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Noncrossing partitions and Milnor fibers

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For a finite real reflection group W we use noncrossing partitions of type W to construct finite cell complexes with the homotopy type of the Milnor fiber of the associated W–discriminant Δ_W and that of the Milnor fiber of the defining polynomial of the associated reflection arrangement. These complexes support natural cyclic group actions realizing the geometric monodromy. Using the shellability of the noncrossing partition lattice, this cell complex yields a chain complex of homology groups computing the integral homology of the Milnor fiber of Δ_W.

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1 Introduction

Suppose $g \in \mathbb{C}[z_1, \dots, z_n]$ is a quasihomogeneous polynomial, defining the hypersurface $V = g^{-1}(0)$ in $\mathbb{C}^n$. Then g restricts to a locally trivial fibration $g\colon \mathbb{C}^n - V \to \mathbb{C}^*$, the global Milnor fibration, with fiber $g^{-1}(1)$, the Milnor fiber of g; see Milnor [25]. The topology of $g^{-1}(1)$ and the monodromy of the bundle are invariants of the singularity type of g at the origin. Of special interest is the case where $g = Q_W$ is a product of complex linear forms defining the arrangement $\mathcal{A} = \mathcal{A}_W$ of reflecting hyperplanes in $\mathbb{C}^n$ of a finite real or complex reflection group W; see Dimca and Lehrer [19], Settepanella [30; 31] and Măcinic and Papadima [27].

In this setting W acts on $\mathbb{C}^n$, preserving $V = \bigcup_{H \in \mathcal{A}} H$, the quotient $W \backslash \mathbb{C}^n$ is homeomorphic to $\mathbb{C}^n$, and under this homeomorphism $W \backslash V$ is carried to a hypersurface Δ_W in $\mathbb{C}^n$. This hypersurface is the zero locus of a quasihomogeneous polynomial P_W, well-defined up to polynomial automorphism of $\mathbb{C}^n$, called the discriminant associated with W (see Section 3).

The fundamental group of $\mathbb{C}^n - \Delta_W$ is the generalized braid group (or Artin group) $B(W)$ associated with W. If W has type A_{d-1} then P_W is the classical discriminant for polynomials of degree d and $B(W)$ is isomorphic to the classical braid group

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on d strands. In this paper, we construct a noncrossing partition (NCP) model for the Milnor fiber $F_P = P_W^{-1}(1)$ and study its structure, including the monodromy action, in the case where W is a real reflection group. We also construct an NCP model for the Milnor fiber $F_Q = Q_W^{-1}(1)$ of the reflection arrangement $\mathcal{A}_W$. Both models arise as subcomplexes of appropriate covering spaces of a finite $K(B(W),1)$ which is defined in terms of noncrossing partitions; see Bessis [4], Brady [7] and Brady and Watt [8].

The NCP model for F_P has a natural filtration by subcomplexes, which are seen to be homotopy equivalent to bouquets of spheres using the lexicographic shellability of the noncrossing partition lattice. This yields a chain complex computing $H_*(F_P, \mathbb{Z})$ whose terms are homology groups of truncations of this lattice.

The standard approach to calculating the (co)homology of F_Q (resp. F_P) is via the co(homology) of $\mathbb{C}^n - V$ (resp. $\mathbb{C}^n - \Delta_W$) with twisted coefficients in $R[t,t^{-1}]$ for some ring R. In the case of F_P, a spectral sequence is often used to compute the cohomology with coefficients in $R[t,t^{-1}]$ of the Salvetti complex, a small cellular model for $\mathbb{C}^n - \Delta_W$ (see De Concini and Salvetti [15], De Concini, Salvetti and Stumbo [16], Callegaro [12], Callegaro and Salvetti [13], Frenkel [20] and Salvetti [29]). Our approach is to construct a small model $\widehat{F}_P$ for F_P and use shellability of the NCP lattice to simplify the direct calculation of homology with integer coefficients. The disadvantage with this approach compared to using the Salvetti complex is the large ranks encountered in our chain complex. For example, the rank of the top-dimensional group in this complex is equal to the number of NCPs which are not contained in proper standard parabolic subgroups (see Athanasiadis, Brady and Watt [3]). The corresponding ranks for the model $\widehat{F}_Q$ of F_Q increase by the factor $|W|/2$.

2 NCP models for subgroups of $B(W)$

2.1 Background

Let W be a finite, irreducible, real reflection group of rank n and let T be the set of all reflections in W. For background on finite reflection groups, see [6; 22]. Equip W with the total reflection length function $w \mapsto |w|$ (with respect to the generating set T) and with the partial order $\leq$ given by $u \leq w$ whenever $|u|+|u^{-1}w| = |w|$ (see [2]). We will use the notation $u \lessdot w$ for the case where w covers u. Fix a specific Coxeter element γ in W and define the noncrossing partitions to be the elements in the interval $[e,\gamma]$ in the poset $(W,\leq)$. The poset of W–noncrossing partitions is a lattice, L (see [9]), whose order complex is denoted $|L|$. If W is of type A_n, then W is isomorphic to the group of permutations of $\{1, \dots, n+1\}$ and the noncrossing partitions are those elements whose cycle structure gives a classical noncrossing partition; see [7].

We define $B(W)$ to be the group with generating set

$$\{[w] : w \in L, w \neq e\}$$

subject to the relations

$$[w_1][w_1^{-1} w_2] = [w_2] \quad \text{whenever } w_1 \leq w_2.$$

It is shown in [4; 7; 8] that $B(W)$ is isomorphic to the generalized braid group of type W.

