ for $j \in \{0, \dots, 4u - 1\}$, with the exception that $s_{3,1} = s_{3,0} + 3$ if $u = 1$. (Notice that by this exception, the trajectory graph defined on $\bigcup_{r \geq 2} S_r$ looks the same for any u, and avoids two different incoming edges at $s_{2,1}$ both marked with x_1.)

Case B For any other w we set $S_r = (s_{r,0}, \dots, s_{r,m})$ with $s_{r,0} = s_{r-1,m}$ and $s_{r,j+1} = s_{r,j} + 1$ for every $r \geq 2$ and $j \in \{0, \dots, m - 1\}$.

Step 3 (partial definition of the maps $\rho'(x_i)$) The sequences S_r define maps g_i on the sets $D_w(S_r, i)$ as in Lemma 1.6, by taking $g_i(s_{r,j-1}) = s_{r,j}$ if $a_j = x_i$ and $g_i(s_{r,j}) = s_{r,j-1}$ if $a_j = x_i^{-1}$. We can set g_i to agree with $\rho(x_i)$ on $(-\infty, m_i]$ and we obtain well-defined maps $g_i \colon (-\infty, m_i] \cup \bigcup_r D_w(S_r, i) \to \mathbb{R}$ that are also injective. (This works by the same arguments used in Lemma 1.6. Here we use that w is cyclically reduced.)

In order to achieve the nontrivial stabilizer, we make a further extension of some of the g_i, which will be different in each case.

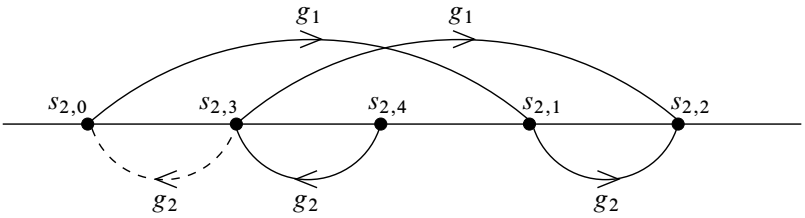


Figure 3: Case A, Step 2: we draw the first four points of the sequence S_2. Step 3: we draw the arrows corresponding to the maps g_i.

- In Case A we set $g_2(s_{2,3}) = s_{2,0}$. (See Figure 3.) Notice there is no arrow coming from $s_{2,3}$ and marked by x_2 in the trajectory graphs, thus g_2 is well defined. Injectivity is given by the fact that there is no incoming edge marked by x_2 at $s_{1,m} = s_{2,0}$, by construction of S_1 and the form of $w = [x_1, x_2]^u$. (There is an outgoing edge marked by x_2 at $s_{2,0}$, but that is compatible.)

- In Case B there is some $j_1 \in \{1, \dots, m\}$ and some $i_1 \in \{1, \dots, n\}$ such that $a_{j_1+1} \neq x_{i_1}^{\pm 1}$, and we do not have both $a_{j_1} = x_{i_1}^{\epsilon}$ and $a_{j_1+2} = x_{i_1}^{-\epsilon}$, where the indexes are taken mod $m = |w|$. Otherwise we would be in Case A.

  If neither a_{j_1} nor a_{j_1+2} is $x_{i_1}^{\pm 1}$, we can set $g_{i_1}(s_{2,j_1}) = s_{2,j_1+1}$ without contradicting the values of g_{i_1} we had defined previously. This is also true if $a_{j_1} = x_{i_1}$ or $a_{j_1+2} = x_{i_1}$. If $a_{j_1} = x_{i_1}^{-1}$ or $a_{j_1+2} = x_{i_1}^{-1}$, then we cannot do that, but we can put $g_{i_1}(s_{2,j_1+1}) = s_{2,j_1}$. (See Figure 4.)

Step 4 (extension of the maps $\rho'(x_i)$) From the previous step we have maps g_i defined on $(-\infty, m_i] \cup X_i$ for X_i a discrete set. As we did for Lemma 1.6, we will check that these maps are increasing, namely that $x < y$ implies $g_i(x) < g_i(y)$, so they can be extended to $\mathbb{R}$ as homeomorphisms. This completes the definition of the maps $\rho'(x_i)$ that give the representation ρ'.

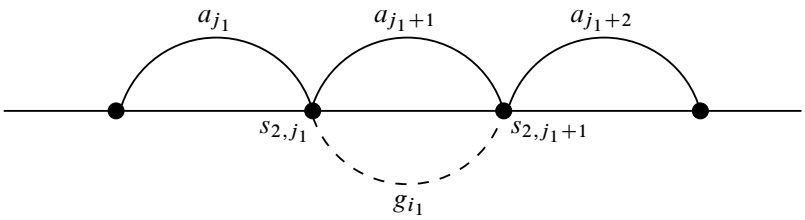


Figure 4: Case B: we have to pick an orientation for the dotted line, ensuring that there is no vertex with two incoming or outgoing arrows marked with g_{i_1}.

Claim $g_i\colon (-\infty, m_i] \cup X_i \to \mathbb{R}$ *is increasing.*

Proof On the sets X_i we can argue as in Lemma 1.6; for Case A it may be helpful to look at Figure 3. It remains to show that $g_i(x) < g_i(y)$ when $x = m_i$ and $y = \min X_i$. The proof splits into two cases depending on i; recall the construction of S_1, and let i_* be such that $a_l = x_{i_*}^{\epsilon}$.

If $i \neq i_*$, then g_i maps $X_i - (-\infty, m_i]$ into $(M, +\infty)$, since we picked $s_{r,j} > M$ for $r > 1$ and $r = 1,\ j > l$. By definition, $M \geq m_i$ for all i, hence g_i is increasing.

For i_* we separate the cases $\epsilon = \pm 1$. If $\epsilon = 1$, then by our definition of l we have $s_{1,l} > m_{i_*}$, and we had set $g_{i_*}(s_{1,l}) = s_{1,l+1} > M > \rho(x_{i_*})(m_{i_*}) = g_{i_*}(m_i)$. We also have $y = s_{1,l}$, so this gives the claim. If $\epsilon = -1$, we have $s_{1,l} > \rho(x_{i_*})(m_{i_*})$ (also by definition of l) and we had set $g_{i_*}(s_{1,l+1}) = s_{1,l} > \rho(x_{i_*})(m_{i_*}) = g_{i_*}(m_{i_*})$. Notice that $y = s_{1,l+1}$ (definition of l, and picking $s_{1,l+1} > M \geq m_{i_*}$), so this case is finished as well. □

Step 5 (verification of the properties of ρ') Point (1) is clear, and implies (2) and the first part of (3). Notice that S_r is the trajectory of $\rho'(w^r)(p)$ by w under the representation ρ'. So we have that $\rho'(w^r)(p) = s_{r,0}$, which tends to $+\infty$ with r, and thus $\rho'(w)$ has no fixed points after p.

For (4) notice that $s_{r,j}$ is in the orbit of p for all $r \geq 1$ and $j = 0, \ldots, m$. Then we have:

- In Case A, $s_{2,0}$ is fixed by $\rho'(x_2 x_1^{-1} x_2 x_1)$.
- In Case B, s_{2,j_1} is fixed by $\rho'(x_{i_1}^{\epsilon} a_{j_1+1})$ for some $\epsilon = \pm 1$.

This gives that the orbit of p has nontrivial stabilizer under ρ'. □

4 Proof of Theorem 1.1

Let $\mathbb{F}_n = \langle x_1, \ldots, x_n \rangle$. Then a group $G = \mathbb{F}_n *_{\mathbb{Z}} H$ as in the statement can be written as $\mathbb{F}_n *_{w=h} H$, where $w \in \mathbb{F}_n$ is a cyclically reduced word and $h \in H$.

Let $<$ be an order on G and ρ a dynamical realization for $<$ (based at 0). We shall define a new representation $\overline{\rho}$ that is a small perturbation of ρ and has nontrivial stabilizer on the orbit of 0. Then the theorem will follow from Proposition 2.2.

Let $g = \rho(h) = \rho(w)$. Let $V_w(g) = \{\rho' \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R})) : \rho'(w) = g\}$. We will find some $\rho' \in V_w(g)$ close to $\rho|_{\mathbb{F}_n}$ such that the representation $\overline{\rho}$ defined on G by gluing it with $\rho|_H$ has nontrivial stabilizer on the orbit of 0.

Take an arbitrarily large compact interval $K \subseteq \mathbb{R}$. Since ρ has no global fixed points, being a dynamical realization [7], we can build a finite ρ–trajectory S, with respect to the generating system $\{H, x_1, \dots, x_n\}$, which is an increasing sequence beginning at 0 and ending outside K. By a ρ–trajectory with respect to $\{H, x_1, \dots, x_n\}$ we mean that each point of S is obtained from the previous one by acting either by $\rho(h)$ for $h \in H$ or by $\rho(x_i)^{\pm 1}$ for some $i \in \{1, \dots, n\}$.

By keeping track of the generators, we can split S into a union of trajectories by $\rho|_{\mathbb{F}_n}$ that are connected by the action of elements of H (where the union of trajectories is sequence concatenation). Namely, we can write $S = \bigcup_{j=1}^{k} S(\rho|_{\mathbb{F}_n}, v_j, p_j)$, where $v_j \in \mathbb{F}_n$, and obtain p_{j+1} by acting on $\rho(v_j)(p_j)$ by some $\rho(h_j)$ for $h_j \in H$. Notice that $p_1 = 0$. Observe also that if $\rho|_H$ has no global fixed point in $K \cap [0, +\infty)$ we may choose $S = (0, \rho(h)(0))$, the trajectories by $\rho|_{\mathbb{F}_n}$ are just points and the v_j are trivial.

Let $p = \rho(v_k)(p_k)$ be the last element of S.

Claim *The representation $\rho|_{\mathbb{F}_n}$, and the $p, p_1, \dots, p_k \in \mathbb{R}$ and $w, v_1, \dots, v_k \in \mathbb{F}_n$ we just constructed, satisfy the hypotheses of Lemma 3.3.*

Proof Recall we are assuming w is cyclically reduced. That $\rho(w)(p) \neq p$ is clear since ρ is the dynamical realization of a total order and p is in the orbit of 0. Then we can apply Lemma 3.5 since $p = \max S$. □

Let ρ_0 be the representation obtained from $\rho|_{\mathbb{F}_n}$ applying Lemma 3.3. Let $q = \min \mathrm{Fix}(g) \cap (p, +\infty)$, with the convention that $\min \varnothing = +\infty$. The case for $q = +\infty$ is simpler, so we will focus on the construction when $q < +\infty$. We conjugate ρ_0 to an action on $(-\infty, q]$ with q as a global fixed point, by a conjugation $\psi\colon \mathbb{R} \to (-\infty, q)$ that restricts to the identity up to $\max\{p, \rho(w)(p)\}$ and such that $\rho(w)|_{(-\infty,q]} = \psi\rho_0(w)\psi^{-1}$. This is possible by Lemma 2.4.

We use Lemma 1.6 to get a representation ρ_1 of $\mathbb{F}_n$ on $(q, +\infty)$ with $\rho_1(w) = g|_{(q,+\infty)}$. We define $\rho' \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$ by

$$\rho'(\gamma)(x) = \begin{cases} \psi\rho_0(\gamma)\psi^{-1}(x) & \text{if } x \leq q, \\ \rho_1(\gamma)(x) & \text{if } x > q. \end{cases}$$

Notice that $\rho'(w) = g$, so we get $\overline{\rho} \in \mathrm{Rep}(G, \mathrm{Homeo}_+(\mathbb{R}))$ that agrees with ρ' on $\mathbb{F}_n$ and with $\rho|_H$ on H. Choosing K large enough we can get $\overline{\rho}$ arbitrarily close to ρ. It remains to show that the orbit of 0 has nontrivial stabilizer. By Lemma 3.3 we have $S(\rho_0, v_j, p_j) = S(\rho, v_j, p_j)$ for all $j = 1, \dots, k$, and since ψ is the identity up to $\max\{p, \rho(w)(p)\}$ these trajectories are not affected by the conjugation, so $S(\rho', v_j, p_j) = S(\rho, v_j, p_j)$. Thus, p is in the $\overline{\rho}$ orbit of 0. On the other hand, the stabilizer of p by ρ_0 is nontrivial by Lemma 3.3, and $\psi(p) = p$, so p has the same stabilizer under ρ'. The stabilizer of p under $\overline{\rho}$ contains it, so it is nontrivial. This gives the theorem by Proposition 2.2.

5 Proof of Theorem 1.3

Let $G = \langle t, x_1, \dots, x_n \mid t w_1 t^{-1} = w_2 \rangle$ as in the statement. We can assume that w_1 and w_2 are cyclically reduced (possibly taking an equivalent presentation).

Let $<$ be an order on G and ρ its dynamical realization. The proof follows the same strategy as that of Theorem 1.1: we shall define a new representation $\overline{\rho}$ that is a small perturbation of ρ and has nontrivial stabilizer on the orbit of 0, then finish by Proposition 2.2.

Take an arbitrarily large compact interval $K \subseteq \mathbb{R}$. As we did for Theorem 1.1, we take a finite increasing ρ–trajectory S, with respect to the generating set $\{t, x_1, \dots, x_n\}$, that begins at 0 and ends outside K. This means each point of S is obtained from the previous one by acting either by $\rho(t)^{\pm 1}$ or by $\rho(x_i)^{\pm 1}$ for some $i \in \{1, \dots, n\}$.

Again as in Theorem 1.1, we split S into a union of trajectories by $\rho|_{\mathbb{F}_n}$. So we write $S = \bigcup_{j=1}^{k} S(\rho|_{\mathbb{F}_n}, v_j, p_j)$, where $p_1 = 0$ and p_{j+1} is obtained by acting on $\rho(v_j)(p_j)$ by $\rho(t)^{\pm 1}$. (It is possible that v_j may be trivial, in which case the trajectory $S(\rho|_{\mathbb{F}_n}, v_j, p_j)$ is the single point p_j.)

Let $p = \rho(v_k)(p_k)$ be the last element of S.

Consider the word $\overline{w} = t^{-1} w_2^{-1} t w_1$ that represents the identity on G. Notice that the trajectory $S(\rho, \overline{w}, p)$ is decomposed as follows: the initial segment is $S(\rho, w_1, p)$, next comes $S(\rho, w_2, \rho(t)(p))$ traversed backwards, and then the final point is p (since $\overline{w}$ is the identity in G, the trajectory must be closed).

As in the statement of Lemma 3.3, we consider

$$d_{ij} = d_{v_j}(S(\rho, v_j, p_j), i) \quad \text{for } i = 1, \dots, n \text{ and } j = 1, \dots, k.$$

Define as well $d_i^{(1)} = d_{w_1}(S(\rho, w_1, p), i)$ and $d_i^{(2)} = d_{w_2}\big(S(\rho, w_2, \rho(t)(p)), i\big)$.

By Lemma 3.5 there exists i_0 with $d_{i_0}^{(1)} \geq \max_j d_{i_0 j}$ (recalling that S is an increasing sequence and p its last point). We will assume that $d_{i_0}^{(1)} \geq d_{i_0}^{(2)}$ for simplicity. Otherwise, we can invert the roles of w_1 and w_2, by replacing S by $S \cup \{\rho(t)(p)\}$ and $\overline{w}$ by $t w_1^{-1} t^{-1} w_2$. This does not affect the $d_i^{(1)}$ and $d_i^{(2)}$ by Remark 3.2.

We will apply Lemma 3.3 to $\rho|_{\mathbb{F}_n}$, with the following setting:

- $v_1, \ldots, v_k$ as constructed above, $v_{k+1} = w_2$ and $w = w_1$.
- $p_1, \ldots, p_k$ and p as constructed above, and $p_{k+1} = \rho(t)(p)$.

Notice the hypotheses of Lemma 3.3 hold for this setting, where the notation in the statement would be $d_{i,k+1} = d_i^{(2)}$ and $d_i = d_i^{(1)}$. From Lemma 3.3 we obtain a representation $\rho_0 \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$.

This ρ_0 is a perturbation of $\rho|_{\mathbb{F}_n}$ that satisfies what we would expect from the restriction to $\mathbb{F}_n$ of our desired representation. However, it is not always possible to extend it to G. We deal with this problem in what follows, splitting the procedure into two cases. For both we will need to notice that $\rho_0(w_1)$ agrees with $\rho(w_1)$ up to p, and $\rho_0(w_2)$ agrees with $\rho(w_2)$ up to $\rho(t)(p)$. These maps are partially conjugated by $\rho(t)$, meaning that for $x \in (-\infty, p]$ we have $\rho(t)\rho_0(w_1)(x) = \rho_0(w_2)\rho(t)(x)$.

Case 1 If $\rho_0(w_2)$ has no fixed points greater than $\rho(t)(p)$, then it is weakly conjugated to $\rho_0(w_1)$ by a map that is strong on $(-\infty, p]$ (see Definition 2.3). Thus, by Lemma 2.4 we can find a homeomorphism φ that conjugates $\rho_0(w_1)$ and $\rho_0(w_2)$, and agrees with $\rho(t)$ on $(-\infty, p]$. In this case we can define $\overline{\rho}(t) = \varphi$ and $\overline{\rho}|_{\mathbb{F}_n} = \rho_0$.

Case 2 On the other hand, assume that $\rho_0(w_2)$ has fixed points greater than $\rho(t)(p)$. Let $M = \max S \cup S(\rho, \overline{w}, p)$. As in the proof of Theorem 1.1, we can conjugate ρ_0 to a representation on $(-\infty, q]$ for some $q > M$ that has q as a global fixed point, by a map that is the identity on $(-\infty, M]$. Call the result of this conjugation ρ_0'.

Let z be the first fixed point of $\rho_0'(w_2)$ on $[\rho(t)(p), q]$. Then $z < q$ by the assumption of this case. Notice that $\rho_0'(w_1)$ and $\rho_0'(w_2)|_{(-\infty,z]}$ are weakly conjugated, because of the definition of z and their partial conjugation up to p by $\rho(t)$. By Lemma 2.4 we can find a homeomorphism $\varphi\colon (-\infty, q] \to (-\infty, z]$ that conjugates $\rho_0'(w_1)$ to $\rho_0'(w_2)|_{(-\infty,z]}$ and agrees with $\rho(t)$ on $(-\infty, p]$.

Extend φ to a homeomorphism of $\mathbb{R}$ that takes $[q, q+1]$ to $[z, q]$ and $[q+m, q+m+1]$ to $[q+m-1, q+m]$ for $m \geq 1$.

Applying Lemma 1.6 recursively, we can define

$$\rho'_m \in \mathrm{Rep}\big(\mathbb{F}_n, \mathrm{Homeo}_+([q+m-1, q+m])\big)$$

so that $\rho'_m(w_1) = \varphi^{-1}\rho'_{m-1}(w_2)\varphi$ for $m \geq 1$, where the base case is $\rho'_1(w_1) = \varphi^{-1}\rho'_0(w_2)|_{[z,q]}\varphi$.

We define $\rho' \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$ so that $\rho'(\gamma)|_{(-\infty,q]} = \rho'_0(\gamma)$ and

$$\rho'(\gamma)|_{(q+m-1,q+m]} = \rho_m(\gamma)$$

for $m \geq 1$. Then we define $\overline{\rho} \in \mathrm{Rep}(G, \mathrm{Homeo}_+(\mathbb{R}))$ by $\overline{\rho}|_{\mathbb{F}_n} = \rho'$ and $\overline{\rho}(t) = \varphi$.

In both cases the resulting $\overline{\rho}$ is a perturbation of ρ that can be made arbitrarily small by choosing K large enough. The stabilizer of p by ρ_0 is nontrivial and this fact is preserved by the modifications in Case 2. On the other hand, the trajectory S is not affected either, since $S(\rho, v_j, p_j) = S(\overline{\rho}, v_j, p_j)$ for all j, and $\overline{\rho}(t)$ agrees with $\rho(t)$ on $(-\infty, p]$. Thus the stabilizer of 0 by $\overline{\rho}$ is nontrivial. The theorem follows from Proposition 2.2.

6 Proof of Theorem 1.4

Throughout this section we consider $\overline{w} = tw_1tw_2tw_3$ as a word that represents the identity in G.

Lemma 6.1 *G can be presented as $G = \langle t, x_1, \dots, x_n \mid tw_1tw_2tw_3 = 1\rangle$, where the strings $w_i w_j^{-1}$ are cyclically reduced for $i \neq j \in \{1, 2, 3\}$.*

Proof Among the presentations of G of the form $\langle t, x_1, \dots, x_n \mid tw_1tw_2tw_3 = 1\rangle$ we choose one so that $|w_1| + |w_2| + |w_3|$ is minimal. We will show that this presentation is the desired one. The string $w_i w_j^{-1}$ is cyclically reduced when w_i and w_j do not have a common initial segment nor a common final segment. If this is not the case, we will give another presentation that contradicts the minimality assumption. Suppose for instance that $w_1 = av_1$ and $w_2 = av_2$, where a is a common initial segment. Then we change the presentation by the Tietze transformation $t' = ta$, which gives $\langle t', x_1, \dots, x_n \mid t'w'_1t'w'_2t'w'_3 = 1\rangle$, where $w'_1 = v_1$, $w'_2 = v_2$ and $w'_3 = a^{-1}w_3$ (after a possible reduction). Then $|w'_1| + |w'_2| + |w'_3| \leq |w_1| + |w_2| + |w_3| - |a|$, contradicting the minimality. For a final segment, or other values of i and j, the argument is very similar. □

Definition 6.2 Given $g_1, g_2, g_3 \in \mathrm{Homeo}_+(\mathbb{R})$, we consider the equation

$$(E) \qquad Xg_1Xg_2Xg_3 = \mathrm{id}$$

in the group $\mathrm{Homeo}_+(\mathbb{R})$. A partial solution is a homeomorphism $h\colon (-\infty, p) \to (-\infty, q)$ such that $hg_1hg_2hg_3(x) = x$ for every x in the domain of the composition.

Remark 6.3 We do not exclude $+\infty$ as a value for p or q in the definition of partial solution. We use the notation $h(p) = q$.

If h is a partial solution of (E) then the domain of the composition $hg_1hg_2hg_3$ is an interval of the form $(-\infty, e_h)$.

Remark 6.4 If $h\colon (-\infty, p) \to (-\infty, h(p))$ and $h_1\colon (-\infty, p_1) \to (-\infty, h_1(p_1))$ are partial solutions of (E) and h_1 extends h (ie $p_1 > p$), then $e_{h_1} > e_h$.

We will need a notion of trajectories for partial solutions of (E). It will also be important to look at the largest point of a trajectory where we apply the partial solution. As we did in the context of representations, we will build partial (and total) solutions via prefixing trajectories.

Definition 6.5 Given (E), a partial solution h and $x < e_h$, we define a sequence

$$\begin{aligned} &S(E, h, x) \\ &\quad = (x, g_3(x), hg_3(x), g_2hg_3(x), hg_2hg_3(x), g_1hg_2hg_3(x), hg_1hg_2hg_3(x) = x). \end{aligned}$$

In turn, for any sequence of the form $S = (a_1, b_1, a_2, b_2, a_3, b_3, a_1) \in \mathbb{R}^7$ we define $d_X S = \max\{b_1, b_2, b_3\}$.

These definitions are very similar to the definition of a trajectory for a representation, given in Section 3, and to Definition 3.1. In fact, when $\rho \in \mathrm{Rep}(G, \mathrm{Homeo}_+(\mathbb{R}))$ and $g_j = \rho(w_j)$, we have that $\rho(t)$ is a solution of (E) and (setting $t = x_{n+1}$, and recalling that $\overline{w} = tw_1tw_2tw_3$) we get $d_{\overline{w}}(S(\rho, \overline{w}, x), n+1) = d_X S(E, \rho(t), x)$.

Notice that $d_X(E, h, x)$ tends to p as x tends to e_h, by our definition of e_h.

Lemma 6.6 *If $h\colon (-\infty, p) \to (-\infty, h(p))$ is a partial solution of (E) with $p < +\infty$, then $h(p) < +\infty$.*

Proof Since h must be defined on $(-\infty, g_3(e_h))$, we have $e_h \le g_3^{-1}(p) < +\infty$. Notice that every entry in $S(E,h,x) = (a_1(x), b_1(x), a_2(x), b_2(x), a_3(x), b_3(x), a_1(x))$ is increasing on $x \in (-\infty, e_h)$. Also $b_j(x) < p$ for every j and $x < e_h$, since h must be defined on $b_j(x)$. Recalling that $d_X S(E,h,x) \to p$ as $x \to e_h$, we see there is some i such that $b_i(x) \to p$ when $x \to e_h$. If $h(p) = +\infty$ then $a_{i+1}(x) = h(b_i(x)) \to +\infty$ as $x \to e_h$. But then $b_{i+1}(x) = g_{i+1}(a_{i+1}(x)) \to +\infty$, which is absurd since $b_j(x) < p$ for every j. □

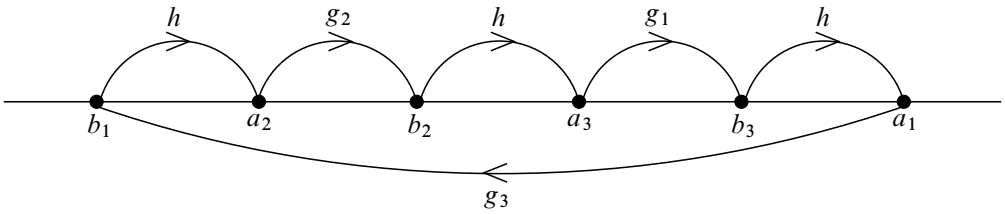


Figure 5: A possible sequence $S(E,h,x)$ that is an instance of Case A in Lemma 6.7

In light of Lemma 6.6 we can assume that partial solutions are defined on $(-\infty, p]$, and so we have $d_X S(E,h,e_h) = p$.

We consider the cyclic permutations of (E),

$$(E') \qquad Xg_3Xg_1Xg_2 = \mathrm{id},$$

$$(E'') \qquad Xg_2Xg_3Xg_1 = \mathrm{id}.$$

Observe that (E), (E') and (E'') share the same partial solutions, though the domains of the composition may be different.

Lemma 6.7 *If $h\colon (-\infty, p] \to (-\infty, h(p)]$ is a partial solution of (E) and there exist i and j such that $g_i(e_h) \ne g_j(e_h)$, then we can extend h to a partial solution h' defined on $(-\infty, p+\epsilon]$ for some $\epsilon > 0$.*

Proof Let $S(E,h,e_h) = (a_1, b_1, a_2, b_2, a_3, b_3, a_1)$. The condition that $g_i(e_h) \ne g_j(e_h)$ for some i and j rules out that $b_1 = b_2 = b_3$. That is because $a_{i+1} = h(b_i)$ (where the indices are taken mod 3), so if $b_1 = b_2 = b_3 = b$ then $b = p$ and $a_1 = a_2 = a_3 = h(p) = e_h$, thus we would have $g_1(e_h) = g_2(e_h) = g_3(e_h) = p$.

Hence we must have one of the following two cases (taking indices mod 3):

Case A $b_j > b_{j\pm1}$ for some j.

Case B $b_j = b_{j-1} > b_{j+1}$ for some j.

Possibly exchanging (E) by a suitable cyclic permutation we can assume that $j = 3$. Thus, $b_3 = d_X S(E, h, e_h) = p$.

Case A Consider the map $\psi = g_1 h g_2 h g_3$ where this composition makes sense. It certainly is defined at $a_1 = e_h$, with $\psi(e_h) = p$. Since the g_i are defined on $\mathbb{R}$ and h is defined up to $p > b_1, b_2$, this composition is also defined up to $e_h + \delta$ for $\delta > 0$. Then $h' = \psi^{-1}$ is defined on $(-\infty, p + \epsilon)$ for some $\epsilon > 0$. It agrees with h on $(-\infty, p]$ since h satisfies (E). It is also a partial solution of (E): if $x < e_{h'}$, then $h' g_1 h' g_2 h' g_3(x) = \psi^{-1} g_1 h g_2 h g_3(x) = x$.

Notice that in this case the local extension h' on $(-\infty, p + \epsilon)$ is unique.

Case B Since $b_3 = b_2$, we get $a_1 = a_3$ by applying h, thus $a_1 = e_h$ is fixed by $h g_1$. Writing $\varphi_0 = h g_1$, which is defined on $(-\infty, e_h]$, we get that $\varphi_0^2 g_1^{-1} g_2 h g_3(x) = x$ for $x < e_h$. If we take $\psi = g_1^{-1} g_2 h g_3$, we see that it is defined on $(-\infty, e_h + \delta]$ for some $\delta > 0$, since $b_1 < p$. Let φ be a square root of ψ^{-1} that agrees with φ_0 up to a_1. This map exists by Lemma 2.6 and Remark 2.7, since a_1 is fixed. Then $h' = \varphi g_1^{-1}$ is the desired extension, which is defined on $(-\infty, p + \epsilon]$ for some $\epsilon > 0$. □

Lemma 6.8 *If $h\colon (-\infty, p) \to (-\infty, h(p))$ is a partial solution of (E) and there exist i and j such that $g_i(x) \neq g_j(x)$ for all $x \geq e_h$, then we can extend h to a solution $h' \in \mathrm{Homeo}_+(\mathbb{R})$.*

Proof A maximal extension of h is an homeomorphism of the line by Lemmas 6.6 and 6.7. □

Proof of Theorem 1.4 Let $<$ be a left order on G with dynamical realization ρ. We will repeat the strategy of the previous theorems, that is, to construct a small perturbation of ρ that has nontrivial stabilizer on the orbit of 0.

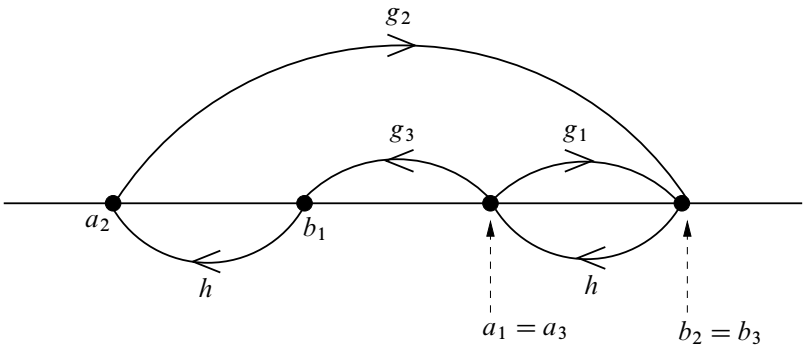


Figure 6: An instance of Case B in Lemma 6.7

Take an arbitrarily large compact interval $K \subseteq \mathbb{R}$. As in the previous theorems, we can find a finite increasing ρ–trajectory S, with respect to the generating set $\{t, x_1, \dots, x_n\}$, that begins at 0 and ends outside K. Recall that each point of S is obtained from the previous one by acting either by $\rho(t)^{\pm 1}$ or by $\rho(x_i)^{\pm 1}$ for some $i \in \{1, \dots, n\}$. As in Theorem 1.3, write $S = \bigcup_{j=1}^{k} S(\rho|_{\mathbb{F}_n}, v_j, p_j)$, where $p_1 = 0$ and p_{j+1} is obtained by acting on $\rho(v_j)(p_j)$ by $\rho(t)^{\pm 1}$. (If v_j is trivial, then $S(\rho|_{\mathbb{F}_n}, v_j, p_j)$ is just the point p_j.)

Let $q = \rho(v_k)(p_k)$ be the last element of S.

Choosing S so that q is large enough, we can assume that $\min S(\rho, \overline{w}, q) > \max K$, where we recall that $\overline{w} = tw_1tw_2tw_3$. We consider, for $i = 1, \dots, n$:

- $d_{ij} = d_{v_j}(S(\rho, v_j, p_j), i)$ for $j = 1, \dots, k$.
- $d_i^{(3)} = d_{w_3}(S(\rho, w_3, q_3), i)$, where $q_3 = q$.
- $d_i^{(2)} = d_{w_2}(S(\rho, w_2, q_2), i)$, where $q_2 = \rho(tw_3)(q)$.
- $d_i^{(1)} = d_{w_1}(S(\rho, w_1, q_1), i)$, where $q_1 = \rho(tw_2tw_3)(q)$.

By Lemma 3.5 there exists i_0 with $d_{i_0}^{(3)} \geq \max_j d_{i_0 j}$. Let $l \in \{1, 2, 3\}$ be such that $d_{i_0}^{(l)} = \max\{d_{i_0}^{(1)}, d_{i_0}^{(2)}, d_{i_0}^{(3)}\}$.

Let $w = w_{l+1}^{-1} w_l$ where $l + 1$ is taken mod 3. Writing $d_i = d_w(S(\rho, w, q_l), i)$, we have that $d_{i_0}^{(l)} \leq d_{i_0}$ since w_l is the final segment of w. That is because the string $w_{l+1}^{-1} w_l$ is reduced, which can be assumed by Lemma 6.1.

We apply Lemma 3.3 to $\rho|_{\mathbb{F}_n}$, setting:

- $v_1, \dots, v_k$ as given above, $v_{k+1} = w_1$, $v_{k+2} = w_2$, $v_{k+3} = w_3$ and $w = w_{l+1}^{-1} w_l$ (also as above).
- $p_1 \dots, p_k$ as given above, $p_{k+1} = q_1$, $p_{k+2} = q_2$, $p_{k+3} = q_3$ and $p = q_l$.

The hypotheses of Lemma 3.3 hold by the previous discussion. Let

$$\rho' \in \mathrm{Rep}(\mathbb{F}_n, \mathrm{Homeo}_+(\mathbb{R}))$$

be the representation given by Lemma 3.3.

Let $g_j = \rho'(w_j)$ for $j = 1, 2, 3$. We consider the equation

$$(E) \qquad Xg_1Xg_2Xg_3 = \mathrm{id}.$$

Lemma 3.3(1) guarantees that g_j agrees with $\rho(w_j)$ up to q_j. Thus, $\rho(t)$ is a partial solution of (E) when restricted to $(-\infty, z)$, where $z = \max\{g_1(q_1), g_2(q_2), g_3(q_3)\}$.

Let $h = \rho(t)|_{(-\infty,z)}$. We consider a cyclic permutation of (E) such that $e_h = q_l$ (for (E) itself we have $e_h = q$).

By Lemma 3.3(3) we have $g_l(x) \neq g_{l+1}(x)$ for all $x \geq q_l$. This allows us to apply Lemma 6.8, so h can be extended to a maximal solution $\psi \in \mathrm{Homeo}_+(\mathbb{R})$. This gives rise to a representation $\overline{\rho} \in \mathrm{Rep}(G, \mathrm{Homeo}_+(\mathbb{R}))$ with $\overline{\rho}|_{\mathbb{F}_n} = \rho'$ and $\overline{\rho}(t) = \psi$.

This $\overline{\rho}$ is a small perturbation of ρ because of Lemma 3.3(1) and the fact that ψ agrees with $\rho(t)$ on $(-\infty, z]$. Also by this fact and Lemma 3.3(2) we see that neither S nor $S(\rho, \overline{w}, q)$ are affected by this perturbation, so $p = q_l$ is in the orbit of 0 under $\overline{\rho}$. Then we use Lemma 3.3(4) to conclude that the stabilizer of 0 under $\overline{\rho}$ is nontrivial. Thus, Proposition 2.2 finishes the proof. □

References

[1] **J Alonso**, **J Brum**, **C Rivas**, *Orderings and flexibility of some subgroups of* $\mathrm{Homeo}_+(\mathbb{R})$, J. Lond. Math. Soc. 95 (2017) 919–941 MR

[2] **G Baumslag**, *Topics in combinatorial group theory*, Birkhäuser, Basel (1993) MR

[3] **V V Bludov**, **A M W Glass**, *On free products of right ordered groups with amalgamated subgroups*, Math. Proc. Cambridge Philos. Soc. 146 (2009) 591–601 MR

[4] **S D Brodskiĭ**, *Equations over groups, and groups with one defining relation*, Sibirsk. Mat. Zh. 25 (1984) 84–103 MR In Russian; translated in Siberian Math. J. 25 (1984) 235–251

[5] **A Clay**, **D Rolfsen**, *Ordered groups and topology*, Graduate Studies in Mathematics 176, Amer. Math. Soc., Providence, RI (2016) MR

[6] **P Dehornoy**, *Monoids of O–type, subword reversing, and ordered groups*, J. Group Theory 17 (2014) 465–524 MR

[7] **B Deroin**, **A Navas**, **C Rivas**, *Groups, orders, and dynamics*, preprint (2014) arXiv

[8] **E Ghys**, *Groups acting on the circle*, Enseign. Math. 47 (2001) 329–407 MR

[9] **T Ito**, *Dehornoy-like left orderings and isolated left orderings*, J. Algebra 374 (2013) 42–58 MR

[10] **T Ito**, *Construction of isolated left orderings via partially central cyclic amalgamation*, Tohoku Math. J. 68 (2016) 49–71 MR

[11] **V M Kopytov**, **N Y Medvedev**, *Right-ordered groups*, Consultants Bureau, New York (1996) MR

[12] **P A Linnell**, *The space of left orders of a group is either finite or uncountable*, Bull. Lond. Math. Soc. 43 (2011) 200–202 MR

[13] **D Malicet**, **K Mann**, **C Rivas**, **M Triestino**, *Ping-pong configurations and circular orders on free groups*, preprint (2017) arXiv

[14] **A Navas**, *On the dynamics of (left) orderable groups*, Ann. Inst. Fourier (Grenoble) 60 (2010) 1685–1740 MR

[15] **A Navas**, *A remarkable family of left-ordered groups: central extensions of Hecke groups*, J. Algebra 328 (2011) 31–42 MR

[16] **C Rivas**, *Left-orderings on free products of groups*, J. Algebra 350 (2012) 318–329 MR

[17] **C Rivas**, **R Tessera**, *On the space of left-orderings of virtually solvable groups*, Groups Geom. Dyn. 10 (2016) 65–90 MR

[18] **A S Sikora**, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. 36 (2004) 519–526 MR

Facultad de Ciencias, Universidad de la República
Montevideo, Uruguay

Facultad de Ingenieria, Universidad de la República
Montevideo, Uruguay

juan@cmat.edu.uy, joaquinbrum@fing.edu.uy

Received: 17 December 2017 Revised: 20 July 2018

msp

Algebraic & Geometric Topology 18 (2018) 4187–4274

On the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained by integral surgery along the figure-eight knot

TOMOTADA OHTSUKI

It is known that the quantum SU(2) invariant of a closed 3–manifold at $q = \exp(2\pi\sqrt{-1}/N)$ is of polynomial order as $N \to \infty$. Recently, Chen and Yang conjectured that the quantum SU(2) invariant of a closed hyperbolic 3–manifold at $q = \exp(4\pi\sqrt{-1}/N)$ is of order $\exp(N \cdot \varsigma(M))$, where $\varsigma(M)$ is a normalized complex volume of M. We can regard this conjecture as a kind of "volume conjecture", which is an important topic from the viewpoint that it relates quantum topology and hyperbolic geometry.

In this paper, we give a concrete presentation of the asymptotic expansion of the quantum SU(2) invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained from the 3–sphere by integral surgery along the figure-eight knot. In particular, the leading term of the expansion is $\exp(N \cdot \varsigma(M))$, which gives a proof of the Chen–Yang conjecture for such 3–manifolds. Further, the semiclassical part of the expansion is a constant multiple of the square root of the Reidemeister torsion for such 3–manifolds. We expect that the higher-order coefficients of the expansion would be "new" invariants, which are related to "quantization" of the hyperbolic structure of a closed hyperbolic 3–manifold.

57M27; 57M50

1. Introduction 4188
2. Physical background 4191
3. Preliminaries 4194
4. The quantum SU(2) invariant 4199
5. Proof of Theorem 1.1 when $|p| \geq 6$ 4201
6. Contributions from Δ_1 and Δ_2 4218
7. Proof of Theorem 1.1 when $|p| = 5$ 4227

Appendix A. Gauss sum 4238
Appendix B. Critical points of $\widehat{V}(t, s)$ 4240

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4187

Appendix C. The hyperbolic structure of M_p 4249
Appendix D. Estimate of the hyperbolic volume of M_p 4254
Appendix E. Proof of Lemma 5.1 4257
Appendix F. Restriction of Δ_0 to Δ_0'' 4261
Appendix G. Proof of Lemma 7.1 4269
References 4272

1 Introduction

In the late 1980s, Witten [26] proposed topological invariants of a closed 3–manifold M for a simple compact Lie group G, what we call the quantum G invariant. This invariant is formally presented by a path integral whose Lagrangian is the Chern–Simons functional of G connections on M. There are two approaches to obtain mathematically rigorous information from a path integral: the operator formalism and the perturbative expansion. Motivated by the operator formalism of the Chern–Simons path integral, we obtain a rigorous mathematical construction of quantum invariants by using linear sums of quantum invariants of links. In particular, the quantum SU(2) invariant $\tau_N(M;q)$ of a closed 3–manifold M is defined to be a linear sum of the colored Jones polynomials $J_n(K;q)$ of a link L at q, where q is a primitive N^{th} root of unity, and L is a link such that M is obtained from S^3 by an integral surgery along L; for details, see eg a book of the author [17]. We note that, as mentioned in [17], when N is odd, $\tau_N(M;q)$ can be defined at $q=e^{4\pi\sqrt{-1}/N}$; we denote it by $\widehat{\tau}_N(M)$.

The volume conjecture of Kashaev, Murakami and Murakami [9; 14] is an important topic, which relates quantum topology and hyperbolic geometry. A complexification of the volume conjecture of Murakami, Murakami, Okamoto, Takata and Yokota [15] states that, for a hyperbolic knot K, $(2\pi\sqrt{-1}/N)\log J_N(K;q^{2\pi\sqrt{-1}/N})$ goes to a (normalized) complex volume of K as $N\to\infty$. Further, it has been expected by Murakami [13] that the quantum invariant of a closed 3–manifold has a similar property, though it is known that $\tau_N(M;q)$ at $q=e^{2\pi\sqrt{-1}}$ is of polynomial order as $N\to\infty$. Recently, Chen and Yang [2] observed that $\widehat{\tau}_N(M)$ is of exponential order as $N\to\infty$ for some hyperbolic 3–manifolds obtained by surgery along the figure-eight knot and the 5_2 knot, and conjectured that

$$4\pi\sqrt{-1}\cdot\lim_{\substack{N\to\infty\\ N\ \text{odd}}}\frac{\log\widehat{\tau}_N(M)}{N}=\mathrm{cs}(M)+\sqrt{-1}\,\mathrm{vol}(M)$$

for a closed hyperbolic 3–manifold M, where $\mathrm{cs}(M)$ and $\mathrm{vol}(M)$ denote the Chern–Simons invariant and the hyperbolic volume of M, respectively. We define a normalized *complex volume* of M by

$$\varsigma(M) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)).$$

From the viewpoint of mathematical physics, we can regard this conjecture as a perturbative expansion of the Chern–Simons path integral; see Section 2, for details.

Let p be an integer, and let M_p be the 3–manifold obtained from S^3 by p surgery along the figure-eight knot. It is known that M_p is hyperbolic if and only if $|p| > 4$. The aim of this paper is to show the following theorem, as a refinement of the above-mentioned conjecture:

Theorem 1.1 *Let N be an odd integer ≥ 3, and let p be an integer with $|p| > 4$. Then the quantum invariant $\widehat{\tau}_N(M_p)$ of M_p is expanded as $N \to \infty$ in the form*

$$\widehat{\tau}_N(M_p) = (-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{\mathrm{sign}(p)(N-3)/2} e^{N\varsigma(M_p)} N^{3/2}\omega(M_p) \times\left(1 + \sum_{i=1}^{d}\kappa_i(M_p)\cdot\left(\frac{4\pi\sqrt{-1}}{N}\right)^i + O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any integer $d \geq 1$, where $\omega(M_p)$ and $\kappa_i(M_p)$ are constants determined by M_p.

We conjecture that, similarly to in the theorem, $\widehat{\tau}_N(M)$ of any closed hyperbolic 3–manifold can be expanded in the form

$$\widehat{\tau}_N(M) = (\text{some root of unity}) e^{N\varsigma(M)} N^{3/2}\omega(M) \times\left(1 + \sum_{i=1}^{d}\kappa_i(M)\cdot\left(\frac{4\pi\sqrt{-1}}{N}\right)^i + O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any integer $d \geq 1$, with some constants $\omega(M)$ and $\kappa_i(M)$ determined by M.

We can numerically observe, for example, the behavior of $\widehat{\tau}_N(M_8)$, as follows:

N	$\widehat{\tau}_N(M_8)\sqrt{-1}^{(p/2)N-(N-3)/2} e^{-N\varsigma(M_8)} N^{-3/2}$
101	$0.0033167246\ldots - \sqrt{-1}\cdot 0.0219539338\ldots$
201	$0.0050414223\ldots - \sqrt{-1}\cdot 0.0215864601\ldots$
501	$0.0060677858\ldots - \sqrt{-1}\cdot 0.0213013492\ldots$
1001	$0.0064080099\ldots - \sqrt{-1}\cdot 0.0211954944\ldots$

Here, $\varsigma(M_8)$ and $\omega(M_8)$ are numerically given by

$$\begin{aligned}\varsigma(M_8) &= 0.1259843998\ldots - \sqrt{-1}\cdot 0.0858243597\ldots \\ &= \frac{1.0785007120\ldots + \sqrt{-1}\cdot 1.5831666606\ldots}{4\pi\sqrt{-1}}, \\ \omega(M_8) &= 0.0067471463\ldots - \sqrt{-1}\cdot 0.0210842217\ldots,\end{aligned}$$

and we can numerically observe that the complex numbers in the above table converge to $\omega(M_8)$ as $N \to \infty$.

The values of $\omega(M_p)$ are numerically given for some p, as follows:

p	$\omega(M_p)$
5	$0.0081594261\ldots - \sqrt{-1}\cdot 0.0558388944\ldots$
6	$0.0078610660\ldots - \sqrt{-1}\cdot 0.0356626288\ldots$
7	$0.0072993993\ldots - \sqrt{-1}\cdot 0.0265443774\ldots$
8	$0.0067471463\ldots - \sqrt{-1}\cdot 0.0210842217\ldots$
9	$0.0062386239\ldots - \sqrt{-1}\cdot 0.0173836407\ldots$

It is shown by the author and Takata in [21] that $\omega(M_p)^2$ is equal to a constant multiple of the twisted Reidemeister torsion of M_p. We note that a similar statement holds for the asymptotic expansion of the Kashaev invariant for the two-bridge knots, as shown by the author and Takata [20]. We also expect that $\kappa_i(M)$ are new invariants of a closed hyperbolic 3–manifold M.

Remark 1.2 There has been recent progress on the Chen–Yang conjectures. Chen and Yang [2] (whose first version was written in March 2015) gave two conjectures based on numerical observations: one is the "volume conjecture" for the quantum SU(2) invariant (the Reshetikhin–Turaev invariant) for closed 3–manifolds, and the other is the "volume conjecture" for the Turaev–Viro invariant for closed or cusped 3–manifolds. Since it is known (see eg a book of Turaev [25, Section VII.4]) that the Turaev–Viro invariant for closed 3–manifolds is determined from the Reshetikhin–Turaev invariant, the Chen–Yang conjecture for the Turaev–Viro invariant for closed 3–manifolds is a consequence of the Chen–Yang conjecture for the Reshetikhin–Turaev invariant. This paper (whose first version was written in August 2016) gives a rigorous proof for the Chen–Yang conjecture for the Reshetikhin–Turaev invariant for closed hyperbolic 3–manifolds obtained by integral surgery along the figure-eight knot. Further, there has been recent progress on the Chen–Yang conjecture for the Turaev–Viro invariant

for some cusped 3–manifolds. We note that the Turaev–Viro invariant for a cusped 3–manifold is defined for an ideal triangulation of the cusped 3–manifold, and it might not be known so far whether there is a direct relation between this invariant and the Reshetikhin–Turaev invariant. Recently, Detcherry, Kalfagianni and Yang [3] have proved the Chen–Yang conjecture for the Turaev–Viro invariant for the complements of the figure-eight knot and the Borromean rings. Further, Belletti, Detcherry, Kalfagianni and Yang [1] have proved the Chen–Yang conjecture for the Turaev–Viro invariant for the complements of infinitely many hyperbolic links called "fundamental shadow links".

The paper is organized as follows. In Section 2, we explain a physical background of this topic. In Section 3, we explain preliminaries, which we need in the proof of Theorem 1.1. In Section 4, we review the definition of the quantum SU(2) invariant of closed 3–manifolds, and give a concrete presentation of the value of the quantum SU(2) invariant of M_p. In Sections 5 and 7, we give a proof of Theorem 1.1, when $|p| \geq 6$ and when $|p| = 5$, respectively. In Section 6, we show some propositions, which we need in the proof of Theorem 1.1. In the appendices, we show some lemmas, which we use in the proof of Theorem 1.1.

The author would like to thank Qingtao Chen, Stavros Garoufalidis, Kazuo Habiro, Rinat Kashaev, Akishi Kato, Sadayoshi Kojima, Hitoshi Murakami, Jun Murakami, Toshie Takata, Akihiro Tsuchiya, Tian Yang, Yoshiyuki Yokota and Don Zagier for helpful comments and discussions. The author would also like to thank the referee for helpful comments. The author is partially supported by JSPS KAKENHI Grant Numbers JP16H02145 and JP16K13754.

2 Physical background

A physical background of quantum invariants of 3–manifolds is the Chern–Simons field theory of Witten [26]; for details, see eg a book of the author [17]. Further, we can regard the volume conjecture as a perturbative expansion of the Chern–Simons path integral. We explain these in this section.

Let M be an oriented closed 3–manifold. Let $\mathcal{A}$ denote the set of connections on the trivial SU(2) bundle $\mathrm{SU}(2) \times M \to M$. We identify $\mathcal{A}$ with the set $\Omega^1(M; \mathfrak{sl}_2)$ of $\mathfrak{sl}_2$–valued 1–forms on M. The Chern–Simons functional $\mathrm{CS}\colon \mathcal{A} \to \mathbb{R}$ is defined by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2} \int_M \mathrm{trace}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big)$$

for a connection A. The gauge group $\mathcal{G}$ is the group of automorphisms of the bundle $\mathrm{SU}(2)\times M\to M$, and it is known that, when $g\in\mathcal{G}$ takes $A\in\mathcal{A}$ to g^*A, $\mathrm{CS}(g^*A)$ differs from $\mathrm{CS}(A)$ by an integer. Hence, the Chern–Simons functional induces the map $\mathrm{CS}\colon \mathcal{A}/\mathcal{G}\to\mathbb{R}/\mathbb{Z}$. The *Chern–Simons path integral* is formally given by

$$Z_N(M)=\int_{\mathcal{A}/\mathcal{G}}\exp(2\pi\sqrt{-1}\,N\,\mathrm{CS}(A))\,\mathcal{D}A \tag{1}$$

for any positive integer N. We note that, since $\mathcal{A}/\mathcal{G}$ is infinite-dimensional, the path integral has not been defined mathematically, but the Chern–Simons path integral gives many interesting suggestions to mathematics. Witten [26] proposed that this formal integral $Z_N(M)$ gives a topological invariant of a closed 3–manifold M; this is a physical background of the quantum invariant of M. In physics, there are two approaches available to obtain observables of a path integral: the operator formalism or the perturbative expansion.

The operator formalism induces a formulation of the invariant in terms of the topological quantum field theory; that is, we can compute the invariant by cutting the 3–manifold along surfaces. In particular, we can formulate the invariant of a closed 3–manifold M as a linear sum of quantum invariants of a link L such that M is obtained from S^3 by an integral surgery along L. In Section 4, we review a mathematical definition of the quantum $\mathrm{SU}(2)$ invariant of closed 3–manifolds along this formulation.

The volume conjecture is a recent important topic, which relates quantum topology and hyperbolic geometry. We briefly review the volume conjecture, and explain the volume conjecture from the viewpoint of the perturbative expansion of the Chern–Simons path integral, in this paragraph. Kashaev [7; 8] defined the Kashaev invariant $\langle L\rangle_N\in\mathbb{C}$ of a link L for $N=2,3,\dots$ by using the quantum dilogarithm. He conjectured [9] that, for any hyperbolic link L, $\frac{2\pi}{N}\log|\langle L\rangle_N|$ goes to the hyperbolic volume of S^3-L as $N\to\infty$. In 1999, H Murakami and J Murakami [14] proved that the Kashaev invariant $\langle L\rangle_N$ of any link L is equal to the colored Jones polynomial $J_N(L;e^{2\pi\sqrt{-1}/N})$ of L at $e^{2\pi\sqrt{-1}/N}$, and conjectured that, for any knot K, $\frac{2\pi}{N}\log|J_N(K;e^{2\pi\sqrt{-1}/N})|$ goes to a normalized simplicial volume of S^3-K, as an extension of Kashaev's conjecture. This is called *the volume conjecture*. As a complexification of the volume conjecture, it is conjectured by Murakami, Murakami, Okamoto, Takata and Yokota [15] that, for a hyperbolic link L,

$$2\pi\sqrt{-1}\cdot\lim_{N\to\infty}\frac{J_N(L;e^{2\pi\sqrt{-1}/N})}{N}=\mathrm{cs}(S^3-L)+\sqrt{-1}\,\mathrm{vol}(S^3-L),$$

where "cs" and "vol" denote the Chern–Simons invariant and the hyperbolic volume, respectively. We define a normalized complex volume by

$$\varsigma(L) = \frac{1}{2\pi\sqrt{-1}}(\mathrm{cs}(S^3 - L) + \sqrt{-1}\,\mathrm{vol}(S^3 - L)).$$

Further, as a refinement of the above conjecture, it is shown by the author and Yokota [18; 22; 19] that, for hyperbolic knots K with up to seven crossings, the asymptotic expansion of the Kashaev invariant is presented by the form

$$\langle K \rangle_N = e^{N\varsigma(K)} N^{3/2} \omega(K) \cdot \left(1 + \sum_{i=1}^{d} \kappa_i(K) \cdot \left(\frac{2\pi\sqrt{-1}}{N}\right)^i + O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any integer $d \geq 1$, where $\omega(K)$ and the $\kappa_i(K)$ are some scalars determined by K. As for the quantum invariant of closed 3–manifolds, it has been expected by Murakami [13] that the quantum invariant of a closed 3–manifold has a similar property, though it is known that $\tau_N(M;q)$ at $q = e^{2\pi\sqrt{-1}}$ is of polynomial order as $N \to \infty$. As we mentioned in the introduction, recently, Chen and Yang [2] observed that $\hat{\tau}_N(M)$ is of exponential order as $N \to \infty$ for some hyperbolic 3–manifolds, and conjectured that

$$4\pi\sqrt{-1} \cdot \lim_{\substack{N\to\infty \\ N \text{ odd}}} \frac{\log \hat{\tau}_N(M)}{N} = \mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)$$

for a closed hyperbolic 3–manifold M. We define a normalized complex volume of M by

$$\varsigma(M) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{cs}(M) + \sqrt{-1}\,\mathrm{vol}(M)).$$

We can regard this conjecture as a perturbative expansion of the Chern–Simons path integral, as follows. Let $\mathcal{A}_{\mathbb{C}}$ be the set of connections on the trivial $\mathrm{SL}(2;\mathbb{C})$ bundle on a closed 3–manifold M. We identify $\mathcal{A}_{\mathbb{C}}$ with the set $\Omega^1(M;\mathfrak{sl}_2\mathbb{C})$ of $\mathfrak{sl}_2\mathbb{C}$–valued 1–forms on M. The gauge group $\mathcal{G}_{\mathbb{C}}$ is the group of automorphisms of the bundle $\mathrm{SL}(2;\mathbb{C}) \times M \to M$. For a closed hyperbolic 3–manifold M, we apply the saddle-point method formally to the integral (1), by moving the domain $\mathcal{A}/\mathcal{G}$ in $\mathcal{A}_{\mathbb{C}}/\mathcal{G}_{\mathbb{C}}$ in such a way that the new domain contains the $\mathrm{SL}(2;\mathbb{C})$ flat connection corresponding to the holonomy representation of the hyperbolic structure of M. By expanding the path integral at this flat connection, we obtain the complex volume in the leading term. Further, in the second term (the part of the semiclassical limit), we obtain the Reidemeister torsion as the determinant of the quadratic part of the Chern–Simons functional (see Witten [26]). Furthermore, in the higher-order terms, we obtain a power

series in $\frac{1}{N}$ by coupling the quadratic part and the higher-order part of the Chern–Simons functional (see eg a book of the author [17]). Hence, we can expect that the quantum invariant is expanded in such a form, and this is a physical background of the expansion of Theorem 1.1. We note that this expansion is obtained from contributions of a neighborhood of the flat $\mathrm{SL}(2,\mathbb{C})$ connection corresponding to the holonomy representation of the hyperbolic structure of M, and we can regard them as perturbations of the hyperbolic structure of M. So we expect that the higher-order coefficients $\kappa_i(M)$ in the expansion of Theorem 1.1 might be related to "quantization" of the hyperbolic structure of M in some sense.

3 Preliminaries

3.1 Integral presentation of $(q)_n$

In this section, we review integral presentations of $(q)_n$ and some of their properties.

We put

$$(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$$

for $n \geq 0$.

Let N be an integer ≥ 3. It is known [5; 27] that

$$\begin{aligned}(e^{2\pi\sqrt{-1}/N})_n &= \exp\Big(\varphi\Big(\frac{1}{2N}\Big) - \varphi\Big(\frac{2n+1}{2N}\Big)\Big),\\ (e^{-2\pi\sqrt{-1}/N})_n &= \exp\Big(\varphi\Big(1-\frac{2n+1}{2N}\Big) - \varphi\Big(1-\frac{1}{2N}\Big)\Big).\end{aligned} \tag{2}$$

Here, following Faddeev [4], we define a holomorphic function $\varphi(t)$ on the domain $\{t \in \mathbb{C} \mid 0 < \mathrm{Re}\, t < 1\}$ by

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{e^{(2t-1)x}dx}{4x\sinh x \sinh\frac{x}{N}}.$$

We note that this integrand has poles at $n\pi\sqrt{-1}$ $(n \in \mathbb{Z})$, and, to avoid the pole at 0, we choose the contour of the integral

$$\gamma = (-\infty,-1] \cup \{z \in \mathbb{C} \mid |z| = 1,\ \mathrm{Im}\, z \geq 0\} \cup [1,\infty).$$

Further, it is known (due to Kashaev — see [18]) that

$$\varphi\Big(\frac{1}{2N}\Big) - \varphi\Big(1-\frac{1}{2N}\Big) = \log N. \tag{3}$$

Furthermore, it is known [18] that

$$\varphi(t) + \varphi(1-t) = 2\pi\sqrt{-1}\Big(-\frac{N}{2}\Big(t^2 - t + \frac{1}{6}\Big) + \frac{1}{24N}\Big) \tag{4}$$

for $0 < \operatorname{Re} t < 1$. Moreover,

$$\frac{1}{N}\varphi(t) \quad \text{uniformly converges to} \quad \frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}t}) \tag{5}$$

in the domain

$$\{t \in \mathbb{C} \mid \delta \le \operatorname{Re} t \le 1-\delta,\ |\operatorname{Im} t| \le M\} \tag{6}$$

for any sufficiently small $\delta > 0$ and any $M > 0$.

Let N be an odd integer ≥ 3. We put $q = e^{4\pi\sqrt{-1}/N}$. It follows from (2), replacing N with $\frac{N}{2}$, that

$$\begin{aligned}(q)_n &= \exp\Big(\widehat{\varphi}\Big(\frac{1}{N}\Big) - \widehat{\varphi}\Big(\frac{2n+1}{N}\Big)\Big),\\ (\overline{q})_n &= \exp\Big(\widehat{\varphi}\Big(1 - \frac{2n+1}{N}\Big) - \widehat{\varphi}\Big(1 - \frac{1}{N}\Big)\Big)\end{aligned} \tag{7}$$

for $0 \le n < \frac{N}{2}$, where we define $\widehat{\varphi}(t)$ on $\{t \in \mathbb{C} \mid 0 < \operatorname{Re} t < 1\}$ by

$$\widehat{\varphi}(t) = \int_{-\infty}^{\infty} \frac{e^{(2t-1)x}\,dx}{4x \sinh x \sinh \frac{2x}{N}}.$$

We note that (7) holds only for $0 \le n < \frac{N}{2}$ since $\widehat{\varphi}(t)$ is defined for $0 < \operatorname{Re} t < 1$, though $(q)_n$ and $(\overline{q})_n$ themselves are defined for $0 \le n < N$. Further, it follows from (3) and (4) that

$$\widehat{\varphi}\Big(\frac{1}{N}\Big) - \widehat{\varphi}\Big(1 - \frac{1}{N}\Big) = \log\frac{N}{2}, \tag{8}$$

$$\widehat{\varphi}(t) + \widehat{\varphi}(1-t) = 2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big(t^2 - t + \frac{1}{6}\Big) + \frac{1}{12N}\Big) \tag{9}$$

for $0 < \operatorname{Re} t < 1$. Furthermore, it follows from (5) that

$$\frac{1}{N}\widehat{\varphi}(t) \quad \text{uniformly converges to} \quad \frac{1}{4\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t}) \tag{10}$$

in the domain (6). Moreover, we have the following lemma, which is a modification of a formula of Murakami and Murakami [14]:

Lemma 3.1 (modification of a formula in [14]) *We have that*

$$(q)_n(\overline{q})_{N-n-1} = N \tag{11}$$

for $0 \le n < N$.

Proof We have that

$$\begin{aligned}(q)_n(\overline{q})_{N-n-1} &= (1-q)(1-q^2)\cdots(1-q^n)\times(1-\overline{q}^{N-n-1})(1-\overline{q}^{N-n-2})\cdots(1-\overline{q})\\ &= (1-q)(1-q^2)\cdots(1-q^{N-1}) = (q)_{N-1}.\end{aligned}$$

Hence, it is sufficient to show that $(q)_{N-1} = N$. We have that

$$\begin{aligned}(q)_{N-1} &= \prod_{\substack{j\in\mathbb{Z}/N\mathbb{Z}\\ j\neq 0}}(1-e^{(4\pi\sqrt{-1}/N)\cdot j}) = \prod_{\substack{k\in\mathbb{Z}/N\mathbb{Z}\\ k\neq 0}}(1-e^{(2\pi\sqrt{-1}/N)\cdot k})\\ &= \prod_{1\le k\le (N-1)/2}(1-e^{(2\pi\sqrt{-1}/N)\cdot k})(1-e^{-(2\pi\sqrt{-1}/N)\cdot k})\\ &= \prod_{1\le k\le (N-1)/2}|1-e^{(2\pi\sqrt{-1}/N)\cdot k}|\cdot|1-e^{-(2\pi\sqrt{-1}/N)\cdot k}|\\ &= \prod_{1\le k\le N-1} 2\sin\frac{k\pi}{N} = N,\end{aligned}$$

where we obtain the second equality by replacing k with $2j$, and obtain the last equality by a formula in the proof of Proposition 4.2 of [14]. Therefore, we obtain the lemma. □

3.2 The saddle-point method

In this section, we review a proposition obtained from the saddle-point method.

Proposition 3.2 (see [18]) *Let A be a nonsingular symmetric complex 2×2 matrix, and let $\psi(z_1,z_2)$ and $r(z_1,z_2)$ be holomorphic functions of the forms*

$$\begin{aligned}\psi(z_1,z_2) &= \boldsymbol{z}^T A\boldsymbol{z} + r(z_1,z_2),\\ r(z_1,z_2) &= \sum_{i,j,k} b_{ijk}z_iz_jz_k + \sum_{i,j,k,l} c_{ijkl}z_iz_jz_kz_l + \cdots,\end{aligned}\tag{12}$$

defined in a neighborhood of $\mathbf{0}\in\mathbb{C}^2$. Suppose that the restriction of the domain

$$\{(z_1,z_2)\in\mathbb{C}^2 \mid \operatorname{Re}\psi(z_1,z_2)<0\}\tag{13}$$

to a neighborhood of $\mathbf{0}\in\mathbb{C}^2$ is homotopy equivalent to S^1. Let D be an oriented disk embedded in $\mathbb{C}^2$ such that ∂D is included in the domain (13) *whose inclusion is*

homotopic to a homotopy equivalence to the above S^1 *in the domain* (13)*. Then*

$$\int_D e^{N\psi(z_1,z_2)} dz_1\, dz_2 = \frac{\pi}{N\sqrt{\det(-A)}}\left(1 + \sum_{i=1}^{d} \frac{\lambda_i}{N^i} + O\left(\frac{1}{N^{d+1}}\right)\right)$$

for any d*, where we choose the sign of* $\sqrt{\det(-A)}$ *as explained in* [18]*, and the* λ_i *are constants presented by using coefficients of the expansion of* $\psi(z_1, z_2)$*; such presentations are obtained by formally expanding the formula*

$$1 + \sum_{i=1}^{\infty} \frac{\lambda_i}{N^i} = \exp\left(Nr\left(\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}\right)\right) \exp\left(-\frac{1}{4N}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}^T A^{-1} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)\Bigg|_{w_1=w_2=0}.$$

For a proof of the proposition, see [18].

Remark 3.3 As mentioned in [18, Remark 3.6], we can extend Proposition 3.2 to the case where $\psi(z_1, z_2)$ depends on N in such a way that $\psi(z_1, z_2)$ is of the form

$$\psi(z_1, z_2) = \psi_0(z_1, z_2) + \psi_1(z_1, z_2)\frac{1}{N} + \psi_2(z_1, z_2)\frac{1}{N^2} + \cdots + \psi_m(z_1, z_2)\frac{1}{N^m} + r_m(z_1, z_2)\frac{1}{N^{m+1}},$$

where the $\psi_i(z_1, z_2)$ are holomorphic functions independent of N, and we assume that $\psi_0(z_1, z_2)$ satisfies the assumption of the proposition and $|r_m(z_1, z_2)|$ is bounded by a constant which is independent of N.

3.3 The Poisson summation formula

In this section, we review the Poisson summation formula and a proposition obtained from it.

Recall (see eg the book of Stein and Weiss [23]) that the Poisson summation formula states that

$$\sum_{\boldsymbol{m}\in\mathbb{Z}^n} f(\boldsymbol{m}) = \sum_{\boldsymbol{m}\in\mathbb{Z}^n} \widehat{f}(\boldsymbol{m}) \tag{14}$$

for a continuous integrable function f on $\mathbb{R}^n$ which satisfies that

$$|f(\boldsymbol{z})| \le C(1 + |\boldsymbol{z}|)^{-n-\delta}, \quad |\widehat{f}(\boldsymbol{z})| \le C(1 + |\boldsymbol{z}|)^{-n-\delta} \tag{15}$$

for some $C, \delta > 0$, where $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(\boldsymbol{w}) = \int_{\mathbb{R}^n} f(\boldsymbol{z}) e^{-2\pi\sqrt{-1}\,\boldsymbol{w}^T\boldsymbol{z}}\, d\boldsymbol{z}.$$

The following proposition is obtained from the Poisson summation formula:

Proposition 3.4 (see [18]) *For $(c_1, c_2) \in \mathbb{C}^2$ and an oriented disk D' in $\mathbb{R}^2$, we put*

$$\Lambda = \left\{ \left(\frac{i}{N} + c_1, \frac{j}{N} + c_2 \right) \in \mathbb{C}^2 \mid i, j \in \mathbb{Z},\ \left(\frac{i}{N}, \frac{j}{N} \right) \in D' \right\},$$
$$D = \{(t + c_1, s + c_2) \in \mathbb{C}^2 \mid (t, s) \in D' \subset \mathbb{R}^2\}.$$

Let $\psi(t, s)$ be a holomorphic function defined in a neighborhood of $\mathbf{0} \in \mathbb{C}^2$ including D. We assume that ∂D is included in the domain

$$\{(t, s) \in \mathbb{C}^2 \mid \operatorname{Re} \psi(t, s) < -\varepsilon_0\}$$

for some $\varepsilon_0 > 0$. Further, we assume that ∂D is null-homotopic in each of the domains

$$\{(t + \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t + \delta\sqrt{-1}, s) < 2\pi\delta\}, \tag{16}$$

$$\{(t - \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t - \delta\sqrt{-1}, s) < 2\pi\delta\}, \tag{17}$$

$$\{(t, s + \delta\sqrt{-1}) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t, s + \delta\sqrt{-1}) < 2\pi\delta\}, \tag{18}$$

$$\{(t, s - \delta\sqrt{-1}) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t, s - \delta\sqrt{-1}) < 2\pi\delta\}. \tag{19}$$

Then

$$\frac{1}{N^2} \sum_{(t,s) \in \Lambda} e^{N\psi(t,s)} = \int_D e^{N\psi(t,s)}\, dt\, ds + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

Remark 3.5 In the assumption of the proposition, if we use the formula

$$\{(t + \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t + \delta\sqrt{-1}, s) < 4\pi\delta\} \tag{20}$$

instead of (16), then the following formula holds instead of the resulting formula of the proposition:

$$\frac{1}{N^2} \sum_{(t,s) \in \Lambda} e^{N\psi(t,s)} = \int_D e^{N\psi(t,s)}\, dt\, ds + \int_D e^{N(\psi(t,s) + 2\pi\sqrt{-1}t)}\, dt\, ds + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

Remark 3.6 In the assumption of the proposition, if we use (20) and the formula

$$\{(t - \delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t, s) \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t - \delta\sqrt{-1}, s) < 4\pi\delta\}, \tag{21}$$

instead of (16) and (17), then the following formula holds instead of the resulting formula of the proposition:

$$\frac{1}{N^2}\sum_{(t,s)\in\Lambda} e^{N\psi(t,s)} = \int_D e^{N(\psi(t,s)-2\pi\sqrt{-1}\,t)}dt\,ds + \int_D e^{N\psi(t,s)}dt\,ds + \int_D e^{N(\psi(t,s)+2\pi\sqrt{-1}\,t)}dt\,ds + O(e^{-N\varepsilon})$$

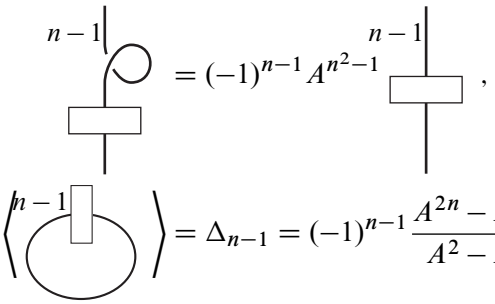
for some $\varepsilon > 0$.

