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Some extensions in the Adams spectral sequence and the 51–stem

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We show a few nontrivial extensions in the classical Adams spectral sequence. In particular, we compute that the 2–primary part of π_{51} is $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$. This was the last unsolved 2–extension problem left by the recent work of Isaksen and the authors through the 61–stem.

The proof of this result uses the RP^∞ technique, which was introduced by the authors to prove $\pi_{61} = 0$. This paper advertises this technique through examples that have simpler proofs than in our previous work.

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1 Introduction

The computation of the stable homotopy groups of spheres is both a fundamental and a difficult problem in homotopy theory. Recently, using Massey products and Toda brackets, Isaksen [5] extended the 2–primary Adams spectral sequence computations to the 59–stem, with a few $2, \eta, \nu$–extensions unsettled.

Based on the algebraic Kahn–Priddy theorem [7] (see also Lin [8]), the authors [12] compute a few differentials in the Adams spectral sequence, and proved that $\pi_{61} = 0$. The 61–stem result has the geometric consequence that the 61–sphere has a unique smooth structure, and it is the last odd-dimensional case. In the article [12], it took the authors more than 40 pages to introduce the method and prove one Adams differential, $d_3(D_3) = B_3$. Here B_3 and D_3 are certain elements in the 60– and 61–stem. Our notation will be consistent with [5; 12].

In this paper, we show that our technique can also be used to solve extension problems in the Adams spectral sequence. We establish a nontrivial 2–extension in the 51–stem, together with a few other extensions left unsolved by Isaksen [5]. As a result, we have the following proposition:

Proposition 1.1 *There is a nontrivial 2–extension from $h_0h_3g_2$ to gn in the 51–stem.*

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We'd like to point out that this is also a nontrivial 2–extension in the Adams–Novikov spectral sequence.

Combining with Theorem 1.1 of Isaksen and Xu [6], which describes the group structure of π_{51} up to this 2–extension, we have the following corollary:

Corollary 1.2 *The* 2*–primary* π_{51} *is* $\mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$*, generated by elements that are detected by* $h_3 g_2$*,* $P^6 h_2$ *and* $h_2 B_2$*.*

Using a Toda bracket argument, Proposition 1.1 is deduced from the following σ–extension in the 46–stem:

Proposition 1.3 (1) *There is a nontrivial* σ*–extension from* $h_3 d_1$ *to* N *in the* 46*–stem.*

(2) *There is a nontrivial* η*–extension from* $h_1 g_2$ *to* N *in the* 46*–stem.*

As a corollary, we prove a few more extensions.

Corollary 1.4 (1) *There is a nontrivial* η*–extension from* C *to* gn *in the* 51*–stem.*

(2) *There is a nontrivial* ν*–extension from* $h_2 h_5 d_0$ *to* gn *in the* 51*–stem.*

(3) *There is a nontrivial* σ*–extension from* $h_0^2 g_2$ *to* gn *in the* 51*–stem. In particular, the element* gn *detects* $\sigma^3 \theta_4$*.*

Remark 1.5 In [5], Isaksen had an argument that implies the nonexistence of the two η–extensions on $h_1 g_2$ and C, which is contrary to our results in Proposition 1.3 and Corollary 1.4. Isaksen's argument fails because of neglect of the indeterminacy of a certain Massey product in a subtle way. For more details, see Remark 2.3.

The proof of the σ–extension in Proposition 1.3 is the major part of this article: we prove it by the RP^∞ technique as a demonstration of the effectiveness of our method.

The rest of this paper is organized as follows:

In Section 2, we deduce Proposition 1.1 and Corollary 1.4 from Proposition 1.3. We also show the two statements in Proposition 1.3 are equivalent. In Section 3, we recall some notation from [12]. We also give a brief review of how to use the RP^∞ technique to prove differentials and to solve extension problems. In Section 4, we present the proof of Proposition 1.3. In Section 5, we prove a lemma which is used in Section 4. The lemma gives a general connection that relates Toda brackets and extension problems in 2 cell complexes. In the appendix, we use cell diagrams as intuition for statements of the lemmas in Section 5.

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2 The 51–stem and some extensions

We first establish the following lemma:

Lemma 2.1 *In the Adams E_2–page, we have the following Massey products in the 46–stem:*

$$gn = \langle N, h_1, h_2 \rangle = \langle N, h_2, h_1 \rangle.$$

Proof By Bruner's computation [4], there is a relation in bidegree $(t-s, s) = (81, 15)$,

$$gnr = mN.$$

We have $\mathrm{Ext}^{15,81+15} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, generated by gnr and $h_1 x_{14,42}$. Moreover, the element gnr is not divisible by h_1, and neither of the generators is divisible by h_2.

By Tangora's computation [10], we have a Massey product in the Adams E_2–page,

$$m = \langle r, h_1, h_2 \rangle.$$

Therefore,

$$gn \cdot r = m \cdot N = N \cdot \langle r, h_1, h_2 \rangle = \langle N \cdot r, h_1, h_2 \rangle = r \cdot \langle N, h_1, h_2 \rangle$$

with zero indeterminacy. This implies

$$gn = \langle N, h_1, h_2 \rangle.$$

Because of the relation $h_2 \cdot N = 0$ in $\mathrm{Ext}^{9,49+9} = 0$, we also have

$$gn \cdot r = m \cdot N = \langle r, h_1, h_2 \rangle \cdot N = r \cdot \langle h_1, h_2, N \rangle.$$

This implies

$$gn = \langle N, h_2, h_1 \rangle.$$ □

Based on Proposition 1.3, we prove Corollary 1.4(1):

Proof By Proposition 1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_2, h_1 \rangle$$

and Moss's theorem [9] imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle\sigma^2\{d_1\}, \nu, \eta\rangle.$$

The indeterminacy of this Toda bracket is

$$\eta\cdot\pi_{50}+\sigma^2\{d_1\}\cdot\pi_5=\eta\cdot\pi_{50},$$

since $\pi_5=0$. Shuffling this bracket, we have

$$\langle\sigma^2\{d_1\}, \nu, \eta\rangle \supseteq \sigma\{d_1\}\cdot\langle\sigma, \nu, \eta\rangle = 0,$$

since $\langle\sigma, \nu, \eta\rangle \subseteq \pi_{12}=0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by η.