We recall from [4; 7; 8] the contractible, n–dimensional, simplicial complex X whose k–simplices are ordered $(k+1)$–tuples from $B(W)$ of the form $(g_0, g_1, \dots, g_k)$ with $g_i = g_0[w_i]$ for some chain $e < w_1 < w_2 < \dots < w_k$ in L. It is convenient to use the notation $(g_0, e < w_1 < \dots < w_k)$ for such a simplex. Thus the simplices of X are identified with pairs (g, σ), where $g \in B(W)$ and σ is an *initialized chain* in L, that is, σ is a chain of the form $e < w_1 < \dots < w_k$. As $B(W)$ acts freely on X, the quotient $K := B(W)\backslash X$ is a $K(B(W), 1)$ and X is its universal cover. The action of $B(W)$ on X is given by

$$g \cdot (g_0, g_1, \dots, g_k) = (gg_0, gg_1, \dots, gg_k),$$

or, in terms of the pair notation,

$$g \cdot (g_0, e < w_1 < \dots < w_k) = (gg_0, e < w_1 < \dots < w_k). \tag{2-1}$$

It is immediate that the simplex $(g_0, e < w_1 < \dots < w_k)$ has k faces of the form $(g_0, e < w_1 < \dots < \widehat{w_i} < \dots < w_k)$, for $1 \leq i \leq k$, each obtained by deleting one of $w_1, w_2, \dots, w_k$ from σ. The remaining face is obtained by deleting e from σ and hence is given by the ordered set

$$(g_0[w_1], g_0[w_2], \dots, g_0[w_k]) = (g_0[w_1]) \cdot (e, [w_1]^{-1}[w_2], \dots, [w_1]^{-1}[w_k]).$$

In pair notation, this is denoted $(g_0[w_1], e < w_1^{-1} w_2 < \dots < w_1^{-1} w_k)$.

2.2 Quotients of X

If H is a normal subgroup of $B(W)$ then we can form a CW–complex X_H whose cells are of the form (Hg, σ), where σ is an initialized chain in L and the first component is a right H coset. If $\sigma = e < w_1 < \cdots < w_k$, then this cell has k boundary faces of the form

$$(Hg, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k) \quad \text{for } 1 \le i \le k,$$

with the remaining face given by $(Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$. It will be convenient to refer to $(Hg[w_1], e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$ as the *top* face of (Hg,σ) and to $(Hg, e < w_1 < \cdots < w_{k-1})$ as the *bottom* face of (Hg,σ). Since X is contractible, X_H is a $K(H,1)$. The action of the quotient group $H\backslash B(W)$ on X_H is given by $(Hg_1)(Hg_2,\sigma) = (Hg_1g_2,\sigma)$.

We now highlight a particular feature of these complexes X_H.

Lemma 2.1 *In X_H, each k–cell of the form*

$$c_k = (Hg, e < w_1 < w_2 < \cdots < w_k) \quad \textit{with } |w_k| < n$$

is incident on precisely two $(k+1)$–cells of the form

$$c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).$$

Proof Suppose that the cell

$$c_k = (Hg, e < w_1 < w_2 < \cdots < w_k),$$

with $|w_k| < n$, is incident on a $(k+1)$–cell of the form

$$c_{k+1} = (Hg', e < u_1 < \cdots < u_k < \gamma).$$

Since the chain of c_k does not contain γ, the cell c_k must be obtained by deleting either e or γ from the chain of c_{k+1}. In the latter case, c_k is the bottom face of c_{k+1}, forcing $Hg' = Hg$ and $u_i = w_i$ for $i = 1, \dots, k$. In the former case, c_k is the top face

$$(Hg'[u_1], e < u_1^{-1}u_2 < \cdots < u_1^{-1}u_k < u_1^{-1}\gamma),$$

so that $Hg' = Hg[w_1]^{-1}$ and $u_1 = \gamma w_k^{-1}, u_2 = \gamma w_k^{-1}w_1, \dots, u_k = \gamma w_k^{-1}w_{k-1}$. □

In our examples, H will arise as the kernel of a specific homomorphism ϕ with domain $B(W)$. In this case we will denote $X_{\ker(\phi)}$ by X_ϕ. We can then identify the coset Hg with the element $\phi(g)$ and denote the cells of X_H as pairs $(\phi(g),\sigma)$, where σ is an initialized chain in L. If $\sigma = e < w_1 < \cdots < w_k$, then the cell $(\phi(g),\sigma)$ has top face given by $\big(\phi(g)\phi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\big)$.

Example 2.2 If ϕ is the trivial homomorphism, then $H = B(W)$ and $X_\phi = K$ is the $K(B(W), 1)$ introduced earlier. The cells of K are of the form (e, σ), where σ is an initialized chain in L. If $\sigma = e < w_1 < \cdots < w_k$, then this cell has top face given by

$$\big(e\phi([w_1]), e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\big) = (e, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k).$$

Thus $X = X_\phi$ can be identified with the quotient of $|L|$ under the equivalence relation generated by identifying $w_1 < w_2 < \cdots < w_k$ with $e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k$.

Example 2.3 The standard projection $s\colon B(W) \to W$, $[w] \mapsto w$, which takes each NCP generator of $B(W)$ to the corresponding NCP in W, is a homomorphism by our presentation of $B(W)$. The kernel of s is the pure braid group $PB(W)$ associated to W, and X_s is a $K(\pi, 1)$ for $\pi = PB(W)$. (By [11; 18], X_s is homotopy equivalent to the complement $M = \mathbb{C}^n - \bigcup_{H \in \mathcal{A}} H$, where $\mathcal{A}$ is the complexification of the associated real reflection arrangement.) The cells of X_s can be identified with pairs (w, σ), for $w \in W$ and σ an initialized chain in L. The top face of the cell $(w, e < w_1 < \cdots < w_k)$ is $(ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$.