4 The quantum SU(2) invariant

In this section, we review the definition of the quantum SU(2) invariant, and calculate it for the 3–manifold M_p obtained from S^3 by p surgery along the figure-eight knot K_{4_1} for a positive integer p.

Let N be an odd integer ≥ 3. We review the definition of the quantum SU(2) invariant following the notation of Lickorish [12]. In this notation, we usually put A to be a $4N^{\text{th}}$ root of unity, but we note that, when N is odd, we can also put A to be a $2N^{\text{th}}$ root of unity; see [17]. We put $A = e^{\pi\sqrt{-1}/N}$ and $q = A^4 = e^{4\pi\sqrt{-1}/N}$. Let p be a positive integer, and let M_p be the 3–manifold obtained from S^3 by p surgery along the figure-eight knot K_{4_1}. We recall that

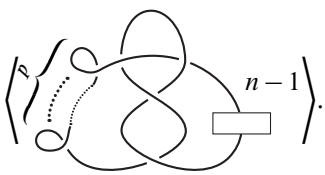
(22)



in the linear skein of [12]. Then, as in [12, Chapter 13], the quantum SU(2) invariant of M_p at $q = e^{4\pi\sqrt{-1}/N}$ is defined by

(23)

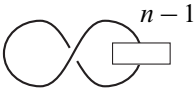
$$\hat{\tau}_N(M_p) = \frac{1}{c_+}\sum_{n=1}^{N-1}\Delta_{n-1}\left\langle\ \right.$$



$$\left.\ \right\rangle.$$

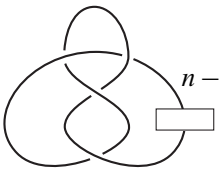
Here, c_+ is a constant given by

$$c_+ = \sum_{n=1}^{N-1} \Delta_{n-1}\left\langle \right. \quad \left. \right\rangle = \sum_{n=1}^{N-1} (-1)^{n-1} A^{n^2-1} \frac{(A^{2n}-A^{-2n})^2}{(A^2-A^{-2})^2},$$



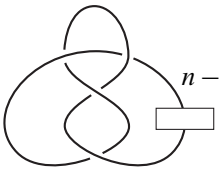
where we obtain the second equality by (22). Further, by (22), the formula (23) is rewritten as

$$\widehat{\tau}_N(M_p) = \frac{1}{c_+} \sum_{n=1}^{N-1} ((-1)^{n-1} A^{n^2-1})^p \cdot (-1)^{n-1} \frac{A^{2n}-A^{-2n}}{A^2-A^{-2}} \left\langle \right. \quad \left. \right\rangle.$$



We put

$$J_n(K_{4_1}) = \left\langle \right. \quad \left. \right\rangle.$$



Here, $J_n(K_{4_1})$ is the n^{th} colored Jones polynomial of the figure-eight knot K_{4_1}, where we normalize the colored Jones polynomial in such a way that

$$J_n(\text{trivial knot}) = (-1)^{n-1} \frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}.$$

Moreover, it is known [6; 11] that the colored Jones polynomial of the figure-eight knot can be presented by

$$J_n(K_{4_1}) = \frac{(-1)^{n-1}}{A^2-A^{-2}} \sum_{j=0}^{n-1} (A^{2(n+j)}-A^{-2(n+j)})(A^{2(n+j-1)}-A^{-2(n+j-1)}) \cdots (A^{2(n-j)}-A^{-2(n-j)}).$$

We calculate c_+, as follows:

$$\begin{aligned} -(q^{1/2}-q^{-1/2})^2 c_+ &= \sum_{n=1}^{N-1} (-1)^n q^{(n^2-1)/4} (q^{n/2}-q^{-n/2})^2 \\ &= 2q^{-1/4}(q^{-1}-1) \sum_{n \in \mathbb{Z}/n\mathbb{Z}} (-A)^{n^2} \\ &= 2q^{-1/4}(q^{-1}-1)(-\sqrt{-1})^{(N-1)/2}\sqrt{N}, \end{aligned}$$

where we obtain the last equality by Lemma A.1. Further, we calculate $J_n(K_{4_1})$, as follows:

$$\begin{aligned} J_n(K_{4_1}) &= \frac{(-1)^n}{q^{1/2}-q^{-1/2}}\sum_{j=0}^{n-1}(1-q^{n+j})(1-q^{n+j-1})\cdots(1-q^{n-j})\cdot q^{-n(2j+1)/2} \\ &= \frac{(-1)^n}{q^{1/2}-q^{-1/2}}\sum_{j=0}^{n-1} q^{-n(2j+1)/2}\frac{(q)_{n+j}}{(q)_{n-j-1}}. \end{aligned}$$

Hence, we calculate $\widehat{\tau}_N(M_p)$, as follows:

$$\begin{aligned} \widehat{\tau}_N(M_p) &= \frac{-1}{c_+(q^{1/2}-q^{-1/2})}\sum_{j=0}^{n-1}((-1)^{n-1}q^{(n^2-1)/4})^p(-1)^n(q^{n/2}-q^{-n/2})J_n(K_{4_1}) \\ &= \frac{-1}{c_+(q^{1/2}-q^{-1/2})^2} \\ &\qquad \times \sum_{0\le j<n<N}(-1)^{p(n-1)}q^{p(n^2-1)/4}q^{-n(2j+1)/2}(q^{n/2}-q^{-n/2})\frac{(q)_{n+j}}{(q)_{n-j-1}} \\ &= \frac{(-1)^p q^{(5-p)/4}}{2(1-q)}\sqrt{-1}^{(N-1)/2}N^{-1/2} \\ &\qquad \times \sum_{\substack{0\le j<n<N \\ n+j<N}}(-1)^{pn}q^{pn^2/4-nj}(1-q^{-n})\frac{(q)_{n+j}}{(q)_{n-j-1}}, \end{aligned}$$

where we obtain the restriction "$n+j<N$" in the sum of the last line since $(q)_{n+j}=0$ for $n+j\ge N$.

5 Proof of Theorem 1.1 when $|p|\ge 6$

In this section, we give a proof of Theorem 1.1 when $|p|\ge 6$. (When $|p|=5$, we give a proof of the theorem in Section 7, where there is the technical difficulty that Lemma 5.1 fails for $|p|=5$, and we need an additional procedure there.)

Since the figure-eight know is isotopic to its mirror image, M_{-p} is homeomorphic to M_p with opposite orientation, and $\widehat{\tau}_N(M_{-p})=\overline{\widehat{\tau}_N(M_p)}$. Hence, it is sufficient to show the theorem for $p>4$, since the theorem for a negative p can be obtained from the theorem for a positive p. We assume that p is an integer ≥ 6 in this section.

As we calculated in the previous section, $\widehat{\tau}_N(M_p)$ is presented by

$$\begin{aligned}\widehat{\tau}_N(M_p) &= \frac{(-1)^p q^{(5-p)/4}}{2(1-q)}\sqrt{-1}^{(N-1)/2}N^{-1/2} \\ &\qquad\times \sum_{\substack{0\le j<i<N\\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij}(1-q^{-i})\frac{(q)_{i+j}}{(q)_{i-j-1}} \\ &= \frac{(-1)^p q^{(5-p)/4}}{2(1-q)}\sqrt{-1}^{(N-1)/2}N^{-1/2} \\ &\qquad\times \sum_{\substack{0\le j<i<N\\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij}(1-q^{-i})\frac{N}{(q)_{i-j-1}(\overline{q})_{N-i-j-1}},\end{aligned}$$

where we obtain the second equality by (11). Further, by (7) and (8),

$$\begin{aligned}\widehat{\tau}_N(M_p) &= \frac{(-1)^p q^{(5-p)/4}}{1-q}\sqrt{-1}^{(N-1)/2}N^{-1/2}\sum_{\substack{0\le j<i<N\\ i+j<N}} (-1)^{pi}(1-q^{-i}) \\ &\qquad\times\exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)+\widehat{\varphi}\Big(\frac{2(i-j)-1}{N}\Big)-\widehat{\varphi}\Big(1-\frac{2(N-i-j)-1}{N}\Big)\Big).\end{aligned}$$

Hence,

$$\begin{aligned}\widehat{\tau}_N(M_p) &= \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{-1/2} \\ &\qquad\times \sum_{\substack{0\le j<i<N\\ i+j<N}} (1-q^{-i})\exp\Big(N\cdot\widetilde{V}\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}\Big)\Big),\end{aligned}$$

where we put

$$\widetilde{V}(t,s)=\frac{1}{N}\Big(\widehat{\varphi}\Big(2s-2t-\frac{1}{N}\Big)-\widehat{\varphi}\Big(1-2t-2s+\frac{1}{N}\Big)\Big)+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts\Big),$$

since, putting $t=\frac{1}{2}-\frac{i}{N}$ and $s=\frac{1}{2}-\frac{j}{N}$,

$$\begin{aligned}&\exp\Big(N\cdot 4\pi\sqrt{-1}\,\Big(\frac{p}{4}t^2-ts\Big)\Big)\\ &=\exp\Big(N\cdot 4\pi\sqrt{-1}\Big(\frac{p}{4}\Big(\frac{i^2}{N^2}-\frac{i}{N}+\frac{1}{4}\Big)-\Big(\frac{ij}{N^2}-\frac{i}{2N}-\frac{j}{2N}+\frac{1}{4}\Big)\Big)\Big)\\ &=\exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)\Big)\exp\Big(4\pi\sqrt{-1}\,\Big(-\frac{p}{4}i+\frac{i}{2}+\frac{j}{2}\Big)\Big)\exp\Big(\pi\sqrt{-1}\,N\Big(\frac{p}{4}-1\Big)\Big)\\ &=\exp\Big(\frac{4\pi\sqrt{-1}}{N}\Big(\frac{p}{4}i^2-ij\Big)\Big)(-1)^{pi}e^{(\pi\sqrt{-1}/4)pN}(-1).\end{aligned}$$

Further, we replace s of $\widetilde{V}(t,s)$ with $s+\frac{1}{2N}$ in the following way:

$$\begin{aligned}\widetilde{V}\Big(t, s+\frac{1}{2N}\Big) &= \frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\left(\frac{p}{4}t^2-t\Big(s+\frac{1}{2N}\Big)\right)\\ &= \frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\left(\frac{p}{4}t^2-ts\right)-\frac{2\pi\sqrt{-1}}{N}t.\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}(24)\quad \widehat{\tau}_N(M_p) = {}& \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{-1/2}\\ &\times\sum_{\substack{0\le j<i<N\\ i+j<N}}(q^{i/2}-q^{-i/2})\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big),\end{aligned}$$

where we put

$$V(t,s)=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts\Big).$$

By (10), $V(t,s)$ converges to

$$\widehat{V}(t,s)=\frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)})-\mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)}))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts\Big).$$

We note that the summand of (24) contributes to the formula of Theorem 1.1 only when $\mathrm{Re}\,V(t,s)\ge\varsigma_R(M_p)$, where $\varsigma_R(M_p)$ is as we define in Section 5.1. Hence, in order to prove the theorem, it is sufficient to consider the domain $\{\mathrm{Re}\,\widehat{V}(t,s)\ge\varsigma_R(M_p)\}$, instead of the whole domain

$$\Delta=\big\{(t,s)\in\mathbb{R}^2 \mid s\ge t,\ s\ge -t,\ s\le\tfrac{1}{2}\big\}.$$

As shown in Figure 1, the domain $\{\mathrm{Re}\,\widehat{V}(t,s)\ge\varsigma_R(M_p)\}$ has three connected components. Corresponding to these three components, we decompose Δ into the three parts

$$\begin{aligned}\Delta_0&=\big\{(t,s)\in\Delta \mid t+s\le\tfrac{1}{2},\ s-t\le\tfrac{1}{2}\big\}=\big\{(t,s)\in\mathbb{R}^2 \mid 0\le t+s\le\tfrac{1}{2},\ 0\le s-t\le\tfrac{1}{2}\big\},\\ \Delta_1&=\big\{(t,s)\in\Delta \mid s-t>\tfrac{1}{2}\big\},\\ \Delta_2&=\big\{(t,s)\in\Delta \mid t+s>\tfrac{1}{2}\big\}.\end{aligned}$$

In Section 6, we show that the contributions from Δ_1 and Δ_2 are sufficiently small, and we can ignore them. So it is sufficient to calculate the contribution from Δ_0.

In fact, we can further restrict Δ_0 to Δ_0' of the following lemma:

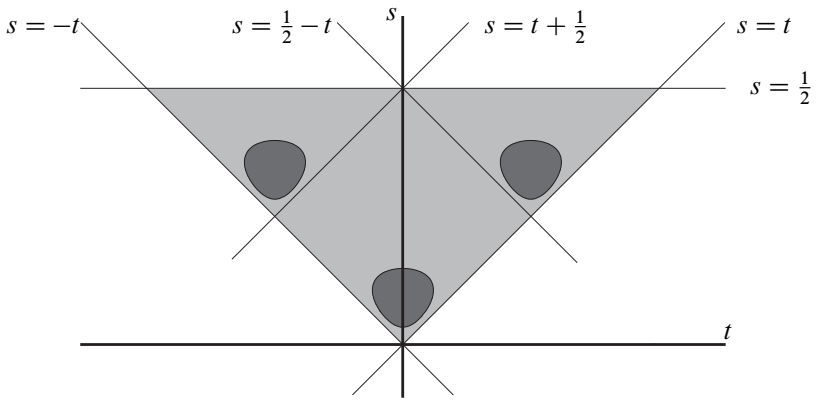


Figure 1: The light gray area is Δ and the dark gray area is $\{\mathrm{Re}\,\widehat{V}(t,s)\geq \varsigma_R(M_p)\}$ for $p=8$

Lemma 5.1 *We put*

$$\Delta_0'=\Big\{(t,s)\in\mathbb{R}^2\;\Big|\;0.005\leq t+s\leq 0.24,\ 0.005\leq s-t\leq 0.24,\ |t|\leq\frac{0.74}{p}\Big\}.$$

Then the domain

$$\{(t,s)\in\Delta_0\mid \mathrm{Re}\,\widehat{V}(t,s)\geq \varsigma_R(M_p)-\varepsilon\}$$

is included in Δ_0' for sufficiently small $\varepsilon>0$.

We give a proof of the lemma in Appendix E. See Figure 2 for graphical representations of the inclusion of the lemma for $p=6,\,12$.

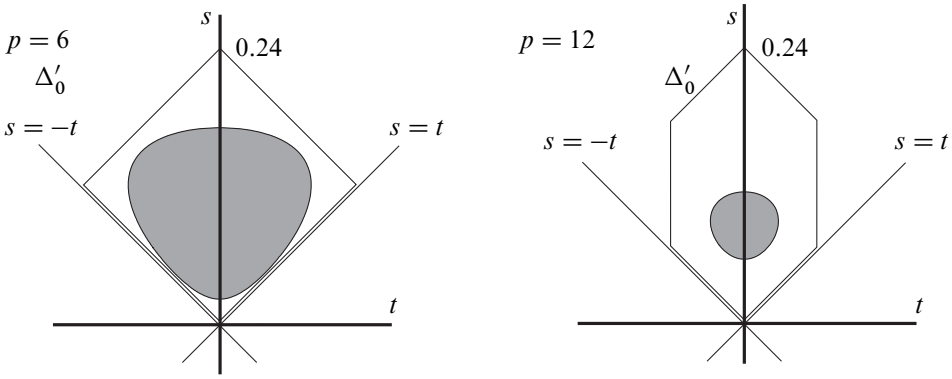


Figure 2: The domain $\{\mathrm{Re}\,\widehat{V}(t,s)\geq \varsigma_R(M_p)\}$ is included in Δ_0'

Proof of Theorem 1.1 We recall that $\widehat{\tau}_N(M_p)$ is presented by the sum (24). Hence, by the above argument, we obtain that

$$\begin{aligned}(25)\quad \widehat{\tau}_N(M_p) = {}& \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{-1/2}\\ &\times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'} (q^{i/2}-q^{-i/2})\\ &\qquad \times\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)\\ &\qquad + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).\end{aligned}$$

By Proposition 3.4 (Poisson summation formula) (see also Remark 3.5), this sum is expressed by the integrals

$$\begin{aligned}\widehat{\tau}_N(M_p) = {}& \frac{q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{3/2}\\ &\times\Big(\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(NV(t,s))\,dt\,ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)})\\ &\quad + \int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\big(N(V(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds\\ &\qquad + O(e^{N(\varsigma_R(M_p)-\varepsilon)})\Big)\end{aligned}$$

for some $\varepsilon>0$, noting that we verify the assumptions of Proposition 3.4 in Section 5.3. We note that, by Lemma 5.2 below, the second integral of the above formula is equal to

$$\begin{aligned}&\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(NV(-t,s))\,dt\,ds\\ &\qquad = \int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t'}-e^{2\pi\sqrt{-1}\,t'})\exp(NV(t',s))\,dt'\,ds\end{aligned}$$

putting $t'=-t$, which is equal to the first integral of the above formula of $\widehat{\tau}_N(M_p)$. Therefore,

$$\begin{aligned}(26)\quad \widehat{\tau}_N(M_p) = {}& \frac{2q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{3/2}\\ &\times\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(NV(t,s))\,dt\,ds\\ &\qquad + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).\end{aligned}$$

In order to apply the saddle-point method (Proposition 3.2), we consider a critical point of $\widehat{V}(t,s)$. The differentials of $\widehat{V}(t,s)$ are given by

$$\begin{aligned}\frac{\partial\widehat{V}}{\partial t} &= \log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s\Big),\\ \frac{\partial\widehat{V}}{\partial s} &= -\log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) - 4\pi\sqrt{-1}\,t.\end{aligned} \tag{27}$$

Further, putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, their differentials are given by

$$\begin{aligned}\frac{\partial^2\widehat{V}}{\partial t^2} &= 4\pi\sqrt{-1}\Big(\frac{e^{4\pi\sqrt{-1}(s-t)}}{1-e^{4\pi\sqrt{-1}(s-t)}} - \frac{e^{-4\pi\sqrt{-1}(t+s)}}{1-e^{-4\pi\sqrt{-1}(t+s)}} + \frac{p}{2}\Big)\\ &= 4\pi\sqrt{-1}\Big(\frac{z/w}{1-z/w} - \frac{zw}{1-zw} + \frac{p}{2}\Big) = 4\pi\sqrt{-1}\Big(\frac{z}{w-z} - \frac{zw}{1-zw} + \frac{p}{2}\Big),\\ \frac{\partial^2\widehat{V}}{\partial t\,\partial s} &= 4\pi\sqrt{-1}\Big(-\frac{z}{w-z} - \frac{zw}{1-zw} - 1\Big),\\ \frac{\partial^2\widehat{V}}{\partial s^2} &= 4\pi\sqrt{-1}\Big(\frac{z}{w-z} - \frac{zw}{1-zw}\Big).\end{aligned}$$

Let (t_0,s_0) be the critical point of $\widehat{V}(t_0,s_0)$ given in Section 5.1. We put

$$V_{tt} = \frac{\partial^2\widehat{V}}{\partial t^2}(t_0,s_0), \quad V_{ts} = \frac{\partial^2\widehat{V}}{\partial t\,\partial s}(t_0,s_0), \quad V_{ss} = \frac{\partial^2\widehat{V}}{\partial s^2}(t_0,s_0).$$

Then, by applying Proposition 3.2 (see also Remark 3.3) to (26), we obtain that

$$\begin{aligned}\widehat{\tau}_N(M_p) = \frac{2q^{(5-p)/4}}{q-1}(-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-1)/2}N^{3/2}\\ \times(e^{-2\pi\sqrt{-1}t_0} - e^{2\pi\sqrt{-1}\,t_0})\exp(N\widehat{V}(t_0,s_0))\cdot\frac{2\pi}{N}(V_{tt}V_{ss}-V_{ts}^2)^{-1/2}\\ \times\Big(1+O\Big(\frac{1}{N}\Big)\Big),\end{aligned}$$

noting that we verify the assumption of Proposition 3.2 in Section 5.2.

Further, noting that $q-1 = 4\pi\sqrt{-1}/N + O(1/N^2)$, we obtain that

$$\begin{aligned}\widehat{\tau}_N(M_p) = (-1)^p e^{-(\pi\sqrt{-1}/4)pN}\sqrt{-1}^{(N-3)/2}\exp(N\widehat{V}(t_0,s_0))N^{3/2}\omega(M_p)\\ \times\Big(1+O\Big(\frac{1}{N}\Big)\Big),\end{aligned}$$

where

$$\omega(M_p) = (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0})(V_{tt}V_{ss}-V_{ts}^2)^{-1/2}.$$

Hence, we obtain the required formula of the theorem. □

We used the following lemma in the above proof of the theorem:

Lemma 5.2 *We suppose that* $0 < \mathrm{Re}(t+s) < \frac{1}{4}$ *and* $0 < \mathrm{Re}(s-t) < \frac{1}{4}$. *Then*

$$V(-t,s) = V(t,s) + 2\pi\sqrt{-1}\,t.$$

Proof From the definition of $V(t,s)$, we have that

$$V(-t,s) = \frac{1}{N}(\widehat{\varphi}(2s+2t) - \widehat{\varphi}(1-2s+2t)) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 + ts\Big).$$

Further, by (9), we have that

$$\widehat{\varphi}(2s+2t) = -\widehat{\varphi}(1-2s-2t) + 2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big((2t+2s)^2 - (2t+2s) + \frac{1}{6}\Big) + \frac{1}{12N}\Big),$$
$$\widehat{\varphi}(1-2s+2t) = -\widehat{\varphi}(2s-2t) + 2\pi\sqrt{-1}\Big(-\frac{N}{4}\Big((2s-2t)^2 - (2s-2t) + \frac{1}{6}\Big) + \frac{1}{12N}\Big).$$

Hence,

$$\begin{aligned} V(-t,s) &= \frac{1}{N}(\widehat{\varphi}(2s-2t) - \widehat{\varphi}(1-2t-2s)) \\ &\quad + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 + ts - \frac{1}{4}(2t+2s)^2 + \frac{1}{4}(2s-2t)^2 + \frac{1}{4}(2t+2s) \\ &\qquad\qquad - \frac{1}{4}(2s-2t)\Big) \\ &= \frac{1}{N}(\widehat{\varphi}(2s-2t) - \widehat{\varphi}(1-2t-2s)) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts + \frac{1}{2}t\Big) \\ &= V(t,s) + 2\pi\sqrt{-1}\,t, \end{aligned}$$

as required. □

5.1 A critical point of $\widehat{V}(t,s)$

In this section, we characterize a critical point (t_0, s_0) of $\widehat{V}(t,s)$, which we use in the proof of Theorem 1.1.

As shown in Appendix B, there exists a single critical point (t_0, s_0) of $\widehat{V}(t,s)$ in the domain

$$\big\{(t,s) \in \mathbb{C}^2 \mid 0 < \mathrm{Re}(t+s) < \tfrac{1}{4},\ 0 < \mathrm{Re}(s-t) < \tfrac{1}{4},\ \mathrm{Re}\,t \geq 0\big\}. \tag{28}$$

We calculate this critical point concretely. By (27), a critical point of $\widehat{V}(t,s)$ is a solution of the equations

$$\begin{aligned} &\log(1 - e^{4\pi\sqrt{-1}(s-t)}) - \log(1 - e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s\Big) = 0, \\ &-\log(1 - e^{4\pi\sqrt{-1}(s-t)}) - \log(1 - e^{-4\pi\sqrt{-1}(t+s)}) - 4\pi\sqrt{-1}\,t = 0, \end{aligned} \tag{29}$$

where we choose the branch of the log in the way that $-\pi < \operatorname{Im}\log(\cdot) < \pi$, noting that $\operatorname{Im}(1-e^{4\pi\sqrt{-1}(s-t)}) > 0$ and $\operatorname{Im}(1-e^{-4\pi\sqrt{-1}(t+s)}) < 0$. Putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, the above equations are rewritten as

$$z^{p/2}(1-zw) = w-z, \quad (1-zw)(w-z) = zw. \tag{30}$$

Further, as shown in [13], they are rewritten as

$$z^2 - \left(\frac{z+z^{p/2}}{z^{p/2}z+1} + 1 + \frac{z^{p/2}z+1}{z+z^{p/2}}\right)z + 1 = 0, \quad w = \frac{z+z^{p/2}}{z^{p/2}z+1}.$$

Furthermore, since

$$z - \left(\frac{z+z^{p/2}}{z^{p/2}z+1} + 1 + \frac{z^{p/2}z+1}{z+z^{p/2}}\right) + \frac{1}{z} = -\frac{z^{p/2}z(z^{p/2}+z^{-p/2}-z^2-z^{-2}+z+z^{-1}+2)}{(z^{p/2}z+1)(z+z^{p/2})},$$

the above equations are rewritten as

$$z^{p/2}+z^{-p/2} = z^2+z^{-2}-z-z^{-1}-2, \quad w = \frac{z+z^{p/2}}{z^{p/2}z+1}. \tag{31}$$

By Lemma B.1, there exists a single solution of these equation which satisfies (28) and (29); we denote it by (t_0, s_0). For a concretely given p, we can obtain a numerical solution by calculating solutions of the above equations; we show some of concrete values of such numerical solutions below:

p	(t_0, s_0)
6	$(0.0743075\ldots - \sqrt{-1}\cdot 0.0382219\ldots, 0.1128050\ldots - \sqrt{-1}\cdot 0.0314723\ldots)$
7	$(0.0640105\ldots - \sqrt{-1}\cdot 0.0283809\ldots, 0.1065380\ldots - \sqrt{-1}\cdot 0.0212048\ldots)$
8	$(0.0566257\ldots - \sqrt{-1}\cdot 0.0221934\ldots, 0.1022661\ldots - \sqrt{-1}\cdot 0.0152090\ldots)$
9	$(0.0509104\ldots - \sqrt{-1}\cdot 0.0179265\ldots, 0.0991274\ldots - \sqrt{-1}\cdot 0.0113510\ldots)$
10	$(0.0462978\ldots - \sqrt{-1}\cdot 0.0148180\ldots, 0.0967225\ldots - \sqrt{-1}\cdot 0.0087183\ldots)$

Further, we put

$$\varsigma(M_p) = \widehat{V}(t_0, s_0), \quad \varsigma_R(M_p) = \operatorname{Re}\widehat{V}(t_0, s_0).$$

We note that $\varsigma(M_p)$ is a normalized complex volume of M_p; see [13; 16] for concrete formulas of the complex volume of M_p. In fact, as shown in Appendix B, we can give the hyperbolic structure of M_p by using the solution (t_0, s_0) of Lemma B.1, and a normalized hyperbolic volume of M_p is given by $\operatorname{Re}\widehat{V}(t_0, s_0)$. We show some

numerical values of the complex volume of M_p, as follows:

p	$\varsigma(M_p)$	$4\pi\sqrt{-1}\varsigma(M_p)=\mathrm{cs}(M_p)+\sqrt{-1}\mathrm{vol}(M_p)$
6	$0.102216092\ldots-\sqrt{-1}\cdot 0.106706823\ldots$	$1.340917487\ldots+\sqrt{-1}\cdot 1.284485300\ldots$
7	$0.116483644\ldots-\sqrt{-1}\cdot 0.095216750\ldots$	$1.196528981\ldots+\sqrt{-1}\cdot 1.463776644\ldots$
8	$0.125984399\ldots-\sqrt{-1}\cdot 0.085824359\ldots$	$1.078500712\ldots+\sqrt{-1}\cdot 1.583166660\ldots$
9	$0.132719661\ldots-\sqrt{-1}\cdot 0.078024388\ldots$	$0.980483376\ldots+\sqrt{-1}\cdot 1.667804452\ldots$
10	$0.137695916\ldots-\sqrt{-1}\cdot 0.071457335\ldots$	$0.897959363\ldots+\sqrt{-1}\cdot 1.730337923\ldots$

5.2 Verifying the assumption of the saddle-point method for V

In this section, in Lemma 5.6, we verify the assumption of the saddle-point method (Proposition 3.2 and Remark 3.3) when we apply Proposition 3.2 and Remark 3.3 to (26).

We note that, by (10), $V(t,s)$ uniformly converges to $\widehat{V}(t,s)$ on Δ_0' as $N\to\infty$. So we verify the assumption of the saddle-point method for $\widehat{V}(t,s)$. To simplify the calculation of the behavior of $\widehat{V}(t,s)$, we change the variables (t,s) to (u,v) by putting $u=s+t$ and $v=s-t$. They are in the ranges that $0<u<\frac{1}{4}$ and $0<v<\frac{1}{4}$. Then $\widehat{V}(t,s)$ is rewritten as

$$\check{V}(u,v)=\frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\,v})-\mathrm{Li}_2(e^{-4\pi\sqrt{-1}\,u}))+4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{(u-v)^2}{4}+\frac{v^2-u^2}{4}\Big).$$

In order to show Lemmas 5.3 and 5.5 below, we calculate the behavior of the function

$$f_{u,v}(\delta_1,\delta_2)=\mathrm{Re}\,\check{V}(u+\delta_1\sqrt{-1},v+\delta_2\sqrt{-1}).$$

The differentials of this function are given by

$$\begin{aligned}(32)\qquad \frac{\partial}{\partial\delta_1}f_{u,v}(\delta_1,\delta_2)&=\mathrm{Re}\Big(\sqrt{-1}\frac{\partial}{\partial u}\check{V}(u+\delta_1\sqrt{-1},v+\delta_2\sqrt{-1})\Big)\\&=\mathrm{Im}\Big(\log\Big(1-\frac{1}{x}\Big)\Big)+4\pi\Big(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\Big)\\&=\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\Big),\\(33)\qquad \frac{\partial}{\partial\delta_2}f_{u,v}(\delta_1,\delta_2)&=\mathrm{Re}\Big(\sqrt{-1}\frac{\partial}{\partial v}\check{V}(u+\delta_1\sqrt{-1},v+\delta_2\sqrt{-1})\Big)\\&=\mathrm{Im}(\log(1-y))+4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\Big)\\&=\mathrm{Arg}(1-y)+4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\Big),\end{aligned}$$

where $x=e^{4\pi\sqrt{-1}(u+\delta_1\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(v+\delta_2\sqrt{-1})}$.

Lemma 5.3 *Fixing $(u,v)\in\Delta_0'$ and $\delta_2\in\mathbb{R}$, we regard $f_{u,v}(X,\delta_2)$ as a function of $X\in\mathbb{R}$.*

(1) *If $v\geq\left(1-\frac{4}{p}\right)u$, then $f_{u,v}(X,\delta_2)$ is monotonically increasing for $X\in\mathbb{R}$.*

(2) *If $\left(1+\frac{4}{p}\right)u-\frac{2}{p}<v<\left(1-\frac{4}{p}\right)u$, then $f_{u,v}(X,\delta_2)$ has a unique minimal point at $X=g_1(u,v)$, where*

$$g_1(u,v)=\frac{1}{4\pi}\log\frac{\sin 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\right)}{\sin 4\pi\left(\frac{1}{4}-\frac{p}{4}\cdot\frac{u-v}{2}-\frac{u}{2}\right)},$$

ie $f_{u,v}(X,\delta_2)$ is monotonically decreasing for $X<g_1(u,v)$, and is monotonically increasing for $X>g_1(u,v)$.

(3) *If $v\leq\left(1+\frac{4}{p}\right)u-\frac{2}{p}$, then $f_{u,v}(X,\delta_2)$ is monotonically decreasing for $X\in\mathbb{R}$.*

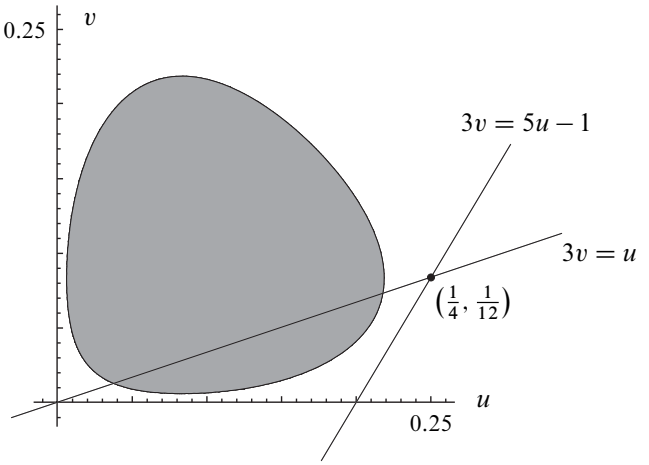


Figure 3: The domain $\{\mathrm{Re}\,\check{V}(u,v)\geq\varsigma_R(M_6)\}$ and the lines $3v=u$ and $3v=5u-1$

Remark 5.4 When $p=6$ (for example), the domain $\{\mathrm{Re}\,\check{V}(u,v)\geq\varsigma_R(M_6)\}$ and the lines $3v=u$ and $3v=5u-1$ (which appear in the statement of the lemma) are located as shown in Figure 3. We note that the crossing point $(u,v)=\left(\frac{1}{4},\frac{1}{12}\right)$ of these two lines does not belong to Δ_0', since it does not satisfy that $u=t+s\leq 0.24$; this is an important point, because the gradient flow of $-\mathrm{Re}\,\check{V}$ does not behave well only at this point. We also note that some cases in the statement of the lemma might not be realized if we were to choose Δ_0' sufficiently close to the domain $\{\mathrm{Re}\,\check{V}(u,v)\geq\varsigma_R(M_p)\}$.

Proof of Lemma 5.3 We put $x = e^{4\pi\sqrt{-1}(u+X\sqrt{-1})}$. Then $1/x = e^{4\pi X}e^{-4\pi\sqrt{-1}\,u}$. We put $\theta = \mathrm{Arg}\big(1-\frac{1}{x}\big)$ in this proof. Since $0 < u < \frac{1}{4}$, θ is in the range

$$0 < \theta < 4\pi\Big(\frac{1}{4} - u\Big).$$

When $v \geq \big(1-\frac{4}{p}\big)u$, we show the lemma, as follows. By (32),

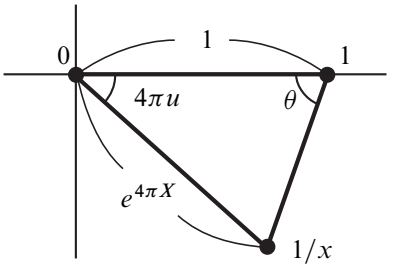
$$\frac{\partial}{\partial X} f_{u,v}(X, \delta_2) = \theta + 4\pi\Big(-\frac{p}{4}\cdot\frac{u-v}{2} + \frac{u}{2}\Big) > 0.$$

Therefore, $f_{u,v}(X, \delta_2)$ is monotonically increasing, and (1) holds.

When $\big(1+\frac{4}{p}\big)u - \frac{2}{p} < v < \big(1-\frac{4}{p}\big)u$, we show the lemma, as follows. In this case, by (32),

$$\frac{\partial}{\partial X} f_{u,v}(X, \delta_2) \begin{cases} > 0 & \text{if } \theta > 4\pi\big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\big), \\ = 0 & \text{if } \theta = 4\pi\big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\big), \\ < 0 & \text{if } \theta < 4\pi\big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\big). \end{cases}$$

Further, θ and X are related as shown in the following picture:



Hence, X is monotonically increasing as a function of θ, and they satisfy that

$$\frac{e^{4\pi X}}{\sin\theta} = \frac{1}{\sin(\pi - 4\pi u - \theta)}.$$

This is rewritten as

$$X = \frac{1}{4\pi}\log\frac{\sin\theta}{\sin(\pi - 4\pi u - \theta)}.$$

Therefore,

$$\frac{\partial}{\partial X} f_{u,v}(X, \delta_2) \begin{cases} > 0 & \text{if } X > g_1(u,v), \\ = 0 & \text{if } X = g_1(u,v), \\ < 0 & \text{if } X < g_1(u,v), \end{cases}$$

where we put

$$g_1(u,v) = \frac{1}{4\pi}\log\frac{\sin 4\pi\big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\big)}{\sin 4\pi\big(\frac{1}{4} - \frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2}\big)}.$$

Hence, (2) holds.

When $v \le \left(1+\frac{4}{p}\right)u - \frac{2}{p}$, we show the lemma, as follows. By (32),

$$\frac{\partial}{\partial X} f_{u,v}(X,\delta_2) = \theta + 4\pi\left(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\right) < 4\pi\left(\frac{1}{4}-u\right)+4\pi\left(-\frac{p}{4}\cdot\frac{u-v}{2}+\frac{u}{2}\right) < 0.$$

Therefore, $f_{u,v}(X,\delta_2)$ is monotonically decreasing, and (3) holds. □

Lemma 5.5 *Fixing $(u,v)\in\Delta_0'$ and $\delta_1\in\mathbb{R}$, we regard $f_{u,v}(\delta_1,Y)$ as a function of $Y\in\mathbb{R}$.*

(1) *If $u \le \left(1+\frac{4}{p}\right)v$, then $f_{u,v}(\delta_1,Y)$ is monotonically decreasing for $Y\in\mathbb{R}$.*

(2) *If $\left(1+\frac{4}{p}\right)v < u < \left(1-\frac{4}{p}\right)v+\frac{2}{p}$, then $f_{u,v}(\delta_1,Y)$ has a unique minimal point at $Y = g_2(u,v)$, where*

$$g_2(u,v) = \frac{1}{4\pi}\log\frac{\sin 4\pi\left(\frac{1}{4}-\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)}{\sin 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right)},$$

ie $f_{u,v}(\delta_1,Y)$ is monotonically decreasing for $Y < g_2(u,v)$, and is monotonically increasing for $Y > g_2(u,v)$.

(3) *If $u \ge \left(1-\frac{4}{p}\right)v+\frac{2}{p}$, then $f_{u,v}(\delta_1,Y)$ is monotonically increasing for $Y\in\mathbb{R}$.*

Proof We put $y = e^{4\pi\sqrt{-1}(v+Y\sqrt{-1})}$. We put $\theta = -\mathrm{Arg}(1-y)$ in this proof. Since $0<v<\frac{1}{4}$, θ is in the range

$$0 < \theta < 4\pi\left(\tfrac{1}{4}-v\right).$$

When $u \le \left(1+\frac{4}{p}\right)v$, we show the lemma, as follows. By (33),

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1,Y) = -\theta + 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right) < 0.$$

Therefore, $f_{u,v}(\delta_1,Y)$ is monotonically decreasing, and (1) holds.

When $\left(1+\frac{4}{p}\right)v < u < \left(1-\frac{4}{p}\right)v+\frac{2}{p}$, we show the lemma, as follows. In this case, by (33),

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1,Y)\begin{cases} >0 & \text{if } \theta < 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right),\\ =0 & \text{if } \theta = 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right),\\ <0 & \text{if } \theta > 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2}-\frac{v}{2}\right).\end{cases}$$

Further, Y is monotonically decreasing as a function of θ, and they satisfy that

$$\frac{e^{-4\pi Y}}{\sin\theta} = \frac{1}{\sin(\pi-4\pi v-\theta)}.$$

This is rewritten as

$$Y = \frac{1}{4\pi}\log\frac{\sin(\pi - 4\pi v - \theta)}{\sin\theta}.$$

Therefore,

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1, Y) \begin{cases} > 0 & \text{if } Y > g_2(u,v), \\ = 0 & \text{if } Y = g_2(u,v), \\ < 0 & \text{if } Y < g_2(u,v), \end{cases}$$

where we put

$$g_2(u,v) = \frac{1}{4\pi}\log\frac{\sin 4\pi\left(\frac{1}{4} - \frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\right)}{\sin 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\right)}.$$

Hence, (2) holds.

When $u \geq \left(1 - \frac{4}{p}\right)v + \frac{2}{p}$, we show the lemma, as follows. By (33),

$$\frac{\partial}{\partial Y} f_{u,v}(\delta_1, Y) = -\theta + 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\right) > -4\pi\left(\frac{1}{4} - v\right) + 4\pi\left(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{v}{2}\right) > 0.$$

Therefore, $f_{u,v}(\delta_1, Y)$ is monotonically increasing, and (3) holds. □

The argument of the proof of the following lemma is due to Yokota [28].

Lemma 5.6 *When we apply Proposition 3.2 (saddle-point method) to* (26)*, the assumption of Proposition 3.2 holds.*

Proof We show that there exists a homotopy $\Delta'_{(\delta)}$ $(0 \leq \delta \leq 1)$ between $\Delta'_{(0)} = \Delta'_0$ and $\Delta'_{(1)}$ such that

$$(t_c, s_c) \in \Delta'_{(1)}, \tag{34}$$

$$\Delta'_{(1)} - \{(t_c, s_c)\} \subset \{(t,s) \in \mathbb{C}^2 \mid \operatorname{Re}\widehat{V}(t,s) < \varsigma_R\}, \tag{35}$$

$$\partial\Delta'_{(\delta)} \subset \{(t,s) \in \mathbb{C}^2 \mid \operatorname{Re}\widehat{V}(t,s) < \varsigma_R\}. \tag{36}$$

For a sufficiently large $R > 0$, we put

$$\widehat{g}_1(t,s) = \begin{cases} -R & \text{if } v \geq \left(1 - \frac{4}{p}\right)u, \\ \max\{-R, g_1(t,s)\} & \text{if } \left(1 + \frac{4}{p}\right)u - \frac{2}{p} < v < \left(1 - \frac{4}{p}\right)u, \\ -R & \text{if } v \leq \left(1 + \frac{4}{p}\right)u - \frac{2}{p}, \end{cases}$$

$$\widehat{g}_2(t,s) = \begin{cases} -R & \text{if } u \leq \left(1 + \frac{4}{p}\right)v, \\ \max\{-R, g_2(t,s)\} & \text{if } \left(1 + \frac{4}{p}\right)v < u < \left(1 - \frac{4}{p}\right)v + \frac{2}{p}, \\ -R & \text{if } u \geq \left(1 - \frac{4}{p}\right)v + \frac{2}{p}. \end{cases}$$

We note that, since $g_1(t,s)\to-\infty$ as $v\to\left(1+\frac{4}{p}\right)u-\frac{2}{p}, \left(1-\frac{4}{p}\right)u$, the function $\widehat{g}_1(t,s)$ is continuous, and similarly, since $g_2(t,s)\to-\infty$ as $u\to\left(1+\frac{4}{p}\right)v, \left(1-\frac{4}{p}\right)v+\frac{2}{p}$, the function $\widehat{g}_2(t,s)$ is continuous. We put

$$\Delta'_{(\delta)}=\{(t+\delta\cdot\widehat{g}_1(t,s)\sqrt{-1},\ s+\delta\cdot\widehat{g}_2(t,s)\sqrt{-1})\in\mathbb{C}^2\mid(t,s)\in\Delta'_0\}.$$

We show (36), as follows. From the definition of Δ'_0,

$$\partial\Delta'_0\subset\{(t,s)\in\mathbb{C}^2\mid \operatorname{Re}\widehat{V}(t,s)<\varsigma_R\}.$$

Further, by Lemmas 5.3 and 5.5,

$$\operatorname{Re}\widehat{V}(t+\delta\cdot\widehat{g}_1(t,s)\sqrt{-1},s+\delta\cdot\widehat{g}_2(t,s)\sqrt{-1})\le\operatorname{Re}\widehat{V}(t,s)$$

for any $\delta\in[0,1]$ and any $(t,s)\in\Delta'_0$. Hence, (36) holds.

We show (34) and (35), as follows. Consider the functions

$$F(t,s,X,Y)=\operatorname{Re}\widehat{V}(t+X\sqrt{-1},s+Y\sqrt{-1}),$$
$$h(t,s)=F(t,s,\widehat{g}_1(t,s),\widehat{g}_2(t,s)).$$

When $v\ge\left(1-\frac{4}{p}\right)u$ or $v\le\left(1+\frac{4}{p}\right)u-\frac{2}{p}$ or $u\le\left(1+\frac{4}{p}\right)v$ or $u\ge\left(1-\frac{4}{p}\right)v+\frac{2}{p}$, $-h(t,s)$ is sufficiently large (because we let R be sufficiently large), and (35) holds in this case. When $\left(1+\frac{4}{p}\right)u-\frac{2}{p}<v<\left(1-\frac{4}{p}\right)u$ and $\left(1+\frac{4}{p}\right)v<u<\left(1-\frac{4}{p}\right)v+\frac{2}{p}$, it follows from the definitions of $g_1(t,s)$ and $g_2(t,s)$ that $\frac{\partial F}{\partial X}=0$ at $X=g_1(t,s)$ and $\frac{\partial F}{\partial Y}=0$ at $Y=g_2(t,s)$. Hence,

$$\operatorname{Im}\frac{\partial\widehat{V}}{\partial t}=\operatorname{Im}\frac{\partial\widehat{V}}{\partial s}=0$$

at $(t+g_1(t,s)\sqrt{-1},s+g_2(t,s)\sqrt{-1})$. Further, $\frac{\partial h}{\partial t}=\operatorname{Re}\frac{\partial\widehat{V}}{\partial t}$ and $\frac{\partial h}{\partial s}=\operatorname{Re}\frac{\partial\widehat{V}}{\partial s}$ at $(t+g_1(t,s)\sqrt{-1},s+g_2(t,s)\sqrt{-1})$. Therefore, when (t,s) is a critical point of $h(t,s)$, $(t+g_1(t,s)\sqrt{-1},s+g_2(t,s)\sqrt{-1})$ is a critical point of $\widehat{V}$. It follows that $h(t,s)$ has a unique maximal point at $(t,s)=(\operatorname{Re}t_c,\operatorname{Re}s_c)$. Therefore, (34) and (35) hold. □

5.3 Verifying the assumption of the Poisson summation formula for V

In this section, in Lemma 5.7, we verify the assumption of the Poisson summation formula (Proposition 3.4 and Remark 3.5), when we apply them to (25). As in the previous section, we consider $\widehat{V}(t,s)$ instead of $V(t,s)$. We assume that $0<t+s<\frac{1}{4}$ and $0<s-t<\frac{1}{4}$ in this section.

We calculate an upper bound of $\widehat{V}(t,s) - \varsigma_R(M_p)$ for $(t,s) \in \Delta_0'$, as follows. From the definition of $\widehat{V}(t,s)$, we have that

$$\begin{aligned}\operatorname{Re}\widehat{V}(t,s) &= \operatorname{Re}\left(\frac{1}{4\pi\sqrt{-1}}(\operatorname{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \operatorname{Li}_2(e^{-4\pi\sqrt{-1}(t+s)}))\right)\\ &= \frac{1}{2}\Lambda(2s-2t) + \frac{1}{2}\Lambda(2t+2s) \le \Lambda\left(\frac{1}{6}\right).\end{aligned}$$

Hence, by Lemma D.1,

$$\operatorname{Re}\widehat{V}(t,s) - \varsigma_R(M_p) \le \Lambda\left(\frac{1}{6}\right) - \varsigma_R(M_p) \le \frac{3}{p^2} \le \frac{1}{12} = 0.0833\ldots. \tag{37}$$

Further, the following inequalities hold for $\delta \in \mathbb{R}$, $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$, which we use in the proofs of lemmas in this section:

$$\begin{aligned}&0 < \operatorname{Arg}\left(1 - \frac{1}{x}\right) < 4\pi\left(\frac{1}{4} - t - s\right),\\ &-4\pi\left(\frac{1}{4} - s + t\right) < \operatorname{Arg}(1-y) < 0.\end{aligned} \tag{38}$$

Lemma 5.7 *When we apply Proposition 3.4 and Remark 3.5 to* (25)*, their assumptions hold.*

Proof We verify (20), (17), (18) and (19) in Lemmas 5.8, 5.9, 5.10 and 5.11, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Lemma 5.8 *The assumption* (20) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_p)$.

Proof As for the assumption (20), we show that $\partial\Delta_0'$ is null-homotopic in

$$\{(t+\delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \Delta_0',\ \delta \ge 0,\ \operatorname{Re}\widehat{V}(t+\delta\sqrt{-1}, s) < \varsigma_R(M_p) + 4\pi\delta\}.$$

To show it, we show that the following disk bounds $\partial\Delta_0'$ in the above domain:

$$\{(t+\delta_0\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \Delta_0'\} \cup \{(t+\delta\sqrt{-1}, s) \in \mathbb{C}^2 \mid (t,s) \in \partial\Delta_0',\ \delta \in [0,\delta_0]\}.$$

We put

$$F_{t,s}(\delta) = \operatorname{Re}\widehat{V}(t+\delta\sqrt{-1}, s) - \varsigma_R(M_p) - 4\pi\delta$$

in this proof. Then it is sufficient to show that

$$F_{t,s}(\delta_0) < 0 \quad \text{for any } (t,s) \in \Delta_0' \tag{39}$$

and

$$F_{t,s}(\delta) < 0 \quad \text{for any } (t,s) \in \partial\Delta_0' \text{ and } \delta \in [0,\delta_0] \tag{40}$$

for some $\delta_0 > 0$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(\sqrt{-1}\frac{\partial}{\partial t}\widehat{V}(t+\delta\sqrt{-1},s)\Big)-4\pi\\ &= \mathrm{Im}\Big(-\log(1-y)+\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(-\frac{p}{2}t+s-1\Big)\Big)\\ &= -\mathrm{Arg}(1-y)+\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-\frac{p}{2}t+s-1\Big),\end{aligned}$$

where we put $x=e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< 4\pi\Big(\frac{1}{4}-s+t\Big)+4\pi\Big(\frac{1}{4}-t-s\Big)+4\pi\Big(-\frac{p}{2}t+s-1\Big)\\ &= 4\pi\Big(-\frac{p}{2}t-s-\frac{1}{2}\Big)<4\pi\Big(-\frac{p}{2}t-\frac{1}{2}\Big)<-4\pi\cdot 0.13,\end{aligned}$$

where we obtain the last inequality since $|t|\le 0.74/p$ by Lemma 5.1.

We show (39), as follows. We have that

$$F_{t,s}(\delta_0)=F_{t,s}(0)+\int_0^{\delta_0}\frac{d}{d\delta}F_{t,s}(\delta)\,d\delta<F_{t,s}(0)-4\pi\cdot\frac{0.1}{2}\cdot\delta_0.$$

Further, by (37),

$$F_{t,s}(0)=\mathrm{Re}\,\widehat{V}(t,s)-\varsigma_R\le\frac{1}{12}=0.0833\ldots.$$

Hence, (39) is satisfied for a sufficiently large δ_0.

We show (40), as follows. From the definition of Δ', we have that $F_{t,s}(0)<0$ for any $(t,s)\in\partial\Delta'$. Since $\frac{d}{d\delta}F_{t,s}(\delta)<0$ as shown above, it is shown similarly as above that $F_{t,s}(\delta)<0$ for any $\delta\ge 0$. Hence, (40) is satisfied. □

Lemma 5.9 *The assumption* (17) *holds for* $\widehat{V}(t,s)-\varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta)=\mathrm{Re}\,\widehat{V}(t-\delta\sqrt{-1},s)-\varsigma_R(M_p)-2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{41}$$

for any $(t,s)\in\Delta'_0$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial t}\widehat{V}(t-\delta\sqrt{-1},s)\Big) - 2\pi \\ &= \mathrm{Im}\Big(\log(1-y) - \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s - \frac{1}{2}\Big)\Big) \\ &= \mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(\frac{p}{2}t - s - \frac{1}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0 + 0 + 4\pi\Big(\frac{p}{2}t - s - \frac{1}{2}\Big) < 4\pi\Big(\frac{p}{2}t - \frac{1}{2}\Big) < -4\pi\cdot 0.13,$$

where we obtain the last inequality since $|t| \le 0.74/p$ by Lemma 5.1. Hence, (41) holds, as required. □

Lemma 5.10 *The assumption* (18) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{V}(t, s+\delta\sqrt{-1}) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{42}$$

for any $(t,s) \in \Delta_0'$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s}\widehat{V}(t,s+\delta\sqrt{-1})\Big) - 2\pi \\ &= \mathrm{Im}\Big(\log(1-y) + \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\,t\Big) - 2\pi \\ &= \mathrm{Arg}(1-y) + \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(t - \frac{1}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0 + 4\pi\Big(\frac{1}{4} - t - s\Big) + 4\pi\Big(t - \frac{1}{2}\Big) = 4\pi\Big(-s - \frac{1}{4}\Big) < -\pi.$$

Hence, (42) holds, as required. □

Lemma 5.11 *The assumption* (19) *holds for* $\widehat{V}(t,s)-\varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta)=\operatorname{Re}\widehat{V}(t,s-\delta\sqrt{-1})-\varsigma_R(M_p)-2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{43}$$

for any $(t,s)\in\Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta)&=\operatorname{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial s}\widehat{V}(t,s-\delta\sqrt{-1})\Big)-2\pi\\&=\operatorname{Im}\Big(-\log(1-y)-\log\Big(1-\frac{1}{x}\Big)-4\pi\sqrt{-1}\,t\Big)-2\pi\\&=-\operatorname{Arg}(1-y)-\operatorname{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-t-\frac{1}{2}\Big),\end{aligned}$$

where we put $x=e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta)<4\pi\Big(\frac{1}{4}-s+t\Big)+0+4\pi\Big(-t-\frac{1}{2}\Big)=4\pi\Big(-s-\frac{1}{4}\Big)<-\pi.$$

Hence, (43) holds, as required. □

6 Contributions from Δ_1 and Δ_2

In this section, in the following two propositions, we show that, when we restrict the range of the sum (24) to Δ_1 and Δ_2, the values of the restricted sums are of sufficiently small order; this fact is used in the proof of Theorem 1.1 in Section 5.

The aim of this section is to show the following two propositions; the second proposition is immediately obtained from the first proposition, as we show below.

Proposition 6.1 *For any integer* p *with* $|p|\geq 6$,

$$\sum_{\substack{0\leq j<i<N\\ i+j<N/2}}(-1)^{pi}q^{pi^2/4-ij}(1-q^{-i})\frac{(q)_{i+j}}{(q)_{i-j-1}}=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon>0$.

Proposition 6.2 *For any integer* p *with* $|p| \geq 6$,

$$\sum_{\substack{0\leq j<i-N/2\\ i+j<N}} (-1)^{pi} q^{pi^2/4-ij}(1-q^{-i})\frac{(q)_{i+j}}{(q)_{i-j-1}} = O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon > 0$.

Proof By putting $i' = N - i$ and $p' = -p$, the left-hand side of the formula of the proposition is calculated as follows:

$$\begin{aligned}
&\sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'(N-i')} q^{-(p'/4)(N-i')^2-(N-i')j}(1-q^{i'})\frac{(q)_{N-i'+j}}{(q)_{N-i'-j-1}}\\
&\quad = (-1)^{p'N} q^{-(p'/4)N^2} \sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'i'} q^{-(p'/4)i'^2+i'j}(1-q^{i'})\frac{(\overline{q})_{i'+j}}{(\overline{q})_{i'-j-1}}\\
&\quad = \sum_{\substack{0\leq j<i'<N\\ i'+j<N/2}} (-1)^{p'i'} q^{-(p'/4)i'^2+i'j}(1-q^{i'})\frac{(\overline{q})_{i'+j}}{(\overline{q})_{i'-j-1}}.
\end{aligned}$$

Since this is equal to the complex conjugate of the left-hand side of the formula of Proposition 6.1, we obtain the required formula of the proposition. □

The rest of this section is devoted to the proof of Proposition 6.1.

Proof of Proposition 6.1 In a similar way as in Section 5, putting $t = \frac{1}{4} - \frac{i}{N}$ and $s = \frac{1}{4} - \frac{j}{N}$, the sum of Proposition 6.1 can be rewritten as

$$\sum_{\substack{0\leq j<i<N\\ i+j<N/2}} \exp\Big(N\widetilde{U}\Big(\frac{1}{4}-\frac{i}{N},\frac{1}{4}-\frac{j}{N}\Big)\Big) = O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}), \tag{44}$$

where we put

$$\widetilde{U}(t,s) = \frac{1}{N}\Big(\widehat{\varphi}\Big(2s-2t-\frac{1}{N}\Big)-\widehat{\varphi}\Big(1-2t-2s+\frac{1}{N}\Big)\Big)+4\pi\sqrt{-1}\,\Big(\frac{p}{4}t^2-ts+\alpha t+\frac{1}{4}s\Big),$$

and we put

$$\alpha = \begin{cases} -\frac{1}{8} & \text{if } p \equiv 1 \mod 4,\\ 0 & \text{if } p \equiv 2 \mod 4,\\ \frac{1}{8} & \text{if } p \equiv 3 \mod 4,\\ \frac{1}{4} & \text{if } p \equiv 0 \mod 4, \end{cases}$$

To be precise, there should be a factor $(1-q^{-i})$ in the sum of (44), but such a factor does not contribute the resulting statement of Proposition 6.1, and so we omit such a factor in the following argument. Further, modifying $\widetilde{U}(t,s)$ to

$$U(t,s)=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\alpha t+\frac{1}{4}s\Big),$$

it is sufficient to show that

$$\sum_{\substack{0\le j<i<N\\ i+j<N/2}}\exp\Big(NU\Big(\frac{1}{4}-\frac{i}{N},\frac{1}{4}-\frac{j}{N}\Big)\Big)=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}). \tag{45}$$

We apply the Poisson summation formula (Proposition 3.4 and Remarks 3.5 and 3.6) to (45), noting that we verify the assumptions of Proposition 3.4 in Section 6.1. Then it follows that the resulting formula is a linear sum of the following formulas. Hence, it is sufficient to show that

$$\begin{aligned}\int_{\Delta_0'}\exp(NU(t,s))\,dt\,ds&=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}),\\ \int_{\Delta_0'}\exp\big(N(U(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds&=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}),\\ \int_{\Delta_0'}\exp\big(N(U(t,s)-2\pi\sqrt{-1}\,t)\big)\,dt\,ds&=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})\end{aligned}$$

for some $\varepsilon>0$, noting that we need the third formula only when $\alpha=\frac{1}{4}$. Therefore, from the definition of $U(t,s)$, putting

$$U'(t,s)=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\alpha' t+\frac{1}{4}s\Big),$$

it is sufficient to show the following formula for $\alpha'=-\frac{2}{8},-\frac{1}{8},0,\dots,\frac{5}{8},\frac{6}{8}$:

$$\int_{\Delta_0'}\exp(NU'(t,s))\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)}) \tag{46}$$

for some $\varepsilon>0$. Further, we note that, in the same way as Lemma 5.2, we can obtain that

$$U'(-t,s)=\frac{1}{N}(\widehat{\varphi}(2s-2t)-\widehat{\varphi}(1-2t-2s))+4\pi\sqrt{-1}\Big(\frac{p}{4}t^2-ts+\Big(\frac{1}{2}-\alpha'\Big)t+\frac{1}{4}s\Big).$$

By using this formula, we can show that it is sufficient to show (46) only for $\alpha'=-\frac{1}{4},-\frac{1}{8},0,\frac{1}{8},\frac{1}{4}$. This is shown in Lemma 6.10, in a similar way as the procedure of the saddle-point method. Hence, we obtain the proposition. □

6.1 Verifying the assumptions of the Poisson summation formula for U

In this section, in Lemma 6.3, we verify the assumption of the Poisson summation formula (Proposition 3.4 and Remarks 3.5 and 3.6), when we apply them to (45).

We note that $U(t,s)$ uniformly converges on Δ_0' as $N \to \infty$ to

$$\widehat{U}(t,s) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts + \alpha t + \frac{1}{4}s\Big).$$

As in Section 5.3, we consider $\widehat{U}(t,s)$ instead of $U(t,s)$.

The differentials of $\widehat{U}(t,s)$ are given by

$$\frac{\partial\widehat{U}}{\partial t} = \log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s + \alpha\Big),$$
$$\frac{\partial\widehat{U}}{\partial s} = -\log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(-t + \frac{1}{4}\Big).$$

We note that similar estimates as (38) hold also in this section.

Lemma 6.3 *When we apply Proposition 3.4 and Remarks 3.5 and 3.6 to* (45)*, their assumptions hold.*

Proof We verify (20), (17), (21), (18) and (19) in Lemmas 6.4, 6.5(1), 6.5(2), 6.6 and 6.7, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Lemma 6.4 *The assumption* (20) *holds for* $\widehat{U}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{U}(t+\delta\sqrt{-1}, s) - \varsigma_R(M_p) - 4\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{47}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial t}\widehat{U}(t+\delta\sqrt{-1},s)\Big)-4\pi\\ &= \mathrm{Im}\Big(-\log(1-y)+\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(-\frac{p}{2}t+s-\alpha\Big)\Big)-4\pi\\ &= -\mathrm{Arg}(1-y)+\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(-\frac{p}{2}t+s-\alpha-1\Big),\end{aligned}$$

where we put $x=e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< 4\pi\Big(\frac{1}{4}-s+t\Big)+4\pi\Big(\frac{1}{4}-t-s\Big)+4\pi\Big(-\frac{p}{2}t+s-\alpha-1\Big)\\ &= 4\pi\Big(-\frac{p}{2}t-s-\alpha-\frac{1}{2}\Big)<4\pi\Big(-\frac{p}{2}t-\alpha-\frac{1}{2}\Big)<4\pi\Big(-\frac{p}{2}t-\frac{3}{8}\Big)\\ &< 4\pi\Big(\frac{0.74}{2}-\frac{3}{8}\Big)=-0.02\pi.\end{aligned}$$

Hence, (47) holds, as required. □

Lemma 6.5 (1) *When $\alpha=\frac{1}{4}$, the assumption* (21) *holds for $\widehat{U}(t,s)-\varsigma_R(M_p)$.*

(2) *When $\alpha=0,\pm\frac{1}{8}$, the assumption* (17) *holds for $\widehat{U}(t,s)-\varsigma_R(M_p)$.*

Proof We put

$$F_{t,s}(\delta)=\mathrm{Re}\,\widehat{U}(t-\delta\sqrt{-1},s)-\varsigma_R(M_p)-2k\pi\delta$$

in this proof, where we put $k=1$ if $\alpha=\frac{1}{4}$, and $k=2$ if $\alpha=0,\pm\frac{1}{8}$. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{48}$$

for any $(t,s)\in\Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \mathrm{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial t}\widehat{U}(t-\delta\sqrt{-1},s)\Big)-2k\pi\\ &= \mathrm{Im}\Big(\log(1-y)-\log\Big(1-\frac{1}{x}\Big)+4\pi\sqrt{-1}\Big(\frac{p}{2}t-s+\alpha\Big)\Big)-2k\pi\\ &= \mathrm{Arg}(1-y)-\mathrm{Arg}\Big(1-\frac{1}{x}\Big)+4\pi\Big(\frac{p}{2}t-s+\alpha-\frac{k}{2}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< 0 + 0 + 4\pi\Big(\frac{p}{2}t - s + \alpha - \frac{k}{2}\Big) < 4\pi\Big(\frac{p}{2}t + \alpha - \frac{k}{2}\Big) \le 4\pi\Big(\frac{p}{2}t - \frac{3}{8}\Big)\\ &< 4\pi\Big(\frac{0.74}{2} - \frac{3}{8}\Big) = -0.02\pi.\end{aligned}$$

Hence, (48) holds, as required. □

Lemma 6.6 *The assumption* (18) *holds for* $\widehat{U}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\widehat{U}(t, s+\delta\sqrt{-1}) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{49}$$

for any $(t,s) \in \Delta_0'$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &= \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial s}\widehat{U}(t, s+\delta\sqrt{-1})\Big) - 2\pi\\ &= \operatorname{Im}\Big(\log(1-y) + \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\Big(t - \frac{1}{4}\Big)\Big) - 2\pi\\ &= \operatorname{Arg}(1-y) + \operatorname{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(t - \frac{3}{4}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 0 + 4\pi\Big(\frac{1}{4} - t - s\Big) + 4\pi\Big(t - \frac{3}{4}\Big) = 4\pi\Big(-s - \frac{1}{2}\Big) < -2\pi.$$

Hence, (49) holds, as required. □

Lemma 6.7 *The assumption* (19) *holds for* $\widehat{U}(t,s) - \varsigma_R(M_p)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re}\widehat{U}(t, s-\delta\sqrt{-1}) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{50}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. The differential of $F_{t,s}(\delta)$ is given by

$$\begin{aligned}\frac{d}{d\delta} F_{t,s}(\delta) &= \mathrm{Re}\Big(-\sqrt{-1}\,\frac{\partial}{\partial s}\widehat{U}(t, s-\delta\sqrt{-1})\Big) - 2\pi \\ &= \mathrm{Im}\Big(-\log(1-y) - \log\Big(1-\frac{1}{x}\Big) + 4\pi\sqrt{-1}\Big(-t+\frac{1}{4}\Big)\Big) - 2\pi \\ &= -\mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-t-\frac{1}{4}\Big),\end{aligned}$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (38), we can estimate it as follows:

$$\frac{d}{d\delta} F_{t,s}(\delta) < 4\pi\Big(\frac{1}{4} - s + t\Big) + 0 + 4\pi\Big(-t-\frac{1}{4}\Big) = -4\pi s < -4\pi\cdot 0.005,$$

since $s \geq 0.005$ by Lemma 5.1. Hence, (50) holds, as required. □

6.2 The integral (46) is sufficiently small

In this section, in Lemma 6.10, we show that the integral (46) is of sufficiently small order. We use a similar procedure as the saddle-point method of Section 5.2, but we have no critical point in the domain of the integral in this case. So, as a consequence, we can show that the value of the integral is of sufficiently small order.

We note that $U'(t,s)$ uniformly converges on Δ_0' as $N \to \infty$ to

$$\begin{aligned}\widehat{U}'(t,s) = \frac{1}{4\pi\sqrt{-1}}&(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) \\ &+ 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts + \alpha' t + \frac{1}{4}s\Big).\end{aligned}$$

To simplify the calculation of the behavior of $\widehat{U}(t,s)$, we change the variables (t,s) to (u,v) by $u = s+t$ and $v = s-t$. They are in the ranges that $0 < u < \frac{1}{4}$ and $0 < v < \frac{1}{4}$. Then $\widehat{U}(t,s)$ is rewritten as

$$\begin{aligned}\check{U}(u,v) = \frac{1}{4\pi\sqrt{-1}}&(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\,v}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}\,u})) \\ &+ 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{(u-v)^2}{4} + \frac{v^2-u^2}{4} + \alpha'\frac{u-v}{2} + \frac{u+v}{8}\Big).\end{aligned}$$

Its differentials are given by

$$\frac{\partial \check{U}}{\partial u} = -\log(1 - e^{-4\pi\sqrt{-1}\,u}) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big),$$
$$\frac{\partial \check{U}}{\partial v} = -\log(1 - e^{4\pi\sqrt{-1}\,v}) + 4\pi\sqrt{-1}\Big(\frac{p}{4}\cdot\frac{v-u}{2} + \frac{v}{2} - \frac{\alpha'}{2} + \frac{1}{8}\Big).$$

In order to show Lemmas 6.8 and 6.9 below, we calculate the behavior of the function

$$f_{u,v}(\delta_1, \delta_2) = \operatorname{Re}\check{U}(u + \delta_1\sqrt{-1}, v + \delta_2\sqrt{-1}).$$

The differentials of this function are given by

$$\frac{\partial}{\partial\delta_1} f_{u,v}(\delta_1, \delta_2) = \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial u}\check{U}(u + \delta_1\sqrt{-1}, v + \delta_2\sqrt{-1})\Big) \tag{51}$$
$$= \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) - 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big),$$
$$\frac{\partial}{\partial\delta_2} f_{u,v}(\delta_1, \delta_2) = \operatorname{Re}\Big(\sqrt{-1}\,\frac{\partial}{\partial v}\check{U}(u + \delta_1\sqrt{-1}, v + \delta_2\sqrt{-1})\Big) \tag{52}$$
$$= \operatorname{Arg}(1 - y) - 4\pi\Big(\frac{p}{4}\cdot\frac{v-u}{2} + \frac{v}{2} - \frac{\alpha'}{2} + \frac{1}{8}\Big),$$

where $x = e^{4\pi\sqrt{-1}(u+\delta_1\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(v+\delta_2\sqrt{-1})}$.