For filtration reasons, the only possibilities are C and h_5c_1. However, Lemma 4.2.51 of [5] states that there is no η–extension from h_5c_1 to gn. Therefore, we must have an η–extension from C to gn. □

Based on Proposition 1.3, we prove Corollary 1.4(2):

Proof By Proposition 1.3, N detects the homotopy class $\sigma^2\{d_1\}$. Then the Massey product

$$gn = \langle N, h_1, h_2\rangle$$

and Moss's theorem [9] imply that gn detects a homotopy class that is contained in the Toda bracket

$$\langle\sigma^2\{d_1\}, \eta, \nu\rangle.$$

The indeterminacy of this Toda bracket is

$$\nu\cdot\pi_{48}+\sigma^2\{d_1\}\cdot\pi_5=\nu\cdot\pi_{48},$$

since $\pi_5=0$. Shuffling this bracket, we have

$$\langle\sigma^2\{d_1\}, \eta, \nu\rangle \supseteq \sigma\cdot\langle\sigma\{d_1\}, \eta, \nu\rangle = \sigma\{d_1\}\cdot\langle\eta, \nu, \sigma\rangle = 0,$$

since $\langle\eta, \nu, \sigma\rangle \subseteq \pi_{12}=0$.

Therefore, gn detects a homotopy class that lies in the indeterminacy, and hence is divisible by ν.

For filtration reasons, the only possibility is $h_2h_5d_0$, which completes the proof. □

Now we prove Corollary 1.4(3), and Proposition 1.1:

Proof Lemma 4.2.31 from Isaksen's computation [5] states that the 2–extension from $h_0h_3g_2$ to gn is equivalent to the ν–extension from $h_2h_5d_0$ to gn. This proves Proposition 1.1.

It is clear that Proposition 1.1 is equivalent to Corollary 1.4(3), since σ is detected by h_3 and $\sigma^2\theta_4$ is detected by $h_0^2g_2$. (See [2; 5] for the second fact.) □

In the following lemma, we show that the two statements in Proposition 1.3 are equivalent:

Lemma 2.2 *There is a σ–extension from h_3d_1 to N if and only if there is an η–extension from h_1g_2 to N.*

Proof First note that there are relations in Ext,

$$h_3d_1 = h_1e_1, h_3e_1 = h_1g_2.$$

By Bruner's differential [3, Theorem 4.1]

$$d_3(e_1) = h_1t = h_2^2n,$$

we have Massey products in the Adams E_4–page

$$h_3d_1 = h_1e_1 = \langle h_2n, h_2, h_1\rangle, \quad h_1g_2 = h_3e_1 = \langle h_3, h_2n, h_2\rangle.$$

Then Moss's theorem implies that they converge to Toda brackets

$$\langle \nu\{n\}, \nu, \eta\rangle, \quad \langle \sigma, \nu\{n\}, \nu\rangle.$$

Therefore, the lemma follows from the shuffling

$$\sigma\cdot\langle \nu\{n\}, \nu, \eta\rangle = \langle \sigma, \nu\{n\}, \nu\rangle\cdot\eta.$$ □

We give a remark on the two η–extensions we proved:

Remark 2.3 In Lemmas 4.2.47 and 4.2.52 of [5], Isaksen showed that there are no η–extensions from h_1g_2 to N or from C to gn. Both arguments are based the statement of Lemma 3.3.45 of [5], whose proof implicitly studied the motivic Massey product

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle \ni Ph_1^3h_5e_0$$

in the 59–stem of the motivic Adams E_3–page, which therefore converges to a motivic Toda bracket. However, in the motivic Adams E_3–page, the element $Ph_1^3h_5e_0$ is in the indeterminacy of this motivic Massey product, since $Ph_1h_5e_0$ is present in the motivic E_3–page (it supports a d_3 differential). Therefore, we have

$$\langle h_1^2, Ph_1h_5c_0, c_0\rangle = \{Ph_1^3h_5e_0, 0\}$$

instead. The statement of Moss's theorem gives us the convergence of *only one* permanent cycle in the motivic Massey product; therefore, in this case, it is inconclusive.

3 The method and notation

In this section, we recall some notation from [12] and set up terminology that will be used in Section 4.

Notation 3.1 All spectra are localized at the prime 2. Suppose Z is a spectrum. Let $\mathrm{Ext}(Z)$ denote the Adams E_2–page $\mathrm{Ext}_A(\mathbb{F}_2, H_*(Z;\mathbb{F}_2))$ that converges to the 2–primary homotopy groups of Z. Here A is the mod 2 dual Steenrod algebra.

We now introduce some spectral sequence terminology. A permanent cycle is a class that does not support any nontrivial differential. A surviving cycle is a permanent cycle that is also not the target of any differential.

For spectra, let S^0 be the sphere spectrum and P_1^∞ be the suspension spectrum of RP^∞. In general, write P_n^{n+k} for the suspension spectrum of RP^{n+k}/RP^{n-1}.

Let α be a class in the stable homotopy groups of spheres. We use $C\alpha$ to denote the cofiber of α.