We will consider two further examples of this construction in Sections 4 and 5. These will give the models for the fibers mentioned in the introduction. The next section establishes the homotopy types of these fibers.

3 Discriminants and Milnor fibers

In this section we recall some definitions and basic facts about discriminants and Milnor fibers.

3.1 The W–discriminant

Recall that W is a real reflection group whose action on $\mathbb{R}^n$ has been complexified to an action on $\mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$. Let $\{f_1, \ldots, f_n\}$ be a set of basic invariants for the ring of W–invariant, complex polynomials [14]. The function $f = (f_1, \ldots, f_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ induces a homeomorphism of $W \backslash \mathbb{C}^n$ with $\mathbb{C}^n$. Recall that T denotes the set of reflections in W, and for each $t \in T$, let $H_t \subset \mathbb{C}^n$ denote its fixed complex hyperplane and $\lambda_t\colon \mathbb{C}^n \to \mathbb{C}$ be a complex linear form with kernel H_t. The polynomial $Q = \prod_{t \in T} \lambda_t\colon \mathbb{C}^n \to \mathbb{C}$ has the property that $Q(wx) = \det(w)Q(x)$ for all $w \in W$ and $x \in \mathbb{C}^n$ and hence Q^2 is invariant under the action of W. It follows that $Q^2 = P(f_1, \ldots, f_n)$ for some quasihomogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ whose weights are equal to the degrees of the f_i. The polynomial P is called the discriminant of W. It is unique up to polynomial automorphism of $\mathbb{C}^n$. The action of W on $\mathbb{C}^n$ leaves the affine algebraic hypersurface $V = Q^{-1}(0) = \bigcup_{t\in T} H_t$ invariant and its quotient $\Delta_W = W\backslash V$ is identified with the affine algebraic hypersurface $\Delta := P^{-1}(0)$. The space $M = \mathbb{C}^n - V$ is a $K(PB(W), 1)$, and W acts freely on M, so the space $f(M) = W\backslash M \cong \mathbb{C}^n - \Delta$ is a $K(B(W), 1)$ [10; 18]. In what follows, we identify $PB(W)$ with $\pi_1(M, z_0)$ and $B(W)$ with $\pi_1(f(M), f(z_0))$.

3.2 Milnor fibers of P and Q

The restriction to $\mathbb{C}^n - g^{-1}(0)$ of any quasihomogeneous polynomial $g\colon \mathbb{C}^n \to \mathbb{C}$ is a locally trivial fibration whose fiber $g^{-1}(1)$ is called the Milnor fiber of g [25]. The Milnor fibers of P and Q will be denoted by F_P and F_Q. These spaces are determined up to polynomial diffeomorphism by W. Then $(Q^2)^{-1}(1) = Q^{-1}(1) \cup Q^{-1}(-1)$ is invariant under the action of W and $F_P = W\backslash((Q^2)^{-1}(1)) \cong W^+\backslash F_Q$, where $W^+ = \{w \in W : \det(w) = 1\}$. The space F_Q is a connected, regular, W^+–cover of F_P since the action of W^+ on F_Q is free.

We will show that each of F_P and F_Q is homotopy equivalent to a complex of the form X_ϕ. Our proofs will make use of the following general result.

Proposition 3.1 *Suppose $g\colon E \to \mathbb{C}^*$ is a fibration with fiber F, where each of E and F has the homotopy type of a connected CW–complex. Then F is homotopy equivalent to the cover of E corresponding to the kernel of $g_*\colon \pi_1(E) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}$.*

Proof Let i denote the inclusion of F into E. The exact sequence of the fibration implies that $i_*\colon \pi_1(F) \to \pi_1(E)$ is an injection whose image is $\ker(g_*)$. Let $p\colon F' \to E$ denote the connected cover of E corresponding to $\ker(g_*)$. Then the inclusion i lifts to a map $h\colon F \to F'$ with $p \circ h = i$. We show that h is a homotopy equivalence. Since p_* and i_* are injections, it follows that $h_*\colon \pi_1(F) \to \pi_1(F')$ is an injection and hence is an isomorphism. Since $p_*\colon \pi_k(F') \to \pi_k(E)$ and $i_*\colon \pi_k(F) \to \pi_k(E)$ are isomorphisms for all $k \geq 2$ (the latter by the exact sequence of the fibration), $h_*\colon \pi_k(F) \to \pi_k(F')$ is also. As each of F and F' has the homotopy type of a CW–complex, h is a homotopy equivalence. □

Since algebraic sets are homotopy equivalent to CW–complexes, Proposition 3.1 yields the following corollary.

Corollary 3.2 (i) *The Milnor fiber F_P of P is homotopy equivalent to the cover of $f(M) = W\backslash M$ corresponding to the kernel of the map $P_*\colon B(W) \to \mathbb{Z}$.*

(ii) *The Milnor fiber F_Q of Q is homotopy equivalent to the cover of $f(M) = W\backslash M$ corresponding to the group $f_*(\ker(Q_*))$.*

Proof (ii) By Proposition 3.1, F_Q is homotopy equivalent to the connected domain Y of a covering map $\rho\colon Y \to M$ for which $\rho_*(\pi_1(Y, y_0)) = \ker(Q_*)$. Since the map $f\colon M \to W\backslash M$ is a finite cover, the map $f \circ \rho$ is a covering map with $(f \circ \rho)_*(\pi_1(Y, y_0)) = f_*(\ker(Q_*))$, as required. □

3.3 The characteristic homomorphism

It remains to identify the homomorphisms

$$Q_*\colon PB(W) = \pi_1(M) \to \mathbb{Z} \quad \text{and} \quad P_*\colon B(W) = \pi_1(W\backslash M) \to \mathbb{Z}.$$