Lemma 6.8 *Fixing* $(u, v) \in \Delta_0'$ *and* $\delta_2 \in \mathbb{R}$, *we regard* $f_{u,v}(X, \delta_2)$ *as a function of* $X \in \mathbb{R}$. *If* $\frac{p}{2}t + \alpha' \geq \frac{1}{4}$, *then* $f_{u,v}(X, \delta_2)$ *is monotonically decreasing for* $X \in \mathbb{R}$, *and* $f_{u,v}(X, \delta_2) \to -\infty$ *as* $X \to \infty$.

Proof Since $0 < u < \frac{1}{4}$, we have that

$$0 < \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) < 4\pi\Big(\frac{1}{4} - u\Big).$$

Hence,

$$\begin{aligned}\frac{d}{dX} f_{u,v}(X, \delta_2) &< 4\pi\Big(\frac{1}{4} - u\Big) - 4\pi\Big(\frac{p}{4}\cdot\frac{u-v}{2} - \frac{u}{2} + \frac{\alpha'}{2} + \frac{1}{8}\Big)\\ &= -4\pi\Big(\frac{pt}{4} + \frac{u}{2} - \frac{1}{8} + \frac{\alpha'}{2}\Big) = -2\pi\Big(\frac{pt}{2} + \alpha' - \frac{1}{4} + u\Big)\\ &\leq -2\pi u \leq -2\pi\cdot 0.005,\end{aligned}$$

since $u \geq 0.005$ by Lemma 5.1. Therefore, $f_{u,v}(X, \delta_2)$ is monotonically decreasing, as required. □

Lemma 6.9 *Fixing $(u,v)\in\Delta_0'$ and $\delta_1\in\mathbb{R}$, we regard $f_{u,v}(\delta_1,Y)$ as a function of $Y\in\mathbb{R}$. If $\frac{p}{2}t+\alpha'\le\frac{1}{4}$, then $f_{u,v}(\delta_1,Y)$ is monotonically decreasing for $Y\in\mathbb{R}$, and $f_{u,v}(\delta_1,Y)\to-\infty$ as $Y\to\infty$.*

Proof Since $0<v<\frac{1}{4}$, we have that

$$-4\pi\Big(\frac{1}{4}-v\Big)<\operatorname{Arg}(1-y)<0.$$

Hence,

$$\begin{aligned}\frac{d}{dY}f_{u,v}(\delta_1,Y)&<-4\pi\Big(\frac{p}{4}\cdot\frac{v-u}{2}+\frac{v}{2}-\frac{\alpha'}{2}+\frac{1}{8}\Big)\\&=4\pi\Big(\frac{pt}{4}-\frac{v}{2}-\frac{1}{8}+\frac{\alpha'}{2}\Big)=2\pi\Big(\frac{pt}{2}+\alpha'-\frac{1}{4}-v\Big)\\&\le-2\pi v\le-2\pi\cdot0.005,\end{aligned}$$

since $v\ge0.005$ by Lemma 5.1. Therefore, $f_{u,v}(\delta_1,Y)$ is monotonically decreasing, as required. □

Lemma 6.10 *For $\alpha'=0,\pm\frac{1}{8},\pm\frac{1}{4}$,*

$$\int_{\Delta_0'}\exp(NU'(t,s))\,dt\,ds=O(\varepsilon^{N(\varsigma_R(M_p)-\varepsilon)})$$

for some $\varepsilon>0$.

Proof We show that there exists a homotopy $\Delta'_{(\delta)}$ $(0\le\delta\le\delta_0)$ between $\Delta'_{(0)}=\Delta_0'$ and $\Delta'_{(\delta_0)}$ such that

$$\Delta'_{(\delta_0)}\subset\{(t,s)\in\mathbb{C}^2\mid\operatorname{Re}U'(t,s)<\varsigma_R(M_p)-\varepsilon\},\tag{53}$$

$$\partial\Delta'_{(\delta)}\subset\{(t,s)\in\mathbb{C}^2\mid\operatorname{Re}U'(t,s)<\varsigma_R(M_p)-\varepsilon\}.\tag{54}$$

We note that $U'(t,s)$ uniformly converges to $\widehat{U}'(t,s)$ on Δ_0' and the error term is of order $O(1/N^2)$. So we show the existence of such a homotopy for $\widehat{U}'(t,s)$, instead of $U'(t,s)$. We recall that we defined $\check{U}(u,v)$ to be $\widehat{U}'(t,s)$ changing variables by putting $u=s+t$ and $v=s-t$. We construct such a homotopy by using $\check{U}(u,v)$.

For each fixed (u,v), we move (X,Y) from $(0,0)$ along the gradient flow of the function $-\operatorname{Re}\check{U}(u+X\sqrt{-1},v+Y\sqrt{-1})$. Then the value of $\operatorname{Re}\check{U}(u+X\sqrt{-1},v+Y\sqrt{-1})$ monotonically decreases, and in particular, by Lemmas 6.8 and 6.9, it goes to $-\infty$. As for (54), since $\partial\Delta_0'$ is originally included in this domain and the value of $\operatorname{Re}\check{U}$ monotonically decreases, (54) holds. As for (53), since the value of $\operatorname{Re}\check{U}$ uniformly goes to $-\infty$, (53) is satisfied for sufficiently large δ_0. Hence, such a homotopy exists, as required. □

7 Proof of Theorem 1.1 when $|p| = 5$

We gave a proof of Theorem 1.1 when $|p| \geq 6$ in Section 5. Unlike that case, we need additional procedure when $|p| = 5$ in the proof of Theorem 1.1. In this case, the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$ slightly intersects the lines $s = t$ and $s = -t$ (see Figure 4), and we must remove neighborhoods of these lines from this domain before we apply the Poisson summation formula and the saddle-point method. We show this procedure and give a proof of Theorem 1.1 when $|p| = 5$ in this section. Since M_{-5} is homeomorphic to M_5 with opposite orientation as mentioned at the beginning of Section 5, it is sufficient to show the proof only for $p = 5$.

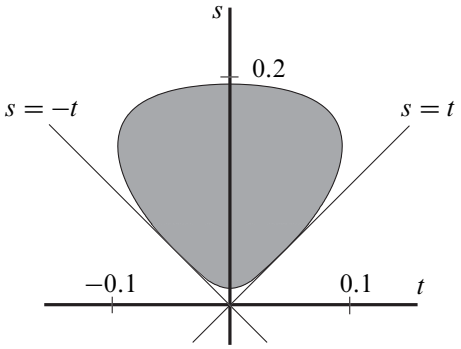


Figure 4: The domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$

We verify that the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$ intersects the lines $s = t$ and $s = -t$, as follows. As shown in Appendix E, $\mathrm{Re}\,\widehat{V}(t,s)$ is presented by

$$\mathrm{Re}\,\widehat{V}(t,s) = \tfrac{1}{2}\Lambda(2s-2t) + \tfrac{1}{2}\Lambda(2s+2t) \geq \varsigma_R(M_5),$$

where $\Lambda(\cdot)$ is as defined in Appendix E. As mentioned in Appendix E, the maximal value of $\Lambda(\cdot)$ is given by $\Lambda\left(\frac{1}{6}\right)$, and $\frac{1}{2}\Lambda\left(\frac{1}{6}\right) = 0.0807665\ldots$, while $\varsigma_R(M_5)$ is given by

$$\varsigma_R(M_5) = 0.07809485\ldots,$$

which is smaller than $\frac{1}{2}\Lambda\left(\frac{1}{6}\right)$. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, the values of $s-t$ and $s+t$ can be zero in the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$. (On the other hand, we note that $\varsigma_R(M_p) > \frac{1}{2}\Lambda\left(\frac{1}{6}\right)$ for $p \geq 6$.) Therefore, the domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$ intersects the lines $s = t$ and $s = -t$, unlike the case of $p \geq 6$.

We recall that $\widehat{\tau}_N(M_5)$ is presented by the following sum, by (24) putting $p=5$:

$$\begin{aligned}(55)\quad \widehat{\tau}_N(M_5) = \frac{1}{1-q}e^{-(5\pi\sqrt{-1}/4)N}\sqrt{-1}^{(N-1)/2}N^{-1/2}\\ \times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0}(q^{i/2}-q^{-i/2})\\ \times\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)\\ +O(e^{N(\varsigma_R(M_p)-\varepsilon)}),\end{aligned}$$

where we recall that Δ_0 is given by

$$\Delta_0=\big\{(t,s)\in\mathbb{R}^2 \mid 0\le t+s\le\tfrac{1}{2},\ 0\le s-t\le\tfrac{1}{2}\big\}.$$

We restrict Δ_0 to

$$\begin{aligned}\Delta_0'' &= \{(t,s)\in\Delta_0 \mid 0.001\le t+s,\ 0.001\le s-t\}\\ &= \big\{(t,s)\in\mathbb{R}^2 \mid 0.001\le t+s\le\tfrac{1}{2},\ 0.001\le s-t\le\tfrac{1}{2}\big\}.\end{aligned}$$

In fact, in the sum (55), the summand itself is too large for our purpose, but the values of the restricted sums along the lines $s=t+\rho$ and $s=-t+\rho$ are of sufficiently small order for any fixed ρ with $0\le\rho\le 0.001$; we show this in Lemma F.1. Hence, we obtain that

$$\begin{aligned}(56)\quad \widehat{\tau}_N(M_5) = \frac{1}{1-q}e^{-(5\pi\sqrt{-1}/4)N}\sqrt{-1}^{(N-1)/2}N^{-1/2}\\ \times \sum_{(1/2-i/N,1/2-j/N)\in\Delta_0''}(q^{i/2}-q^{-i/2})\\ \times\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)\\ +O(e^{N(\varsigma_R(M_p)-\varepsilon)}).\end{aligned}$$

Further, in order to apply the Poisson summation formula and the saddle-point method later, we consider to restrict Δ_0'' to Δ_0' of the following lemma; we will use the defining inequalities of Δ_0' later when we verify the assumption of the Poisson summation formula and the saddle-point method in Sections 7.2 and 7.3.

Lemma 7.1 *We put*

$$\Delta_0'=\{(t,s)\in\mathbb{R}^2 \mid 0.001\le t+s\le 0.26,\ 0.001\le s-t\le 0.26,\ |t|\le 0.099,\ s\le 0.2\}.$$

Then the domain

$$\{(t,s)\in\Delta_0'' \mid \operatorname{Re}\widehat{V}(t,s)\ge\varsigma_R(M_5)-\varepsilon\}$$

is included in Δ_0' for sufficiently small $\varepsilon>0$.

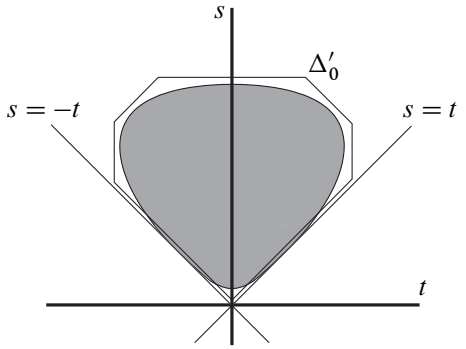


Figure 5: The domain $\{\mathrm{Re}\,\widehat{V}(t,s) \geq \varsigma_R(M_5)\}$ (the gray area) and the domain Δ_0'

We give a proof of the lemma in Appendix G. See Figure 5 for a graphical representation of the inclusion of the lemma.

Proof of Theorem 1.1 when $p = 5$ We recall that $\widehat{\tau}_N(M_5)$ is presented by (56). Hence, by Lemma 7.1, we obtain that

$$\begin{aligned}(57)\quad \widehat{\tau}_N(M_5) = {}& \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{-1/2} \\ & \times \sum_{(1/2-i/N,\,1/2-j/N)\in\Delta_0'} (q^{i/2} - q^{-i/2}) \\ & \times \exp\Big(N \cdot V\Big(\frac{1}{2} - \frac{i}{N}, \frac{1}{2} - \frac{j}{N} - \frac{1}{2N}\Big)\Big) \\ & + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).\end{aligned}$$

By the Poisson summation formula (Proposition 7.2), this sum is expressed by the integrals

$$\begin{aligned}\widehat{\tau}_N(M_5) = {}& \frac{1}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \\ & \times \Bigg(\int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N V(t,s))\, dt\, ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)}) \\ & \quad + \int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp\big(N(V(t,s) + 2\pi\sqrt{-1}\,t)\big)\, dt\, ds \\ & \quad + O(e^{N(\varsigma_R(M_p)-\varepsilon)}) \Bigg)\end{aligned}$$

for some $\varepsilon > 0$. Further, similarly as in Section 5, by Lemma 5.2, the second integral

is equal to the first integral, and we can rewrite the above formula as

$$\widehat{\tau}_N(M_5) = \frac{2}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \times \int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N V(t,s))\, dt\, ds + O(e^{N(\varsigma_R(M_p)-\varepsilon)}).$$

Let (t_0, s_0) be the critical point of $\widehat{V}(t,s)$ given in Section 7.1. Then, by applying the saddle-point method (Proposition 7.8), we obtain that

$$\widehat{\tau}_N(M_5) = \frac{2}{1-q} e^{-(5\pi\sqrt{-1}/4)N} \sqrt{-1}^{(N-1)/2} N^{3/2} \times (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0}) \exp(N\widehat{V}(t_0,s_0)) \cdot \frac{2\pi}{N} (V_{tt}V_{ss} - V_{ts}^2)^{-1/2} \times \Big(1 + O\Big(\frac{1}{N}\Big)\Big),$$

in a similar way as in Section 5. Hence, we obtain the required formula of the theorem. □

7.1 A critical point of $\widehat{V}(t,s)$

In this section, we characterize a critical point (t_0, s_0) of $\widehat{V}(t,s)$, which we use in the proof of Theorem 1.1.

In the same way as in Section 5.1, we show that there is a single critical point in the domain

$$\{(t,s) \in \mathbb{C}^2 \mid 0 < \mathrm{Re}(t+s) < \tfrac{1}{4},\ 0 < \mathrm{Re}(s-t) < \tfrac{1}{4},\ \mathrm{Re}\, t \geq 0\}. \tag{58}$$

As in Section 5.1, putting $z = e^{-4\pi\sqrt{-1}\,t}$ and $w = e^{-4\pi\sqrt{-1}\,s}$, such a critical point is obtained from a solution of

$$z^2 - \left(\frac{z+z^{5/2}}{z^{5/2}z+1} + 1 + \frac{z^{5/2}z+1}{z+z^{5/2}}\right)z + 1 = 0, \quad w = \frac{z+z^{5/2}}{z^{5/2}z+1}.$$

We can verify that there is a single solution which satisfies (58); we denote it by (t_0, s_0). It is numerically given by

$$(t_0, s_0) = (0.0916106\ldots - \sqrt{-1}\cdot 0.0574205\ldots, 0.1238288\ldots - \sqrt{-1}\cdot 0.0530897\ldots).$$

Further, we put

$$\varsigma(M_5) = \widehat{V}(t_0, s_0), \quad \varsigma_R(M_5) = \mathrm{Re}\,\widehat{V}(t_0, s_0) = 0.07809485\ldots.$$

7.2 The Poisson summation formula for V

The aim of this section is to show the following proposition, which is obtained from the Poisson summation formula (Proposition 3.4 and Remark 3.5).

Proposition 7.2 *We have*

$$\begin{aligned}\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'} &(q^{i/2}-q^{-i/2})\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)\\ &= \int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(NV(t,s))\,dt\,ds\\ &\quad+\int_{\Delta_0'}(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\big(N(V(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds\\ &\quad+O(e^{N(\varsigma_R(M_5)-\varepsilon)}).\end{aligned}$$

Proof We put a smooth function $g\colon\mathbb{R}\to\mathbb{R}$ by

$$g(t)=\begin{cases}1 & \text{if } t\geq 0.001,\\ 0 & \text{if } t\leq 0.0005,\end{cases}\qquad 0\leq g(t)\leq 1 \quad \text{if } 0.0005<t<0.001$$

for $t\in\mathbb{R}$. Further, by extending Δ_0', we put

$$\Delta_0'''=\{(t,s)\in\mathbb{R}^2 \mid 0.0005\leq t+s\leq 0.26,\ 0.0005\leq s-t\leq 0.26,\ |t|\leq 0.1,\ s\leq 0.2\}.$$

Then, by applying the Poisson summation formula (Proposition 3.4 and Remark 3.5) to $g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}t}-e^{2\pi\sqrt{-1}t})V(t,s)$, we obtain that

$$\begin{aligned}\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0'''} &g\Big(1-\frac{i}{N}-\frac{j}{N}\Big)g\Big(\frac{i}{N}-\frac{j}{N}\Big)(q^{i/2}-q^{-i/2})\\ &\times\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)\\ =\int_{\Delta_0'''} &g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp(NV(t,s))\,dt\,ds\\ &+\int_{\Delta_0'''}g(s+t)g(s-t)(e^{-2\pi\sqrt{-1}\,t}-e^{2\pi\sqrt{-1}\,t})\exp\big(N(V(t,s)+2\pi\sqrt{-1}\,t)\big)\,dt\,ds\\ &+O(e^{N(\varsigma_R(M_5)-\varepsilon)}),\end{aligned}$$

noting that we verify the assumption of the Poisson summation formula in Lemma 7.3 below. As we show in Appendix F, we can ignore the error term, which is derived from the function g. Hence, from the above formula, we obtain the required formula of the proposition. □

Lemma 7.3 *When we apply Proposition 3.4 and Remark 3.5 in the proof of Proposition 7.2, their assumptions hold.*

Proof It is sufficient to show the assumptions for $\widehat{V}(t,s)$. We verify (20), (17), (18) and (19) in Lemmas 7.4, 7.5, 7.6 and 7.7, respectively. The other assumptions of Proposition 3.4 can be verified easily. □

Before we show Lemmas 7.4, 7.5, 7.6 and 7.7, we note some inequalities, which we use in the proofs of the lemmas. Let $(t,s)\in\Delta_0'$. The following inequalities hold for $\delta\in\mathbb{R}$ and $x=e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$:

$$\operatorname{Arg}\Big(1-\frac{1}{x}\Big)\begin{cases}<4\pi\big(\frac{1}{4}-t-s\big)<\pi & \text{if } 0<t+s<\frac{1}{4},\\ \le 0 & \text{if } \frac{1}{4}\le t+s\le 0.26,\end{cases}$$

$$-\operatorname{Arg}\Big(1-\frac{1}{x}\Big)\begin{cases}\le 0 & \text{if } 0<t+s\le\frac{1}{4},\\ <4\pi\big(t+s-\frac{1}{4}\big)\le 4\pi\cdot 0.01 & \text{if } \frac{1}{4}<t+s\le 0.26.\end{cases}$$

Hence,

$$\operatorname{Arg}\Big(1-\frac{1}{x}\Big)<\pi,\quad -\operatorname{Arg}\Big(1-\frac{1}{x}\Big)<4\pi\cdot 0.01. \tag{59}$$

Further, the following inequalities hold for $\delta\in\mathbb{R}$ and $y=e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$:

$$\operatorname{Arg}(1-y)\begin{cases}\le 0 & \text{if } 0<s-t\le\frac{1}{4},\\ <4\pi\big(s-t-\frac{1}{4}\big)<4\pi\cdot 0.01 & \text{if } \frac{1}{4}<s-t\le 0.26,\end{cases}$$

$$-\operatorname{Arg}(1-y)\begin{cases}<4\pi\big(\frac{1}{4}-s+t\big)<\pi & \text{if } 0<s-t<\frac{1}{4},\\ \le 0 & \text{if } \frac{1}{4}\le s-t\le 0.26.\end{cases}$$

Hence,

$$\operatorname{Arg}(1-y)<4\pi\cdot 0.01,\quad -\operatorname{Arg}(1-y)<\pi. \tag{60}$$

Lemma 7.4 *The assumption* (20) *holds for* $\widehat{V}(t,s)-\varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta)=\operatorname{Re}\widehat{V}(t+\delta\sqrt{-1},s)-\varsigma_R(M_5)-4\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon>0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta)<-\varepsilon \tag{61}$$

for any $(t,s)\in\Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.8, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta}F_{t,s}(\delta) = -\mathrm{Arg}(1-y) + \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(-\frac{5}{2}t+s-1\Big),$$

where we put $x = e^{4\pi\sqrt{-1}\,(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\begin{aligned}\frac{d}{d\delta}F_{t,s}(\delta) &< \pi + \pi + 4\pi\Big(-\frac{5}{2}t+s-1\Big)\\ &= 4\pi\Big(-\frac{5}{2}t+s-\frac{1}{2}\Big) < 4\pi\Big(\frac{5}{2}\cdot 0.1 + 0.2 - \frac{1}{2}\Big) = -4\pi\cdot 0.05,\end{aligned}$$

where we obtain the second inequality since $|t| \le 0.1$ and $s \le 0.2$ by Lemma 7.1. Hence, (61) holds, as required. □

Lemma 7.5 *The assumption* (17) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \mathrm{Re}\,\widehat{V}(t-\delta\sqrt{-1}, s) - \varsigma_R(M_p) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_{t,s}(\delta) < -\varepsilon \tag{62}$$

for any $(t,s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.9, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta}F_{t,s}(\delta) = \mathrm{Arg}(1-y) - \mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(\frac{5}{2}t-s-\frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < 4\pi\cdot 0.01 + 4\pi\cdot 0.01 + 4\pi\Big(\frac{5}{2}t-s-\frac{1}{2}\Big) \le 4\pi\Big(0.02+\frac{5}{2}\cdot 0.1-\frac{1}{2}\Big) = -4\pi\cdot 0.23,$$

where we obtain the second inequality since $|t| \le 0.1$ and $s > 0$ by Lemma 7.1. Hence, (62) holds, as required. □

Lemma 7.6 *The assumption* (18) *holds for* $\widehat{V}(t,s) - \varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re} \widehat{V}(t, s + \delta\sqrt{-1}) - \varsigma_R(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{63}$$

for any $(t, s) \in \Delta_0'$.

To show these, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.10, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta} F_{t,s}(\delta) = \operatorname{Arg}(1-y) + \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) + 4\pi\Big(t - \frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s+\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t+\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta} F_{t,s}(\delta) < 4\pi \cdot 0.01 + \pi + 4\pi\Big(t - \frac{1}{2}\Big) = 4\pi\Big(0.01 + \frac{1}{4} + t - \frac{1}{2}\Big) \le -4\pi \cdot 0.14,$$

where we obtain the last inequality since $|t| \le 0.1$ by Lemma 7.1. Hence, (63) holds, as required. □

Lemma 7.7 *The assumption* (19) *holds for* $\widehat{V}(t, s) - \varsigma_R(M_5)$.

Proof We put

$$F_{t,s}(\delta) = \operatorname{Re} \widehat{V}(t, s - \delta\sqrt{-1}) - \varsigma_R(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta} F_{t,s}(\delta) < -\varepsilon \tag{64}$$

for any $(t, s) \in \Delta_0'$.

To show this, we estimate the differential of $F_{t,s}(\delta)$, as follows. As shown in the proof of Lemma 5.11, the differential of $F_{t,s}(\delta)$ is given by

$$\frac{d}{d\delta} F_{t,s}(\delta) = -\operatorname{Arg}(1-y) - \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) + 4\pi\Big(-t - \frac{1}{2}\Big),$$

where we put $x = e^{4\pi\sqrt{-1}(t+s-\delta\sqrt{-1})}$ and $y = e^{4\pi\sqrt{-1}(s-t-\delta\sqrt{-1})}$. By (59) and (60), we can estimate it as follows:

$$\frac{d}{d\delta}F_{t,s}(\delta) < \pi + 4\pi\cdot 0.01 + 4\pi\Big(-t-\frac{1}{2}\Big) = 4\pi\Big(\frac{1}{4}+0.01-t-\frac{1}{2}\Big) \le -4\pi\cdot 0.14,$$

where we obtain the last inequality since $|t| \le 0.1$ by Lemma 7.1. Hence, (64) holds, as required. □

7.3 Verifying the assumption of the saddle-point method for V

The aim of this section is to show the following proposition, which is obtained from the saddle-point method (Proposition 3.2 and Remark 3.3).

Proposition 7.8 *We have*

$$\int_{\Delta_0'} (e^{-2\pi\sqrt{-1}\,t} - e^{2\pi\sqrt{-1}\,t}) \exp(N V(t,s))\, dt\, ds$$
$$= (e^{-2\pi\sqrt{-1}\,t_0} - e^{2\pi\sqrt{-1}\,t_0}) \exp(N\widehat{V}(t_0,s_0))\cdot\frac{2\pi}{N}(V_{tt}V_{ss}-V_{ts}^2)^{-1/2}\Big(1+O\Big(\frac{1}{N}\Big)\Big).$$

Proof When we apply the saddle-point method, unlike the case of Section 5, it is a problem that the boundary of Δ_0' is not included in the domain $\{\mathrm{Re}\,\widehat{V}(t,s) < \varsigma_R(M_5)\}$. To make an appropriate boundary, we consider to extend Δ_0' by adding some additional parts. We note that Δ_0' intersects $\{\mathrm{Re}\,\widehat{V}(t,s) \ge \varsigma_R(M_5)\}$ along the lines $s = t+\rho$ and $s = -t+\rho$, where we put $\rho = 0.001$. We put $\partial_1\Delta_0'$ to be the edge of Δ_0' on $\{s = t+\rho\}$, and put $\partial_2\Delta_0'$ to be the edge of Δ_0' on $\{s = -t+\rho\}$. As in the proof of Lemma 7.12, we move $\partial_1\Delta_0'$ and $\partial_2\Delta_0'$ along the gradient flow of $-\mathrm{Re}\,\widehat{V}$, and denote the resulting segments by $\partial_1^{(\delta)}\Delta_0'$ and $\partial_2^{(\delta)}\Delta_0'$ for $\delta \ge 0$. As shown in the proof of Lemma 7.12, $\partial_1^{(\delta_0)}\Delta_0'$ and $\partial_2^{(\delta_0)}\Delta_0'$ are included in

$$\{(t,s)\in\mathbb{C}^2 \mid \mathrm{Re}\,\widehat{V}(t,s) < \varsigma_R(M_5)-\varepsilon\} \tag{65}$$

for a sufficiently large δ_0 and a sufficiently small $\varepsilon > 0$. We define a new domain $\widehat{\Delta}_0'$ by

$$\widehat{\Delta}_0' = \Delta_0' \cup \bigcup_{0\le\delta\le\delta_0} (\partial_1^{(\delta)}\Delta_0' \cup \partial_2^{(\delta)}\Delta_0').$$

Then the boundary of $\widehat{\Delta}_0'$ is included in (65) for a sufficiently small $\varepsilon > 0$. When we extend Δ_0' to $\widehat{\Delta}_0'$, we have the error term of the value of the integral of the proposition. We note that, as in Appendix F, we can show that this error term is of sufficiently small order.

Hence, in a similar way as in Section 5.2, we can apply the saddle-point method (Proposition 3.2 and Remark 3.3) for $\widehat{\Delta}'_0$, noting that we verify the assumption of the saddle-point method in Lemma 7.12. Therefore, we obtain the required formula of the proposition. □

As in Section 5.2, putting $u = t + s$ and $v = s - t$, we put $\check{V}(u, v)$ to be $\widehat{V}(t, s)$, in order to simplify the calculation.

Lemma 7.9 *Fixing $(u, v) \in \Delta'_0$ and $\delta_2 \in \mathbb{R}$, we regard $f_{u,v}(X, \delta_2)$ as a function of $X \in \mathbb{R}$.*

(1) *If $5v \geq u$ and $5v \geq 9u - 2$, then $f_{u,v}(X, \delta_2)$ is monotonically increasing for $X \in \mathbb{R}$.*

(2) *If $9u - 2 < 5v < u$ or $u < 5v < 9u - 2$, then $f_{u,v}(X, \delta_2)$ has a unique minimal point at $X = g_1(u, v)$, where*

$$g_1(u, v) = \frac{1}{4\pi} \log \frac{\sin \frac{\pi}{2}(u - 5v)}{\sin \frac{\pi}{2}(2 - 9u + 5v)},$$

ie $f_{u,v}(X, \delta_2)$ is monotonically decreasing for $X < g_1(u, v)$, and is monotonically increasing for $X > g_1(u, v)$.

(3) *If $5v \leq u$ and $5v \leq 9u - 2$, then $f_{u,v}(X, \delta_2)$ is monotonically decreasing for $X \in \mathbb{R}$.*

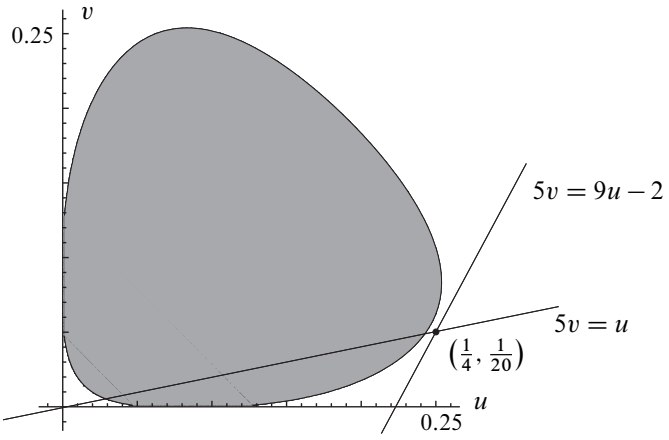


Figure 6: The domain $\{\mathrm{Re}\,\check{V}(u, v) \geq \varsigma_R(M_5)\}$ and the lines $5v = u$ and $5v = 9u - 2$

Remark 7.10 The point $\left(\frac{1}{4}, \frac{1}{20}\right)$ is the intersection of the two lines $5v = u$ and $5v = 9u - 2$, which appear in the statement of the lemma. This point is in the exterior of the domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_5)\}$, since

$$\begin{aligned}\check{V}\left(\tfrac{1}{4}, \tfrac{1}{20}\right) - \varsigma_R(M_5) &= \tfrac{1}{2}\Lambda\left(\tfrac{1}{2}\right) + \tfrac{1}{2}\Lambda\left(\tfrac{1}{10}\right) - \varsigma_R(M_5) \\ &= 0 + 0.0735101\ldots - 0.0780948\ldots = -0.0045847\ldots < 0.\end{aligned}$$

In fact, the domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_5)\}$, and the two lines are located as shown in Figure 6. Hence, some cases in the statement of the lemma might not be realized if we were to choose Δ_0' sufficiently close to the domain $\{\mathrm{Re}\,\check{V}(u,v) \geq \varsigma_R(M_5)\}$.

Proof of Lemma 7.9 We can show the lemma in a similar way as the proof of Lemma 5.3. A different point is that "which cases are actually realized"; we note that there are 4 cases depending on the signs of $5v - u$ and $5v - 9u + 2$. It is also an important point that the point $(u,v) = \left(\frac{1}{4}, \frac{1}{20}\right)$ $\left(\text{ie } (t,s) = \left(\frac{1}{10}, \frac{3}{20}\right)\right)$ does not belong to Δ_0', because the gradient flow of $-\mathrm{Re}\,\check{V}$ does not behave well only at this point. Hence, we can show the lemma in a similar way as the proof of Lemma 5.3. □

Lemma 7.11 *Fixing $(u,v) \in \Delta_0'$ and $\delta_1 \in \mathbb{R}$, we regard $f_{u,v}(\delta_1, Y)$ as a function of $Y \in \mathbb{R}$.*

(1) *If $5u \leq 9v$, then $f_{u,v}(\delta_1, Y)$ is monotonically decreasing for $Y \in \mathbb{R}$.*

(2) *If $5u > 9v$, then $f_{u,v}(\delta_1, Y)$ has a unique minimal point at $Y = g_2(u,v)$, where*

$$g_2(u,v) = \frac{1}{4\pi}\log\frac{\sin\frac{\pi}{2}(v - 5u + 2)}{\sin\frac{\pi}{2}(5u - 9v)},$$

ie $f_{u,v}(\delta_1, Y)$ is monotonically decreasing for $Y < g_2(u,v)$, and is monotonically increasing for $Y > g_2(u,v)$.

Proof We can show the lemma in a similar way as the proof of Lemma 5.5. The main difference is that (3) of Lemma 5.5 does not happen in this lemma. □

Lemma 7.12 *When we apply Proposition 3.2 (saddle-point method) in the proof of Proposition 7.8, the assumption of Proposition 3.2 holds.*

Proof We can show the lemma in a similar way as the proof of Lemma 5.6. A difference is that we use $\widehat{\Delta}_0'$ instead of Δ_0' in this proof. When we make a homotopy of $\widehat{\Delta}_0'$, we move $\widehat{\Delta}_0'$ along the gradient flow of $-\mathrm{Re}\,\check{V}$. Then the resulting domain after moving by the homotopy is the same as that of $\widehat{\Delta}_0'$. Hence, we can show the lemma in a similar way as the proof of Lemma 5.6. □

Appendix A Gauss sum

The aim of this section is to show Lemma A.1 below. For a textbook of number theory, see eg [10].

Lemma A.1 *Let N be a positive odd integer, and let A be $e^{\pi\sqrt{-1}/N}$. Then*

$$\sum_{n\in\mathbb{Z}/N\mathbb{Z}} (-A)^{n^2} = (-\sqrt{-1})^{(N-1)/2}\sqrt{N}.$$

We note that

$$-A = e^{\pi\sqrt{-1}+\pi\sqrt{-1}/N} = e^{(2\pi\sqrt{-1}/N)\cdot(N+1)/2} = e^{(2\pi\sqrt{-1}/N)\cdot\bar{2}},$$

where we denote by $\bar{2}$ the inverse of 2 in $\mathbb{Z}/N\mathbb{Z}$. Hence, the sum of the lemma is a Gauss sum.

It is known (see eg [10]) that we can calculate the value of a Gauss sum concretely. We review this procedure in the remainder of this section to show Lemma A.1.

Let b be a positive odd integer, and let a to an integer coprime to b. We put

$$G(a,b) = \sum_{n\in\mathbb{Z}/b\mathbb{Z}} e^{(2\pi\sqrt{-1}/b)\cdot an^2},$$

and call it a *Gauss sum*. It is known (see [10]) that

$$G(a,b) = \Big(\frac{a}{b}\Big)G(1,b),$$

where $\big(\frac{a}{b}\big)$ is a natural generalization of the Legendre symbol. Our aim is to calculate $G(\bar{2},N)$ for any positive odd integer N.

The simplest case of Lemma A.1 is the case where N is an odd prime. We show this case in the following lemma:

Lemma A.2 *For an odd prime ℓ,*

$$G(\bar{2},\ell) = (-\sqrt{-1})^{(\ell-1)/2}\sqrt{\ell}.$$

Proof We have that

$$G(\bar{2},\ell) = G(2,\ell) = \Big(\frac{2}{\ell}\Big)G(1,\ell).$$

Further, it is known (see [10]) that

$$\left(\frac{2}{\ell}\right) = (-1)^{(\ell^2-1)/8}, \qquad G(1,\ell) = \begin{cases} \sqrt{\ell} & \text{if } \ell \equiv 1 \mod 4, \\ \sqrt{-1}\sqrt{\ell} & \text{if } \ell \equiv 3 \mod 4. \end{cases}$$

Hence, we can verify the lemma for each $\ell \equiv 0, 1, \dots, 7 \mod 8$. □

The second simplest case of Lemma A.1 is the case where N is an odd prime power. We show this case in the following lemma:

Lemma A.3 *For an odd prime ℓ and a positive integer r,*

$$G(\bar{2}, \ell^r) = (-\sqrt{-1})^{(\ell^r-1)/2}\sqrt{\ell^r}.$$

Proof It is known (see [10]) that

$$G(a, \ell^r) = \ell G(a, \ell^{r-2}).$$

By Lemma A.2, it is sufficient to show the lemma recursively, assuming the lemma for $r-2$. We have that

$$G(\bar{2}, \ell^r) = \ell G(\bar{2}, \ell^{r-2}) = (-\sqrt{-1})^{(\ell^{r-2}-1)/2}\sqrt{\ell^r}.$$

Hence, it is sufficient to show that

$$(-\sqrt{-1})^{(\ell^r-1)/2} = (-\sqrt{-1})^{(\ell^{r-2}-1)/2}. \tag{66}$$

We have that

$$\frac{\ell^r-1}{2} - \frac{\ell^{r-2}-1}{2} = \frac{\ell^r-\ell^{r-2}}{2} = \frac{\ell^{r-2}(\ell-1)(\ell+1)}{2}.$$

This is divisible by 4, since $(\ell-1)(\ell+1)$ is divisible by 8. Hence, (66) holds, as required. □

Let b_1 and b_2 be coprime odd positive integers, and let a be an integer coprime to $b_1 b_2$. Then it is known (see [10]) that

$$G(a, b_1 b_2) = G(ab_2, b_1)G(ab_1, b_2).$$

Further, it is known (see [10]) that, for coprime odd positive integers b_1 and b_2,

$$\left(\frac{b_1}{b_2}\right)\left(\frac{b_2}{b_1}\right) = (-1)^{((b_1-1)/2)((b_2-1)/2)},$$

as a natural generalization of the reciprocity law.

Proof of Lemma A.1 We show the lemma by induction on N. It is sufficient to show that, for coprime odd positive integers N_1 and N_2, the lemma holds for N_1N_2, assuming the lemma for N_1 and for N_2. Hence, it is sufficient to show that

$$G(\bar{2}, N_1N_2) = (-\sqrt{-1})^{(N_1N_2-1)/2}\sqrt{N_1N_2},$$

assuming that

$$G(\bar{2}, N_1) = (-\sqrt{-1})^{(N_1-1)/2}\sqrt{N_1} \quad \text{and} \quad G(\bar{2}, N_2) = (-\sqrt{-1})^{(N_2-1)/2}\sqrt{N_2}.$$

We have that

$$\begin{aligned} G(\bar{2}, N_1N_2) &= G(\bar{2}N_2, N_1)G(\bar{2}N_1, N_2) \\ &= \left(\frac{N_2}{N_1}\right)G(\bar{2}, N_1)\left(\frac{N_1}{N_2}\right)G(\bar{2}, N_2) \\ &= (-1)^{((N_1-1)/2)((N_2-1)/2)}G(\bar{2}, N_1)G(\bar{2}, N_2) \\ &= (-1)^{((N_1-1)/2)((N_2-1)/2)}(-\sqrt{-1})^{(N_1-1)/2}(-\sqrt{-1})^{(N_2-1)/2}\sqrt{N_1N_2}. \end{aligned}$$

Hence, it is sufficient to show that

$$(-1)^{((N_1-1)/2)((N_2-1)/2)}(-\sqrt{-1})^{(N_1-1)/2}(-\sqrt{-1})^{(N_2-1)/2}\sqrt{N_1N_2} = (-\sqrt{-1})^{(N_1N_2-1)/2}.$$

This formula holds, since

$$\begin{aligned} &2\cdot\frac{N_1-1}{2}\cdot\frac{N_2-1}{2} + \frac{N_1-1}{2} + \frac{N_2-1}{2} - \frac{N_1N_2-1}{2} \\ &= \frac{1}{2}((N_1-1)(N_2-1) + N_1 - 1 + N_2 - 1 - N_1N_2 + 1) = 0. \end{aligned}$$

Therefore, the lemma holds. □

Appendix B Critical points of $\widehat{V}(t, s)$

The aim of this section is to show the following lemma, which implies that there exists a single critical point of $\widehat{V}(t, s)$ in the domain (28).

Lemma B.1 *The system of equations* (29) *has a single solution in the domain* (28).

Before showing Lemma B.1, we show the following lemma:

Lemma B.2 *Let p be an integer > 4. Then the equation*

$$z^{p/2} + z^{-p/2} = z^2 + z^{-2} - z - z^{-1} - 2$$

has exactly two solutions in the domain

$$D = \left\{ e^{-4\pi\sqrt{-1}\,t} \in \mathbb{C} \;\middle|\; 0 < \operatorname{Re} t < \frac{1}{p} \right\}.$$

Proof Since the equation of the lemma is a polynomial equation of degree $\leq 2p$, we can verify the lemma by calculating all solutions concretely for a concrete p. So we assume that $p \geq 100$ in the remainder of this proof.

We put $z = e^{-4\pi\sqrt{-1}\,t}$. Then the equation of the lemma is rewritten as

$$e^{p\cdot 2\pi\sqrt{-1}\,t} + e^{-p\cdot 2\pi\sqrt{-1}\,t} = e^{8\pi\sqrt{-1}\,t} + e^{-8\pi\sqrt{-1}\,t} - e^{4\pi\sqrt{-1}\,t} - e^{-4\pi\sqrt{-1}\,t} - 2. \tag{67}$$

We note that, when t_0 is a solution of this equation, $\overline{t_0}$ is also a solution. Hence, it is sufficient to show that there is a single solution of (67) in the domain

$$D' = \left\{ t \in \mathbb{C} \;\middle|\; 0 < \operatorname{Re} t < \frac{1}{p},\ 0 \leq \operatorname{Im} t \right\}.$$

We put $t = x + y\sqrt{-1}$. Then $z = e^{-4\pi\sqrt{-1}\,x} e^{4\pi y}$ and $z^{p/2} = e^{-p\cdot 2\pi\sqrt{-1}\,x} e^{p\cdot 2\pi y}$. In the remainder of this proof, we show that there is a single solution of (67) in D' by decomposing D' into seven subdomains in the following seven cases, respectively; there is no solution in Cases 1–6 and a single solution in Case 7.

Case 1 We show that, in $\left\{ x + y\sqrt{-1} \in D' \mid y \geq \frac{0.3}{p} \right\}$, there is no solution of (67), as follows.

Putting $u = z^{p/2}$, since

$$|u + u^{-1}|^2 = (u + u^{-1})(\overline{u} + \overline{u}^{-1}) = |u|^2 + |u|^{-2} + \frac{u}{\overline{u}} + \frac{\overline{u}}{u} \geq |u|^2 + |u|^{-2} - 2,$$

we have that

$$|z^{p/2} + z^{-p/2}| \geq \sqrt{|z^{p/2}|^2 + |z^{-p/2}|^2 - 2} \geq \sqrt{e^{p\cdot 4\pi y} + e^{-p\cdot 4\pi y} - 2} = e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}.$$

Further,

$$|z^2 + z^{-2} - z - z^{-1} - 2| \leq |z|^2 + |z|^{-2} + |z| + |z|^{-1} + 2 = e^{8\pi y} + e^{-8\pi y} + e^{4\pi y} + e^{-4\pi y} + 2.$$

Hence, putting

$$f(y) = e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y} - (e^{8\pi y} + e^{-8\pi y} + e^{4\pi y} + e^{-4\pi y} + 2),$$

it is sufficient to show that $f(y) > 0$ for $y \geq \frac{0.3}{p}$. Since

$$\begin{aligned} f'(y) &= p \cdot 2\pi e^{p\cdot 2\pi y} - 8\pi e^{8\pi y} - 4\pi e^{4\pi y} + p \cdot 2\pi e^{-p\cdot 2\pi y} + 8\pi e^{-8\pi y} + 4\pi e^{-4\pi y} \\ &> (p-6) \cdot 2\pi e^{p\cdot 2\pi y} + 8\pi(e^{p\cdot 2\pi y} - e^{8\pi y}) - 4\pi(e^{p\cdot 2\pi y} - e^{4\pi y}) > 0, \end{aligned}$$

$f(y)$ is monotonically increasing. Therefore, it is sufficient to show that $f\left(\frac{0.3}{p}\right) > 0$. In fact,

$$\begin{aligned} f\Big(\frac{0.3}{p}\Big) &= e^{2\pi\cdot 0.3} - e^{-2\pi\cdot 0.3} - (e^{8\pi\cdot 0.3/p} + e^{-8\pi\cdot 0.3/p} + e^{4\pi\cdot 0.3/p} + e^{-4\pi\cdot 0.3/p} + 2) \\ &> e^{2\pi\cdot 0.3} - e^{-2\pi\cdot 0.3} - (e^{8\pi\cdot 0.003} + e^{-8\pi\cdot 0.003} + e^{4\pi\cdot 0.003} + e^{-4\pi\cdot 0.003} + 2) \\ &= 0.427117\ldots > 0, \end{aligned}$$

since $p \geq 100$. Hence, there is no solution in the domain of this case, as required.

Case 2 We show that, in $\left\{x + y\sqrt{-1} \in D' \mid \frac{0.01}{p} \leq x \leq \frac{0.49}{p},\ 0 < y < \frac{0.3}{p}\right\}$, there is no solution of (67), as follows.

Since $z = e^{-4\pi\sqrt{-1}\,x} e^{p\cdot 2\pi y}$, we have that

$$\begin{aligned} \operatorname{Im}(z^{p/2} + z^{-p/2}) &= -(e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}) \sin(p \cdot 2\pi x), \\ \operatorname{Im}(z^2 + z^{-2} - z - z^{-1} - 2) &= -(e^{8\pi y} - e^{-8\pi y}) \sin 8\pi x + (e^{4\pi y} - e^{-4\pi y}) \sin 4\pi x. \end{aligned}$$

Hence,

$$\begin{aligned} |\operatorname{Im}(z^{p/2} + z^{-p/2})| &= (e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}) \sin(p \cdot 2\pi x), \\ |\operatorname{Im}(z^2 + z^{-2} - z - z^{-1} - 2)| &\leq (e^{8\pi y} - e^{-8\pi y}) \sin 8\pi x. \end{aligned}$$

Therefore,

$$\sin(p \cdot 2\pi x) \leq \frac{e^{8\pi y} - e^{-8\pi y}}{e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y}} \cdot \sin 8\pi x.$$

Further, since

$$\begin{aligned} \frac{e^{8\pi y} - e^{-8\pi y}}{16\pi y} &\leq \frac{e^{8\pi\cdot 0.003} - e^{-8\pi\cdot 0.003}}{16\pi \cdot 0.003} = 1.00094775\ldots < 1.01, \\ e^{p\cdot 2\pi y} - e^{-p\cdot 2\pi y} &\geq 2p \cdot 2\pi y, \end{aligned}$$

we have that

$$\begin{aligned} \sin(p \cdot 2\pi x) &\leq \frac{16\pi y \cdot 1.01}{2p \cdot 2\pi y} \cdot \sin 8\pi \frac{1}{2p} \leq \frac{4\cdot 1.01}{p} \cdot \frac{4\pi}{p} \leq \frac{4 \cdot 1.01 \cdot 4\pi}{100^2} \\ &= 0.005076\ldots < 0.01. \end{aligned}$$

On the other hand, since $0.01 \le px \le \frac{1}{2} - 0.01$, we have that

$$\sin(p \cdot 2\pi x) \ge \sin(2\pi \cdot 0.01) = 0.0627905\ldots > 0.05.$$

Hence, we have no solution in the domain of this case, as required.

Case 3 We can show that, in $\{x + y\sqrt{-1} \in D' \mid \frac{0.51}{p} \le x \le \frac{0.99}{p},\ 0 < y < \frac{0.3}{p}\}$, there is no solution of (67), in a similar way as in Case 2.

Case 4 We show that, in $\{x + y\sqrt{-1} \in D' \mid 0 < x < \frac{0.01}{p},\ 0 \le y < \frac{0.3}{p}\}$, there is no solution of (67), as follows.

Since $z = e^{4\pi y}(\cos 4\pi x - \sqrt{-1}\sin 4\pi x)$, we have that

$$\begin{aligned}|z^2 - z| &= ((e^{8\pi y}\cos 8\pi x - e^{4\pi y}\cos 4\pi x)^2 + (e^{8\pi y}\sin 8\pi x - e^{4\pi y}\sin 4\pi x)^2)^{1/2} \\ &\le \left((e^{8\pi\cdot 0.003} - e^{-4\pi\cdot 0.003}\cos(4\pi\cdot 0.0001))^2 + (e^{8\pi\cdot 0.003}\sin(8\pi\cdot 0.0001))^2\right)^{1/2} \\ &= 0.1153434\ldots < 0.12.\end{aligned}$$

Similarly, we can obtain that

$$|z^{-2} - z^{-1}| < 0.12.$$

Further, we have that

$$\begin{aligned}|z^{p/2} - 1| &= \left((e^{p\cdot 2\pi y}\cos(p\cdot 2\pi x) - 1)^2 + (e^{p\cdot 2\pi y}\sin(p\cdot 2\pi x))^2\right)^{1/2} \\ &\le \left((e^{-2\pi\cdot 0.3}\cos(2\pi\cdot 0.01) - 1)^2 + (e^{2\pi\cdot 0.3}\sin(2\pi\cdot 0.01))^2\right)^{1/2} \\ &= 0.9438792\ldots < 1.\end{aligned}$$

Similarly, we can obtain that

$$|z^{-p/2} - 1| < 1.$$

Hence,

$$4 = |(z^{p/2} - 1) + (z^{-p/2} - 1) - (z^2 - z) - (z^{-2} - z^{-1})| \le 2 + 2\cdot 0.12 = 2.24,$$

and this is a contradiction. Therefore, there is no solution in the domain of this case, as required.

Case 5 We can show that, in $\{x + y\sqrt{-1} \in D' \mid \frac{0.99}{p} < x < \frac{1}{p},\ 0 \le y < \frac{0.3}{p}\}$, there is no solution of (67), in a similar way as in Case 4.

Case 6 We show that, in $\{x \in D' \mid \frac{0.01}{p} < x < \frac{0.99}{p}\}$, there is no solution of (67), as follows.

The equation of the lemma is rewritten as

$$\cos(p\cdot 2\pi x)+1=\cos 8\pi x-\cos 4\pi x.$$

Since $\cos(p\cdot 2\pi x)+1\geq 0$ and $\cos 8\pi x-\cos 4\pi x<0$, there is no solution in the domain of this case, as required.

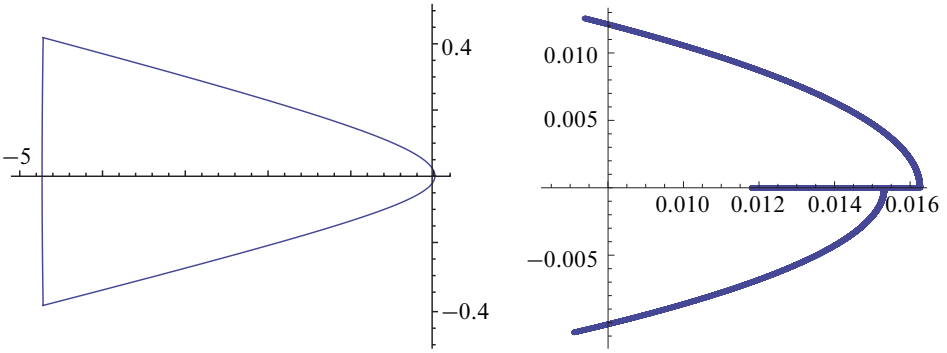


Figure 7: The image of the domain of Case 7 under the map ψ (left) and a local enlargement around the origin (right) for $p=100$

Case 7 We show that, in $\left\{x+y\sqrt{-1}\in D' \mid \frac{0.49}{p}<x<\frac{0.51}{p},\ 0<y<\frac{0.3}{p}\right\}$, there is a single solution of (67), as follows.

A solution of (67) is a zero of the holomorphic function

$$\psi(t)=e^{p\cdot 2\pi\sqrt{-1}\,t}+e^{-p\cdot 2\pi\sqrt{-1}\,t}-e^{8\pi\sqrt{-1}\,t}-e^{-8\pi\sqrt{-1}\,t}+e^{4\pi\sqrt{-1}\,t}+e^{-4\pi\sqrt{-1}\,t}+2.$$

This function maps the four edges of the domain of this case in the following way:

$$\begin{aligned}
\left\{x\in D' \,\middle|\, \frac{0.49}{p}\leq x\leq\frac{0.51}{p}\right\} &\to \{u\in\mathbb{C}\mid \operatorname{Re}u>0\},\\
\left\{\frac{0.51}{p}+y\sqrt{-1}\in D' \,\middle|\, 0<y\leq\frac{0.3}{p}\right\} &\to \{u\in\mathbb{C}\mid \operatorname{Im}u>0\},\\
\left\{x+\frac{0.3}{p}\sqrt{-1}\in D' \,\middle|\, \frac{0.49}{p}\leq x\leq\frac{0.51}{p}\right\} &\to \{u\in\mathbb{C}\mid \operatorname{Re}u<0\},\\
\left\{\frac{0.49}{p}+y\sqrt{-1}\in D' \,\middle|\, 0<y\leq\frac{0.3}{p}\right\} &\to \{u\in\mathbb{C}\mid \operatorname{Im}u<0\}.
\end{aligned}$$

See Figure 7 for a graphical representation of this image. Hence, the boundary of the domain is taken to a closed path, which goes around the origin once. Therefore, since ψ is holomorphic, ψ has a single zero in the domain. Hence, there is a single solution of (67) in the domain of this case, as required. □

Proof of Lemma B.1 As mentioned at the beginning of the proof of Lemma B.2, we can verify the lemma for a concrete p, and so we assume that $p \geq 100$ in the remainder of this proof.

We consider a solution of (29) in the domain (28). Since $0 < \mathrm{Re}(t+s) < \frac{1}{4}$ and $0 < \mathrm{Re}(s-t) < \frac{1}{4}$, we have that

$$-4\pi\Big(\frac{1}{4} - s_R + t_R\Big) < \mathrm{Arg}(1 - e^{4\pi\sqrt{-1}(s-t)}) < 0,$$
$$0 < \mathrm{Arg}(1 - e^{-4\pi\sqrt{-1}(t+s)}) < 4\pi\Big(\frac{1}{4} - t_R - s_R\Big),$$

where we put $t_R = \mathrm{Re}\, t$ and $s_R = \mathrm{Re}\, s$. Further, as the imaginary part of the first equation of (29),

$$\mathrm{Arg}(1 - e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Arg}(1 - e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\Big(\frac{p}{2}t_R - s_R\Big) = 0.$$

Hence,

$$0 < 4\pi\Big(\frac{p}{2}t_R - s_R\Big) < 4\pi\Big(\frac{1}{4} - s_R + t_R\Big) + 4\pi\Big(\frac{1}{4} - t_R - s_R\Big) = 4\pi\Big(\frac{1}{2} - 2s_R\Big).$$

Therefore,

$$s_R < \frac{p}{2}t_R < \frac{1}{2} - s_R.$$

Hence, since $s_R = \mathrm{Re}\, s > 0$ by the condition of (28), we obtain that

$$0 < \mathrm{Re}\, t < \frac{1}{p}.$$

Further, putting $z = e^{-4\pi\sqrt{-1}\,t}$, a solution (29) satisfies (31). Hence, by Lemma B.2, the first equation of (31) has exactly two solutions. As shown in the proof of Lemma B.2, putting $t = x + y\sqrt{-1}$, these two solutions belong to the following domains, respectively:

$$\Big\{x + y\sqrt{-1} \in \mathbb{C} \;\Big|\; \frac{0.49}{p} < x < \frac{0.51}{p},\ 0 < y < \frac{0.3}{p}\Big\}, \tag{68}$$

$$\Big\{x + y\sqrt{-1} \in \mathbb{C} \;\Big|\; \frac{0.49}{p} < x < \frac{0.51}{p},\ -\frac{0.3}{p} < y < 0\Big\}. \tag{69}$$

We let t_0 be the solution in (69). Then the other solution $\overline{t}_0$ belongs to (68).

We show that the solution $\overline{t}_0$ does not induce a solution of (29) in the domain (28), as follows. By (31), putting $z = e^{-4\pi\sqrt{-1}\,\overline{t}_0}$, we have that

$$w = \frac{z^{p/2} + z}{z^{p/2}z + 1} = \frac{(z^{p/2} + z)(\overline{z}^{p/2}\overline{z} + 1)}{|z^{p/2}z + 1|^2} = \frac{|z|^p\overline{z} + z^{p/2} + \overline{z}^{p/2} + z}{|z^{p/2}z + 1|^2}.$$

Further, putting $x=\frac{1}{2p}+\frac{\varepsilon}{p}$ for $-0.01<\varepsilon<0.01$, the numerator of the last term is calculated as

$$\begin{aligned}&e^{4\pi\sqrt{-1}\,x}e^{p\cdot 4\pi y}-e^{2\pi\sqrt{-1}\,\varepsilon}e^{p\cdot 2\pi y}-e^{-2\pi\sqrt{-1}\,\varepsilon}e^{p\cdot 2\pi y}+e^{-4\pi\sqrt{-1}\,x}e^{4\pi y}\\&\quad=(e^{(p+1)4\pi y}+e^{4\pi y})\cos 4\pi x-2e^{p\cdot 2\pi y}\cos 2\pi\varepsilon+\sqrt{-1}(e^{(p+1)4\pi y}-e^{4\pi y})\sin 4\pi x.\end{aligned}$$

We note that $0<x<\frac{0.51}{100}$. Since $e^{(p+1)4\pi y}-e^{4\pi y}>0$ for $y>0$, we have that $\operatorname{Im} w>0$. However, since $0<\operatorname{Re} s<\frac{1}{4}$ by the condition of (28), the imaginary part of $w=e^{-4\pi\sqrt{-1}\,s}$ must be negative, and this is a contradiction. Therefore, $\overline{t}_0$ does not induce a solution of (29) in the domain (28).

Hence, in order to show the lemma, it is sufficient to show that t_0 induces a single solution of (29) in the domain (28). We show that a solution of (29) induced by t_0 belongs to the domain (28), as follows. We put $z_0=e^{-4\pi\sqrt{-1}\,t_0}$ and

$$w_0=\frac{z_0^{p/2}+z_0}{z_0^{p/2}z_0+1}.$$

Then, in a similar way as above, we can show that $\operatorname{Im} w_0<0$, since $y<0$ in this case. We put s_0 to be the value satisfying that $w_0=e^{-4\pi\sqrt{-1}\,s_0}$ and $0<\operatorname{Re} s_0<\frac{1}{4}$. It is sufficient to show that this (t_0,s_0) satisfies (28) and (29). We recall that $\frac{0.49}{p}<\operatorname{Re} t_0<\frac{0.51}{p}$; in particular, $\operatorname{Re} t_0>0$. Further, in order to show that $0<\operatorname{Re}(t_0+s_0)<\frac{1}{4}$ and $0<\operatorname{Re}(s_0-t_0)<\frac{1}{4}$, we estimate the value of $\operatorname{Re} s_0$. By the second equation of (30), we have that $(1-z_0w_0)\big(\frac{1}{z_0}-\frac{1}{w_0}\big)=1$. Hence,

$$w_0+\frac{1}{w_0}=z_0+\frac{1}{z_0}-1.$$

We put $t_0=x+y\sqrt{-1}$, noting that $\frac{0.49}{p}<x<\frac{0.51}{p}$ and $-\frac{0.3}{p}<y<0$. Further, we put

$$c=z_0+\frac{1}{z_0}-1.$$

Then we have that

$$\begin{aligned}|c-1|&=|z_0^{1/2}-z_0^{-1/2}|^2=(z_0^{1/2}-z_0^{-1/2})(\overline{z}_0^{1/2}-\overline{z}_0^{-1/2})\\&=|z_0|+|z_0|^{-1}-\frac{z_0^{1/2}}{\overline{z}_0^{1/2}}-\frac{\overline{z}_0^{1/2}}{z_0^{1/2}}=e^{4\pi y}+e^{-4\pi y}-e^{-4\pi\sqrt{-1}\,x}-e^{4\pi\sqrt{-1}\,x}\\&=e^{4\pi y}+e^{-4\pi y}-2\cos 4\pi x\le e^{4\pi\cdot 0.003}+e^{-4\pi\cdot 0.003}-2\cos(4\pi\cdot 0.0051)\\&=0.00552732\ldots<0.006.\end{aligned}$$

Since $w_0 + \frac{1}{w_0} = c$, we have that

$$w_0 = \frac{1}{2}(c - \sqrt{-1}\,\sqrt{4-c^2}),$$

where we choose the sign of $\sqrt{4-c^2}$ in the way that $\operatorname{Re}\sqrt{4-c^2} > 0$, noting that c is close to 1, as shown above. This term is calculated as

$$\sqrt{4-c^2} = \sqrt{3+(1-c^2)} = \sqrt{3}\sqrt{1+\frac{1-c^2}{3}} = \sqrt{3}\sqrt{1+\delta},$$

where we put $\delta = \frac{1-c^2}{3}$. Further, the last term is estimated by

$$|\sqrt{1+\delta}-1| = \frac{|\delta|}{|\sqrt{1+\delta}+1|} \le \frac{\delta_0}{\sqrt{1-\delta_0}+1} = 0.00200802\ldots < 0.003,$$

where, noting that $\delta = \frac{1-c^2}{3}$, we define δ_0 by

$$|\delta| = \frac{1}{3}|c-1|\cdot|c+1| < \frac{1}{3}\cdot 0.006\cdot 2.006 = \delta_0.$$

Since $|\sqrt{4-c^2}-\sqrt{3}| = \sqrt{3}\cdot|\sqrt{1+\delta}-1|$, we have that

$$\left|w_0 - \frac{1-\sqrt{-1}\sqrt{3}}{2}\right| \le \frac{1}{2}|c-1| + \frac{1}{2}|\sqrt{c^2-1}-\sqrt{3}| \le \frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003,$$

which means that w_0 is close to $e^{-\pi\sqrt{-1}/3}$. Hence,

$$\left|\operatorname{Arg}(w_0) + \frac{\pi}{3}\right| \le \arcsin\Big(\frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003\Big).$$

Therefore, noting that $w_0 = e^{-4\pi\sqrt{-1}\,s_0}$, we obtain that

$$\left|\operatorname{Re} s_0 - \frac{1}{12}\right| \le \frac{1}{4\pi}\arcsin\Big(\frac{1}{2}\cdot 0.006 + \frac{\sqrt{3}}{2}\cdot 0.003\Big) = 0.000445483\ldots < 0.001.$$

Hence,

$$\frac{1}{12} - 0.001 < \operatorname{Re} s_0 < \frac{1}{12} + 0.001. \tag{70}$$

Further, since $0 < \operatorname{Re} t_0 < \frac{0.51}{p} \le 0.0051$, we obtain that

$$0 < \operatorname{Re}(t_0+s_0) < \frac{1}{4}, \quad 0 < \operatorname{Re}(s_0-t_0) < \frac{1}{4}, \tag{71}$$

and (t_0, s_0) satisfies (28), as required. In order to complete the proof of the lemma, it is sufficient to show that (t_0, s_0) satisfies (29). Since (z_0, w_0) satisfies (30), we have

that

$$\log(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\log(1-e^{-4\pi\sqrt{-1}\,(t_0+s_0)})+4\pi\sqrt{-1}\Big(\frac{p}{2}\,t_0-s_0\Big)\in 2\pi\sqrt{-1}\,\mathbb{Z},$$
$$-\log(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\log(1-e^{-4\pi\sqrt{-1}\,(t_0+s_0)})-4\pi\sqrt{-1}\,t_0\in 2\pi\sqrt{-1}\,\mathbb{Z}.$$

Hence, it is sufficient to show that their imaginary part is in the range $-2\pi<\cdot<2\pi$. That is, it is sufficient to show that

$$-2\pi<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+4\pi\ \operatorname{Re}\Big(\frac{p}{2}t_0-s_0\Big)<2\pi,$$
$$-2\pi<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})+\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+4\pi\ \operatorname{Re}t_0<2\pi.$$

Hence, putting $t_R=\operatorname{Re}t_0$, $s_R=\operatorname{Re}s_0$ and

$$c_1=\frac{1}{4\pi}\operatorname{Arg}(1-e^{4\pi\sqrt{-1}\,(s_0-t_0)})-\frac{1}{4\pi}\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+\Big(\frac{p}{2}t_R-s_R\Big),$$
$$c_2=\frac{1}{4\pi}\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})+\frac{1}{4\pi}\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})+t_R,$$

it is sufficient to show that

$$-\frac{1}{2}<c_1<\frac{1}{2},\quad -\frac{1}{2}<c_2<\frac{1}{2}. \tag{72}$$

Since (71) holds, we have that

$$-4\pi\Big(\frac{1}{4}-s_R+t_R\Big)<\operatorname{Arg}(1-e^{4\pi\sqrt{-1}(s_0-t_0)})<0,$$
$$0<\operatorname{Arg}(1-e^{-4\pi\sqrt{-1}(t_0+s_0)})<4\pi\Big(\frac{1}{4}-t_R-s_R\Big).$$

Further, we recall that $\frac{0.49}{p}<t_R<\frac{0.51}{p}$ and $\frac{1}{12}-0.001<s_R<\frac{1}{12}+0.001$. Hence, we obtain that

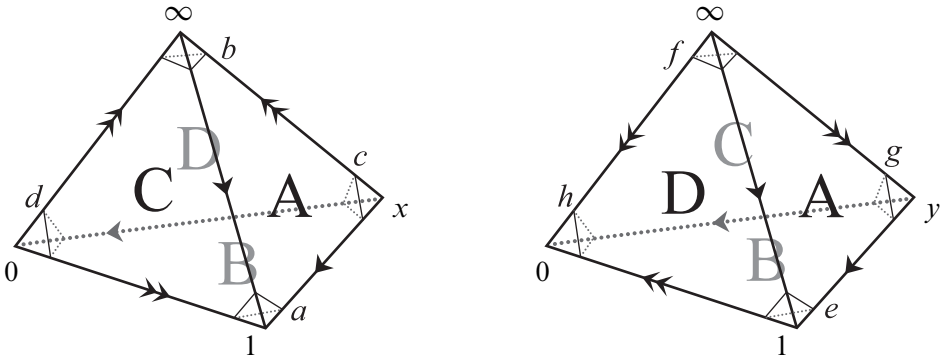
$$c_1>-\Big(\frac{1}{4}-s_R+t_R\Big)-\Big(\frac{1}{4}-t_R-s_R\Big)+\frac{p}{2}t_R-s_R=-\frac{1}{2}+\frac{p}{2}t_R+s_R$$
$$>-\frac{1}{2}+\frac{0.49}{2}+\frac{1}{12}-0.001=-0.172667\ldots>-0.2,$$
$$c_1<\frac{p}{2}t_R-s_R<\frac{0.51}{2}-\frac{1}{12}+0.001=0.172667\ldots<0.2,$$
$$c_2>-\Big(\frac{1}{4}-s_R+t_R\Big)+t_R=-\frac{1}{4}+s_R>-\frac{1}{4}+\frac{1}{12}-0.001=-0.167667\ldots>-0.2,$$
$$c_2<\Big(\frac{1}{4}-t_R-s_R\Big)+t_R=\frac{1}{4}-s_R<\frac{1}{4}-\frac{1}{12}-0.001=0.165667\ldots<0.2.$$

Therefore, (72) holds, as required. This completes the proof of the lemma. □

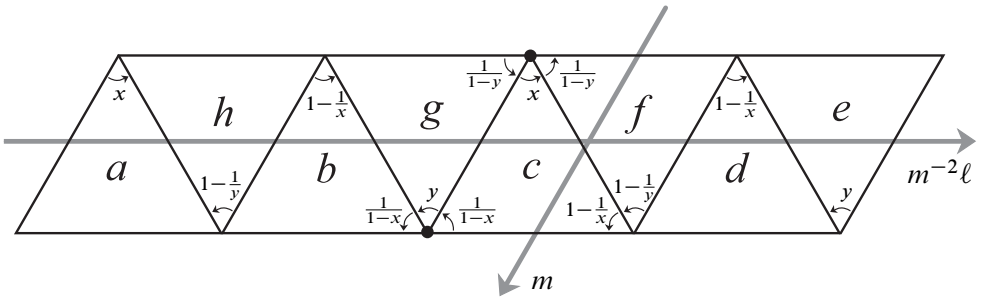
Appendix C The hyperbolic structure of M_p

In this section, we review the hyperbolic structure of M_p, and, in Lemma C.1, we explain how it is related to the critical point (t_0, s_0) of $\widehat{V}(t,s)$ given in Appendix B. For a textbook of hyperbolic geometry, see [24].