Definition 3.2 Let A, B, C and D be CW spectra and i and q be maps

$$A \xhookrightarrow{i} B, \quad B \xrightarrow{q} \!\!\!\!\rightarrow C.$$

We say that (A,i) is an $H\mathbb{F}_2$–subcomplex of B if the map i induces an injection on mod 2 homology. We denote an $H\mathbb{F}_2$–subcomplex by a hooked arrow as above.

We say that (C,q) is an $H\mathbb{F}_2$–quotient complex of B if the map q induces a surjection on mod 2 homology. We denote an $H\mathbb{F}_2$–quotient complex by a double-headed arrow as above.

When the maps involved are clear in the context, we also say A is an $H\mathbb{F}_2$–subcomplex of B, and C is an $H\mathbb{F}_2$–quotient complex of B.

Furthermore, we say D is an $H\mathbb{F}_2$–subquotient of B if D is an $H\mathbb{F}_2$–subcomplex of an $H\mathbb{F}_2$–quotient complex of B, or an $H\mathbb{F}_2$–quotient complex of an $H\mathbb{F}_2$–subcomplex of B.

One example is that S^1, S^3 and S^7 are $H\mathbb{F}_2$–subcomplexes of P_1^∞, due to the solution of the Hopf invariant one problem. Another example is that $\Sigma^7 C\eta$ is an $H\mathbb{F}_2$–subcomplex of P_7^9.

We use the following way to denote the elements in the Adams E_2–page of P_1^∞ and its $H\mathbb{F}_2$–subquotients. One way to compute $\mathrm{Ext}(P_1^\infty)$ is to use the algebraic Atiyah–Hirzebruch spectral sequence

$$E_1 = \bigoplus_{n=1}^{\infty} \mathrm{Ext}(S^n) \Rightarrow \mathrm{Ext}(P_1^\infty).$$

Notation 3.3 We denote any element in $\mathrm{Ext}(S^n)$ by $a[n]$, where $a \in \mathrm{Ext}(S^0)$ and n indicates that it comes from $\mathrm{Ext}(S^n)$. We will abuse notation and write the same symbol $a[n]$ for an element of $\mathrm{Ext}(P_1^\infty)$ detected by the element $a[n]$ of the Atiyah–Hirzebruch E_∞–page. Thus, there is indeterminacy in the notation $a[n]$ that is detected by Atiyah–Hirzebruch E_∞ elements in lower filtration. If $a[n]$ is the element of lowest Atiyah–Hirzebruch filtration in the Atiyah–Hirzebruch E_∞–page in a given bidegree (s,t), then $a[n]$ also is a well-defined element of $\mathrm{Ext}(P_1^\infty)$.

We use similar notation for homotopy classes.

Remark 3.4 In [11], we computed differentials in the algebraic Atiyah–Hirzebruch spectral sequence that converges to the Adams E_2–page of P_1^∞ in the range of $t < 72$.

By truncating the algebraic Atiyah–Hirzebruch spectral sequence for P_1^∞, one can also read off information about $\mathrm{Ext}(P_n^{n+k})$. For details, see [11].

Remark 3.5 Despite the indeterminacy in Notation 3.3, there is a huge advantage of it. Suppose $f\colon Q \to Q'$ is a map between two $H\mathbb{F}_2$–subquotients of P_1^∞ and there exists an element $a[n]$ which is a generator of both $\mathrm{Ext}^{s,t}(Q)$ and $\mathrm{Ext}^{s,t}(Q')$ for some bidegree (s,t) (this implies both Q and Q' have a cell in dimension n). We must have that, with the right choices, $a[n]$ in $\mathrm{Ext}^{s,t}(Q)$ maps to $a[n]$ in $\mathrm{Ext}^{s,t}(Q')$:

$$\begin{array}{ccc}
\bigoplus_{i\in I}\mathrm{Ext}(S^i) & \longrightarrow & \bigoplus_{i\in I'}\mathrm{Ext}(S^i) \\
\Downarrow & & \Downarrow \\
\mathrm{Ext}(Q) & \longrightarrow & \mathrm{Ext}(Q') \\
 & & \\
a[n] & \longmapsto & a[n]
\end{array}$$

This property follows from the naturality of the algebraic Atiyah–Hirzebruch spectral sequence.

4 The σ–extension on h_3d_1

In this section, we prove Proposition 1.3(1). The proof can be summarized in the following "road map" with four main steps:

$$\begin{array}{ccccccc}
S^0 & \overset{\sigma}{\dashleftarrow} & S^7 & & N & & h_3d_1[7] \\
\uparrow & \swarrow & \downarrow & & \uparrow & & \\
P_1^\infty & & \Sigma^7 C\eta & & h_1t[9] & & h_2\cdot h_2n[9]=h_1t[9] \\
\uparrow & & \downarrow & & \uparrow & & \downarrow \\
P_1^9 & \twoheadrightarrow & P_7^9 & & h_1t[9] & \longmapsto & h_1t[9]
\end{array}$$

Here the elements in the right side of the "road map" are elements in the 46–stem of the E_∞–page of the Adams spectral sequences of the spectra in the corresponding positions.

Step 1 We show that the element $h_1t[9]$ is a permanent cycle in the Adams spectral sequence of $\Sigma^7C\eta$, and hence a permanent cycle in the Adams spectral sequence of P_7^9. This is stated as Proposition 4.3.

Step 2 Under the inclusion map $S^7 \hookrightarrow \Sigma^7C\eta$, we show that the element $h_1t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7C\eta)$. By naturality, the same statement is true after we further map it to $\pi_{46}(P_7^9)$. This is stated as Proposition 4.6.

Step 3 Under any map $S^7 \hookrightarrow P_1^9$ lifting the inclusion $S^7 \hookrightarrow P_7^9$, we show that the element $h_1t[9]$ in $\mathrm{Ext}(P_1^9)$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(P_1^9)$. This is stated as Proposition 4.7.