First we describe convenient generating sets for their respective domains. Fix $\epsilon > 0$. For each reflection $t \in T$, choose a point $z_t \in H_t - \bigcup_{t' \in T, t' \neq t} H_{t'}$ and let D_t be the closed disc of radius ϵ centered at z_t in the complex line L_t which passes through z_t and is orthogonal to H_t. The complex structure induces a natural orientation on each L_t. By shrinking ϵ, if necessary, we may assume that $D_t \cap \bigcup_{t' \in T, t' \neq t} (H_{t'} \cup D_{t'}) = \varnothing$ for all $t \in T$. Now choose a basepoint z_0 in M and for each $t \in T$ choose a path σ_t in M which starts at z_0 and ends on the boundary of D_t. Let γ_t be the loop which travels along σ_t, then around the boundary of D_t in a positive orientation and finally back to z_0 along σ_t. The set of homotopy classes $\overline{\gamma}_t$ for $t \in T$ generates $\pi_1(M, z_0)$ [10]; see [1] for a recent generalization.

Since W acts freely on M, the restriction of f to M is a regular covering map onto $f(M) = W\backslash M$. We specify a set of elements of $\pi_1(W\backslash M, f(z_0))$ which corresponds to the set of reflections in $W \cong \pi_1(W\backslash M, f(z_0))/f_*\pi_1(M, z_0)$. Under this isomorphism, if δ is any path in M which starts at z_0 and ends at $w(z_0)$ (or which starts at $w(z_0)$ and ends at $w(w(z_0))$), then the homotopy class of $f \circ \delta$ corresponds to $w \in W$. If $t \in T$ is a reflection, let δ_t be the path in M which travels first along σ_t, then half way around the boundary of D_t in the positive sense and finally along the reverse of $t\sigma_t$ to $t(z_0)$. Similarly, let δ'_t be the path from $t(z_0)$ to z_0 in M which travels first along $t\sigma_t$, then around the other half of the boundary of D_t in the positive sense and finally along the reverse of σ_t to $t(t(z_0)) = z_0$. Note that $f \circ \delta_t = f \circ \delta'_t$, let Γ_t be the corresponding homotopy class and note that the composition Γ_t followed by Γ_t is represented by the path $f \circ \gamma_t$. From the short exact sequence

$$\pi_1(M, z_0) \to \pi_1(W\backslash M, f(z_0)) \to W$$

the group $\pi_1(W\backslash M, f(z_0))$ is generated by $\{\overline{f \circ \gamma_t}, \Gamma_t : t \in T\}$ and, in fact, by the smaller set $\{\Gamma_t : t \in T\}$, because $\overline{f \circ \gamma_t} = \Gamma_t^2$. For $t \in T$, the generator $\Gamma_t \in \pi_1(W\backslash M, f(z_0))$ corresponds to the generator $[t]$ in the presentation for $B(W)$ from Section 2.1.

Proposition 3.3 (i) *The map $Q_*\colon \pi_1(M, z_0) \to \pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ is given by*

$$Q_*(\overline{\gamma}_t) = 1 \quad \text{for each } t \in T.$$

(ii) *The map $P_*\colon \pi_1(W\backslash M, f(z_0)) \to \pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ is given by*

$$P_*(\Gamma_t) = 1 \quad \text{for each } t \in T.$$

Proof (i) By scaling each λ_t if necessary, we may assume that $\lambda_t(z_0) = 1$. Then $Q_* = \left(\prod_{t\in T} \lambda_t\right)_* = \sum_{t\in T}(\lambda_{t*})$ (since $\mathbb{C}^*$ is a topological group). The result now follows since our choices ensure that the winding number of $(\lambda_t)_*(\overline{\gamma}_{t'})$ about the origin is one if $t = t'$ and zero otherwise.

(ii) First note that

$$P_*(\Gamma_t^2) = P_*(\overline{f \circ \gamma_t}) = (Q^2)_*(\overline{\gamma}_t) = Q_*(\overline{\gamma}_t) + Q_*(\overline{\gamma}_t) = 2.$$

Since P_* is a group homomorphism, it follows that $P_*(\Gamma_t) = 1$. □

Remark 3.4 Our calculation of P_* from Q_* can be modified to apply when W is a Shephard group, the symmetry group of a complex polytope. The construction of a model for the Milnor fiber of the W–discriminant (in the next section) will also be valid in this case.

4 NCP model for the Milnor fiber of the discriminant

Consider the homomorphism $P_*\colon B(W) \to \mathbb{Z}$ and the cover of K given by $X_{P_*} := \ker(P_*)\backslash X$.

Proposition 4.1 *X_{P_*} is homotopy equivalent to the Milnor fiber of the discriminant P.*

Proof This follows from Corollary 3.2(i) and Proposition 3.3, since corresponding covers of K and $W\backslash M$ are homotopy equivalent. □

Remark 4.2 By Section 2.2, X_{P_*} can be identified with the CW–complex whose cells are pairs (m, σ) for $m \in \mathbb{Z}$ and σ an initialized chain in L. Since $P_*([t]) = 1$ for each reflection generator $[t]$ of $B(W)$, it follows that $P_*([w]) = |w|$, the reflection length of w, for each NCP $w \in W$. Hence, the top face of the cell $(m, e < w_1 < \cdots < w_k)$ is the cell

$$(m + |w_1|, e < w_1^{-1} w_2 < \cdots < w_1^{-1} w_k).$$

Definition 4.3 We define the *small* NCP *model* $\widehat{F}_P$ of the Milnor fiber of P_W to be the finite subcomplex of X_{P_*} consisting of the cells of the form

$$(m, e < w_1 < w_2 < \cdots < w_k) \quad \text{with } 0 \le m < n - |w_k|.$$

Remark 4.4 We observe that $\widehat{F}_P$ is the union of cells of the form

$$(0, e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1})$$

together with their faces. In particular, $\widehat{F}_P$ is $(n-1)$–dimensional.