We review the hyperbolic structure of M_p given in [24], where we recall that M_p is the 3–manifold obtained from S^3 by p surgery along the figure-eight knot. As shown in [24], the complement of the figure-eight knot can be expressed as the union of the following two ideal tetrahedra:



Here, the four faces "A", "B", "C" and "D" are glued, respectively. Then the resulting space has two edges: the edge of a single arrow and the edge of double arrows. The shapes of the tetrahedra are given by parameters written at the vertices. Further, the boundary torus of a tubular neighborhood of the figure-eight knot is expressed as the union of eight triangles "a", "b", ..., "h", which appear in neighborhoods of the vertices of the above ideal tetrahedra:



As shown in [24], the holonomy m of the meridian and the holonomy ℓ of the longitude are given by

$$m = \frac{1-\frac{1}{x}}{\frac{1}{1-y}} = -\frac{(1-x)(1-y)}{x}, \quad m^{-2}\ell = \frac{x^2\left(1-\frac{1}{x}\right)^2}{y^2\left(1-\frac{1}{y}\right)^2} = \frac{(1-x)^2}{(1-y)^2}.$$

We have some requirements to obtain the hyperbolic structure from these tetrahedra. We require that

$$\operatorname{Im} x > 0, \quad \operatorname{Im} y > 0, \tag{73}$$

which implies that the two tetrahedra have positive orientations. Further, we require the gluing equation,

$$xy\left(\frac{1}{1-x}\right)^2\left(\frac{1}{1-y}\right)^2 = 1,$$

$$\operatorname{Arg}(x) + \operatorname{Arg}(y) + 2\operatorname{Arg}\left(\frac{1}{1-x}\right) + 2\operatorname{Arg}\left(\frac{1}{1-y}\right) = 2\pi, \tag{74}$$

which we obtain as the product of moduli around the vertex of the dot. The first equation is rewritten as

$$(1-x)^2(1-y)^2 = xy. \tag{75}$$

We note that, for the gluing equations around the other vertices, we obtain equivalent conditions. Furthermore, we require that

$$m^p\ell = 1, \tag{76}$$

$$(p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell) = 2\pi, \tag{77}$$

which is necessary in order that M_p be obtained as the closure of the union of the two tetrahedra; see Figure 8 for a graphical representation of this condition.

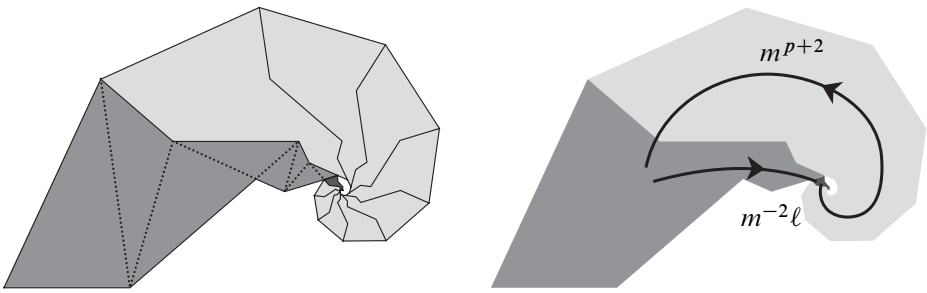


Figure 8: A fundamental domain of the boundary torus (the dark gray area) and its image by the action of $m^{-2}\ell$ (the black area) and its images by the actions of m^k for $k = 1, 2, \ldots, p+1$ (the light gray area) for $p = 8$

The following lemma implies that the critical point of $\widehat{V}(t,s)$ given in Appendix B is related to the hyperbolic structure of M_p.

Lemma C.1 *Let* (t_0, s_0) *be the critical point of* $\widehat{V}(t,s)$ *given in Lemma B.1. We put* $x_0 = e^{4\pi\sqrt{-1}\,(s_0-t_0)}$ *and* $y_0 = e^{4\pi\sqrt{-1}\,(t_0+s_0)}$. *Then* (x_0, y_0) *gives the hyperbolic structure of* M_p *in the above-mentioned way.*

Proof It is sufficient to show that (x_0, y_0) satisfies (73), (74), (75), (76) and (77). We put $z_0 = e^{-4\pi\sqrt{-1}\,t_0}$ and $w_0 = e^{-4\pi\sqrt{-1}\,s_0}$ as in the proof of Lemma B.1. Then $x_0 = z_0/w_0$ and $y_0 = 1/(z_0 w_0)$.

We obtain (73) from the definitions of x_0 and y_0, since $0 < \mathrm{Re}(s_0 - t_0) < \frac{1}{4}$ and $0 < \mathrm{Re}(t_0 + s_0) < \frac{1}{4}$ by (28).

We show (75), as follows. We have that

$$\begin{aligned}\frac{(1-x_0)^2(1-y_0)^2}{x_0 y_0} &= \frac{(1-z_0/w_0)^2(1-1/(z_0w_0))^2}{(z_0/w_0)\cdot 1/(z_0w_0)}\\ &= \frac{1}{z_0^2 w_0^2}(w_0 - z_0)^2(1-z_0w_0)^2 = 1,\end{aligned}$$

where we obtain the last equality by (30). Hence, (x_0, y_0) satisfies (75), as required.

We show (74), as follows. By (75), we have that

$$\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) + 2\,\mathrm{Arg}\Big(\frac{1}{1-x_0}\Big) + 2\,\mathrm{Arg}\Big(\frac{1}{1-y_0}\Big) \in 2\pi\mathbb{Z}.$$

Hence, putting

$$\begin{aligned}c_1 &= \frac{1}{2\pi}\Big(\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) + 2\,\mathrm{Arg}\Big(\frac{1}{1-x_0}\Big) + 2\,\mathrm{Arg}\Big(\frac{1}{1-y_0}\Big)\Big)\\ &= \frac{1}{2\pi}\big(\mathrm{Arg}(x_0) + \mathrm{Arg}(y_0) - 2\,\mathrm{Arg}(1-x_0) - 2\,\mathrm{Arg}(1-y_0)\big),\end{aligned}$$

it is sufficient to show that

$$0 < c_1 < 2.$$

Since $x_0 = e^{4\pi\sqrt{-1}(s_0-t_0)}$ and $y_0 = e^{4\pi\sqrt{-1}(t_0+s_0)}$,

$$\begin{aligned}&\mathrm{Arg}(x_0) = 4\pi(s_R - t_R), \quad -4\pi\big(\tfrac{1}{4} - s_R + t_R\big) < \mathrm{Arg}(1-x_0) < 0,\\ &\mathrm{Arg}(y_0) = 4\pi(t_R + s_R), \quad -4\pi\big(\tfrac{1}{4} - t_R - s_R\big) < \mathrm{Arg}(1-y_0) < 0,\end{aligned}$$

where we put $t_R = \operatorname{Re} t_0$ and $s_R = \operatorname{Re} s_0$. It follows that

$$c_1 > 2(s_R - t_R) + 2(t_R + s_R) = 4s_R,$$
$$c_1 < 2(s_R - t_R) + 2(t_R + s_R) + 2\left(\tfrac{1}{4} - s_R + t_R\right) + 2\left(\tfrac{1}{4} - t_R - s_R\right) = 2 - 8s_R.$$

Since $\frac{1}{12} - 0.001 < s_R < \frac{1}{12} + 0.001$ by (70), we obtain that $0 < c_1 < 2$, as required.

We show (76), as follow. We have that

$$\begin{aligned} m &= -\frac{(1-x_0)(1-y_0)}{x_0} = -\frac{(1-z_0/w_0)(1-1/(z_0w_0))}{z_0/w_0} \\ &= \frac{1}{z_0}(w_0 - z_0)\frac{1-z_0w_0}{z_0w_0} = \frac{1}{z_0}, \end{aligned}$$

where we obtain the last equality by (30). Further,

$$m^{-2}\ell = \frac{(1-x_0)^2}{(1-y_0)^2} = \frac{(1-z_0/w_0)^2}{(1-1/(z_0w_0))^2} = \frac{z_0^2(w_0-z_0)^{2^2}}{1-z_0w_0} = z_0^p \cdot z_0^2,$$

where we obtain the last equality by (30). Hence,

$$m^p\ell = \frac{z_0^p \cdot z_0^2}{z_0^p \cdot z_0^2} = 1.$$

Therefore, (x_0, y_0) satisfies (76), as required.

We show (77), as follows. By (76), we have that

$$(p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell) \in 2\pi\mathbb{Z}.$$

Hence, putting

$$\begin{aligned} c_2 &= \frac{1}{2\pi}((p+2)\operatorname{Arg}(m) + \operatorname{Arg}(m^{-2}\ell)) \\ &= \frac{1}{2\pi}\left(-(p+2)\operatorname{Arg}(z_0) + \operatorname{Arg}\left(\frac{(1-x_0)^2}{(1-y_0)^2}\right)\right) \\ &= (p+2)\cdot 2t_R + \frac{1}{2\pi}(2\operatorname{Arg}(1-x_0) - 2\operatorname{Arg}(1-y_0)), \end{aligned}$$

it is sufficient to show that

$$0 < c_2 < 2.$$

Since $\frac{0.49}{p}<t_R<\frac{0.51}{p}$ and $\frac{1}{12}-0.001<s_R<\frac{1}{12}+0.001$, as shown in the proof of Lemma B.1 (for $p\geq 100$), it follows that

$$c_2>(p+2)\cdot 2t_R-4\big(\tfrac{1}{4}-s_R+t_R\big)=2pt_R-1+4s_R>0.98-1+4\big(\tfrac{1}{12}-0.001\big)>0,$$
$$c_2<(p+2)\cdot 2t_R+4\big(\tfrac{1}{4}-t_R-s_R\big)=2pt_R+1-4s_R<1.02+1-4\big(\tfrac{1}{12}+0.001\big)<2.$$

Hence (since we can verify the inequalities for $p<100$ by calculating (t_0,s_0) concretely), we obtain that $0<c_2<2$, as required. □

It is known (see [13; 16]) that the critical value $\widehat{V}(t_0,s_0)$ is equal to a normalized complex volume. We verify the real part of this fact, as follows. It is known that the hyperbolic volume of an ideal tetrahedron of modulus z is presented by the Bloch–Wigner function,

$$D(z)=\operatorname{Im}\operatorname{Li}_2(z)+\log|z|\cdot\operatorname{Arg}(1-z).$$

Hence,

$$\begin{aligned}\operatorname{vol}(M_p)&=D(x_0)+D(y_0)=D\Big(\frac{z_0}{w_0}\Big)+D\Big(\frac{1}{z_0w_0}\Big)=D\Big(\frac{z_0}{w_0}\Big)-D(z_0w_0)\\&=\operatorname{Im}\Big(\operatorname{Li}_2\Big(\frac{z_0}{w_0}\Big)-\operatorname{Li}_2(z_0w_0)\Big)+\log\Big|\frac{z_0}{w_0}\Big|\cdot\operatorname{Arg}\Big(1-\frac{z_0}{w_0}\Big)\\&\qquad-\log|z_0w_0|\cdot\operatorname{Arg}(1-z_0w_0)\\&=\operatorname{Im}\Big(\operatorname{Li}_2\Big(\frac{z_0}{w_0}\Big)-\operatorname{Li}_2(z_0w_0)\Big)+\log|z_0|\Big(\operatorname{Arg}\Big(1-\frac{z_0}{w_0}\Big)-\operatorname{Arg}(1-z_0w_0)\Big)\\&\qquad+\log|w_0|\Big(-\operatorname{Arg}\Big(1-\frac{z_0}{w_0}\Big)-\operatorname{Arg}(1-z_0w_0)\Big).\end{aligned}$$

Further, by using the imaginary part of (29), the last line is calculated as

$$\begin{aligned}&\operatorname{Re}(-4\pi\sqrt{-1}\,t_0)\Big(-4\pi\ \operatorname{Re}\Big(\frac{p}{2}t_0-s_0\Big)\Big)+\operatorname{Re}(-4\pi\sqrt{-1}\,s_0)(4\pi\operatorname{Re}t_0)\\&\qquad=(4\pi)^2\Big(-\operatorname{Im}t_0\ \operatorname{Re}\Big(\frac{p}{2}t_0-s_0\Big)+\operatorname{Im}s_0\ \operatorname{Re}t_0\Big)\\&\qquad=-(4\pi)^2\ \operatorname{Re}\Big(\frac{p}{2}t_0^2-t_0s_0\Big).\end{aligned}$$

Therefore,

$$\frac{1}{4\pi}\operatorname{vol}(M_p)=\frac{1}{4\pi}\ \operatorname{Im}\Big(\operatorname{Li}_2\Big(\frac{z_0}{w_0}\Big)-\operatorname{Li}_2(z_0w_0)\Big)-4\pi\ \operatorname{Re}\Big(\frac{p}{2}t_0^2-t_0s_0\Big)=\operatorname{Re}\widehat{V}(t_0,s_0).$$

Hence, $\operatorname{Re}\widehat{V}(t_0,s_0)$ is equal to a normalized hyperbolic volume of M_p.

Appendix D Estimate of the hyperbolic volume of M_p

In this section, in Lemma D.1, we give a lower bound of $\varsigma_R(M_p)$, which is 4π times the hyperbolic volume of M_p. We use this lemma to show Lemma 5.1 in Appendix E. The aim of this section is to show Lemma D.1.

Lemma D.1 *For an integer $p \geq 6$,*

$$\varsigma_R(M_p) \geq \Lambda\Big(\frac{1}{6}\Big) - \frac{3}{p^2}.$$

We give a proof of this lemma later in this section.

We recall the definition of $\varsigma_R(M_p)$, as follows. We recall that the potential function is given by

$$\widehat{V}(t,s) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{p}{4}t^2 - ts\Big),$$

and its differentials are given by

$$\frac{\partial\widehat{V}}{\partial t} = \log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) + 4\pi\sqrt{-1}\Big(\frac{p}{2}t - s\Big) = 0,$$
$$\frac{\partial\widehat{V}}{\partial s} = -\log(1-e^{4\pi\sqrt{-1}(s-t)}) - \log(1-e^{-4\pi\sqrt{-1}(t+s)}) - 4\pi\sqrt{-1}\,t = 0.$$

As mentioned in Section 5.1, we have a single solution of these equations in the domain (28). Putting $\gamma = \frac{1}{p}$, we regard it as a function of γ, and denote it by $(t(\gamma), s(\gamma))$. We recall that $\varsigma_R(M_p)$ is defined by

$$\varsigma_R(M_p) = \mathrm{Re}\,\widehat{V}(t(\gamma), s(\gamma)).$$

By expanding the above equations, we can obtain the expansions of $t(\gamma)$ and $s(\gamma)$ as

$$t(\gamma) = \frac{1}{2}\gamma - \sqrt{3}\sqrt{-1}\,\gamma^2 - 6\gamma^3 + O(\gamma^4),$$
$$s(\gamma) = \frac{1}{12} + \frac{\pi}{\sqrt{3}}\gamma^2 - 4\pi\sqrt{-1}\gamma^3 + O(\gamma^4).$$

Hence, we can obtain that

$$\widehat{V}(t(\gamma), s(\gamma))$$
$$= \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{\pi\sqrt{-1}/3}) - \mathrm{Li}_2(e^{-\pi\sqrt{-1}/3})) - \frac{\pi\sqrt{-1}}{4}\gamma - \frac{\sqrt{3}\pi}{2}\gamma^2 + O(\gamma^3).$$

Therefore,

$$\varsigma_R(M_p) = \operatorname{Re} \widehat{V}(t(\gamma), s(\gamma)) = \Lambda\Big(\frac{1}{6}\Big) - \frac{\sqrt{3}\pi}{2}\gamma^2 + O(\gamma^3).$$

Since $\frac{\sqrt{3}\pi}{2} = 2.72069904\ldots$, we can verify that Lemma D.1 holds for sufficiently large p. (In fact, it probably holds that $\varsigma_R(M_p) \geq \Lambda\big(\frac{1}{6}\big) - \frac{\sqrt{3}\pi}{2}\gamma^2$.)

Proof of Lemma D.1 We put

$$\begin{aligned} f(\gamma) &= \widehat{V}(t(\gamma), s(\gamma), \gamma) \\ &= \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}(s-t)}) - \mathrm{Li}_2(e^{-4\pi\sqrt{-1}(t+s)})) + 4\pi\sqrt{-1}\Big(\frac{1}{4\gamma}t^2 - ts\Big), \end{aligned}$$

regarding $\widehat{V}$ as a function of t, s and γ. Then

$$\begin{aligned} f'(\gamma) &= \frac{\partial \widehat{V}}{\partial t}(t(\gamma), s(\gamma), \gamma)\cdot t'(\gamma) + \frac{\partial \widehat{V}}{\partial s}(t(\gamma), s(\gamma), \gamma)\cdot s'(\gamma) - 4\pi\sqrt{-1}\,\frac{1}{4\gamma^2}t(\gamma)^2 \\ &= -\pi\sqrt{-1}\,\widehat{t}(\gamma)^2, \end{aligned}$$

where we put

$$\widehat{t}(\gamma) = \frac{t(\gamma)}{\gamma} = \frac{1}{2} - \sqrt{3}\sqrt{-1}\,\gamma - 6\gamma^2 + O(\gamma^3).$$

Hence,

$$f''(\gamma) = -2\pi\sqrt{-1}\,\widehat{t}(\gamma)\widehat{t}\,'(\gamma).$$

By the Taylor expansion, we have that

$$\operatorname{Re} f(\gamma) = f(0) + \operatorname{Re} f'(0)\gamma + \operatorname{Re} f''(c)\frac{\gamma^2}{2} = \Lambda\Big(\frac{1}{6}\Big) + \gamma^2\pi\ \operatorname{Im}(\widehat{t}(c)\widehat{t}\,'(c))$$

for some c with $0 \leq c \leq \gamma$. Therefore, it is sufficient to show that

$$\operatorname{Im}(\widehat{t}(\gamma)\widehat{t}\,'(\gamma)) \geq -\frac{\sqrt{3}}{2}. \tag{78}$$

Since $\operatorname{Im}(\widehat{t}(\gamma)\widehat{t}\,'(\gamma)) = \operatorname{Re}\widehat{t}(\gamma)\cdot\operatorname{Im}\widehat{t}\,'(\gamma) + \operatorname{Im}\widehat{t}(\gamma)\cdot\operatorname{Re}\widehat{t}\,'(\gamma)$, it is sufficient to show that

$$0 \leq \operatorname{Re}\widehat{t}(\gamma) \leq \frac{1}{2}, \quad -\sqrt{3} \leq \operatorname{Im}\widehat{t}\,'(\gamma) \leq 0, \quad \operatorname{Re}\widehat{t}\,'(\gamma) \leq 0, \quad \operatorname{Im}\widehat{t}(\gamma) \leq 0. \tag{79}$$

We can verify that, for sufficiently large p, the inequalities (79) hold, since

$$\operatorname{Re}\widehat{t}(\gamma) = \frac{1}{2} - 6\gamma^2 + O(\gamma^4), \quad \operatorname{Im}\widehat{t}(\gamma) = -\sqrt{3}\,\gamma + \frac{4}{\sqrt{3}}(\pi^2+9)\gamma^3 + O(\gamma^5).$$

Further, we can numerically verify (79) for $p \leq 100$; see Remark D.2 below.

We explain how we can give a rigorous proof of (79), as follows. As shown in Section 5.1, putting $z = e^{-4\pi\sqrt{-1}\,t}$, z satisfies that

$$z^{p/2} + z^{-p/2} - z^2 - z^{-2} + z + z^{-1} + 2 = 0.$$

Hence,

$$e^{2\pi\sqrt{-1}\,pt} + e^{-2\pi\sqrt{-1}\,pt} - e^{8\pi\sqrt{-1}\,t} - e^{-8\pi\sqrt{-1}\,t} + e^{4\pi\sqrt{-1}\,t} + e^{-4\pi\sqrt{-1}\,t} + 2 = 0.$$

Further, putting $\widehat{t} = tp = \frac{t}{\gamma}$, we have that

$$e^{2\pi\sqrt{-1}\,\widehat{t}} + e^{-2\pi\sqrt{-1}\,\widehat{t}} - e^{8\pi\sqrt{-1}\,\gamma\widehat{t}} - e^{-8\pi\sqrt{-1}\,\gamma\widehat{t}} + e^{4\pi\sqrt{-1}\,\gamma\widehat{t}} + e^{-4\pi\sqrt{-1}\,\gamma\widehat{t}} + 2 = 0.$$

We put

$$\widehat{t}(\gamma) = t_R(\gamma) + \sqrt{-1}\,\gamma t_I(\gamma),$$

where

$$t_R(\gamma) = \frac{1}{2} - 6\gamma^2 + O(\gamma^4), \quad t_I(\gamma) = -\sqrt{3} + \frac{4}{\sqrt{3}}(\pi^2 + 9)\gamma^2 + O(\gamma^4).$$

We note that $\widehat{t}(-\gamma) = \overline{\widehat{t}(\gamma)}$, since M_{-p} is homeomorphic to M_p with the opposite orientation, and we choose $\overline{z}^{-1}$ instead of z when we change the sign of p. Hence, $t_R(\gamma)$ and $t_I(\gamma)$ are real-valued even functions. From the above equation, we have that

$$\begin{aligned}(80)\quad & e^{2\pi\sqrt{-1}\,t_R - 2\pi\gamma t_I} + e^{-2\pi\sqrt{-1}\,t_R + 2\pi\gamma t_I} - e^{8\pi\sqrt{-1}\,\gamma t_R - 8\pi\sqrt{-1}\,\gamma^2 t_I} \\ & - e^{-8\pi\sqrt{-1}\,\gamma t_R + 8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{4\pi\sqrt{-1}\,\gamma t_R - 4\pi\gamma^2 t_I} \\ & + e^{-4\pi\sqrt{-1}\,\gamma t_R + 4\pi\gamma^2 t_I} + 2 = 0.\end{aligned}$$

By changing the sign of γ, we have that

$$\begin{aligned}(81)\quad & e^{2\pi\sqrt{-1}\,t_R + 2\pi\gamma t_I} + e^{-2\pi\sqrt{-1}\,t_R - 2\pi\gamma t_I} - e^{-8\pi\sqrt{-1}\,\gamma t_R - 8\pi\sqrt{-1}\,\gamma^2 t_I} \\ & - e^{8\pi\sqrt{-1}\,\gamma t_R + 8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{-4\pi\sqrt{-1}\,\gamma t_R - 4\pi\gamma^2 t_I} \\ & + e^{4\pi\sqrt{-1}\,\gamma t_R + 4\pi\gamma^2 t_I} + 2 = 0.\end{aligned}$$

By adding the above two equations, we obtain that

$$\begin{aligned}&(e^{2\pi\sqrt{-1}\,t_R} + e^{-2\pi\sqrt{-1}\,t_R})(e^{2\pi\gamma t_I} + e^{-2\pi\gamma t_I}) \\ &\quad - (e^{8\pi\sqrt{-1}\,\gamma t_R} + e^{-8\pi\sqrt{-1}\,\gamma t_R})(e^{8\pi\sqrt{-1}\,\gamma^2 t_I} + e^{-8\pi\sqrt{-1}\,\gamma^2 t_I}) \\ &\quad + (e^{4\pi\sqrt{-1}\,\gamma t_R} + e^{-4\pi\sqrt{-1}\,\gamma t_R})(e^{4\pi\gamma^2 t_I} + e^{-4\pi\gamma^2 t_I}) + 4 = 0.\end{aligned}$$

Hence,

$$\cos 2\pi t_R \cosh 2\pi\gamma t_I - \cos 8\pi\gamma t_R \cosh 2\pi\gamma^2 t_I + \cos 4\pi\gamma t_R \cosh 4\pi\gamma^2 t_I + 1 = 0.$$

Similarly, by subtracting (80) from (81), we can obtain that

$$\sin 2\pi t_R \sinh 2\pi\gamma t_I - \sin 8\pi\gamma t_R \sinh 2\pi\gamma^2 t_I + \sin 4\pi\gamma t_R \sinh 4\pi\gamma^2 t_I = 0.$$

The above two equations are real-valued equations for real-valued functions $t_R(\gamma)$ and $t_I(\gamma)$, and we can estimate the functions sin, cos, sinh and cosh as precisely as we need by the Taylor expansion. Hence, we can estimate $t_R(\gamma)$ and $t_I(\gamma)$ as precisely as we need, and we can (in principle) show (79). □

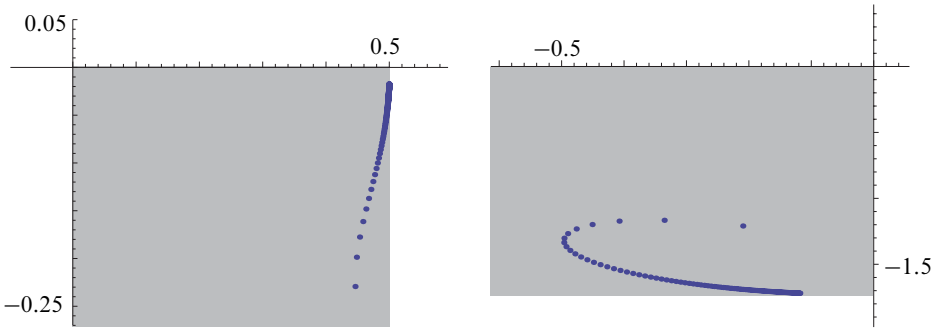


Figure 9: $\widehat{t} \in \mathbb{C}$ for $p = 6, 7, \ldots, 100$ (left) and $\widehat{t}' \in \mathbb{C}$ for $p = 7, 8, \ldots, 100$ (right). The gray areas are the areas of (79).

Remark D.2 We can numerically verify that $\widehat{t}$ satisfies (79) for $p = 6, 7, \ldots, 100$ and $\widehat{t}'$ satisfies (79) for $p = 7, 8, \ldots, 100$; see Figure 9 for their graphical representations. When $p = 6$, $\operatorname{Re}\widehat{t}' > 0$ and it does not satisfy (79), but we can directly verify that (78) holds in this case.

Appendix E Proof of Lemma 5.1

The aim of this section is to give a proof of Lemma 5.1.

Proof of Lemma 5.1 We recall that

$$\Delta_0 = \left\{(t, s) \in \mathbb{R}^2 \mid 0 \le t + s \le \tfrac{1}{2},\ 0 \le s - t \le \tfrac{1}{2}\right\}.$$

In order to prove Lemma 5.1, it is sufficient to show that the domain

$$\{(t,s)\in\Delta_0 \mid \operatorname{Re}\widehat{V}(t,s)\geq \varsigma_R(M_p)\} \tag{82}$$

is included in the interior of Δ_0', since the domain (82) is a compact domain. Supposing that $\operatorname{Re}\widehat{V}(t,s)\geq\varsigma_R(M_p)$ for $(t,s)\in\Delta_0$, it is sufficient to show that

$$0.005<t+s<0.24,\quad 0.005<s-t<0.24,\quad |t|<\frac{0.74}{p}.$$

We show the first and second formulas in Lemma E.1 below, and show the third formula in Lemma E.3 below. Hence, we obtain the lemma. □

Before showing Lemmas E.1 and E.3, we review some behavior of the dilogarithm function. We put

$$\Lambda(t)=\operatorname{Re}\left(\frac{1}{2\pi\sqrt{-1}}\operatorname{Li}_2(e^{2\pi\sqrt{-1}\,t})\right).$$

Since

$$\Lambda'(t)=-\log 2\sin\pi t,\quad \Lambda''(t)=-\pi\cot\pi t,$$

the behavior of $\Lambda(t)$ is as follows:

t	0	$\cdots$	$\frac{1}{6}$	$\cdots$	$\frac{1}{2}$	$\cdots$	$\frac{5}{6}$	$\cdots$	1
$\Lambda(t)$	0	$\nearrow$	$\Lambda\left(\frac{1}{6}\right)$	$\searrow$	0	$\searrow$	$-\Lambda\left(\frac{1}{6}\right)$	$\nearrow$	0
$\Lambda'(t)$		$+$	0	$-$	$-$	$-$	0	$+$	
$\Lambda''(t)$		$-$	$-$	$-$	0	$+$	$+$	$+$	

Here, $\Lambda\left(\frac{1}{6}\right)=0.161533\ldots$. For the graph of $\Lambda(t)$, see Figure 10.

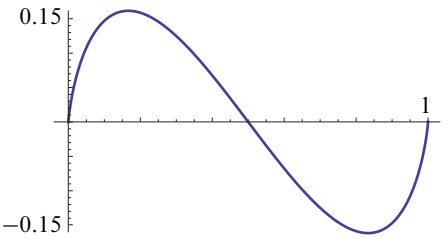


Figure 10: The graph of $\Lambda(t)$ for $0\leq t\leq 1$

We note that, from the definition of $\widehat{V}$,

$$\operatorname{Re}\widehat{V}(t,s)=\tfrac{1}{2}\Lambda(2s-2t)+\tfrac{1}{2}\Lambda(2s+2t)$$

for $(t,s)\in\Delta_0$.

Lemma E.1 *Let p be an integer ≥ 6. We suppose that* $\operatorname{Re}\widehat{V}(t,s) \geq \varsigma_R(M_p)$ *for* $(t,s) \in \Delta_0$. *Then*

$$0.005 < t+s < 0.24, \quad 0.005 < s-t < 0.24.$$

Proof We put $u = t+s$ and $v = s-t$. Then

$$\operatorname{Re}\widehat{V} = \tfrac{1}{2}\Lambda(u) + \tfrac{1}{2}\Lambda(v).$$

It is sufficient to show that

$$0.005 < u < 0.24, \quad 0.005 < v < 0.24.$$

We have that

$$\tfrac{1}{2}\Lambda(2u) + \tfrac{1}{2}\Lambda(2v) = \operatorname{Re}\widehat{V} \geq \varsigma_R(M_p) \geq \varsigma_R(M_6).$$

Hence,

$$\begin{aligned}\Lambda(2u) &\geq 2\varsigma_R(M_6) - \Lambda(2v) \geq 2\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\\ &= 2\cdot 0.102216\ldots - 0.161533\ldots = 0.042899\ldots > 0.\end{aligned}$$

Therefore, from the above-mentioned behavior of $\Lambda(t)$, we have that

$$0 < 2u < 0.5.$$

More precisely, since

$$\begin{aligned}\Lambda(0.01) - \left(2\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\right) &= -0.00522573\ldots < 0,\\ \Lambda(0.48) - \left(2\varsigma_R(M_6) - \Lambda\left(\tfrac{1}{6}\right)\right) &= -0.0290494\ldots < 0,\end{aligned}$$

we obtain that

$$0.01 < 2u < 0.48.$$

Hence, $0.005 < u < 0.24$, as required. We can obtain that $0.005 < v < 0.24$ in the same way. □

Before we show Lemma E.3, we show the following lemma:

Lemma E.2 *For* $0 \leq \frac{1}{6} + t \leq 0.5$,

$$\Lambda\left(\tfrac{1}{6}+t\right) \leq \Lambda\left(\tfrac{1}{6}\right) - 1.4t^2.$$

Proof We put

$$f(t) = \Lambda\left(\tfrac{1}{6}\right) - 1.4t^2 - \Lambda\left(\tfrac{1}{6} + t\right).$$

Then its differentials are given by

$$f'(t) = -2.8t + \log 2 \sin \pi\left(\tfrac{1}{6} + t\right), \quad f''(t) = -2.8 + \pi \cot \pi\left(\tfrac{1}{6} + t\right).$$

For $-\frac{1}{6} \le t \le \frac{1}{3}$, the solution of $f''(t) = 0$ is given by

$$t = t_1 = -\frac{1}{6} + \frac{1}{\pi} \arctan\left(\frac{\pi}{2.8}\right) = 0.101613\ldots.$$

Further, for $-\frac{1}{6} \le t \le \frac{1}{3}$, the solutions of $f'(t) = 0$ are given by

$$t = 0, t_2, \quad \text{where } t_2 = 0.227398\ldots.$$

Hence, the behavior of $f(t)$ is given as follows:

t	$-\frac{1}{6}$	$\cdots$	0	$\cdots$	t_1	$\cdots$	t_2	$\cdots$	$\frac{1}{3}$
$f(t)$	$f\left(-\frac{1}{6}\right)$	$\searrow$	0	$\nearrow$	$f(t_1)$	$\nearrow$	$f(t_2)$	$\searrow$	$f\left(\frac{1}{3}\right)$
$f'(t)$		$-$	0	$+$	$+$	$+$	0	$-$	
$f''(t)$		$+$	$+$	$+$	0	$-$	$-$	$-$	

Here,

$$f\left(\tfrac{1}{3}\right) = \Lambda\left(\tfrac{1}{6}\right) - \tfrac{1.4}{9} = 0.00597742\ldots > 0.$$

Therefore, $f(t) \ge 0$, and we obtain the lemma. □

Lemma E.3 *Let p be an integer ≥ 6. We suppose that $\operatorname{Re} \widehat{V}(t,s) \ge \varsigma_R(M_p)$ for $(t,s) \in \Delta_0$. Then*

$$|t| < \frac{0.74}{p}.$$

Proof By Lemma D.1, we have that

$$\operatorname{Re} \widehat{V}(t,s) \ge \varsigma_R(M_p) \ge \Lambda\left(\frac{1}{6}\right) - \frac{3}{p^2}.$$

Further, putting $s = \frac{1}{12} + \widehat{s}$,

$$\begin{aligned}\operatorname{Re} \widehat{V}(t,s) &= \frac{1}{2}\Lambda(2s-2t) + \frac{1}{2}\Lambda(2s+2t) \le \frac{1}{2}\Lambda\left(\frac{1}{6} + 2\widehat{s} - 2t\right) + \frac{1}{2}\Lambda\left(\frac{1}{6} + 2\widehat{s} + 2t\right) \\ &\le \Lambda\left(\frac{1}{6}\right) - \frac{1.4}{2}(2\widehat{s} - 2t)^2 - \frac{1.4}{2}(2\widehat{s} + 2t)^2 \le \Lambda\left(\frac{1}{6}\right) - 5.6(\widehat{s}^2 + t^2),\end{aligned}$$

where we obtain the second last inequality by Lemma E.2. Hence,

$$\Lambda\Big(\frac{1}{6}\Big)-5.6(\hat{s}^2+t^2)\geq\Lambda\Big(\frac{1}{6}\Big)-\frac{3}{p^2}.$$

Therefore,

$$\frac{3}{5.6p^2}\geq\hat{s}^2+t^2\geq t^2.$$

It follows that

$$|t|\leq\frac{\sqrt{\frac{3}{5.6}}}{p}<\frac{0.74}{p},$$

since $\sqrt{\frac{3}{5.6}}=0.731925\ldots$. Hence, we obtain the lemma. □

Appendix F Restriction of Δ_0 to Δ_0''

In Section 7, we restrict Δ_0 to Δ_0'', and it is necessary to show that the following error term is sufficiently small:

$$\sum_{(1/2-i/N,1/2-j/N)\in\Delta_0-\Delta_0''}(q^{i/2}-q^{-i/2})\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)$$
$$=O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

We can reduce this formula to the following lemma; the aim of this section is to show this lemma.

Lemma F.1 *For any fixed k with $0<\frac{k}{N}\leq 0.001$, we have that*

$$\sum_{\substack{(1/2-i/N,1/2-j/N)\in\Delta_0\\ j-i=k}}(q^{i/2}-q^{-i/2})\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)$$
$$=O(e^{N(\varsigma_R(M_5)-\varepsilon)}),$$

$$\sum_{\substack{(1/2-i/N,1/2-j/N)\in\Delta_0\\ i+j=N-k}}(q^{i/2}-q^{-i/2})\exp\Big(N\cdot V\Big(\frac{1}{2}-\frac{i}{N},\frac{1}{2}-\frac{j}{N}-\frac{1}{2N}\Big)\Big)$$
$$=O(e^{N(\varsigma_R(M_5)-\varepsilon)})$$

for sufficiently small $\varepsilon>0$.

We let $\rho=\frac{k}{N}$ in the remainder of this section. Then ρ is a fixed real number with $0\leq\rho\leq 0.001$. We note that we can ignore a factor such as $(q^{i/2}-q^{-i/2})$ in the formulas of the lemma, because the aim of this section is to estimate the exponential order of the

formulas of the lemma and such a factor does not contribute to the exponential order, as we see in the calculation in Section 5, while the behavior of V does contribute to the exponential order. Hence, in order to show the lemma, it is sufficient to show that

$$\sum_{1/2-i/N\in\Delta_{0,1}} \exp\Big(N\cdot W_1\Big(\frac{1}{2}-\frac{i}{N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}), \tag{83}$$

$$\sum_{1/2-i/N\in\Delta_{0,2}} \exp\Big(N\cdot W_2\Big(\frac{1}{2}-\frac{i}{N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}), \tag{84}$$

where we put

$$W_1(t) = \frac{1}{N}(\widehat{\varphi}(2\rho)-\widehat{\varphi}(1-4t-2\rho))+4\pi\sqrt{-1}\Big(\frac{1}{4}t^2-\rho t\Big),$$
$$W_2(t) = \frac{1}{N}(\widehat{\varphi}(-4t+2\rho)-\widehat{\varphi}(1-2\rho))+4\pi\sqrt{-1}\Big(\frac{9}{4}t^2-\rho t\Big),$$

and

$$\Delta_{0,1} = \big\{t\in\mathbb{R}\mid 0\le t\le \tfrac{1}{8}\big\},$$
$$\Delta_{0,2} = \big\{t\in\mathbb{R}\mid -\tfrac{1}{8}\le t\le 0\big\}.$$

We note that, by Lemma 5.2, we can obtain that

$$W_2(-t) = W_1(t)+2\pi\sqrt{-1}t. \tag{85}$$

The functions W_i converge to the following functions as $N\to\infty$:

$$\widehat{W}_1(t) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{4\pi\sqrt{-1}\,\rho})-\mathrm{Li}_2(e^{-4\pi\sqrt{-1}(2t+\rho)}))+4\pi\sqrt{-1}\Big(\frac{1}{4}t^2-\rho t\Big),$$
$$\widehat{W}_2(t) = \frac{1}{4\pi\sqrt{-1}}(\mathrm{Li}_2(e^{-4\pi\sqrt{-1}(2t-\rho)})-\mathrm{Li}_2(e^{-4\pi\sqrt{-1}\,\rho}))+4\pi\sqrt{-1}\Big(\frac{9}{4}t^2-\rho t\Big).$$

We note again that, by (85), we obtain that

$$\widehat{W}_2(-t) = \widehat{W}_1(t)+2\pi\sqrt{-1}\,t. \tag{86}$$

Similarly as the calculation of Section 5, we can restrict $\Delta_{0,1}$ to $\Delta'_{0,1}$ of the following lemma:

Lemma F.2 *We recall that ρ is a fixed real number with $0\le\rho\le 0.001$. We put*

$$\Delta'_{0,1} = \{t\in\mathbb{R}\mid 0.02\le t\le 0.07\}.$$

Then the domain

$$\{t\in\Delta_{0,1}\mid \mathrm{Re}\,\widehat{W}_1(t)\ge\varsigma_R(M_5)-\varepsilon\}$$

is included in Δ'_0 for sufficiently small $\varepsilon>0$.

Proof Since the domain $\{\operatorname{Re}\widehat{W}_1(t) \geq \varsigma_R(M_5)\}$ is compact, assuming that $\operatorname{Re}\widehat{W}_1(t) \geq \varsigma_R(M_5)$, it is sufficient to show that

$$0.02 < t < 0.07.$$

From the definition of $\widehat{W}_1(t)$, we have that

$$\operatorname{Re}\widehat{W}_1(t) = \tfrac{1}{2}\Lambda(4t + 2\rho) + \tfrac{1}{2}\Lambda(2\rho) \geq \varsigma_R(M_5).$$

Hence,

$$\Lambda(4t + 2\rho) \geq 2\varsigma_R(M_5) - \Lambda(2\rho) \geq 2\varsigma_R(M_5) = 0.1561897\ldots,$$

and $\Lambda(0.09) = 0.141707\ldots$ and $\Lambda(0.26) = 0.142161\ldots$, which do not satisfy the above condition. Therefore, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that

$$0.09 \leq 4t + 2\rho \leq 0.26.$$

Further, since $0 \leq \rho \leq 0.001$, we obtain that

$$0.02 < t < 0.07,$$

as required. □

Proof of Lemma F.1 We recall that it is sufficient to show (83) and (84). For simplicity, we show a proof of (83). (We can show (84) similarly.) By Lemma F.2, we can restrict the range of the sum of (83) to $\Delta'_{0,1}$. Hence, it is sufficient to show that

$$\sum_{1/2 - i/N \in \Delta'_{0,1}} \exp\Big(N \cdot W_1\Big(\frac{1}{2} - \frac{i}{N}\Big)\Big) = O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

Further, by Proposition F.4 (Poisson summation formula), it is sufficient to show that

$$\int_{\Delta'_{0,1}} \exp(N \cdot W_1(t))\, dt = O(e^{N(\varsigma_R(M_5)-\varepsilon)}). \tag{87}$$

We show this formula in Proposition F.7 by a similar procedure to the saddle-point method. Therefore, we obtain the lemma. □

F.1 Critical points of $\widehat{W}_i$

In this section, we calculate critical points of $\widehat{W}_1$ and $\widehat{W}_2$, which we use in Section F.3 later.

From the definitions of $\widehat{W}_1$ and $\widehat{W}_2$, their differentials are given by

$$\frac{d\widehat{W}_1}{dt} = -2\log(1-e^{-4\pi\sqrt{-1}\,(2t+\rho)}) + 4\pi\sqrt{-1}\left(\frac{1}{2}t-\rho\right), \tag{88}$$

$$\frac{d\widehat{W}_2}{dt} = 2\log(1-e^{-4\pi\sqrt{-1}(2t-\rho)}) + 4\pi\sqrt{-1}\left(\frac{9}{2}t-\rho\right). \tag{89}$$

We consider a critical point of $\widehat{W}_1$ in the domain

$$\{t\in\mathbb{C} \mid 0.02 \le \operatorname{Re} t \le 0.08\}. \tag{90}$$

We note that this range is more extended than the range of Lemma F.2, because there is a critical point in the extended part and we use it in Section F.3. The equation $d\widehat{W}_1/dt = 0$ is rewritten as

$$\log(1-e^{-4\pi\sqrt{-1}(2t+\rho)}) = 2\pi\sqrt{-1}\left(\frac{t}{2}-\rho\right).$$

Further, it is rewritten as

$$1-e^{-4\pi\sqrt{-1}(2t+\rho)} = e^{2\pi\sqrt{-1}(t/2-\rho)},$$

and

$$1-x^8e^{-4\pi\sqrt{-1}\,\rho} = \frac{1}{x}e^{-2\pi\sqrt{-1}\,\rho},$$

where we put $x = e^{-\pi\sqrt{-1}\,t}$. We can verify that there is a single solution of this equation such that the corresponding t is in (90); we put this solution t_0. We show some numerical values of t_0, as follows:

ρ	t_0
0	$0.0787594\ldots - \sqrt{-1}\cdot 0.0467182\ldots$
0.0002	$0.0788007\ldots - \sqrt{-1}\cdot 0.0468217\ldots$
0.0004	$0.0788424\ldots - \sqrt{-1}\cdot 0.0469251\ldots$
0.0006	$0.0788846\ldots - \sqrt{-1}\cdot 0.0470283\ldots$
0.0008	$0.0789272\ldots - \sqrt{-1}\cdot 0.0471313\ldots$
0.001	$0.0789703\ldots - \sqrt{-1}\cdot 0.0472342\ldots$

We consider a critical point of $\widehat{W}_2$ in the domain

$$\{t\in\mathbb{C} \mid -0.08 \le \operatorname{Re} t \le -0.02\}.$$

Then, noting (86), we can show that $-t_0$ is a single critical point of $\widehat{W}_2$ in the above domain.

F.2 The Poisson summation formula for W_i

The aim of this section is to show Proposition F.4, which we use in the proof of Lemma F.2.

In order to show Proposition F.4, we review the following proposition, which corresponds to the 1–dimensional case of Proposition 3.4:

Proposition F.3 (see [18]) *For $c \in \mathbb{C}$ and a closed interval D' in $\mathbb{R}$, we put*

$$\Lambda = \Big\{ \frac{i}{N} + c_1 \in \mathbb{C} \;\Big|\; i \in \mathbb{Z},\ \frac{i}{N} \in D' \Big\},$$
$$D = \{ t + c_1 \in \mathbb{C} \mid t \in D' \subset \mathbb{R} \}.$$

Let $\psi(t)$ be a holomorphic function defined in a neighborhood of $0 \in \mathbb{C}$ including D. We assume that ∂D is included in the domain

$$\{ t \in \mathbb{C} \mid \operatorname{Re} \psi(t) < -\varepsilon_0 \}$$

for some $\varepsilon_0 > 0$. Further, we assume that the two points of ∂D can be connected by a path in each of the domains

$$\{ t + \delta\sqrt{-1} \in \mathbb{C} \mid t \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t + \delta\sqrt{-1}) < 2\pi\delta \}, \tag{91}$$

$$\{ t - \delta\sqrt{-1} \in \mathbb{C} \mid t \in D',\ \delta \geq 0,\ \operatorname{Re} \psi(t - \delta\sqrt{-1}) < 2\pi\delta \}. \tag{92}$$

Then

$$\frac{1}{N} \sum_{t \in \Lambda} e^{N\psi(t)} = \int_D e^{N\psi(t)} dt + O(e^{-N\varepsilon})$$

for some $\varepsilon > 0$.

The aim of this section is to show the following proposition, by using the above proposition:

Proposition F.4 *We have*

$$\frac{1}{N} \sum_{1/2 - i/N \in \Delta'_{0,1}} \exp\Big(N \cdot W_1\Big(\frac{1}{2} - \frac{i}{N}\Big)\Big) = \int_{\Delta'_{0,1}} \exp(N \cdot W_1(t))\, dt + O(e^{N(\varsigma_R(M_5) - \varepsilon)}).$$

Proof By applying Proposition F.3 to $\widehat{W}_1(t) - \varsigma_R(M_5)$, we obtain the proposition. We verify the assumptions (91) and (92) in Lemmas F.5 and F.6 below. The other assumptions of Proposition F.3 can be verified easily. □

We note that the following inequalities hold for $x = e^{4\pi\sqrt{-1}\,(2t+\rho+2\delta\sqrt{-1})}$ and $\delta \in \mathbb{R}$:

$$0 < \operatorname{Arg}\Big(1 - \frac{1}{x}\Big) < 4\pi\Big(\frac{1}{4} - 2t - \rho\Big); \tag{93}$$

we use these in the proofs of the following two lemmas:

Lemma F.5 *The assumption* (91) *holds in the proof of Proposition F.4.*

Proof We let

$$F_t(\delta) = \operatorname{Re}\widehat{W}_1(t + \delta\sqrt{-1}) - \varsigma_{\boldsymbol{R}}(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_t(\delta) < -\varepsilon, \tag{94}$$

for any $t \in \Delta'_{0,1}$.

To show this, we estimate the differential of $F_t(\delta)$, as follows. By (88), the differential of $F_t(\delta)$ is given by

$$\frac{dF_t}{d\delta} = \operatorname{Re}\Big(\sqrt{-1}\,\frac{d}{dt}\widehat{W}_1(t + \delta\sqrt{-1})\Big) - 2\pi = 2\operatorname{Arg}\Big(1 - \frac{1}{x}\Big) - 4\pi\Big(\frac{1}{2}t - \rho\Big) - 2\pi,$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho+2\delta\sqrt{-1})}$. By (93), we can estimate it as follows:

$$\frac{dF_t}{d\delta} < 2\cdot 4\pi\Big(\frac{1}{4} - 2t - 2\rho\Big) - 4\pi\Big(\frac{1}{2}t - \rho\Big) - 2\pi = -4\pi\Big(\frac{9}{2}t + \rho\Big) \le -4\pi\cdot 0.09,$$

where we obtain the last inequality since $t \ge 0.02$ by Lemma F.2. Hence, (94) holds, as required. □

Lemma F.6 *The assumption* (92) *holds in the proof of Proposition F.4.*

Proof We let

$$F_t(\delta) = \operatorname{Re}\widehat{W}_1(t - \delta\sqrt{-1}) - \varsigma_{\boldsymbol{R}}(M_5) - 2\pi\delta$$

in this proof. Similarly as the proof of Lemma 5.8, it is sufficient to show that there exists $\varepsilon > 0$ such that

$$\frac{d}{d\delta}F_t(\delta) < -\varepsilon \tag{95}$$

for any $t \in \Delta'_{0,1}$.

To show this, we estimate the differential of $F_t(\delta)$, as follows. By (88), the differential of $F_t(\delta)$ is given by

$$\frac{dF_t}{d\delta} = \mathrm{Re}\Big(-\sqrt{-1}\frac{d}{dt}\widehat{W}_1(t+\delta\sqrt{-1})\Big) - 2\pi = -2\,\mathrm{Arg}\Big(1-\frac{1}{x}\Big) + 4\pi\Big(\frac{1}{2}t-\rho\Big) - 2\pi,$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho-2\delta\sqrt{-1})}$. By (93), we can estimate it as follows:

$$\frac{dF_t}{d\delta} < 0 + 4\pi\Big(\frac{1}{2}t-\rho\Big) - 2\pi = 4\pi\Big(\frac{1}{2}t-\rho-\frac{1}{2}\Big) \le -4\pi\cdot\frac{0.93}{2},$$

where we obtain the last inequality since $t \le 0.07$ by Lemma F.2. Hence, (95) holds, as required. □

F.3 The integral (87) is sufficiently small

In this section, we show that the integral (87) is of sufficiently small order in the following proposition. The aim of this section is to show this proposition.

Proposition F.7 *We have*

$$\int_{\Delta'_{0,1}} \exp(N\cdot W_1(t))\,dt = O(e^{N(\varsigma_R(M_5)-\varepsilon)}).$$

We give a proof of the proposition later in this section.

We use a similar procedure as the saddle-point method, but we have no critical point whose real part belongs to $\Delta'_{0,1}$ in this case. Hence, the conclusion is that the integral of the proposition is of sufficiently small order. We put

$$f_t(X) = \mathrm{Re}\,\widehat{W}_1(t+X\sqrt{-1}).$$

Then, in the same way as in the previous section, we have that

$$\frac{df_t}{dX} = 2\,\mathrm{Arg}\Big(1-\frac{1}{x}\Big) - 4\pi\Big(\frac{1}{2}t-\rho\Big), \tag{96}$$

where we put $x = e^{4\pi\sqrt{-1}(2t+\rho+2X\sqrt{-1})}$.

Lemma F.8 *We fix a real number t with $0.02 \le t \le 0.08$. We recall that ρ is a fixed real number with $0 \le \rho \le 0.001$. Then $f_t(X)$ has a unique minimal point at $X = g(t)$, where*

$$g(t) = \frac{1}{8\pi}\log\frac{\sin 2\pi\big(\frac{1}{2}t-\rho\big)}{\sin 2\pi\big(\frac{1}{2}-\frac{9}{2}t-\rho\big)},$$

ie $f_t(X)$ is monotonically decreasing for $X < g(t)$ and is monotonically increasing for $X > g(t)$.

Proof We put $x = e^{4\pi\sqrt{-1}(2t+\rho+2X\sqrt{-1})}$. Then $\frac{1}{x} = e^{8\pi X}e^{-4\pi\sqrt{-1}(2t+\rho)}$. We put $\theta = \operatorname{Arg}\big(1-\frac{1}{x}\big)$ in this proof. Since $0.02 \le t \le 0.08$, θ is in the range

$$0 < \theta < 4\pi\big(\tfrac{1}{4} - 2t - \rho\big).$$

By (96),

$$\frac{\partial}{\partial X} f_t(X) \begin{cases} > 0 & \text{if } \theta > 2\pi\big(\frac{1}{2}t - \rho\big), \\ = 0 & \text{if } \theta = 2\pi\big(\frac{1}{2}t - \rho\big), \\ < 0 & \text{if } \theta < 2\pi\big(\frac{1}{2}t - \rho\big). \end{cases}$$

Further, in the same way as in the proof of Lemma 5.3, we can show that X is monotonically increasing as a function of θ, and they satisfy that

$$\frac{e^{8\pi X}}{\sin\theta} = \frac{1}{\sin\big(4\pi\big(\frac{1}{4} - 2t - \rho\big) - \theta\big)}.$$

This is rewritten as

$$X = \frac{1}{8\pi}\log\frac{\sin\theta}{\sin\big(4\pi\big(\frac{1}{4} - 2t - \rho\big) - \theta\big)}.$$

Therefore,

$$\frac{\partial}{\partial X} f_t(X) \begin{cases} > 0 & \text{if } X > g(t), \\ = 0 & \text{if } X = g(t), \\ < 0 & \text{if } X < g(t), \end{cases}$$

where we put

$$g(t) = \frac{1}{8\pi}\log\frac{\sin 2\pi\big(\frac{1}{2}t - \rho\big)}{\sin 2\pi\big(\frac{1}{2} - \frac{9}{2}t - \rho\big)}.$$

Hence, we obtain the lemma. □

Proof of Proposition F.7 We put

$$\Delta''_{0,1} = \{t \in \mathbb{R} \mid 0.02 \le t \le 0.08\}.$$

We show that there exists a homotopy $\Delta''_{(\delta)}$ $(0 \le \delta \le 1)$ between $\Delta''_{(0)} = \Delta''_{0,1}$ and $\Delta''_{(1)}$ such that

$$\Delta''_{(1)} \subset \{t \in \mathbb{C} \mid \operatorname{Re} W_1(t) < \varsigma_R(M_5) - \varepsilon\}, \tag{97}$$

$$\partial\Delta''_{(\delta)} \subset \{t \in \mathbb{C} \mid \operatorname{Re} W_1(t) < \varsigma_R(M_5) - \varepsilon\}. \tag{98}$$

We note that $W_1(t)$ uniformly converges to $\widehat{W}_1(t)$ on $\Delta''_{0,1}$ and the error term is of order $O(1/N^2)$. So we show the existence of such a homotopy for $\widehat{W}_1(t)$, instead of $W_1(t)$.

For each fixed t, we move X from 0 along the gradient flow of $-\mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1})$. Then the value of $\mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1})$ monotonically decreases. By Lemma F.8, X arrives at $g(t)$. We put $\Delta''_{(\delta)}$ to be

$$\Delta''_{(\delta)} = \{t + \delta \cdot g(t)\sqrt{-1} \in \mathbb{C} \mid 0.02 \leq t \leq 0.08\}.$$

It is sufficient to show that this homotopy satisfies (98) and (97).

We show (98), as follows. Since $\partial\Delta''_{0,1}$ is originally included in this domain and the value of $\mathrm{Re}\,\widehat{W}_1$ monotonically decreases, $\partial\Delta''_{(\delta)}$ is also included in this domain. Hence, (98) holds.

We show (97), as follows. We consider the functions

$$F(t, X) = \mathrm{Re}\,\widehat{W}_1(t + X\sqrt{-1}), \quad h(t) = F(t, g(t)) = \mathrm{Re}\,\widehat{W}_1(t + g(t)\sqrt{-1}).$$

It follows from the definition of $g(t)$ that $\frac{\partial F}{\partial X} = 0$ at $X = g(t)$. Hence, $\mathrm{Im}\,d\widehat{W}_1/dt = 0$ at $(t + g(t)\sqrt{-1})$. Further, $dh/dt = \mathrm{Re}\,d\widehat{W}_1/dt$ at $(t + g(t)\sqrt{-1})$. Therefore, when t is a critical point of $h(t)$, $(t + g(t)\sqrt{-1})$ is a critical point of $\widehat{W}_1$. Further, as shown in Section F.1, $\widehat{W}_1$ has a single critical point at t_0 in (90). Hence, putting $t_R = \mathrm{Re}\,t_0$, $h(t)$ has a unique maximal point at t_R and no other critical points. Originally, t_R belongs to the domain of (97). Further, since $\mathrm{Re}\,\widehat{W}_1$ monotonically decreases by the above-mentioned gradient flow, t_0 belongs to the domain of (97). Furthermore, since $h(t)$ has a unique maximal point at t_R, $(t + g(t)\sqrt{-1})$ also belongs to the domain of (97). Therefore, (97) holds, as required. □

Appendix G Proof of Lemma 7.1

The aim of this section is to give a proof of Lemma 7.1.

Proof of Lemma 7.1 Since the domain $\{\mathrm{Re}\,\widehat{V}(t, s) \geq \varsigma_R(M_5)\}$ is compact, assuming that $\mathrm{Re}\,\widehat{V}(t, s) \geq \varsigma_R(M_5)$, it is sufficient to show that

$$t + s < 0.26, \quad s - t < 0.26, \quad |t| < 0.099, \quad s < 0.2.$$

We show these inequalities in this proof. We recall that

$$\mathrm{Re}\,\widehat{V}(t, s) = \tfrac{1}{2}\Lambda(2t + 2s) + \tfrac{1}{2}\Lambda(2s - 2t),$$

as shown in Appendix E. We put $u = t + s$ and $v = s - t$.

We show that $t+s<0.26$ and $s-t<0.26$, as follows. Assuming that

$$\tfrac{1}{2}\Lambda(2u)+\tfrac{1}{2}\Lambda(2v)\geq \varsigma_R(M_5),$$

it is sufficient to show that $u<0.26$ and $v<0.26$. We have that

$$\begin{aligned}\Lambda(2u)&\geq 2\,\varsigma_R(M_5)-\Lambda(2v)\geq 2\,\varsigma_R(M_5)-\Lambda\left(\tfrac{1}{6}\right)\\&=2\cdot 0.078094\ldots-0.161533\ldots=-0.005343\ldots,\end{aligned}$$

and $\Lambda(2\cdot 0.26)=-0.0138498$, which does not satisfy the above condition. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that $u<0.26$. We also obtain that $v<0.26$ in the same way.

We show that $s<0.2$, as follows. By Lemma G.1 below, $\{\operatorname{Re}\widehat{V}(t,s)\geq \varsigma_R(M_5)\}$ is a convex domain. Further, this domain is symmetric with respect to the change of the sign of t. Hence, s has a maximal value when $t=0$. Putting $t=0$, we have that

$$\operatorname{Re}\widehat{V}(0,s)=\Lambda(2s)\geq \varsigma_R(M_5)=0.078094\ldots,$$

and $\Lambda(2\cdot 0.2)=0.0676532\ldots$, which does not satisfy the above condition. Hence, since the behavior of $\Lambda(\cdot)$ is as shown in Appendix E, we obtain that $s<0.2$.

We show that $|t|<0.099$, as follows. Noting that the domain $\{\operatorname{Re}\widehat{V}(t,s)\geq \varsigma_R(M_5)\}$ is symmetric with respect to the change of the sign of t, it is sufficient to show that $t<0.099$. We calculate the maximal value of $t_{\max}$ in this domain. It satisfies the equations

$$\begin{cases}\frac{1}{2}\Lambda(2t+2s)+\frac{1}{2}\Lambda(2s-2t)=\varsigma_R(M_5),\\ \frac{\partial}{\partial s}\left(\frac{1}{2}\Lambda(2t+2s)+\frac{1}{2}\Lambda(2s-2t)\right)=0.\end{cases}$$

They are rewritten as

$$\begin{cases}\Lambda(2t+2s)+\Lambda(2s-2t)=2\varsigma_R(M_5),\\ \Lambda'(2t+2s)+\Lambda'(2s-2t)=0.\end{cases}$$

We note that this system of equations has exactly two solutions, corresponding to the maximal and minimal values of t, since the domain $\{\operatorname{Re}\widehat{V}(t,s)\geq \varsigma_R(M_5)\}$ is a convex domain whose boundary curve is smooth by Lemma G.1 below. By calculating a solution of them by Newton's method from $(t,s)=(0.1,0.2)$, we obtain $t_{\max}=0.097454002299\ldots$. Hence, $t<0.099$, as required. □

Lemma G.1 *The domain*

$$\left\{(t,s)\in\mathbb{R}^2 \mid \operatorname{Re}\widehat{V}(t,s)\geq \varsigma_R(M_5),\ 0\leq t+s<\tfrac{1}{2},\ 0\leq s-t<\tfrac{1}{2}\right\}$$

is a convex domain.

See Figure 6 for a graphical representation of this lemma.

Proof Putting $u = t + s$ and $v = s - t$, it is sufficient to show that the domain

$$\{(u, v) \in \mathbb{R}^2 \mid \Lambda(2u) + \Lambda(2v) \geq 2\varsigma_R(M_5),\ 0 \leq u < \tfrac{1}{2},\ 0 \leq v < \tfrac{1}{2}\} \tag{99}$$

is a convex domain. Hence, it is sufficient to show that the curvature of the boundary curve of this domain with respect to the outward normal vector is negative everywhere except for the lines $\{u = 0\}$ and $\{v = 0\}$.

When $0 \leq u \leq \frac{1}{4}$ and $0 \leq v \leq \frac{1}{4}$, $\Lambda(2u)$ and $\Lambda(2v)$ are concave functions, and it follows that the domain of the lemma is convex, and the curvature of the boundary curve with respect to the outward normal vector is negative in this area.

When $0 \leq u \leq \frac{1}{4}$ and $\frac{1}{4} < v < \frac{1}{2}$, we show the lemma, as follows. As we can observe in Figure 6, this area is relatively small. We can obtain that

$$0.062 < u < 0.107,$$

since $\Lambda(2u) \geq 2\varsigma_R(M_5)$. Further, we can obtain that

$$\tfrac{1}{4} < v < 0.254,$$

since $\Lambda(2v) \geq 2\varsigma_R(M_5) - \Lambda(2u) \geq 2\varsigma_R(M_5) - \Lambda\left(\frac{1}{6}\right)$. Hence, noting the behavior of $\Lambda(\cdot)$ shown in Appendix E, we can obtain that

$$-0.22 < \Lambda'(2u) < -0.28,$$

since $\Lambda'(2\cdot 0.062) = 0.275018\ldots$ and $\Lambda'(2\cdot 0.107) = -0.219598\ldots$, and can obtain that

$$-0.7 < \Lambda'(2v) < -0.69,$$

since $\Lambda'\left(2\cdot\frac{1}{4}\right) = -0.693147\ldots$ and $\Lambda'(2\cdot 0.254) = -0.692831\ldots$, and can obtain that

$$-8 < \Lambda''(2u) < -2,$$

since $\Lambda''(2\cdot 0.062) = -7.65238\ldots$ and $\Lambda''(2\cdot 0.107) = -3.94669\ldots$, and can obtain that

$$0 < \Lambda''(2v) < 0.1,$$

since $\Lambda'\left(2\cdot\frac{1}{4}\right) = 0$ and $\Lambda'(2\cdot 0.254) = 0.0789735\ldots$. We consider the boundary curve of the domain (99). It is given by $F(u, v) = 0$, where we put

$$F(u, v) = \Lambda(2u) + \Lambda(2v) - 2\varsigma_R(M_5).$$

When $0.062 < u < 0.107$ and $\frac{1}{4} < v < 0.254$, this curve can be presented by $v = f(u)$ with some function f satisfying that $F(u, f(u)) = 0$. Its differentials are given by

$$\frac{d}{du}F(u, f(u)) = 2\big(\Lambda'(2u) + \Lambda'(2f(u))f'(u)\big) = 0,$$
$$\frac{d^2}{du^2}F(u, f(u)) = 2\big(2\Lambda''(2u) + 2\Lambda''(2f(u))f'(u)^2 + \Lambda'(2f(u))f''(u)\big) = 0.$$

Since

$$f'(u) = -\frac{\Lambda'(2u)}{\Lambda'(2v)},$$

we obtain that $|f'(u)| < 1$ from the above inequalities. Further, since

$$f''(u) = -\frac{1}{\Lambda'(2v)}(2\Lambda''(2u) + 2\Lambda''(2v)f'(u)^2),$$

we obtain that $f''(u) < 0$ from the above inequalities. Therefore, the curvature of the boundary curve with respect to the outward normal vector is negative in the area that $0.062 < u < 0.107$ and $\frac{1}{4} < v < 0.254$.

When $0 \le v \le \frac{1}{4}$ and $\frac{1}{4} < u < \frac{1}{2}$, we can show that the above-mentioned curvature is negative in the area that $0.062 < v < 0.107$ and $\frac{1}{4} < u < 0.254$, in the same way as above, exchanging u and v.

Therefore, the curvature of the boundary curve of the domain (99) with respect to the outward normal vector is negative everywhere except for the lines $\{u = 0\}$ and $\{v = 0\}$, as required. □

References

[1] **G Belletti**, **R Detcherry**, **E Kalfagianni**, **T Yang**, *Growth of quantum $6j$–symbols and applications to the volume conjecture*, preprint (2018) arXiv

[2] **Q Chen**, **T Yang**, *Volume conjectures for the Reshetikhin–Turaev and the Turaev–Viro invariants*, Quantum Topol. 9 (2018) 419–460 MR

[3] **R Detcherry**, **E Kalfagianni**, **T Yang**, *Turaev–Viro invariants, colored Jones polynomials, and volume*, Quantum Topol. 9 (2018) 775–813

[4] **L D Faddeev**, *Discrete Heisenberg–Weyl group and modular group*, Lett. Math. Phys. 34 (1995) 249–254 MR

[5] **L D Faddeev**, **R M Kashaev**, **A Y Volkov**, *Strongly coupled quantum discrete Liouville theory, I: Algebraic approach and duality*, Comm. Math. Phys. 219 (2001) 199–219 MR

[6] **K Habiro**, *On the colored Jones polynomials of some simple links*, from "Recent progress towards the volume conjecture" (H Murakami, editor), Sūrikaisekikenkyūsho Kōkyūroku 1172, RIMS, Kyoto (2000) 34–43 MR

[7] **R M Kashaev**, *Quantum dilogarithm as a* $6j$*–symbol*, Modern Phys. Lett. A 9 (1994) 3757–3768 MR

[8] **R M Kashaev**, *A link invariant from quantum dilogarithm*, Modern Phys. Lett. A 10 (1995) 1409–1418 MR

[9] **R M Kashaev**, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. 39 (1997) 269–275 MR

[10] **S Lang**, *Algebraic number theory*, 2nd edition, Graduate Texts in Mathematics 110, Springer (1994) MR

[11] **T T Q Le**, *Quantum invariants of* 3*–manifolds: integrality, splitting, and perturbative expansion*, Topology Appl. 127 (2003) 125–152 MR

[12] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Mathematics 175, Springer (1997) MR

[13] **H Murakami**, *Optimistic calculations about the Witten–Reshetikhin–Turaev invariants of closed three-manifolds obtained from the figure-eight knot by integral Dehn surgeries*, from "Recent progress towards the volume conjecture" (H Murakami, editor), Sūrikaisekikenkyūsho Kōkyūroku 1172, RIMS, Kyoto (2000) 70–79 MR

[14] **H Murakami**, **J Murakami**, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. 186 (2001) 85–104 MR

[15] **H Murakami**, **J Murakami**, **M Okamoto**, **T Takata**, **Y Yokota**, *Kashaev's conjecture and the Chern–Simons invariants of knots and links*, Experiment. Math. 11 (2002) 427–435 MR

[16] **H Murakami**, **Y Yokota**, *The colored Jones polynomials of the figure-eight knot and its Dehn surgery spaces*, J. Reine Angew. Math. 607 (2007) 47–68 MR

[17] **T Ohtsuki**, *Quantum invariants: a study of knots,* 3*–manifolds, and their sets*, Series on Knots and Everything 29, World Sci., River Edge, NJ (2002) MR

[18] **T Ohtsuki**, *On the asymptotic expansion of the Kashaev invariant of the* 5_2 *knot*, Quantum Topol. 7 (2016) 669–735 MR

[19] **T Ohtsuki**, *On the asymptotic expansions of the Kashaev invariant of hyperbolic knots with seven crossings*, Internat. J. Math. 28 (2017) art. id. 1750096 MR

[20] **T Ohtsuki**, **T Takata**, *On the Kashaev invariant and the twisted Reidemeister torsion of two-bridge knots*, Geom. Topol. 19 (2015) 853–952 MR

[21] **T Ohtsuki**, **T Takata**, *On the quantum* SU(2) *invariant at* $q = \exp(4\pi\sqrt{-1}/N)$ *and the twisted Reidemeister torsion for some closed* 3*–manifolds*, preprint, RIMS (2017) `http://www.kurims.kyoto-u.ac.jp/preprint/file/RIMS1877.pdf`

[22] **T Ohtsuki**, **Y Yokota**, *On the asymptotic expansions of the Kashaev invariant of the knots with* 6 *crossings*, Math. Proc. Cambridge Philos. Soc. 165 (2018) 287–339 MR

[23] **E M Stein**, **G Weiss**, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series 32, Princeton Univ. Press (1971) MR

[24] **W P Thurston**, *The geometry and topology of three-manifolds*, lecture notes, Princeton University (1979) `http://msri.org/publications/books/gt3m`

[25] **V G Turaev**, *Quantum invariants of knots and* 3*–manifolds*, De Gruyter Studies in Mathematics 18, de Gruyter, Berlin (1994) MR

[26] **E Witten**, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989) 351–399 MR

[27] **S L Woronowicz**, *Quantum exponential function*, Rev. Math. Phys. 12 (2000) 873–920 MR

[28] **Y Yokota**, *On the Kashaev invariant of twist knots*, from "Intelligence of low-dimensional topology" (T Ohtsuki, M Wakui, editors), Sūrikaisekikenkyūsho Kōkyūroku 1766, RIMS, Kyoto (2011) 45–51

Research Institute for Mathematical Sciences, Kyoto University
Sakyo-ku, Kyoto, Japan

`tomotada@kurims.kyoto-u.ac.jp`

`http://www.kurims.kyoto-u.ac.jp/~tomotada/`

Received: 15 January 2018 Revised: 29 June 2018

msp

Algebraic & Geometric Topology 18 (2018) 4275–4303

Notes on open book decompositions for Engel structures

VINCENT COLIN
FRANCISCO PRESAS
THOMAS VOGEL

We relate open book decompositions of a 4–manifold M with its Engel structures. Our main result is, given an open book decomposition of M whose binding is a collection of 2–tori and whose monodromy preserves a framing of a page, the construction of an Engel structure whose isotropic foliation is transverse to the interior of the pages and tangent to the binding.