Step 4 Using the inclusion map $P_1^9 \to P_1^\infty$ and the transfer map $P_1^\infty \to S^0$, we push forward the element $h_1t[9]$ in the E_∞–page of P_1^9 to the element N in the E_∞–page of S^0. Since the composition

$$S^7 \hookrightarrow P_1^9 \to P_1^\infty \to S^0$$

is just σ, we have the desired σ–extension from h_3d_1 to N in the Adams spectral sequence for S^0.

Remark 4.1 Step 2 is the essential step. Intuitively, it comes from the zigzag of the following two differentials:

$$d_3(e_1) = h_1t$$

in the Adams spectral sequence of S^0, and

$$d_2(e_1[9]) = h_1e_1[7] = h_3d_1[7]$$

in the algebraic Atiyah–Hirzebruch spectral sequence of $\Sigma^7 C\eta$, which comes from the η–attaching map. Here

$$h_1e_1 = h_3d_1$$

is a relation in Ext. This zigzag suggested that we consider the possibility that $h_1t[9]$ detects $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$.

We start with Step 1. Proposition 4.3 is a consequence of the following lemma:

Lemma 4.2 *The element $h_2n[9]$ is a permanent cycle in $\Sigma^7 C\eta$, which detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map*

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

Proof The cofiber sequence

$$S^7 \xhookrightarrow{i} \Sigma^7 C\eta \xrightarrow{p} S^9 \xrightarrow{\eta} S^8$$

gives us a long exact sequence of homotopy groups

$$\pi_{43}(S^7) \xrightarrow{i_*} \pi_{43}(\Sigma^7 C\eta) \xrightarrow{p_*} \pi_{43}(S^9) \xrightarrow{\eta} \pi_{43}(S^8).$$

Since $h_2 n$ detects $\nu\{n\}$ and

$$\eta \cdot \nu\{n\} = 0,$$

there is an element α in $\pi_{43}(\Sigma^7 C\eta)$ such that $p_* \alpha = \nu\{n\}[9]$.

The element $h_2 n[9]$ in $\mathrm{Ext}(S^9)$ has Adams filtration 6, therefore by naturality, if it were not detected by $h_2 n[9]$ in $\mathrm{Ext}(\Sigma^7 C\eta)$, it would be detected by an element with Adams filtration at most 5.

From the same cofiber sequence, we have a short exact sequence on cohomology

$$0 \to H^*(S^9) \xrightarrow{p^*} H^*(\Sigma^7 C\eta) \xrightarrow{i^*} H^*(S^7) \to 0$$

and therefore a long exact sequence of Ext groups

$$\mathrm{Ext}^{s-1,t-1}(S^8) \xrightarrow{h_1} \mathrm{Ext}^{s,t}(S^7) \xrightarrow{i_\sharp} \mathrm{Ext}^{s,t}(\Sigma^7 C\eta) \xrightarrow{p_\sharp} \mathrm{Ext}^{s,t}(S^9).$$

This gives us the Adams E_2–page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \le 6$ in Table 1.

$s\backslash t-s$	42	43
6		$h_2 n[9]$, $t[7]$
5	$h_0 p[9]$, $h_2 d_1[7]$	
4	$p[9]$	$h_0^2 h_2 h_5[9]$
3		$h_0 h_2 h_5[9]$
2		$h_2 h_5[9]$

Table 1: The Adams E_2–page of $\Sigma^7 C\eta$ in the 42 and 43 stems for $s \le 6$

The element $h_2 h_5[9]$ must support a nontrivial differential, since its image $p_\sharp(h_2 h_5[9])$ supports a d_3 differential that kills $h_0 p[9]$ in the Adams spectral sequence of S^9.

The elements $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ survive and detect homotopy classes that map to $\{h_0 h_2 h_5\}[9]$ and $\{h_0^2 h_2 h_5\}[9]$ in $\pi_{43}(S^9)$. In fact, since there is no η–extension on $h_0 h_2 h_5$ and $h_0^2 h_2 h_5$, we can choose homotopy classes in $\pi_{43}(S^9)$, which are detected by $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ and are zero after multiplying by η. Therefore, they have nontrivial preimages under the map p_* in the long exact sequence of homotopy groups. For filtration reasons, their preimages must be detected by $h_0 h_2 h_5[9]$ and $h_0^2 h_2 h_5[9]$ in the Adams spectral sequence of $\Sigma^7 C\eta$.

Therefore, the only possibility left is $h_2 n[9]$, which completes the proof. □

We prove Proposition 4.3 in Step 1.

Proposition 4.3 *The elements $h_2n[9]$ and $h_1t[9]$ are permanent cycles in the Adams spectral sequence of $\Sigma^7 C\eta$, and hence also in that of P_7^9.*

Proof We have a relation in Ext,

$$h_2 \cdot h_2 n = h_1 t.$$

Therefore, $h_1t[9]$ is product of permanent cycles. The second claim follows from the naturality of the Adams spectral sequences. □

For Step 2, we first show the following lemma:

Lemma 4.4 *The element $h_1t[9]$ is not a boundary in the Adams spectral sequences of $\Sigma^7 C\eta$ and P_7^9.*

Proof The element $h_1t[9]$ is hit by a d_3 differential on $e_1[9]$.

In the Adams spectral sequence of S^9, we have the Bruner differential

$$d_3(e_1[9]) = h_1t[9].$$

However, the element $e_1[9]$ is not present in either $\mathrm{Ext}(\Sigma^7 C\eta)$ or $\mathrm{Ext}(P_7^9)$.

Therefore, by naturality, the element $h_1t[9]$ cannot be hit by any d_r differential for $r \leq 3$ in the Adams spectral sequence of $\Sigma^7 C\eta$ and P_7^9.