Theorem 4.5 *The subcomplex $\widehat{F}_P$ is a strong deformation retract of X_{P_*}.*

Proof We construct an acyclic matching (see Chapter 11 of [24]) which pairs cells of $X_{P_*} - \widehat{F}_P$. Suppose $c_{k+1} = (m, \sigma)$ and that the chain σ ends in γ. Then this matching pairs c_{k+1} with its top (resp. bottom) face if $m \ge 0$ (resp. $m < 0$). In particular, the matching pairs cells whose chains end in γ with cells whose chains do not end in γ.

To show that this matching is acyclic, consider an alternating path

$$(l_1, \sigma_1) \succ_m (l_2, \sigma_2) \prec (l_3, \sigma_3) \succ_m (l_4, \sigma_4) \prec \cdots$$

in this matching, where $(l_i, \sigma_i) \prec (l_{i+1}, \sigma_{i+1})$ means (l_i, σ_i) is a facet of (l_{i+1}, σ_{i+1}) and $(l_j, \sigma_j) \succ_m (l_{j+1}, \sigma_{j+1})$ means (l_j, σ_j) is matched with its facet (l_{j+1}, σ_{j+1}). By definition of the matching, if i is odd, the chain σ_i ends in γ and σ_{i+1} does not end in γ. By Lemma 2.1, each such (l_{i+1}, σ_{i+1}) is incident on precisely two $(k+1)$–cells with chains ending in γ. These must be (l_i, σ_i) and (l_{i+2}, σ_{i+2}). If $l_i < 0$ it follows that (l_{i+1}, σ_{i+1}) is the bottom face of (l_i, σ_i) and hence $l_{i+2} > l_i$ for all i. Similarly, if $l_i \ge 0$ it follows that (l_{i+1}, σ_{i+1}) is the top face of (l_{i+2}, σ_{i+2}) and hence $l_{i+2} < l_i$ for all i. In particular, the path cannot form a cycle and the matching is acyclic.

It remains to show that the set of critical cells is precisely the set of cells of $\widehat{F}_P$. Let $c_k = (m, e < w_1 < \cdots < w_k)$ be a cell of X_{P_*}. When $w_k = \gamma$, the cell c_k is matched with its top or bottom face according to whether $m \geq 0$ or $m < 0$ and, hence, is not critical. When $w_k \neq \gamma$, two cases arise. If $m < 0$ then c_k is matched as the bottom face of $(m, e < w_1 < \cdots < w_k < \gamma)$ and is not critical. On the other hand, if $m \geq 0$, then c_k is matched as the top face of

$$(m - n + |w_k|, e < \gamma w_k^{-1} < \gamma w_k^{-1} w_1 < \cdots < \gamma w_k^{-1} w_{k-1} < \gamma)$$

if and only if $m - n + |w_k| \geq 0$. □

Corollary 4.6 *The finite complex $\widehat{F}_P$ is homotopy equivalent to the Milnor fiber F_P and is a $K(\pi, 1)$ for $\pi = \ker(P_*)$.*

Example 4.7 Let W be a dihedral group acting on $\mathbb{R}^2$ with t reflections, denoted by $R_1, R_2, \ldots, R_t$. Thus the NCPs are $\{e, R_1, R_2, \ldots, R_t, \gamma\}$, where γ can be taken to be a rotation through twice the angle between adjacent lines of symmetry. Here $n = 2$ and $\widehat{F}_P$ is 1–dimensional with two vertices, namely $(0, e)$ and $(1, e)$. The complex $\widehat{F}_P$ has a 1–cell $(0, e < R_i)$ for each chain $e < R_i$ and the endpoints of this 1–cell are $(0, e)$ and $(1, e)$. Thus $\widehat{F}_P$ has the homotopy type of the suspension of a 0–dimensional subcomplex on t points and $\ker(P_*)$ is free of rank $t - 1$. This agrees with [26, Theorem 1]; P has weights 2 and t.

Example 4.8 Let W be the group $A_3 \cong \Sigma_4$, the symmetric group; the polynomial P is the classical discriminant for univariate quartics. Choose γ to be the four-cycle (1234). Here $n = 3$ and $\widehat{F}_P$ is 2–dimensional. (See Figure 1.) $\widehat{F}_P$ has three vertices, namely $(0, e)$, $(1, e)$ and $(2, e)$. Each transposition R contributes a 1–cell $(0, e < R)$ with endpoints $(0, e)$ and $(1, e)$ together with a 1–cell $(1, e < R)$ with endpoints $(1, e)$ and $(2, e)$. Each length-two NCP $\alpha \in \{(123), (124), (134), (234), (12)(34), (14)(23)\}$ contributes a single 1–cell $(0, e < \alpha)$ with endpoints $(0, e)$ and $(2, e)$. Finally, for each of the 16 chains of the form $e < R < \alpha$ corresponding to factorizations of γ by transpositions, we have a 2–cell $(0, e < R < \alpha)$ whose boundary is glued along the three 1–cells $(0, e < R)$, $(0, e < \alpha)$ and $(1, e < R^{-1}\alpha)$.

Remark 4.9 The cells of $\widehat{F}_P$ are simplices, but $\widehat{F}_P$ is not a simplicial complex; rather it is a Δ–complex in the sense of [21]. It can be realized as the nerve of the germ (in the sense of category theory; see [17]) with objects $\{0, 1, \ldots, n-1\}$ and morphisms $i \xrightarrow{\alpha} j$, where α is an NCP of length $j - i$. The composition, when defined, is determined by multiplication of NCPs.