In particular, the pages are contact manifolds and the monodromy is a compactly supported contactomorphism. As a consequence, on a parallelizable closed 4–manifold, every open book with toric binding carries in the previous sense an Engel structure. Moreover, we show that among the supported Engel structures we construct, there are loose Engel structures.

58A30

1 Introduction

One particularly fruitful approach to the study of contact structures is the use of open book decompositions, that was introduced by E Giroux [13]. He showed that every cooriented contact structure on a closed $(2n-1)$–manifold is supported by an open book decomposition of the underlying manifold, ie it can be defined by a 1–form α such that α gives a contact structure on the binding and α turns the pages into Weinstein domains. If $n=2$, then there is a one-to-one correspondence between open book decompositions of M up to positive stabilization and isotopy classes of contact structures on M (see for example Etnyre [9]).

The simpler part of this correspondence — the construction of a contact structure starting from an open book decomposition — was established by W Thurston and H Winkelnkemper [23].

Engel structures form a class of plane fields on 4–manifolds which have attracted some interest recently, although our understanding of their properties is relatively underdeveloped. We refer to Montgomery [19] and Vogel [24] for explanations concerning

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4275

the motivation of their study. Here we note that Engel structures are closely related to contact structures in dimension 3. This fact was applied successfully in [24] to prove that every parallelizable 4–manifold admits an Engel structure.

The purpose of this note is to prove an analogue of the Thurston–Winkelnkemper theorem for Engel structures: under the assumptions that the binding is a collection of 2–tori and that the monodromy preserves a framing of a page, we will obtain Engel structures starting from an open book decomposition of the manifold that make the pages contact; see Theorem 5.3. This follows from Corollary 5.4 for every open book decomposition with toric binding of a parallelizable 4–manifold.

We also address stabilization constructions and the uniqueness question. In Theorem 5.8 we show that amongst the supported Engel structures we construct there is a special class of loose ones which is invariant under stabilization. We also raise the problem of studying Engel structures supported by an open book with tight contact pages, a property that is invariant under stabilization by Theorem 6.3.

Whether an Engel structure is homotopic — through Engel structures — to an Engel structure which is adapted to an open book decomposition is unknown. Notice that it is unknown whether nonloose Engel structures exist. Finding a supporting open book decomposition for a loose Engel structure up to Engel homotopy amounts, thanks to our construction, to finding an open book decomposition and a framing (e_1, e_2, e_3, e_4) such that the pages are transverse to a vector field e_1' homotopic to e_1 and the binding tangent to e_1' and linearly foliated by e_1'.

Also, our definition of an Engel structure being adapted to an open book decomposition is not tightly suited since the requirements in the definition only use the associated even contact structure.

Acknowledgements It is a pleasure for us to thank the American Institute of Mathematics for organizing a conference on Engel structures, where we had the most recent opportunity to discuss the subject of this paper among ourselves and with others. We also would like to thank the referee for the swift reading of this manuscript, and for the suggestions that helped to improve the presentation and for pointing out several references.

Colin thanks the ERC Geodycon and the ANR Quantact for their support. Presas is supported by the Spanish Research Projects SEV-2015-0554, MTM2016-79400-P and MTM2015-72876-EXP.

2 Engel and even contact structures

We assume that the reader is familiar with the basics of contact topology in dimension 3; a lot of the material we use is covered in [11]. The Giroux correspondence for contact manifolds in dimension 3 is the subject of [9].

We now give definitions of Engel structures and associated distributions.

Definition 2.1 Let M be a $2n$–dimensional manifold. A (cooriented) even contact structure on M is a hyperplane field $\mathcal{E}$ defined as the kernel of a 1–form α such that $\alpha \wedge (d\alpha)^{n-1}$ never vanishes.

To each even contact structure one can associate a line field $\mathcal{W} \subset \mathcal{E}$, called its *isotropic foliation* or *kernel foliation*, which is the kernel of $d\alpha$ restricted to $\mathcal{E}$. When $n = 2$, the even contact condition is equivalent to $[\mathcal{E}, \mathcal{E}] = TM$.

Definition 2.2 An Engel structure $\mathcal{D}$ is a smooth plane field on a 4–manifold M such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is an even contact structure.

Notice that if an even contact structure $\mathcal{E}$ is induced by an Engel structure, ie $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$, then $\mathcal{W}$ is tangent to $\mathcal{D}$. Thus, an Engel structure $\mathcal{D}$ induces a flag of distributions

$$\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM.$$

Moreover, if $\mathcal{E}$ comes from an Engel structure, then it is canonically oriented, and an orientation of TM induces an orientation of $\mathcal{W}$ and vice versa. If we assume that both $\mathcal{D}$ and M are oriented, then this shows that $\mathcal{D}$ induces a framing of TM which is well defined up to homotopy. In particular, two Engel structures cannot be homotopic through Engel structures if the associated framings are not homotopic. Quite recently it was shown in [2] that every framing of a parallelizable 4–manifold is homotopic to the framing induced by an Engel structure on M.

Example 2.3 On $\mathbb{R}^4$ the 1–form $dz - y\,dx$ defines an even contact structure $\mathcal{E}$. The isotropic foliation is spanned by ∂_w (where w is the fourth coordinate). Locally, every even contact structure is diffeomorphic to the one given here.

The plane field $\mathcal{D} = \ker(dy - w\,dx) \cap \mathcal{E}$ is an Engel structure such that $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$. Every Engel structure is locally diffeomorphic to $\mathcal{D}$ (this fact is due to F Engel [8]). □

The fact that there is a line field associated to an even contact structure/Engel structure means that Gray's stability theorem cannot hold for these distributions: a varying family of even contact structures or Engel structures will induce a varying family of isotropic foliations. Hence, the even contact structures/Engel structures in the family are not diffeomorphic to each other since dynamical properties of the isotropic foliation can change. This is explored in [19].

If $\mathcal{E}$ is an even contact structure on M and if Y is a hypersurface in M transverse to $\mathcal{W}$, then the plane field

$$\xi = TY \cap \mathcal{E}$$

is a contact structure on Y.

Recall also that even contact structures satisfy the h-principle: a formal (oriented) even contact structure on M is a pair $(\mathcal{E}, \mathcal{W})$, where $\mathcal{E}$ is an oriented hyperplane field on M and $\mathcal{W} \subset \mathcal{E}$ is an oriented line field. According to [17], the space of even contact structures is homotopy equivalent to the space of formal even contact structures.

Remark 2.4 Once the even contact structure is fixed, the homotopy class of $\mathcal{D}$ is also fixed as a plane field in $\mathcal{E}$. Indeed, let W be a vector field generating $\mathcal{W}$ and N be a vector field orthogonal to $\mathcal{D}$ in $\mathcal{E}$ (assuming we have fixed a metric). The path of vector fields $\cos s W + \sin s N$ for $s \in \left[0, \frac{\pi}{2}\right]$ interpolates between W and N, and thus gives a path between their normal plane fields in $\mathcal{E}$.

However, for a fixed pair $\mathcal{W} \subset \mathcal{E}$, the homotopy class of $\mathcal{D}$ as a distribution sitting on the sequence $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E}$ is not fixed and it corresponds to the homotopy classes of sections of the S^1–bundle $S(\mathcal{E}/\mathcal{W}) \to M$.

The main motivating example of an even contact structure is the preimage of the contact structure under the projection $M = Y \times [0,1] \to Y$ for a contact structure ξ on Y. The fibers of the projection are the leaves of the isotropic foliation. A slight modification of this example proves the following lemma:

Lemma 2.5 *Let (Y, ξ) be a contact manifold and φ the coordinate on $[0,1]$. Then the hyperplane field $\mathcal{E} = \xi \oplus \mathbb{R}(\partial_\varphi + L)$ on $Y \times [0,1]$ is an even contact structure whose isotropic foliation is spanned by $\partial_\varphi + L$ if and only if L is a contact vector field on (Y, ξ).*

In order to find an Engel structure $\mathcal{D}$ such that the even contact structure $\mathcal{E}$ using Lemma 2.5 satisfies $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ we require, in addition, that ξ admits a trivialization

C_1, C_2. The following lemma was certainly known to H Geiges [10] and we are not aware of an earlier reference.

Lemma 2.6 *Let L be a contact vector field on (Y,ξ). For a sufficiently big positive integer k the plane field spanned by*

$$W = \partial_\varphi + L \quad \text{and} \quad X_k = \cos(k\varphi)C_1 + \sin(k\varphi)C_2$$

is an Engel structure $\mathcal{D}_k$ such that $[\mathcal{D}_k,\mathcal{D}_k]=\mathcal{E}$. In particular, the isotropic foliation is spanned by W.

3 Open book decomposition of a 4–manifold

Open book decompositions — their definition can be found below — have been probably introduced into differential topology by J Alexander, who showed that every oriented 3–manifold has an open book decomposition. This led to the short proof of the fact that every oriented 3–manifold has a contact structure [23] that we will adapt to Engel structures.

Existence results for open book decompositions in dimension ≥ 6 were found by Winkelnkemper [25] and T Lawson [16]. Later, F Quinn [22] found necessary and sufficient conditions for the existence of an open book decomposition on closed manifolds of dimension ≥ 5. To the best of our knowledge there is no reference dealing with open book decompositions of 4–manifolds specifically. The (simple) fact that if a closed 4–manifold admits an open book decomposition then its signature has to vanish was certainly known to Winkelnkemper [25].

For background on the relationship between contact structures and open book decompositions we refer to [9] when the underlying manifold has dimension 3, and to [13; 5] for the case of contact manifolds in higher dimension.

3.1 Open books with torus binding

In this section we summarize basic definitions concerning open book decompositions.

Definition 3.1 Let M be a closed manifold. An *open book decomposition* of M is a pair (K,θ) where

- $K \subset M$ is a nonempty, not necessarily connected, submanifold of codimension two with trivial normal bundle,

- $\theta\colon M \setminus K \to S^1$ is a fibration, and
- one can choose coordinates on a neighborhood

(1) $$N(K) \simeq K \times D^2 = \{(x,(r,\phi)) \mid x \in K \text{ and } (r,\phi) \in D^2 \text{ are polar coordinates}\}$$

of K such that $\theta(x,(r,\phi)) = \phi$ for $r \neq 0$.

The submanifold K is called the *binding* of the open book decomposition. For $t \in S^1$, the preimage $\theta^{-1}(t)$ will be called a *fiber* of (K,θ). The compactification of any fiber obtained by addition of K is called a *page*. The natural orientation of S^1 lifts through θ to a natural coorientation of the pages. If the ambient manifold M is oriented, then we obtain a natural orientation on the pages, and K is naturally oriented as the boundary of a page.

The simplest example of a manifold with an open book decomposition is the 2–sphere. In this case, the binding consists of a pair of points, the complementary annulus fibers over the circle and the fibers are intervals.

Recall that every oriented 3–manifold admits an open book decomposition [1].

Definition 3.2 (Giroux [13]) Let (M,ξ) be a contact 3–manifold and (K,θ) an open book decomposition of M. Then (K,θ) *supports* ξ if there is a defining contact form α for ξ such that

(i) $\alpha|_K$ is positive, and

(ii) the restriction of $d\alpha$ to fibers of (K,θ) is a positive area form.

According to [13], in any odd dimension, every contact structure has a supporting open book decomposition. Details of the proof can be found in [21].

When M has dimension four, K is a disjoint union of embedded closed surfaces. From a vector field transverse to the fibers of $M \setminus K \to S^1$ and a vector field V_K with isolated zeroes on K one can obtain a vector field on M whose zeroes are precisely the zeroes of K such that the zeroes have the same index. Hence, the Euler characteristic of K equals the Euler characteristic of M by the Poincaré–Hopf index theorem.

For 4–manifolds we will require in addition that K is a union of tori. Then $\chi(M) = 0$, and this is a condition Engel manifolds satisfy.

Moreover, it follows from Novikov additivity [20] that the signature of a 4–manifold admitting an open book decomposition vanishes. However, 4–manifolds which admit

an open book decomposition do not always admit a Spin–structure. Consider for example the nontrivial S^2–bundle $S^2 \hat{\times} S^2$ over S^2. This manifold does not have a Spin–structure, but an open book decomposition can be obtained by pulling back the open book decomposition of S^2.

It seems to be unknown whether all oriented 4–manifolds with vanishing signature admit an open book decomposition [22]. Below we describe simple examples of open book decompositions on 4–manifolds which fiber over 3–manifolds.

So far we have considered open book decomposition from an intrinsic point of view. From a more extrinsic point of view, an open book decomposition of an n–manifold M is a triple (Y, h, ϕ) where

- Y is a compact oriented $(n-1)$–manifold with nonempty boundary,
- $h\colon Y \to Y$ is a diffeomorphism which is the identity on a neighborhood of ∂Y, and
- $\phi\colon \Sigma_0(Y, h) \to M$ is a homeomorphism, where $\Sigma_0(Y, h)$ is the relative mapping torus of (Y, h).

This determines the manifold only up to homeomorphism. Recall that the relative mapping torus of (Y, h) is

$$\Sigma_0(Y, h) = Y \times [0, 1]/\sim_h,$$

where $\sim_h$ is the equivalence relation

$$\begin{aligned} (h(x), 0) &\sim_h (x, 1) \quad \text{for all } x \in Y, \\ (x, t) &\sim_h (x, t') \quad \text{for all } x \in \partial Y,\ t, t' \in [0, 1]. \end{aligned}$$

The triple (Y, h, ϕ) gives rise to an open book decomposition of M whose binding is $\phi(\partial Y \times [0, 1]/\sim_h)$ and whose pages are $Y_t = \phi(Y \times \{t\})$. Conversely, any open book decomposition can be seen as the relative mapping torus of some (Y, h) (notice that we usually do not mention the identification ϕ).

Example 3.3 Let (K_0, θ_0) be an open book decomposition of a 3–manifold Y, also given by the relative mapping torus of (S_0, h_0). Here S_0 is a compact oriented surface with nonempty boundary. Then $(S^1 \times K_0, \mathrm{id} \times \theta_0)$ is an open book decomposition of $S^1 \times Y$, also described by the relative mapping torus of $(S^1 \times S_0, \mathrm{id} \times h_0)$. □

We close this section with an example of an open book decomposition of S^3.

Example 3.4 On $S^3 \subset \mathbb{C}^2$ we consider the contact form $\alpha = r_1^2\, d\varphi_1 + r_2^2\, d\varphi_2$. Here r_1, φ_1 and r_2, φ_2 are polar coordinates on the first and second factors of $\mathbb{C}^2$, respectively.

Then the Hopf link $K_0 = \{r_1 = 0\} \cup \{r_2 = 0\}$ is the binding of an open book decomposition which supports $\ker(\alpha)$. The complement of the Hopf link K_0 fibers over S^1 by

$$\theta_0 \colon S^3 \setminus K \to S^1, \quad ((r_1, \varphi_1), (r_2, \varphi_2)) \mapsto \varphi_1 + \varphi_2.$$

One can check that (K_0, θ_0) is an open book decomposition of S^3 and that (K_0, θ_0) supports $\xi = \ker(\alpha)$. The page $A_0 = \theta_0^{-1}(0)$ of the open book is the annulus

$$\big\{(r, \varphi, \sqrt{1-r^2}, -\varphi) \in S^3 \mid 0 < r < 1 \text{ and } \varphi \in [0, 2\pi]\big\}.$$

The monodromy of this open book decomposition is a positive Dehn twist along the circle $\{r = \frac{1}{2}\} \subset \theta_0^{-1}(0)$. □

The following description of the stabilization of open book decompositions supporting a contact structure on a 3–manifold M can be found in [14]; see also [9]. Let (K, θ) be an open book which carries the contact 3–manifold (M, ξ). On a page Σ we pick a properly embedded Legendrian arc β. Now choose a vector field on $M \setminus K$ which is transverse to the pages such that

- on $N(K) \setminus K \simeq K \times (D^2 \setminus \{0\})$ the vector field is tangent to the second factor,
- the time-1 flow is defined and it induces a self-diffeomorphism ψ_1 of Σ whose support lies in the interior of the page, and
- $\Sigma \cup \psi_{1/2}(\Sigma)$ is a surface whose interior is smooth.

Let $N(a) \subset M$ be a closed ball which intersects K in two arcs and contains $a = \psi_{1/2}(\beta)$ in its interior such that $\partial N(a) \setminus K$ has precisely two tangencies with pages of the open book decomposition. On a page of the open book decomposition of (S^3, ξ_{st}) described in Example 3.4 we choose an arc connecting the two components of the binding (a Hopf link) and a neighborhood N_{S^3} of that arc with properties analogous to those of $N(a)$.

By carefully performing the connected sum of (M, ξ) with (S^3, ξ_{st}) using the balls $N(a)$ and N_{S^3} such that the oriented singular foliations are induced by the open book decompositions on $\partial N(a)$ and ∂N_{S^3}, one obtains a new open book decomposition of $M \simeq M \# S^3$. This yields a contact structure supported by the resulting open book

decomposition which is isotopic to ξ by the classification of tight contact structures on the ball (see [7]).

Figure 1 shows the intersection of the pages with $\partial N(a)$. The thickened solid arcs K and K' represent the binding before and after stabilization, respectively. These arcs lie in the interior of the ball. Finally, the dashed arc represents pieces of β, a and γ.

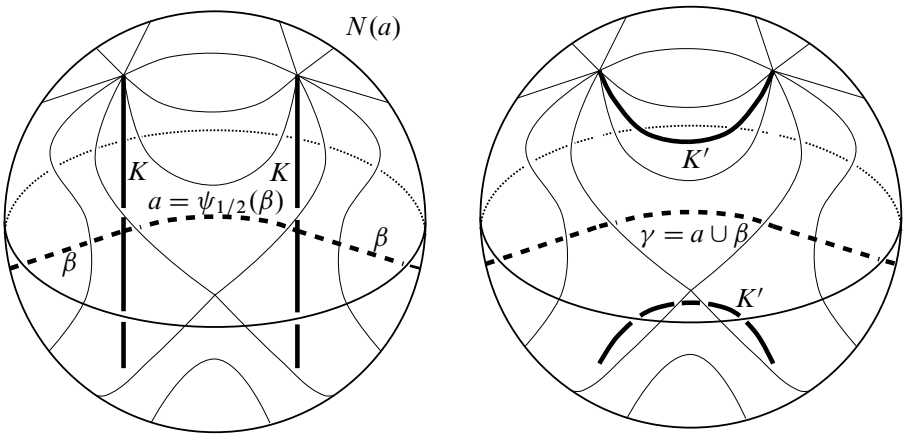


Figure 1: Stabilization of open book decompositions of 3–manifolds

3.2 Stabilization

As we have stated in the introduction, isotopy classes of contact structures on 3–manifolds are in one-to-one correspondence with open book decompositions up to positive stabilization. In this section we suggest a definition of stabilization for open book decompositions of 4–manifolds as a fiber connected sum along an annulus A in a page of the open book by taking the product of the stabilization in dimension 3 with S^1, ie the annulus A will be of the form $a \times S^1$.

The stabilization of open books on 4–manifolds can be thought of as removing neighborhoods of annuli in M and $S^3 \times S^1$ and identifying the boundaries. In the following we also want to ensure that the binding is a union of tori after stabilization when this is true for the original open book decomposition.

Let M be a 4–manifold equipped with an open book decomposition (K, θ) obtained as the relative mapping torus of (Y, h). We pick a properly embedded annulus $(A, \partial A) \subset (Y, \partial Y)$ such that no component of ∂A is contractible in ∂Y.

Consider the 3–manifold Y' obtained by gluing $S^1\times[0,1]\times[-1,1]$ to Y, where the gluing map takes $S^1\times\{0,1\}\times[-1,1]$ to ∂Y so that the two curves $S^1\times\{0,1\}\times\{0\}$ are identified with ∂A and $T=A\cup(S^1\times[0,1]\times\{0\})$ is a torus. The boundary of Y' is still a union of tori.

Let $\gamma\subset T$ be an embedded closed curve intersecting the circle $S^1\times\{\frac{1}{2}\}\times\{0\}$ exactly once. We then take a small tubular neighborhood $N(T)$ of T in Y' and consider the fibered Dehn twist τ_γ along T in the direction of γ, supported in $N(T)$. Let $h'=\tau_\gamma\circ h\in\mathrm{Diff}(Y')$.

Proposition 3.5 *The relative mapping torus of (Y',h') is diffeomorphic to M by a diffeomorphism which induces an injection from a page Y of $\Sigma_0(Y,h)$ to a page Y' of $\Sigma_0(Y',h')$.*

Proof Recall that an elementary positive stabilization of a 3–dimensional open book is obtained by performing the connected sum with S^3 on a suitable neighborhood $N(a)$ of a properly embedded arc a [13; 14].

We start from the open book decomposition (K_0,θ_0) of S^3 given by the relative mapping torus of a positive Dehn twist on an annulus as described in Example 3.4. Its binding is a positive Hopf link. Let A_0 be a page of this open book, and a a properly embedded arc on A_0 connecting the two boundary components.

Next, we consider the manifold $S^1\times S^3$ together with the open book $(S^1\times K_0,\mathrm{id}\times\theta_0)$. It contains the annulus $S^1\times a$ and the associated neighborhood $S^1\times N(a)$, where $N(a)$ is a neighborhood of a as in [13]. The complement of $S^1\times N(a)$ in $S^1\times S^3$ is diffeomorphic to $S^1\times B^3$.

Now we take coordinates $S^1\times[0,1]$ on $A\subset Y$ such that $\gamma=\{0\}\times[0,1]\subset A$. As in [13], we can find a neighborhood $N(A)$ of A in M with coordinates $S^1\times B^3$, so that the pages of (K,θ) define a "partial" open book decomposition on $S^1\times B^3$. This open book is conjugate to $S^1\times(N(a),K_0,\theta_0)$.

Since both restrictions $(S^1\times B^3,K,\theta)$ and $S^1\times(N(a),K_0,\theta_0)$ are S^1–invariant, we can form an S^1–invariant connected sum of open books, ie we glue

$$\big(S^1\times S^3\setminus\mathrm{int}(S^1\times N(a)),S^1\times K_0,\mathrm{id}\times\theta_0\big)$$

to $(M\setminus\mathrm{int}N(A),K,\theta)$ in an S^1–invariant way. In this manner we get a manifold diffeomorphic to M, with an open book of page Y' and monodromy h'. □

Another way to argue in the previous proof would be as follows: The stabilized manifold is obtained from the original manifold by removing a copy of $S^1 \times D^3$ and gluing it back in. According to [15] every diffeomorphism of $S^1 \times \partial D^3$ extends to $S^1 \times D^3$. Hence, the diffeomorphism type of the manifold does not change under this surgery operation.

4 Open book decompositions and even contact structures

Let M be an oriented closed 4–manifold together with an open book decomposition (K, θ).

Definition 4.1 An even contact structure $\mathcal{E}$ with kernel $\mathcal{W}$ is *adapted to* (or *supported by*) (K, θ) if the following conditions are satisfied:

- K is a union of tori.
- $\mathcal{W}$ is transverse to the fibers of θ. In particular, the interior of the pages Y_θ are naturally contact manifolds for the contact plane $\xi_\theta = \mathcal{E} \cap TY_\theta$.
- ξ_θ is a positive contact structure for the canonical orientation of Y_θ.
- $\mathcal{E}$ is transverse to K.
- $\mathcal{W}$ restricts to a linear vector field on each connected component of K.
- For each $\theta \in S^1$ there is a collar $U \simeq \partial Y_\theta \times (-1, 0]$ of ∂Y_θ in Y_θ such that $K = \partial Y_\theta \times \{0\}$ and the characteristic foliation on each connected component of $\partial Y_\theta \times \{r\}$ for $r \in (-1, 0]$ is linear.

Remark 4.2 Since all linear vector fields on T^2 are homotopic, two even contact structures supported by an open book are homotopic through even contact structures if the contact structures in the pages are homotopic as 1–parametric families of plane fields: the associated kernel foliations are homotopic through foliations which are transverse to the interiors of the pages and induce linear vector fields on the binding.

Two even contact structures on M supported by the same open book (K, θ) are not necessarily homotopic as formal even contact structures. A sufficient condition for two different even contact structures to be homotopic through formal even contact structures is that the associated gluing contactomorphisms are isotopic through formal contactomorphisms. In particular, if the return maps are contact isotopic, then the

even contact structures are formally homotopic and therefore homotopic through even contact structures according to [17].

Note that uniqueness holds in the 3–dimensional contact case, because the return map is an exact symplectomorphism of a surface, and two symplectomorphisms of a surface are isotopic through symplectomorphisms if they are isotopic through diffeomorphisms.

Example 4.3 Let ξ be a contact structure on Y carried by an open book (K,θ), and α an adapted contact form. Its Reeb vector field X is transverse to the pages and tangent to the binding. We assume that ξ is trivial as a vector bundle.

Let $M=\widetilde{\mathbb{P}\xi}$ be the manifold consisting of 1–dimensional oriented subspaces of the contact planes and

$$\mathrm{pr}\colon \widetilde{\mathbb{P}\xi}\to M$$

the S^1–fibration which sends a line in ξ_p to p. Then $\mathrm{pr}^*\alpha$ defines an orientable even contact structure $\mathcal{E}$ whose isotropic foliation is tangent to the fibers of pr.

We want to modify $\mathcal{E}$ so that the open book decomposition $(\mathrm{pr}^{-1}(K),\theta\circ\mathrm{pr})$ of $\widetilde{\mathbb{P}\xi}$ is adapted to the resulting even contact structure $\mathcal{E}_\varepsilon$. For this we identify $S^1\times Y$ with $\widetilde{\mathbb{P}\xi}$, denote the coordinate in S^1 by t, and let $\widetilde{X}$ be the lift of the Reeb field X to $S^1\times Y$ using the connection $\{0\}\oplus TY$.

For $\varepsilon>0$ small enough the 1–form

$$\alpha_\varepsilon := \mathrm{pr}^*\alpha-\varepsilon\,dt$$

still defines an even contact structure $\mathcal{E}_\varepsilon$. Let ∂_t denote the vector field tangent to the fibers dual to dt. Then $\partial_t+\varepsilon\widetilde{X}$ is tangent to $\mathcal{E}_\varepsilon$ and this vector field preserves $\mathcal{E}_\varepsilon$. The isotropic foliation of $\mathcal{E}_\varepsilon$ is therefore spanned by $\partial_t+\varepsilon\widetilde{X}$, and the open book $(\mathrm{pr}^{-1}(K),\theta\circ\mathrm{pr})$ supports $\mathcal{E}_\varepsilon$ provided that $\varepsilon>0$. □

When one views an open book decomposition supporting an even contact structure as a mapping torus, then the monodromy is a contact diffeomorphism of a page. Using Eliashberg's classification of overtwisted contact structures [6] it is easy to obtain contact diffeomorphisms from diffeomorphisms which preserve a given plane field up to homotopy.

Proposition 4.4 *Let Y be a closed 3–manifold, $e=(e_0,e_1,e_2)$ be a trivialization of TY and h a diffeomorphism of Y such that $h_*(e)$ is homotopic to e. Then there*

exists a contact structure ξ on Y, together with a contact diffeomorphism ψ of (Y,ξ) isotopic to h, such that ξ contains a nonvanishing vector field X with $\psi_* X$ homotopic to X in ξ.

Proof Write $e_i^0 = e_i$ and $e_i^1 = h_*(e_i)$ for $i = 0, 1, 2$. By hypothesis, there exists a family (e_0^t, e_1^t, e_2^t) interpolating between these two framings. Let $\xi_t = \langle e_0^t, e_1^t\rangle$. Since the ball is contractible, we may assume that, after a homotopy of the framing e_i and an isotopy of the diffeomorphism h, there exists a ball $U \subset Y$ such that $\xi_t|_U = \xi_0|_U$ is contact and $\xi_0|_U$ contains an overtwisted disk. By [6], there exists a continuous family $\xi_{t,s}$ such that $\xi_{t,0} = \xi_t$ and $\xi_{t,1}$ are contact structures. We apply Eliashberg's classification theorem [6] as follows:

(1) First, for $t = 0$, find $\xi_{0,s}$.

(2) Then we have a solution for $t = 1$, given by $\xi_{1,s} = h_*\xi_{0,s}$.

(3) The relative character with respect to the parameter of [6] allows us to extend it to $\xi_{t,s}$.

By contractibility of the interval, there is an extension $(e_0^{t,s}, e_1^{t,s})$ that coincides with (e_0^t, e_1^t) for $s = 0$. Moreover, $\xi_{t,s} = \langle e_0^{t,s}, e_1^{t,s}\rangle$, $h_* e_0^{0,1} = e_0^{1,1}$ and $h_* e_1^{0,1} = e_1^{1,1}$.

By Gray stability, there exists a flow $\varphi_t\colon Y \to Y$ such that $(\varphi_t)_*\xi_{0,1} = \xi_{t,1}$. Therefore, $(\varphi_1^{-1} \circ h)_*\xi_{0,1} = \xi_{0,1}$ shows that $(\varphi_1^{-1} \circ h)$ is the required contactomorphism. Finally, $X = e_0^{0,1}$ satisfies the stated hypothesis. □

The same proof works for compact manifolds with boundary when the diffeomorphism h is the identity on the boundary and $h_*(e)$ is homotopic to e relative to the boundary. We then obtain a contact diffeomorphism ψ supported in the interior of Y and a nowhere-vanishing Legendrian vector field X such that $\psi_*(X)$ is homotopic (relative to the boundary) to X through nowhere-vanishing Legendrian vector fields.

The assumption in Proposition 4.4 that h preserves a framing up to homotopy turns out to be not too restrictive by the following observation of Geiges [10]:

Lemma 4.5 *Let M be a closed oriented 4–manifold with trivial tangent bundle and X a nowhere-vanishing vector field on M. Then there are vector fields e_1, e_2, e_3 such that X, e_1, e_2, e_3 is a framing.*

Proof We identify $TM \simeq M \times \mathbb{R}^4$ with $M \times \mathbb{H}$. Then X together with $e_1 = iX$, $e_2 = jX$ and $e_3 = kX$ is a framing of TM. □

In particular, we can apply this lemma to the suspension vector field on the mapping torus of a diffeomorphism $h\colon N \to N$ when the mapping torus is parallelizable. Projecting e_1, e_2 and e_3 to the fibers of a mapping torus, we obtain a framing of N with the desired properties.

Theorem 4.6 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and ξ a positive contact structure on Y which prints a linear characteristic linear foliation on ∂Y. If h is a contact diffeomorphism of (Y,ξ) which is the identity near the boundary, then the relative mapping torus of (Y,h) carries an even contact structure which induces ξ on (a small retraction of) the pages. This even contact structure is uniquely determined up to homotopy through even contact structures carried by the triple (Y,ξ,h).*

Proof For simplicity, we assume that ∂Y is connected. Let $Y_s = Y\times[0,1]/\sim$ be the mapping torus of (Y,h). We denote the suspension coordinate on Y_s by φ and fix a coordinate system (x,y) on $T^2 \simeq \partial Y$ such that ∂Y is linear in terms of the coordinates. The suspension vector field is ∂_φ and $D^2_{r_0}\subset\mathbb{R}^2$ is a disc of radius r_0.

We view M as the mapping torus $Y_s = Y\times[0,1]/\sim$ of (Y,h), with a neighborhood $N(K) = \partial Y\times D^2_{r_0}$ of the binding K of the binding attached along the boundary such that $\{y\}\times\partial D^2_{r_0}$ is mapped to $\{y\}\times S^1$ in $\partial Y_s = \partial Y\times S^1$ for all $y\in\partial Y$.

Because h preserves ξ we get a well-defined 2–plane field ξ_s on Y_s which is tangent to the fibers of $\theta\colon Y_s\to S^1$. Together with the suspension vector field ∂_φ, this plane field spans an even contact structure $\mathcal{E}_s$ on Y_s whose isotropic foliation is directed by ∂_φ.

Since h is the identity on a neighborhood of ∂Y and the characteristic foliation $\xi(\partial Y)$ is linear, we can choose a collar $U\simeq\partial Y\times[0,2\varepsilon)$, with $\varepsilon>0$, such that $h|_U = \mathrm{id}_U$ and

$$\xi = \ker(\cos(r+a)\,dx - \sin(r+a)\,dy).$$

where the parameter $a\in\mathbb{R}$ is determined by the slope of the linear characteristic foliation $\xi(\partial Y)$. Since we are free to choose the coordinates (x,y), we may assume that the slope is positive in terms of these coordinates. On $U\times S^1\subset Y_s$ we consider the even contact structure $\mathcal{E}_0 = \xi\oplus\mathbb{R}\partial_\varphi$. The isotropic foliation is of course spanned by ∂_φ. We want to homotope $\mathcal{E}_0$ through even contact structures so that isotropic foliation of the resulting even contact structure $\mathcal{E}_1$ is $\partial_\varphi+\partial_y$.

Recall that the space of contact vector fields on a given contact manifold is in one-to-one correspondence to the space of smooth functions once a contact form is fixed (see [18, Section I.3.4]). This implies the existence of a contact vector field L with support in U and $L = \partial_y$ on a collar $V = \partial Y \times [0, \varepsilon] \subset U$ of ∂Y.

We use the same notation for L and its lift to the product $Y \times S^1$. The desired homotopy of even contact structures is $\mathcal{E}_s = \xi_s \oplus \mathbb{R}(\partial_\varphi + sL)$ with $s \in [0, 1]$.

Finally, we extend $\mathcal{E}_s$ to an even contact structure over $V \cup N(K)$: let (r, φ) be polar coordinates on the second factor of $N(K) = \partial Y_s \times \mathbb{R}^2$ with $r \in [0, 1]$ and

$$\mathcal{E}_b = \ker(dx + r^2\, d\varphi - r^2 dy).$$

One can check by a simple computation that $\mathcal{E}_b$ is an even contact structure whose isotropic foliation is spanned by $\partial_\varphi + \partial_y$. In particular, $\mathcal{E}_b$ induces a contact structure on $\{\varphi = \varphi_0,\ 0 < r \le r_0\}$. The characteristic foliation on $T^2 = T^2 \times \{r = r_0,\ \varphi = \varphi_0\}$ is linear and the slope is $1/r^2 > 0$.

Therefore, we can choose r_0 so that the even contact structures $\mathcal{E}_b$ and $\mathcal{E}_1$ on Y_s can be glued, and we obtain an even contact structure supported by the resulting open book on $Y_s \cup_\partial (\partial Y \times D^2_{r_0})$. The pages of this open book are formed by fibers of $Y_s \to S^1$ with $\{\varphi = \varphi_0,\ 0 < r \le r_0\}$ attached along ∂Y.

We now prove the uniqueness part of the theorem. Let $\mathcal{E}$ and $\mathcal{E}'$ be two even contact structures carried by the open book decomposition (Y, ξ, h). Their isotropic foliations $\mathcal{W}$ and $\mathcal{W}'$ are spanned by vector fields of the form $\partial_\varphi + L$ and $\partial_\varphi + L'$ on both $N(K)$ and Y_s, where L and L' are tangent to the pages and to K.

The path of hyperplanes fields $\mathcal{E}_s = \xi \oplus \mathbb{R}(\partial_\varphi + (1-s)L + sL')$ for $s \in [0, 1]$ interpolates between $\mathcal{E}$ and $\mathcal{E}'$, satisfies the even contact condition and is carried by (Y, ξ, h). □

Remark 4.7 We are able to prove that the built even contact structure is unique (see Remark 4.2) because we start with an open book given by a triple (Y, ξ, h), and not by a pair (K, θ).

Using the flexibility properties of overtwisted contact structures [6], we can show the existence of a contact structure satisfying the assumptions of Theorem 4.6 for a given open book decomposition (Y, h) such that ∂Y is a union of tori and h preserves a plane field up to homotopy relative to the boundary.

Theorem 4.8 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and h a diffeomorphism of Y which is the identity near the boundary such that there exists a plane field P of Y such that h_*P is homotopic to P relative to ∂Y.*

Then the relative mapping torus of h carries an even contact structure.

The proof of Theorem 4.8 follows from a combination of Theorem 4.6 and a variant of Proposition 4.4. Under the hypothesis that the monodromy h preserves a plane field $\mathcal{P}$ up to homotopy relative to the boundary, a slightly modified version of Proposition 4.4 gives an overtwisted contact structure homotopic to $\mathcal{P}$ and preserved by h. Theorem 4.6 then gives the conclusion.

The following theorem shows that one can always arrange that the page of an open book supporting an even contact structure is overtwisted if one allows homotopies through even contact structures which are not necessarily supported by the open book decomposition. In particular, ξ' can be taken overtwisted even if ξ is tight.

Theorem 4.9 *If $\mathcal{E}$ is carried by (Y,h) and $\mathcal{E}$ induces a contact structure ξ on Y. Then, for every overtwisted contact structure ξ' on Y which is homotopic to ξ as oriented plane field relative to the boundary, the even contact structures $\mathcal{E}$ and $\mathcal{E}'$ associated to ξ and ξ' are homotopic through even contact structures.*

In particular, ξ' can be taken overtwisted even if ξ is tight.

Proof It is enough to prove the result when ξ' is overtwisted. In this situation, $h_*\xi'$ is overtwisted and homotopic to ξ' rel ∂Y. By Eliashberg's classification of overtwisted contact structures [6], ξ' is isotopic to $h_*\xi'$.

We can thus deform h to h' to make ξ' an h'–invariant contact structure. Then we construct an even contact structure $\mathcal{E}'$ carried by (Y,ξ',h'). The pair $(\mathcal{E}',\mathcal{W}')$ is homotopic to $(\mathcal{E},\mathcal{W})$ by construction. The h-principle [17] implies that $\mathcal{E}$ and $\mathcal{E}'$ are homotopic through even contact structures: On the interior of the suspension of (Y,h), the homotopy of plane fields ζ_t for $t\in[0,1]$ interpolating between ξ and ξ' induces a homotopy of formal even contact structures $(\zeta_t\oplus\mathbb{R}\partial_\varphi,\mathbb{R}\partial_\varphi)$, and both $\mathcal{W}$ and $\mathcal{W}'$ are homotopic to $\mathbb{R}\partial_\varphi$ as line fields transverse to the pages. □

Still, rigidity could come from tightness of the pages.

Question 4.10 Which even contact structures are carried by at least one open book decomposition with a tight page?

5 Open book decompositions and Engel structures

Let M be a closed oriented 4–manifold and (K,θ) an open book decomposition of M.

Definition 5.1 An Engel structure $\mathcal{D}$ is *adapted to* (or *supported by*) (K,θ) if the associated even contact structure $\mathcal{E}=[\mathcal{D},\mathcal{D}]$ is adapted to (K,θ).

As we have stated in the introduction, this is a somewhat unsatisfactory definition. For example, it is possible that different Engel structures are adapted to the same open book even if the associated framings are not homotopic.

Example 5.2 Let ξ be a contact structure on a compact 3–manifold Y which is trivial as a plane field and is supported by an open book (K,θ). We fix an adapted contact form α together with the corresponding Reeb vector field R and a trivialization C_1, C_2 of ξ. Let pr: $M=S^1\times Y\to Y$ be the projection. We denote the horizontal lifts of the trivialization C_1, C_2 and of the Reeb vector field R to $S^1\times Y$ with the same letters, and the coordinate on S^1 will be t.

For $\varepsilon>0$ small enough, the plane field $\mathcal{D}_k$, with $k\geq 1$, spanned by

$$W=\partial_t+\varepsilon R,\quad X=\cos(2\pi k t)C_1+\sin(2\pi k t)C_2 \tag{2}$$

is an Engel structure supported by $(S^1\times K,\theta\circ\mathrm{pr})$.

This can be applied to the contact structure on $S(T^*T^2)$ for different k. The homotopy class of the induced framing of $S(T^*T^2)\times S^1$ depends on the parity of k. Thus, an open book can generate Engel structures which cannot be homotopic through Engel structures. □

5.1 Invariants of Engel structures supported by open books

Fix $\mathcal{D}$ an Engel structure carried by (Y,h); associate an integer k to $\mathcal{D}$ in the following way: Pick a page Y_0 and a collar neighborhood $\partial Y_0\times(-1,0]$ of ∂Y_0 in Y_0 such that the characteristic foliation $\xi_0(\partial Y_0\times\{s\})$ for $s\in(-1,0]$ is linear and where the monodromy h is the identity. Let $x_0\in\partial Y_0\times\{s_0\}$ be a point in that neighborhood.

Along the path $x_0\times S^1\subset\Sigma_0(Y,h)$, we have a trivialization of $\xi_t=\mathcal{E}\cap TY_t$ at the point (x_0,t) for $t\in S^1$ by the line field $C=\xi_t\cap T(\partial Y_t\times\{s_0\})$. There is another

line field $C' = \mathcal{D} \cap \xi_t$. We define k to be the rotation number of C' with respect to C in ξ_t when we go around $x_0 \times S^1$.

Now, if $\mathcal{D}$ and $\mathcal{D}'$ are two Engel structures which are carried by the same open book and induce the same contact structure ξ_0 on one page Y_0, then there are two Legendrian line fields $L = \xi_0 \cap \mathcal{D}$ and $L' = \xi_0 \cap \mathcal{D}'$. The degree of L' with respect to L induces a map which defines a homomorphism

$$\delta(\mathcal{D}, \mathcal{D}')\colon H_1(Y_0, \mathbb{Z}) \to \mathbb{Z}.$$

If $\mathcal{W}$ and $\mathcal{W}'$ are the respective kernels of $\mathcal{D}$ and $\mathcal{D}'$ and are equal near K, their first return maps ϕ and ϕ' of the kernel foliations are contactomorphisms on the interior of (Y_0, ξ_0) which coincide near ∂Y_0. The even contact structures $\mathcal{E}$ and $\mathcal{E}'$ are homotopic through even contact structures supported by the open book if and only if there is a family (ϕ_t, ξ_t) such that ϕ_t is a contactomorphism of the contact structure ξ_t on Y_0 such that $\phi_0 = \phi$, $\phi_1 = \phi'$, ϕ_t has support in the interior of Y_0 and ξ_t is constant on a neighborhood of ∂Y_0.

If we moreover assume $\mathcal{W} = \mathcal{W}'$, then $\xi_t = \xi_t'$ for all $t \in S^1$, and $\delta(\mathcal{D}, \mathcal{D}') \circ \phi_* = \delta(\mathcal{D}, \mathcal{D}')$.

We now define *twisting numbers*, which will be used to keep track of the homotopy class of plane fields in even contact structures such that the plane field is tangent to the isotropic foliation of the even contact structure.

The restriction ξ_φ of $\mathcal{E}$ to each page is a contact structure. We assume that there is a section X of $\mathcal{E}$ which is tangent to the pages. In the situations we are going to study, X will be $-\partial_r$ on collars of the boundary of $T^2 \times D^2$ and of $Y_s = Y \times [0,1]/\sim$.

Let C_1, C_2 be an oriented framing of ξ_φ. For a closed oriented curve γ there are functions g_1 and g_2 such that $X(\gamma(t)) = g_1(t)C_1(\gamma(t)) + g_2(t)C_2(\gamma(t))$. We define the twisting number $\mathrm{tw}(C_1, \gamma)$ to be the degree of the map

$$S^1 \to S^1, \quad t \mapsto \frac{(g_1(t), g_2(t))}{\|(g_1(t), g_2(t))\|}.$$

Obviously, $\mathrm{tw}(C_1, \gamma)$ depends only on the homology class of γ and on the homotopy class of C_1 as a nowhere-vanishing section of ξ_φ.

5.2 Construction of Engel structures

In this section we adapt the construction from Section 4 to obtain Engel structures and prove the following theorem:

Theorem 5.3 *Let Y be a compact 3–manifold bounded by a nonempty union of tori, and h a diffeomorphism of Y which is the identity near the boundary and for which there exists a framing e of Y such that h_*e is homotopic to e relatively to ∂Y. Then the relative mapping torus of h carries an Engel structure.*

We will prove this theorem in Section 5.2. Using Lemma 4.5 we obtain the following corollary:

Corollary 5.4 *If M is an oriented closed parallelizable 4–manifold, every open book decomposition with toric binding (K, θ) of M carries an Engel structure.*

Proposition 5.5 *Let $\mathcal{D}$ be an Engel structure carried by (Y, h) and $\delta\colon H_1(Y, \mathbb{Z}) \to \mathbb{Z}$ be a morphism. If $\delta \circ h_* = \delta$, then there exists en Engel structure $\mathcal{D}'$ supported by (Y, h) which induces the same contact structure as $\mathcal{D}$ on $\operatorname{int}(Y_0)$ (with $\mathcal{W}' = \mathcal{W}$ and $\mathcal{E}' = \mathcal{E}$ outside a neighborhood of K) such that $\delta(\mathcal{D}, \mathcal{D}') = \delta$.*

Question 5.6 *Is every Engel structure homotopic to an Engel structure carried by an open book?*

Question 5.7 *Which Engel structures are carried by an open book decomposition with tight pages?*

Let $\mathcal{E}$ be an even contact structure on the total space of a fibration over S^1 such that the isotropic foliation is transverse to the fibers of θ. (We are going to consider two cases, $\theta\colon Y_s \to S^1$ and $\theta\colon T^2 \times (D^2 \setminus \{0\}) \to S^1$.) We denote the vertical tangent vectors in $\mathcal{E}$ by $\mathcal{E}^v$. The analogous notation $\mathcal{D}^v$ will be used later for Engel structures $\mathcal{D}$ whose isotropic foliation is transverse to a fibration over S^1.

In what follows, γ_x, γ_y and γ_φ are simple closed curves in $\partial Y \times S^1$ (or, equivalently, in $T^2 \times \partial D^2 \subset T^2 \times D^2$) which correspond to the coordinates x, y and φ.

5.2.1 Engel structure on Y_s Let S_ξ be vector field directing the characteristic foliation on $T^2 \times \{r\} \times \{\varphi\}$ such that ∂_r, S_ξ is an oriented framing of ξ on a neighborhood of ∂Y and ∂_r points into Y_s along ∂Y_s. We choose coordinates x, y on $T^2 = \partial Y$ such that

$$\operatorname{tw}(C_1, \gamma_x) = 0, \quad \operatorname{tw}(C_1, \gamma_y) = \lambda, \tag{3}$$

where λ is an integer. This determines the homotopy class of C_1 as a nowhere-vanishing section of ξ, and we assume

$$\begin{aligned} C_1 &= \cos(\lambda y)\,\partial_r + \sin(\lambda y) S_\xi, \\ C_2 &= -\sin(\lambda y)\,\partial_r + \cos(\lambda y) S_\xi \end{aligned} \tag{4}$$

on a collar of $T^2 \subset \partial Y$. A simple computation shows that C_1 has the right twisting numbers (see (3)). Note in particular that, along $\{y = 0\}$, $C_1 = -\partial_r$ and $C_2 = S_\xi$.

On $Y \times [0,2]$ with coordinates (p, φ) consider the plane field $\mathcal{D}_k$ spanned by the isotropic foliation of the even contact structure defined as in Section 4 using the contact structure ξ, and

$$X_k := \cos(2\pi k\varphi) C_1 + \sin(2\pi k\varphi) C_2 \tag{5}$$

with $k \in \mathbb{N}$. According to Lemma 2.6, $\mathcal{D}_k$ is an Engel structure when $k > 0$ is large enough and $[\mathcal{D}_k, \mathcal{D}_k] = \mathcal{E}$ is independent of k. The orientation of $\mathcal{E}$ induced by $\mathcal{D}_k$ is the one determined by the framing C_1, C_2. Moreover, assuming again that k is large enough, there is a smooth function $f\colon Y \to (0,2)$ with $f \equiv 1$ on U (where $h = \mathrm{id}$) such that

$$h\colon Y \simeq \{(p, f(p)) \mid p \in Y\} \to Y = Y \times \{0\} \tag{6}$$

preserves the line field spanned by X_k. At this point we use the assumption that the framing $(h_*(C_1), h_*(C_2))$ is homotopic to (C_1, C_2) together with the fact that the holonomy of the characteristic foliation of $\mathcal{D}_k$ is the identity on U. Thus, we obtain an Engel structure $\mathcal{D}_k$ on Y_s. Notice that near ∂Y_s the Engel structure $\mathcal{D}_k$ is $\langle \partial_\varphi + \partial_y \rangle \oplus \langle X_k \rangle$.

By construction, $\mathrm{tw}(X_k, \gamma_\varphi) = k$. In order to make subsequent constructions possible, we require that k is odd.

5.2.2 Even contact structure on $T^2 \times D^2$ We need to extend the Engel structure $\mathcal{D}_k$ to $T^2 \times D^2$. For that we will first extend the associated even contact structure $\mathcal{E}$ in such a way that the plane field $\mathcal{D}_k$ also extends to a plane field in $\mathcal{E}$ containing $\mathcal{W}$. This extra condition requires a modification of the construction in Section 4.

We will choose the even contact structure on $T^2 \times D^2 = S^1_x \times S^1_y \times D^2$ so that it has a section C_1' extending X_k over $(0,0) \times D^2$ which is never tangent to $\mathcal{W}$. Note that the homotopy type of X_k as a section of $\mathcal{E}^v$ along $(0,0) \times \partial D^2$ depends only on k (and not on the choice of C_1 in Y since it is φ–invariant). This is done as follows.

On $S^1_x \times D^2$ consider an x–invariant contact structure ξ' with the following properties:

- $\xi' = \ker(dx + r^2\, d\varphi)$ near the boundary of $S^1_x \times D^2$. In particular, $\xi' = \xi$ along $\partial Y_s = \partial N(K)$.
- The characteristic foliation on $\{x = 0\}$ has only nondegenerate singularities. Let $e_\pm$ and $h_\pm$ denote the number of elliptic and hyperbolic singularities, respectively. The subscript refers to the sign of the singularities. We require

$$\begin{aligned} e_+ = \tfrac{1}{2}(k+1), \quad e_- = 0, \\ h_+ = 0, \quad h_- = \tfrac{1}{2}(k-1). \end{aligned} \tag{7}$$

  Since k is odd, these numbers are all integers.

Such a foliation exists since $e_+ + e_- - h_+ - h_- = 1 = \chi(D^2)$ and standard results from the theory of convex surfaces imply that the singular foliation on $\{x = 0\}$ determines a x–invariant contact structure ξ' on $S^1_x \times D^2$. Let $\mathcal{E}'$ be the extension of $\mathcal{E}$ given by $\mathcal{E}' = \xi' \oplus \langle \partial_y + \partial_\varphi \rangle$.

We let X'_k be a vector field along ∂Y_s generating $\mathcal{D}_k \cap \xi'$. Note that

$$e(\xi')[D^2] = e_+ - e_- - h_+ + h_- = k = \mathrm{tw}(X'_k, \partial D^2).$$

By the Poincaré–Hopf theorem, the Legendrian vector field X'_k of $\xi'|_{\partial D^2}$ extends to a nowhere-vanishing section C'_1 of $\mathcal{E}'$ on $T^2 \times (0,0)$ such that C'_1 is nowhere tangent to $\mathcal{W}' = \partial_y + \partial_\varphi$.

This extension is unique up to homotopy and there is a Legendrian vector field C'_2 such that (C'_1, C'_2) is a framing of ξ' on $(0,0) \times D^2$.

Near $T^2 \times \partial D^2$ we require C_i and C'_i have the same image in $\mathcal{E}/\mathcal{W}$ for $i = 1, 2$. Then there is a homotopy between C_i and C'_i in $\mathcal{D}_k \setminus \mathcal{W}$ such that C'_i can be viewed as an extension of C_i. Near $T^2 \times \partial D^2$, we have

$$\begin{aligned} C'_1 &= \cos(k\varphi)\, \partial_r + \sin(k\varphi) S_\xi, \\ C'_2 &= -\sin(k\varphi)\, \partial_r + \cos(k\varphi) S_\xi. \end{aligned} \tag{8}$$

This is consistent with the choice of X_k near ∂Y_s, ie the C_i together with C'_i form a smooth vector field on a neighborhood of $T^2 \times \partial D^2$ in $M = Y_s \cup_\partial (T^2 \times D^2)$ for $i = 1, 2$.

5.2.3 Engel structure on $T^2 \times D^2$ Now we define $\widehat{\mathcal{D}}_l$ on $T^2 \times D^2$ as the span of ∂_y and $\widetilde{X}_l = \cos(ly) C'_1 + \sin(ly) C'_2$. This is an Engel structure for all $l \in \mathbb{N}$. The

isotropic foliation is spanned by ∂_y, so it is not transverse to the fibers of θ. However, the Engel structure $\mathcal{D}'_l$ obtained by pushing forward $\widehat{\mathcal{D}}_l$ with the diffeomorphism

$$f\colon S^1\times(S^1\times D^2)\to S^1\times(S^1\times D^2),\quad (x,y,r,\varphi)\mapsto(x,y,r,\varphi+y),$$

has the desired property since its isotropic foliation is spanned by $\partial_y+\partial_\varphi$. Let

$$X'_l=f_*(\widetilde{X}_l)=\cos(ly)\,f_*(C'_1)+\sin(ly)\,f_*(C'_2). \tag{9}$$

By construction, the gluing map g maps the isotropic foliation (which is tangent to ∂Y_s) to the isotropic foliation of $\mathcal{D}'_l$ (which is also tangent to $T^2\times\partial D^2$). As in Section 4 we can arrange that the gluing map preserves the singular foliation on the fibers of θ.

5.2.4 Gluing the pieces together The even contact structures induced by $\mathcal{D}_k$ and $\mathcal{D}'_l$ coincide with the even contact structures near $\partial Y\times S^1$ and $T^2\times\partial D^2$, respectively, which arose in Section 4. Therefore, the only remaining problem is to ensure that the gluing maps the line field $\mathcal{D}'^v_l$ to $\mathcal{D}^v_k$. From (5) and (4) we obtain

$$X_k=\cos(k\varphi+\lambda y)\,\partial_r+\sin(k\varphi+\lambda y)S_\xi$$

and, according to (8) and (9), X'_l is given by

$$X'_l=\cos((l-k)y+k\varphi)\,\partial_r+\sin((l-k)y+k\varphi)S_\xi$$

near $T^2\times\partial D^2$. The conditions on the parameters l and k are

- k is a positive odd integer,
- l is a positive integer.

Thus, we obtain a smooth Engel structure on $M=(T^2\times D^2)\cup_\partial Y_s$ if $l-k=\lambda$ when k is large enough.

5.2.5 Uniqueness Using recent results about flexibility of Engel structures [3] we can prove the following:

Theorem 5.8 *For a fixed choice of framing e, the Engel structures constructed in Theorem 5.3 are unique up to homotopy through Engel structures for k large enough.*

According to [3], we only need to check that

- any pair of them are homotopic through formal Engel structures, ie the induced framings of TM are homotopic, and
- the Engel structure is loose in the sense of [3] when k is big enough.

Recall that k is odd, therefore the Engel structures on the mapping torus associated to k and $k+2$ are formally homotopic, ie the associated framings are homotopic. Near the boundary, we can be more specific using the coordinates x, y and φ on the boundary of the mapping torus: The homotopy of framings can be chosen to be:

- Constant on the neighborhood $D_{1/2} \times T^2$ of the binding, and we may assume that the characteristic foliation on $D_{1/2}$ contains all singular points introduced during the construction (7) for a fixed, odd value of k at the beginning of the homotopy. At the end of the homotopy, $D_1 \setminus \mathring{D}_{1/2}$ contains an additional pair of singularities, so that, at the end of the homotopy, the characteristic foliation on D_1 has the number of singular points prescribed by (7) for $k+2$ instead of k.
- The hyperplane field on $D_1 \setminus D_{1/2}$ spanned by the first three components of the framing is invariant under rotations in the (x, y)–direction throughout the homotopy.
- Throughout the homotopy, the framing is invariant under translations in the x–direction; the first and the last component of the framing are also y–invariant.
- The second and the third components are not y–invariant in general, but they are twisted and the twisting is determined by λ as in (4).

A homotopy with all these properties can be found by first fixing it on the annulus $D_1 \setminus \mathring{D}_{1/2}$ and then extending the homotopy using the desired invariance properties.

Since $\pi_2(\mathrm{SO}(4)) = \{0\}$, this homotopy is unique, and we may assume that the at the end of the homotopy, the characteristic foliation on $D_1 \setminus \mathring{D}_{1/2} \times \{(0,0)\} \subset D^2 \times T^2$ has precisely one elliptic and one hyperbolic singularity (as in (4) for $k+2$ instead of k).

The end result of the homotopy of framings coincides (up to homotopy) with the Engel framing obtained by our construction for $k+2$. In particular, the twisting number along γ_y is fixed.

We use the following definition for looseness of Engel structures. A chart in $(M, \mathcal{D})$ is an N–Darboux chart for $N \in \mathbb{N}$ if there is an Engel embedding of the ball $D^3 \times (0, N)$ with coordinates (x, y, z, θ) and with Engel structure

$$\ker(dy - z\,dx) \cap \ker\bigl(\cos(2\pi(\theta - \theta_0))\,dz - \sin(2\pi(\theta - \theta_0))\,dx\bigr)$$

for some $\theta_0 \in \mathbb{R}$.

In other words, the chart is a Darboux chart of an even contact structure in which the vector transverse to $\mathcal{W}$ rotates N times. We say that $\mathcal{D}$ is N–loose if any point of the manifold can be placed in the interior (not necessarily in the middle) of an N–Darboux chart. The main result of [3] implies that there is an integer N_0 with the following property: if $\mathcal{D}_0$ and $\mathcal{D}_1$ are N–loose with $N > N_0$ and formally homotopic, then they are homotopic through Engel structures.

The Engel structure on Y_s is $(k-L)$–loose, where L depends only on the gluing map φ and the framing (see (6)). The Engel structure $\mathcal{D}'_l$ on $T^2 \times D^2$ is l–loose. Finally, recall that $l = k + \lambda$. Since both λ and L are independent of k, this implies Theorem 5.8.

6 Stabilization

Let (Y, ξ) be a contact manifold whose boundary is a nonempty union of tori such that the characteristic foliation of ξ on each component of ∂Y is linear.

Let A be a properly embedded annulus in $(Y, \partial Y)$ such that

(i) ∂A is not contractible in ∂Y,

(ii) the characteristic foliation ξA is nonsingular and made of intervals going from one boundary component to the other, and

(iii) the characteristic foliations on the two 2–tori formed by A and the two annuli cut by ∂A in ∂Y are linear.

Like in the topological case explained in Section 3.2, we glue a handle $S^1 \times [0,1] \times [-1,1]$ to ∂Y and consider the torus $T = A \cup S^1 \times [0,1] \times \{0\}$. Recall that the stabilization procedure requires the choice of a curve γ in T intersecting the cocore $S^1 \times \{\frac{1}{2}\} \times \{0\}$ exactly once.

We extend ξ to the handle $S^1 \times [0,1] \times [-1,1]$ as follows. First, we extend the characteristic foliation ξA of A to a linear foliation of T such that γ is an integral curve. This determines the germ of the contact structure ξ' along T. After reducing the thickness of the handle, we obtain a contact structure ξ' on $Y' = Y \cup S^1 \times [0,1] \times [-1,1]$. Because of condition (iii), the characteristic foliations of the two tori formed by gluing $S^1 \times [0,1] \times \{0\}$ and the two annuli cut by $\partial A = S^1 \times \{0,1\} \times \{0\}$ in ∂Y are linear. This implies that we can choose the boundary of Y' so that the characteristic foliation of ξ' is linear.

There exists a neighborhood $T\times[-\varepsilon,\varepsilon]$ of $T\simeq T\times\{0\}$ such that each torus $T\times\{s\}$ for $s\in[-\varepsilon,\varepsilon]$ has a linear characteristic foliation for ξ'. We now choose a representative of τ_γ which preserves each torus together with its foliation. Following Giroux [13; 9], we then obtain another representative τ_γ which preserves ξ'. Let $h'=\tau_\gamma\circ h$.

Continuing Example 4.3, we give an example of a situation where the above conditions are satisfied.

Example 6.1 Let (K,θ) be an open book decomposition of a closed 3–manifold Y supporting a contact structure ξ which is trivial as a vector bundle. In Example 4.3 we showed that for $\varepsilon>0$ small enough there is an even contact structure $\mathcal{E}_\varepsilon$ on $S^1\times Y$ which is homotopic to $\mathrm{pr}_*^{-1}(\xi)$ through even contact structures. Recall that pr denotes the projection onto the second factor of $S^1\times Y$.

We now consider a stabilization of (K,θ). For this we choose a properly embedded arc a inside a page S. By Giroux's realization lemma [12], it is always possible to isotope ξ through contact structures carried by (K,θ) so that a is Legendrian. Then the open book decomposition $(\mathrm{pr}^{-1}(K),\theta\circ\mathrm{pr})$ is adapted to $\mathcal{E}_\varepsilon$ and the annulus $S^1\times a$ verifies the conditions stated at the beginning of this section. □

Theorem 6.2 *The even contact structures $\mathcal{E}$ and $\mathcal{E}'$ determined by the mapping tori of (Y,ξ,h) and (Y',ξ',h') are homotopic.*

Proof Recall first that if an open book decomposition supporting a contact structure ξ on a closed 3–manifold Z is stabilized along an arc a properly embedded in a page, then the stabilized open book decomposition also carries a contact structure ξ' and that ξ and ξ' are isotopic. More precisely, after Giroux and Goodman [14] (see also [9]), one chooses a to be Legendrian. Then

$$(Z,\xi')=(Z,\xi)\#(S^3,\xi_{\mathrm{st}}),$$

where the sphere separating the two summands bounds a small neighborhood of a in Z. The connected sum with S^3 does of course not change the manifold and the contact structure ξ' is isotopic to ξ since one Darboux ball (the neighborhood $N(a)$ of a) is removed and another Darboux ball is glued in. But then ξ' is homotopic to ξ relative to the complement of $N(a)$ since all tight contact structures on a ball which induce the same characteristic foliation on the boundary are isotopic by Eliashberg [7] (in this case it can be checked directly).

Now let $A \simeq S^1 \times [0,1]$ be an annulus satisfying the conditions (i), (ii) and (iii) stated at the beginning of this section. Then one can find a neighborhood $N_0(A) \simeq A \times (-\delta, \delta)$ of A in the page Y such that the plane field ξ is defined by $dt - f(y)\,dx$, where (t,x) are product coordinates on $A = S^1 \times [0,1]$ and $y \in (-\delta,\delta)$. Remember that ξ is contact in the interior of Y, so $f'(y) > 0$ when $x \in (0,1)$ and $f'(y) = 0$ on the binding $\partial Y \cap N_0(A) = \{x = 0,1\}$.

In order to describe the even contact structure $\mathcal{E}'$ one can now refer to Example 6.1 when we view $A = S^1 \times a$, where a is a leaf of the characteristic foliation of a fixed annular page of the open book of S^3 described in Example 3.4, joining the two components of the Hopf link binding. In particular, there is a neighborhood $N(A)$ of $N_0(A)$ in M where $\mathcal{E}$ is, possibly after homotopy, conjugate to $(S^1 \times N(a), \mathcal{E}_\varepsilon) \subset (S^1 \times S^3, \mathcal{E}_\varepsilon)$.

Notice that the contact structure ξ_{st} on S^3 supported by this open book is not trivial as a vector bundle, but it is over trivial $N(a)$, where we can apply the construction of Example 4.3. The even contact structure $\mathcal{E}'$ is obtained from $\mathcal{E}$ by first removing $(S^1 \times N(a), \mathcal{E})$ and then attaching $\big(S^1 \times (S^3 \setminus N(a)), \mathcal{E}_\varepsilon\big)$. Here, again, ξ_{st} is trivial over $S^3 \setminus N(a)$, so the construction makes sense, and there is an orientation-reversing identification sending the boundary of $(S^1 \times N(a), \mathcal{E}_\varepsilon)$ to the boundary of $(S^1 \times (S^3 \setminus N(a)), \mathcal{E}_\varepsilon)$.

It now follows that $(\mathcal{E}', \mathcal{W}')$ and $(\mathcal{E}, \mathcal{W})$ are homotopic as formal even contact structures, and hence as even contact structures [17]. □

In view of Questions 4.10 and 5.7, a nice feature of the stabilization operation is the following:

Theorem 6.3 *If (Y,ξ) is universally tight then so is (Y',ξ').*

Proof The contact manifold (Y',ξ') is obtained by gluing a universally tight contact structure on $T^2 \times [0,1]$ to (Y,ξ) along the prelagrangian torus $\partial Y \subset (Y,\xi)$. The result is universally tight by [4]. □

Finally, we describe how to obtain an Engel structure $\mathcal{D}'$ through a stabilization procedure on an open book decomposition supporting an Engel structure $\mathcal{D}$ on a 4–manifold M. In many situations, as in the case of even contact structures, if an open book (Y,ξ,h) supports an Engel structure $\mathcal{D}$ and if $A \subset (Y,\xi)$ is an annulus satisfying conditions (i), (ii) and (iii) on page 4298, then using Theorem 5.3 one can construct an

Engel structure $\mathcal{D}'$ carried by the stabilization (Y', ξ', h') of (Y, ξ, h) along A. We can check that they are formally homotopic and by [3] they are homotopic for k large enough, since we have shown that they are loose. However, we lose control of the geometry.

We present a more intrinsic operation below. Unfortunately, this construction involves a specific rigid normal form for $\mathcal{D}$ near the annulus A along which we stabilize. We do not know how restrictive it is.

More explicitly, in Example 5.2, where ξ_v is a contact structure on some closed 3–manifold Y supported by an open book (K, θ), we pick a Legendrian arc a properly embedded in a page S of (K, θ) and consider the annulus $S^1 \times a = \mathrm{pr}^{-1}(a)$ together with an S^1–invariant neighborhood of it $S^1 \times N(a)$. We assume that the Engel structure $\mathcal{D}$ on a neighborhood $N(A)$ of A is conjugated with some Engel structure $\mathcal{D}_k$ from Example 5.2 on $S^1 \times N(a)$. Recall the construction of $\mathcal{D}_k$ involves the choice of a trivialization and of a (K, θ)–adapted Reeb vector field of ξ_v on $N(a)$.

In order to obtain an Engel structure carried by the open book of M stabilized along A, we replace the Engel structure $\mathcal{D} \simeq \mathcal{D}_k$ on $N(A) \simeq S^1 \times N(a)$ with an Engel structure $\mathcal{D}'$ obtained in the following way.

We stabilize the open book (K, θ) of Y along a using Giroux and Goodman's stabilization construction [14]. This is given, as mentioned in the proof of Theorem 6.2, by replacing ξ_v and the partial open book induced by (K, θ) on $N(a)$ by a structure ξ_v' — contactomorphic to ξ_v — and a stabilized partial open book (K', θ') supporting ξ_v' on $N(a)$. The trivialization and the Reeb vector field of ξ_v along $\partial N(a)$ extend to a trivialization and a (K', θ')–adapted Reeb vector field of $\xi_v' \simeq \xi_v$ on $N(a)$.

Then we again apply the construction of Example 5.2 to ξ_v' and (K', θ') to get $\mathcal{D}'$ as an Engel structure $\mathcal{D}_k'$ on $N(A) \simeq S^1 \times N(a)$. It is supported by the original open book of M stabilized along A.