We have the Adams E_2–page of $\Sigma^7 C\eta$ and P_7^9 in the 46 and 47 stems for $s \leq 7$ in Table 2.

We need to rule out two candidates: $h_0h_3h_5[9]$ and $h_1h_3h_5[8]$.

In the Adams spectral sequence of S^9, we have a d_4 differential,

$$d_4(h_0h_3h_5[9]) = h_0^2 x[9].$$

By naturality of the quotient map to S^9, the element $h_0h_3h_5[9]$ cannot support a d_4 differential that kills $h_1t[9]$.

For the element $h_1h_3h_5[8]$, it is straightforward to check it is a permanent cycle in the Adams spectral sequence of P_7^8, and hence a permanent cycle in that of P_7^9. This rules out the candidate $h_1h_3h_5[8]$ and completes the proof. □

	$\mathrm{Ext}(\Sigma^7 C\eta)$		$\mathrm{Ext}(P_7^9)$	
$s\backslash t-s$	46	47	46	47
7	$h_1t[9]$ $h_0^2x[9]$	$\bullet$	$h_1t[9]$ $h_0^2x[9]$	$\bullet$ $\bullet$
6	$h_0x[9]$	$\bullet$ $\bullet$	$h_0x[9]$ $h_1x[8]$	$\bullet$ $\bullet$
5		$\bullet$ $\bullet$	$\bullet$	$\bullet$ $\bullet$
4	$\bullet$	$\bullet$ $\bullet$	$\bullet$ $\bullet$	$\bullet$ $\bullet$ $\bullet$
3	$\bullet$	$h_0h_3h_5[9]$	$\bullet$	$h_0h_3h_5[9]$ $h_1h_3h_5[8]$

Table 2: The Adams E_2–page of $\Sigma^7 C\eta$ and P_7^9 in the 46 and 47 stems for $s \le 7$

Remark 4.5 In $\mathrm{Ext}^{6,6+46}(P_7^9)$, the element $h_1x[8]$ is clearly a surviving cycle. There are two possibilities for the other element $h_0x[9]$: it is either killed by a d_3 differential from $h_0h_3h_5[9]$, or it survives and detects $\{h_1h_3h_5\}[7]$. We will leave it to the reader to figure out which way it goes.

We prove Proposition 4.6 in Step 2.

Proposition 4.6 *Under the inclusion map $S^7 \hookrightarrow \Sigma^7 C\eta$, the element $h_1t[9]$ detects the image of $\sigma\{d_1\}[7]$ in $\pi_{46}(\Sigma^7 C\eta)$. By naturality, the same statement is true after we further map it to $\pi_{46}(P_7^9)$.*

Proof By Lemma 4.2 and Proposition 4.3, the element $h_2n[9]$ survives in the Adams spectral sequence of $\Sigma^7 C\eta$, and detects a homotopy class that maps to $\nu\{n\}[9]$ under the quotient map

$$\Sigma^7 C\eta \twoheadrightarrow S^9.$$

By Lemma 4.4, the element $h_1t[9] = h_2 \cdot h_2n[9]$ survives and detects the homotopy class $\nu\{n\}[9]\cdot\nu$. As shown in the proof of Lemma 2.2, the element $h_3d_1 = h_1e_1$

detects an element in the Toda bracket

$$\langle \eta, \nu\{n\}, \nu \rangle.$$

Therefore, by Lemma 5.3, we have

$$\nu\{n\}[9] \cdot \nu = \langle \eta, \nu\{n\}, \nu \rangle[7] = \sigma\{d_1\}[7]$$

in $\pi_{46}(\Sigma^7 C\eta)$. □

Now we prove Step 3:

Proposition 4.7 *Under the inclusion map* $S^7 \hookrightarrow P_1^9$*, the element* $h_1 t[9]$ *in* $\mathrm{Ext}(P_1^9)$ *detects the image of* $\sigma\{d_1\}[7]$ *in* $\pi_{46}(P_1^9)$*.*

The idea of the proof of Proposition 4.7 is to make use of naturality of the Adams filtrations:

$$\begin{array}{ccccc} S^7 & \hookrightarrow & P_1^9 & \twoheadrightarrow & P_7^9 \\ h_3 d_1[7] & & & & h_1 t[9] \end{array}$$

The homotopy class $\sigma\{d_1\}[7]$ is detected by $h_3 d_1[7]$ in S^7, which has Adams filtration 5, while its image in $\pi_{46}(P_7^9)$ is detected by $h_1 t[9]$ by Proposition 4.6, which has Adams filtration 7. Therefore, to prove Proposition 4.7, we only need to rule out surviving cycles in the Adams filtration 6, which also lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞–page. Note that the element $h_3 d_1[7]$ is not present in $\mathrm{Ext}(P_1^9)$.

Proof We have the Adams E_2–page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \le 8$ in Table 3.

There are four elements in $\mathrm{Ext}^{6,6+46}(P_1^9)$:

$$Ph_1h_5[6], \quad h_0^2 g_2[2], \quad h_1 x[8], \quad h_0 x[9].$$

Remark 4.5 rules out the last two candidates, since they do not lie in the kernel of the map

$$P_1^9 \twoheadrightarrow P_7^9$$

in the Adams E_∞–page.