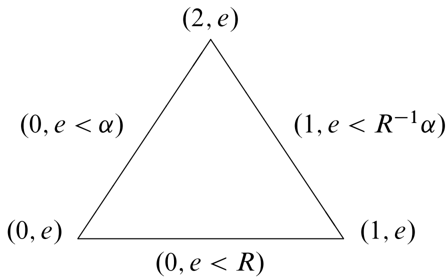

Figure 1: Typical 2–cell $(0, e < R < \alpha < \gamma)$ of $\widehat{F}_P$ for $W = A_3$

Remark 4.10 The monodromy action on $\widehat{F}_P$ is obtained by composing the monodromy action on X_{P_*} with the retraction defined by the acyclic matching. The monodromy action is determined by the action of either generator of $\mathbb{Z}$. Since one of the two cells identified in Lemma 2.1 has a simpler formula than the other we will describe the action of -1. Explicitly a cell of the form

$$(m, e < w_1 < w_2 < \cdots < w_k)$$

for $0 < m < n - |w_k|$ is taken to

$$(m-1, e < w_1 < w_2 < \cdots < w_k),$$

while

$$(0, e < w_1 < w_2 < \cdots < w_k)$$

for $|w_k| < n$ is taken to

$$(|w_1| - 1, e < w_1^{-1} w_2 < w_1^{-1} w_3 < \cdots < w_1^{-1} w_k < w_1^{-1} \gamma),$$

since, in the second case, the action of -1 on the k–cell $(0, e < w_1 < w_2 < \cdots < w_k)$ as a cell in X_{P_*} takes it to the k–cell $(-1, e < w_1 < w_2 < \cdots < w_k)$, which is the bottom face of the $(k+1)$–cell $(-1, e < w_1 < w_2 < \cdots < w_k < \gamma)$, which in turn has top face $(|w_1| - 1, e < w_1^{-1} w_2 < w_1^{-1} w_3 < \cdots < w_1^{-1} w_k < w_1^{-1} \gamma)$.

5 NCP model for the Milnor fiber of the arrangement

The homomorphisms s and P_* from Example 2.3 and Section 4 combine to give the homomorphism

$$\psi = (P_*, s)\colon B(W) \to \mathbb{Z} \times W, \quad x \mapsto (P_*(x), s(x)).$$

The space X_ψ (constructed as in Section 2.2) is then a $K(\pi, 1)$ for $\pi = \ker(\psi)$.

Proposition 5.1 *The complex X_ψ is homotopy equivalent to the Milnor fiber F_Q of Q.*

Proof The map $f\colon M \to W\backslash M \cong \mathbb{C}^n - \Delta$ induces an injection $f_*\colon \pi_1(M) \to B(W)$ whose image is the kernel of $s\colon B(W) \to W$, by [10]. Now

$$\begin{aligned}\ker(\psi) &= \ker(P_*) \cap \ker(s)\\ &= \ker(P_*) \cap \operatorname{im}(f_*)\\ &= f_*(\ker(P_* \circ f_*))\\ &= f_*\big(\ker((Q^2)_*)\big) \quad \text{since } P \circ f = Q^2\\ &= f_*(\ker(Q_*)) \quad \text{since } (Q^2)_* = 2(Q_*).\end{aligned}$$

It follows that X_ψ is homotopy equivalent to the cover of K corresponding to $f_*(\ker(Q_*))$ and hence to the cover of $W\backslash M$ corresponding to $f_*(\ker(Q_*))$. However, this latter cover is homotopy equivalent to F_Q by Corollary 3.2(ii). □

Remark 5.2 The cover X_ψ is a simplicial complex. The vertices of the k–cell $c_k = \big((P_*(x), s(x)), e < w_1 < \cdots < w_k\big)$ are

$$\big((P_*(x), s(x)), e\big) \qquad \text{and} \qquad \big((P_*(x) + |w_i|, s(xw_i)), e\big) \quad \text{for } 1 \le i \le k.$$

This set of $k+1$ vertices uniquely determines c_k.

Remark 5.3 The map $\psi = (P_*, s)$ is not onto. For each $x \in B(W)$, the integer $P_*(x)$ is even if and only if $s(x)$ belongs to the subgroup $W^+ < W$ of Section 3. Thus X_ψ can be identified with the CW–complex whose cells are triples (m, w, σ), where σ is an initialized chain in L and the parity of $m \in \mathbb{Z}$ is the same as that of $w \in W$. The top face of $(m, w, e < w_1 < \cdots < w_k)$ is $(m + |w_1|, ww_1, e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k)$, while the other faces are given by $(m, w, e < w_1 < \cdots < \widehat{w_i} < \cdots < w_k)$ for $1 \le i \le k$.

Definition 5.4 We define the *small* NCP *model* $\widehat{F}_Q$ of the Milnor fiber of Q to be the finite subcomplex of X_ψ which is the preimage of $\widehat{F}_P$ under the covering $X_\psi \to X_{P_*}$ determined by the subgroup inclusion $\ker(\psi) \subseteq \ker(P_*)$.

Remark 5.5 The complex $\widehat{F}_Q$ is the union of all cells of the form

$$(0, w, e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1}), \quad \text{where } w \in W^+,$$

together with their faces.