This concludes the description of $\mathcal{D}'$. As in the case of even contact structures it follows that $\mathcal{D}'$ is homotopic to $\mathcal{D}$ through Engel structures: the space of framings of ξ_v' which coincide with a fixed framing on $\partial N(a)$ is contractible. Moreover, notice that ξ_v and ξ_v' are both tight near $\partial N(a)$. According to a theorem of Eliashberg [7], any two contact structures on the ball which coincide near $\partial N(a)$ are isotopic relative to the boundary of the ball. Finally, any two contact vector fields which are transverse to a fixed contact structure are homotopic through contact vector fields which are transverse

to that contact structure (this can be seen using convex combinations of two such contact vector fields).

Remark 6.4 Given an annulus A in a page (Y, ξ) satisfying conditions (i), (ii) and (iii), it is not clear whether one can find a neighborhood $N(A)$ of A and a homotopy of $\mathcal{D}$ such that the resulting Engel structure is conjugate to one of our models.

References

[1] **J Alexander**, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. USA 9 (1923) 93–95

[2] **R Casals**, **J L Pérez**, **A del Pino**, **F Presas**, *Existence h–principle for Engel structures*, Invent. Math. 210 (2017) 417–451 MR

[3] **R Casals**, **A del Pino**, **F Presas**, *Loose Engel structures*, preprint (2017) arXiv

[4] **V Colin**, *Recollement de variétés de contact tendues*, Bull. Soc. Math. France 127 (1999) 43–69 MR

[5] **V Colin**, *Livres ouverts en géométrie de contact (d'après Emmanuel Giroux)*, from "Séminaire Bourbaki Vol. 2006/2007", Astérisque 317, Soc. Math. France, Paris (2008) Exposé 969, 91–117 MR

[6] **Y Eliashberg**, *Classification of overtwisted contact structures on 3–manifolds*, Invent. Math. 98 (1989) 623–637 MR

[7] **Y Eliashberg**, *Contact 3–manifolds twenty years since J Martinet's work*, Ann. Inst. Fourier (Grenoble) 42 (1992) 165–192 MR

[8] **F Engel**, *Zur Invariantentheorie der Systeme Pfaffscher Gleichungen*, Berichte über d. Verh. d. Sachsischen Gesell. der Wiss. 41, Hirzel, Leipzig (1889) 157–176

[9] **J B Etnyre**, *Lectures on open book decompositions and contact structures*, from "Floer homology, gauge theory, and low-dimensional topology" (D A Ellwood, P S Ozsváth, A I Stipsicz, Z Szabó, editors), Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 103–141 MR

[10] **H Geiges**, review of [19], MathSciNet MR

[11] **H Geiges**, *An introduction to contact topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge Univ. Press (2008) MR

[12] **E Giroux**, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) 637–677 MR

[13] **E Giroux**, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from "Proceedings of the International Congress of Mathematicians" (L Tatsien, editor), volume II, Higher Ed. Press, Beijing (2002) 405–414 MR

[14] **E Giroux**, **N Goodman**, *On the stable equivalence of open books in three-manifolds*, Geom. Topol. 10 (2006) 97–114 MR

[15] **F Laudenbach**, **V Poénaru**, *A note on* 4*–dimensional handlebodies*, Bull. Soc. Math. France 100 (1972) 337–344 MR

[16] **T Lawson**, *Open book decompositions for odd dimensional manifolds*, Topology 17 (1978) 189–192 MR

[17] **D McDuff**, *Applications of convex integration to symplectic and contact geometry*, Ann. Inst. Fourier (Grenoble) 37 (1987) 107–133 MR

[18] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, 2nd edition, Oxford Univ. Press (1998) MR

[19] **R Montgomery**, *Engel deformations and contact structures*, from "Northern California symplectic geometry seminar" (Y Eliashberg, D Fuchs, T Ratiu, A Weinstein, editors), Amer. Math. Soc. Transl. Ser. 2 196, Amer. Math. Soc., Providence, RI (1999) 103–117 MR

[20] **S P Novikov**, *Pontrjagin classes, the fundamental group and some problems of stable algebra*, from "Essays on topology and related topics" (A Haefliger, R Narasimhan, editors), Springer (1970) 147–155 MR

[21] **F Presas**, *Geometric decompositions of almost contact manifolds*, from "Contact and symplectic topology" (F Bourgeois, V Colin, A Stipsicz, editors), Bolyai Soc. Math. Stud. 26, János Bolyai Math. Soc., Budapest (2014) 137–172 MR

[22] **F Quinn**, *Open book decompositions, and the bordism of automorphisms*, Topology 18 (1979) 55–73 MR

[23] **W P Thurston**, **H E Winkelnkemper**, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975) 345–347 MR

[24] **T Vogel**, *Existence of Engel structures*, Ann. of Math. 169 (2009) 79–137 MR

[25] **H E Winkelnkemper**, *Manifolds as open books*, Bull. Amer. Math. Soc. 79 (1973) 45–51 MR

Laboratoire de Mathématiques Jean Leray, Université de Nantes
Nantes, France

Instituto de Ciencias Matemáticas, CSIC–UAM–UC3M–UCM
Madrid, Spain

Mathematisches Institut der Ludwig-Maximilians-Universität, Universität München
München, Germany

vincent.colin@univ-nantes.fr, fpresas@icmat.es, tvogel@math.lmu.de

Received: 21 February 2018 Revised: 12 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 4305–4328

Anick spaces and Kac–Moody groups

STEPHEN THERIAULT
JIE WU

For primes $p \geq 5$ we prove an approximation to Cohen, Moore and Neisendorfer's conjecture that the loops on an Anick space retracts off the double loops on a mod-p Moore space. The approximation is then used to answer a question posed by Kitchloo regarding the topology of Kac–Moody groups. We show that, for certain rank-2 Kac–Moody groups K, the based loops on K is p–locally homotopy equivalent to the product of the loops on a 3–sphere and the loops on an Anick space.

55P35; 55P15, 57T20

1 Introduction

This paper has two purposes. The first is to address an important conjecture in homotopy theory regarding the homotopy type of the double loops on an odd primary Moore space. The second is to establish a connection between rank-2 Kac–Moody groups and Anick spaces.

Let p be an odd prime and $r \geq 1$. Take homology with mod-p coefficients. For $m \geq 1$ the *Moore space* $P^{m+1}(p^r)$ is the cofibre of the degree p^r map on S^m. Its homotopy theory was investigated in depth by Cohen, Moore and Neisendorfer [4; 3; 5] and additional properties were proved by Neisendorfer [13; 14]. In the case of an odd-dimensional Moore space $P^{2n+1}(p^r)$, a related space was constructed by Anick [2] for $p \geq 5$, and reconstructed in a much simpler way by Gray and Theriault [9] for $p \geq 3$. For each $n, r \geq 1$ there is a space $T^{2n+1}(p^r)$ which fits in a homotopy fibration

$$S^{2n-1} \to T^{2n+1}(p^r) \to \Omega S^{2n+1}$$

and has the property that there is a coalgebra isomorphism

$$H_*(T^{2n+1}(p^r)) \cong \Lambda(u_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[v_{2n}]$$

with $\beta^r(v_{2n}) = u_{2n-1}$, where β^r is the r^{th} Bockstein. Cohen, Moore and Neisendorfer conjectured that $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$. Neisendorfer [14] proved this for $p \geq 3$ and $r \geq 2$, but the critical case of $r = 1$ remains open.

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4305

Our first result is to prove an approximation to this remaining open case, although we state the result for all $r \geq 1$. Define $C^{2n+1}(p^r)$ by the homotopy cofibration

$$P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \to C^{2n+1}(p^r),$$

where $[\nu, \mu]$ is the mod-p^r Whitehead product of the identity map ν on $P^{2n+1}(p^r)$ and the Bockstein map μ.

Theorem 1.1 *Let $p \geq 5$, $r \geq 1$ and $n > 1$. Then $\Omega T^{2n+1}(p^r)$ is a retract of $\Omega^2 C^{2n+1}(p^r)$.*

Philosophically, Theorem 1.1 says that if one gets rid of mod-p^r Whitehead products on $P^{2n+1}(p^r)$ (by coning them out in $C^{2n+1}(p^r)$) then all obstructions to a splitting involving $\Omega T^{2n+1}(p^r)$ vanish. This may or may not be helpful in trying to show that $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$. However, it is interesting to note that if $T_0^{2n+1}(p^r)$ is the bottom indecomposable factor of $\Omega P^{2n+1}(p^r)$, then Anick [1] showed that $T_0^{2n+1}(p^r)$ retracts off ΩL, where L is the $4n$–skeleton of $C^{2n+1}(p^r)$. So the obstruction to retracting $\Omega T^{2n+1}(p^r)$ off $\Omega^2 P^{2n+1}(p^r)$ is encoded in the attaching map of the top-dimensional cell of $C^{2n+1}(p^r)$.

Theorem 1.1 has practical applications, which leads to the second purpose of the paper. Fix a prime p. Let $k \in \{p, 2p\}$ or let k be a divisor of $p-1$ or $p+1$. Kitchloo [10; 11] showed that for each such k there is a nonempty set $\mathcal{V}_k$ of positive integers with the property that if $r \in \mathcal{V}_k$ then there is a rank-2 Kac–Moody group K such that

$$H_*(K) \cong \Lambda(z_3, y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}] \tag{1}$$

and $\beta^r(x_{2k}) = y_{2k-1}$, where β^r is the r^{th} Bockstein. Further, K has an S^3 subgroup whose inclusion induces an isomorphism onto the subalgebra $\Lambda(z_3)$ in homology. Taking classifying spaces, this results in a homotopy fibration sequence

$$S^3 \to K \xrightarrow{\delta} X \to BS^3 \to BK,$$

where

$$H_*(X) \cong \Lambda(y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$$

and δ_* is the projection. Observe that X has the same homology as the Anick space $T^{2k+1}(p^r)$.

Kitchloo conjectured that there is a p–local homotopy fibration $S^{2k-1} \to X \to \Omega S^{2k+1}$ that is equivalent to Anick's fibration. A weaker conjecture is that there is a p–local homotopy equivalence $X \simeq T^{2k+1}(p^r)$. We prove that the weaker conjecture

holds after looping if $1 < k < p-1$. Moreover, the method results in a homotopy decomposition for ΩK.

Theorem 1.2 *Let $p \geq 5$ and let K be a rank-2 Kac–Moody group satisfying* (1). *If $1 < k < p-1$ and $r \in \mathcal{V}_k$ then there are p–local homotopy equivalences*

$$\Omega X \simeq \Omega T^{2k+1}(p^r) \quad \text{and} \quad \Omega K \simeq \Omega S^3 \times \Omega T^{2k+1}(p^r).$$

The approach to proving Theorem 1.2 involves four steps. First, we lift the inclusion of the bottom Moore space $P^{2k}(p^r) \to X$ to K. Second, we show that its adjoint $P^{2k+1}(p^r) \to BK$ extends to a map $C^{2k+1}(p^r) \to BK$. Third, Theorem 1.1 is applied to produce a map $\Omega T^{2k+1}(p^r) \to \Omega K$. Finally, an atomicity-style argument is used to show that the composite $\Omega T^{2k+1}(p^r) \to \Omega K \to \Omega X$ is a p–local homotopy equivalence, from which Theorem 1.2 follows.

The decomposition of ΩK in Theorem 1.2 implies exponent information about K. The *p–primary homotopy exponent* of a space Y is the least power of p that annihilates the p–torsion in the homotopy groups of Y. If this power is r, write $\exp_p(Y) = p^r$. Selick [16] showed that $\exp_p(S^3) = p$ for $p \geq 3$, and Gray [8, Corollary 7.28] showed that $\exp_p(T^{2n+1}(p^r)) = p^r$ for $p \geq 5$. Since looping simply shifts homotopy groups down one dimension, Theorem 1.2 immediately implies the following.

Corollary 1.3 *If K is a Kac–Moody group as in Theorem 1.2, then* $\exp_p(K) = p^r$. □

In particular, if $r = 1$ then one obtains the remarkable outcome that $\exp_p(K) = p$.

Acknowledgements The authors would like to thank the referee for pointing out an error in an earlier version of the paper, and for several comments that have improved the exposition.

Wu is partially supported by a Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-190-112) and a grant (No. 11329101) from the NSFC of China.

2 Background information on the homotopy theory of Moore spaces

In this section we record some of the material from [4; 14] that will be needed later. From here on it will be assumed that all spaces and maps have been localized at an odd prime p.

If A is a co-H–space, let $\underline{p^r}\colon A \to A$ be the map of degree p^r. If B is an H–space, let $p^r\colon B \to B$ be the p^r–power map. For $m \geq 1$, the Moore space $P^{m+1}(p^r)$ is defined by the homotopy cofibration

$$S^m \xrightarrow{\underline{p^r}} S^m \to P^{m+1}(p^r).$$

The sphere S^{2n+1} is an H–space localized at an odd prime p and its p^r–power map is homotopic to the map of degree p^r. Define the space $S^{2n+1}\{p^r\}$ by the homotopy fibration

$$S^{2n+1}\{p^r\} \to S^{2n+1} \xrightarrow{p^r} S^{2n+1}.$$

One key result in [4] is that there is a map

$$S^{2n+1}\{p^r\} \to \Omega P^{2n+2}(p^r)$$

which has a left homotopy inverse.

It will be necessary to relate Moore spaces of different torsion orders. For $r, s \geq 1$, there is a homotopy pushout diagram

$$\begin{array}{ccccc}
S^m & \xrightarrow{p^r} & S^m & \longrightarrow & P^{m+1}(p^r) \\
\| & & \downarrow{\scriptstyle p^s} & & \downarrow{\scriptstyle \omega_r^{r+s}} \\
S^m & \xrightarrow{p^{r+s}} & S^m & \longrightarrow & P^{m+1}(p^{r+s}) \\
 & & \downarrow & & \downarrow{\scriptstyle \rho^s_{r+s}} \\
 & & P^{m+1}(p^s) & = & P^m(p^s)
\end{array}$$

that defines the maps ω_r^{r+s} and ρ^s_{r+s}. For fibres of degree maps there is an analogous homotopy pullback diagram

$$\begin{array}{ccccc}
S^{2n+1}\{p^r\} & \xrightarrow{\varpi_r^{r+s}} & S^{2n+1}\{p^{r+s}\} & \xrightarrow{\varrho^s_{r+s}} & S^{2n+1}\{p^s\} \\
\| & & \downarrow & & \downarrow \\
S^{2n+1}\{p^r\} & \longrightarrow & S^{2n+1} & \xrightarrow{p^r} & S^{2n+1} \\
 & & \downarrow{\scriptstyle p^{r+s}} & & \downarrow{\scriptstyle p^s} \\
 & & S^{2n+1} & = & S^{2n+1}
\end{array}$$

defining the maps ϖ_r^{r+s} and ϱ^s_{r+s}. As in [14, Diagram 1.10], the map $S^{2n+1}\{p^r\} \to \Omega P^{2n+2}(p^r)$ with a left homotopy inverse may be chosen so that it is natural with

respect to changes in torsion order. That is, there is a homotopy commutative diagram

$$\begin{array}{ccccc} S^{2n+1}\{p^r\} & \xrightarrow{\varpi_r^{r+s}} & S^{2n+1}\{p^{r+s}\} & \xrightarrow{\varrho_{r+s}^{s}} & S^{2n+1}\{p^s\} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega P^{2n+2}(p^r) & \xrightarrow{\Omega\omega_r^{r+s}} & \Omega P^{2n+2}(p^{r+s}) & \xrightarrow{\Omega\rho_{r+s}^{r}} & \Omega P^{2n+2}(p^s) \end{array} \tag{2}$$

For odd-dimensional Moore spaces, consider the homotopy fibration sequence

$$\Omega S^{2n+1} \xrightarrow{\partial_r} F^{2n+1}(p^r) \to P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1},$$

where q is the pinch map to the top cell, and the fibration sequence defines the space $F^{2n+1}(p^r)$ and the map ∂_r. In [4], it was shown that there is a homotopy equivalence

$$\kappa\colon S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \to \Omega F^{2n+1}(p^r). \tag{3}$$

There may be choices of the homotopy equivalence κ, and in Lemma 3.3 a lift of κ will be produced that depends on making a specific choice. To this end, κ will now be described in more detail.

Let $f\colon \Sigma P^s(p^r) \to P^m(p^r)$ and $g\colon \Sigma P^t(p^r) \to P^m(p^r)$ be maps. Let

$$w(f,g)\colon \Sigma P^s(p^r) \wedge P^t(p^r) \to P^m(p^r)$$

be the Whitehead product of f and g. By [12], as p is odd there is a homotopy equivalence $P^s(p^r) \wedge P^t(p^r) \simeq P^{s+t}(p^r) \vee P^{s+t-1}(p^r)$. The mod-$p^r$ Whitehead product of f and g is the composite

$$[f,g]\colon P^{s+t+1}(p^r) \hookrightarrow P^{s+t+1}(p^r) \vee P^{s+t}(p^r) \simeq \Sigma P^s(p^r) \wedge P^t(p^r) \xrightarrow{w(f,g)} P^m(p^r).$$

Mod-p^r Whitehead products play an important role, and certain ones are distinguished. Let $\nu\colon P^{2n+1}(p^r) \to P^{2n+1}(p^r)$ be the identity map and let $\mu\colon P^{2n}(p^r) \to P^{2n+1}(p^r)$ be the composite $P^{2n}(p^r) \xrightarrow{q} S^{2n} \to P^{2n+1}(p^r)$, where the right map is the inclusion of the bottom cell. Let $\mathrm{ad}^1(\nu)(\mu) = [\nu,\mu]$ and, for $k > 1$, recursively define $\mathrm{ad}^k(\nu)(\mu)$ by $\mathrm{ad}^k(\nu,\mu) = [\nu, \mathrm{ad}^{k-1}(\nu)(\mu)]$. In [4] it was shown that there is an extension

$$\begin{array}{ccc} & & P^{2np^j}(p^r) \\ & & \downarrow{\scriptstyle \omega_r^{r+1}} \\ & \swarrow{\scriptstyle e_j} & P^{2np^j}(p^{r+1}) \\ P^{2n+1}(p^r) & & \downarrow{\scriptstyle \mathrm{ad}^{p^j-1}(\nu)(\mu)} \end{array} \tag{4}$$

for some map e_j.

We now describe the spaces and maps appearing in (3). First, there is the inclusion $i\colon S^{2n-1} \to \Omega F^{2n+1}(p^r)$ of the bottom cell. Second, the space $R^{2n+1}(p^r)$ is a wedge of mod-p^r Moore spaces and there is a map $R^{2n+1}(p^r) \to P^{2n+1}(p^r)$ which is a wedge sum of iterated mod-p^r Whitehead products. Each mod-p^r Whitehead product composes trivially with the pinch map $P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1}$ because S^{2n+1} is an H–space and any Whitehead product on an H–space is null-homotopic. Thus there is a lift $\psi\colon R^{2n+1}(p^r) \to F^{2n+1}(p^r)$. Looping gives a map $\Omega R^{2n+1}(p^r) \xrightarrow{\Omega\psi} \Omega F^{2n+1}(p^r)$. Third, as above, the mod-p^r Whitehead product $\mathrm{ad}^{p^j-1}(\nu)(\mu)$ lifts to a map $\ell_j\colon P^{2np^j}(p^r) \to F^{2n+1}(p^r)$ and in [4] it is shown that the extension property in (4) occurs at the lifted level as well. That is, there is a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^j}(p^r) & \xrightarrow{\omega_r^{r+1}} & P^{2np^j}(p^{r+1}) \\ {\scriptstyle \ell_j}\big\downarrow & \swarrow_{e'_j} & \\ F^{2n+1}(p^r) & & \end{array} \tag{5}$$

for some map e'_j. Thus for each $j \geq 1$ there is a composite

$$S^{2np^j-1}\{p^{r+1}\} \to \Omega P^{2np^j}(p^r) \xrightarrow{\Omega e'_j} \Omega F^{2n+1}(p^r).$$

Letting $V^{2n+1}(p^{r+1}) = \prod_{j=1}^{\infty} S^{2np^j-1}\{p^{r+1}\}$ and using the loop space structure on $\Omega F^{2n+1}(p^r)$ to multiply we obtain the map $\epsilon\colon V^{2n+1}(p^{r+1}) \to \Omega F^{2n+1}(p^r)$. The map κ in (3) is the result of multiplying together the maps i, ϵ and $\Omega\psi$.

Remark 2.1 There may have been choices of the lifts ψ and ℓ_j. Any choice of ψ and any choice of ℓ_j that satisfied (5) would do to produce a choice of the homotopy equivalence κ.

Let b_r be the composite

$$b_r\colon \Omega^2 S^{2n+1} \xrightarrow{\Omega\partial_r} \Omega F^{2n+1}(p^r) \xrightarrow{\kappa^{-1}} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r).$$

Then there is a homotopy fibration sequence

$$\Omega^2 S^{2n+1} \xrightarrow{b_r} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \to \Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}.$$

There is a factorization of b_r proved by Neisendorfer. Changes in torsion order will play a role. For any $t \geq 1$, let $V^{2n+1}(p^t) = \prod_{j=1}^{\infty} S^{2np^j-1}\{p^t\}$. Abusing notation, let

$$\varpi_r^{r+s}\colon V^{2n+1}(p^r) \to V^{2n+1}(p^{r+s}) \quad \text{and} \quad \varrho_{r+s}^{s}\colon V^{2n+1}(p^{r+s}) \to V^{2n+1}(p^s)$$

also denote, respectively, the products of the maps $S^{2np^j-1}\{p^r\}\xrightarrow{\varpi_r^{r+s}}S^{2np^j-1}\{p^{r+s}\}$ and $S^{2np^j-1}\{p^{r+s}\}\xrightarrow{\varpi_{r+s}^{s}}S^{2np^j-1}\{p^s\}$. In [15], Neisendorfer proved the following.

Lemma 2.2 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} V^{2n+1}(p^{r+1}) & \longrightarrow & \Omega P^{2n+1}(p^r) \\ \downarrow{\scriptstyle \varrho_{r+1}^r} & \swarrow{\scriptstyle \zeta} & \\ V^{2n+1}(p^r) & & \end{array}$$

for some map ζ. □

From the homotopy fibration

$$S^{2n+1}\{p^r\}\xrightarrow{\varpi_r^{r+s}}S^{2n+1}\{p^{r+s}\}\xrightarrow{\varrho_{r+s}^{s}}S^{2n+1}\{p^s\}$$

we obtain a product homotopy fibration

$$V^{2n+1}(p^r)\xrightarrow{\varpi_r^{r+s}}V^{2n+1}(p^{r+s})\xrightarrow{\varrho_{r+s}^{s}}V^{2n+1}(p^s).$$

Therefore, Lemma 2.2 implies that there is a lift

$$\begin{array}{ccc} & & S^{2n-1}\times V^{2n+1}(p)\times\Omega R^{2n+1}(p^r) \\ & \nearrow & \downarrow{\scriptstyle 1\times\varpi_1^{r+1}\times 1} \\ \Omega^2 S^{2n+1} & \xrightarrow{b_r} & S^{2n-1}\times V^{2n+1}(p^{r+1})\times\Omega R^{2n+1}(p^r) \end{array}$$

In what follows, we only require a weaker lift. The map ϖ_1^{r+1} factors as the composite $\varpi_r^{r+1}\circ\varpi_1^r$. Therefore we obtain the following.

Lemma 2.3 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} & & S^{2n-1}\times V^{2n+1}(p^r)\times\Omega R^{2n+1}(p^r) \\ & \nearrow & \downarrow{\scriptstyle 1\times\varpi_r^{r+1}\times 1} \\ \Omega^2 S^{2n+1} & \xrightarrow{b_r} & S^{2n-1}\times V^{2n+1}(p^{r+1})\times\Omega R^{2n+1}(p^r) \end{array}$$

□

3 A retraction of $\Omega T^{2n+1}(p^r)$ off $\Omega^2 C^{2n+1}(p^r)$

In this section we prove Theorem 1.1 by constructing maps $a\colon \Omega T^{2n+1}(p^r)\to\Omega^2 C^{2n+1}(p^r)$ and $b\colon \Omega^2 C^{2n+1}(p^r)\to\Omega T^{2n+1}(p^r)$ with the property that $b\circ a$ is a homotopy equivalence. We begin with a description of the properties of Anick spaces that will be needed.

3.1 Properties of Anick spaces

As in the introduction, for each odd prime p and $n, r \geq 1$ there is a homotopy fibration

$$S^{2n-1} \to T^{2n+1}(p^r) \to \Omega S^{2n+1} \tag{6}$$

and a coalgebra isomorphism

$$H_*(T^{2n+1}(p^r)) \cong \Lambda(u_{2n-1}) \otimes \mathbb{Z}/p\mathbb{Z}[v_{2n}]$$

with $\beta^r(v_{2n}) = u_{2n-1}$, where β^r is the r^{th} Bockstein.

A simply connected space X is *atomic* if any self-map $f\colon X \to X$ which induces an isomorphism in the least nonvanishing degree in homology is a homotopy equivalence. Atomicity is used to detect indecomposable spaces, those for which no nontrivial product decompositions exist.

Theorem 3.1 *The space $T^{2n+1}(p^r)$ and the homotopy fibration* (6) *have the following properties:*

(a) *There is a factorization*

$$\begin{array}{ccc} \Omega P^{2n+1}(p^r) & \xrightarrow{\Omega q} & \Omega S^{2n+1} \\ \downarrow{\scriptstyle t} & & \| \\ T^{2n+1}(p^r) & \longrightarrow & \Omega S^{2n+1} \end{array}$$

for some map t.

(b) *The fibration connecting map for* (6) *is homotopic to $\Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1}$.*

(c) *If $r \geq 2$ then the map $\Omega^2 P^{2n+1}(p^r) \xrightarrow{\Omega t} \Omega T^{2n+1}(p^r)$ has a right homotopy inverse.*

(d) *If $p \geq 5$ then $T^{2n+1}(p^r)$ is a homotopy associative, homotopy commutative H–space and t is an H–map.*

(e) *$\Omega T^{2n+1}(p^r)$ is atomic.*

Proof Part (a) is proved in [2] for $p \geq 5$ and in [9] for $p \geq 3$, and part (b) — also established in both papers — is a consequence of part (a). Part (c) is proved in [14], part (d) in [8], and part (e) in [17]. □

3.2 Constructing a map $\Omega T^{2n+1}(p^r) \to \Omega^2 C^{2n+1}(p^r)$

In general, if X is a path-connected space, let $J_2(\Sigma X)$ be the second stage of the James construction on ΣX. There is a homotopy cofibration

$$\Sigma X \wedge X \xrightarrow{[1,1]} \Sigma X \xrightarrow{j} J_2(\Sigma X),$$

where $[1,1]$ is the Whitehead product of the identity map on ΣX with itself and j can be regarded as the inclusion of $J_1(\Sigma X) = \Sigma X$ into $J_2(\Sigma X)$. In our case take $X = P^{2n}(p^r)$. Let $T\colon X \wedge X \to X \wedge X$ be the map that swaps factors. As we are localized at an odd prime, the self-map

$$\tfrac{1}{2}(1-T)\colon \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \to \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r)$$

exists, and as in [6], it is an idempotent because $P^{2n}(p^r)$ is a suspension since $n \geq 1$. Moreover, as in [6], the Whitehead product $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{[1,1]} P^{2n+1}(p^r)$ factors through the telescope of $\frac{1}{2}(1-T)$, which is $P^{4n}(p^r)$, giving a factorization of $[1,1]$ as a composite $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{\mathfrak{t}} P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r)$, where $\mathfrak{t}$ is the map to the telescope and has a right homotopy inverse. Consequently, there is a homotopy pushout diagram

$$\begin{array}{ccccc}
\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) & \xrightarrow{\mathfrak{t}} & P^{4n}(p^r) & \xrightarrow{*} & P^{4n+2}(p^r) \\
\| & & \downarrow{\scriptstyle [\nu,\mu]} & & \downarrow \\
\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) & \xrightarrow{[1,1]} & P^{2n+1}(p^r) & \xrightarrow{j} & J_2(P^{2n}(p^r)) \\
 & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \varrho} \\
 & & C^{2n+1}(p^r) & = & C^{2n+1}(p^r)
\end{array} \tag{7}$$

that defines the map ϱ.

In general, if Y and Z are simply connected spaces, let ev_1 and ev_2 be the composites

$$\mathrm{ev}_1\colon \Sigma\Omega Y \xrightarrow{\mathrm{ev}} Y \xrightarrow{i_1} Y \vee Z \quad \text{and} \quad \mathrm{ev}_2\colon \Sigma\Omega Z \xrightarrow{\mathrm{ev}} Z \xrightarrow{i_2} Y \vee Z,$$

where i_1 and i_2 are the inclusions of the left and right wedge summands respectively. By [7], there is a homotopy fibration

$$\Sigma\Omega Y \wedge \Omega Z \xrightarrow{[\mathrm{ev}_1,\mathrm{ev}_2]} Y \vee Z \to Y \times Z,$$

where the right map is the inclusion of the wedge into the product. When $Y = Z$ there is a fold map $\nabla\colon Y \vee Y \to Y$. The *universal Whitehead product* on Y is the composite

$$\Psi\colon \Sigma\Omega Y \wedge \Omega Y \xrightarrow{[\mathrm{ev}_1,\mathrm{ev}_2]} Y \vee Y \to Y.$$

It is universal because any Whitehead product on Y factors through Ψ. In [18] it was shown that if $Y = \Sigma X$ then the composite $\Sigma\Omega\Sigma X \wedge \Omega\Sigma X \xrightarrow{\Psi} \Sigma X \xrightarrow{j} J_2(\Sigma X)$ is null-homotopic. In our case, taking $X = P^{2n}(p^r)$, the factorization of c through j in (7) immediately implies the following.

Lemma 3.2 *The composite* $\Sigma\Omega P^{2n+1}(p^r) \wedge \Omega P^{2n+1}(p^r) \xrightarrow{\Psi} P^{2n+1}(p^r) \xrightarrow{c} C^{2n+1}(p^r)$ *is null-homotopic.* □

Next, since S^{2n+1} is an H–space when localized at a prime $p \geq 3$, the composite $P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \xrightarrow{q} S^{2n+1}$ is null-homotopic. Thus q extends to a map $q'\colon C^{2n+1}(p^r) \to S^{2n+1}$. From this extension we obtain a homotopy fibration diagram

$$\begin{array}{ccccccc} & & M^{2n+1}(p^r) & = & M^{2n+1}(p^r) & & \\ & & \downarrow & & \downarrow & & \\ \Omega S^{2n+1} & \xrightarrow{\partial_r} & F^{2n+1}(p^r) & \longrightarrow & P^{2n+1}(p^r) & \xrightarrow{q} & S^{2n+1} \\ \| & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} & & \| \\ \Omega S^{2n+1} & \xrightarrow{\overline{\partial}_r} & D^{2n+1}(p^r) & \longrightarrow & C^{2n+1}(p^r) & \xrightarrow{q'} & S^{2n+1} \end{array} \tag{8}$$

that defines the spaces $D^{2n+1}(p^r)$ and $M^{2n+1}(p^r)$ and the maps d and $\overline{\partial}_r$.

Lemma 3.3 *There is a choice of the homotopy equivalence*

$$S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \xrightarrow{\kappa} \Omega F^{2n+1}(p^r)$$

with the property that there is a homotopy commutative diagram

$$\begin{array}{ccc} V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) & \xrightarrow{\xi} & \Omega M^{2n+1}(p^r) \\ \downarrow{\scriptstyle *\times\varpi_r^{r+1}\times 1} & & \downarrow \\ S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) & \xrightarrow{\kappa} & \Omega F^{2n+1}(p^r) \end{array}$$

for some map ξ.

Proof Start with the homotopy cofibration

$$P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \xrightarrow{c} C^{2n+1}(p^r).$$

By definition, $[\nu,\mu] = \mathrm{ad}^1$, so $c \circ \mathrm{ad}^1$ is null-homotopic. Since $\mathrm{ad}^k = [\nu, \mathrm{ad}^{k-1}]$ for $k > 1$, the naturality of the mod-p^r Whitehead product implies that $c \circ \mathrm{ad}^k$ is

null-homotopic for all $k \geq 1$. Thus each ad^k lifts to the homotopy fibre $M^{2n+1}(p^r)$ of c. Moreover, any iterated mod-p^r Whitehead product in which $[\nu, \mu]$ appears has the property that it composes trivially with c and so lifts to $M^{2n+1}(p^r)$.

Recall the construction of κ in Section 2. The map $R^{2n+1}(p^r) \to P^{2n+1}(p^r)$ was a wedge sum of mod-p^r Whitehead products. Each such Whitehead product factors through the universal Whitehead product on $P^{2n+1}(p^r)$, so Lemma 3.2 implies that the composite $R^{2n+1}(p^r) \to P^{2n+1}(p^r) \xrightarrow{c} C^{2n+1}(p^r)$ is null-homotopic. Thus the map $R^{2n+1}(p^r) \to P^{2n+1}(p^r)$ lifts to $M^{2n+1}(p^r)$, and the lift $R^{2n+1}(p^r) \xrightarrow{\psi} F^{2n+1}(p^r)$ used in forming κ may be chosen to be the composite $R^{2n+1}(p^r) \to M^{2n+1}(p^r) \to F^{2n+1}(p^r)$. Therefore we obtain a homotopy commutative diagram

$$\begin{array}{ccc} & & \Omega M^{2n+1}(p^r) \\ & \nearrow & \downarrow \\ \Omega R^{2n+1}(p^r) & \xrightarrow{\Omega\psi} & \Omega F^{2n+1}(p^r) \end{array} \tag{9}$$

Similarly, each ad^{p^j-1} has $[\nu, \mu]$ appearing in it and so can be chosen to lift to $F^{2n+1}(p^r)$ through $M^{2n+1}(p^r)$. The extension through ω_r^{r+1} may not exist as a map to $M^{2n+1}(p^r)$, but we do not require this. We obtain, for each $j \geq 1$, a homotopy commutative diagram

$$\begin{array}{ccc} P^{2np^j}(p^r) & \longrightarrow & M^{2n+1}(p^r) \\ \downarrow{\scriptstyle \omega_r^{r+1}} & & \downarrow \\ P^{2np^j}(p^{r+1}) & \xrightarrow{e'_j} & F^{2n+1}(p^r) \end{array}$$

Looping to take products and using (2) we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} V^{2n+1}(p^r) & \longrightarrow & \prod_{j=1}^{\infty} \Omega P^{2np^j}(p^r) & \longrightarrow & \Omega M^{2n+1}(p^r) \\ \downarrow{\scriptstyle \varpi_r^{r+1}} & & \downarrow{\scriptstyle \prod_{j=1}^{\infty} \Omega\omega_r^{r+1}} & & \downarrow \\ V^{2n+1}(p^{r+1}) & \longrightarrow & \prod_{j=1}^{\infty} \Omega P^{2np^j}(p^{r+1}) & \longrightarrow & \Omega F^{2n+1}(p^r) \end{array} \tag{10}$$

Let $\epsilon\colon V^{2n+1}(p^{r+1}) \to \Omega F^{2n+1}(p^r)$ be the composition along the bottom row of (10). By Remark 2.1, we may choose κ to be the product of the inclusion i of the bottom cell S^{2n-1} into $\Omega F^{2n+1}(p^r)$, the map $\Omega\psi$ in (9), and the map ϵ. Then κ is a homotopy equivalence and from (9) and (10) we obtain the homotopy commutative diagram asserted in the statement of the lemma. □

Consider the map $\Omega S^{2n+1} \xrightarrow{\overline{\partial}_r} D^{2n+1}(p^r)$ appearing in (8). We give a factorization of $\Omega\overline{\partial}_r$. Let φ_r be the composite

$$\varphi_r\colon \Omega^2 S^{2n+1} \xrightarrow{b_r} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \xrightarrow{\text{proj}} S^{2n-1},$$

where the right map is the projection.

Proposition 3.4 *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{\varphi_r} & S^{2n-1} \\ \| & & \downarrow \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega\overline{\partial}_r} & \Omega D^{2n+1}(p^r) \end{array}$$

Proof Let κ' be the composite

$$\kappa'\colon S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) \xrightarrow{1\times\varpi_r^{r+1}\times 1} S^{2n-1} \times V^{2n+1}(p^{r+1}) \times \Omega R^{2n+1}(p^r) \xrightarrow{\kappa} \Omega F^{2n+1}(p^r).$$

Consider the homotopy fibration $\Omega M^{2n+1}(p^r) \to \Omega F^{2n+1}(p^r) \xrightarrow{\Omega d} \Omega D^{2n+1}(p^r)$ from (8). Since Ωd is an H–map and κ is defined by using the loop multiplication on $\Omega F^{2n+1}(p^r)$ to multiply the factors together, the composite $\Omega d \circ \kappa$ is determined by the restriction to each of the factors. The restriction to $V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r)$ is null-homotopic by Lemma 3.3. Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) & \xrightarrow{\kappa'} & \Omega F^{2n+1}(p^r) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \Omega d} \\ S^{2n-1} & \longrightarrow & \Omega D^{2n+1}(p^r) \end{array} \tag{11}$$

where π_1 is the projection onto the first factor.

Now consider the diagram

$$\begin{array}{ccccc} \Omega^2 S^{2n+1} & \longrightarrow & S^{2n-1} \times V^{2n+1}(p^r) \times \Omega R^{2n+1}(p^r) & & \\ \| & & \downarrow{\scriptstyle \kappa'} & \searrow^{\pi_1} & \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega\partial_r} & \Omega F^{2n+1}(p^r) & & S^{2n-1} \\ \| & & \downarrow{\scriptstyle \Omega d} & \nearrow_{\iota} & \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega\overline{\partial}_r} & \Omega D^{2n+1}(p^r) & & \end{array}$$

where ι is the inclusion of the bottom cell. By Lemma 2.3 and the definition of κ', the upper left square homotopy commutes. The lower left square homotopy commutes by (8) and the right triangle homotopy commutes by (11). The diagram as a whole therefore states that $\Omega\overline{\partial}_r$ factors through S^{2n-1}. It remains to identify the map $\varphi'_r\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ along the upper direction of the diagram as φ_r. But, by definition, κ' is the identity on the S^{2n-1} factor so φ'_r can be identified as the composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega\overline{\partial}_r} \Omega F^{2n+1}(p^r) \xrightarrow{\text{proj}} S^{2n-1}$, which is the definition of φ_r. □

By Theorem 3.1(b) there is a homotopy fibration $\Omega T^{2n+1}(p^r) \to \Omega^2 S^{2n+1} \xrightarrow{\varphi_r} S^{2n-1}$. Proposition 3.4 therefore implies that the map $\Omega T^{2n+1}(p^r) \to \Omega^2 S^{2n+1}$ lifts to the homotopy fibre of $\Omega^2 S^{2n+1} \xrightarrow{\Omega\overline{\partial}_r} \Omega D^{2n+1}(p^r)$, which by (8) is $\Omega^2 C^{2n+1}(p^r)$. Hence we have shown the following, where we explicitly remember that everything done so far holds for all odd primes.

Corollary 3.5 *If $p \geq 3$ then there is a lift*

$$\begin{array}{ccc} & & \Omega T^{2n+1}(p^r) \\ & \overset{\lambda}{\swarrow} & \downarrow \\ \Omega^2 C^{2n+1}(p^r) & \longrightarrow & \Omega^2 S^{2n+1} \end{array}$$

for some map λ. □

3.3 Constructing a map $\Omega^2 C^{2n+1}(p^r) \to \Omega T^{2n+1}(p^r)$

This will be done for $p \geq 5$, and the map will in fact be a loop map. Recall that there is a homotopy cofibration $P^{4n}(p^r) \xrightarrow{[\nu,\mu]} P^{2n+1}(p^r) \xrightarrow{c} C^{2n+1}(p^r)$. As $[\nu,\mu]$ factors through the Whitehead product $\Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) \xrightarrow{[1,1]} P^{2n+1}(p^r)$, there is a homotopy pushout diagram

$$\begin{array}{ccccc} P^{4n}(p^r) & \longrightarrow & \Sigma P^{2n}(p^r) \wedge P^{2n}(p^r) & \longrightarrow & P^{4n+1}(p^r) \\ \| & & \downarrow{\scriptstyle [1,1]} & & \downarrow \\ P^{4n}(p^r) & \xrightarrow{[\nu,\mu]} & P^{2n+1}(p^r) & \xrightarrow{c} & C^{2n+1}(p^r) \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} \\ & & J_2(P^{2n+1}(p^r)) & = & J_2(P^{2n+1}(p^r)) \end{array} \tag{12}$$

that defines the map j'.

By Theorem 3.1(a), the loops on the pinch map $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2n+1}$ factors as a composite $\Omega P^{2n+1}(p^r) \xrightarrow{t} T^{2n+1}(p^r) \to \Omega S^{2n+1}$ for some map t.

Lemma 3.6 *If $p \geq 5$ then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega P^{2n+1}(p^r) & \xrightarrow{t} & T^{2n+1}(p^r) \\ \downarrow{\scriptstyle \Omega c} & \nearrow_{\widetilde{t}} & \\ \Omega C^{2n+1}(p^r) & & \end{array}$$

for some map $\widetilde{t}$.

Proof In [18, Lemma 2.6] it was shown that if Z is a homotopy associative and homotopy commutative H–space and $f\colon \Omega P^{2n+1}(p^r) \to Z$ is an H–map then there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega P^{2n+1}(p^r) & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle \Omega j} & \nearrow_{\bar{f}} & \\ \Omega J_2(P^{2n+1}(p^r)) & & \end{array}$$

for some map $\bar{f}$. By (12), the map Ωj factors as the composite $\Omega P^{2n+1}(p^r) \xrightarrow{\Omega c} \Omega C^{2n+1}(p^r) \xrightarrow{\Omega j'} \Omega J_2(P^{2n+1}(p^r))$. Thus if we take $\widetilde{f} = \bar{f} \circ \Omega j'$ then there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega P^{2n+1}(p^r) & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle \Omega c} & \nearrow_{\widetilde{f}} & \\ \Omega C^{2n+1}(p^r) & & \end{array} \tag{13}$$

By Theorem 3.1(d), if $p \geq 5$ then $T^{2n+1}(p^r)$ is homotopy associative and homotopy commutative, and the map t is an H–map. Thus the assertion of the lemma follows by applying (13) to t. □

3.4 The proof of Theorem 1.1 and an application

We first put the pieces together to obtain a retraction of $\Omega T^{2n+1}(p^r)$ off $\Omega^2 C^{2n+1}(p^r)$, proving Theorem 1.1.

Proof of Theorem 1.1 We will show that if $p \geq 5$ and $n > 1$ then the composite $\Omega T^{2n+1}(p^r) \xrightarrow{\lambda} \Omega^2 C^{2n+1}(p^r) \xrightarrow{\Omega \widetilde{t}} \Omega T^{2n+1}(p^r)$ is a homotopy equivalence. The

homotopy commutativity of the diagrams in Corollary 3.5 and Lemma 3.6 imply that both λ and $\Omega\widetilde{t}$ induce an isomorphism in mod-p homology in degree $2n-2$, the least nonvanishing degree. Thus $\Omega\widetilde{t}\circ\lambda$ is a self-map of $\Omega T^{2n+1}(p^r)$ which induces an isomorphism in the least nonvanishing degree in homology. By Theorem 3.1(e), $\Omega T^{2n+1}(p^r)$ is atomic if $p\geq 5$ and $n>1$, so $\Omega\widetilde{t}\circ\lambda$ is a homotopy equivalence. □

Next, Theorem 1.1 is applied to produce maps from $\Omega T^{2n+1}(p^r)$ into certain spaces by checking that minimal requirements hold.

Theorem 3.7 *Let $p\geq 5$ and $r\geq 1$. Suppose there is a map $f\colon P^{2n+1}(p^r)\to Z$ for some space Z. If the composite $P^{4n}(p^r)\xrightarrow{[\nu,\mu]}P^{2n+1}(p^r)\xrightarrow{f}Z$ is null-homotopic then there is a map $\Omega T^{2n+1}(p^r)\to\Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f.*

Proof The hypothesis that $f\circ[\nu,\mu]$ is null-homotopic is equivalent to saying that the map f extends to a map $C^{2n+1}(p^r)\to Z$. Theorem 1.1 therefore implies that there is a composite $\Omega T^{2n+1}(p^r)\to\Omega^2 C^{2n+1}(p^r)\to\Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f. □

By Theorem 3.1(c), if $r\geq 2$ then $\Omega T^{2n+1}(p^r)$ retracts off $\Omega^2 P^{2n+1}(p^r)$, so in this case, given a map $P^{2n+1}(p^r)\xrightarrow{f}Z$, one automatically obtains a map $\Omega T^{2n+1}(p^r)\to\Omega^2 Z$ whose restriction to the bottom Moore space is the double adjoint of f. The additional hypothesis in Theorem 3.7 regarding $f\circ[\nu,\mu]$ being null-homotopic is not necessary.

However, the $r=1$ case is often the vital one, and it is an open conjecture as to whether $\Omega T^{2n+1}(p)$ retracts off $\Omega^2 P^{2n+1}(p)$. So Theorem 3.7 can be thought of as a way of producing a consequence of the conjecture without having to prove it first. Moreover, it is practical in the sense that one can hope to check that a given map $P^{2n+1}(p^r)\xrightarrow{f}Z$ has $f\circ[\nu,\mu]$ null-homotopic. In fact, this criterion will be used in the next section in the context of Kac–Moody groups.

4 Kac–Moody groups

As in [10; 11], Kac–Moody groups of rank 2 correspond to generalized Cartan matrices of the form

$$A=\begin{pmatrix}2 & -a\\ -b & 2\end{pmatrix}.$$

Given such a matrix A, the Kac–Moody group K is the semisimple factor inside the corresponding unitary form. If $ab < 4$ then K is a compact Lie group. We are interested in the case when $ab \geq 4$. Define integers c_i and d_i recursively by

$$c_0 = d_0 = 1, \quad c_1 = d_1 = 1, \quad c_{j+1} = ad_j - c_{j-1}, \quad d_{j+1} = bc_j - d_{j-1}.$$

Let $g_i = (c_i, d_i)$ be the greatest common divisor of c_i and d_i. Fix an odd prime p and take homology with mod-p coefficients. Let k be the smallest positive integer such that p divides g_k. Then there is an isomorphism of Hopf algebras

$$H_*(K) \cong \Lambda(z_3, y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}], \tag{14}$$

where the generators are primitive, and if r is the exponent of p in g_k, then $\beta^r(x_{2k}) = y_{2k-1}$. Further, it is known that if k exists then it is either p, $2p$ or a nontrivial divisor of $p+1$ or $p-1$, and in each case there are choices of a and b which produce such a k.

Given such a k, let $\mathcal{V}_k$ be the collection of all possible integers r that arise as the exponent of p in g_k for some choice of integers a and b. In general it is known that $\mathcal{V}_k$ is nonempty but no precise description is known. However, to give some examples, we show that if $p \geq 5$ and $k \in \{2, 3, 4\}$ then $\mathcal{V}_k = \mathbb{N}$. Note that each of 2, 3 and 4 is a proper divisor of either $p+1$ or $p-1$, and so is a valid value of k. Observe that

$$c_2 = a, \quad d_2 = b, \quad c_3 = d_3 = ab - 1, \quad c_4 = a(ab-2), \quad d_4 = b(ab-2).$$

If $k = 2$ then taking $a = b = p^r$ gives $g_2 = (c_2, d_2) = p^r$. If $k = 3$ then taking $a = 1$ and $b = p^r + 1$ gives $g_2 = (c_2, d_2) = 1$ and $g_3 = (c_3, d_3) = p^r$. If $k = 4$ then taking $a = 1$ and $b = p^r + 2$ gives $g_2 = (c_2, d_2) = 1$, $g_3 = (c_3, d_3) = p^r + 1$, and $g_4 = (c_4, d_4) = p^r$. Thus, in all cases, any $r \geq 1$ will do, so $\mathcal{V}_k = \mathbb{N}$.

Recall from the introduction that K has an S^3 subgroup whose inclusion induces an isomorphism onto the sub-Hopf-algebra $\Lambda(z_3)$ in homology, resulting in a homotopy fibration sequence

$$S^3 \to K \xrightarrow{\delta} X \to BS^3 \to BK, \tag{15}$$

where

$$H_*(X) \cong \Lambda(y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$$

and δ_* is the projection. In particular, this is an isomorphism of coalgebras (as X may not be an H–space).

Now localize all spaces and maps at p. We aim to show that for $p \geq 5$ and $1 < k < p-1$ there is a p–local homotopy equivalence $\Omega X \simeq \Omega T^{2n+1}(p^r)$. Any approach to the

problem is limited by the fact that almost nothing is known about the homotopy theory, or homotopy groups, of K. The method is to first produce a map $P^{2k+1}(p^r) \to BK$ whose adjoint induces an isomorphism in homology onto the generators x_{2k} and y_{2k-1}, and then to extend it to a map $C^{2k+1}(p^r) \to BK$. At both stages certain values of k have to be eliminated in order to ensure that potential obstructions vanish. This will require some information about the homotopy groups of spheres proved by Toda [19].

Remark 4.1 It is worth pointing out beforehand that the only properties of Kac–Moody groups used in showing $\Omega T^{2n+1}(p^r) \simeq \Omega X$ are the existence of the homotopy fibration sequence (15) and the description of $H_*(X)$. The rest of the argument is based on the homotopy theory of spheres, Moore spaces and Anick spaces.

Lemma 4.2 *Let $p \geq 5$. If $3 < m \leq 4p$ then $\pi_m(S^3) \cong 0$ unless $m \in \{2p, 4p-2\}$.* □

Lemma 4.3 *Let $p \geq 5$. If $k \leq p$ and $r \in \mathcal{V}_k$ then there is a map $P^{2k+1}(p^r) \to BK$ whose adjoint induces an isomorphism onto the generators x_{2k} and y_{2k-1} in $H_*(K)$.*

Proof Let $P^{2k}(p^r) \to X$ be the inclusion of the bottom Moore space. We aim to show that there is a lift

$$\begin{array}{ccccc} & & P^{2k}(p^r) & & \\ & \swarrow & \downarrow & & \\ K & \longrightarrow & X & \longrightarrow & BS^3 \end{array}$$

The adjoint of this lift is the map asserted in the lemma. The lift will certainly exist when $[P^{2k}(p^r), BS^3] \cong 0$.

The homotopy cofibration $S^{2k-1} \to P^{2k}(p^r) \to S^{2k}$ induces an exact sequence $[S^{2k}, BS^3] \to [P^{2k}(p^r), BS^3] \to [S^{2k-1}, BS^3]$. By Lemma 4.2, the first nontrivial torsion homotopy group of BS^3 occurs in dimension $2p+1$. So if $k \leq p$ then the homotopy groups $\pi_{2k}(BS^3)$ and $\pi_{2k-1}(BS^3)$ are trivial. Therefore, by exactness, $[P^{2k+1}(p^r), BS^3] \cong 0$, and the asserted lift exists. (Note that when $k = p+1$ the map α_1 generating $\pi_{2k-1}(BS^3)$ is a potential obstruction to a lift, and when $k = 2p$ the map α_2 generating $\pi_{2k-1}(BS^3)$ is a potential obstruction.) □

Lemma 4.4 *Let $p \geq 5$. Suppose that k is a proper divisor of $p-1$ or $p+1$, but $k \neq \frac{1}{2}(p+1)$. If $r \in \mathcal{V}_k$ then the composite $P^{4k}(p^r) \xrightarrow{[\nu,\mu]} P^{2k+1}(p^r) \to BK$ is null-homotopic.*

Proof Let $\xi\colon P^{2k+1}(p^r) \to BK$ be the map in Lemma 4.3. Consider the map $\Omega P^{2k+1}(p^r) \xrightarrow{\Omega\xi} K$. Since $P^{2n+1}(p^r)$ is a suspension, the Bott–Samelson theorem implies that $H_*(\Omega P^{2k+1}(p^r)) \cong T(u_{2k-1}, v_{2k})$, where $\beta^r(v_{2k}) = u_{2k-1}$. The homology statement in Lemma 4.3 implies that $(\Omega\xi)_*$ sends u_{2k-1} and v_{2k} to y_{2k-1} and x_{2k} respectively in $H_*(K) \cong \Lambda(z_3, y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$. Since $(\Omega\xi)_*$ is an algebra map, this implies that the image of $(\Omega\xi)_*$ is the subalgebra of $H_*(K)$ generated by y_{2k-1} and x_{2k}. Recall that the map $K \xrightarrow{\delta} X$ in (15) induces a projection in homology onto $H_*(X) \cong \Lambda(y_{2k-1}) \otimes \mathbb{Z}/p\mathbb{Z}[x_{2k}]$. Therefore the composite $\theta\colon \Omega P^{2n+1}(p^r) \xrightarrow{\Omega\xi} K \xrightarrow{\delta} X$ has the property that θ_* is the abelianization of the tensor algebra.

Let N be the homotopy fibre of θ. Since θ_* is the abelianization of the tensor algebra, the Serre exact sequence implies that in degrees $\leq 4k-1$, the homology group $H_*(N)$ is the kernel of θ_*. Therefore, in this degree range, $H_*(N)$ consists of the brackets $\langle u_{2n-1}, u_{2n-1}\rangle$ and $\langle v_{2n}, u_{2n-1}\rangle$, which are connected by a Bockstein β^r. Thus there is an inclusion of a bottom Moore space into N which gives a composite

$$\gamma\colon P^{4k-1}(p^r) \to N \to \Omega P^{2k+1}(p^r). \tag{16}$$

Now we compare γ to known elements in the group $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$. Consider the homotopy fibration

$$\Omega F^{2k+1}(p^r) \to \Omega P^{2k+1}(p^r) \xrightarrow{\Omega q} \Omega S^{2k+1}.$$

Applying the functor $[P^{4k-1}(p^r), -]$ to this fibration we obtain an exact sequence

$$\begin{aligned}[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \to [P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \\ \to [P^{4k-1}(p^r), \Omega S^{2k+1}].\end{aligned} \tag{17}$$

By Lemma 4.2, the hypothesis that $k \neq p-1$ implies that $[P^{4k-1}(p^r), \Omega S^{2k+1}] \cong 0$. On the other hand, by (3) there is a homotopy equivalence

$$\Omega F^{2k+1}(p^r) \simeq S^{2k-1} \times \Omega R^{2k+1}(p^r) \times \prod_{j=1}^{\infty} S^{2kp^j-1}\{p^{r+1}\}.$$

Recall that $R^{2k+1}(p^r)$ is a wedge of mod-p^r Moore spaces mapping to $P^{2k+1}(p^r)$ by a wedge sum of mod-p^r Whitehead products. In particular, by [4], the least-dimensional Moore space in $R^{2k+1}(p^r)$ is $P^{4k}(p^r)$, which maps to $P^{2k+1}(p^r)$ by $[\nu, \mu]$, and for $p \geq 5$ the second-least-dimensional Moore space in $R^{2k+1}(p^r)$

is $P^{6k}(p^r)$. This, together with the fact that each factor $S^{2kp^j-1}\{p^{r+1}\}$ is more than $(4k-1)$–connected, implies that

$$[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), S^{2k-1} \times \Omega P^{4k}(p^r)].$$

Lemma 4.2 and the hypothesis that $k \neq p$ ensure that $[P^{4k-1}(p^r), S^{2k-1}] \cong 0$. Since the suspension map $P^{4k-1}(p^r) \xrightarrow{E} \Omega P^{4k}(p^r)$ is $(8k-6)$–connected, which is greater than the dimension of $P^{4k-1}(p^r)$, it induces an isomorphism between $[P^{4k-1}(p^r), P^{4k-1}(p^r)]$ and $[P^{4k-1}(p^r), \Omega P^{4k}(p^r)]$. Therefore

$$[P^{4k-1}(p^r), \Omega F^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z},$$

where the generator of the group $[P^{4k-1}(p^r), P^{4k-1}(p^r)]$ is the identity map. Thus the exact sequence (17) simplifies to an exact sequence

$$\mathbb{Z}/p^r\mathbb{Z} \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \to [P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \to 0. \tag{18}$$

By [3], the factor $\Omega R^{2k+1}(p^r)$ of $\Omega F^{2k+1}(p^r)$ also retracts off $\Omega P^{2k+1}(p^r)$. Therefore there is a retraction of $[P^{4k-1}(p^r), P^{4k-1}(p^r)]$ off $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$, implying that there is an isomorphism

$$[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \cong [P^{4k-1}(p^r), P^{4k-1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z}.$$

Hence if $k \notin \{p-1, p\}$ then $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)]$ is isomorphic to $\mathbb{Z}/p^r\mathbb{Z}$, and is generated by the adjoint of the mod-p^r Whitehead product $[\nu, \mu]$.

The adjoint of $[\nu, \mu]$ is equivalently described as the mod-p^r Samelson product $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$, where $\widetilde{\nu}$ and $\widetilde{\mu}$ are the adjoints of ν and μ respectively. Thus $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$ generates $[P^{4k-1}(p^r), \Omega P^{2k+1}(p^r)] \cong \mathbb{Z}/p^r\mathbb{Z}$. In particular, the map γ in (16) must be some multiple of $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$. To see which multiple, we look at homology. In degree $4k-1$ in mod-p homology, the mod-p^r Samelson product $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$ has image $\langle v_{2k}, u_{2k-1} \rangle$. Recall that the composite $\gamma\colon P^{4k-1}(p^r) \to M \to \Omega P^{2k+1}(p^r)$ has the same image in homology. Thus, as γ is a multiple of $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$, we must have $\gamma \simeq u \cdot \langle \widetilde{\nu}, \widetilde{\mu} \rangle$ for some unit u in $\mathbb{Z}/p^r\mathbb{Z}$.

Consequently, as γ factors through the homotopy fibre of θ, the map $\langle \widetilde{\nu}, \widetilde{\mu} \rangle$ also factors through the homotopy fibre of θ. Therefore the composite $P^{4k-1}(p^r) \xrightarrow{\langle \widetilde{\nu}, \widetilde{\mu} \rangle} \Omega P^{2k+1}(p^r) \to K \xrightarrow{\delta} X$ is null-homotopic. So $\Omega\xi \circ \langle \widetilde{\nu}, \widetilde{\mu} \rangle$ lifts to the homotopy fibre of δ, which is S^3. By Lemma 4.2, if $k \notin \{\frac{1}{2}(p+1), p\}$ then $[P^{4k-1}(p^r), S^3] \cong 0$, implying that $\Omega\xi \circ \langle \widetilde{\nu}, \widetilde{\mu} \rangle$ is null-homotopic. Taking adjoints, this is equivalent to

saying that $\xi\circ[\nu,\mu]$ is null-homotopic. Summarizing, we have shown that if $k\notin\{\frac{1}{2}(p+1),p-1,p\}$ then $\xi\circ[\nu,\mu]$ is null-homotopic, as asserted. □

The case when $k=\frac{1}{2}(p+1)$ can be recovered using a special argument.

Lemma 4.5 *Let $p\geq 5$. If $k=\frac{1}{2}(p+1)$ and $r\in\mathcal{V}_k$ then the composite $P^{4k}(p^r)\xrightarrow{[\nu,\mu]}P^{2k+1}(p^r)\to BK$ is null-homotopic.*

Proof The potential obstruction in the case when $k=\frac{1}{2}(p+1)$ in the proof of Lemma 4.4 arose from the composite $P^{2p+1}(p^r)\xrightarrow{\langle\widetilde{\nu},\widetilde{\mu}\rangle}\Omega P^{2k+1}(p^r)\xrightarrow{\Omega\xi}K$ lifting to a map $P^{2p+1}(p^r)\to S^3$ which could be an extension of the homotopy class $\alpha_1\colon S^{2p}\to S^3$ that generates $\pi_{2p}(S^3)\cong\mathbb{Z}/p\mathbb{Z}$. Assume that this occurs. Taking adjoints, we obtain a homotopy commutative diagram

(19)
$$\begin{array}{ccc} & & P^{2p+2}(p^r)\\ & \swarrow^{\overline{\alpha}_1} & \downarrow{\scriptstyle [\nu,\mu]}\\ & & P^{2k+1}(p^r)\\ & & \downarrow{\scriptstyle \xi}\\ BS^3 & \longrightarrow & BK \end{array}$$

where $\overline{\alpha}_1$ is an extension of the adjoint of α_1. Observe that $\Sigma[\nu,\mu]$ is null-homotopic since the suspension of any mod-p^r Whitehead product is null-homotopic. Therefore if we restrict (19) to S^{2p+1} and suspend we obtain an extension

$$\begin{array}{ccccc} S^{2p+2} & \xrightarrow{\alpha_1} & S^5 & \longrightarrow & A\\ & & \downarrow & & \vdots\\ & & & & \downarrow{\scriptstyle \zeta}\\ & & \Sigma BS^3 & \longrightarrow & \Sigma BK \end{array}$$

for some map ζ, where A is the homotopy cofibre of α_1. The class α_1 is detected in mod-p cohomology by the Steenrod operation $\mathcal{P}^1$, so the two-cell complex A has its bottom cell attached to its top cell by $\mathcal{P}^1$. The homotopy commutativity of the square in the preceding diagram implies that ζ^* is an isomorphism in degree 5. Therefore, as $\mathcal{P}^1$ is nonzero on $H^5(A)$, it must also be nonzero on $H^5(\Sigma BK)$. By stability, this implies that it is nontrivial in $H^4(BK)$. As the generator of $H^4(BK)$ is the transgression of the generator in $H^3(K)$ in the cohomology Serre spectral sequence for the path-loop fibration $K\to *\to BK$, and as Steenrod operations commute with the transgression, we obtain that $\mathcal{P}^1$ is nontrivial on $H^3(K)$.

On the other hand, as $k = \frac{1}{2}(p+1)$, we have $H_*(K) \cong \Lambda(z_3, y_p) \otimes \mathbb{Z}/p\mathbb{Z}[x_{p+1}]$, and as this coalgebra is primitively generated, we can dualize to obtain an algebra isomorphism $H^*(K) \cong \Lambda(\overline{z}_3, \overline{y}_p) \otimes \Gamma[\overline{x}_{p+1}]$, where $\overline{z}_3$, $\overline{y}_p$ and $\overline{x}_{p+1}$ are dual to z_3, y_p and x_{p+1} and $\Gamma[-]$ is the divided power algebra. The only element in degree $2p+1$ in this algebra is $y_p \cup x_{2p+1}$. The nontriviality of $\mathcal{P}^1$ on $H^3(K)$ therefore implies that $\mathcal{P}^1(z_3) = u \cdot (y_p \cup x_{p+1})$ for some unit $u \in \mathbb{Z}/p\mathbb{Z}$. But $\mathcal{P}^1$ sends primitives to primitives, giving a contradiction. Thus it cannot have been the case that $\Omega\xi \circ \langle \widetilde{\nu}, \widetilde{\mu} \rangle$ lifted nontrivially to S^3. Thus $\Omega\xi \circ \langle \widetilde{\nu}, \widetilde{\mu} \rangle$ is null-homotopic, as required. □

Let $k < p-1$. Start with the map $f\colon P^{2k+1}(p^r) \to BK$ in Lemma 4.3. By Lemmas 4.4 and 4.5, the hypotheses of Theorem 3.7 are satisfied. Therefore there is a map $\Omega T^{2k+1}(p^r) \to \Omega K$ whose restriction to the bottom Moore space is the double adjoint of f. The composite

$$g\colon \Omega T^{2k+1}(p^r) \to \Omega K \to \Omega X$$

therefore induces an isomorphism in the least nonvanishing degree in homology. We claim that this is enough to show that g induces an isomorphism in homology in all degrees, and so is a homotopy equivalence. This is an atomicity-style argument, and requires a preliminary lemma.

Lemma 4.6 *There is an abstract isomorphism of vector spaces* $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$.

Proof Since $H_*(T^{2k+1}(p^r)) \cong H_*(X)$ as coalgebras, there is an induced isomorphism between cobar constructions. This implies that there is an isomorphism between the outputs of the homology Eilenberg–Moore spectral sequences for the path-loop fibrations $\Omega A \to * \to A$ and $\Omega B \to * \to B$ which converge to $H_*(\Omega A)$ and $H_*(\Omega B)$. That is, there are coalgebra isomorphisms between the associated graded modules $E^0(H_*(\Omega T^{2k+1}(p^r)))$ and $E^0(H_*(\Omega X))$. As we are taking homology with coefficients in a field, this implies that there is a vector space isomorphism $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$. (A coalgebra isomorphism would require resolving potential extension problems, which we do not address.) □

Proposition 4.7 *If $k \geq 2$ then the composite $g\colon \Omega T^{2k+1}(p^r) \to \Omega K \to \Omega X$ is a homotopy equivalence.*

Proof In general, suppose that Y is a simply connected space and there is a self-map $e\colon Y \to Y$. In [17, Lemma 2.2] it is shown that if x is an element of least nontrivial degree in the kernel of e_* then x is (i) primitive, (ii) annihilated by all dual Steenrod operations and higher Bocksteins, and (iii) in the image of the Hurewicz homomorphism or the mod-p Hurewicz homomorphism. Let $HMH(Y)$ be the submodule of $H_*(Y)$ that consists of elements satisfying (i), (ii) and (iii). The argument in [17, Lemma 2.2] did not really require a self-map of spaces, but only a map $e'\colon Y \to Z$, where $H_*(Y)$ is known as a coalgebra over the Steenrod algebra, and $H_*(Z)$ is abstractly isomorphic to $H_*(Y)$ as vector spaces (in order to show at the appropriate moment that an injection $H_m(Y) \to H_m(Z)$ is an isomorphism). This fits our case as we have a map $\Omega T^{2k+1}(p^r) \xrightarrow{g} \Omega X$ and, by Lemma 4.6, there is an abstract isomorphism of vector spaces $H_*(\Omega T^{2k+1}(p^r)) \cong H_*(\Omega X)$.

Let $k > 2$. Instead of determining $HMH(\Omega T^{2k+1}(p^r))$ directly, we follow [17] by making use of the calculation

$$HMH(\Omega^2 T^{2k+1}(p^r)) = \{a_{2k-3}\}$$

for $k > 2$, where a_{2k-3} is a generator of $H_{2k-3}(\Omega^2 T^{2k+1}(p^r)) \cong \mathbb{Z}/p\mathbb{Z}$, the least-dimensional nontrivial homology group. Let $x \in HMH(\Omega T^{2k+1}(p^r))$ and suppose x is of degree m. As x is in the image of the Hurewicz homomorphism there is a map $h\colon S^m \to \Omega T^{2k+1}(p^r)$ such that $h_*(\iota_m) = x$, where $\iota_m \in H_m(S^m)$ represents a generator. Let $\widetilde{h}\colon S^{m-1} \to \Omega^2 T^{2k+1}(p^r)$ be the adjoint of h. We claim that $\widetilde{h}(\iota_{m-1}) \in HMH(\Omega^2 T^{2k+1}(p^r))$. By definition, $\widetilde{h}_*(\iota_{m-1})$ is in the image of the Hurewicz homomorphism (although we have not yet checked if it is nonzero), and as it is a Hurewicz image, it is also primitive and is annihilated by all dual Steenrod operations. It remains to show that $\widetilde{h}(\iota_{m-1})$ is nonzero. Consider the composite $S^m \xrightarrow{\Sigma\widetilde{h}} \Sigma\Omega^2 T^{2k+1}(p^r) \xrightarrow{\mathrm{ev}} \Omega T^{2k+1}(p^r)$, where ev is the canonical evaluation map. This composite is the adjoint of $\widetilde{h}$, which is the map h. Therefore, as $h_*(\iota_m)$ is nonzero so is $(\Sigma\widetilde{h})_*(\iota_m)$, and hence so is $\widetilde{h}_*(\iota_{m-1})$. The calculation $HMH(\Omega^2 T^{2k+1}(p^r)) = \{a_{2k-3}\}$ therefore implies that $HMH(\Omega T^{2k+1}(p^r)) = \{\overline{a}_{2k-2}\}$, where $\overline{a}_{2k-2}$ transgresses to a_{2k-3} in the Serre spectral sequence for the path-loop fibration.

Thus, for the map $\Omega T^{2k+1}(p^r) \xrightarrow{g} \Omega X$, if $x \in \mathrm{Ker}\,(g_*)$ is a nontrivial element of least degree it must be a multiple of $\overline{a}_{2k-2}$. But we have shown that g_* is an isomorphism in degree $2k-2$. Therefore $\mathrm{Ker}\,(g_*) = 0$ and so g_* is an injection. Since $H_*(\Omega X)$ is isomorphic to $H_*(\Omega T^{2k+1}(p^r))$ as vector spaces, they have identical Euler–Poincaré

series, so g_* being an injection implies that it is an isomorphism. Hence g is a homotopy equivalence by Whitehead's theorem.

The $k = 2$ case is different in that $HMH(\Omega^2 T^5(p^r))$ may not be just $\{a_1\}$. This was dealt with separately in [17, Proof of Theorem 1.1], where it was noted that $HMH(\Omega T^5(p^r)) = \{\overline{a}_2\}$. The rest of the argument in the present proof now goes through as before. □

Finally, we prove the second main statement in the paper.