	$\mathrm{Ext}(P_1^9)$		$\mathrm{Ext}(P_1^\infty)$	
$s\backslash t-s$	46	47	46	47
8	$Ph_1^3h_5[4]$	$\bullet$	$Ph_1^3h_5[4]$	$\bullet$
	$\bullet$	$\bullet$		
	$\bullet$	$\bullet$		
7	$Ph_1^2h_5[5]$	$\bullet$	$Ph_1^2h_5[5]$	$\bullet$
	$h_1t[9]$	$\bullet$	$h_1t[9]$	$\bullet$
	$h_0^2x[9]$	$\bullet$		$\bullet$
		$\bullet$		
6	$Ph_1h_5[6]$	$\bullet$	$Ph_1h_5[6]$	$h_1h_5d_0[1]$
	$h_0^2g_2[2]$	$\bullet$	$h_0^2g_2[2]$	$h_1x[9]$
	$h_1x[8]$	$\bullet$	$h_1x[8]$	
	$h_0x[9]$			
5	$h_0^3h_3h_5[8]$	$\bullet$	$h_0^3h_3h_5[8]$	$h_1g_2[2]$
	$\bullet$	$\bullet$	$\bullet$	$h_1f_1[6]$
	$\bullet$	$\bullet$	$\bullet$	
			$\bullet$	
4	$\bullet$	$\bullet$	$h_1^3h_5[12]$	$h_0h_4^3[2]$
		$\bullet$		$g_2[3]$
		$\bullet$		$f_1[7]$
		$\bullet$		
3	$\bullet$	$\bullet$	$h_1^2h_5[13]$	
	$\bullet$		$\bullet$	
			$\bullet$	
2			$h_1h_5[14]$	$h_2h_5[13]$
1			$h_5[15]$	

Table 3: The Adams E_2–page of P_1^9 and P_1^∞ in the 46 and 47 stems for $s \leq 8$

In the table for the transfer map in [11], we have that the element $h_0^2g_2[2]$ maps to B_1. If the image of the homotopy class $\sigma\{d_1\}[7]$ were detected by $h_0^2g_2[2]$, then we would have a σ–extension from h_3d_1 to B_1 in $\pi_{46}S^0$, which by Lemma 2.2 is equivalent to an η–extension from h_1g_2 to B_1 in $\pi_{46}S^0$. However, the proof of Lemma 4.2.47 of [5] shows the latter is not true.

The only candidate left is $Ph_1h_5[6]$. To rule it out, we notice there is a long h_0 tower in the 46 stem of P_1^∞: from $h_5[15]$ to $Ph_1^3h_5[4]$. In particular, we have

$$h_0 \cdot Ph_1h_5[6] = Ph_1^2h_5[5], \quad h_0 \cdot Ph_1^2h_5[5] = Ph_1^3h_5[4].$$

Since

$$2 \cdot \sigma\{d_1\} = 0,$$

the image of the homotopy class $\sigma\{d_1\}[7]$ must have order 2. Therefore, we only need to show the element $Ph_1^2h_5[5]$ is not a boundary. In the following Lemma 4.8, we show that the elements in Adams filtration 4 to 6 of $\mathrm{Ext}(P_1^\infty)$ are all permanent cycles. This only leaves the possibility that $h_2h_5[13]$ kills $Ph_1^3h_5[4]$, but not $Ph_1^2h_5[5]$, and hence completes the proof. □

Lemma 4.8 *The elements in Adams filtration* 4 *to* 6 *of the* 47*–stem of* $\mathrm{Ext}(P_1^\infty)$ *are all permanent cycles.*

Proof There are seven elements:

$$h_1h_5d_0[1], \quad h_1x[9], \quad h_1g_2[2], \quad h_1f_1[6], \quad h_0h_4^3[2], \quad g_2[3], \quad f_1[7].$$

The spheres S^1, S^3 and S^7 are $H\mathbb{F}_2$–subcomplexes of P_1^∞ by the solution of the Hopf invariant one problem. Since the elements $h_1h_5d_0$, g_2 and f_1 are permanent cycles in the Adams spectral sequence for S^0, The elements $h_1h_5d_0[1]$, $h_1g_2[2]$ and $f_1[7]$ are permanent cycles.

The element $h_1f_1[6] = h_0 \cdot f_1[7]$ is therefore also a permanent cycle.

It is straightforward to show that the elements $h_1g_2[2]$ and $h_0h_4^3[2]$ are permanent cycles in the Adams spectral sequence of P_1^2. By naturality, they are permanent cycles in that of P_1^∞.

For the element $h_1x[9]$, one uses the $H\mathbb{F}_2$–subcomplex of P_1^∞ which contains cells in dimensions 3, 5, 7 and 9 to show that it is a permanent cycle. In fact, by comparing the Atiyah–Hirzebruch spectral sequence with the Adams spectral sequence of this 4 cell complex, it follows from the following relations in the stable homotopy groups of spheres:

$$0 \in \eta \cdot \{h_1x\}, \quad 0 \in \langle \nu, \eta, \{h_1x\} \rangle.$$

The homotopy class $\{h_1x\}[9]$ survives in the Atiyah–Hirzebruch spectral sequence, and is detected by $h_1x[9]$ in its Adams E_2–page. In particular, $h_1x[9]$ is a permanent cycle in the Adams spectral sequence of this 4 cell complex, and therefore also a permanent cycle in the Adams spectral sequence of P_1^∞. □

Now we prove Step 4.

Lemma 4.9 *The element $h_1t[9]$ maps to N under the transfer map.*

Proof We check the two tables in the appendix of [11]. See [11] for more details of the Lambda algebra notation we use here. The element N is in $\mathrm{Ext}^{8,8+46}(S^0) = \mathbb{Z}/2$. Checking the table for P_1^∞, we have that

$$\mathrm{Ext}^{7,7+46}(P_1^\infty) = (\mathbb{Z}/2)^2, \text{ generated by (5) 11 12 4 5 3 3 3 and (9) 3 5 7 3 5 7 7,}$$

which means $\mathrm{Ext}^{7,7+46}(P_1^\infty)$ is generated by $Ph_1^2h_5[5]$ and $h_1t[9]$. Since $Ph_1^2h_5[5]$ is divisible by h_0 in $\mathrm{Ext}(P_1^\infty)$, while N is not divisible by h_0 in $\mathrm{Ext}(S^0)$, $Ph_1^2h_5[5]$ cannot map to N under the transfer map. By the algebraic Kahn–Priddy theorem, the other generator $h_1t[9]$ has to map to N. □

5 A lemma for extensions in the Atiyah–Hirzebruch spectral sequence

Let $\alpha\colon Y \to X$ and $\beta\colon Z \to Y$ be homotopy classes of maps between spectra. Suppose that the composite $\alpha\beta = 0$. Let $C\alpha$ and $C\beta$ be the cofiber of α and β, respectively.