Theorem 5.6 *The subcomplex $\widehat{F}_Q$ is a strong deformation retract of the cover X_ψ.*

Proof The simplicial complex X_ψ is the cover of X_{P_*} determined by the subgroup inclusion $\ker(P_*, s) < \ker(P_*)$. By the homotopy lifting property, the strong deformation retraction of X_{P_*} onto $\widehat{F}_P$ (Theorem 4.5) is covered by a strong deformation retraction of X_ψ onto $\widehat{F}_Q$. □

Corollary 5.7 *The finite simplicial complex $\widehat{F}_Q$ is homotopy equivalent to the Milnor fiber F_Q and is a $K(\pi, 1)$ for the group $\ker(P_*, s)$.*

Example 5.8 When W is the dihedral group acting on $\mathbb{R}^2$ with t reflections, the subcomplex $\widehat{F}_Q$ has the following description. Each vertex $(0, w, e)$ has $w \in W^+$, the rotation subgroup of W, while each vertex $(1, w, e)$ has $w \in W - W^+$, the set of reflections of W. Furthermore, for each rotation $w \in W^+$ and each reflection R there is another reflection S with $w = RS$ giving an edge in $\widehat{F}_Q$ labeled $(0, w, e < S)$, starting at $(0, w, e)$ and ending at $(1, R, e)$. Thus $\widehat{F}_Q$ is the complete bipartite graph $K_{t,t}$, which is homotopy equivalent to a bouquet of $(t-1)^2$ circles. This is consistent with the calculations in [26, Theorem 1].

Remark 5.9 The monodromy action on $\widehat{F}_Q$ is obtained as in Remark 4.10 by composing the monodromy action on X_ψ with the retraction defined by the acyclic matching. In this case, the action on X_ψ is by shifting the height of cells by multiples of 2. However, since $\widehat{F}_Q$ is a simplicial complex, it is sufficient to compute the action on 0–cells. We describe the action of $-2 \in \mathbb{Z}$.

Explicitly, we define the action of -2 on 0–cells by

$$((m, w), e) \mapsto \begin{cases} ((m-2, w), e) & \text{if } 2 \le m \le n-1, \\ ((n-1, w\gamma), e) & \text{if } m = 1, \\ ((n-2, w\gamma), e) & \text{if } m = 0. \end{cases}$$

The second case is explained as follows. The action of -2 on the 0–cell $((1, w), e)$ as a cell in X_ψ takes it to the 0–cell $((-1, w), e)$. This last 0–cell is the bottom face of the 1–cell $((-1, w), e < \gamma)$, which in turn has top face $((n-1, w\gamma), e)$. The third case is similar.

6 Structure and homology of $\widehat{F}_P$

Although our model $\widehat{F}_P$ of the Milnor fiber of P is not a simplicial complex, it does have a combinatorial description as a sequence of mapping cones. The domains of the mappings in question are order complexes of truncations of the noncrossing partition lattice L and the lexicographic shellability of L yields a chain complex which computes $H_*(\widehat{F}_P,\mathbb{Z})$.

Let $L_{[i,j]}=\{w\in L : i\le |w|\le j\}$ and let $A_i=\{(m,\sigma)\in \widehat{F}_P : m\ge n-i-1\}$, where i and j are integers with $0\le i\le j\le n$. Note that A_i has dimension i and that $A_0\subset A_1\subset\cdots\subset A_{n-1}=\widehat{F}_P$. Define $g_i\colon |L_{[1,n-i-1]}|\to A_{n-i-2}$ by

$$w_1<w_2<\cdots<w_k\mapsto (i+|w_1|, e<w_1^{-1}w_2<\cdots<w_1^{-1}w_k).$$

Proposition 6.1 *The mapping cone of g_i is cellularly isomorphic to A_{n-i-1}.*

Proof $|L_{[0,k]}|$ is simplicially isomorphic to the cone on $|L_{[1,k]}|$ since L has a unique minimal element e. Under this identification, the subcomplex $|L_{[1,k]}|$ of $|L_{[0,k]}|$ corresponds to $|L_{[1,k]}|\times\{1\}\subset |L_{[1,k]}|\times[0,1]$. The map $|L_{[0,n-i-1]}|\to A_{n-i-1}$ given by

$$e<w_1<w_2<\cdots<w_k\mapsto (i, e<w_1<w_2<\cdots<w_k)$$

combines with the inclusion of A_{n-i-2} into A_{n-i-1} to give a map

$$\widehat{g}_i\colon (|L_{[1,n-i-1]}|\times[0,1])\amalg A_{n-i-2}\to A_{n-i-1}.$$

One can show that $\widehat{g}_i$ is an identification map which identifies precisely the pair $(x,0)$ with $(x',0)$ for each $x,x'\in |L_{[1,n-i-1]}|$ and $(x,1)$ with $g_i(x)$. □

Lemma 6.2 $H_q(A_p,A_{p-1})\cong \widetilde{H}_{q-1}(|L_{[1,p]}|)$ *for all* $q\ge 1$.

Proof The filtration $A_0\subset A_1\subset\cdots\subset A_{n-1}=\widehat{F}_P$ yields

$$H_q(A_p,A_{p-1})\cong \widetilde{H}_q(A_p/A_{p-1})\cong \widetilde{H}_q(\Sigma(|L_{[1,p]}|))\cong \widetilde{H}_{q-1}(|L_{[1,p]}|),$$

where Σ denotes suspension. The second isomorphism follows from Proposition 6.1. □

Definition 6.3 For each i, define the i^{th} face map $d_i\colon C_{p-1}(L_{[1,p]})\to C_{p-2}(L_{[1,p]})$ by

$$d_i(w_1\lessdot w_2\lessdot\cdots\lessdot w_p)=(-1)^{i-1}(w_1\lessdot w_2\lessdot\cdots\lessdot\widehat{w_i}\lessdot\cdots\lessdot w_p)$$

and the top face map $\Omega\colon C_{p-1}(L_{[1,p]})\to C_{p-2}(L_{[1,p-1]})$ by

$$\Omega(w_1\lessdot w_2\lessdot\cdots\lessdot w_p)=(w_1^{-1}w_2\lessdot w_1^{-1}w_3\lessdot\cdots\lessdot w_1^{-1}w_p).$$