Proof of Theorem 1.2 The p–local homotopy equivalence for ΩX is given by Proposition 4.7. The p–local homotopy decomposition for ΩK now follows since g factors through $\Omega K \xrightarrow{\Omega\delta} \Omega X$. □

References

[1] **D J Anick**, *Single loop space decompositions*, Trans. Amer. Math. Soc. 334 (1992) 929–940 MR

[2] **D Anick**, *Differential algebras in topology*, Research Notes in Mathematics 3, A K Peters, Wellesley, MA (1993) MR

[3] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. 110 (1979) 549–565 MR

[4] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Torsion in homotopy groups*, Ann. of Math. 109 (1979) 121–168 MR

[5] **F R Cohen**, **J C Moore**, **J A Neisendorfer**, *Exponents in homotopy theory*, from "Algebraic topology and algebraic K–theory" (W Browder, editor), Ann. of Math. Stud. 113, Princeton Univ. Press (1987) 3–34 MR

[6] **F R Cohen**, **J Wu**, *A remark on the homotopy groups of* $\Sigma^n \mathbb{R}\mathrm{P}^2$, from "The Čech centennial" (M Cenkl, H Miller, editors), Contemp. Math. 181, Amer. Math. Soc., Providence, RI (1995) 65–81 MR

[7] **T Ganea**, *A generalization of the homology and homotopy suspension*, Comment. Math. Helv. 39 (1965) 295–322 MR

[8] **B Gray**, *Abelian properties of Anick spaces*, Mem. Amer. Math. Soc. 1162, Amer. Math. Soc., Providence, RI (2017) MR

[9] **B Gray**, **S Theriault**, *An elementary construction of Anick's fibration*, Geom. Topol. 14 (2010) 243–275 MR

[10] **N R Kitchloo**, *Topology of Kac–Moody groups*, PhD thesis, Massachusetts Institute of Technology (1998) MR Available at `http://hdl.handle.net/1721.1/47466`

[11] **N Kitchloo**, *On the topology of Kac–Moody groups*, Math. Z. 276 (2014) 727–756 MR

[12] **J Neisendorfer**, *Primary homotopy theory*, Mem. Amer. Math. Soc. 232, Amer. Math. Soc., Providence, RI (1980) MR

[13] **J A Neisendorfer**, *The exponent of a Moore space*, from "Algebraic topology and algebraic K–theory" (W Browder, editor), Ann. of Math. Stud. 113, Princeton Univ. Press (1987) 35–71 MR

[14] **J Neisendorfer**, *Product decompositions of the double loops on odd primary Moore spaces*, Topology 38 (1999) 1293–1311 MR

[15] **J Neisendorfer**, *Odd primary exponents of Moore spaces*, preprint (2017) Available at `https://web.math.rochester.edu/people/faculty/jnei/mooreexp.pdf`

[16] **P Selick**, *Odd primary torsion in $\pi_k(S^3)$*, Topology 17 (1978) 407–412 MR

[17] **S Theriault**, *Atomicity for Anick's spaces*, J. Pure Appl. Algebra 219 (2015) 2346–2358 MR

[18] **S Theriault**, **J Wu**, *Relative homotopy abelian H–spaces*, Manuscripta Math. (online publication August 2018)

[19] **H Toda**, *On iterated suspensions, I*, J. Math. Kyoto Univ. 5 (1966) 87–142 MR

Mathematical Sciences, University of Southampton
Southampton, United Kingdom

Department of Mathematics, National University of Singapore
Singapore

`s.d.theriault@soton.ac.uk, matwuj@nus.edu.sg`

Received: 31 March 2018 Revised: 10 June 2018

msp

Algebraic & Geometric Topology 18 (2018) 4329–4358

Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$

ANNA BELIAKOVA
CHRISTIAN BLANCHET
NATHAN GEER

We construct a Hennings-type logarithmic invariant for restricted quantum $\mathfrak{sl}(2)$ at a $2p^{\text{th}}$ root of unity. This quantum group U is not quasitriangular and hence not ribbon, but factorizable. The invariant is defined for a pair: a 3–manifold M and a colored link L inside M. The link L is split into two parts colored by central elements and by trace classes, or elements in the 0^{th} Hochschild homology of U, respectively. The two main ingredients of our construction are the universal invariant of a string link with values in tensor powers of U, and the *modified trace* introduced by the third author with his collaborators and computed on tensor powers of the regular representation. Our invariant is a colored extension of the logarithmic invariant constructed by Jun Murakami.

57M27; 17B37, 57M25

1 Introduction

In the 1990s, M Hennings [13] came up with a construction of 3–manifold invariants out of a factorizable ribbon Hopf algebra H. In his construction the right integral $\mu \in H^*$ satisfying

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)1_H \quad \text{for all } x \in H \tag{1}$$

plays the role of the Kirby color. If the category of H–modules is semisimple, Hennings recovers the Reshetikhin–Turaev invariant. However in the nonsemisimple case, his invariant vanishes for manifolds with positive first Betti number (see Kerler [15]). A TQFT based on the Hennings invariant was constructed by Kerler [14] and Lyubashenko [19]. It provides a representation of the mapping class group for closed surfaces with one boundary component, but in general, it has weak functoriality and monoidality properties.

More recently, a completely different nonsemisimple TQFT based on the unrolled quantum $\mathfrak{sl}(2)$ was defined by Blanchet, Costantino, Geer and Patureau-Mirand [3]. This construction uses the logarithmic 3–manifold invariant constructed previously

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4329

by Costantino, Geer and Patureau-Mirand (CGP) in [5] by generalizing the Kashaev invariant. One of the most innovative ingredients of the CGP construction is the so-called *modified trace* which contrary to the usual quantum trace does not vanish on the projective modules. Motivated by extension of Kashaev's volume conjecture, Jun Murakami [20] has constructed a logarithmic invariant of knots in 3–manifolds with value in the center of the quantum group. This invariant is obtained from the universal invariant of links in the 3–sphere by evaluating surgery components with the right integral defined in (1).

In this paper we combine the Hennings approach with the modified trace methods of Geer, Kujawa and Patureau-Mirand [10] and Geer, Patureau-Mirand and Virelizier [11] and construct a family of invariants of 3–manifolds with colored links inside. In the case of a knot we recover Murakami's logarithmic invariant. Our construction for links is more general, and its formulation is well suited for TQFT extension (compare De Renzi, Geer and Patureau-Mirand [7]).

Another novelty of our approach is that we study the modified trace as a symmetric function on U rather than on its category of modules and compute this function explicitly. These ideas were further developed by Beliakova, Blanchet and Gainutdinov [2] and led to a conceptual understanding of a modified trace as a shifted integral in the context of finite-dimensional pivotal Hopf algebras. Finally, this paper shows a way to extend link and 3–manifold invariants outside of ribbon categories, giving an explicit example of constructions by Bruguières [4].

Results on modified trace

Let us denote by U the restricted quantum $\mathfrak{sl}(2)$ at a $2\mathfrak{p}^{\text{th}}$ root of unity $q = e^{i\pi/\mathfrak{p}}$, explicitly defined in the next section. The Hopf algebra U does not contain an R–matrix, but only monodromy or double braiding. However, U is a subalgebra of a ribbon Hopf algebra D obtained by adjoining a square root of the generator K. Since D is not factorizable and U is not quasitriangular, neither U nor D supports the Hennings–Kerler–Lyubashenko TQFT construction.

Let U–mod be the category of finite-dimensional U–modules. This is a finite pivotal tensor category. Hence, for any morphism $f\colon V \to V$ in U–mod there is a notion of a categorical (or quantum) left and right traces, denoted by $\mathrm{tr}_l^V(f)$ and $\mathrm{tr}_r^V(f)$, respectively.

Let U–pmod be its full subcategory of projective U–modules. An explicit description of U–pmod is given by Gainutdinov, Semikhatov, Tipunin and Feĭgin [9]. Let us denote

by $\mathcal{P}_j^{\pm}$, with $j = 1, \dots, \mathfrak{p}$, the indecomposable projective modules. Here $\mathcal{P}_{\mathfrak{p}}^{\pm}$ is a simple module with highest weight $\pm q^{\mathfrak{p}-1}$. The module $\mathcal{P}_1^{+}$ is the projective cover of the trivial module. The space of endomorphisms $\mathrm{End}_U(\mathcal{P}_j^{\pm})$, where $1 \le j \le \mathfrak{p}-1$, is two-dimensional with basis given by the identity $\mathrm{id}_{\mathcal{P}_j^{\pm}}$ and a nilpotent endomorphism $x_j^{\pm}$, defined in Section 3 .

The subcategory U–pmod is an ideal of U–mod in the sense of Geer, Kujawa and Patureau-Mirand [10] and Geer, Patureau-Mirand and Virelizier [11]. A *modified trace* on U–pmod is a family of linear functions

$$\{\mathrm{t}_V\colon \mathrm{End}_U V \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

such that the following two conditions hold:

Cyclicity If $X, V \in U$–pmod, then for any morphisms $f\colon V \to X$ and $g\colon X \to V$ in U–mod we have

$$\mathrm{t}_V(gf) = \mathrm{t}_X(fg).$$

Partial trace properties If $X \in U$–pmod and $W \in U$–mod, then for any morphisms $f \in \mathrm{End}_U(X \otimes W)$ and $g \in \mathrm{End}_U(W \otimes X)$ we have

$$\mathrm{t}_{X\otimes W}(f) = \mathrm{t}_X(\mathrm{tr}_r^W(f)) \quad \text{and} \quad \mathrm{t}_{W\otimes X}(g) = \mathrm{t}_X(\mathrm{tr}_l^W(g)),$$

where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined using the pivotal structure in (6).

Our first main result is the following.

Theorem 1 *There exists a unique family of linear functions*

$$\{\mathrm{t}_V\colon \mathrm{End}_U(V) \to \mathbb{C}\}_{V \in U\text{–pmod}},$$

satisfying cyclicity and the partial trace properties, normalized by

$$\mathrm{t}_{\mathcal{P}_{\mathfrak{p}}^{+}}(\mathrm{id}_{\mathcal{P}_{\mathfrak{p}}^{+}}) = (-1)^{\mathfrak{p}-1}.$$

Moreover, $\mathrm{t}_{\mathcal{P}_{\mathfrak{p}}^{-}}(\mathrm{id}_{\mathcal{P}_{\mathfrak{p}}^{-}}) = 1$ *and for* $1 \le j \le \mathfrak{p}-1$ *we have*

$$\mathrm{t}_{\mathcal{P}_j^{\pm}}(\mathrm{id}_{\mathcal{P}_j^{\pm}}) = (\pm 1)^{\mathfrak{p}-1}(-1)^j(q^j + q^{-j}) \quad \textit{and} \quad \mathrm{t}_{\mathcal{P}_j^{\pm}}(x_j^{\pm}) = (\pm 1)^{\mathfrak{p}}(-1)^j[j]^2.$$

This family is called the modified trace on U–pmod. The proof uses the fact that U–mod is unimodular (ie the projective cover of the trivial module is self-dual) and it has a simple projective object. In this case, there exist unique left and right modified

traces on U–pmod by [10, Corollary 3.2.1]. We actually compute these traces on the algebra of endomorphisms of indecomposable projectives and show that they are equal.

Observe that Theorem 1 applies to U, considered as a free left module over itself, called the regular representation, and its tensor powers. Recall from [9] that the regular representation decomposes as

$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^+ \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^-.$$

The algebra $\mathrm{End}_U(U)$ of the U–endomorphisms of U can be identified with U^{op} (ie U with the opposite multiplication). The isomorphism is given by sending an element x of U^{op} to the operator r_x of the right multiplication by x on the regular representation. By definition r_x commutes with the left action.

The space of characters (or symmetric functions; see Arike [1]) on U is defined as

$$\mathrm{Char}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(yx) \text{ for any } x, y \in U\}.$$

This space is dual to the 0^{th} Hochschild homology $\mathrm{HH}_0(U)$, which is

$$\mathrm{HH}_0(U) := \frac{U}{[U,U]}, \quad \text{where } [U,U] = \mathrm{Span}\{xy - yx \mid x, y \in U\}.$$

There is an obvious action of the center $Z(U)$ on $\mathrm{Char}(U)$ by setting $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$.

Modified trace provides us with symmetric linear maps

$$\mathtt{t}_{U^{\otimes m}}\colon \mathfrak{C}_m \to \mathbb{C},$$

where $\mathfrak{C}_m = \mathrm{End}_U(U^{\otimes m})$ for $m \geq 1$ are known as centralizer algebras. For $m = 1$, we obtain a special symmetric function $\mathsf{T} \in \mathrm{Char}(U)$ defined by $\mathsf{T}(x) = \mathtt{t}_U(r_x)$, which we call the *modified trace* of U.

Theorem 2 *The pairing $\langle\ ,\ \rangle\colon Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C}$ defined from the modified trace by $\langle z, u\rangle = \mathsf{T}(zu)$ is nondegenerate.*

The proof is by direct computation of the pairing in the basis of the center and the basis of $\mathrm{HH}_0(U)$, defined in Sections 2 and 4, respectively.

Note that similarly to the space $\mathrm{Char}(U)$ we can define the space of quantum characters

$$\mathrm{qChar}(U) := \{\phi \in U^* \mid \phi(xy) = \phi(S^2(y)x) \text{ for any } x, y \in U\}.$$

Evaluation of quantum characters on central elements defines a pairing

$$\mathrm{qChar}(U) \times Z(U) \to \mathbb{C}.$$

This pairing is degenerate. Indeed, for any $1 \le j \le \mathfrak{p}$ the quantum character $\mathrm{tr}_r^{\mathcal{P}_j^\pm}$ associated with an indecomposable projective $\mathcal{P}_j^\pm$ has zero evaluation on all central elements. However, its evaluation on the pivotal element gives the dimension of the module. This is why quantum characters cannot detect Murakami's invariant while the modified trace does.

Construction of invariants

For any ribbon Hopf algebra there exists a *universal* invariant associated to an oriented framed tangle T. This invariant is obtained by assigning the R–matrix to crossings and evaluation and coevaluation maps to maxima and minima (see Section 4). Although our restricted quantum group U is not ribbon, it has a ribbon extension D, which produces a universal invariant $J_T \in D^{\otimes m}$ for any tangle T with m components. If T is an upwards oriented string link, we argue that J_T actually belongs to the subspace $(U^{\otimes m})^U \subset U^{\otimes m}$ of invariants under left action.

We are now ready to state our main result. Let $\mu \in U^*$ be the right integral of U. It is unique up to a normalization, which we fix in Section 2. For $m \ge 1$, the function $\mathrm{t}_{U^{\otimes m}}$ defines a bilinear pairing $\langle\ ,\ \rangle\colon (U^{\otimes m})^U \times U^{\otimes m} \to \mathbb{C}$ by

$$\langle z, x\rangle = \mathrm{t}_{U^{\otimes m}}(l_z r_x).$$

Here l_z and r_x are the left and right multiplications, respectively. This bilinear pairing factorizes on the right through $\mathrm{HH}_0(U^{\otimes m})$.

Assume that a closed 3–manifold M with an (m_+, m_-)–component oriented framed link (L^+, L^-) inside is represented by surgery in S^3 along the m_0–component oriented framed link L^0. Let us color the components of L^+ (resp. L^-) by central elements $z_j \in Z(U)$ for $1 \le j \le m_+$ (resp. by trace classes $h_k \in \mathrm{HH}_0(U)$ for $1 \le k \le m_-$). Let $T = T^+ \cup T_0 \cup T^-$ be an upward oriented string link in S^3 obtained by opening all $m = m_+ + m_0 + m_-$ components. Let s be the signature of the linking matrix for L^0 and $\delta := \frac{1}{\sqrt{2}}(1-i)q^{\frac{1}{2}(3-\mathfrak{p}^2)}$. We set $z^+ = \bigotimes_j z_j$ $\big($resp. $h^- = \bigotimes_k h_k\big)$, and denote by L the colored link $((L^+, z^+), (L^-, h^-))$. Note that to define z^+ and h^- as well as J_T, we need to fix an order on the components of L; change of this order will result in an obvious permutation of the entries. For each j, the central element z_j acting on the right integral μ defines the quantum character $\mu_{z_j}\colon x \mapsto \mu(z_j x)$. We

denote the tensor product $\bigotimes_j \mu_{z_j}$ by $z^+\mu^{\otimes m_+}$. We define a number associated to the pair (M, L) as follows.

Theorem 3 *With the notation as above,*

$$\mathrm{H}^{\log}(M, L) := \delta^s \langle (z^+\mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id})(J_T),\ h^- \rangle$$

is a topological invariant of the pair (M, L).

When L^- is empty, the colored Hennings invariants [13] are recovered. When L^- is a knot, the collection of our invariants for all colors of L^- is equivalent to the Murakami center-valued logarithmic invariant (see Proposition 11). Thus $\mathrm{H}^{\log}$ can be understood as a colored extension of the Murakami invariants. An action of the modular group $\mathrm{SL}(2, \mathbb{Z})$ on the center $Z(U)$ was studied by Feigin, Gainutdinov, Semikhatov and Tipunin [8]. We expect that $\mathrm{H}^{\log}$ can be used to extend these mapping class group representations in genus one to refined TQFTs with full functorial and monoidal properties.

The paper is organized as follows. In Section 2 we define the restricted quantum group U and its braided extension D. In Section 3 we discuss their categories of finite-dimensional modules. The universal tangle invariant is constructed in Section 4, where we also compute a basis for the space of trace classes $\mathrm{HH}_0(U)$. Our main theorems are proved in the two last sections.

Acknowledgements The authors are grateful to Azat Gainutdinov and Thomas Kerler for helpful discussions. The work was supported by NCCR SwissMAP, founded by the Swiss National Science Foundation. Geer was partially supported by NSF grants DMS-1308196 and MS-1452093. He also would like to thank CNRS for its support.

2 Restricted quantum $\mathfrak{sl}(2)$ and its braided extension

Definition of U

Fix an integer $\mathfrak{p} \geq 2$ and let $q = e^{i\pi/\mathfrak{p}}$ be a $2\mathfrak{p}^{\text{th}}$ root of unity. Let $U = \overline{U}_q(\mathfrak{sl}(2))$ be the $\mathbb{C}$–algebra given by generators E, F, K, K^{-1} and relations

$$E^{\mathfrak{p}} = F^{\mathfrak{p}} = 0, \qquad K^{2\mathfrak{p}} = 1, \qquad KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The algebra U is a Hopf algebra where the coproduct, counit and antipode are defined by

$$\begin{aligned}
\Delta(E) &= 1\otimes E + E\otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1},\\
\Delta(F) &= K^{-1}\otimes F + F\otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF,\\
\Delta(K) &= K\otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1},\\
\Delta(K^{-1}) &= K^{-1}\otimes K^{-1}, & \varepsilon(K^{-1}) &= 1, & S(K^{-1}) &= K.
\end{aligned}$$

In what follows we will use Sweedler notation. For $x\in U$ we write

$$\Delta(x)=\sum x_{(1)}\otimes x_{(2)} \qquad \text{and} \qquad \Delta^{[n]}(x)=\sum x_{(1)}\otimes x_{(2)}\otimes\cdots\otimes x_{(n)} \quad \text{for } n\geq 1.$$

The center of U

The dimension of the center $Z(U)$ is $3\mathsf{p}-1$. A basis consists of $\mathsf{p}+1$ central idempotents $\boldsymbol{e}_j$ $(0\leq j\leq \mathsf{p})$ and $2\mathsf{p}-2$ elements $\boldsymbol{w}_j^{\pm}$ $(1\leq j\leq \mathsf{p}-1)$ in the radical [8]. These elements satisfy the following relations:

$$\begin{aligned}
\boldsymbol{e}_s\boldsymbol{e}_t &= \delta_{s,t}\boldsymbol{e}_s && \text{for } 0\leq s,t\leq \mathsf{p},\\
\boldsymbol{e}_s\boldsymbol{w}_t^{\pm} &= \delta_{s,t}\boldsymbol{w}_t^{\pm} && \text{for } 0\leq s\leq \mathsf{p} \text{ and } 1\leq t\leq \mathsf{p}-1,\\
\boldsymbol{w}_s^{\pm}\boldsymbol{w}_t^{\pm} &= \boldsymbol{w}_s^{\pm}\boldsymbol{w}_t^{\mp}=0 && \text{for } 1\leq s,t\leq \mathsf{p}-1.
\end{aligned}$$

Braided extension

The Hopf algebra U is not braided; see [16]. However, it can be realized as a Hopf subalgebra of the following braided Hopf algebra. Let D be the Hopf algebra generated by e, ϕ, k and k^{-1} with the relations

$$e^{\mathsf{p}}=\phi^{\mathsf{p}}=0, \qquad k^{4\mathsf{p}}=1, \qquad kk^{-1}=k^{-1}k=1,$$
$$kek^{-1}=qe, \quad k\phi k^{-1}=q^{-1}\phi, \quad [e,\phi]=\frac{k^2-k^{-2}}{q-q^{-1}},$$

and Hopf algebra structure

$$\begin{aligned}
\Delta(e) &= 1\otimes e + e\otimes k^2, & \varepsilon(e) &= 0, & S(e) &= -ek^{-2},\\
\Delta(\phi) &= k^{-2}\otimes\phi + \phi\otimes 1, & \varepsilon(\phi) &= 0, & S(\phi) &= -k^2\phi,\\
\Delta(k) &= k\otimes k, & \varepsilon(k) &= 1, & S(k) &= k^{-1},\\
\Delta(k^{-1}) &= k^{-1}\otimes k^{-1}, & \varepsilon(k^{-1}) &= 1, & S(k^{-1}) &= k.
\end{aligned}$$

The Hopf algebra D has two special invertible elements: the R–matrix

$$R = \frac{1}{4\mathsf{p}} \sum_{m=0}^{\mathsf{p}-1} \sum_{n,j=0}^{4\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{m(m-1)/2+m(n-j)-nj/2} e^m k^n \otimes \phi^m k^j$$

and the ribbon element

$$r = \frac{1-i}{2\sqrt{\mathsf{p}}} \sum_{m=0}^{\mathsf{p}-1} \sum_{j=0}^{2\mathsf{p}-1} \frac{(q-q^{-1})^m}{[m]!} q^{-m/2+mj+(j+p+1)^2/2} \phi^m e^m k^{2j},$$

where $q^{\frac{1}{2}} = e^{\frac{1}{2}i\pi/\mathsf{p}}$, $[m] = (q^m - q^{-m})/(q-q^{-1})$ and $[m]! = [m][m-1]\cdots[1]$. The following theorem is well known; see [8].

Theorem 4 *The triple (D, R, r) is a ribbon Hopf algebra.*

Let us call $M = R_{21}R$ the double braiding or *monodromy*, where $R_{21} = \sum_i \beta_i \otimes \alpha_i$ with $R = \sum_i \alpha_i \otimes \beta_i$. A Hopf algebra A is called *factorizable* if its monodromy matrix can be written as

$$M = \sum_i m_i \otimes n_i,$$

where m_i and n_i are two bases of A. The Hopf algebra D is not factorizable. There is a Hopf algebra embedding $U \to D$ given by

$$E \mapsto e, \quad F \mapsto \phi \quad \text{and} \quad K \mapsto k^2.$$

It is easy to check that $r \in U$, and the monodromy $M = R_{21}R$ is in $U \otimes U$. Moreover, U is factorizable.

Ribbon and balancing elements of U

Let $u = \sum_i S(\beta_i)\alpha_i$ be the canonical element implementing the inner automorphism S^2, ie $S^2(x) = uxu^{-1}$ for any $x \in D$, and satisfying

$$\Delta(u) = M^{-1}(u \otimes u).$$

Using the formula for the R–matrix, it is easy to check that $u \in U$. The ribbon element $r \in U$ is central and invertible, so that

$$r^2 = uS(u), \quad S(r) = r, \quad \varepsilon(r) = 1, \quad \Delta(r) = M^{-1}(r \otimes r). \tag{2}$$

We then define

$$g := r^{-1}u = k^{2\mathsf{p}+2} = K^{\mathsf{p}+1} \in U,$$

the *balancing* element. This element is grouplike, ie

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \quad \text{and} \quad gxg^{-1} = S^2(x) \tag{3}$$

for any $x \in U$. The balancing element will be used to define the pivotal structure.

Remark As was shown by Drinfeld, the equations (3) determine g^2 only. Hence $g' = K$ is another balancing element, which will not be considered in this paper.

Right integral

Recall that a *right integral* $\mu \in U^*$ is defined by the system of equations

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)1_U \quad \text{for all } x \in U.$$

For any finite-dimensional Hopf algebra over a field of characteristic zero, there is a unique solution to these equations up to a scalar. In our case in the PBW basis, it is given by the formula

$$\mu(E^m F^n K^l) = \zeta \delta_{m,\mathsf{p}-1}\delta_{n,\mathsf{p}-1}\delta_{l,\mathsf{p}+1}.$$

We fix a normalization as in [20] by setting

$$\zeta = -\sqrt{\frac{2}{\mathsf{p}}}([\mathsf{p}-1]!)^2.$$

The evaluation on the ribbon element and its inverse are given by

$$\mu(r) = \frac{1-i}{\sqrt{2}} q^{\frac{1}{2}(3-\mathsf{p}^2)} = \frac{1}{\mu(r^{-1})} = \delta.$$

Dually, a left (resp. right) *cointegral* is an element $\boldsymbol{c} \in U$ such that $x\boldsymbol{c} = \varepsilon(x)\boldsymbol{c}$ (resp. $\boldsymbol{c}x = \varepsilon(x)\boldsymbol{c}$) for all $x \in U$. Nontrivial left and right cointegrals exist and are unique up to scalar. A Hopf algebra is *unimodular* if its right cointegral is also left. Our Hopf algebra U is unimodular with cointegral

$$\boldsymbol{c} = F^{\mathsf{p}-1}E^{\mathsf{p}-1}\sum_{j=0}^{2\mathsf{p}-1} K^j.$$

The integral μ belongs to the space of quantum characters $\mathrm{qChar}(U)$. This is a general fact for a finite-dimensional unimodular Hopf algebra [22, Theorem 3]. Here it's enough to check the defining relation when x is in the PBW basis and y is a generator.

The center $Z(U)$ acts on $\mathrm{qChar}(U)$ by $z\phi(x) := \phi(zx)$ for any $z \in Z(U)$ and $x \in U$. Under this action $\mathrm{qChar}(U)$ is a free module of dimension one with basis given by the right integral μ. Hence, as a vector space $\mathrm{qChar}(U)$ has dimension $3\mathfrak{p}-1$.

The space of quantum characters $\mathrm{qChar}(U)$ is naturally isomorphic to the space of characters. Indeed, we can define the map

$$Q\colon \mathrm{qChar}(U) \to \mathrm{Char}(U) \quad \text{by sending } \phi \mapsto \phi_g, \tag{4}$$

where $\phi_g(x) := \phi(gx)$ and g is the balancing element. Cyclicity can be verified as follows:

$$\phi_g(xy) = \phi(gxy) = \phi(S^2(y)gx) = \phi(gyx) = \phi_g(yx).$$

The inverse map is given by sending $\psi \in \mathrm{Char}(U)$ to $\psi_{g^{-1}} \in \mathrm{qChar}(U)$. Hence Q is an isomorphism. We get that $\mathrm{HH}_0(U)$ has dimension $3\mathfrak{p}-1$ since its dual vector space is isomorphic to $\mathrm{Char}(U)$.

3 Categories of modules

In this section we will study the $\mathbb{C}$–linear categories of finite-dimensional modules over D and U, which we denote by D–mod and U–mod, respectively.

Category D–mod

The category D–mod is ribbon, with the usual braiding

$$c_{V,W}\colon V \otimes W \to W \otimes V, \quad \text{given by } u \otimes v \mapsto \tau R(u \otimes v),$$

where $\tau(x \otimes y) = y \otimes x$, with self-dual twist

$$\theta_V\colon V \to V, \quad \text{given by } v \mapsto r^{-1}v,$$

and with compatible duality

$$\begin{aligned}
\mathrm{coev}_V&\colon \mathbb{C} \to V \otimes V^*, && \text{given by } 1 \mapsto \textstyle\sum_i v_i \otimes v_i^*,\\
\mathrm{ev}_V&\colon V^* \otimes V \to \mathbb{C}, && \text{given by } f \otimes v \mapsto f(v),\\
\widetilde{\mathrm{coev}}_V&\colon \mathbb{C} \to V^* \otimes V, && \text{given by } 1 \mapsto \textstyle\sum_i v_i^* \otimes g^{-1}v_i,\\
\widetilde{\mathrm{ev}}_V&\colon V \otimes V^* \to \mathbb{C}, && \text{given by } v \otimes f \mapsto f(gv),
\end{aligned} \tag{5}$$

where g is the balancing element. The ribbon structure implies that the twist is *self-dual*, ie $\theta_{V^*} = \theta_V^*$.

The duality morphisms (5) define the pivotal structure on D–mod (see eg [11]). In particular, in the pivotal setting, one can define left and right (categorical) traces of any endomorphism $f\colon V \to V$ as

$$\mathrm{tr}_l(f) = \mathrm{ev}_V(\mathrm{id}_{V^*} \otimes f)\,\widetilde{\mathrm{coev}}_V \quad \text{and} \quad \mathrm{tr}_r(f) = \widetilde{\mathrm{ev}}_V(f \otimes \mathrm{id}_{V^*})\,\mathrm{coev}_V$$

and dimensions of objects.

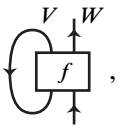
A *spherical* category is a pivotal category whose left and right traces are equal, ie $\mathrm{tr}_l(f) = \mathrm{tr}_r(f)$ for any endomorphism f. It is easy to see that any ribbon category is spherical. We call $\mathrm{tr}_l(\mathrm{id}_V) = \mathrm{tr}_r(\mathrm{id}_V)$ the *quantum* dimension of V.

We will use a standard graphical calculus to represent morphisms in D–mod by diagrams in the plane which are read from the bottom to the top, with the convention that an X–colored vertical arc oriented upwards represents the identity id_X.

In what follows, we will need *partial* categorical traces of endomorphisms. Given $V, W \in D$–mod and $f\colon V \otimes W \to V \otimes W$, let tr_l^V and tr_r^W be the left and right *partial* traces defined as follows:

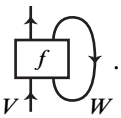
(6)

$$\mathrm{tr}_l^V(f) = (\mathrm{ev}_V \otimes \mathrm{id}_W)(\mathrm{id}_{V^*} \otimes f)(\widetilde{\mathrm{coev}}_V \otimes \mathrm{id}_W) =$$



$$,$$

$$\mathrm{tr}_r^W(f) = (\mathrm{id}_V \otimes \widetilde{\mathrm{ev}}_W)(f \otimes \mathrm{id}_{W^*})(\mathrm{id}_V \otimes \mathrm{coev}_W) =$$



$$.$$

Category U–mod

Let us call a module *simple* if its endomorphism ring is one-dimensional. A module is *projective* if it is a direct summand of a free module.

The category U–mod includes the s–dimensional simple modules $\mathscr{X}_s^{\pm}$, and their projective covers $\mathcal{P}_s^{\pm}$ for $1 \le s \le \mathsf{p}$, which are $2\mathsf{p}$–dimensional for $1 \le s < \mathsf{p}$. The simple module $\mathscr{X}_s^{\pm}$ is determined by its highest-weight vector v with the action $Ev = 0$ and $Kv = \pm q^{s-1}v$. It is projective if and only if $s = p$. A category is called *unimodular* if the projective cover of the trivial module is self-dual. It is shown in [18, Section 2.5] that a finite-dimensional Hopf algebra is unimodular if and only if its category of finite-dimensional modules is unimodular, hence the category U–mod is unimodular.

The category U–mod inherits the pivotal structure, twist and double braiding from D–mod. The double braiding is

$$M_{V,W}\colon V\otimes W\to V\otimes W,\quad \text{given by } x\otimes y\mapsto M(x\otimes y),$$

where M is the monodromy matrix; the self-dual twist and duality are given by (5). One can check that U–mod is a *twisted* category with duality in the sense of Bruguières [4].

Let U–pmod be the full subcategory of U–mod consisting of projective modules. This category is nonabelian. To compute the modified trace on U–pmod, we will need an explicit structure of this category.

Structure of U–pmod

A module is *indecomposable* if it does not decompose as a direct sum of two modules. The indecomposable projective U–modules are classified up to isomorphism in [9]: they are precisely the projective covers $\mathcal{P}_j^{\pm}$ of the simple modules, where $j=1,\dots,\mathsf{p}$. In particular, $\mathcal{P}_{\mathsf{p}}^{\pm}$ is a simple module with highest weight $\pm q^{\mathsf{p}-1}$. The module $\mathcal{P}_1^{+}$ is the projective cover of the trivial one.

We will recall some facts about these projective modules. We reproduce in the appendix the defining relations for the modules $\mathcal{P}_j^{+}$ and $\mathcal{P}_{\mathsf{p}-j}^{-}$ for $1\le j\le \mathsf{p}-1$, in the form given in [8, Section C.2]. They have respective bases $\boldsymbol{B}^{+}$ and $\boldsymbol{B}^{-}$, given by

$$\boldsymbol{B}^{\pm}=\{\boldsymbol{x}_k^{\pm},\boldsymbol{y}_k^{\pm}\}_{0\le k\le \mathsf{p}-j-1}\cup\{\boldsymbol{a}_n^{\pm},\boldsymbol{b}_n^{\pm}\}_{0\le n\le j-1}. \tag{7}$$

Following [6] we call a weight vector v *dominant* if $(FE)^2v=0$. The vector $\boldsymbol{b}_0^{+}$ (resp. $\boldsymbol{y}_0^{-}$) is a generating dominant vector of $\mathcal{P}_j^{+}$ (resp. $\mathcal{P}_j^{-}$) with weight $\pm q^{j-1}$. Analyzing the possible images for this generating dominant weight vector of $\mathcal{P}_j^{\epsilon}$, where $\epsilon\in\{-,+\}$, we can completely determine the nontrivial morphisms between indecomposable projective modules. Let x_j^{+} (resp. x_j^{-}) be the nilpotent endomorphism of $\mathcal{P}_j^{+}$ (resp. $\mathcal{P}_j^{-}$) determined by $\boldsymbol{b}_0^{+}\mapsto\boldsymbol{a}_0^{+}$ (resp. $\boldsymbol{y}_0^{-}\mapsto\boldsymbol{x}_0^{-}$), and let $a_j^{+},b_j^{+}\colon\mathcal{P}_j^{+}\to\mathcal{P}_{\mathsf{p}-j}^{-}$ and $a_j^{-},b_j^{-}\colon\mathcal{P}_j^{-}\to\mathcal{P}_{\mathsf{p}-j}^{+}$ be the morphisms defined by

$$a_j^{+}(\boldsymbol{b}_0^{+})=\boldsymbol{a}_0^{-},\quad b_j^{+}(\boldsymbol{b}_0^{+})=\boldsymbol{b}_0^{-},\quad a_j^{-}(\boldsymbol{y}_0^{-})=\boldsymbol{x}_0^{+}\quad\text{and}\quad b_j^{-}(\boldsymbol{y}_0^{-})=\boldsymbol{y}_0^{+}.$$

The nontrivial Hom–spaces between indecomposable projective modules are

- the endomorphism ring $\mathrm{End}_U(\mathcal{P}_j^{\pm})$, which is one-dimensional for $j=p$ and two-dimensional with basis $\{\mathrm{id}_{\mathcal{P}_j^{\pm}},x_j^{\pm}\}$ for $1\le j<\mathsf{p}$,
- the Hom–spaces $\mathrm{Hom}_U(\mathcal{P}_j^{+},\mathcal{P}_{\mathsf{p}-j}^{-})$ and $\mathrm{Hom}_U(\mathcal{P}_j^{-},\mathcal{P}_{\mathsf{p}-j}^{+})$, which are two-dimensional, with respective bases $\{a_j^{+},b_j^{+}\}$ and $\{a_j^{-},b_j^{-}\}$, for $1\le j<\mathsf{p}$.

Proposition 5 [8, Proposition 4.4.4] *The action of the center on the indecomposable projective modules is as follows, where* $1 \le j < p$*:*

	$\mathcal{P}_p^-$	$\mathcal{P}_p^+$	$\mathcal{P}_j^+$	$\mathcal{P}_{p-j}^-$
$\boldsymbol{e}_0$	$\mathrm{id}_{\mathcal{P}_p^-}$	0	0	0
$\boldsymbol{e}_p$	0	$\mathrm{id}_{\mathcal{P}_p^+}$	0	0
$\boldsymbol{e}_j$	0	0	$\mathrm{id}_{\mathcal{P}_j^+}$	$\mathrm{id}_{\mathcal{P}_{p-j}^-}$
$\boldsymbol{w}_j^+$	0	0	x_j^+	0
$\boldsymbol{w}_j^-$	0	0	0	x_{p-j}^-

4 Tangle invariants and the trace of U

Our links and tangles are always assumed to be framed and oriented. The diagrams are going from bottom to top. A *string link* is a tangle without closed component whose arcs end at the same order as they start with upwards orientation. A pure braid is an example.

Reshetikhin–Turaev invariant

Given any ribbon category $\mathcal{C}$, there exists a unique ribbon functor

$$F_{\mathcal{C}}\colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C},$$

where $\mathrm{Rib}_{\mathcal{C}}$ is the category of $\mathcal{C}$–colored ribbon graphs, which send an upwards oriented vertical string colored by $V \in \mathcal{C}$ to the identity id_V. This is shown in [24], with opposite orientation convention. Applying this construction to D–mod, for an m–component upwards string link T, colored with the regular representation D, we obtain $F_D(T) \in \mathrm{End}_D(D^{\otimes m})$. Here we use the shorthand F_D for $F_{D\text{–mod}}$.

The Reshetikhin–Turaev invariant of the colored link $(L, V_1, \dots, V_m)$ is obtained by evaluating the categorical right traces on $F_D(T)$, ie

$$\begin{aligned} J_L(V_1, \dots, V_m) &:= (\mathrm{tr}_r^{V_1} \otimes \cdots \otimes \mathrm{tr}_r^{V_m}) F_D(T) \\ &= (\mathrm{trace}^{V_1} \otimes \cdots \otimes \mathrm{trace}^{V_m})(g \otimes \cdots \otimes g) F_D(T), \end{aligned}$$

where T is a string link with braid closure isotopic to L. Here trace^V means the (usual) trace of a linear endomorphism. Recall also that $\mathrm{tr}_r^V = \mathrm{tr}_l^V$ since both closures are isotopic. Hence, we can replace g by g^{-1} in the last line. If one of the V_i is projective, this invariant vanishes.

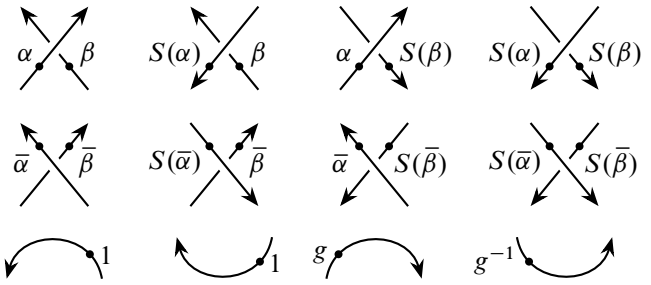


Figure 1: Local formulas for the universal invariant

Universal invariant

Associated with a ribbon Hopf algebra there is another powerful invariant: the *universal* invariant of links and tangles introduced by Lawrence [17] for some quantum groups and defined by Hennings [13] in the general case. For a string link T with m components, its universal invariant J_T is obtained by pasting together pieces shown in Figure 1. Here we write $R = \sum \alpha \otimes \beta$ and $R^{-1} = \sum \bar{\alpha} \otimes \bar{\beta}$.

More precisely, for each arc of T we obtain an element of D by writing a word from right to left with labels read using the order following the orientation. Thus we get $J_T \in D^{\otimes m}$. This element does not change by Reidemeister moves. The original proof of the invariance was stated for a link. The argument was extended to tangles in [12, Section 7.3].

Note that in [12] Habiro uses different conventions. His tangles are depicted from top to bottom and orientations are reversed. Hence, our model can be recovered from his after reflecting over a horizontal axis. The universal invariants coincide. In [21], Ohtsuki defines the universal invariant using orientation opposite to our convention, but the word is written there from left to right when following the orientation. Again the universal invariant is finally the same as ours.

Relation between them

The universal invariant is known to dominate Reshetikhin–Turaev invariants in the following sense (the proof is given in [21, Theorem 4.9]):

$$J_L(V_1, \dots, V_m) = (\mathrm{tr}_r^{V_1} \otimes \cdots \otimes \mathrm{tr}_r^{V_m})(J_T).$$

Let us denote by $(U^{\otimes m})^U \subset U^{\otimes m}$ the submodule centralizing the left action, ie

$$x \in (U^{\otimes m})^U \qquad \Longleftrightarrow \qquad \Delta^{[m]}(h)x = x\Delta^{[m]}(h) \quad \text{for all } h \in U.$$

The following lemma is folklore, but we are adding the proof for completeness.

Lemma 6 *The Reshetikhin–Turaev intertwiner* $F_D(T)$ *is equal to left multiplication by* J_T. *In addition, for an* m*–component string link* T, *its universal invariant* J_T *belongs to* $(U^{\otimes m})^U$.

Proof The fact that $F_D(T)$ is the left multiplication by J_T follows directly by comparing the definitions of these two invariants. More details are given in the proof of [21, Theorem 4.9]. Hence, multiplication by J_T has to commute with left action, and we conclude $J_T \in (D^{\otimes m})^D$.

Let us show that for a string link the universal invariant J_T actually belongs to $U^{\otimes m}$. This implies the claim, since $(U^{\otimes m})^D \subset (U^{\otimes m})^U$.

The linking matrix of a string link is diagonal mod 2. In [4, Section 1.3], Bruguières shows that *any* tangle with this property can be obtained as compositions and tensor products of evaluations, coevaluations and twists. Thus the universal invariant J_T is built up from the ribbon element r, the balancing element g, and their inverses by applying the Hopf algebra operations. We obtain $J_T \in U^{\otimes m}$. □

Evaluations of the universal invariant

Assume for simplicity that T is a $(1,1)$–tangle whose closure is the knot K. Then for any $\phi \in \mathrm{qChar}(U)$, the evaluation $\phi(J_T) \in \mathbb{C}$ is a knot invariant. To prove this fact, we need to show that this evaluation does not change by cyclic permutations of the word $J_K = g^{-1} J_T$ obtained by applying the algorithm described above to the left closure of T. Using (4), we get

$$\phi(J_T) = \phi_g(g^{-1} J_T) = \phi_g(J_K).$$

The last expression does not change by cyclic permutations since $\phi_g \in \mathrm{Char}(U)$. Applying this argument to the k leftmost components of a string link with m strands, we will get the following.

Lemma 7 *Let* T *be an* m*–component string link and* $1 \le k \le m$. *Let* $\Phi = \bigotimes_{j=1}^{k} \phi_j$, *with* $\phi_j \in \mathrm{qChar}(U)$ *a sequence of quantum characters. Then*

$$(\Phi \otimes \mathrm{id}_{U^{\otimes(m-k)}})(J_T) \in (U^{\otimes(m-k)})^U$$

is an invariant of the tangle obtained from T *by closing the first* k *components.*

Further observe that given $\phi \in \mathrm{qChar}(U)$ we can obtain another quantum character ϕ_z by twisting ϕ with a central element $z \in Z(U)$, where $\phi_z(x) := \phi(zx)$ for any $x \in U$.

In the next section we will define a family of linear functions

$$\{\mathtt{t}_V \colon \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V \in U\text{–pmod}}$$

satisfying cyclicity and the partial trace property. For any $m > 0$, we can use $\mathtt{t}_{U^{\otimes m}}$ to define a bilinear pairing

$$\langle\ ,\ \rangle \colon (U^{\otimes m})^U \otimes U^{\otimes m} \to \mathbb{C},$$

by the formula

$$\langle x, y\rangle = \mathtt{t}_{U^{\otimes m}}(l_x \circ r_y). \tag{8}$$

Here l_x and r_x are the operators of the left and right multiplications by x.

From the cyclicity of $\mathtt{t}$, we obtain an induced pairing

$$\langle\ ,\ \rangle \colon (U^{\otimes m})^U \otimes \mathrm{HH}_0(U^{\otimes m}) \to \mathbb{C},$$

which we call the modified trace pairing. To achieve the full evaluation, besides the basis for the center, we will need a basis for $\mathrm{HH}_0(U)$.

The trace $\mathrm{HH}_0(U)$

Let us construct a basis of the trace of U.

Recall that 0^{th} Hochschild homology or trace of a linear category $\mathcal{C}$ is defined by

$$\mathrm{HH}_0(\mathcal{C}) := \frac{\bigoplus_{x \in \mathcal{C}} \mathrm{End}_{\mathcal{C}}(x)}{fg - gf} \quad \text{for any } f \colon x \to y \text{ and } g \colon y \to x.$$

The image of an endomorphism $f \in \mathrm{End}_{\mathcal{C}}(x)$ in $\mathrm{HH}_0(\mathcal{C})$ will be called its trace class and denoted by $[f]$.

For an algebra A (ie a category with one object) this reduces to

$$\mathrm{HH}_0(A) := \frac{A}{[A, A]} \quad \text{with } [A, A] = \mathrm{Span}\{xy - yx \mid x, y \in A\}.$$

This space supports a natural action of the center defined by $z[x] = [zx]$ for any $z \in Z(A)$. Recall that the dual of $\mathrm{HH}_0(A)$ is isomorphic to $\mathrm{Char}(A)$.

Proposition 8 *The trace $\mathrm{HH}_0(U)$ has dimension $3\mathsf{p} - 1$, with basis consisting of*

- *$\boldsymbol{h}_k^{\pm}$ for $1 \le k \le \mathsf{p}$, represented by the minimal (noncentral) idempotent projecting onto a copy of the module $\mathcal{P}_k^{\pm}$, and*
- *$\boldsymbol{h}_j = \boldsymbol{w}_j^{+}\boldsymbol{h}_j^{+} = \boldsymbol{w}_j^{-}\boldsymbol{h}_{p-j}^{-}$ for $1 \le j \le \mathsf{p} - 1$.*

The proof will use the following well-known fact.

Proposition 9 *For any finite-dimensional algebra* A,

$$\mathrm{HH}_0(A\text{–pmod}) \simeq \mathrm{HH}_0(A).$$

Let us give a proof for completeness.

Proof Assume M is projective. Then there exists another projective module N such that $M \oplus N = A^{\oplus n}$ for some n. Hence in A–pmod there are morphisms

$$i\colon M \to A^{\oplus n} \quad \text{and} \quad p\colon A^{\oplus n} \to M$$

with $p \circ i = \mathrm{id}_M$ and $i \circ p$ an idempotent. For $1 \le k \le n$, let us denote by $a_k\colon M \to A$ the composition of i with the k^{th} projection, and by $b_k\colon A \to M$ the composition of the k^{th} inclusion with p. We have

$$\mathrm{id}_M = \sum_{k=1}^{n} b_k \,\mathrm{id}_A\, a_k. \tag{9}$$

This can be used to define a map $\mathrm{HH}_0(A\text{–pmod}) \to \mathrm{HH}_0(A)$ in the following manner: Given $f \in \mathrm{End}_A(M)$ and $M \in A$–pmod, we then associate to f the trace class $\left[\sum_k (a_k f b_k)(1)\right] \in \mathrm{HH}_0(A)$. We can check that this trace class depends neither on the choice giving identity (9) nor on the representative of $[f] \in \mathrm{HH}_0(A\text{–pmod})$. Hence we have a map

$$\Phi\colon \mathrm{HH}_0(A\text{–pmod}) \to \mathrm{HH}_0(A)$$

defined by $\Phi([f]) = \left[\sum_k (a_k f b_k)(1)\right]$. The inverse of this map sends $[x] \in \mathrm{HH}_0(A)$ to the trace class $[r_x] \in \mathrm{HH}_0(A\text{–pmod})$ of the right multiplication $r_x \in \mathrm{End}_A(A)$. Hence, we have an isomorphism. □

Let us recall that U as a free left U–module decomposes into a direct sum of indecomposable projectives as follows:

$$U \cong \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^{+} \oplus \bigoplus_{j=1}^{\mathsf{p}} j\mathcal{P}_j^{-}.$$

The module $\mathbb{P} = \bigoplus_{j,\pm} \mathcal{P}_j^{\pm}$ is called the projective generator of U and $B = \mathrm{End}(\mathbb{P})$ the *basic* algebra. Using decomposition of any projective module M into indecomposable ones, we show that the map $B \to \mathrm{HH}_0(U\text{–pmod})$ sending f to its trace class induces a bijection

$$\mathrm{HH}_0(B) \cong \mathrm{HH}_0(U\text{–pmod}).$$

Let us use the notation $\mathrm{id}_j^{\pm} = \mathrm{id}_{\mathcal{P}_j^{\pm}}$ for simplicity.

Lemma 10 *A basis for* $\mathrm{HH}_0(B)$ *is represented by* $[\mathrm{id}_k^{\pm}]$ *for* $1 \le k \le \mathsf{p}$, *along with* $[x_j^+] = [x_{\mathsf{p}-j}^-]$ *for* $1 \le j < \mathsf{p}$.

Proof Recall that

$$B = \bigoplus_{j,\epsilon,j',\epsilon'} \mathrm{Hom}(\mathcal{P}_j^{\epsilon}, \mathcal{P}_{j'}^{\epsilon'}).$$

A linear basis for B, as defined in Section 3, consists of

$$\mathrm{id}_k^{\pm}, \quad x_j^{\epsilon}, \quad a_j^{\epsilon}\colon \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon} \quad \text{and} \quad b_j^{\epsilon}\colon \mathcal{P}_j^{\epsilon} \to \mathcal{P}_{\mathsf{p}-j}^{-\epsilon},$$

where $1 \le k \le \mathsf{p}$, $1 \le j < \mathsf{p}$ and $\epsilon \in \{-,+\}$. For $1 \le j < \mathsf{p}$, we have $a_j^{\epsilon}\,\mathrm{id}_j^{\epsilon} = a_j^{\epsilon}$, while $\mathrm{id}_j^{\epsilon}\,a_j^{\epsilon} = 0$, so that a_j^{ϵ} and similarly b_j^{ϵ} vanish in $\mathrm{HH}_0(B)$. We also have $b_{\mathsf{p}-j}^{-\epsilon} a_j^{\epsilon} = x_j^{\epsilon}$, while $a_j^{\epsilon} b_{\mathsf{p}-j}^{-\epsilon} = x_{\mathsf{p}-j}^{-\epsilon}$. We get the relation $[x_j^{\epsilon}] = [x_{\mathsf{p}-j}^{-\epsilon}] \in \mathrm{HH}_0(B)$. Since the resulting set of generators has cardinality $3\mathsf{p}-1 = \dim(\mathrm{HH}_0(U))$, this completes the proof of the lemma. □

Proof of Proposition 8 Combining Proposition 9 with Lemma 10, we conclude that $\mathrm{HH}_0(U)$ has dimension $3\mathsf{p}-1$, with basis consisting of

- $\boldsymbol{h}_k^{\pm}$ for $1 \le k \le \mathsf{p}$, represented by the minimal (noncentral) idempotent projecting onto a copy of the module $\mathcal{P}_k^{\pm}$, and
- $\boldsymbol{h}_j = \boldsymbol{w}_j^+ \boldsymbol{h}_j^+ = \boldsymbol{w}_j^- \boldsymbol{h}_{p-j}^-$ for $1 \le j \le \mathsf{p}-1$.

On the regular representation in U–pmod these elements act by the right multiplication. □

5 Proofs of Theorems 1 and 2

In this section we construct the modified trace on U–pmod and compute it for the regular representation U and its tensor powers. This will provide the main tool to prove Theorems 1 and 2.

Modified trace

The subcategory U–pmod is an ideal of U–mod in the sense of [10; 11], that is:

(1) If $V \in U$–pmod and $W \in U$–mod, then $V \otimes W \in U$–pmod and $V \otimes W \in U$–pmod.

(2) If $V \in U$–pmod and $W \in U$–mod and there exist morphisms $f\colon W \to V$ and $g\colon V \to W$ such that $gf = \mathrm{id}_W$, then $W \in U$–pmod.

Let us recall that a *modified trace* on U–pmod is a family of linear functions

$$\{\mathrm{t}_V\colon \mathrm{End}_{\mathscr{C}}(V) \to \mathbb{C}\}_{V\in U\text{–pmod}}$$

such that the following two conditions hold:

Cyclicity If $X, V \in U$–pmod, then for any morphisms $f\colon V \to X$ and $g\colon X \to V$ in U–mod we have

$$\mathrm{t}_V(gf) = \mathrm{t}_X(fg).$$

Partial trace properties If $X \in U$–pmod and $W \in U$–mod then for any morphisms $f \in \mathrm{End}_U(X \otimes W)$ and $g \in \mathrm{End}_U(W \otimes X)$ we have

$$\mathrm{t}_{X\otimes W}(f) = \mathrm{t}_X(\mathrm{tr}_r^W(f)) \quad \text{and} \quad \mathrm{t}_{W\otimes X}(g) = \mathrm{t}_X(\mathrm{tr}_l^W(g)),$$

where tr_r^W and tr_l^W are the right and left partial categorical traces along W defined by (6).

If only the first (resp. the second) of the two partial trace properties is satisfied, we call the modified trace right (resp. left).

Proof of Theorem 1

Corollary 3.2.1 of [10] implies the existence of a unique (up to global scalar) right modified trace in any unimodular pivotal category with enough projectives and a simple projective object L such that $\widetilde{\mathrm{ev}}_L$ is surjective. The category U–pmod does satisfy all these assumptions. Hence there exists a unique right modified trace

$$\{\mathrm{t}_V^R\colon \mathrm{End}(V) \to \mathbb{C}\}_{V\in U\text{–pmod}}$$

normalized by

$$\mathrm{t}^R_{\mathcal{P}_\mathrm{p}^+}(\mathrm{id}_{\mathcal{P}_\mathrm{p}^+}) = (-1)^{\mathrm{p}-1}.$$

Analogous arguments imply the existence of the unique left trace

$$\{\mathrm{t}_V^L\colon \mathrm{End}(V) \to \mathbb{C}\}_{V\in U\text{–pmod}}$$

normalized by

$$\mathrm{t}^L_{\mathcal{P}_\mathrm{p}^+}(\mathrm{id}_{\mathcal{P}_\mathrm{p}^+}) = (-1)^{\mathrm{p}-1}.$$

We will compute both and show that they coincide. For this, we will use the additivity of trace functions for direct sums, which follows from the cyclicity. Hence, it is enough to compute modified traces on the endomorphisms of the indecomposable projectives.

Identity endomorphisms Recall $\mathscr{K}_1^+$ is the one-dimensional trivial U–module whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = v$. There is another one-dimensional module $\mathscr{K}_1^-$ whose action on any vector v is given by $Ev = Fv = 0$ and $K^{\pm 1}v = -v$. Hence, $\mathrm{tr}_r(\mathrm{id}_{\mathscr{K}_1^-}) = \mathrm{tr}^{\mathscr{K}_1^-}(K^{p+1}) = (-1)^{p+1}$. Using $\mathcal{P}_j^- \cong \mathcal{P}_j^+ \otimes \mathscr{K}_1^-$, we can compute the trace on $\mathcal{P}_{\mathsf{p}}^-$:

$$\mathrm{t}^{R}_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = \mathrm{t}^{R}_{\mathcal{P}_{\mathsf{p}}^+ \otimes \mathscr{K}_1^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^+ \otimes \mathscr{K}_1^-}) = \mathrm{t}^{R}_{\mathcal{P}_{\mathsf{p}}^+}(\mathrm{tr}_r^{\mathscr{K}_1^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^+ \otimes \mathscr{K}_1^-})) = 1.$$

Similarly, $\mathcal{P}_{\mathsf{p}}^- \cong \mathscr{K}_1^- \otimes \mathcal{P}_p^+$ implies that $\mathrm{t}^{L}_{\mathcal{P}_{\mathsf{p}}^-}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^-}) = 1$.

Next we compute some partial traces involving $\mathscr{K}_2^+$. The module $\mathscr{K}_2^+$ has a basis $\{w_0, w_1\}$ with action

$$\begin{aligned} Ew_0 &= 0, \quad Fw_0 = w_1, \quad Kw_0 = qw_0, \\ Ew_1 &= w_0, \quad Fw_1 = w_0, \quad Kw_1 = q^{-1}w_1. \end{aligned}$$

Recall that an endomorphism of $\mathcal{P}_j^+$ is determined by the image of the dominant vector $\boldsymbol{b}_0^+$. We can use this fact to compute the following partial trace:

$$(\mathrm{tr}_l^{\mathscr{K}_2^+}(\mathrm{id}_{\mathscr{K}_2^+ \otimes \mathcal{P}_j^+}))(\boldsymbol{b}_0^+) = w_0^*(K^{\mathsf{p}+1}w_0)\boldsymbol{b}_0^+ + w_1^*(K^{\mathsf{p}+1}w_1)\boldsymbol{b}_0^+ = (q^{\mathsf{p}+1} + q^{-\mathsf{p}-1})\boldsymbol{b}_0^+.$$

Thus, $\mathrm{tr}_l^{\mathscr{K}_2^+}(\mathrm{id}_{\mathscr{K}_2^+ \otimes \mathcal{P}_j^+}) = -(q + q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}$ for $j = 1, \dots, \mathsf{p}-1$. Similarly, for the right partial trace of the identity of $\mathcal{P}_j^+ \otimes \mathscr{K}_2^+$ we get

$$\mathrm{tr}_r^{\mathscr{K}_2^+}(\mathrm{id}_{\mathcal{P}_j^+ \otimes \mathscr{K}_2^+}) = (-q - q^{-1})\,\mathrm{id}_{\mathcal{P}_j^+}.$$

The decomposition of tensor products of a simple module with a projective indecomposable module is given in [23, Proposition 4.1] (also see [16, Theorems 3.1.5 and 3.2.1]). In particular,

$$\mathcal{P}_{\mathsf{p}}^{\pm} \otimes \mathscr{K}_2^+ \cong \mathcal{P}_{p-1}^{\pm} \cong \mathscr{K}_2^+ \otimes \mathcal{P}_{\mathsf{p}}^{\pm}, \tag{10}$$

$$\mathcal{P}_{\mathsf{p}-1}^{\pm} \otimes \mathscr{K}_2^+ \cong \mathcal{P}_{\mathsf{p}-2}^{\pm} \oplus 2\mathcal{P}_{\mathsf{p}}^{\pm} \cong \mathscr{K}_2^+ \otimes \mathcal{P}_{\mathsf{p}-1}^{\pm}, \tag{11}$$

$$\mathcal{P}_j^{\pm} \otimes \mathscr{K}_2^+ \cong \mathcal{P}_{j-1}^{\pm} \oplus \mathcal{P}_{j+1}^{\pm} \cong \mathscr{K}_2^+ \otimes \mathcal{P}_j^{\pm} \quad \text{for } j \in \{2, \dots, \mathsf{p}-2\}. \tag{12}$$

Combining these formulas with properties of the modified trace we have

$$\mathrm{t}^{R}_{\mathcal{P}_j^{\pm} \otimes \mathscr{K}_2^+}(\mathrm{id}_{\mathcal{P}_j^+ \otimes \mathscr{K}_2^+}) = \mathrm{t}^{R}_{\mathcal{P}_j^{\pm}}(\mathrm{tr}_r^{\mathscr{K}_2^+}(\mathrm{id}_{\mathcal{P}_j^+ \otimes \mathscr{K}_2^+})) = (-q - q^{-1})\mathrm{t}^{R}_{\mathcal{P}_j^{\pm}}(\mathrm{id}_{\mathcal{P}_j^+}),$$

where $j \in \{2, \dots, \mathsf{p}\}$. This equality together with the isomorphism on the left-hand side of (10) for $j = \mathsf{p}$ gives

$$\mathrm{t}^{R}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}(\mathrm{id}_{\mathcal{P}_{\mathsf{p}-1}^{\pm}}) = (\mp 1)^{\mathsf{p}-1}(-q - q^{-1}).$$

Then the isomorphism of the left-hand side of (11) implies

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}(\mathrm{id}_{\mathcal{P}^{\pm}_{\mathsf{p}-2}}) = (\mp 1)^{\mathsf{p}-1}((-q-q^{-1})^2-2) = (\mp 1)^{\mathsf{p}-1}(q^2+q^{-2}).$$

Finally, for $j \in \{2,\dots,\mathsf{p}-2\}$, the isomorphism of the left-hand side of (12) implies

$$-(q+q^{-1})\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_j}(\mathrm{id}_{\mathcal{P}^{\pm}_j}) = \mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_{j-1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j-1}}) + \mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_{j+1}}(\mathrm{id}_{\mathcal{P}^{\pm}_{j+1}}). \tag{13}$$

Recursively the last equality implies, for $j \in \{2,\dots,\mathsf{p}-1\}$,

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_j}(\mathrm{id}_{\mathcal{P}^{\pm}_j}) = (\pm 1)^{p-1}(-1)^j(q^j+q^{-j}). \tag{14}$$

Using the right-hand side of the tensor product in each of (10), (11) and (12) we compute similarly the left trace of the identities and get

$$\mathrm{t}^{\boldsymbol{L}}_{\mathcal{P}^{\pm}_j}(\mathrm{id}_{\mathcal{P}^{\pm}_j}) = \mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_j}(\mathrm{id}_{\mathcal{P}^{\pm}_j}) \quad \text{for } 1 \le j \le \mathsf{p}.$$

Endomorphisms $\boldsymbol{x^{\pm}_j}$ For $x \in U$, let us denote by l^V_x the operator of the left multiplication by x on V. Sometimes, we will omit V for simplicity. To compute $\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_j}(x^{\pm}_j)$ we will use the action of the Casimir element

$$C = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}.$$

Since C is central, the action of C commutes with the left U–action, and hence defines an intertwiner $l^{\mathcal{P}}_C \in \mathrm{End}_U(\mathcal{P})$.

On simple modules, C acts by a scalar, hence

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}^{\pm}_{\mathsf{p}}}(l^{\mathcal{P}^{\pm}_{\mathsf{p}}}_C) = \frac{2(\mp 1)^{\mathsf{p}}}{(q-q^{-1})^2}. \tag{15}$$

For $j \in \{1,\dots,\mathsf{p}-1\}$, the dominant vector $\boldsymbol{b}^+_0$ (resp. $\boldsymbol{y}^-_0$) of $\mathcal{P}^+_j$ (resp. $\mathcal{P}^-_j$) has weight $\pm q^{j-1}$. The action of C on this vector is

$$C\boldsymbol{b}^+_0 = \boldsymbol{a}^+_0 + \frac{q^j+q^{-j}}{(q-q^{-1})^2}\boldsymbol{b}^+_0 \quad \left(\text{resp. } C\boldsymbol{y}^-_0 = \boldsymbol{x}^-_0 - \frac{q^j+q^{-j}}{(q-q^{-1})^2}\boldsymbol{y}^-_0\right).$$

Thus, for $j \in \{1,\dots,\mathsf{p}-1\}$ we have

$$l^{\mathcal{P}^{\pm}_j}_C = x^{\pm}_j \pm \frac{q^j+q^{-j}}{(q-q^{-1})^2}\,\mathrm{id}_{\mathcal{P}^{\pm}_j}. \tag{16}$$

To compute the action of C on tensor products we need the formula

$$\Delta(C) = K^{-1}\otimes FE + K^{-1}E\otimes FK + F\otimes E + FE\otimes K + \frac{qK\otimes K + q^{-1}K^{-1}\otimes K^{-1}}{(q-q^{-1})^2}.$$

The second and third terms of $\Delta(C)$ have no diagonal contribution, and hence vanish when computing the partial trace $\mathrm{tr}_r^{\mathscr{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^\pm\otimes\mathscr{X}_2^+}\big)$. In particular, for $v\in\mathcal{P}_i^\pm$ we have

$$\begin{aligned}\big[\mathrm{tr}_r^{\mathscr{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^\pm\otimes\mathscr{X}_2^+}\big)\big](v)\\ &= -q^{-1}K^{-1}v-(q^2+q^{-2})FEv-\frac{(q^2+q^{-2})q}{(q-q^{-1})^2}Kv+\frac{-2q^{-1}}{(q-q^{-1})^2}K^{-1}v.\end{aligned}$$

When $i\in\{1,\dots,\mathsf{p}-1\}$ and $v=\boldsymbol{b}_0^+\in\mathcal{P}_i^+$ or $v=\boldsymbol{y}_0^-\in\mathcal{P}_i^-$ is a generating dominant vector of $\mathcal{P}_i^\pm$ this equality implies

$$\mathrm{tr}_r^{\mathscr{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_i^\pm\otimes\mathscr{X}_2^+}\big)=-(q^2+q^{-2})x_i^\pm\mp\frac{(q^2+q^{-2})}{(q-q^{-1})^2}(q^i+q^{-i})\,\mathrm{id}_{\mathcal{P}_i^\pm}.$$

Similarly, if v is the highest weight of $\mathcal{P}_{\mathsf{p}}^\pm$, we obtain

$$\mathrm{tr}_r^{\mathscr{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_{\mathsf{p}}^\pm\otimes\mathscr{X}_2^+}\big)=\pm2\frac{(q^2+q^{-2})}{(q-q^{-1})^2}\,\mathrm{id}_{\mathcal{P}_{\mathsf{p}}^\pm}. \tag{17}$$

Now we can compute $\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big(x_{\mathsf{p}-1}^\pm\big)$. From the isomorphism in (10), we have

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^\pm\otimes\mathscr{X}_2^+}\big(l_{\Delta(C)}^{\mathcal{P}_{\mathsf{p}}^\pm\otimes\mathscr{X}_2^+}\big)=\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big(l_C^{\mathcal{P}_{\mathsf{p}-1}^\pm}\big).$$

Using (16) and (17) we can simplify the last equality as follows:

$$\pm2\frac{q^2+q^{-2}}{(q-q^{-1})^2}\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}}^\pm}\big(\mathrm{id}^\pm_{\mathcal{P}_{\mathsf{p}}^\pm}\big)=\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big(x_{\mathsf{p}-1}^\pm\big)\pm\frac{q^{\mathsf{p}-1}+q^{-\mathsf{p}+1}}{(q-q^{-1})^2}\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big(\mathrm{id}^\pm_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big),$$

or

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{\mathsf{p}-1}^\pm}\big(x_{\mathsf{p}-1}^\pm\big)=\pm(\mp)^{\mathsf{p}-1}\frac{(q^{\mathsf{p}-1}-q^{-\mathsf{p}+1})^2}{(q-q^{-1})^2}.$$

Using the isomorphism in (12), for $j\in\{2,\dots,\mathsf{p}-2\}$ we obtain the recursive relation

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{j-1}^\pm}\big(x_{j-1}^\pm\big)+\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_{j+1}^\pm}\big(x_{j+1}^\pm\big)+(q^2+q^{-2})\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_j^\pm}\big(x_j^\pm\big)=-2(\pm1)^{\mathsf{p}}(-1)^j. \tag{18}$$

Using (11) we can show that this formula also holds for $j=\mathsf{p}-1$ by setting $x_{\mathsf{p}}^\pm=0$. We deduce the general formula for $j\in\{1,\dots,\mathsf{p}-1\}$,

$$\mathrm{t}^{\boldsymbol{R}}_{\mathcal{P}_j^\pm}\big(x_j^\pm\big)=(\pm1)^{\mathsf{p}}(-1)^j[j]^2,$$

which is compatible with the computation at $j=\mathsf{p}-2$ or $j=\mathsf{p}-1$ and satisfies the recursive relation for $j\in\{2,\dots,\mathsf{p}-2\}$.

With a similar computation we get the same value for $\mathrm{t}^{L}_{\mathcal{P}_j^-}(x_j^{\pm})$. Thus, we have proved that the left and right modified traces are equal on U–pmod. Let us summarize our computations of the modified trace ($0 < j < \mathsf{p}$):

$\mathrm{id}_{\mathcal{P}_\mathsf{p}^-}$	$\mathrm{id}_{\mathcal{P}_\mathsf{p}^+}$	$\mathrm{id}_{\mathcal{P}_j^-}$	$\mathrm{id}_{\mathcal{P}_j^+}$	x_j^-	x_j^+
1	$(-1)^{\mathsf{p}-1}$	$(-1)^{\mathsf{p}+j-1}(q^j+q^{-j})$	$(-1)^j(q^j+q^{-j})$	$(-1)^{\mathsf{p}+j}[j]^2$	$(-1)^j[j]^2$

Proof of Theorem 2

Let us apply the modified trace construction to the regular representation $U \in U$–pmod. Using the isomorphism of algebras

$$r\colon U^{\mathrm{op}} \cong \mathrm{End}_U(U), \quad x \mapsto r_x,$$

where $r_x(y) = yx$ is the right multiplication, we define

$$\mathsf{T}\colon U \to \mathbb{C} \quad \text{by } \mathsf{T}(x) = \mathrm{t}_U(r_x).$$

By Theorem 1, the linear map T is a character and satisfies the partial trace property.