We have cofiber sequences

$$Y \xrightarrow{\alpha} X \xrightarrow{i_\alpha} C\alpha \xrightarrow{\partial_\alpha} \Sigma Y,$$
$$Z \xrightarrow{\beta} Y \xrightarrow{i_\beta} C\beta \xrightarrow{\partial_\beta} \Sigma Z.$$

Denote by $L^\alpha\beta$ be the set of maps in $[\Sigma Z, C\alpha]$ such that the composite

$$\Sigma Z \to C\alpha \xrightarrow{\partial_\alpha} \Sigma Y$$

is $-\Sigma\beta$. The indeterminacy of the set $L^\alpha\beta$ is

$$i_\alpha \cdot [\Sigma Z, X].$$

Similarly, denote by $L_\beta\alpha$ be the set of maps in $[C\beta, X]$ such that the composite

$$Y \xrightarrow{i_\beta} C\beta \to X$$

is α. The indeterminacy of the set $L_\beta\alpha$ is

$$[\Sigma Z, X]\cdot \partial_\beta.$$

Lemma 5.1 *The two sets of maps* $L^\alpha\beta\cdot\partial_\beta$ *and* $i_\alpha\cdot L_\beta\alpha$ *in* $[C\beta, C\alpha]$ *are equal.*

Proof It is clear that the indeterminacy of the two sets are given by the composition

$$i_\alpha\cdot[\Sigma Z, X]\cdot\partial_\beta.$$

We need to show that they contain one common element. We have

$$\begin{array}{ccccccccccc}
 & & Y & \xrightarrow{\alpha} & X & \xrightarrow{i_\alpha} & C\alpha & \xrightarrow{\partial_\alpha} & \Sigma Y & \xrightarrow{-\Sigma\alpha} & \Sigma X \\
 & & \| & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle g} & & \| & & \\
Z & \xrightarrow{\beta} & Y & \xrightarrow{i_\beta} & C\beta & \xrightarrow{\partial_\beta} & \Sigma Z & \xrightarrow{-\Sigma\beta} & \Sigma Y & &
\end{array}$$

Take $f\in L_\beta\alpha$. Since both lines are cofiber sequences, there exists a coextension $g\in L^\alpha\beta$ such that the diagram commutes. The commutativity of the middle square gives the claim. □

Lemma 5.2 *Let*

$$W\xrightarrow{\gamma} Z\xrightarrow{\beta} Y\xrightarrow{\alpha} X$$

be a sequence of homotopy classes of maps. Suppose that $\alpha\beta=0$ *and* $\beta\gamma=0$. *Then the two sets of maps* $-L^\alpha\beta\cdot\Sigma\gamma$ *and* $i_\alpha\cdot L_\beta\alpha\cdot L^\beta\gamma$ *in* $[\Sigma W, C\alpha]$ *are equal.*

Proof First, the indeterminacy of the former is $i_\alpha\cdot[\Sigma Z, X]\cdot\Sigma\gamma$. The indeterminacy of the latter is

$$i_\alpha\cdot[\Sigma Z, X]\cdot\partial_\beta\cdot L^\beta\gamma + i_\alpha\cdot L_\beta\alpha\cdot i_\beta\cdot[\Sigma W, Y].$$

Note that $\partial_\beta\cdot L^\beta\gamma=-\Sigma\gamma$ and $i_\alpha\cdot L_\beta\alpha\cdot i_\beta = i_\alpha\cdot\alpha=0$. So the two sets have the same indeterminacy.

We have the following diagram:

$$\begin{array}{ccccccccccccc}
W & \xrightarrow{\gamma} & Z & \xrightarrow{i_\gamma} & C\gamma & \xrightarrow{\partial_\gamma} & \Sigma W & \xrightarrow{-\Sigma\gamma} & \Sigma Z & & & & \\
\downarrow{\scriptstyle \Sigma^{-1}L^\beta\gamma} & & \| & & \downarrow{\scriptstyle L_\gamma\beta} & & \downarrow{\scriptstyle L^\beta\gamma} & & \| & & & & \\
\Sigma^{-1}C\beta & \xrightarrow{-\Sigma^{-1}\partial_\beta} & Z & \xrightarrow{\beta} & Y & \xrightarrow{i_\beta} & C\beta & \xrightarrow{\partial_\beta} & \Sigma Z & \xrightarrow{-\Sigma\beta} & \Sigma Y & & \\
\downarrow{\scriptstyle \Sigma^{-1}L_\beta\alpha} & & \downarrow{\scriptstyle \Sigma^{-1}L^\alpha\beta} & & \| & & \downarrow{\scriptstyle L_\beta\alpha} & & \downarrow{\scriptstyle L^\alpha\beta} & & \| & & \\
\Sigma^{-1}X & \xrightarrow{-\Sigma^{-1}i_\alpha} & \Sigma^{-1}C\alpha & \xrightarrow{-\Sigma^{-1}\partial_\alpha} & Y & \xrightarrow{\alpha} & X & \xrightarrow{i_\alpha} & C\alpha & \xrightarrow{\partial_\alpha} & \Sigma Y & &
\end{array}$$

By Lemma 5.1, with suitable choices, all the squares commute, so the claim follows. In fact, taking any choices of $L^\beta\gamma$ and $L_\beta\alpha$, Lemma 5.1 says there exist choices for $L_\gamma\beta$ and $L^\alpha\beta$ making the diagrams commute. □

Now we have the following lemma as a corollary of Lemma 5.2 when the spectra X, Y, Z and W are all spheres.