Theorem 6.4 *The homology of $\widehat{F}_P$ is isomorphic to the homology of the chain complex whose p^{th} group is $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ and whose boundary homomorphism is given, at the level of chains, by $\sum a_\sigma\sigma\mapsto\sum a_\sigma\Omega(\sigma)$.*

Proof First note that

$$H_q(A_p, A_{p-1}) \cong \widetilde{H}_{q-1}(|L_{[1,p]}|) \cong \begin{cases} \mathbb{Z}^{n_p} & \text{if } q = p, \\ 0 & \text{if } q \neq p, \end{cases}$$

where the last equality uses the lexicographical shellability of the NCP lattices [3; 5]. By Theorem 39.4 of [28], the homology of $\widehat{F}_P$ is isomorphic to the homology of the chain complex with p^{th} group given by $H_p(A_p, A_{p-1})$ and boundary homomorphism given by the connecting homomorphism of the exact sequence of the triple (A_p, A_{p-1}, A_{p-2}).

It remains for us to compute the boundary homomorphism from $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ to $\widetilde{H}_{p-2}(|L_{[1,p-1]}|)$. The isomorphism from $\widetilde{H}_{p-1}(|L_{[1,p]}|)$ to $H_p(A_p, A_{p-1})$ of Lemma 6.2 is induced by $b_p\colon |L_{[1,p]}| \to A_p$, $\sigma \mapsto (n-p-1, e * \sigma)$, where $e * \sigma$ means the simplex represented by $e < w_1 < \cdots < w_l$ when σ is the simplex represented by $w_1 < \cdots < w_l$. Let $\sum a_\sigma \sigma$ in $C_{p-1}(|L_{[1,p]}|)$ be a cycle, so that

$$0 = \partial\Big(\sum a_\sigma \sigma\Big) = \sum a_\sigma \partial(\sigma) = \sum a_\sigma \sum_i d_i(\sigma).$$

Since each σ is maximal, the $(p-1)$–chains $d_i\sigma$ and $d_j\sigma$ have different length distributions whenever $i \neq j$. (The length distribution of $w_1 < w_2 < \cdots < w_p$ is $(|w_1|, |w_2|, \ldots, |w_p|)$.) It follows that

$$\sum a_\sigma d_i(\sigma) = 0 \quad \text{for each } 1 \leq i \leq p. \tag{6-1}$$

Using the fact that the connecting homomorphism is defined on the level of chains by the boundary map in $\widehat{F}_P$, we get

$$\begin{aligned} \partial\Big(b_p\Big(\sum a_\sigma \sigma\Big)\Big) &= \partial\Big(\sum a_\sigma (n-1-p, e*\sigma)\Big) \\ &= \sum a_\sigma \partial(n-1-p, e*\sigma) \\ &= \sum a_\sigma \Big((n-p, e*\Omega(\sigma)) + \sum_i (n-1-p, e*d_i\sigma)\Big) \\ &= \sum a_\sigma (n-p, e*\Omega(\sigma)) \quad \text{by (6-1)} \\ &= b_{p-1}\Big(\sum a_\sigma \Omega(\sigma)\Big). \end{aligned}$$

□

Remark 6.5 By Proposition 6.1 and Theorem 6.4, the total rank of the cellular chain complex of $\widehat{F}_P$ is the sum of the ranks of the chain groups of $|L_{[1,p]}|$, while the complex introduced in this section has total rank equal to the sum of the ranks of the reduced homology groups of the $|L_{[1,p]}|$, which vanish except in the top dimension.

Example 6.6 In the case $W = C_3$, the symmetry group of the cube, there are nine reflections, which we label $1, \dots, 9$, and nine length-two elements given by the length-two products

$$12,\ 14,\ 18,\ 19,\ 27,\ 29,\ 45,\ 47,\ 78.$$

The complex $|L_{[1,1]}|$ is a set of nine discrete points while $|L_{[1,2]}|$ has the homotopy type of the graph shown in Figure 2; see [23].

The upper face map applied to each chain $R \lessdot \alpha$ yields the reflection $R^{-1}\alpha$. These values are given by the labels along the edges of the graph. Therefore the map $\widetilde{H}_1(|L_{[1,2]}|) \to \widetilde{H}_0(|L_{[1,1]}|)$ is given by reading, for each loop in the graph, the signed sum of the corresponding labels. The resulting 8×10 matrix has rank 7 and hence $H_2(\widehat{F}_{\boldsymbol{P}}) \cong \mathbb{Z}^3$ and $H_1(\widehat{F}_{\boldsymbol{P}}) \cong \mathbb{Z}$.

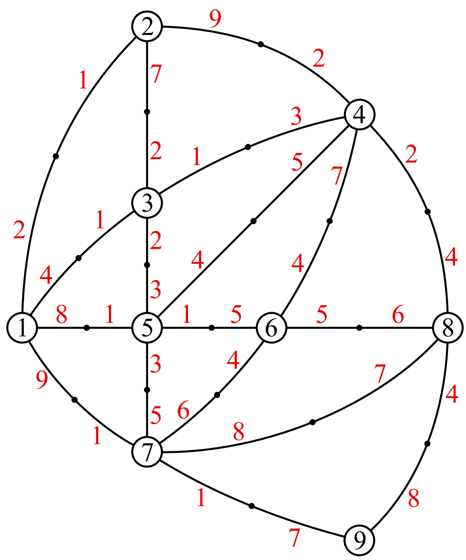

Figure 2: $W = C_3$

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