We can now show by direct computation that the pairing

$$Z(U) \times \mathrm{HH}_0(U) \to \mathbb{C} \quad \text{given by } (z, x) \mapsto \mathsf{T}(zx)$$

is nondegenerate. Using Proposition 5 and Theorem 1 we can explicitly compute this pairing in the basis of the center and the trace. For example

$$\mathsf{T}(\boldsymbol{w}_j^+ \boldsymbol{h}_j^+) = \mathrm{t}^{R}_{\mathcal{P}_j^+}(x_j^+) = (-1)^j[j]^2 \quad \text{or} \quad \mathsf{T}(\boldsymbol{e}_0 \boldsymbol{h}_\mathsf{p}^-) = \mathrm{t}^{R}_{\mathcal{P}_\mathsf{p}^-}(\mathrm{id}_{\mathcal{P}_\mathsf{p}^-}) = 1.$$

Completing the computation, we obtain the pairing shown in Table 1. From the table it is easy to see the pairing is nondegenerate.

	$\boldsymbol{h}_\mathsf{p}^+$	$\boldsymbol{h}_\mathsf{p}^-$	$\boldsymbol{h}_s$	$\boldsymbol{h}_s^+$	$\boldsymbol{h}_{\mathsf{p}-s}^-$
$\boldsymbol{e}_\mathsf{p}$	$(-1)^{\mathsf{p}-1}$	0	0	0	0
$\boldsymbol{e}_0$	0	1	0	0	0
$\boldsymbol{e}_j$	0	0	$(-1)^j[j]^2$	$(-1)^j(q^j+q^{-j})$	$(-1)^j(q^j+q^{-j})$
$\boldsymbol{w}_j^+$	0	0	0	$(-1)^j[j]^2$	0
$\boldsymbol{w}_j^-$	0	0	0	0	$(-1)^j[j]^2$

Table 1: Values of the pairing on a basis, where $1 \le j < \mathsf{p}$

6 Proof of Theorem 3

In this section we will define our logarithmic 3–manifold invariant $\mathrm{H}^{\log}(M,L)$, prove Theorem 3 and compare $\mathrm{H}^{\log}(M,L)$ with the invariant defined by Murakami in [20].

Logarithmic invariant

Assume we are given an oriented framed link (L^+,L^-) with (m_+,m_-) components inside a 3–manifold $M=S^3(L^0)$, where $L^0\subset S^3$ is a surgery oriented framed link for M with m_0 components. Here m_+ or m_- can be zero. We suppose that (L^+,L^-) is in $S^3\setminus L^0$, and choose an upwards oriented string link $T=(T^+,T^0,T^-)$ whose closure is (L^+,L^0,L^-). By Lemma 6, the universal invariant satisfies

$$J_T\in (U^{\otimes(m_++m_0+m_-)})^U.$$

Let us color the components of L^+ by central elements $z_j\in Z(U)$, where $1\le j\le m_+$, and L^- by trace classes $h_k\in \mathrm{HH}_0(U)$, where $1\le k\le m_-$, and write $z^+=\bigotimes_j z_j$ and $h^-=\bigotimes_k h_k$. Let us denote by L the resulting colored link $((L^+,z^+),(L^-,h^-))$. For each j, the central element z_j acting on the right integral μ defines the quantum character $\mu_{z_j}\colon x\mapsto \mu(z_jx)$. We denote the tensor product $\bigotimes_j \mu_{z_j}$ by $z^+\mu^{\otimes m_+}$. Using the map $\mathrm{HH}_0(U)^{\otimes m_-}\to \mathrm{HH}_0(U^{\otimes m_-})$ induced by $\mathrm{id}_{U^{\otimes m_-}}$, the tensor product h^- gives an element in $\mathrm{HH}_0(U^{\otimes m_-})$ which we also denote by h^-. We obtain

$$\mathrm{H}^{\log}(M,L):=\delta^s\langle (z^+\mu^{\otimes m_+}\otimes\mu^{\otimes m_0}\otimes\mathrm{id})(J_T),\ h^-\rangle$$

by evaluating components of J_T corresponding to (L^+,L^0) with the right integral μ twisted with central elements and by applying to the result the modified trace pairing. Here s is the signature of the linking matrix for L^0. Theorem 3 claims that $\mathrm{H}^{\log}(M,L)$ is a topological invariant of the pair (M,L).

Proof of Theorem 3 When L^- is empty the formula recovers the Hennings invariant. We further assume $m_->0$. We first show that $\mathrm{H}^{\log}(M,L)$ is an invariant of the colored link L, ie it does not depend on the choice of T. By applying Lemma 7 we see that the partial closure $(z^+\mu^{\otimes m_+}\otimes\mu^{\otimes m_0}\otimes\mathrm{id})(J_T)$ is invariant of the tangle T_1 obtained by closing the first m_++m_0 components of T.

We may continue the procedure as follows. To an element $h\in U$ we associate the quantum character χ_h defined by

$$\chi_h(x)=\mathrm{tr}_l^H(l_xr_h)\quad \text{for any } x\in U.$$

This character χ_h only depends on the trace class of h in $\mathrm{HH}_0(U)$. Hence to each trace class h_k we associate a quantum character χ_{h_k}. If $m_- > 1$, using the partial trace property of the modified trace, we have the formula

$$\mathrm{H}^{\log}(M,L) = \delta^s \left\langle \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \left(\bigotimes_{k=1}^{m_- -1} \chi_{h_k} \right) \otimes \mathrm{id} \right)(J_T),\, h_{m_-} \right\rangle \tag{19}$$

Now from Lemma 7 we get that the central element

$$J_K = \left(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \left(\bigotimes_{k=1}^{m_- -1} \chi_{h_k} \right) \otimes \mathrm{id} \right)(J_T)$$

is an invariant of the $(1,1)$–tangle K obtained by closing all components but the last one. The main point is that it does not matter which $m_- - 1$ components we choose! Hence $\mathrm{H}^{\log}(M,L)$ is invariant of the $(1,1)$–tangle K obtained from T_1 by closing all but one trace-class-colored component.

But the resulting central element J_K does not depend on the point where we cut L into K. The proof mimics the argument showing that long knots and knots in S^3 are equivalent.

Indeed, let us think about our $(1,1)$–tangle K as being a long knot. We need to show that our central element does not change if we move an arc through "infinity" (or the cutting disc). This move can be alternatively realized by moving the arc in the opposite way through the long knot, which is just a sequence of Reidemeister moves, under which we know J_K to be stable.

It remains to show invariance under Kirby moves: sliding along a component of L^0 and stabilization with a ± 1 framed unknot. The defining property of the right integral ensures the sliding invariance (see eg [15]). Note that if we change the orientation on one of the T^0 components, this will change J_T by applying S at the corresponding position, but now $\mu \circ S$ is a left integral, hence the sliding property holds after rearranging the components.

Adding to L^0 a ± 1 framed unknot multiplies $\mathrm{H}^{\log}(M,L)$ by $\mu(v^{\mp}) = \delta^{\mp}$, and changes the signature s by ± 1, so $\mathrm{H}^{\log}(M,L)$ remains the same. □

Relation with Murakami invariants

Here we show that the logarithmic invariant in [20] is a special case of $\mathrm{H}^{\log}(M,L)$ where L^- has precisely one component.

Murakami's invariant is defined for a knot and a colored link in a 3–manifold. We will adapt our notation to his setting. Let us consider a link (L^+, L^-) in a 3–manifold $M = S^3(L_0)$ as before, but with $L_- = K$ a knot and $m_- = 1$. The link L^+ is colored by z^+ as before. The logarithmic knot invariant defined by Murakami is then

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \delta^s(z^+ \mu^{\otimes m_+} \otimes \mu^{\otimes m_0} \otimes \mathrm{id})(J_T) \in Z(U).$$

Due to different conventions in the definition of J_T, Murakami's original invariant corresponds to the mirror image of the link in our notation.

Murakami further expands his invariant in the basis of the center as follows:

$$\mathrm{J}^{\log}(M, (L^+, z^+), K) = \sum_{j=0}^{p} a_j \boldsymbol{e}_j + \sum_{j=1}^{p-1} b_j^+ \boldsymbol{w}_j^+ + \sum_{j=1}^{p-1} b_j^- \boldsymbol{w}_j^-.$$

The coefficients are clearly topological invariants of the triple $(M, (L^+, z^+), K)$.

Proposition 11 *With the above notation, we have, for* $1 \le j < \mathsf{p}$,

$$\begin{aligned}
a_0 &= \mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^-)),\\
a_{\mathsf{p}} &= (-1)^{\mathsf{p}-1}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^+)),\\
a_j &= \frac{(-1)^j}{[j]^2}\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j)),\\
b_j^+ &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}\big(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_j^+ - (q^j + q^{-j})\boldsymbol{h}_j)\big),\\
b_j^- &= \frac{(-1)^j}{[j]^4}\mathrm{H}^{\log}\big(M, (L^+, z^+), (L^-, [j]^2\boldsymbol{h}_{\mathsf{p}-j}^- - (q^j + q^{-j})\boldsymbol{h}_j)\big).
\end{aligned}$$

Proof For any trace class $h \in \mathrm{HH}_0(U)$ we have

$$\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, h)) = \langle \mathrm{J}^{\log}(M, (L^+, z^+), L^-), h\rangle.$$

From Table 1 we get

$$\begin{aligned}
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^-)) &= a_0,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}}^+)) &= (-1)^{\mathsf{p}-1}a_{\mathsf{p}},\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j)) &= (-1)^j[j]^2 a_j,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_j^+)) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^+,\\
\mathrm{H}^{\log}(M, (L^+, z^+), (L^-, \boldsymbol{h}_{\mathsf{p}-j}^-)) &= (-1)^j(q^j + q^{-j})a_j + (-1)^j[j]^2 b_j^-.
\end{aligned}$$

The claim follows. □

To summarize, Murakami's invariant is an invariant of a link in a 3–manifold together with a choice of one *preferred* component of the link (the one which is not evaluated). Our approach provides a tool to see why this choice is actually irrelevant.

Indeed, to each trace class h we have an associated quantum character χ_h, defined by

$$\chi_h(x) = \mathrm{tr}_l^U(l_x r_h) \quad \text{for any } x \in U.$$

On the other hand, using the Radford isomorphism between $\mathrm{qChar}(U)$ and $Z(U)$ we obtain $z^+ \in Z(U)$ such that $z^+\mu = \chi_h$.

From the proof of Theorem 3 we get the following statement, which allows us to move all components but an *arbitrary* one from the negative to the positive part of the link.

Proposition 12 *Let $L = (L^+, L^-) \subset M$ be a colored link as above, with more than one component in L^-. Assume that L' is obtained from L by moving an h^-–colored component from L^- to L^+ and assigning the color z^+ to it so that $z^+\mu = \chi_{h^-}$. Then*

$$\mathrm{H}^{\log}(M, L) = \mathrm{H}^{\log}(M, L').$$

Appendix Indecomposable projective modules

We reproduce here the defining relations for the modules $\mathcal{P}_j^+$ and $\mathcal{P}_{\mathsf{p}-j}^-$, where $1 \le j \le \mathsf{p}-1$, as given in [8, Section C.2]. The module $\mathcal{P}_j^\pm$, where $1 \le j \le \mathsf{p}-1$, is the indecomposable projective module whose irreducible quotient is given by $\mathscr{X}_j^\pm$.

Let j be an integer such that $1 \le j \le \mathsf{p}-1$. The indecomposable projective module $\mathcal{P}_j^+$ has the basis

$$\{\boldsymbol{x}_k^+, \boldsymbol{y}_k^+\}_{0 \le k \le \mathsf{p}-j-1} \cup \{\boldsymbol{a}_n^+, \boldsymbol{b}_n^+\}_{0 \le n \le j-1},$$

with the U–action given by

$$\begin{aligned}
K\boldsymbol{x}_k^+ &= -q^{\mathsf{p}-j-1-2k}\boldsymbol{x}_k^+, \quad K\boldsymbol{y}_k^+ = -q^{\mathsf{p}-j-1-2k}\boldsymbol{y}_k^+ && \text{for } 0 \le k \le \mathsf{p}-j-1,\\
K\boldsymbol{a}_n^+ &= q^{j-1-2n}\boldsymbol{a}_n^+, \qquad K\boldsymbol{b}_n^+ = q^{j-1-2n}\boldsymbol{b}_n^+ && \text{for } 0 \le n \le j-1,\\
E\boldsymbol{x}_k^+ &= -[k][\mathsf{p}-j-k]\boldsymbol{x}_{k-1}^+ && \text{for } 0 \le k \le \mathsf{p}-j-1 \quad (\text{where } \boldsymbol{x}_{-1}^+ = 0),\\
E\boldsymbol{y}_k^+ &= \begin{cases} -[k][\mathsf{p}-j-k]\boldsymbol{y}_{k-1}^+ & \text{for } 1 \le k \le \mathsf{p}-j-1,\\ \boldsymbol{a}_{j-1}^+ & \text{for } k = 0,\end{cases}\\
E\boldsymbol{a}_n^+ &= [n][j-n]\boldsymbol{a}_{n-1}^+ && \text{for } 0 \le n \le j-1 \quad (\text{where } \boldsymbol{a}_{-1}^+ = 0),\\
E\boldsymbol{b}_n^+ &= \begin{cases} [n][j-n]\boldsymbol{b}_{n-1}^+ + \boldsymbol{a}_{n-1}^+ & \text{for } 1 \le n \le j-1,\\ \boldsymbol{x}_{\mathsf{p}-j-1}^+ & \text{for } n = 0,\end{cases}
\end{aligned}$$

$$
\begin{aligned}
F\boldsymbol{x}_k^+ &= \begin{cases} \boldsymbol{x}_{k+1}^+ & \text{for } 0 \le k \le \mathsf{p}-j-2, \\ \boldsymbol{a}_0^+ & \text{for } k = \mathsf{p}-j-1, \end{cases} \\
F\boldsymbol{y}_k^+ &= \boldsymbol{a}_{k+1}^+ \quad \text{for } 0 \le k \le \mathsf{p}-j-1 \quad (\text{where } \boldsymbol{y}_{\mathsf{p}-j}^+ = 0), \\
F\boldsymbol{a}_n^+ &= \boldsymbol{a}_{n+1}^+ \quad \text{for } 0 \le n \le j-1 \quad (\text{where } \boldsymbol{a}_j^+ = 0), \\
F\boldsymbol{b}_n^+ &= \begin{cases} \boldsymbol{b}_{n+1}^+ & \text{for } 0 \le n \le j-2, \\ \boldsymbol{y}_0^+ & \text{for } n = j-1. \end{cases}
\end{aligned}
$$

Let j be an integer such that $1 \le j \le \mathsf{p}-1$. The indecomposable projective module $\mathcal{P}_{\mathsf{p}-j}^-$ has the basis

$$\{\boldsymbol{x}_k^-, \boldsymbol{y}_k^-\}_{0 \le k \le \mathsf{p}-j-1} \cup \{\boldsymbol{a}_n^-, \boldsymbol{b}_n^-\}_{0 \le n \le j-1},$$

with the U–action given by

$$
\begin{aligned}
K\boldsymbol{x}_k^- &= -q^{\mathsf{p}-j-1-2k}\boldsymbol{x}_k^-, \quad K\boldsymbol{y}_k^- = -q^{\mathsf{p}-j-1-2k}\boldsymbol{y}_k^-, \qquad \text{for } 0 \le k \le \mathsf{p}-j-1, \\
K\boldsymbol{a}_n^- &= q^{j-1-2n}\boldsymbol{a}_n^-, \qquad K\boldsymbol{b}_n^- = q^{j-1-2n}\boldsymbol{b}_n^-, \qquad \text{for } 0 \le n \le j-1, \\
E\boldsymbol{x}_k^- &= -[k][\mathsf{p}-j-k]\boldsymbol{x}_{k-1}^- \quad \text{for } 0 \le k \le \mathsf{p}-j-1 \quad (\text{where } \boldsymbol{x}_{-1}^- = 0), \\
E\boldsymbol{y}_k^- &= \begin{cases} -[k][\mathsf{p}-j-k]\boldsymbol{y}_{k-1}^- + \boldsymbol{x}_{k-1}^- & \text{for } 1 \le k \le \mathsf{p}-j-1, \\ \boldsymbol{a}_{j-1}^- & \text{for } k = 0, \end{cases} \\
E\boldsymbol{a}_n^- &= [n][j-n]\boldsymbol{a}_{n-1}^- \quad \text{for } 0 \le n \le j-1 \quad (\text{where } \boldsymbol{a}_{-1}^- = 0), \\
E\boldsymbol{b}_n^- &= \begin{cases} [n][j-n]\boldsymbol{b}_{n-1}^- & \text{for } 1 \le n \le j-1, \\ \boldsymbol{x}_{\mathsf{p}-j-1}^- & \text{for } n = 0, \end{cases} \\
F\boldsymbol{x}_k^- &= \boldsymbol{x}_{k+1}^- \quad \text{for } 0 \le k \le \mathsf{p}-j-1 \quad (\text{where } \boldsymbol{x}_{\mathsf{p}-j}^- = 0), \\
F\boldsymbol{y}_k^- &= \begin{cases} \boldsymbol{y}_{k+1}^- & \text{for } 0 \le k \le \mathsf{p}-j-2, \\ \boldsymbol{b}_0^- & \text{for } k = \mathsf{p}-j-1, \end{cases} \\
F\boldsymbol{a}_n^- &= \begin{cases} \boldsymbol{a}_{n+1}^- & \text{for } 0 \le n \le j-2, \\ \boldsymbol{x}_0^- & \text{for } n = j-1, \end{cases} \\
F\boldsymbol{b}_n^- &= \boldsymbol{b}_{n+1}^- \quad \text{for } 0 \le n \le j-1 \quad (\text{where } \boldsymbol{b}_j^- = 0).
\end{aligned}
$$

References

[1] **Y Arike**, *A construction of symmetric linear functions on the restricted quantum group $\overline{U}_q(\mathrm{sl}_2)$*, Osaka J. Math. 47 (2010) 535–557 MR

[2] **A Beliakova**, **C Blanchet**, **A M Gainutdinov**, *Modified trace is a symmetrised integral*, preprint (2018) `arXiv:1801.00321`

[3] **C Blanchet**, **F Costantino**, **N Geer**, **B Patureau-Mirand**, *Non-semi-simple TQFTs, Reidemeister torsion and Kashaev's invariants*, Adv. Math. 301 (2016) 1–78 MR

[4] **A Bruguières**, *Double braidings, twists and tangle invariants*, J. Pure Appl. Algebra 204 (2006) 170–194 MR

[5] **F Costantino**, **N Geer**, **B Patureau-Mirand**, *Quantum invariants of* 3*–manifolds via link surgery presentations and non-semi-simple categories*, J. Topol. 7 (2014) 1005–1053 MR

[6] **F Costantino**, **N Geer**, **B Patureau-Mirand**, *Some remarks on the unrolled quantum group of* $\mathfrak{sl}(2)$, J. Pure Appl. Algebra 219 (2015) 3238–3262 MR

[7] **M De Renzi**, **N Geer**, **B Patureau-Mirand**, *Renormalized Hennings invariants and* $2+1$*–TQFTs*, Comm. Math. Phys. 362 (2018) 855–907 MR

[8] **B L Feigin**, **A M Gainutdinov**, **A M Semikhatov**, **I Y Tipunin**, *Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center*, Comm. Math. Phys. 265 (2006) 47–93 MR

[9] **A M Gaĭnutdinov**, **A M Semikhatov**, **I Y Tipunin**, **B L Feĭgin**, *The Kazhdan–Lusztig correspondence for the representation category of the triplet W–algebra in logorithmic conformal field theories*, Teoret. Mat. Fiz. 148 (2006) 398–427 MR In Russian; translated in Theor. Math. Phys. 148 (2006) 1210–1235

[10] **N Geer**, **J Kujawa**, **B Patureau-Mirand**, *Ambidextrous objects and trace functions for nonsemisimple categories*, Proc. Amer. Math. Soc. 141 (2013) 2963–2978 MR

[11] **N Geer**, **B Patureau-Mirand**, **A Virelizier**, *Traces on ideals in pivotal categories*, Quantum Topol. 4 (2013) 91–124 MR

[12] **K Habiro**, *Bottom tangles and universal invariants*, Algebr. Geom. Topol. 6 (2006) 1113–1214 MR

[13] **M Hennings**, *Invariants of links and* 3*–manifolds obtained from Hopf algebras*, J. London Math. Soc. 54 (1996) 594–624 MR

[14] **T Kerler**, *Mapping class group actions on quantum doubles*, Comm. Math. Phys. 168 (1995) 353–388 MR

[15] **T Kerler**, *Genealogy of non-perturbative quantum-invariants of* 3*–manifolds: the surgical family*, from "Geometry and physics" (J E Andersen, J Dupont, H Pedersen, A Swann, editors), Lecture Notes in Pure and Appl. Math. 184, Dekker, New York (1997) 503–547 MR

[16] **H Kondo**, **Y Saito**, *Indecomposable decomposition of tensor products of modules over the restricted quantum universal enveloping algebra associated to* $\mathfrak{sl}_2$, J. Algebra 330 (2011) 103–129 MR

[17] **R J Lawrence**, *A universal link invariant using quantum groups*, from "Differential geometric methods in theoretical physics" (A I Solomon, editor), World Sci., Teaneck, NJ (1989) 55–63 MR

[18] **M Lorenz**, *Representations of finite-dimensional Hopf algebras*, J. Algebra 188 (1997) 476–505 MR

[19] **V V Lyubashenko**, *Invariants of 3–manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Comm. Math. Phys. 172 (1995) 467–516 MR

[20] **J Murakami**, *Generalized Kashaev invariants for knots in three manifolds*, Quantum Topol. 8 (2017) 35–73 MR

[21] **T Ohtsuki**, *Quantum invariants: a study of knots, 3–manifolds, and their sets*, Series on Knots and Everything 29, World Sci., River Edge, NJ (2002) MR

[22] **D E Radford**, *The trace function and Hopf algebras*, J. Algebra 163 (1994) 583–622 MR

[23] **R Suter**, *Modules over $\mathfrak{U}_q(\mathfrak{sl}_2)$*, Comm. Math. Phys. 163 (1994) 359–393 MR

[24] **V G Turaev**, *Quantum invariants of knots and 3–manifolds*, De Gruyter Studies in Mathematics 18, de Gruyter, Berlin (1994) MR

Institut für Mathematik, Universität Zürich
Zurich, Switzerland

Institut de Mathématiques de Jussieu - Paris Rive Gauche, Université Paris 7 (Denis Diderot)
Paris, France

Department of Mathematics and Statistics, Utah State University
Logan, UT, United States

anna@math.uzh.ch, christian.blanchet@imj-prg.fr, nathan.geer@gmail.com

Received: 10 April 2018

msp

Algebraic & Geometric Topology 18 (2018) 4359–4373

Nonarithmetic hyperbolic manifolds and trace rings

OLIVIER MILA

We give a sufficient condition on the hyperplanes used in the Belolipetsky–Thomson inbreeding construction to obtain nonarithmetic manifolds. We explicitly construct infinitely many examples of such manifolds that are pairwise noncommensurable and estimate their volume.

53C23, 22E40; 57M50, 51M25, 20G30

Let M be a finite-volume hyperbolic n–manifold, with $n \geq 2$. If M is complete, it can be written as a quotient $\Gamma\backslash\mathbb{H}^n$ for Γ a torsion-free lattice in the semisimple Lie group $\mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$, the group of isometries of the hyperbolic n–space $\mathbb{H}^n$. The lattice Γ is then uniform (ie cocompact) if and only if M is compact.

A standard way to construct hyperbolic manifolds in higher dimensions is via arithmetic lattices. In Lie groups with rank at least 2, this is actually the only possible construction (by Margulis' arithmeticity theorem); yet it is known that there are nonarithmetic lattices in $\mathrm{PO}(n,1)$ for every $n \geq 2$. Many examples in low dimensions were constructed using Coxeter groups, notably by Vinberg [16], but the first construction in arbitrary dimension was given by Gromov and Piatetski-Shapiro [7]. Roughly, their idea consists in constructing two pieces of noncommensurable arithmetic manifolds with isometric boundaries and gluing them together to form a nonarithmetic manifold. This construction has then been generalized by Raimbault [12] and Gelander and Levit [6] to produce many different commensurability classes of nonarithmetic manifolds.

A similar construction was introduced by Agol [1] in dimension 4 and generalized by Belolipetsky and Thomson [2] in arbitrary dimension to obtain manifolds with short systole. They start with two hyperplanes chosen at distance $\delta > 0$ apart and find a torsion-free arithmetic lattice Γ such that, in $M = \Gamma\backslash\mathbb{H}^n$, the hyperplanes project down to two disjoint hypersurfaces. Then they cut M open along the hypersurfaces and glue it back to a copy of itself along its boundary; as $\delta \to 0$, the systole of such a manifold then becomes arbitrarily small. Manifolds obtained via this construction will be referred to as *doubly cut gluings* and the two corresponding hyperplanes as the *cut hyperplanes* (see Section 1.2).

Published: 11 December 2018 DOI: 10.2140/agt.2018.18.4359

An interesting consequence is that infinitely many of such doubly cut gluings are nonarithmetic and pairwise noncommensurable. Moreover, these are the first examples in arbitrary dimension of nonarithmetic manifolds that are quasiarithmetic (see Thomson [15]). However if one is only interested in constructing nonarithmetic manifolds, their proof is somehow nonexplicit in the sense that it relies on the systole argument for proving both nonarithmeticity and pairwise noncommensurability. Furthermore, it is hard to give an estimate on the volume of one particular nonarithmetic manifold.

In this paper we give a sufficient condition on the cut hyperplanes to obtain nonarithmetic doubly cut gluings. Recall that the group $\mathrm{PO}(n,1) \cong \mathrm{Isom}(\mathbb{H}^n)$ has a natural matrix representation in $\mathrm{O}_f(\mathbb{R}) \subset \mathrm{GL}_{n+1}(\mathbb{R})$ for $f = -x_0^2 + x_1^2 + \cdots + x_n^2$ the standard Lorentzian quadratic form (see Section 1.1).

Proposition 1 *Let M be a doubly cut gluing with cut hyperplanes R_1 and R_2. Let $\rho_1, \rho_2 \in \mathrm{O}_f(\mathbb{R})$ denote the reflections in R_1, R_2 respectively. If the trace of $g = \rho_1\rho_2$ is not an algebraic integer, then M is nonarithmetic.*

In order to study the commensurability classes of doubly cut gluings, we use an invariant called the *adjoint trace ring*. This invariant was introduced by Vinberg [17] as the minimal ring of definition of a lattice Γ (see Section 2.1). We first show that we can realize every finitely generated subring of $\mathbb{Q}$ as the adjoint trace ring of a doubly cut gluing.

Theorem 2 *Let $n \geq 4$. For every square-free integer $d > 1$, there exists a nonarithmetic lattice Γ_d in $\mathrm{PO}(n,1)$ with adjoint trace ring $\mathbb{Z}[1/d]$.*

These lattices are nonuniform by construction (see Remark 2.7). Since the adjoint trace ring is an invariant of the commensurability class, there is an immediate corollary:

Corollary 3 *The lattices Γ_d and $\Gamma_{d'}$ are noncommensurable whenever $d \neq d'$.*

Thus we are able to construct many noncommensurable nonarithmetic doubly cut gluings avoiding the nonexplicit systole argument. We will see in Section 3 that this allows us to give explicit estimates on the volumes of these constructions.

The proof mainly relies on an observation about manifolds which admit a mirror symmetry in two of their embedded hypersurfaces. In the first section we introduce the necessary background and prove Proposition 1. In Section 2 we explain how to proceed to obtain noncommensurable manifolds and prove Theorem 2. Section 3 is devoted to volume computations, and some generalizations of Theorem 2 are discussed in Section 4.

Acknowledgements The author expresses his gratitude towards his PhD advisor Vincent Emery for his guidance and useful comments, as well as to Filip Misev, Jean Raimbault and Scott Thomson for helpful discussions. The author also thanks the referees for their remarks, suggestions and simplifications of the proofs. This work is supported by the Swiss National Science Foundation, project number PP00P2_157583.

1 Background and nonarithmeticity

1.1 Background

Let $k \subset \mathbb{R}$ be a totally real number field with ring of integers $\mathcal{O}_k \subset k$. Let G be an absolutely simple adjoint algebraic k–group, and write $\mathrm{G}(\mathcal{O}_k)$ for $\mathrm{G}(k) \cap \mathrm{GL}_N(\mathcal{O}_k)$ in an arbitrary embedding $\mathrm{G} \subset \mathrm{GL}_N$ (this group is well defined up to commensurability). For $n \geq 4$, we say that G is *admissible* if $\mathrm{G}(\mathbb{R}) \cong \mathrm{PO}(n,1)$ and $\mathrm{G}^\sigma(\mathbb{R})$ is compact for any nontrivial embedding $\sigma\colon k \hookrightarrow \mathbb{R}$. In that case, $\mathrm{G}(\mathcal{O}_k)$ is a lattice in $\mathrm{G}(\mathbb{R})$.

A lattice $\Gamma \subset \mathrm{PO}(n,1)$ is called *arithmetic* if there exists an admissible algebraic k–group G and an isomorphism $\varphi\colon \mathrm{PO}(n,1) \to \mathrm{G}(\mathbb{R})$ such that

$$\varphi(\Gamma) \text{ is commensurable to } \mathrm{G}(\mathcal{O}_k). \tag{1}$$

Since G is adjoint, we have $\varphi(\Gamma) \subset \mathrm{G}(k)$ (see Borel and Prasad [4, Proposition 1.2]). The lattice Γ is called *quasiarithmetic* if condition (1) is replaced by the weaker requirement that $\varphi(\Gamma) \subset \mathrm{G}(k)$. Thus arithmetic implies quasiarithmetic.

Let $n \geq 4$, and let f be a quadratic form of rank $n+1$ defined over k. Assume that f is *admissible*; that is, assume

(i) f has signature $(n,1)$ when seen over $\mathbb{R}$;

(ii) f^σ is positive definite for any nontrivial embedding $\sigma\colon k \hookrightarrow \mathbb{R}$.

Condition (i) implies the existence of an $\mathbb{R}$–isometry $f \cong -x_0^2 + x_1^2 + \cdots + x_n^2$. Therefore we may identify the hyperbolic space $\mathbb{H}^n$ with the "f–hyperbolic space" $\mathbb{H}_f$ which is obtained by choosing one of the two connected components of the space

$$\{x \in \mathbb{R}^{n+1} \mid f(x) = -1\}.$$

Let O_f denote the algebraic k–group of f–orthogonal matrices and $\mathrm{PO}_f = \mathrm{O}_f / Z(\mathrm{O}_f)$ the associated projective orthogonal group (in a matrix representation, we have $\mathrm{PO}_f = \mathrm{O}_f / \{\pm \mathrm{id}\}$). Conditions (i) and (ii) ensure that PO_f is an admissible algebraic group; the corresponding arithmetic groups are called *of the first type*.

Instead of directly working in $\mathrm{PO}_f(\mathbb{R})$, it is more convenient for our purposes to use the Lie group $\mathrm{O}'_f(\mathbb{R})$ consisting of the matrices in $\mathrm{GL}_{n+1}(\mathbb{R})$ which preserve $\mathbb{H}_f$. This group acts on $\mathbb{H}_f$ and may be identified with the group $\mathrm{Isom}(\mathbb{H}^n)$ of isometries of the hyperbolic n–space. Moreover there is an obvious isomorphism $\mathrm{O}'_f(\mathbb{R}) \cong \mathrm{PO}_f(\mathbb{R})$ which allows us to see elements of $\mathrm{PO}_f(\mathbb{R})$ as matrices. Since it is a matrix group, we can define $\mathrm{O}'_f(A)$ for any subring $A \subset \mathbb{R}$ as $\mathrm{O}'_f(\mathbb{R}) \cap \mathrm{GL}_{n+1}(A)$. In particular, the group $\mathrm{O}'_f(\mathcal{O}_k)$ is unambiguously defined and is an arithmetic subgroup of the first type.

With these identifications, a hyperplane of $\mathbb{H}_f$ is simply $\mathbb{H}_f$ intersected with a linear subspace of $\mathbb{R}^{n+1}$ of dimension n. We will say that such a hyperplane is *k–rational* (or more briefly *rational*) if the corresponding subspace in $\mathbb{R}^{n+1}$ is the f–orthogonal complement of a vector $v \in k^{n+1}$ (equivalently, $v \in {\mathcal{O}_k}^{n+1}$). Observe that the reflection in such a hyperplane is an isometry of $\mathbb{H}_f$ which lies in $\mathrm{O}'_f(k)$.

1.2 Nonarithmeticity

We briefly recall the construction of Belolipetsky and Thomson [2] in order to prove Proposition 1. Let f/k be an admissible quadratic form over a totally real number field k. Let R_1 and R_2 be two *disjoint* rational hyperplanes in $\mathbb{H}_f$; ie R_1 and R_2 do not meet in $\mathbb{H}_f$ or at infinity. In that case, there is a unique geodesic segment v orthogonal to both R_1 and R_2, and its length is the distance between R_1 and R_2.

Let $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$ be a finite-index torsion-free subgroup such that

(i) for $i = 1, 2$ and each $\lambda \in \Lambda$, either λR_i is disjoint from R_i or they coincide;

(ii) λR_1 is disjoint from R_2 for any $\lambda \in \Lambda$.

In that case, for $i = 1, 2$, the orbit $\Lambda R_i = \{\lambda R_i \mid \lambda \in \Lambda\}$ forms a collection of disjoint hyperplanes, and $\Lambda R_1 \cap \Lambda R_2 = \varnothing$. It follows from [2, Lemma 3.1] that such a subgroup $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$ always exists.

Let $L = \Lambda \backslash \mathbb{H}_f$. By construction, the two hyperplanes R_1 and R_2 project down in L to two disjoint *hypersurfaces* (by which we mean finite-volume totally geodesic codimension 1 embedded submanifolds); we denote them by N_1 and N_2 respectively. The preimage of $L \setminus (N_1 \cup N_2)$ in $\mathbb{H}_f$ consists of disjoint convex open sets whose boundaries are hyperplane lifts of N_1 or N_2. Let $\widetilde{C}$ be the connected component among them having R_1 and R_2 as boundary components, and let C denote the image of $\widetilde{C}$ in L. Then, since $\widetilde{C}$ contains the geodesic segment v, the set C is a connected component of $L \setminus (N_1 \cup N_2)$ such that the distance separating N_1 and N_2 in C equals

the distance between R_1 and R_2 in $\mathbb{H}_f$. Finally let M denote the manifold obtained by gluing two copies of C to each other by identifying their boundaries (that is, M is the "double" of C). We will call M a *doubly cut gluing*, and R_1 and R_2 the *cut hyperplanes*. Observe that this manifold depends on the choice of the subgroup $\Lambda \subset \mathrm{O}'_f(\mathcal{O}_k)$.

Since the hypersurfaces N_1 and N_2 have finite volume, the manifold M is complete without boundary (even when N_1 and N_2 pass through cusps; see Gromov and Piatetski-Shapiro [7, Section 2.10.B]). Hence we may write it as $M = \Gamma\backslash\mathbb{H}_f$ with $\Gamma \subset \mathrm{O}'_f(\mathbb{R})$. Let D and D' denote the two copies of C, seen as submanifolds of M. A connected component $\widetilde{D}$ of the preimage of D under the covering map $\mathbb{H}_f \twoheadrightarrow M$ is then a universal cover of $D \cong C$. Thus upon conjugating Γ we can assume that $\widetilde{D} = \widetilde{C}$ and that

$$\mathrm{Stab}_\Gamma(\widetilde{C}) = \mathrm{Stab}_\Lambda(\widetilde{C}).$$

Write Λ^+ for this stabilizer; it corresponds to the image of $\pi_1(C)$ in Λ via the isomorphism $\pi_1(L) \cong \Lambda$.

The following lemma gives a generating set for Γ. A similar generating set appears in Thomson [15] in the proof of quasiarithmeticity. We include a proof of this specific version, which we will need in the sequel.

Lemma 1.1 *Let $\rho_1, \rho_2 \in \mathrm{O}'_f(k)$ denote the reflections in R_1, R_2 respectively. There exists $\lambda_1, \lambda_2 \in \Lambda$ such that*

$$\Gamma = \langle \Lambda^+,\ \rho_1\Lambda^+\rho_1,\ \rho_1\rho_2,\ \rho_1\lambda_1\rho_1\lambda_1^{-1},\ \rho_1\lambda_2\rho_2\lambda_2^{-1} \rangle.$$

In particular, $\Gamma \subset \langle \mathrm{O}'_f(\mathcal{O}_k), \rho_1, \rho_2 \rangle$.

Proof Topologically, M consists of D and D' glued together along their boundaries. Now depending on whether N_1 (resp. N_2) separates L or not, ∂D consists of one or two copies of N_1 (resp. N_2). Write $\partial D = N_{1,1} \cup N_{1,2} \cup N_{2,1} \cup N_{2,2}$ with $N_{i,j} \cong N_i$ and possibly $N_{i,1} = N_{i,2}$. By our choice of Γ, we can assume (upon exchanging $N_{i,1}$ and $N_{i,2}$) that the hyperplane R_i is a lift of $N_{i,1}$. Moreover since the boundary components of C corresponding to $N_{i,1}$ and $N_{i,2}$ are identified in L, we can find a (possibly trivial) $\lambda_i \in \Lambda$ such that $\lambda_i R_i$ is a hyperplane lift of $N_{i,2}$ contained in the boundary of $\widetilde{C}$.

Observe that by construction the reflections in the $N_{i,j}$ are isometries of M, and that all these reflections have the same effect on M: they exchange D and D'. Since $\widetilde{C}$

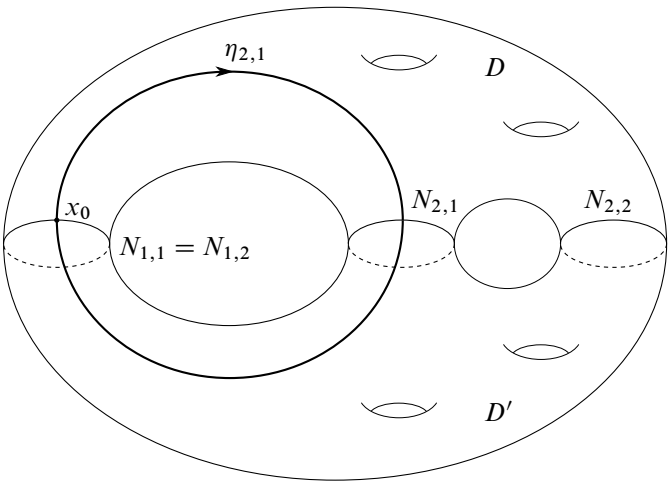


Figure 1: A doubly cut gluing M

contains all R_i and $\lambda_i R_i$ as boundary, it follows that the reflection about $N_{i,j}$ is induced by ρ_i if $j = 1$ and $\lambda_i \rho_i$ if $j = 2$. Thus for $i = 1, 2$ and $\lambda \in \{1, \lambda_i\}$, the elements ρ_1 and $\lambda \rho_i \lambda^{-1}$ induce the same isometry on M. Therefore

$$\rho_1 \lambda \rho_i \lambda^{-1} \in \Gamma.$$

Now pick a basepoint x_0 in $N_{1,1}$, and let $\widetilde{x}_0 \in R_1$ be a preimage of x_0. The geodesic between $\widetilde{x}_0$ and $\rho_2 \circ \rho_1(\widetilde{x}_0)$ is a segment which crosses R_2 exactly once and does not meet any other hyperplane lifts of the $N_{i,j}$ except at its endpoints. Therefore, $\rho_1 \rho_2 = (\rho_2 \rho_1)^{-1} \in \Gamma$ corresponds to the homotopy class $[\eta_{2,1}] \in \pi_1(M, x_0)$ of a loop $\eta_{2,1}$ at x_0 which crosses $N_{2,1}$ exactly once and is contained in D' before this crossing and in D after this crossing.

Similarly, for $i = 1, 2$, the element $\rho_1 \lambda_i \rho_i \lambda_i^{-1}$ corresponds to the class of a loop $\eta_{i,2}$ crossing $N_{i,2}$ exactly once. Consequently, the group $\pi_1(M, x_0)$ is generated by

$$\pi_1(D) \cup \pi_1(D') \cup \{[\eta_{2,1}], [\eta_{1,2}], [\eta_{2,2}]\}.$$

Since $\pi_1(D')$ corresponds to $\rho_1 \Lambda^+ \rho_1$ in Γ, the lemma follows. □

We turn towards the proof of Proposition 1.

Proof of Proposition 1 We prove the contrapositive. Assume M is arithmetic. Since Γ and Λ share the Zariski-dense subgroup Λ^+, it follows from Gromov and Piatetski-Shapiro [7, Section 1.6] that $\Gamma \cap \mathrm{O}'_f(\mathcal{O}_k)$ is a finite-index subgroup of Γ. Therefore

the element $g = \rho_1\rho_2 \in \Gamma$ must have a power g^N in $\mathrm{O}'_f(\mathcal{O}_k)$. Let α be an eigenvalue of g. Then α^N is an eigenvalue of g^N, and is thus an algebraic integer (since it lies in an integral extension of $\mathcal{O}_k$). It is easy to see that the same is true about α. Since α is arbitrary, we get that $\mathrm{tr}(g) = \sum(\text{eigenvalues})$ is also an algebraic integer. □

Remark 1.2 The proof actually only uses the fact that $\rho_1\rho_2$ has an eigenvalue which is not an algebraic integer. Therefore the conclusion of the proposition still holds if the trace condition is replaced by this requirement.

2 Noncommensurability and examples over $\mathbb{Q}$

2.1 The adjoint trace ring

Proposition 1 gives a way to control the nonarithmeticity of doubly cut gluings, but does not say anything about their commensurability. To that end, we will use an invariant introduced by Vinberg called the adjoint trace ring [17].

Let Γ be a Zariski-dense subgroup of a semisimple algebraic group G. The *adjoint trace field* of Γ is the field

$$K(\Gamma) = \mathbb{Q}(\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}),$$

where Ad denotes the adjoint representation. Similarly, the *adjoint trace ring* $A(\Gamma)$ of Γ is the integral closure of the ring $\mathbb{Z}[\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}]$. If $k = K(\Gamma)$ is a number field, we simply have (see Davis [5])

$$A(\Gamma) = \mathcal{O}_k[\{\mathrm{tr}\,\mathrm{Ad}(\gamma) \mid \gamma \in \Gamma\}].$$

In his paper, Vinberg defines the minimal field (resp. ring) of definition of $\mathrm{Ad}\,\Gamma$, and shows that it coincides with $K(\Gamma)$ (resp. $A(\Gamma)$) [17, Corollary of Theorem 1]. The adjoint trace field is an invariant of the commensurability class of Γ, and the same is true for the adjoint trace ring when the adjoint trace field is a number field [17, Theorem 3].

Let us assume that G is defined over $\mathbb{R}$. Suppose further that $\mathrm{G}(\mathbb{R}) \cong \mathrm{PO}(n,1)$ as Lie groups (with $n \geq 4$) and that $\Gamma \subset \mathrm{G}(\mathbb{R})$. For example, one can take $\mathrm{G} = \mathrm{PO}_f$ for f a signature $(n,1)$ quadratic form. In that case, the algebraic adjoint representation and the one coming from the Lie group structure coincide on $\mathrm{G}(\mathbb{R})$. Thus the adjoint trace field (resp. ring) of Γ does not depend on G, and it makes sense to speak of the adjoint trace field $K(M)$ (resp. of the adjoint trace ring $A(M)$) of a hyperbolic manifold M.

Remark 2.1 For $n = 3$ the group $\mathrm{Isom}(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$ has a structure of a complex Lie group. The adjoint trace field — for the complex adjoint representation — of a Zariski-dense $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ coincides with the invariant trace field in the sense of Maclachlan and Reid [8, Definition 3.3.6]. If one uses an algebraic group G such that $\mathrm{G}(\mathbb{R}) \cong \mathrm{Isom}(\mathbb{H}^3)$, one gets a different adjoint trace field. For $n \geq 4$ this ambiguity does not appear, as $\mathrm{PO}(n, 1)$ then does not possess a complex algebraic structure.

By Borel's density theorem, any lattice Γ is Zariski-dense in $\mathrm{G}(\mathbb{R})$ or $\mathrm{G}(\mathbb{R})^0$. Furthermore, it follows from local rigidity that the adjoint trace field of Γ is a number field (even if Γ is nonarithmetic; see Vinberg, Gorbatsevich and Shvartsman [18, Chapter 1, Section 6] or Bergeron and Gelander [3, page 124]).

Remark 2.2 When Γ is arithmetic and commensurable to $\mathrm{G}(\mathcal{O}_k)$, it follows from Prasad and Rapinchuk [11, Lemma 2.6] that $K(\Gamma) = k$ and $A(\Gamma) = \mathcal{O}_k$. In particular, the adjoint trace field of Λ^+ (defined in Section 1.2) is k since it contains the stabilizer of a rational hyperplane which is an arithmetic group of adjoint trace field k.

The following easy lemma is useful to compute the adjoint trace ring.

Lemma 2.3 *Let M be a doubly cut gluing with associated quadratic form f over the field k. Write $M = \Gamma\backslash\mathbb{H}_f$ with $\Gamma \subset \mathrm{O}'_f(\mathbb{R})$. Then the adjoint trace field $K(\Gamma)$ of Γ is k and its adjoint trace ring is*

$$A(\Gamma) = \mathcal{O}_k[\{\mathrm{tr}\,\gamma \mid \gamma \in \Gamma\}].$$

Proof We first look at the adjoint trace field. Choose Γ as in Lemma 1.1. Since Γ contains the Zariski-dense subgroup Λ^+ we get $K(\Gamma) \supset k$ (see Remark 2.2). From Lemma 1.1 we see that $\Gamma \subset \mathrm{O}'_f(k)$, thus establishing the other inclusion.

For the adjoint trace ring, we first apply Theorem 2 of Vinberg [17] to the representation $\Gamma \hookrightarrow \mathrm{O}'_f(k)$ given by the inclusion. We obtain that a conjugate of Γ lies in $\mathrm{O}'_f(A(\Gamma))$. By [17, Lemma 1 and 2] this happens exactly when $\{\mathrm{tr}(\gamma) \mid \gamma \in \Gamma\} \subset A(\Gamma)$ and the proof is complete. □

Remark 2.4 It follows that a doubly cut gluing is arithmetic if and only if the trace of all its elements are algebraic integers (that is, contained in $\mathcal{O}_k$). With the help of Lemma 1.1, Proposition 1 already implied one direction of this equivalence.

Remark 2.5 In the complex orthogonal group, the following formula relates the trace of an element and that of its adjoint:

$$\operatorname{tr}\operatorname{Ad}(g) = \frac{(\operatorname{tr} g)^2 - \operatorname{tr}(g^2)}{2} \quad \text{for any } g \in \mathrm{O}_n(\mathbb{C}).$$

This can be proven by direct computation using a basis of the Lie algebra. Since the formula holds over $\mathbb{C}$, it remains valid in any orthogonal group O_f for an arbitrary quadratic form f, and can therefore be used to compute the adjoint trace ring of hyperbolic manifolds.

2.2 Examples over $\mathbb{Q}$

To prove Theorem 2, we will apply the results of the previous section to the quadratic form $f = -x_0^2 + x_1^2 + \cdots + x_n^2$ defined over $\mathbb{Q}$. In this section, norms, scalar products and orthogonal complements are to be understood with respect to the quadratic form f.

Fix the hyperplane $R_1 = \{x_1 = 0\} = v^\perp$, where $v = (0, 1, 0, \ldots, 0)$. Let $w = (w_0, w_1, \ldots, w_n) \in \mathbb{Z}^{n+1}$ be such that

$$w_1^2 \geq \langle w, w\rangle > 0.$$

Then the hyperplanes R_1 and $R_2 = w^\perp$ do not intersect. Indeed we have $|w_1| = |\langle v, w\rangle| \geq \|v\|\|w\| = \sqrt{\langle w, w\rangle}$, and by [13, Theorem 3.2.7] it follows that $R_1 \cap R_2 = \varnothing$.

Let M_w be a doubly cut gluing with cut hyperplanes R_1 and R_2. Write $M_w = \Gamma_w \backslash \mathbb{H}_f$ with $\Gamma_w \subset \mathrm{O}'_f(\mathbb{R})$. Let ρ_1 and ρ_2 denote the reflections in R_1 and R_2 respectively.

Lemma 2.6 *The adjoint trace ring $A(M_w)$ of the manifold $M_w = \Gamma_w \backslash \mathbb{H}_f$ satisfies*

$$\mathbb{Z}\left[\frac{4w_1^2}{\langle w, w\rangle}\right] \subset A(M_w) \subset \mathbb{Z}\left[\frac{2}{\langle w, w\rangle}\right].$$

Proof By elementary computations, the matrix of ρ_2 is

$$\rho_2 = I - \frac{2}{\langle w, w\rangle} w w^t J, \quad \text{where } J = \operatorname{diag}(-1, 1, \ldots, 1).$$

Multiplying by ρ_1 and using linearity and cyclic invariance of the trace, one gets

$$\operatorname{tr}(\rho_1\rho_2) = (n-1) - 2\frac{\langle w, \rho_1 w\rangle}{\langle w, w\rangle}, \tag{2}$$

which in our case equals $n - 3 + 4w_1^2/\langle w, w\rangle$ by an easy computation. With Lemma 2.3 this establishes the left inclusion. The right inclusion follows from Lemma 1.1 since $\mathrm{O}'_f(\mathbb{Z}) \cup \{\rho_1, \rho_2\} \subset \mathrm{O}'_f(\mathbb{Z}[2/\langle w, w\rangle])$. □

We now dive into the proof Theorem 2.

Proof of Theorem 2 Set $b = d$ if 2 does not divide d, and $b = 4d$ otherwise. Pick $w_1 \in \mathbb{Z}$ such that $w_1^2 > b$ and $\gcd(w_1, b) = 1$. Choose integers $w_0, w_2, \dots, w_n$ such that $-w_0^2 + w_2^2 + \cdots + w_n^2 = (b - w_1^2)$. The latter is possible since the quadratic form $-x_0^2 + x_2^2 + \cdots + x_n^2$ represents any arbitrary integer ($-x_0^2 + x_2^2$ represents every odd integer, and we have at least one more variable to represent 1). If $w = (w_0, w_1, \dots, w_n)$, we have $\langle w, w\rangle = b$ whence $w_1^2 > \langle w, w\rangle > 0$. Thus we can construct $M_w = \Gamma_w \backslash \mathbb{H}_f$ as in the beginning of this section. Since $\gcd(w_1^2, b) = 1$ we have, using standard properties of subrings of $\mathbb{Q}$,

$$\mathbb{Z}\left[\frac{4w_1^2}{\langle w, w\rangle}\right] = \mathbb{Z}[4/b] = \mathbb{Z}[1/d] = \mathbb{Z}\left[\frac{2}{\langle w, w\rangle}\right].$$

Therefore the theorem follows from Lemma 2.6, with $\Gamma_d = \Gamma_w$. □

Remark 2.7 It follows from the proof that the lattices Γ_d are all contained in $\mathrm{O}'_f(\mathbb{Q})$ for the same quadratic form f. Since its restriction to R_1 is simply $-x_0^2 + x_2^2 + \cdots + x_n^2$, it follows (this form being isotropic) that N_1 and thus also M_w are noncompact for any (admissible) w.

More generally, for $n \geq 5$ any doubly cut gluing M with adjoint trace field $K(M) = \mathbb{Q}$ is noncompact. Indeed, the adjoint trace field of the gluing hypersurface must then also be $\mathbb{Q}$, and since it is arithmetic (of the first type) it corresponds to a quadratic form over $\mathbb{Q}$ with at least 5 variables. Such a quadratic form is always isotropic, and thus, by Godement's compactness criterion, the hypersurface, and hence M, are noncompact.

Remark 2.8 Theorem 2 still holds for $n = 3$, but the notion of adjoint trace ring/field in the statement differs from its usual meaning if the lattices are considered in $\mathrm{PSL}_2(\mathbb{C})$ (see Remark 2.1). Thus Theorem 2 has been stated for $n \geq 4$ to avoid confusion.

3 Volume bound

The doubly cut gluings M_w from previous section depend on the choice of a finite-index subgroup $\Lambda \subset \mathrm{O}'_f(\mathbb{Z})$; in this section we will write M_w^Λ for the M_w obtained using the subgroup Λ. Note that it is not clear if two commensurable subgroups $\Lambda, \Lambda' \subset \mathrm{O}'_f(\mathbb{Z})$ give rise to commensurable manifolds M_w^Λ and $M_w^{\Lambda'}$.

Define V_w to be the minimal volume of M_w^Λ for any choice of $\Lambda \subset \mathrm{O}'_f(\mathbb{Z})$ (satisfying the requirements given in the beginning of Section 1.2). Our goal is to give an upper-bound on V_w by constructing explicitly such a Λ. The arguments of this section are inspired by the proof of [9, Lemma 10] of Margulis and Vinberg.

We use the notation of the previous section. The subgroup $\Lambda \subset \mathrm{O}'_f(\mathbb{Z})$ must be torsion-free and satisfy, for all $\lambda \in \Lambda$,

(i) $\lambda R_1 \cap R_1 = \varnothing$,

(ii) $\lambda R_2 \cap R_2 = \varnothing$,

(iii) $\lambda R_1 \cap R_2 = \varnothing$.

We focus on principal congruence subgroups of the form

$$\Lambda_m = \{\lambda \in \mathrm{O}'_f(\mathbb{Z}) \mid \lambda \equiv \mathrm{id} \bmod m\} \quad \text{(for } m \in \mathbb{Z} \text{ not necessarily prime)}.$$

The goal is to find $m \geq 2$ such that Λ_m fulfills (i)–(iii).

To obtain a torsion-free subgroup, it is enough to take a congruence subgroup Λ_m with $m > 2$ (see [10, Theorem IX.7]). Furthermore, it follows from Gromov and Piatetski-Shapiro [7, Section 2.8] that any principal congruence subgroup satisfies (i).

For (ii), we need to ensure that for any $\lambda \in \Lambda_m$ we have (using [13, Theorem 3.2.7], as above)

$$|\langle \lambda w, w\rangle| \geq \langle w, w\rangle. \tag{3}$$

Now if we choose $m \geq 2\langle w, w\rangle$, the fact that

$$\langle \lambda w, w\rangle = \langle w, w\rangle \mod m$$

implies inequality (3).

Finally for (iii) the situation is similar: we just need to ensure that

$$|\langle \lambda v, w\rangle| \geq \|w\|. \tag{4}$$

If $m \geq 2w_1^2$, the fact that $w_1^2 \geq \langle w, w\rangle$ and the equation

$$\langle \lambda v, w\rangle^2 = \langle v, w\rangle^2 = w_1^2 \mod m$$

imply that inequalities (3) and (4) are fulfilled.

To sum up, we define $\Lambda = \Lambda_m$, where $m = \max(3, 2w_1^2)$. This congruence subgroup is then such that $L = \Lambda \backslash \mathbb{H}_f$ is a hyperbolic manifold in which R_1 and R_2 project to

disjoint hypersurfaces, as desired. Now the doubly cut gluing M_w^Λ constructed from Λ is obtained as the double of a piece of L. Thus its volume cannot exceed twice the volume of L, and we have proven:

Proposition 3.1 *The minimal volume V_w of a doubly cut gluing M_w constructed as in Section 2.2 satisfies*

$$V_w \leq 2 \cdot |\mathrm{O}_f(\mathbb{Z}/m\mathbb{Z})| \cdot \operatorname{covol} \mathrm{O}_f(\mathbb{Z}),$$

where $m = \max(3, 2w_1^2)$.

As a corollary we obtain the following examples of relatively small volume:

Corollary 3.2 *For any $n \geq 4$, there exists an n–dimensional nonarithmetic doubly cut gluing M with*

$$\operatorname{vol}(M) \leq 2 \cdot |\mathrm{O}_f(\mathbb{Z}/8\mathbb{Z})| \cdot \operatorname{covol} \mathrm{O}_f(\mathbb{Z}).$$

Proof We pick $w = (1, 2, 0, \ldots, 0)$. Observe that $w_1^2 = 4 > \langle w, w \rangle = 3 > 0$. Therefore, we can construct doubly cut gluings M_w^Λ as before. Let $M = M_w^\Lambda$ be a doubly cut gluing realizing the volume V_w. This manifold has adjoint trace ring $\mathbb{Z}\left[\frac{1}{3}\right]$ (by Lemma 2.6) and is therefore nonarithmetic. The volume bound is a consequence of Proposition 3.1. □

This volume bound is larger than — but still comparable with — the volume bound $|\mathrm{O}_f(\mathbb{Z}/3\mathbb{Z})| \cdot \operatorname{covol}(\mathrm{O}_f(\mathbb{Z}))$ in the arithmetic case (which is obtained using the principal congruence subgroup mod 3). For exact computations of the volumes we refer to the formulas of Ratcliffe and Tschantz [14] and the references therein.

4 About generalizations

There are two easy-to-state generalizations of the doubly cut gluing construction of nonarithmetic manifolds. The first one is to increase the number of cut hyperplanes. Indeed, if one finds n disjoint rational hyperplanes, then it is possible (using again Belolipetsky and Thomson [2, Lemma 3.1]) to find an arithmetic lattice $\Gamma \subset \mathrm{PO}_f$ such that they project down to disjoint hypersurfaces. If one chooses the hyperplanes in such a way that their reflections have interesting rational properties, one might get better results regarding, for example, the minimal volume estimate of a manifold with prescribed adjoint trace ring.

The second possible generalization is to give an analog of Theorem 2 for number fields. One way to proceed would be to replace $\mathbb{Z}[1/d]$ with the ring $\mathcal{O}_S$ of S–integers of a totally real number field k, where S is a finite set of nonarchimedean places. For specific examples of S this is feasible (as suggested in the toy example of the next proposition). However in order to get a general result one has to face the problem that an admissible quadratic form over $k \neq \mathbb{Q}$ is not isotropic, and thus cannot represent any element of k. Moreover, even when we restrict to quadratic extensions, it is likely that more hyperplanes will be needed to generate $\mathcal{O}_S$, thereby making the construction more complicated.

Proposition 4.1 *Let $k = \mathbb{Q}(\alpha)$ be a totally real number field, with α an algebraic integer which is positive at all but one embedding $k \hookrightarrow \mathbb{R}$. Let $d \in \mathbb{Z}$ be a square-free integer. Then there exists $c \in \mathbb{Z}$ having the same prime factors as d and a doubly cut gluing M of arbitrary dimension $n \geq 4$ with adjoint trace ring satisfying*

$$\mathcal{O}_k\left[\frac{1}{d}\right] \subset A(M) \subset \mathcal{O}_k\left[\frac{1}{d}, \frac{1}{c-\alpha}\right].$$

Proof Start with $c = d$. By eventually increasing the powers of the primes occurring in it, we can ensure that $c > \max_\sigma |\sigma(\alpha)|$, where σ ranges over all embeddings $k \hookrightarrow \mathbb{R}$. Consider the quadratic form $f = \alpha x_0^2 + (c-\alpha)x_1^2 + x_2^2 + \cdots + x_n^2$. The conditions on α and c imply that f is admissible. Fix an embedding $k \subset \mathbb{R}$ such that $\alpha < 0$, and set $v = (0, 1, 0, \ldots, 0)$ and $w = (1, 1, 0, \ldots, 0)$. We have

$$f(v) = c - \alpha \quad \text{and} \quad f(w) = c,$$

hence both have positive f–norm, and the hyperplanes they define do not intersect since $\langle v, w\rangle_f^2 = (c-\alpha)^2 > (c-\alpha)c = f(v)f(w)$. Now if ρ_1 and ρ_2 denote the reflections at the hyperplanes defined by v and w respectively, a computation shows that formula (2) in the proof of Lemma 2.6 still holds when one uses the scalar product induced by our f instead. We get

$$\operatorname{tr}(\rho_1\rho_2) = (n-1) - 2 \cdot \frac{2\alpha - c}{c} \in -\frac{4\alpha}{c} + \mathbb{Z}.$$

Let $p(x)$ denote the minimal polynomial of α, and let $N = p(0)$. Then $(p(\alpha) - N)/\alpha$ is in $\mathcal{O}_k$, and thus

$$\mathcal{O}_k[1/c] \supset \mathcal{O}_k[\operatorname{tr}(\rho_1\rho_2)] \supset \mathcal{O}_k\left[\frac{p(\alpha)-N}{\alpha} \cdot \frac{4\alpha}{c}\right] \supset \mathcal{O}_k[4N/c].$$

Now by possibly increasing further the powers of the primes in c that divide $4N$, we get that the rings on both sides of the chain of inclusions coincide and are equal to $\mathcal{O}_k[1/d]$.

This proves the first inclusion of the proposition. The second inclusion follows from the same argument as in Lemma 2.6, observing that $\rho_1 \in \mathrm{O}'_f(\mathcal{O}_k[1/c-\alpha])$ and $\rho_2 \in \mathrm{O}'_f(\mathcal{O}_k[1/d])$. □

References

[1] **I Agol**, *Systoles of hyperbolic 4–manifolds*, preprint (2006) arXiv

[2] **M V Belolipetsky**, **S A Thomson**, *Systoles of hyperbolic manifolds*, Algebr. Geom. Topol. 11 (2011) 1455–1469 MR

[3] **N Bergeron**, **T Gelander**, *A note on local rigidity*, Geom. Dedicata 107 (2004) 111–131 MR

[4] **A Borel**, **G Prasad**, *Finiteness theorems for discrete subgroups of bounded covolume in semi-simple groups*, Inst. Hautes Études Sci. Publ. Math. 69 (1989) 119–171 MR

[5] **E D Davis**, *Overrings of commutative rings, II: Integrally closed overrings*, Trans. Amer. Math. Soc. 110 (1964) 196–212 MR

[6] **T Gelander**, **A Levit**, *Counting commensurability classes of hyperbolic manifolds*, Geom. Funct. Anal. 24 (2014) 1431–1447 MR

[7] **M Gromov**, **I Piatetski-Shapiro**, *Nonarithmetic groups in Lobachevsky spaces*, Inst. Hautes Études Sci. Publ. Math. 66 (1988) 93–103 MR

[8] **C Maclachlan**, **A W Reid**, *The arithmetic of hyperbolic 3–manifolds*, Graduate Texts in Mathematics 219, Springer (2003) MR

[9] **G A Margulis**, **E B Vinberg**, *Some linear groups virtually having a free quotient*, J. Lie Theory 10 (2000) 171–180 MR

[10] **M Newman**, *Integral matrices*, Pure and Applied Mathematics 45, Academic, New York (1972) MR

[11] **G Prasad**, **A S Rapinchuk**, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces*, Publ. Math. Inst. Hautes Études Sci. 109 (2009) 113–184 MR

[12] **J Raimbault**, *A note on maximal lattice growth in* $\mathrm{SO}(1,n)$, Int. Math. Res. Not. 2013 (2013) 3722–3731 MR

[13] **J G Ratcliffe**, *Foundations of hyperbolic manifolds*, 2nd edition, Graduate Texts in Mathematics 149, Springer (2006) MR

[14] **J G Ratcliffe**, **S T Tschantz**, *Volumes of integral congruence hyperbolic manifolds*, J. Reine Angew. Math. 488 (1997) 55–78 MR

[15] **S Thomson**, *Quasi-arithmeticity of lattices in* $\mathrm{PO}(n,1)$, Geom. Dedicata 180 (2016) 85–94 MR

[16] **E B Vinberg**, *Discrete groups generated by reflections in Lobachevskii spaces*, Mat. Sb. 72 (1967) 471–488 MR Correction in 73 (1967) 303–303 In Russian; translated in Math. USSR-Sb. 1 (1967) 429–444

[17] **E B Vinberg**, *Rings of definition of dense subgroups of semisimple linear groups*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971) 45–55 MR In Russian; translated in Math. USSR-Izv. 5 (1971) 45–55

[18] **E B Vinberg**, **V V Gorbatsevich**, **O V Shvartsman**, *Discrete subgroups of Lie groups*, from "Lie groups and Lie algebras, II" (E B Vinberg, editor), Encyclopaedia Math. Sci. 21, Springer (2000) 1–123, 217–223 MR

Mathematisches Institut, Universität Bern
Bern, Switzerland

olivier.mila@math.unibe.ch

Received: 1 June 2018 Revised: 23 July 2018

Guidelines for Authors

Submitting a paper to Algebraic & Geometric Topology

Papers must be submitted using the upload page at the AGT website. You will need to choose a suitable editor from the list of editors' interests and to supply MSC codes.

The normal language used by the journal is English. Articles written in other languages are acceptable, provided your chosen editor is comfortable with the language and you supply an additional English version of the abstract.

Preparing your article for Algebraic & Geometric Topology

At the time of submission you need only supply a PDF file. Once accepted for publication, the paper must be supplied in LaTeX, preferably using the journal's class file. More information on preparing articles in LaTeX for publication in AGT is available on the AGT website.

`arXiv` papers

If your paper has previously been deposited on the `arXiv`, we will need its `arXiv` number at acceptance time. This allows us to deposit the DOI of the published version on the paper's `arXiv` page.

References

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited at least once in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used, but will be converted to the house style (see a current issue for examples).

Figures

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Fuzzy or sloppily drawn figures will not be accepted. For labeling figure elements consider the pinlabel LaTeX package, but other methods are fine if the result is editable. If you're not sure whether your figures are acceptable, check with production by sending an email to graphics@msp.org.

Proofs

Page proofs will be made available to authors (or to the designated corresponding author) in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

ALGEBRAIC & GEOMETRIC TOPOLOGY

Volume 18 Issue 7 (pages 3749–4373) 2018

Poincaré duality complexes with highly connected universal cover 3749
BEATRICE BLEILE, IMRE BOKOR and JONATHAN A HILLMAN

Cubical rigidification, the cobar construction and the based loop space 3789
MANUEL RIVERA and MAHMOUD ZEINALIAN

Noncrossing partitions and Milnor fibers 3821
THOMAS BRADY, MICHAEL J FALK and COLUM WATT

Knot Floer homology and Khovanov–Rozansky homology for singular links 3839
NATHAN DOWLIN

Some extensions in the Adams spectral sequence and the 51–stem 3887
GUOZHEN WANG and ZHOULI XU

Dimension functions for spherical fibrations 3907
CIHAN OKAY and ERGÜN YALÇIN

Weighted sheaves and homology of Artin groups 3943
GIOVANNI PAOLINI and MARIO SALVETTI

Equivariant complex bundles, fixed points and equivariant unitary bordism 4001
ANDRÉS ÁNGEL, JOSÉ MANUEL GÓMEZ and BERNARDO URIBE

Detecting a subclass of torsion-generated groups 4037
EMILY STARK

Cohomology of symplectic groups and Meyer's signature theorem 4069
DAVE BENSON, CATERINA CAMPAGNOLO, ANDREW RANICKI and CARMEN ROVI

On periodic groups of homeomorphisms of the 2–dimensional sphere 4093
JONATHAN CONEJEROS

Algebraic and topological properties of big mapping class groups 4109
PRIYAM PATEL and NICHOLAS G VLAMIS

Equivariant cohomology Chern numbers determine equivariant unitary bordism for torus groups 4143
ZHI LÜ and WEI WANG

Spaces of orders of some one-relator groups 4161
JUAN ALONSO and JOAQUÍN BRUM

On the asymptotic expansion of the quantum $\mathrm{SU}(2)$ invariant at $q = \exp(4\pi\sqrt{-1}/N)$ for closed hyperbolic 3–manifolds obtained by integral surgery along the figure-eight knot 4187
TOMOTADA OHTSUKI

Notes on open book decompositions for Engel structures 4275
VINCENT COLIN, FRANCISCO PRESAS and THOMAS VOGEL

Anick spaces and Kac–Moody groups 4305
STEPHEN THERIAULT and JIE WU

Logarithmic Hennings invariants for restricted quantum $\mathfrak{sl}(2)$ 4329
ANNA BELIAKOVA, CHRISTIAN BLANCHET and NATHAN GEER

Nonarithmetic hyperbolic manifolds and trace rings 4359
OLIVIER MILA

1472-2747(2018)18:7;1-L