Lemma 5.3 *Let α, β and γ be maps between spheres,*

$$\alpha\colon S^{|\alpha|} \to S^0, \quad \beta\colon S^{|\alpha|+|\beta|} \to S^{|\alpha|}, \quad \gamma\colon S^{|\alpha|+|\beta|+|\gamma|} \to S^{|\alpha|+|\beta|}.$$

Then, in the Atiyah–Hirzebruch spectral sequence of $C\alpha$, we have a γ–extension

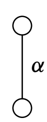
$$\beta[|\alpha|+1]\cdot\Sigma\gamma = \langle\alpha,\beta,\gamma\rangle[0].$$

Proof By definition, the set of classes represented by $\beta[|\alpha|+1]$ in the Atiyah–Hirzebruch spectral sequence is $-L^\alpha\beta$. On the other hand, by definition, $L_\beta\alpha\cdot L^\beta\gamma$ is $\langle\alpha,\beta,\gamma\rangle$, and $i_\alpha\cdot L_\beta\alpha\cdot L^\beta\gamma$ is $\langle\alpha,\beta,\gamma\rangle[0]$. So the claim follows from Lemma 5.2. □

Appendix

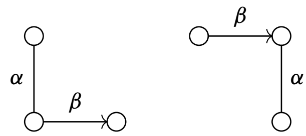
In this appendix, we use cell diagrams as intuition for the statements of the lemmas in Section 5. It is very helpful when thinking of CW spectra. See [1; 12; 13] for example. For simplicity, we restrict to the cases when the spectra X, Y, Z and W are all spheres. For the definition of cell diagrams, see [1].

Let α and β be classes in the stable homotopy groups of spheres such that $\beta\cdot\alpha = 0$. We denote the cofiber of α by

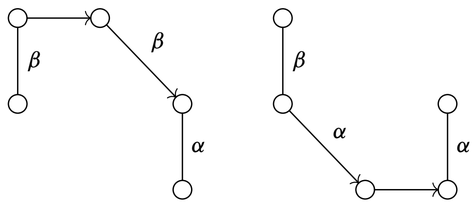


We denote the maps i_α and ∂_α by

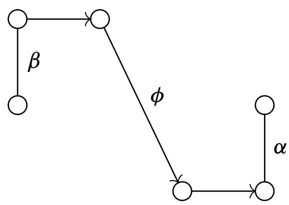
and the extension and coextension maps $L_\alpha\beta$ and $L^\alpha\beta$ by



Then Lemma 5.1 says the following two sets of maps are equal:

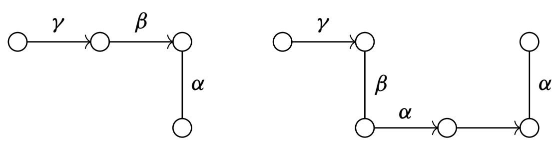


with the same indeterminacy



where $\phi \in \pi_{|\alpha|+|\beta|+1}S^0$ could be any class.

Suppose further that $\beta \cdot \gamma = 0$. Precomposing with $L^\beta\gamma$, Lemma 5.3 says that the following two sets of maps are equal:



References

[1] **M G Barratt**, **J D S Jones**, **M E Mahowald**, *Relations amongst Toda brackets and the Kervaire invariant in dimension* 62, J. London Math. Soc. 30 (1984) 533–550 MR

[2] **M G Barratt**, **M E Mahowald**, **M C Tangora**, *Some differentials in the Adams spectral sequence, II*, Topology 9 (1970) 309–316 MR

[3] **R Bruner**, *A new differential in the Adams spectral sequence*, Topology 23 (1984) 271–276 MR

[4] **R Bruner**, *The cohomology of the mod 2 Steenrod algebra: a computer calculation*, preprint (2006) Available at `http://www.math.wayne.edu/~rrb/papers/cohom.pdf`

[5] **D C Isaksen**, *Stable stems*, preprint (2014) arXiv

[6] **D C Isaksen**, **Z Xu**, *Motivic stable homotopy and the stable* 51 *and* 52 *stems*, Topology Appl. 190 (2015) 31–34 MR

[7] **D S Kahn**, **S B Priddy**, *The transfer and stable homotopy theory*, Math. Proc. Cambridge Philos. Soc. 83 (1978) 103–111 MR

[8] **W H Lin**, *Algebraic Kahn–Priddy theorem*, Pacific J. Math. 96 (1981) 435–455 MR

[9] **R M F Moss**, *Secondary compositions and the Adams spectral sequence*, Math. Z. 115 (1970) 283–310 MR

[10] **M C Tangora**, *Some Massey products in* Ext, from "Topology and representation theory" (E M Friedlander, M E Mahowald, editors), Contemp. Math. 158, Amer. Math. Soc., Providence, RI (1994) 269–280 MR

[11] **G Wang**, **Z Xu**, *The algebraic Atiyah–Hirzebruch spectral sequence of real projective spectra*, preprint (2016) arXiv

[12] **G Wang**, **Z Xu**, *The triviality of the* 61*–stem in the stable homotopy groups of spheres*, Ann. of Math. 186 (2017) 501–580 MR

[13] **Z Xu**, *The strong Kervaire invariant problem in dimension* 62, Geom. Topol. 20 (2016) 1611–1624 MR

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